Title: Wilson loops in 4d SYM, in 2d YM and in 0d

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Abstract: I will present a construction of supersymmetric Wilson loop operators in N=4 SYM for an arbitrary path on an S3 subspace of space-time. I will show how they are evaluated in AdS and in particular that the string world-sheet is a generalized calibration with respect to an almost-complex structure associate to the supersymmetries preserved by the loop. I will present some special examples and in the case when the loop is restricted to an S2, some evidence that the calculation reduces to a perturbative calculation in YM in 2-dimensions on S2. This in turn is exactly soluble in terms of a 0-dimensional matrix model.

Wilson loops in 4d SYM, in 2d YM and in 0d

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based on arXiv:0704.2237, arXiv:0707.2699 and work in progress with
Simone Giombi, Riccardo Ricci and Diego Trancanelli.

Introduction and motivation

In this talk I study supersymmetric Wilson loops in $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions. Like all gauge theories it has the vector fields A_μ, with $\mu = 1, \cdots 4$ (I will work in Euclidean space). In addition there are 4 fermi fields Ψ^A (with $A = 1, \cdots 4$) and six scalars Φ^I (with $I = 1, \cdots 6$).

I will present results both from the gauge theory side and the dual string theory on $AdS_5 \times S^5$, where those Wilson loops are described by macroscopic semi-classical strings.

$\mathcal{N}=4$ SYM has a remarkable amount of symmetry. The Poincaré symmetry is enlarged to a the conformal supergroup $PSU(2,2|4)$, whose even part is $SO(5,1) \times SO(6)$. The symmetry generators are

$J_{\mu\nu}$,	rotations/Lorentz transformations .
P_μ ,	translations .
K_μ ,	special conformal transformations .
D ,	dilation .
L_{AB} ,	SU(4) rotations .
$Q^A_\alpha, \bar{Q}^A_{\dot\alpha}$,	Poincaré supersymmetries .
$S^A_\alpha, \bar{S}^A_{\dot\alpha}$,	superconformal symmetries .

Like in other supersymmetric theories there are certain operators that preserve some of the supersymmetries and will have special properties. Consider the complex combinations of the scalar fields

$$X = \Phi^1 + i\Phi^2 , \quad Y = \Phi^3 + i\Phi^4 , \quad Z = \Phi^5 + i\Phi^6 .$$

Each is charged under a $U(1)$ subgroup of $SO(6)$.

It is easy to show that operators made only from holomorphic combinations of those fields will preserve some supersymmetries, i.e. they will be annihilated by some of the Q generators. For example the variation of Z^J is

$$\delta Z^J \simeq J\,\bar{\Psi}(\rho^5 + i\rho^6)\epsilon(x)\,.$$

where the ρ's are $SO(6)$ gamma matrices and ϵ is made of two 16-component spinors, one constant and one a conformal Killing spinor

$$\epsilon(x) = \epsilon_0 + x^\mu \gamma_\mu \epsilon_1\,.$$

For an operator at the origin the variation will be zero for arbitrary ϵ_1 and half of the ϵ_0. This means that Z^J is annihilated by all the S and by half of the Qs.

Such an operator is a chiral primary, and acting on it with the Q operators (which do not annihilate it) will generate the supermultiplet it belongs to.

A lot is know about local operators in this gauge theory. Those supersymmetric operators have protected dimensions and 3-point functions, which are given completely by their charges. Actually in the study of local operators a lot of progress has been achieved in the past few years and there is a good understanding of the spectrum of very long operators, even those not preserving any supersymmetry and a good agreement between the gauge theory and the dual string theory in $AdS_5 \times S^5$.

In this talk I will not discuss local operators, but rather Wilson loops. Those can be also supersymmetric, as I will show, but a lot less is known about them. Those operators form a very interesting set of observables, in a confining theory they exhibit the famous area-law. In our case they will not confine, but they still provide non-local data on the theory. In the string dual they are described by semi-classical strings (or branes) and therefore touch on stringy properties of the gauge theory.

In any gauge theory one may define Wilson loop operators as

$$W = \mathrm{Tr}\,\mathcal{P}\exp i\oint A_\mu\, dx^\mu\,,$$

along a closed path $x^\mu(s)$. Those operators turn out to be supersymmetric only when the path is light-like (which of course will not occur in the Euclidean theory).

To find supersymmetric operators it's possible to add extra terms, couplings to the Fermi-fields and scalars. I will consider the modification

$$W = \mathrm{Tr}\,\mathcal{P}\exp\oint(iA_\mu\dot{x}^\mu + |\dot{x}|\Theta^I\Phi^I)ds\,,$$

where Θ^I are arbitrary functions of s.

