Title: Manipulating Entanglement

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Abstract: Entanglement plays a fundamental role in quantum information
processing and is regarded as a valuable, fungible resource,
The practical ability to transform (or manipulate) entanglement from one form to another is useful for many applications.
Usually one considers entanglement manipulation of states which are multiple copies of a given bipartite entangled state and requires that the fidelity of the transformation to (or from) multiple copies of
a maximally entangled state approaches unity asymptotically in the
number of copies of the original state. The optimal rates of these protocols yield two asymptotic measures of entanglement, namely, entanglement cost and
distillable entanglement.
It is not always justified, however, to assume that the entanglement resource available, consists of states which are multiple copies, i.e.,tensor products, of a given entangled state. More generally, an entanglement
resource is characterized by an arbitrary sequence of bipartite states which
are not necessarily of the tensor product form. In this seminar, we address the issue of entanglement manipulation
for such general resources and obtain expressions for the entanglement cost and distillable entanglement.

Manipulating Entanglement

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joint work with: Garry Bowen

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- It is a novel resource which can be used to perform tasks which are impossible in the classical realm, e.g., teleportation, superdense coding, quantum cryptography etc.

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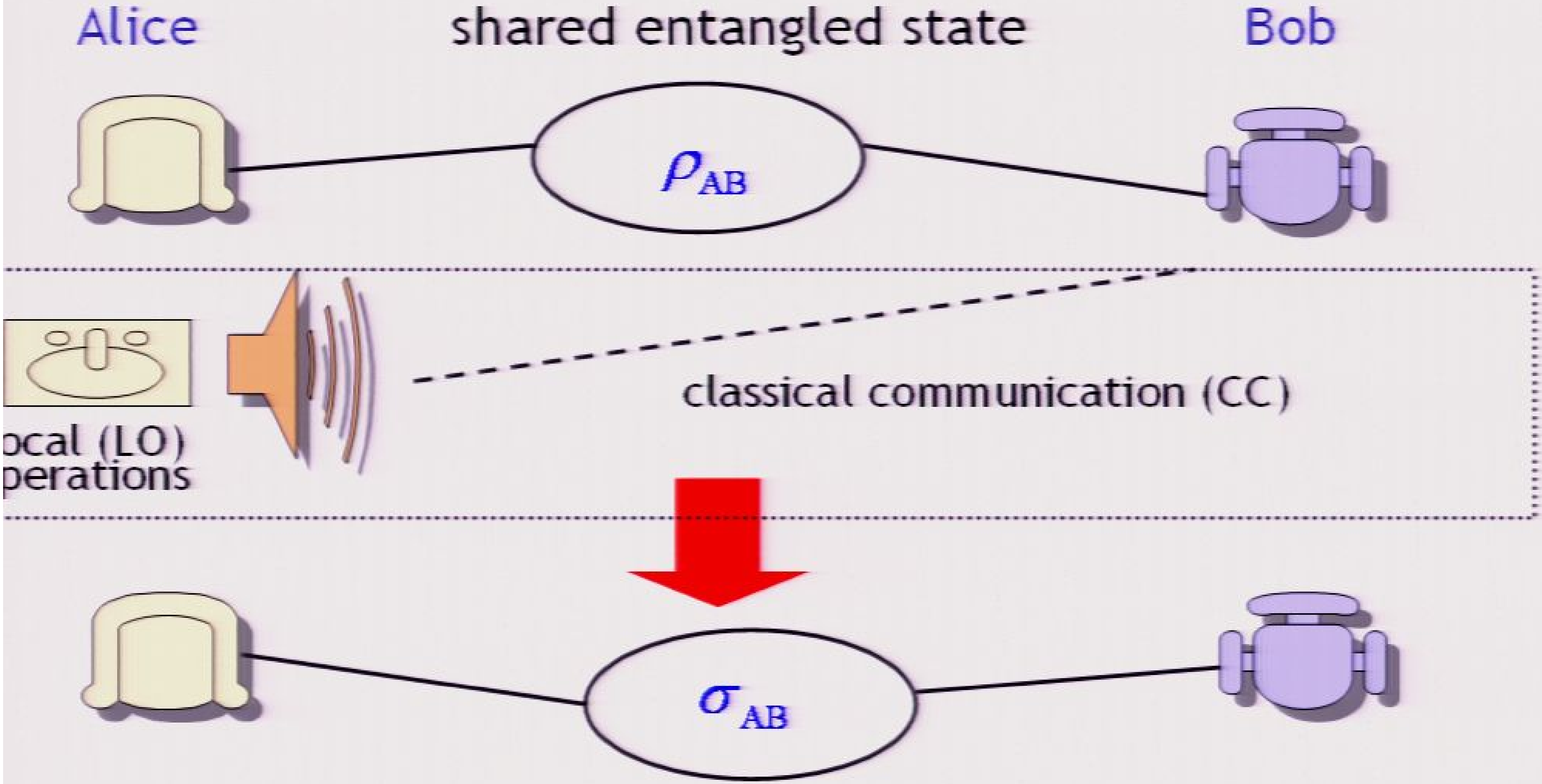
- It is a novel resource which can be used to perform tasks which are impossible in the classical realm, e.g., teleportation, superdense coding, quantum cryptography etc.
- a fundamental property of entanglement: it cannot be created by local operations and classical communications (LOCC) alone.
- However, one can transform one entangled state to another by LOCC alone: this is called as ***entanglement manipulation***


Alice
shared entangled state
Bob
ρ_AB
classical communication (CC)
ocal (LO)
perations
σ_AB

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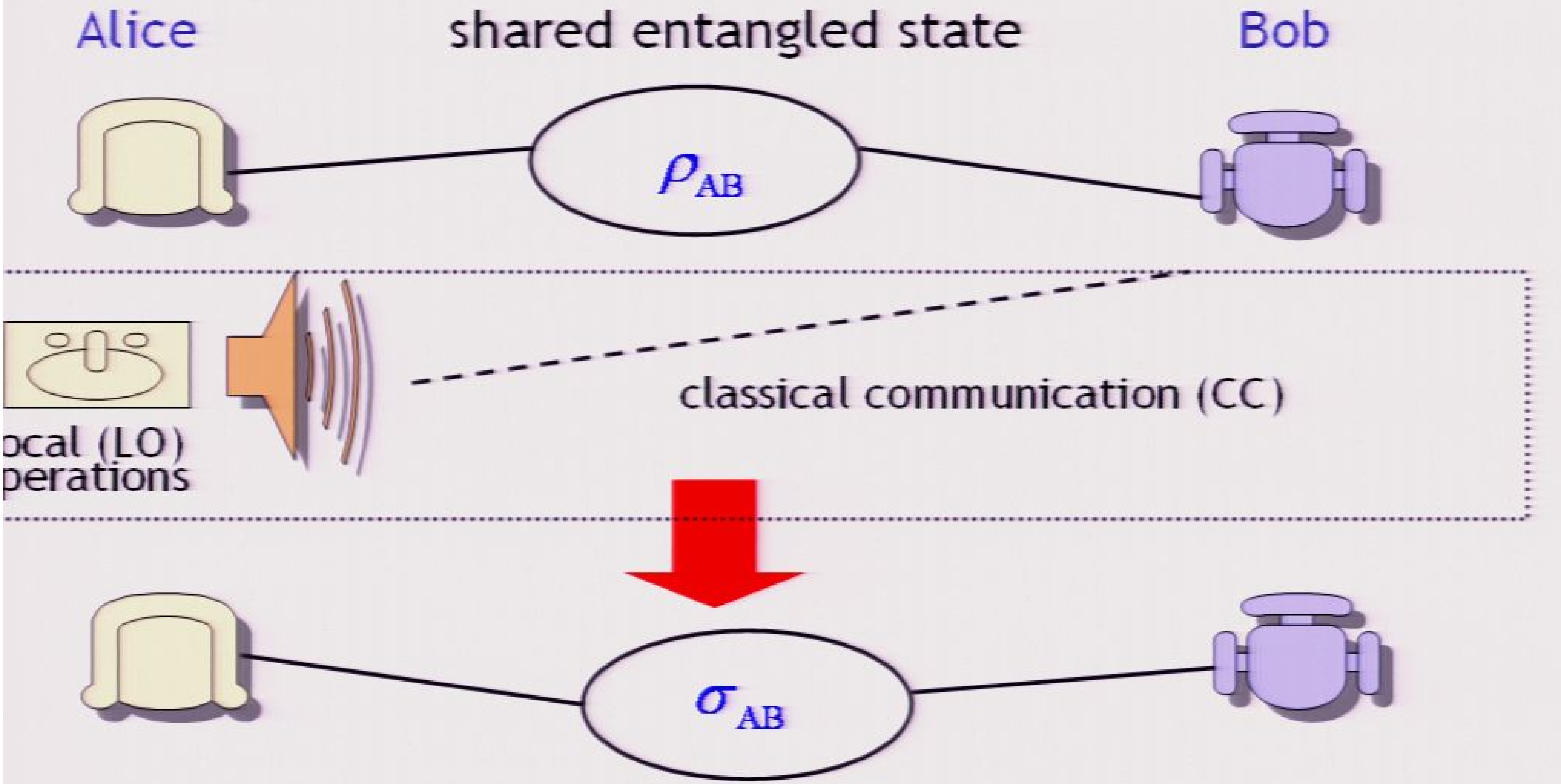

If $E(\rho_{AB})$ denotes entanglement of state ρ_{AB} then :

$$\rho_{AB} \xrightarrow[\text{LOCC}]{} \sigma_{AB} \Rightarrow E(\sigma_{AB}) \leq E(\rho_{AB})$$

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There is no such simple quantity characterising the entanglement of arbitrary bipartite states ρ_{AB} .

However, one can establish asymptotic measures of entanglement for any arbitrary bipartite state ρ_{AB} by considering suitable entanglement manipulations of it.

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To obtain "standard form" or "currency" for entanglement: define the entanglement of maximally entangled state (MES) of rank M

$$|\Psi_{\mathbf{M}}^{+}\rangle = \frac{1}{\sqrt{M}} \sum_{k=1}^{M} |e_k^{\mathbf{A}}\rangle |e_k^{\mathbf{B}}\rangle$$

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[Note: take logarithm in (1) is taken to base 2]

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- Denoting a the density matrix of a Bell pair by ω the above transformations can be denoted as follows:

$$\rho^{\otimes n} \xrightarrow[LOCC]{} \omega^{\otimes m_n} \quad \text{.......(i)} \qquad \omega^{\otimes m'_n} \xrightarrow[LOCC]{} \rho^{\otimes n} \quad \text{.......(ii)}$$

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- Note: transformations (i) and (ii) cannot be achieved perfectly for finite n . Hence one allows imperfections and requires instead that the fidelities of the transformations approach unity asymptotically in n .

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FIDELITY: $$F_n = F(\tau_n(\rho_n), \sigma_n) := \mathrm{Tr}(\tau_n(\rho_n)\sigma_n)$$

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  - (i) the entanglement cost

$$E_C(\rho) = \inf \lim_{n\to\infty} \frac{m_n'}{n} \qquad \begin{cases} \omega^{\otimes m_n'} \xrightarrow[LOCC]{} \rho^{\otimes} \\ F_n \xrightarrow[n\to\infty]{} 1 \end{cases}$$

: the minimum number of Bell pairs needed to create ρ

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: the maximum number of Bell pairs that can be extracted locally from the state ρ.

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- For a bipartite pure state $|\Psi_{AB}\rangle$ it is known that

$$E_D\left(|\Psi_{AB}\rangle\right) = S(\rho_A) = S(\rho_B) = E_C\left(|\Psi_{AB}\rangle\right)$$

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- Hence, locally transforming

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is an asymptotically reversible process.

- Moreover $S(\rho_A)$ is the unique asymptotic entanglement measure for $|\Psi_{AB}\rangle$ since any other entanglement measure E for $|\Psi_{AB}\rangle$ satisfies:

$$E_D \le E \le E_C$$

[Donald et al.]

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More generally, an entanglement resource is characterized by an arbitrary sequence of bipartite states, which are not necessarily of the tensor product form.