How does one choose those Θ^I to get supersymmetric operators?

Outline

- Introduction and motivation.
- The "Zarembo loops"
- The Wilson loops on S^3
- Calibration equation in $AdS_5 \times S^5$.
- Examples:
  - ◇ $1/2$ BPS: Circle
  - ◇ $1/4$ BPS: Hopf fibers, latitude, longitudes
  - ◇ $1/8$ BPS: S^2 observables
  - ◇ $1/8$ BPS: Zarembo limit
- YM_2 and the matrix model
- Discussion.

The "Zarembo loops"

Consider the straight Wilson loop in the x^1 direction coupled to the scalar Φ^1

$$W = \operatorname{Tr}\mathcal{P}\exp\oint(iA_1 + \Phi^1)dx^1\,.$$

Its supersymmetry variation will be proportional to $(i\gamma^1 + \rho^1)\epsilon(x)$. This combination of gamma matrices has half vanishing eigenvalues, so this loop is $1/2$ BPS.

The same will be true if we take the line in the x^2 direction coupled to the scalar Φ^2, and gets the projector $i\gamma^2 + \rho^2$ which commutes with the above one, so the combined system is $1/4$ BPS. More generally, one can consider an **arbitrary** curve, and if A_μ is always accompanied by ${M_\mu}^I\Phi_I$ (with some norm-preserving matrix M), the loop will be supersymmetric. At every point along the loop we will find a linear combination of the equations above.

If the curve is confined to a straight line, the loop will be $1/2$ BPS, in a plane, $1/4$, in $\mathbb{R}^3$ it will be $1/8$, and for an arbitrary curve, $1/16$.

Quite amazingly all those loops have expectation value unity, which can be seen (perturbatively) both in the gauge theory and in string theory. There are also arguments why this would apply to all orders in perturbation theory.

This is an amazing fact, and those operators are quite interesting, but there isn't much to calculate with them... Also this family of loops does not include the $1/2$ BPS circular loop. This is a circle coupled to only one scalar, which turns out to be supersymmetric too. That operator has non-trivial VEV, and we want to find generalizations of it in the rest of the talk.

The Wilson loops on S^3

The previous construction effectively replaced

$$A_\mu \to A_\mu + i\Phi_\mu$$

where four of the scalars are related to a vector in space-time by $\Phi_\mu = M_\mu{}^I \Phi_I$, which can be considered as a topological twist.

We will use a different twist that will relate three of the scalars to a self-dual tensor $\Phi_{\mu\nu} = \sigma^i_{\mu\nu} M_i{}^I \Phi_I$. These σ^i give the decomposition into Pauli matrices of the Lorentz group generators in the anti-chiral representation

$$\gamma_{\mu\nu}\epsilon^- = i\sigma^i_{\mu\nu}\tau^R_i\epsilon^- .$$

They are essentially the same as the 't Hooft symbol η and are also related to the invariant 1-forms on S^3

$$\sigma_1^{R,L} = 2\left[\pm(x^2dx^3 - x^3dx^2) + (x^4dx^1 - x^1dx^4)\right]$$
$$\sigma_2^{R,L} = 2\left[\pm(x^3dx^1 - x^1dx^3) + (x^4dx^2 - x^2dx^4)\right]$$
$$\sigma_3^{R,L} = 2\left[\pm(x^1dx^2 - x^2dx^1) + (x^4dx^3 - x^3dx^4)\right].$$

by $\sigma^{iR} = -2\sigma^i_{\mu\nu}x^\mu dx^\nu$.

Now we can write our loops as

$$W = \frac{1}{N}\mathrm{Tr}\,\mathcal{P}\exp\oint\left(iA + \frac{1}{2}\sigma_i^R M^i{}_I\Phi^I\right).$$

which by the above discussion is the replacement

$$A_\mu \to A_\mu - i\Phi_{\mu\nu}x^\nu$$

(I will take M that identifies i and I).

We should verify that this construction indeed leads to supersymmetric operators. The variation of the loop gives

$$\delta W \simeq \left(i\dot{x}^\mu \gamma_\mu + \sigma^i_{\mu\nu}\dot{x}^\mu x^\nu \rho^i\right) \epsilon(x) \,.$$

Expanding $\epsilon(x)$, the terms with 0 and 2 γ's are

$$\dot{x}^\mu x^\nu (i\gamma_{\mu\nu}\epsilon_1 + \sigma^i_{\mu\nu}\rho^i\epsilon_0) \,.$$

For this to vanish for a general curve we get

$$i\gamma_{\mu\nu}\epsilon_1 = -\sigma^i_{\mu\nu}\rho^i\epsilon_0 \,.$$

This can be solved, since σ is related to the action of γ on anti-chiral spinors. So the two spinors should be anti-chiral and in addition (τ^R are representation matrices for the anti-chiral $SU(2)_R$)

$$\tau^R_i \epsilon_1^- = \rho^i \epsilon_0^- \,.$$

Imposing this condition also guarantees that the terms with a single γ matrix in the equation vanish.