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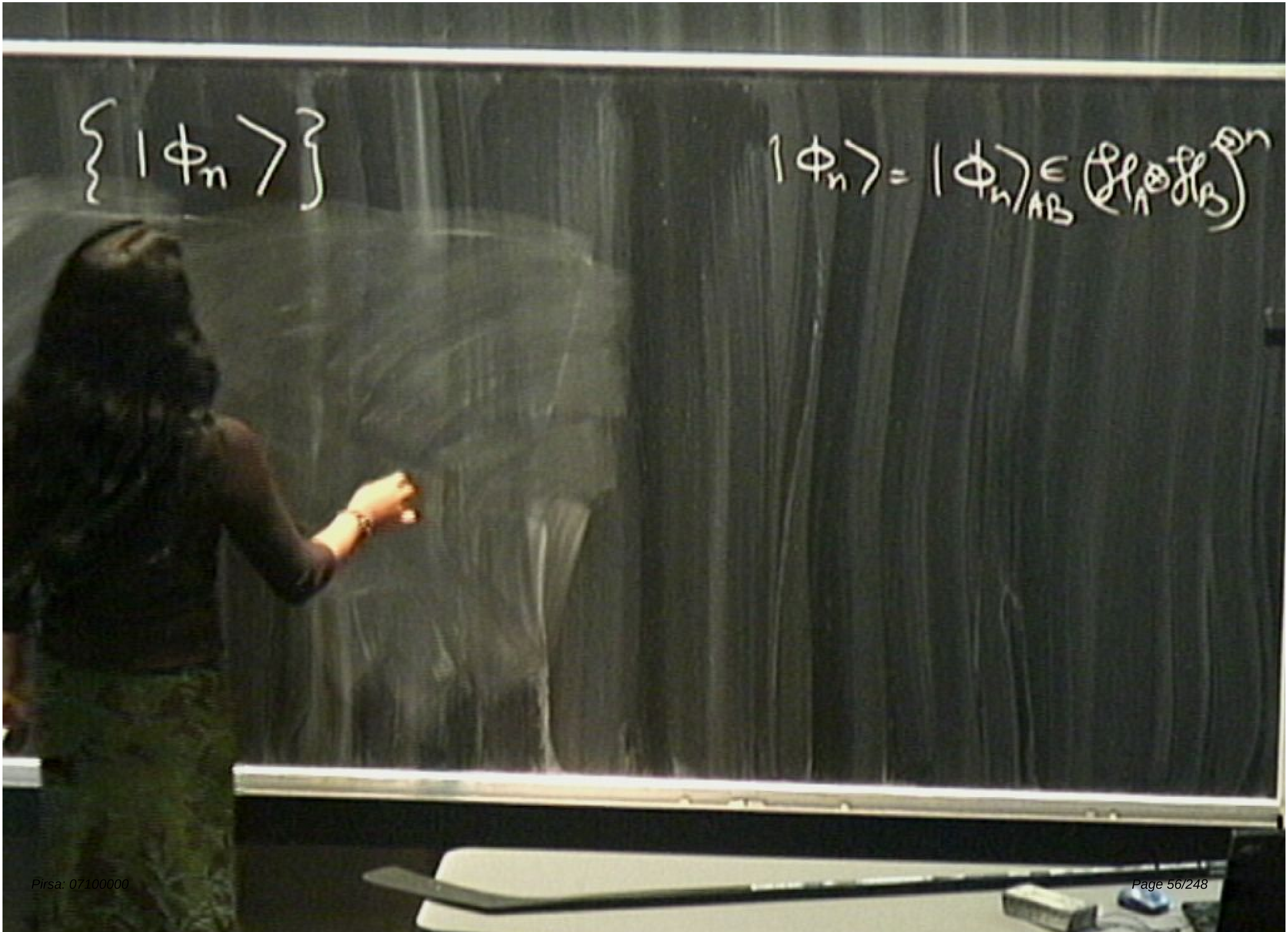
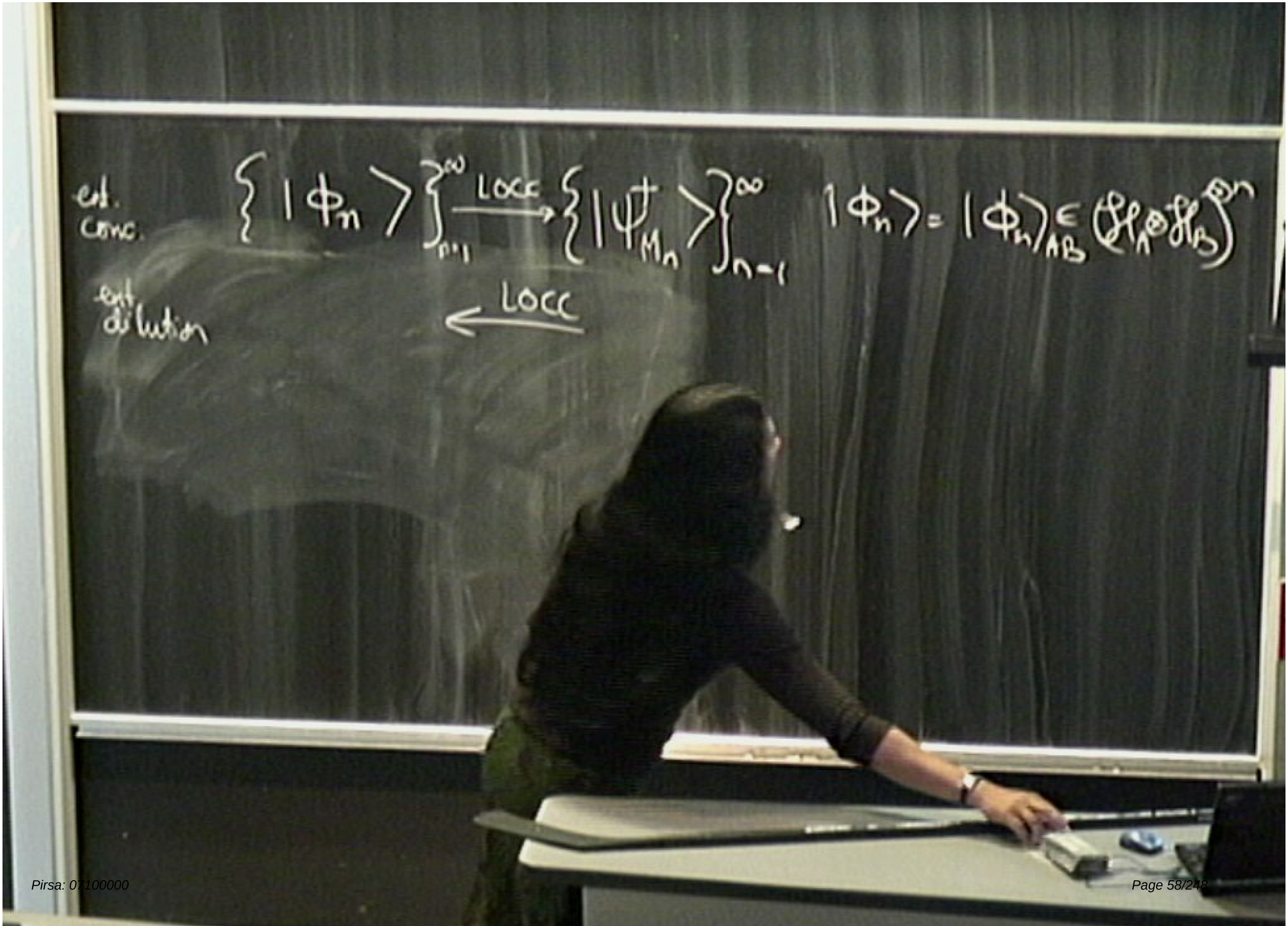
- Our Aim: to establish asymptotic entanglement measures

  for arbitrary sequences of bipartite states : $\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$


$\{|\phi_n\rangle\}$
$|\phi_n\rangle = |\phi_n\rangle_{AB} \in (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$

$$\{|\phi_n\rangle\}_{n=1}^{\infty} \xrightarrow{LOCC} \{|\psi^+_{M_n}\rangle\}_{n=1}^{\infty} \qquad |\phi_n\rangle = |\phi_n\rangle_{AB} \in (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$$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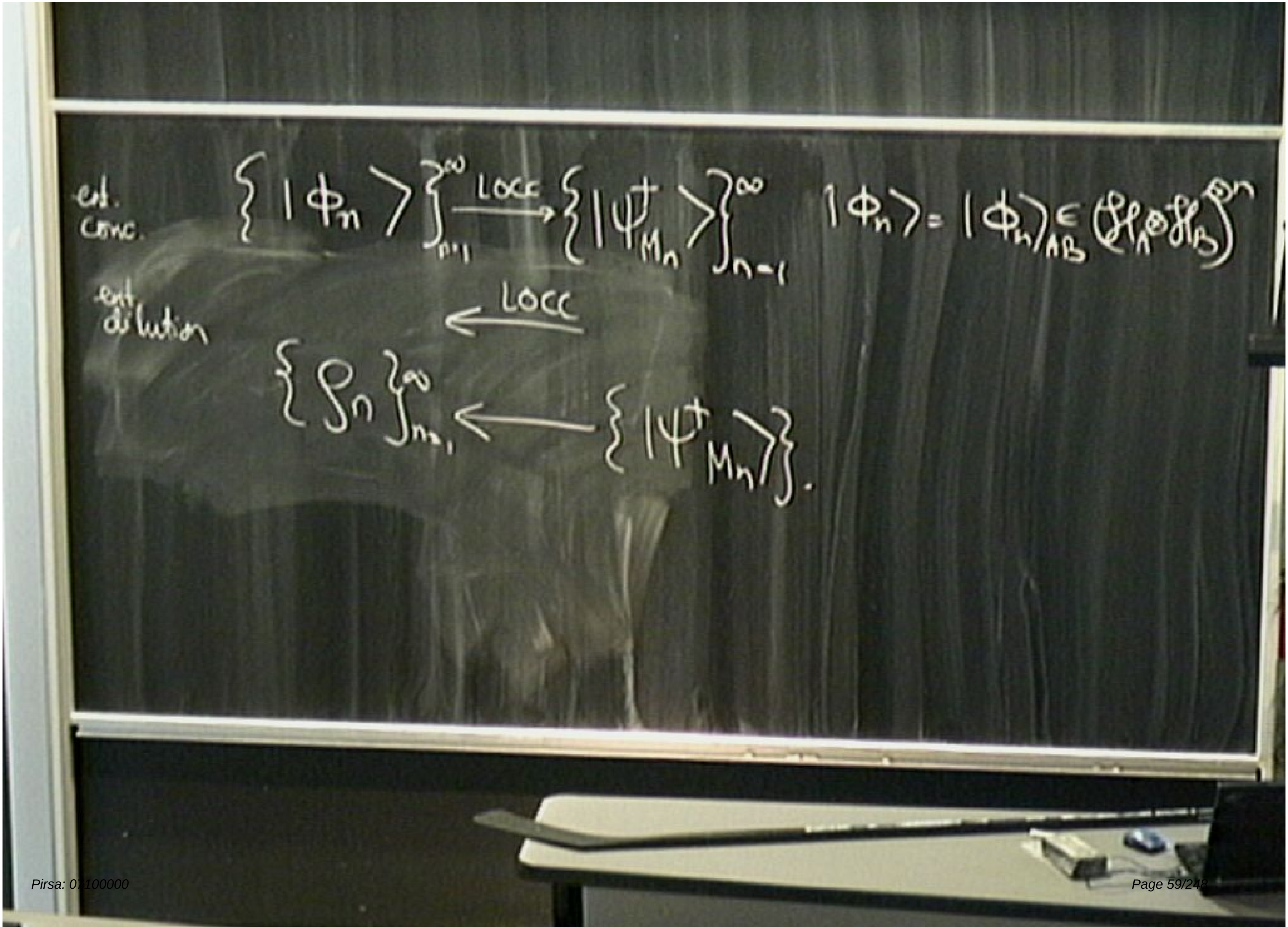
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{|Φn⟩}n=1∞ LOCC→ {|ψ+Mn⟩}n=1∞
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Spectral Projections

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we define the spectral projection on A as

$$\{A \geq 0\} = \sum_{\lambda_i \geq 0} |i\rangle\langle i|$$

:the projector onto the eigenspace of non-negative eigenvalues of A

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Spectral Projections

- The Quantum Information Spectrum approach requires the extensive use of spectral projections.
- For a self-adjoint operator A with spectral decomposition

$$A = \sum_i \lambda_i |i\rangle\langle i|$$

we define the spectral projection on A as

$$\{A \geq 0\} = \sum_{\lambda_i \geq 0} |i\rangle\langle i|$$

:the projector onto the eigenspace of non-negative eigenvalues of A

- For 2 operators A and B we can then define

$$\{A \geq B\} = \{A - B \geq 0\}$$

$$\{A<0\} = \sum_{\substack{i \\ \lambda_i<0}} |i\rangle\langle i|$$

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- For any given constant γ , one can associate with any sequence of states $\hat{\rho}=\{\rho_n\}_{n=1}^{\infty}$, a sequence of orthogonal projectors $\{P_n^\gamma\}_{n=1}^{\infty}$ with $P_n^\gamma=\{\rho_n \geq 2^{-n\gamma} I_n^\gamma\}$

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- If $\rho_n = \sum_i \lambda_i^n |e_i^n\rangle\langle e_i^n|$: spectral decomposition

$$P_n^\gamma = \sum_{i:\lambda_i^n \geq 2^{-n\gamma}} |e_i^n\rangle\langle e_i^n|$$

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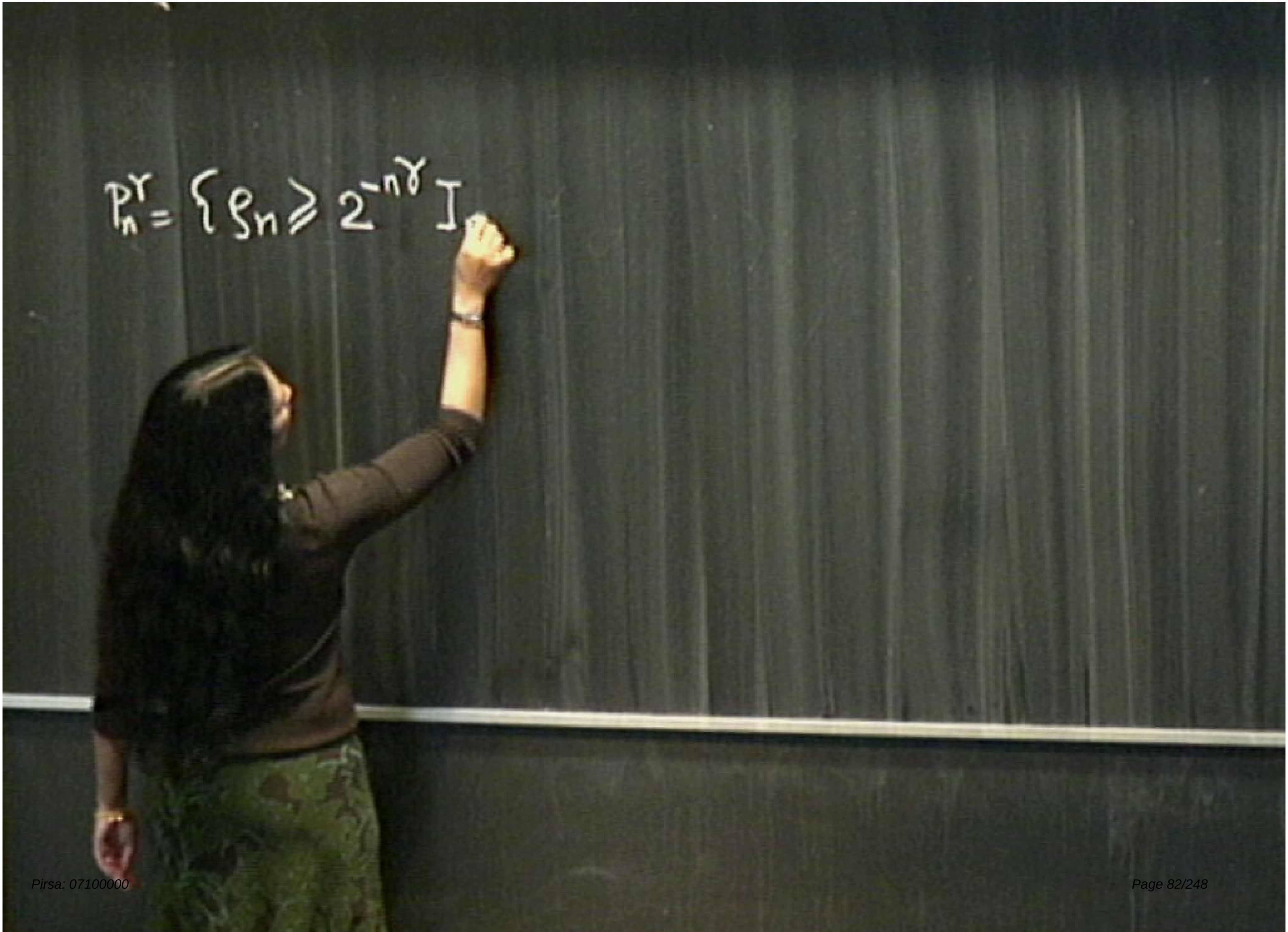
[redacted]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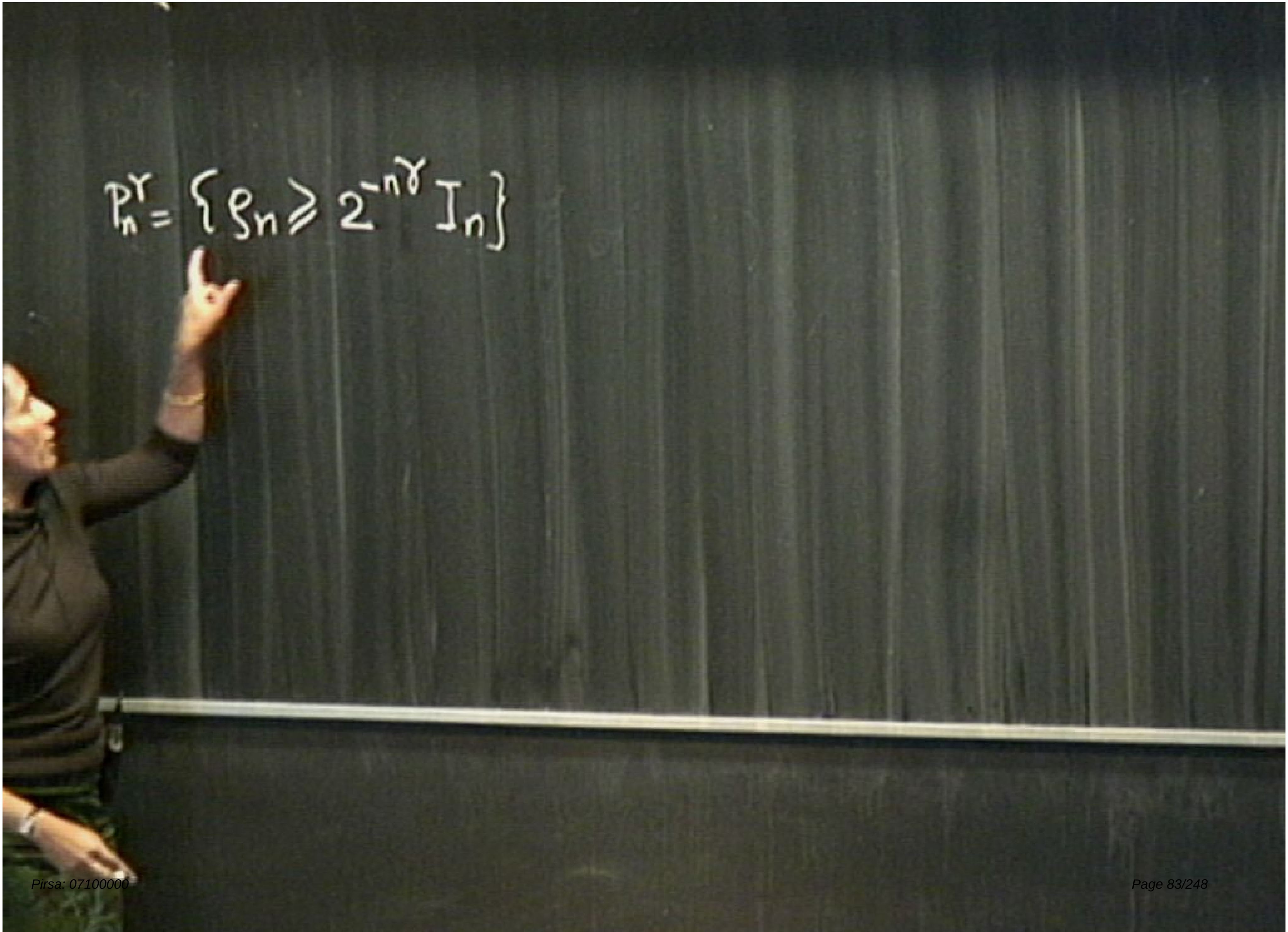
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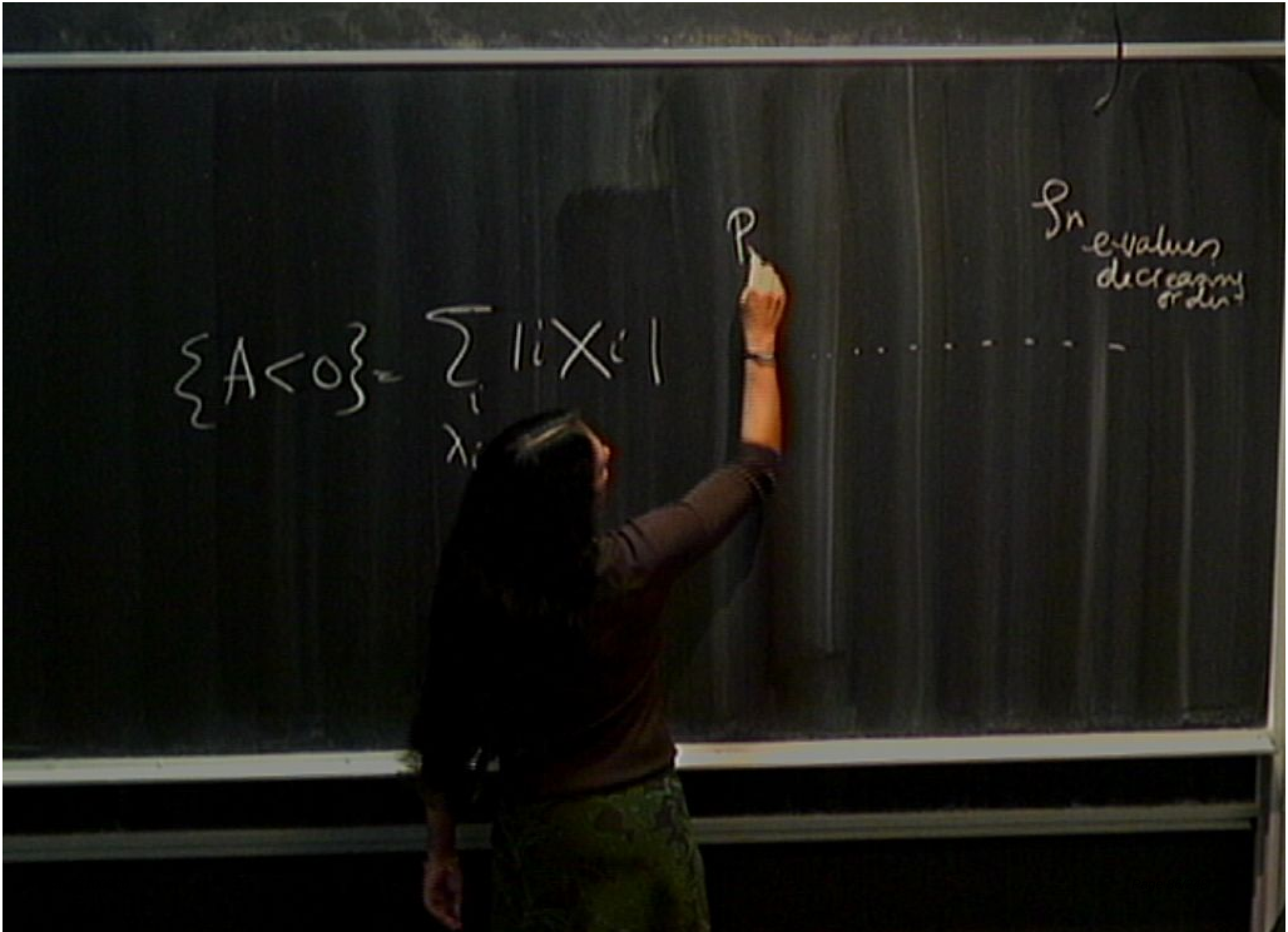
e-values
decreasing
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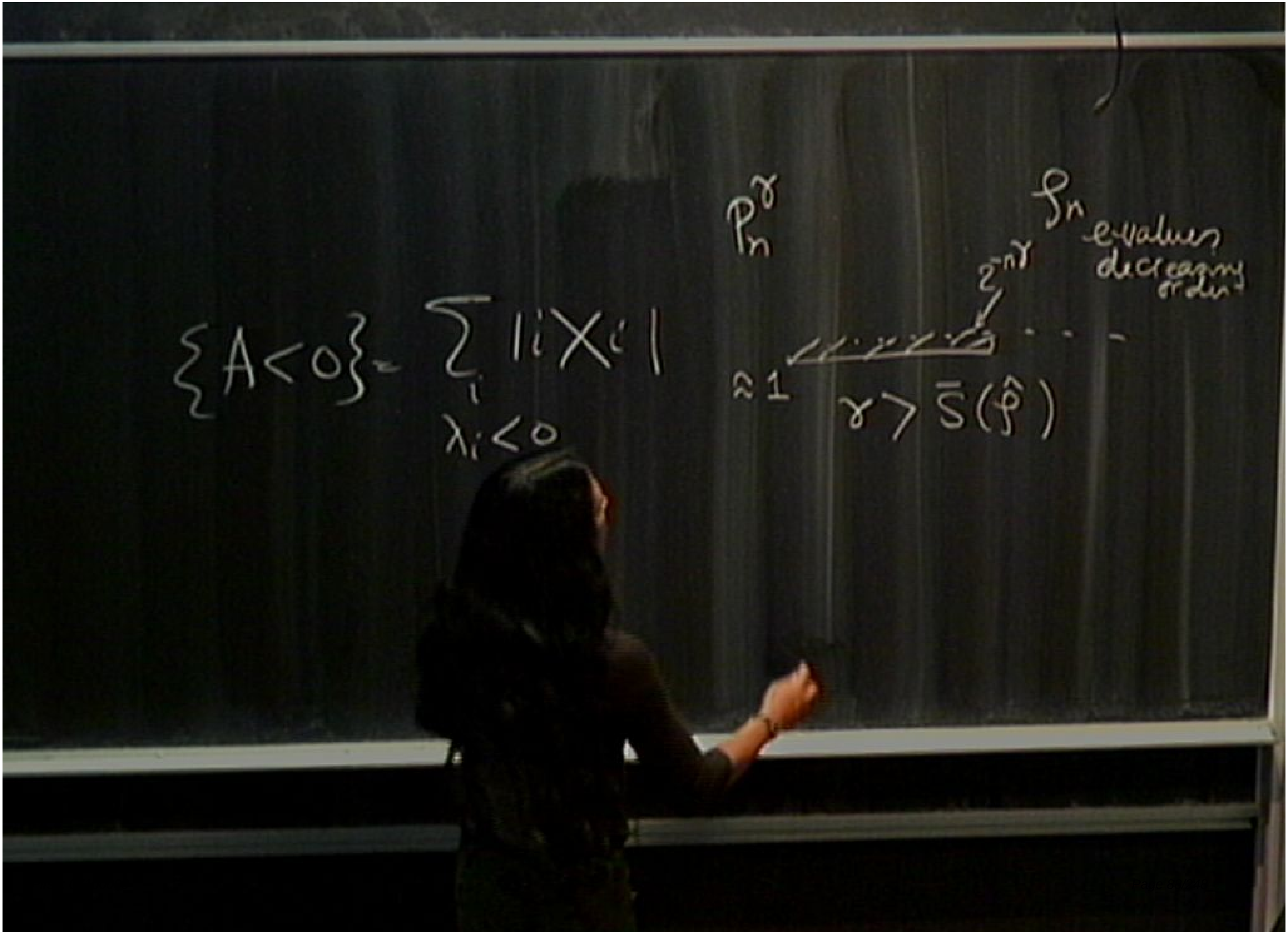
$$\{A<0\} = \sum_{\substack{i \\ \lambda_i<0}} |i\rangle\langle i|$$