Consider the breaking of $SO(6)$ to $SU(2)_A \times SU(2)_B$, where the loop leaves $SU(2)_B$ unbroken but the SUSY equations involve three of the ρ's, which are the generators of $SU(2)_A$

Eliminating ϵ_0^R from the last equation we get the relation

$$(\tau_i^R + \rho_i)\epsilon_1^- = 0\,,$$

and the same is true for ϵ_0^-.

So under the sum of the two groups $SU(2)_R + SU(2)_A$, the supercharges have to be singlets. Using $\dot{a}$, a indices for $SU(2)_A$ and $SU(2)_B$, the supercharges preserved by the loops are

$$\bar{Q}^a = \varepsilon^{\dot{\alpha}\dot{a}}\left(\bar{Q}^a_{\dot{\alpha}\dot{a}} - \bar{S}^a_{\dot{\alpha}\dot{a}}\right).$$

For special curves, when the pull-backs of the forms are not independent, there will be more solutions and the Wilson loops will preserve more supersymmetry. We will demonstrate this in some special cases below.

Calibration equation in $AdS_5 \times S^5$

Imposing SUSY on the gauge theory side introduced the relations

$$i\gamma_{\mu\nu}\epsilon_1 = \sigma^i_{\mu\nu}\rho^i\epsilon_1 \,.$$

And one can also check that

$$i\gamma_\mu\rho_i\epsilon_1 = -\sigma^i_{\mu\nu}\gamma^\nu\epsilon_1 \,.$$

Together with the $SU(2)_A$ algebra

$$i\rho_{ij}\epsilon_1 = -\varepsilon_{ijk}\rho^k\epsilon_1 \,,$$

this can be viewed as a multiplication rule relating the seven gamma matrices.

We will use this as a guide for our construction of supersymmetric string surfaces in $AdS_5 \times S^5$. The seven gamma matrices (with

curved-space indices) will satisfy a similar multiplication rule

$$i\Gamma_{MN}\epsilon_1 = -\Sigma^L_{MN}\Gamma^L\epsilon_1 \,.$$

Now consider the $AdS_4 \times S^2$ subspace with metric

$$ds^2 = \frac{1}{z^2}dx^\mu dx^\mu + z^2 dy^i dy^i \,. \qquad y^i y^i = \frac{1}{z^2} \,.$$

The above product allows us to introduce the linear operator on the tangent space $\mathcal{J}^M_N = \Sigma^M_{NL}X^L$, where X^L represents the seven coordinates. Explicitly

$$\mathcal{J} = \begin{pmatrix} z^2\begin{pmatrix} 0 & y_3 & -y_2 & y_1 \\ -y_3 & 0 & y_1 & y_2 \\ y_2 & -y_1 & 0 & y_3 \\ -y_1 & -y_2 & -y_3 & 0 \end{pmatrix} & z^2\begin{pmatrix} x_4 & -x_3 & x_2 \\ x_3 & x_4 & -x_1 \\ -x_2 & x_1 & x_4 \\ -x_1 & -x_2 & -x_3 \end{pmatrix} \\ z^{-2}\begin{pmatrix} -x_4 & -x_3 & x_2 & x_1 \\ x_3 & -x_4 & -x_1 & x_2 \\ -x_2 & x_1 & -x_4 & x_3 \end{pmatrix} & z^2\begin{pmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{pmatrix} \end{pmatrix}$$

This matrix is in fact an almost complex structure on the $AdS_4 \times S^2$ where the string dual of the Wilson loops should live.

The construction is similar to that of the almost complex structure on S^6 related to the multiplication rules of the imaginary octonions.

Our claim is that the strings describing those Wilson loops will be calibrated with respect to this almost complex structure. So the complex structure on the world-sheet agrees with the pullback from space-time

$$\mathcal{J}^M_N \partial_\alpha X^N = \sqrt{g}\varepsilon_{\alpha\beta}\partial^\beta X^M \,.$$

Some straight-forward algebra allows us to prove that any surface satisfying this first-order equation will preserve the same supersymmetries as the Wilson loops in the gauge theory.

Moreover one can use this to simplify the expression for the classical action, unfortunately, we do not know how to evaluate it without an explicit solution.

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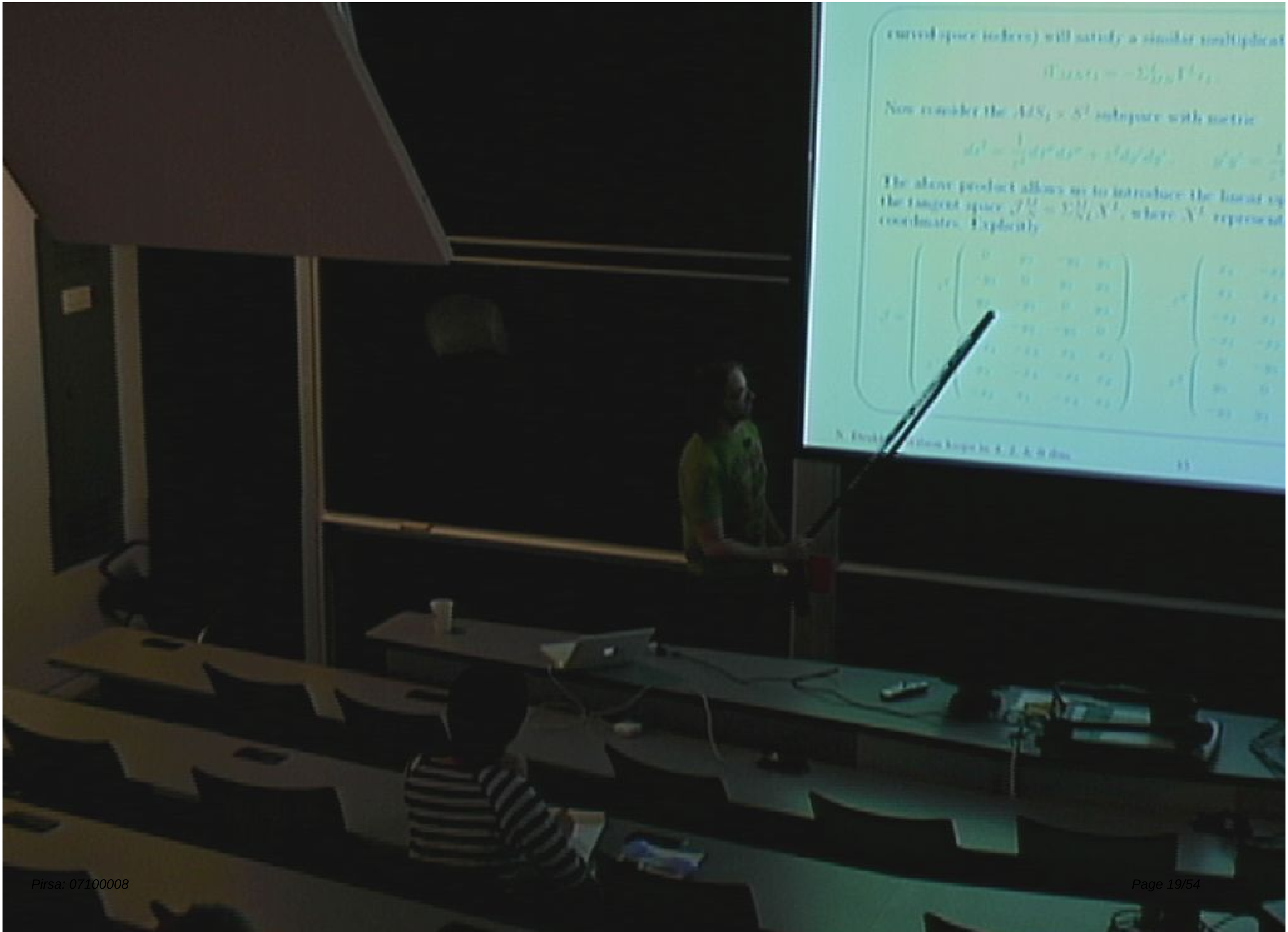
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Moreover one can use this to simplify the expression for the classical action, unfortunately, we do not know how to evaluate it without an explicit solution.

Examples:

$1/2$ BPS: Circle

Consider a circle in the x^1, x^2 plane. Along this curve

$$\sigma_{12}^{1R} = \sigma_{12}^{2R} = 0\,, \qquad \sigma_{12}^{3R} = |\dot{x}|\,.$$

So it will couple to a single scalar Φ^3. Most of the constraints on ϵ derived before for the general curve do not appear now, and this loop is annihilated by half the supersymmetries. Of the $PSU(2,2|4)$ of the vacuum it preserves the supergroup $OSp(4^\star|4)$.