ρ_n^{γ}

ρ_n evalues decreasing order

$2^{-n\gamma}$



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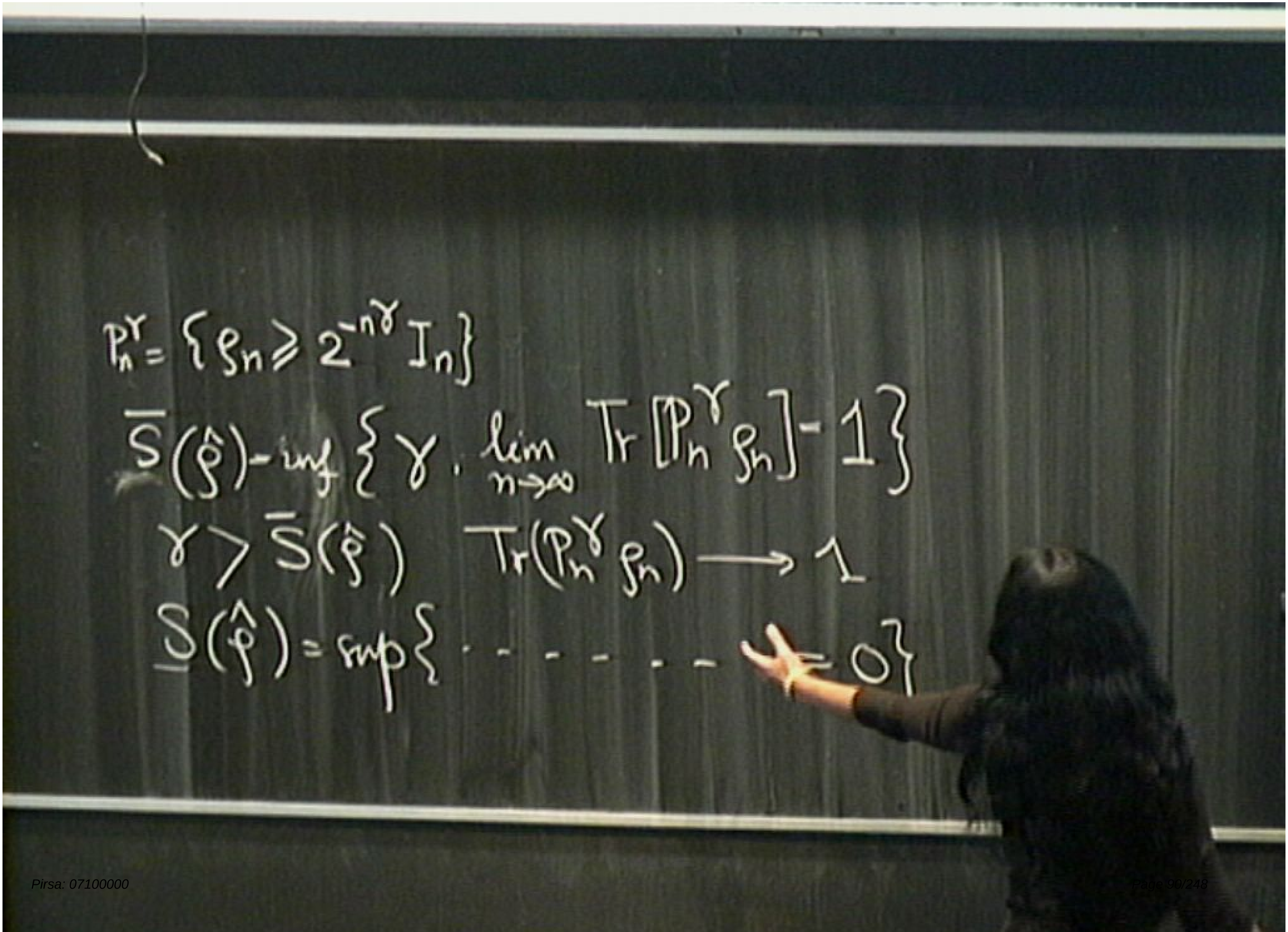
P_n^γ proj. onto h. prob. sub-space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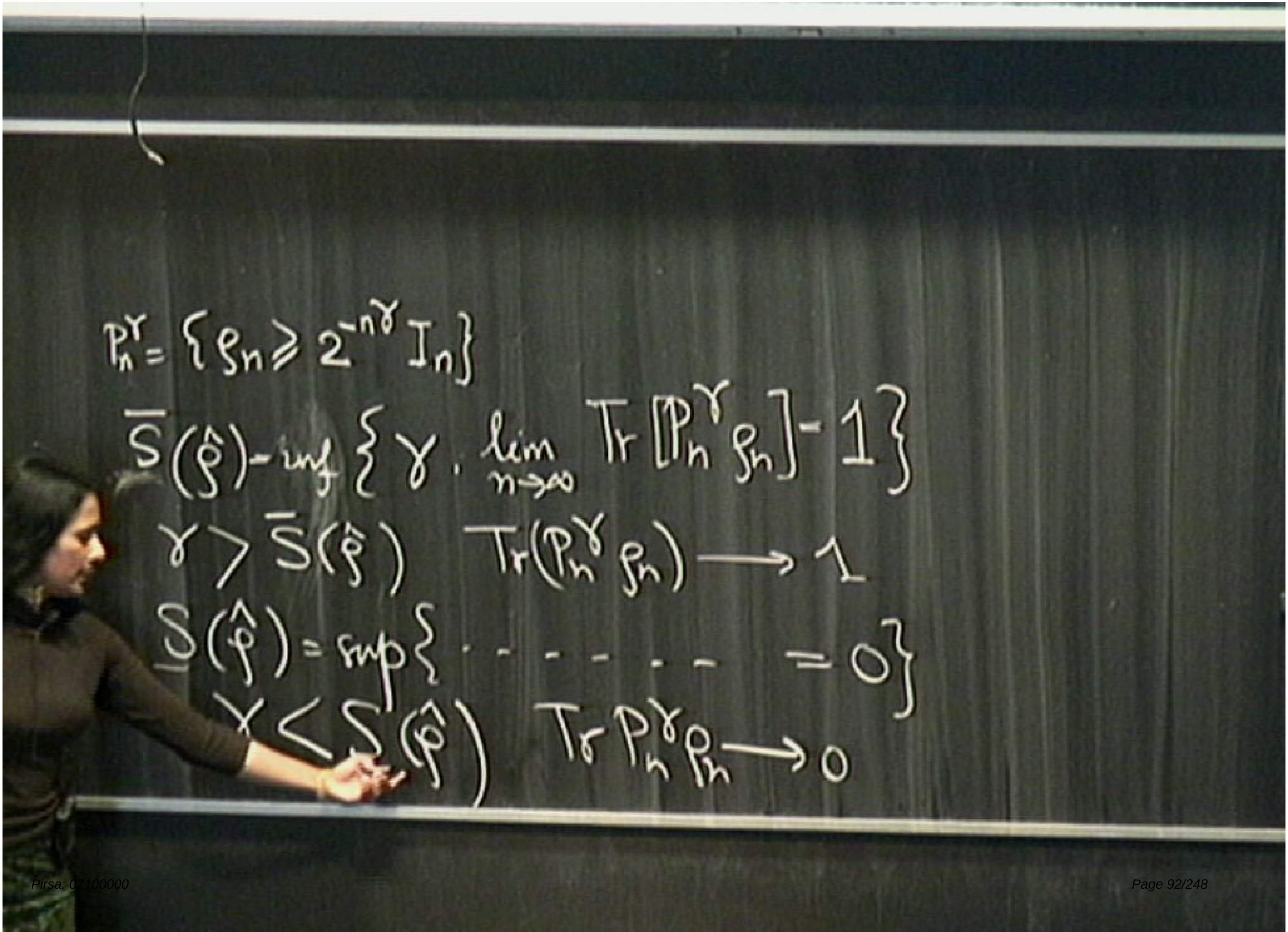
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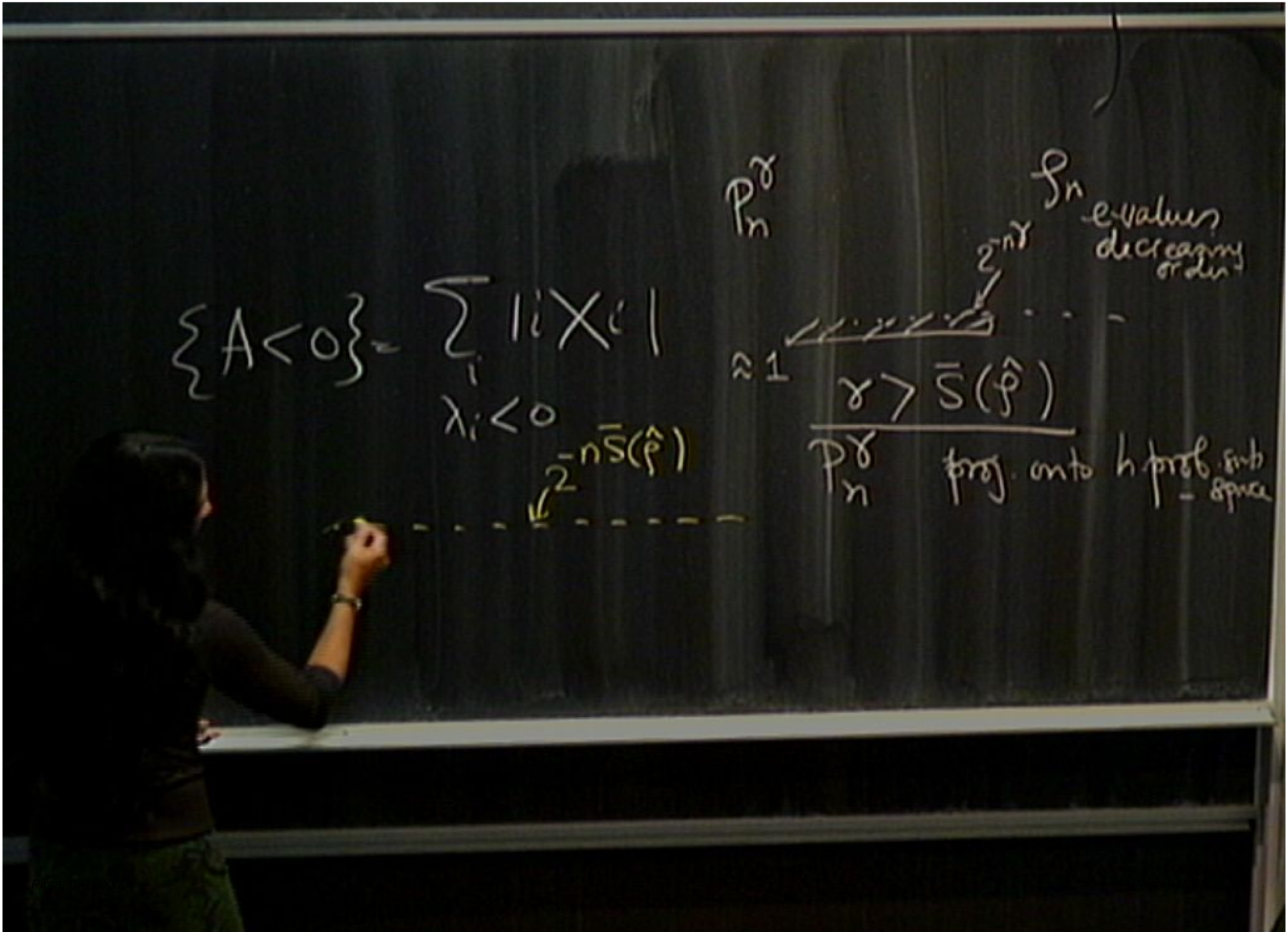
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P_n^γ proj. onto h. prob. sub-space
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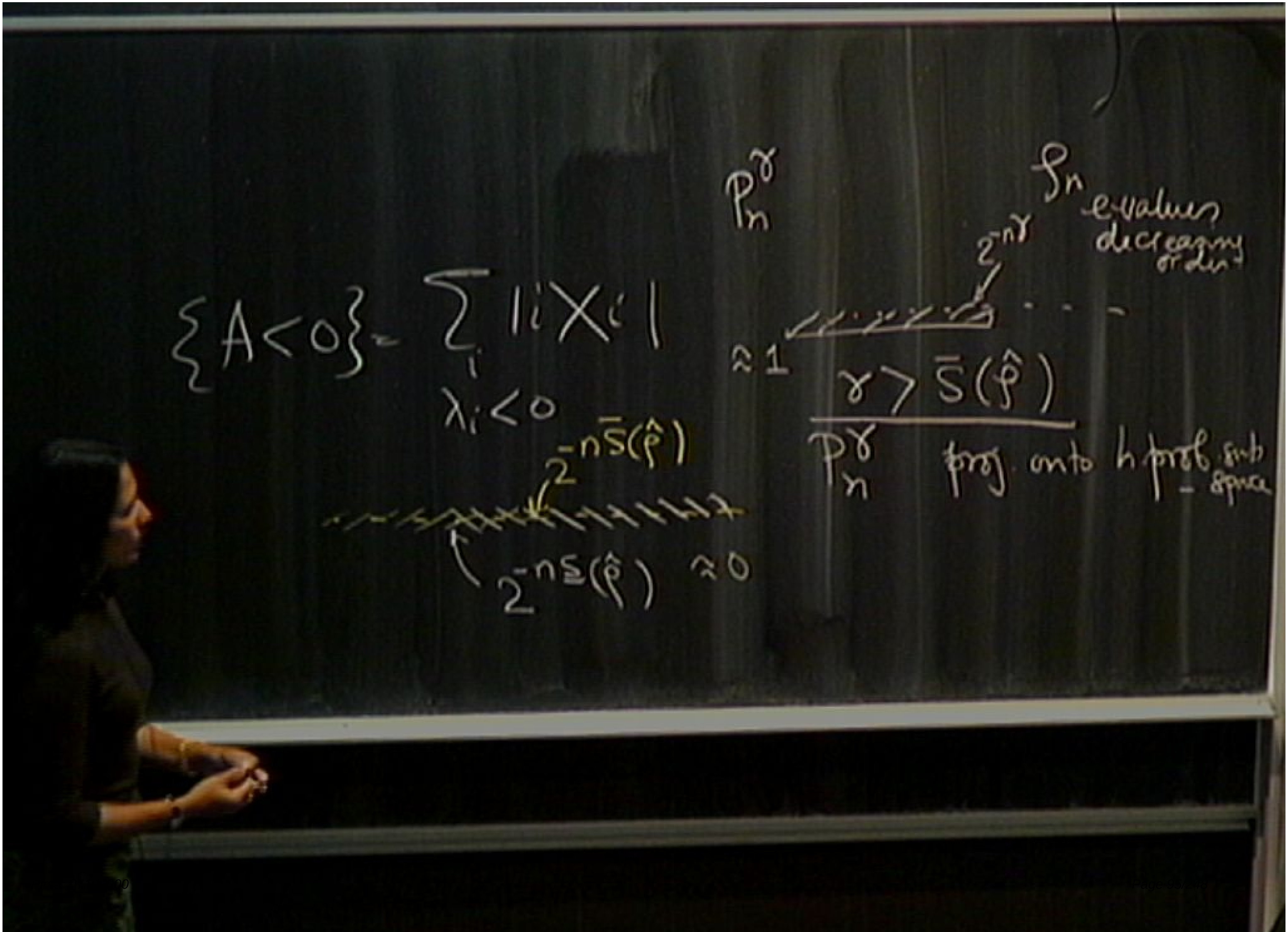
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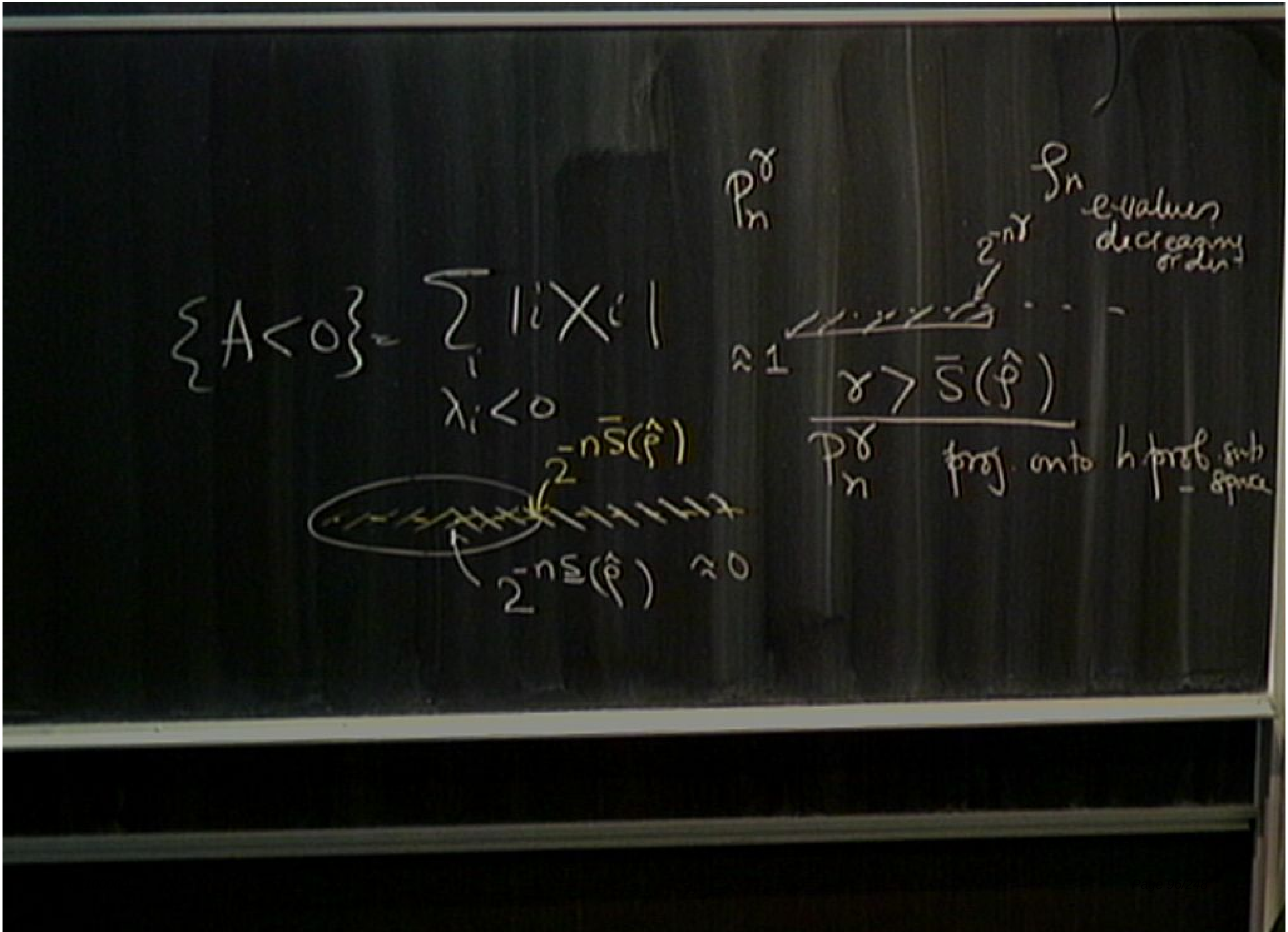
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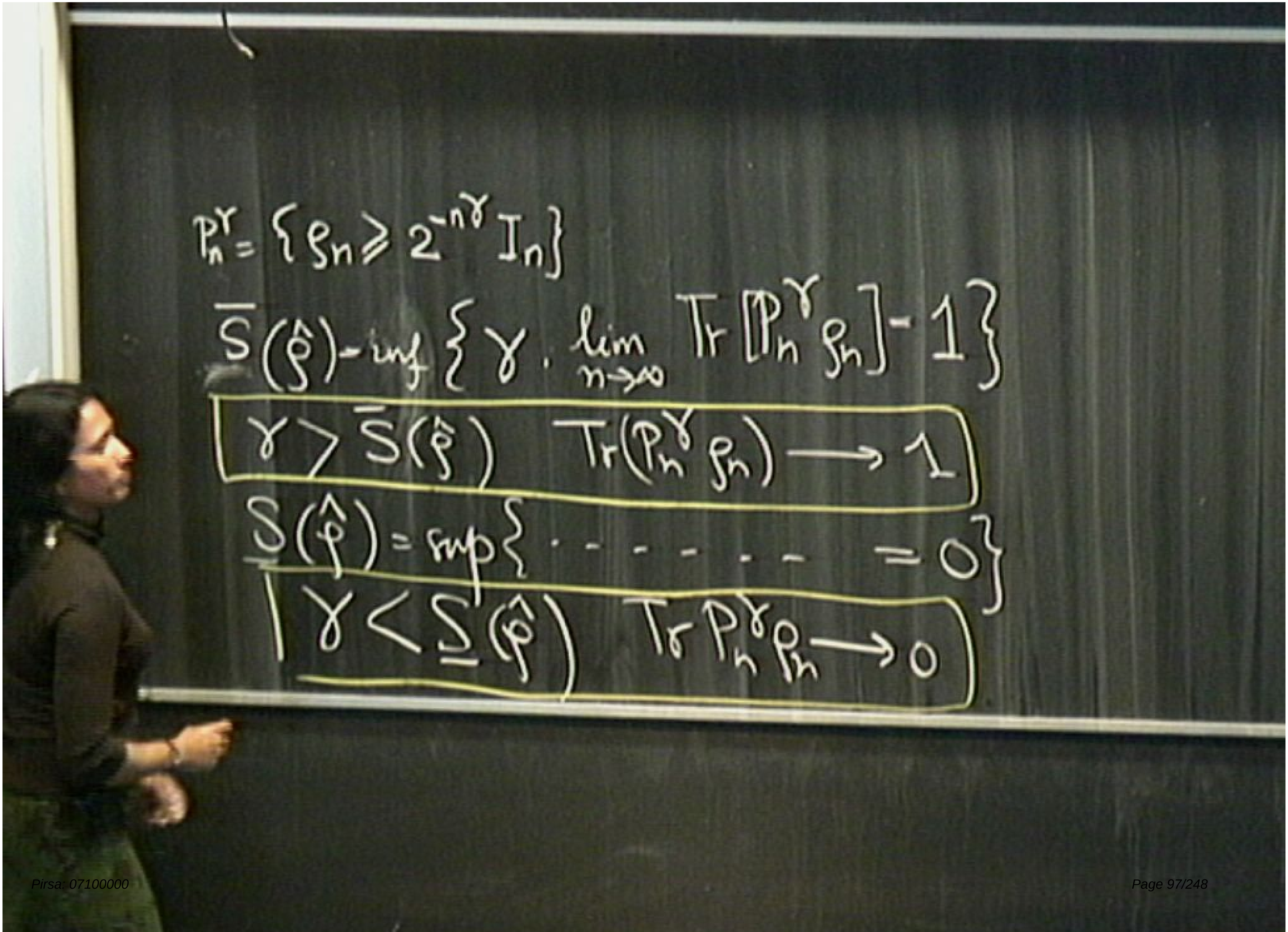