An interesting fact about this loop is that in perturbation theory the combined propagator, including both the scalar and the gauge field

$$\frac{g_{YM}^2\delta_{ab}}{4\pi^2}\frac{1-\dot{x}_{(1)}\cdot\dot{x}_{(2)}}{(x_{(1)}-x_{(2)})^2} = \frac{g_{YM}^2\delta_{ab}}{4\pi^2}\frac{1-\dot{x}_{(1)}\cdot\dot{x}_{(2)}}{(2-2x_{(1)}\cdot x_{(2)})} = \frac{g_{YM}^2\delta_{ab}}{8\pi^2}\,.$$

between two arbitrary points is just a constant.

This allows one to represent all ladder diagrams in terms of a 0-dimensional Gaussian matrix model and calculate it exactly

$$\langle W \rangle_{\text{ladders}} = \frac{1}{Z} \int \mathcal{D}M \, \frac{1}{N} \operatorname{Tr} e^{M} \, e^{-\frac{2}{g^2} \operatorname{Tr} M^2} \, .$$

The result of this integral can be expressed in terms of a Laguerre polynomial and it can then be studied in the large N and/or the large g^2N regime, and compared to string theory on $AdS_5 \times S^5$.

In this calculation the planar result at large g^2N is given by a semiclassical string, whose action agrees with the leading exponent in the strong coupling expansion of the matrix model. It was even possible to go beyond the planar approximation and calculate **all** the $1/N$ corrections by comparing to a certain D3-brane in AdS, and again the results exactly agreed.

1/4 BPS: More circles (1)

I will take now more than one circle, but will make sure that all of them couple to the same scalar Φ^3. To do that within our framework, parameterize the sphere by

$$x^1 = -\sin\frac{\theta}{2}\sin\frac{\psi-\phi}{2}\,, \qquad x^2 = \sin\frac{\theta}{2}\cos\frac{\psi-\phi}{2}\,,$$
$$x^3 = \cos\frac{\theta}{2}\sin\frac{\psi+\phi}{2}\,, \qquad x^4 = \cos\frac{\theta}{2}\cos\frac{\psi+\phi}{2}\,,$$

Each of the circles will have constant θ and ϕ while ψ will vary along them. This construction is related to the writing of S^3 as a Hopf-fibration. Each circle will be along a different fiber.

This combined system of several different circles will preserve eight supercharges, all of them with the same chirality. An amazing fact, that is quite simple to check, is that even between different circles the propagator is the same constant as before.

So two coincident circles and two that are separated in this way have exactly the same interactions. This is analogous to the fact that parallel lines in flat space do not interact. Here they do interact, but this interaction is independent of the relative position.

We can again calculate them using the matrix model and they will be more complicated observables

$$\langle W \rangle_{\text{ladders}} = \left\langle \left(\frac{1}{N} \operatorname{Tr} e^{M} \right)^{k} \right\rangle_{\text{matrix model}} .$$

For k circles.

At the planar level this is just the same as k non-interacting circles, and in $AdS_5 \times S^5$ will be given by k independent string surfaces. We have not calculated the connected part from the string side.

$1/8$ BPS: S^2 observables

Let us consider now a much more general loop, any curve on a maximal S^2 inside our S^3. All those loops will preserve 4 supercharges, since on S^2 the forms satisfy

$$\sigma_i^L = -\sigma_i^R = -2\epsilon_{ijk}x^j\, dx^k\,.$$

So in addition to the two anti-chiral supercharges, those loops preserve two chiral supercharges.

A cute fact is that if we have a curve on $\vec{x}(s)$ on S^2, it will have gauge couplings $\dot{\vec{x}}$ and scalar couplings $\vec{x}\times\dot{\vec{x}}$. Note that this if we take $|\dot{x}|=1$, then this is also a vector on S^2, so we can consider the Wilson loop with that shape. Its scalar coupling would be

$$(\vec{x}\times\dot{\vec{x}})\times(\vec{x}\times\ddot{\vec{x}}) = -\vec{x}(\dot{\vec{x}}\cdot(\vec{x}\times\ddot{\vec{x}}) \propto \vec{x}\,.$$

This suggests a duality between the gauge and scalar couplings. we still do not understand the significance of this relation.