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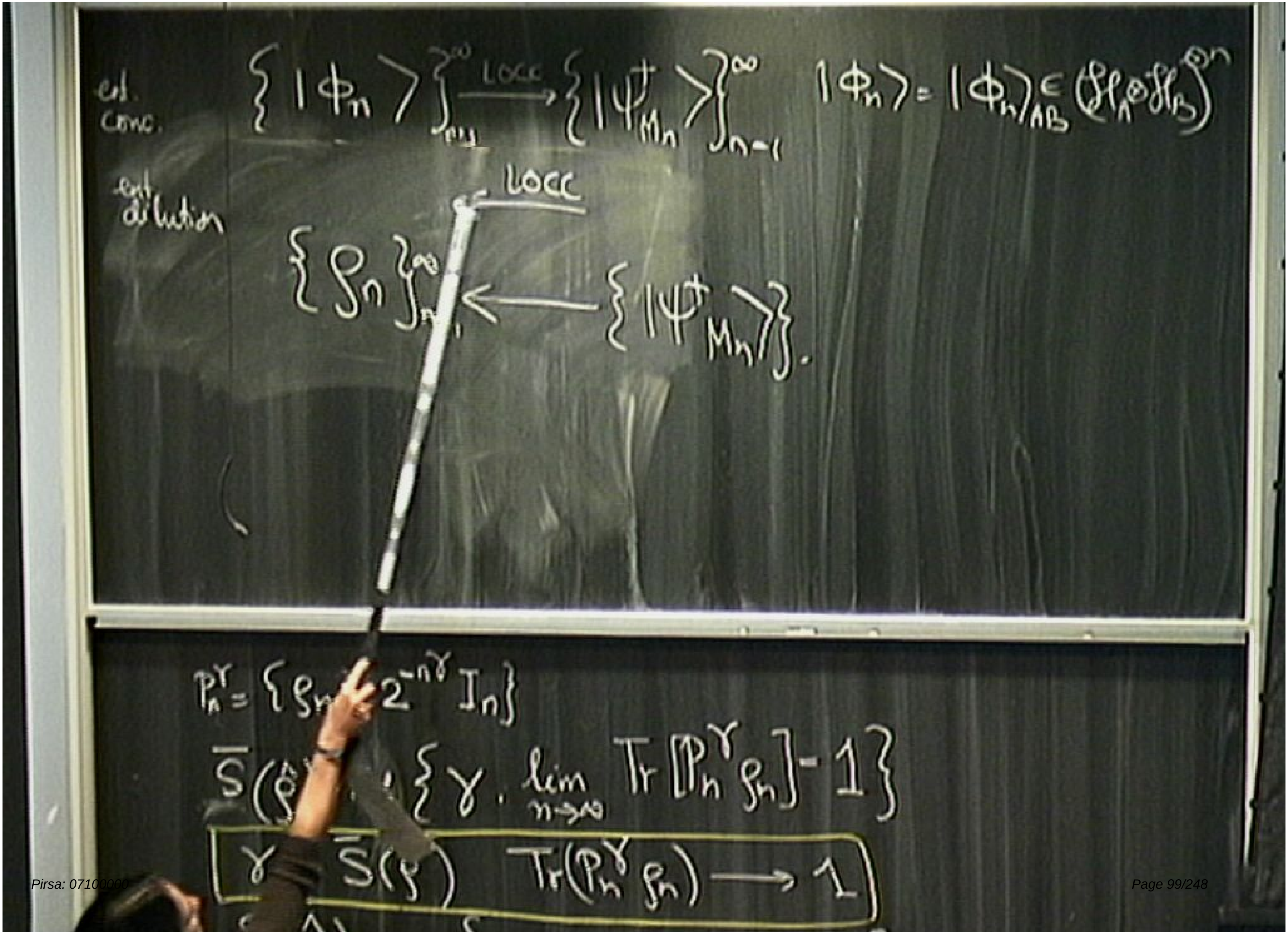
ent. conc.
$\{|\phi_n\rangle\}_{n=1}^{\infty} \xrightarrow{LOCC} \{|\psi^+_{M_n}\rangle\}_{n=1}^{\infty}$
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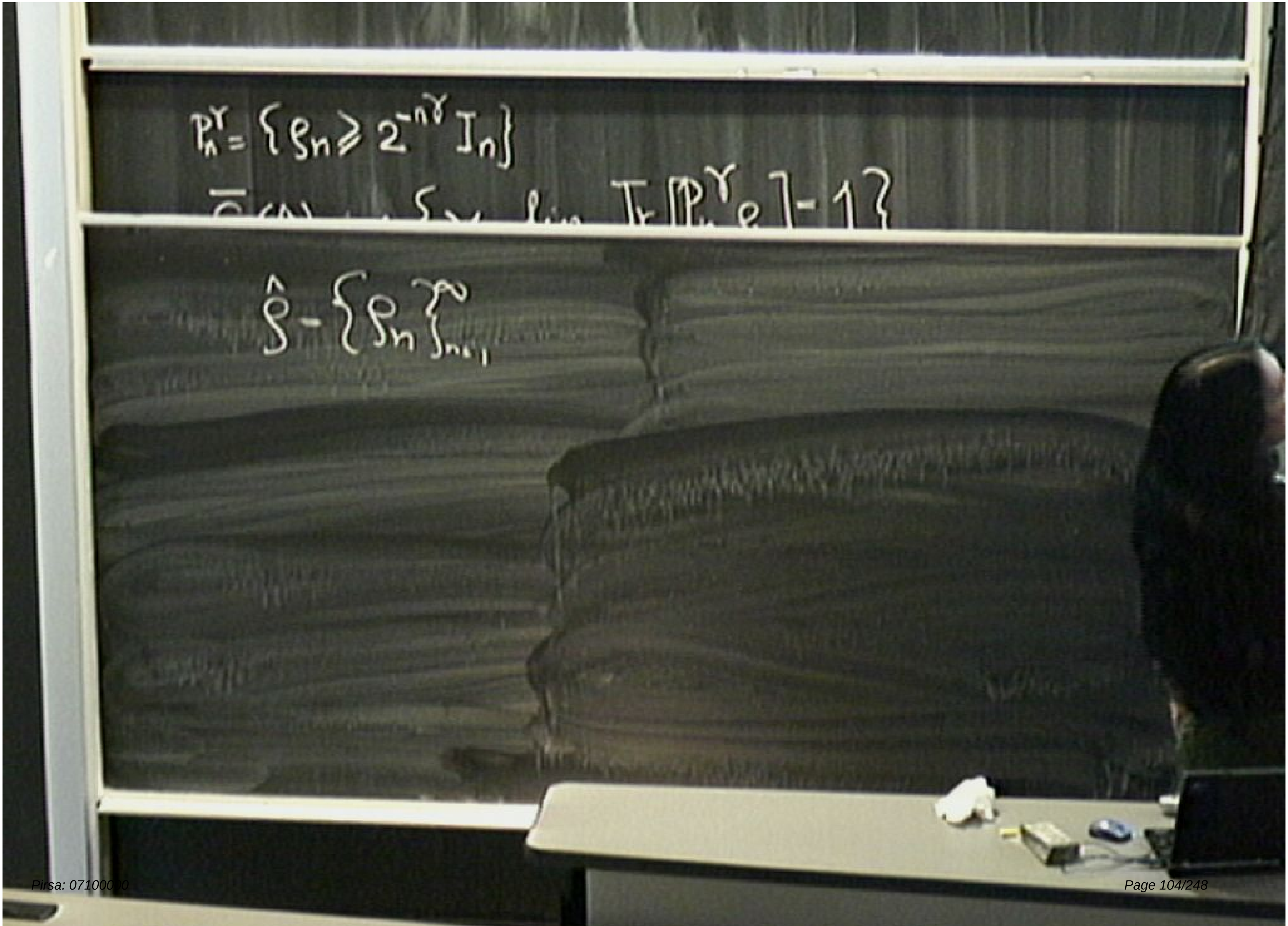
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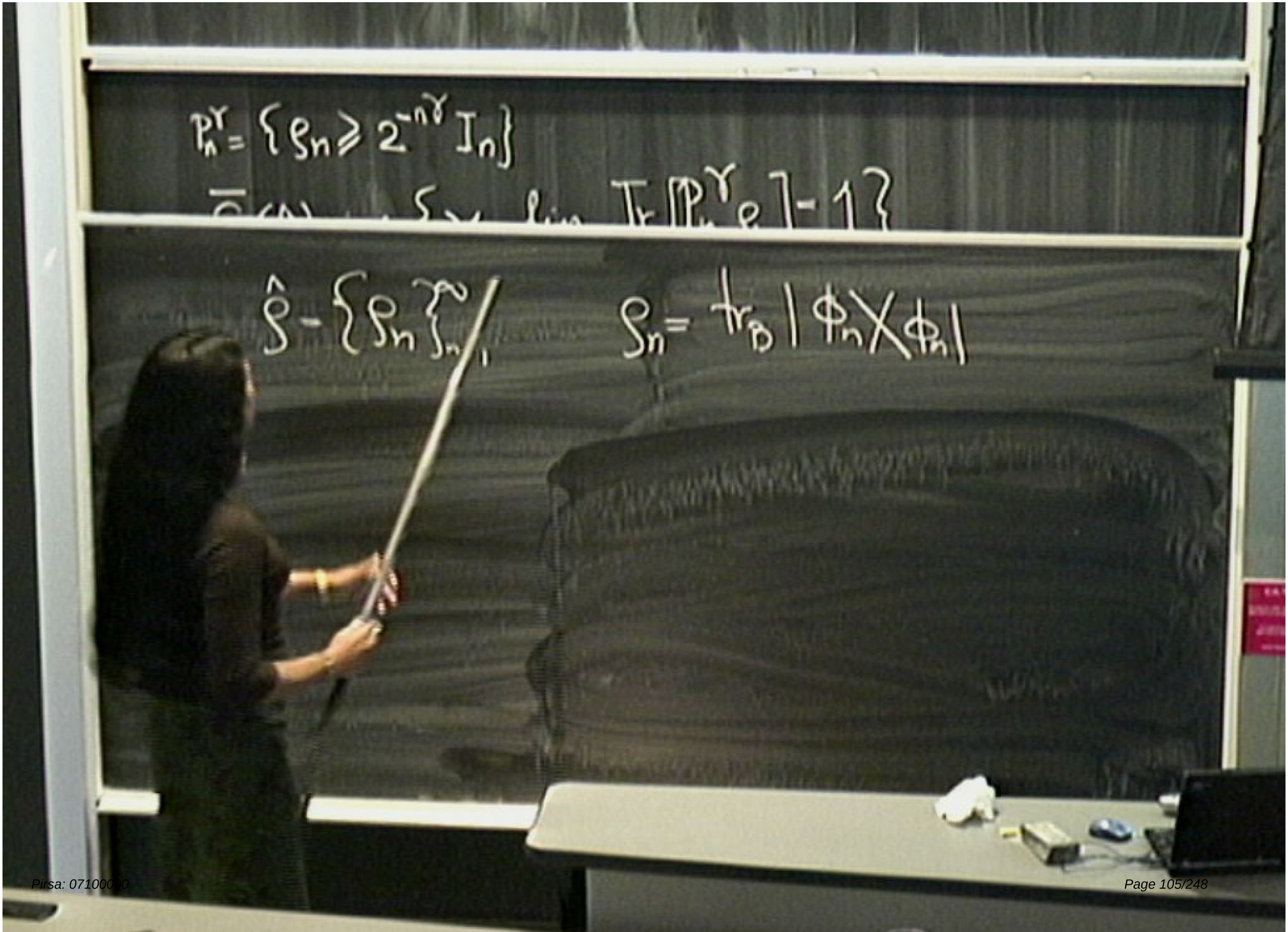
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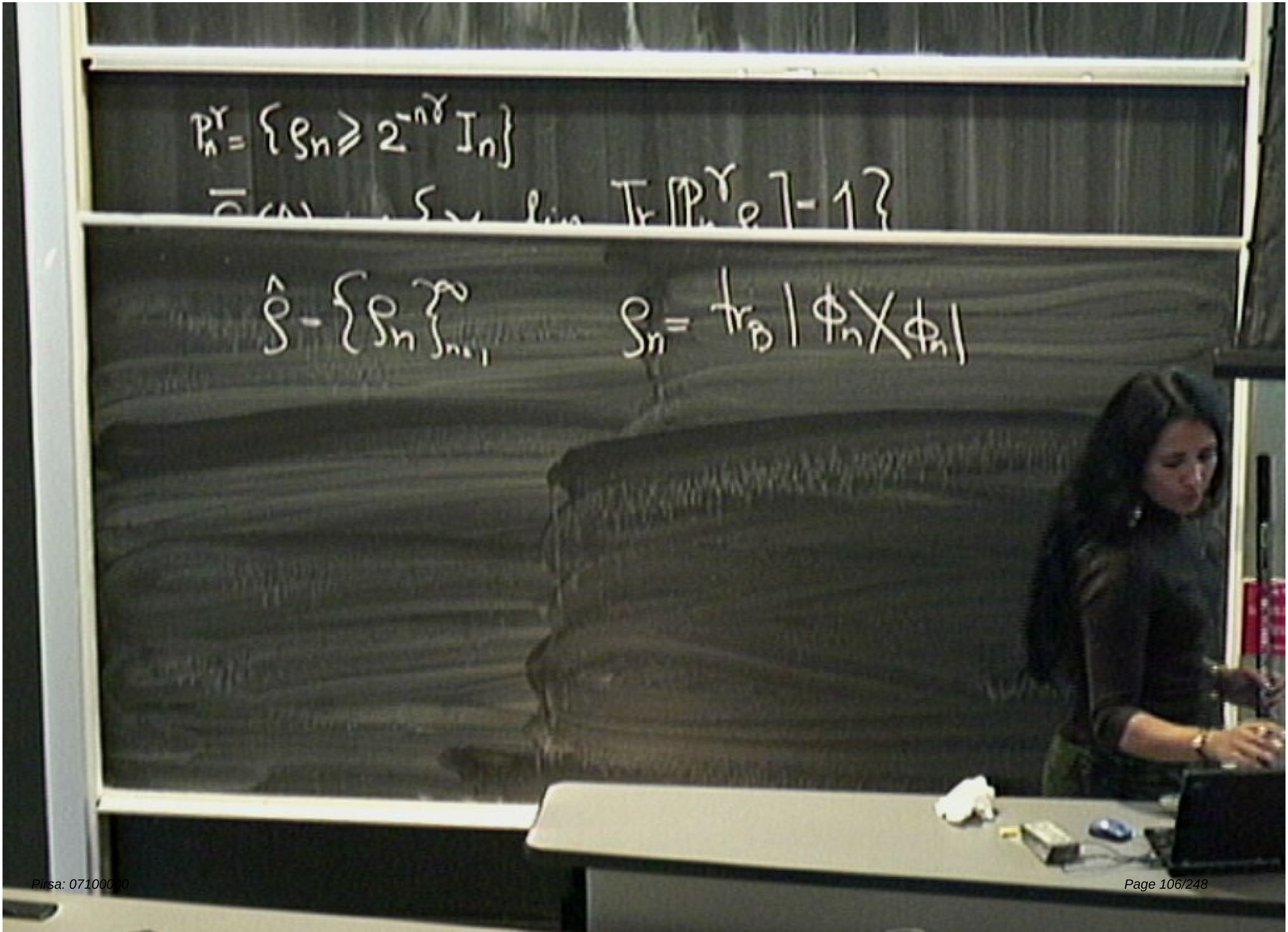
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- For $\hat{\rho} = \{\rho^{\otimes n}\}_{n=1}^{\infty}$ we have $\underline{S}(\hat{\rho}) = S(\rho) = \overline{S}(\hat{\rho})$

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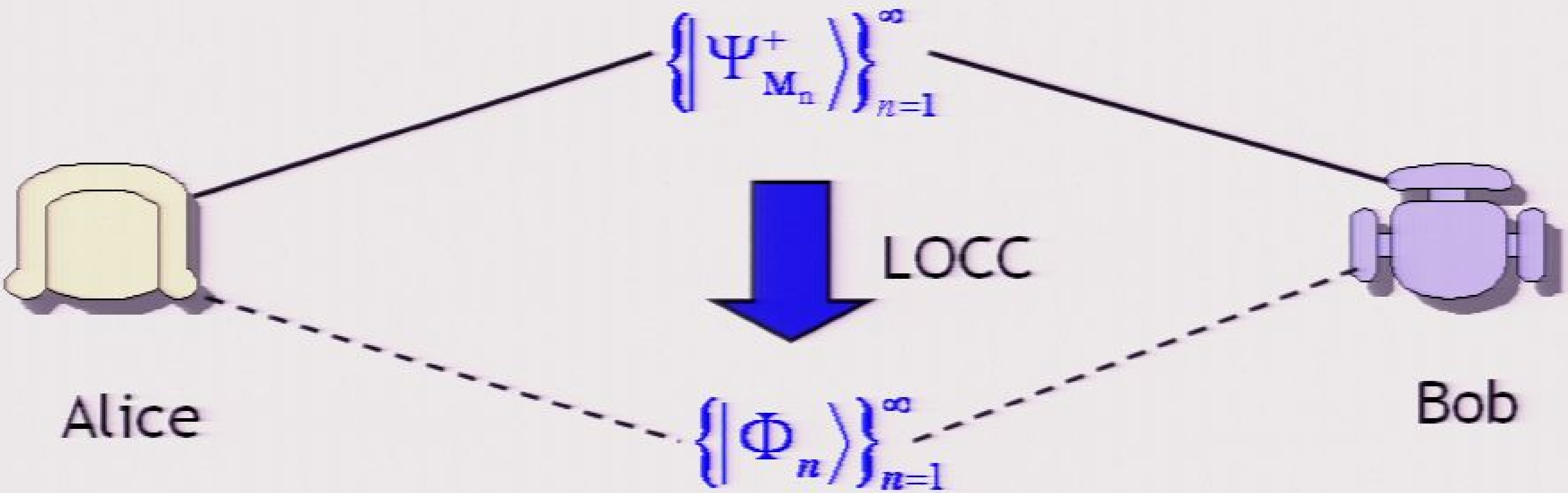
Asymptotic Entanglement Dilution of Pure States