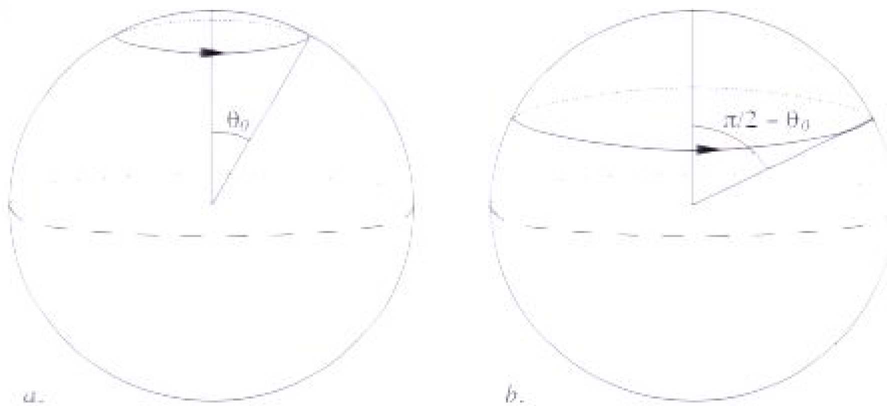
$1/4$ BPS: Small circle, "latitude"

Now take a non-maximal circle, or a latitude on S^2. Explicitly,

$$\vec{x} = (\sin\theta_0 \cos t, \sin\theta_0 \sin t, \cos\theta_0) .$$

$$\vec{x} \times \dot{\vec{x}} = \sin\theta_0(-\cos\theta_0 \cos t, -\cos\theta_0 \sin t, \sin\theta_0) .$$

Here you see an explicit example of this duality, which is just $\theta_0 \leftrightarrow \pi/2 - \theta_0$.



Those loops preserve $1/4$ of the supersymmetry. Here too the propagators are constants, proportional to $\sin^2\theta_0$, leading to the same matrix model as in the $1/2$ BPS case with the replacement

$g^2N \to g^2N\sin^2\theta_0$.

Those loops can also be calculated by a string in $AdS_5 \times S^5$ and the result is that the classical action of the string is $S = -\sqrt{g^2N}\,|\sin\theta_0|$, so the same scaling of g^2N works, and matches the large g^2N result of the matrix model. Furthermore, in string theory there is a second saddle point with the sign of the action reversed, which matches another term in the matrix model.

In the limit of small θ_0, of infinitesimal loops, the string is very "small" and one can calculate its fluctuations. Considering just the contribution of the zero modes broken by the small θ_0 gives an answer that matches with the full planar result of the matrix model.

Finally one can also calculate this loop using a D3-brane rather than a fundamental string and again the result agrees with the perturbative matrix model, this time including all $1/N$ corrections at large g^2N.

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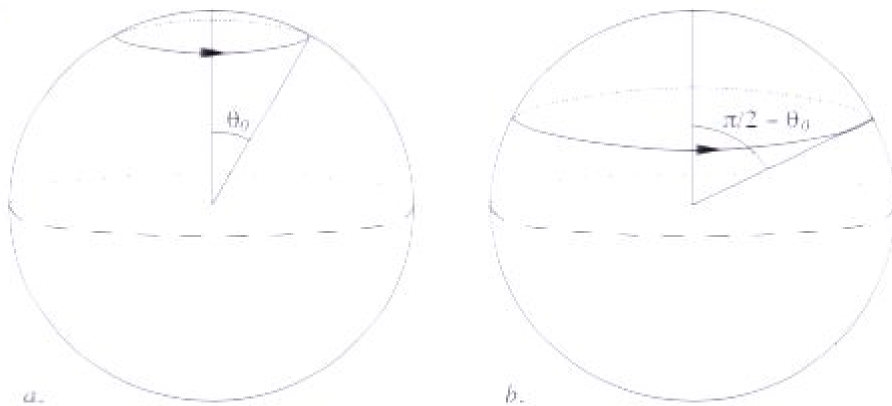


Those loops preserve $1/4$ of the supersymmetry. Here too the propagators are constants, proportional to $\sin^2\theta_0$, leading to the same matrix model as in the $1/2$ BPS case with the replacement

$g^2N \to g^2N\sin^2\theta_0$.

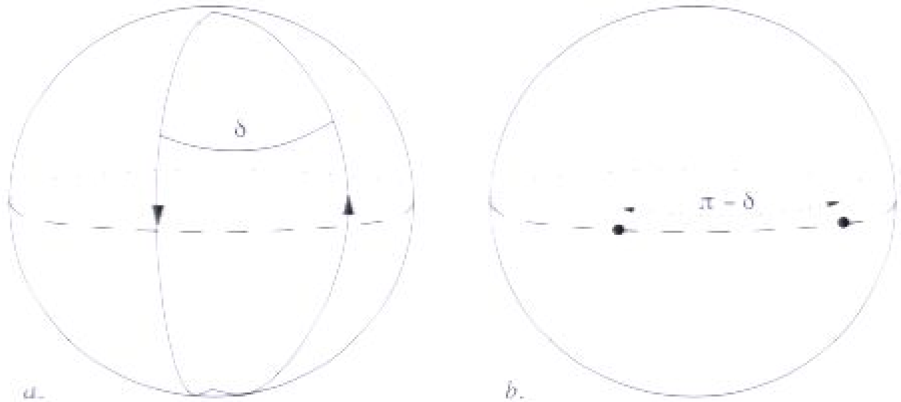
Those loops can also be calculated by a string in $AdS_5 \times S^5$ and the result is that the classical action of the string is $S = -\sqrt{g^2N}\,|\sin\theta_0|$, so the same scaling of g^2N works, and matches the large g^2N result of the matrix model. Furthermore, in string theory there is a second saddle point with the sign of the action reversed, which matches another term in the matrix model.

In the limit of small θ_0, of infinitesimal loops, the string is very "small" and one can calculate its fluctuations. Considering just the contribution of the zero modes broken by the small θ_0 gives an answer that matches with the full planar result of the matrix model.

Finally one can also calculate this loop using a D3-brane rather than a fundamental string and again the result agrees with the perturbative matrix model, this time including all $1/N$ corrections at large g^2N.

1/4 BPS: More circles (2), "longitudes"

Take a loop made of two halves of large circles.



That is, going from the north pole of S^2 to the southern one and then back along a different longitude to the northern one.

$$x^\mu = (\sin t, 0, \cos t, 0)\,, \qquad 0 \le t \le \pi\,,$$

$$x^\mu = (-\cos\delta \sin t, -\sin\delta \sin t, \cos t, 0)\,, \qquad \pi \le t \le 2\pi\,.$$

The loop will couple to Φ^2 along the first arc and to $-\Phi^2 \cos\delta + \Phi^1 \sin\delta$ along the second one.

This loop will preserve also 1/4 of the supersymmetries. By a stereographic map to the plane one gets a cusp with opening angle δ, where along each of the rays it will couple to the combination of scalars written above. This new loop is of the class studied by Zarembo, therefore it has trivial expectation value.

But the longitudes on S^2 is non-trivial. We calculated it at leading order in perturbation theory and found

$$\langle W \rangle = 1 + \frac{g^2 N}{8\pi^2}\delta(2\pi - \delta) + O(g^4)\,.$$

We haven't calculated the next corrections, but we found the string solution for this surface, and the finite part of the action is

$$S = -\frac{\sqrt{g^2 N \delta(2\pi - \delta)}}{\pi}\,.$$

We see once more that both the perturbative and string theory result are related to the usual circle by a rescaling of the coupling. One may hope that this loop is also given by the matrix model, even though the propagators are not constant.

$1/8$ BPS: Zarembo limit

After all those examples of previously unstudied supersymmetric Wilson loops, it's also possible to recover the $1/8$ BPS loops considered by Zarembo.

Consider infinitesimal loops, say around the point $x^4 = 1$. Then $\sigma_i^{R,L} \sim dx^i$ for $i = 1, 2, 3$ and locally the invariant 1-forms on S^3 are exact differentials. So those loops will approximate the loops of Zarembo.

More precisely, we can rescale the sphere as we get closer to that point, keeping the size of the loops finite. In the infinite radius limit, the curves are in flat $\mathbb{R}^3$ and the scalar couplings proportional to the tangent vectors, which is exactly the construction of Zarembo.

Note that we cannot reproduce, by our construction, his $1/16$ BPS loops.

YM_2 and the matrix model

We saw that in all the explicit examples on S^2 the result was the same as the circle with a modified coupling. How does that come about?

We may write the loop on S^2 as

$$W = \operatorname{Tr}\mathcal{P}\exp\oint(iA_\mu\dot{x}^\mu + (x\times\dot{x})_\mu M_I^\mu\Phi^I)ds\,.$$

Let me calculate this again in perturbation theory. For the circle we saw that the propagator was a constant. More generally this will not be true. But I will still combine the vector and scalar terms together. At leading order in perturbation theory one gets the effective propagator

$$\frac{g_{YM}^2\delta_{ab}}{4\pi^2}\,\frac{-\dot{x}_{(1)}\cdot\dot{x}_{(2)} + (x_{(1)}\times\dot{x}_{(1)})\cdot(x_{(2)}\times\dot{x}_{(2)})}{(x_{(1)}-x_{(2)})^2}\,.$$

Evaluating the cross product one finds

$$\dot{x}^{\mu}_{(1)}\dot{x}^{\nu}_{(2)}\,\frac{g^2_{YM}\delta_{ab}}{4\pi^2}\left(\frac{\delta^{\mu\nu}}{2}-\frac{x^{\nu}_{(1)}x^{\mu}_{(2)}}{(x_{(1)}-x_{(2)})^2}\right).$$

Interestingly, though we are in four dimensional space, instead of having mass dimension two, the resulting effective propagator is dimensionless.