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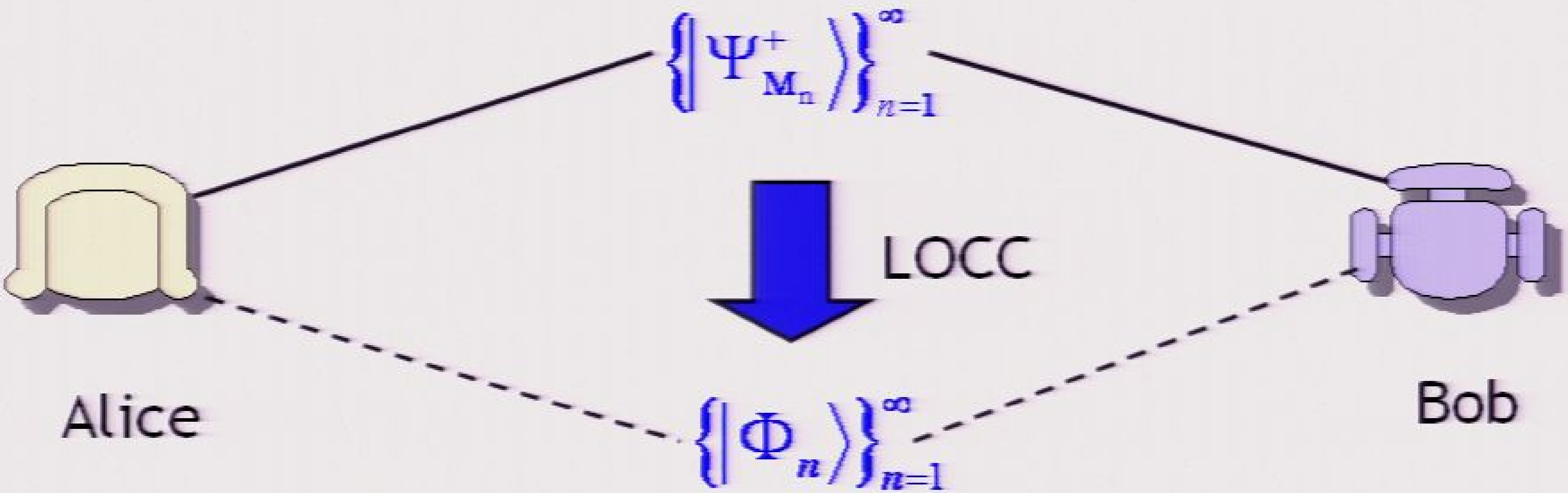

$$\left| \Psi^{+}_{M_n} \right\rangle = \frac{1}{\sqrt{M_n}} \sum_{k=1}^{M_n} \left| i_A^{(n)} \right\rangle \left| i_B^{(n)} \right\rangle$$

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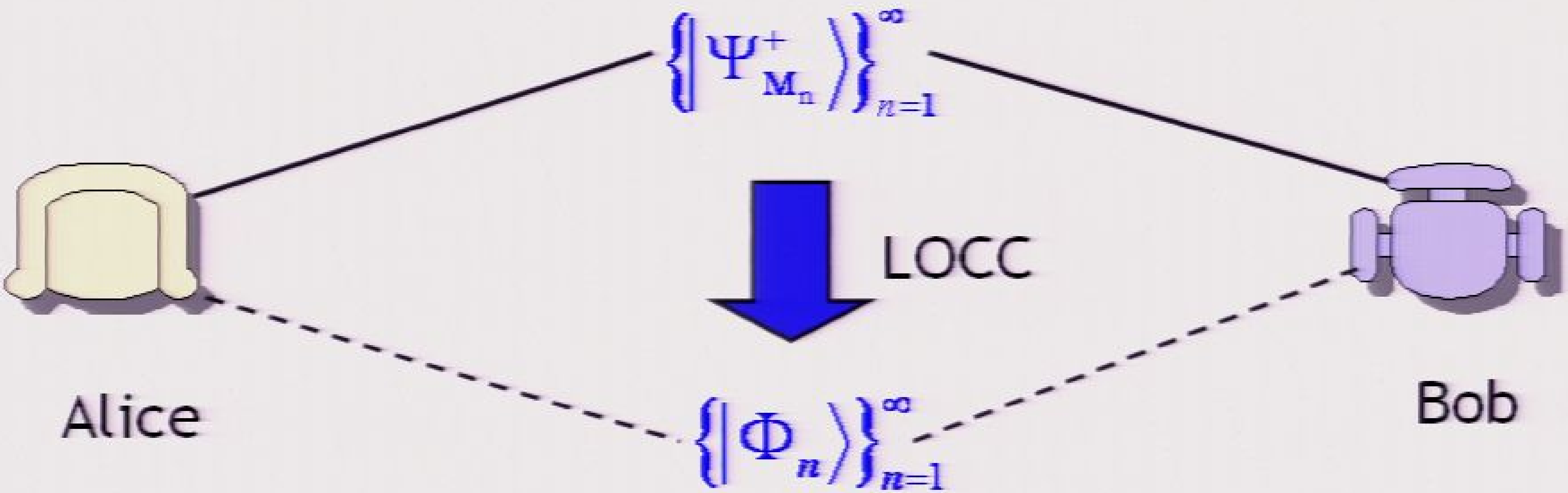