That would be the expected behavior for a vector propagator in two-dimensions. Indeed the last expression can serve as a propagator for YM_2 on the unit two-sphere with coupling $-g^2_{YM}/4\pi^2$.

So the sum of all non-interacting graphs for those Wilson loops on S^2 agrees with that of YM_2. That problem is famously solved and the result is

$$\langle W\rangle_{2d}\sim e^{\frac{g^2_{YM}N}{8\pi^2}A_1}\,,$$

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where A_1 and A_2 are the areas of the two parts of S^2 defined by the curve.

This does not agree with the result of the AdS calculation, not for the $1/2$ BPS circle and not for the other examples. So what did I do wrong?

Such a discrepancy has appeared already in YM_2 as was explored by Staudacher-Krauth and resolved by Bassetto-Griguolo. In perturbation theory one may be missing instanton corrections that are included in the full non-perturbative solution. We are doing a perturbative calculation, so we should compare to the perturbative results, excluding instantons.

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In those papers they found that the perturbative result for YM_2 on the sphere is given by the function

$$\langle W \rangle_{\text{pert. } YM_2} = \frac{1}{N} e^{g^2 \frac{A_1 A_2}{32\pi^2}} L^1_{N-1}\left(-g^2 \frac{A_1 A_2}{16\pi^2}\right)$$

Where L^1_{N-1} is a Laguerre polynomial.

In the planar approximation this is

$$\langle W \rangle_{\text{planar}} = \frac{4\pi}{\sqrt{g^2_{YM} N A_1 A_2}} I_1\left(\frac{1}{2\pi}\sqrt{g^2_{YM} N A_1 A_2}\right).$$

At weak coupling this goes as

$$\langle W \rangle \sim 1 - \frac{g^2_{YM} N}{32\pi^2} A_1 A_2 + O(g^4_{YM} N^2).$$

and at strong coupling

$$\langle W \rangle \sim \exp\left(\frac{1}{2\pi}\sqrt{g^2_{YM} N A_1 A_2}\right).$$

Those above expressions agree with **ALL** explicit calculations done so far!

For the $1/2$ BPS circle, $A_1 = A_2 = 2\pi$, and this Laguerre polynomial is exactly the result of the Gaussian matrix model.

For the $1/4$ BPS latitude $A_{1,2} = 2\pi(1 \pm \cos\theta_0)$, so the full perturbative result is given by the regular circle with $g^2 \to g^2 \sin^2\theta_0$, in agreement with the sum of ladders.

For the latitude example we do not have an all-order calculation, only the first terms at weak and strong coupling. There $A_1 = 2\delta$ and $A_2 = 2(2\pi - \delta)$ and the weak and strong coupling results are reproduced by $g^2 \to g^2(2\pi - \delta)\delta/\pi^2$.

From those calculations it seems that Wilson loops on S^2 are given by the perturbative expansion of 2-dimensional Yang-Mills on the sphere. This provides a subsector of $\mathcal{N} = 4$ which is invariant under area-preserving diffeomophisms.

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Discussion

- I presented an infinite family of new supersymmetric Wilson loops by adjusting the scalar couplings for any curve on S^3.
- Unlike Zarembo's loops, they have non-trivial expectation values, giving many new interesting observables that may be calculated in the gauge theory or $AdS_5 \times S^5$.
- We found a very elegant structure in $AdS_5 \times S^5$, where the string surfaces describing those loops are calibrated with regard to a novel almost complex structure on $AdS_4 \times S^2$. These calibration equations imply κ-symmetry for the strings and also give a simple expression for the action.
- A new configuration I described are several circles following the Hopf-fibers of S^3. They preserve eight chiral supercharges and seem like a natural generalization of parallel lines in flat space. The interaction between them is independent of the distance.

- When the curves are restricted to an S^2, the supersymmetry is doubled. Specific curves (longitudes, the latitude) were $1/4$ BPS.
- For curves on S^2 the perturbative series seems to agree with 2-dimensional Yang-Mills. Providing a subsector of $\mathcal{N} = 4$ SYM which is invariant under area-preserving diffeomorphisms.

The end