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: MES of rank M_n

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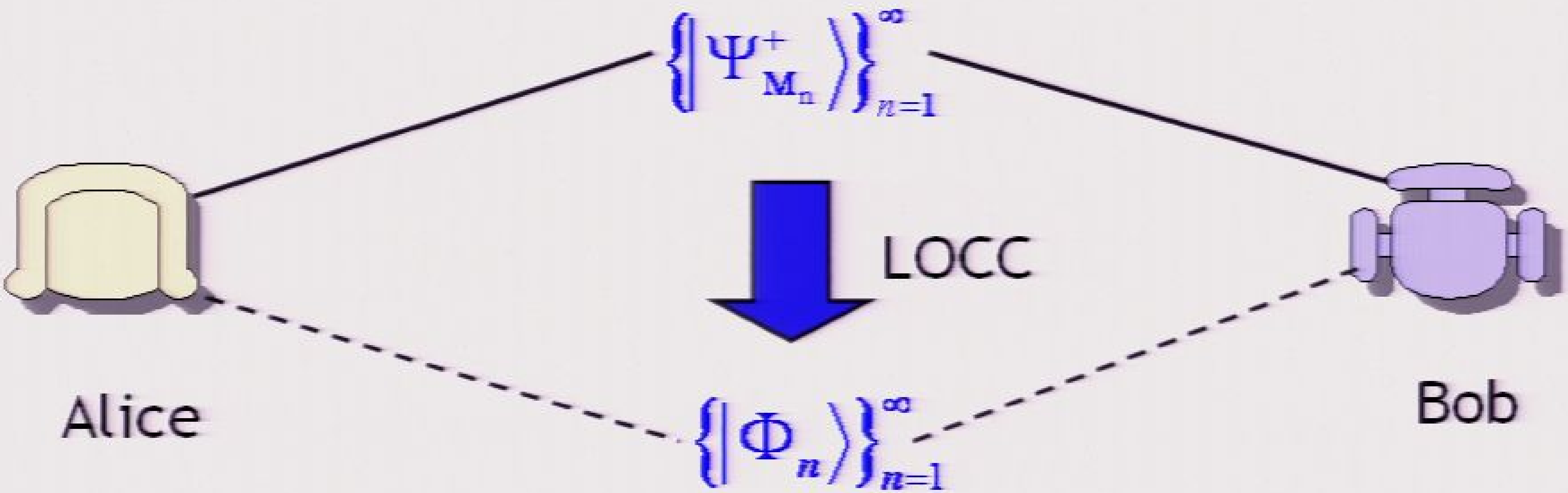

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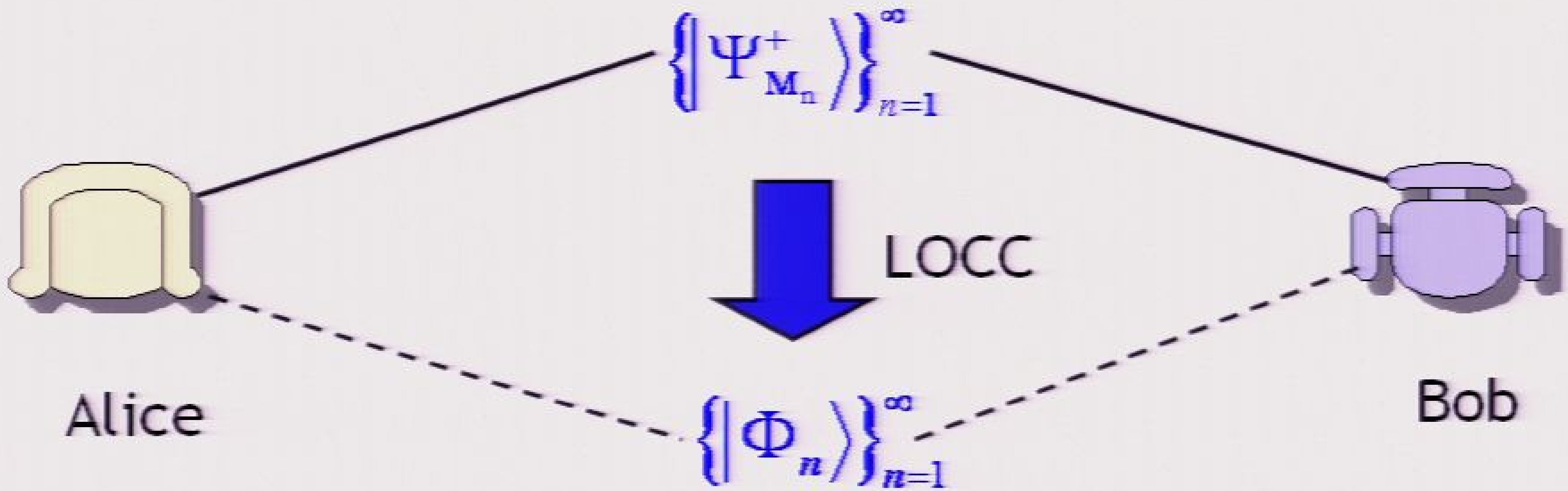

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Aim:

$$\left\{ \left| \Psi^{+}_{M_n} \right\rangle \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ \left| \Phi_n \right\rangle \right\}_{n=1}^{\infty}$$

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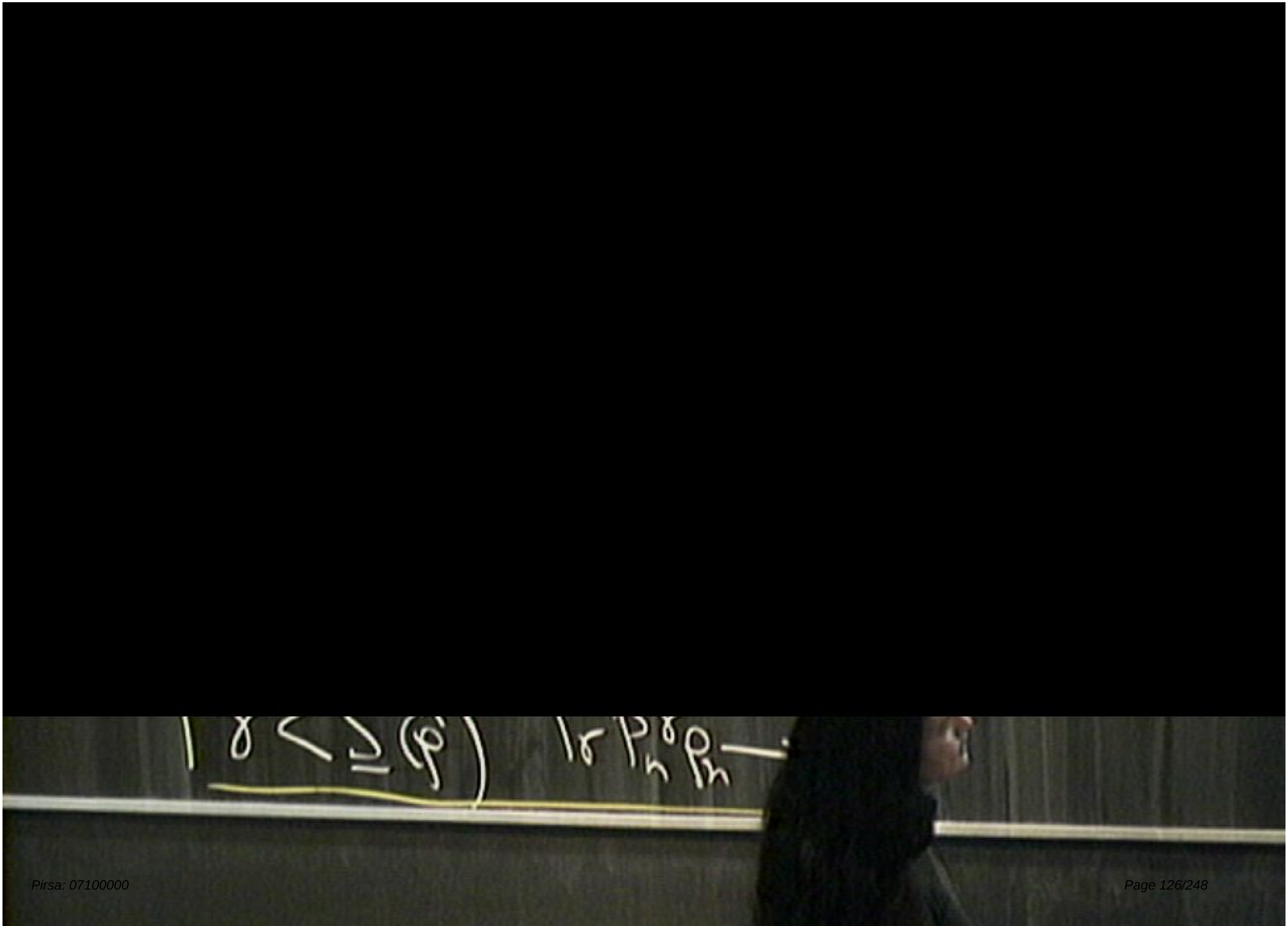

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$$\gamma > \overline{S}(\hat{\rho}) \quad \mathrm{Tr}(P_n^\gamma \rho_n) \longrightarrow 1$$

$$\underline{S}(\hat{\rho}) = \sup\{ \cdots \}$$

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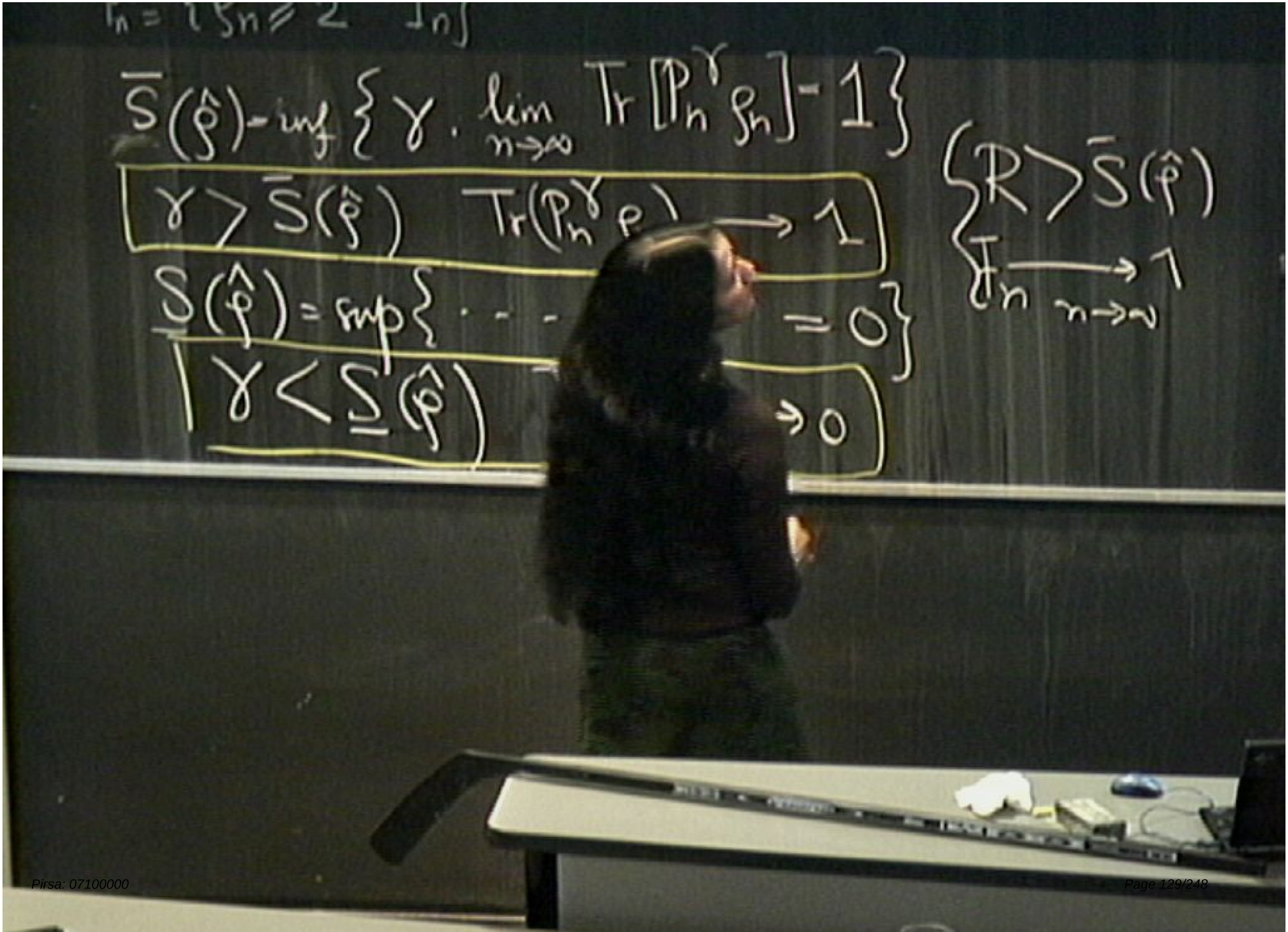
- i.e., the eigenspace corrs.to eigenvalues of ρ_n^A which are $\geq 2^{-n\overline{S}(\hat{\rho})}$ is a *high probability subspace*


S̄(ρ̂) = inf { γ . lim Tr [P_n^γ ρ_n] = 1 }
n→∞
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= 0 }
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R > S̄(ρ̂)
F_n → 1
n→∞

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- PROOF: [$R > \bar{S}(\hat{\rho})$ is achievable] :
- Let the target state $|\Phi_n\rangle$ have N_n non-zero Schmidt coefficients.

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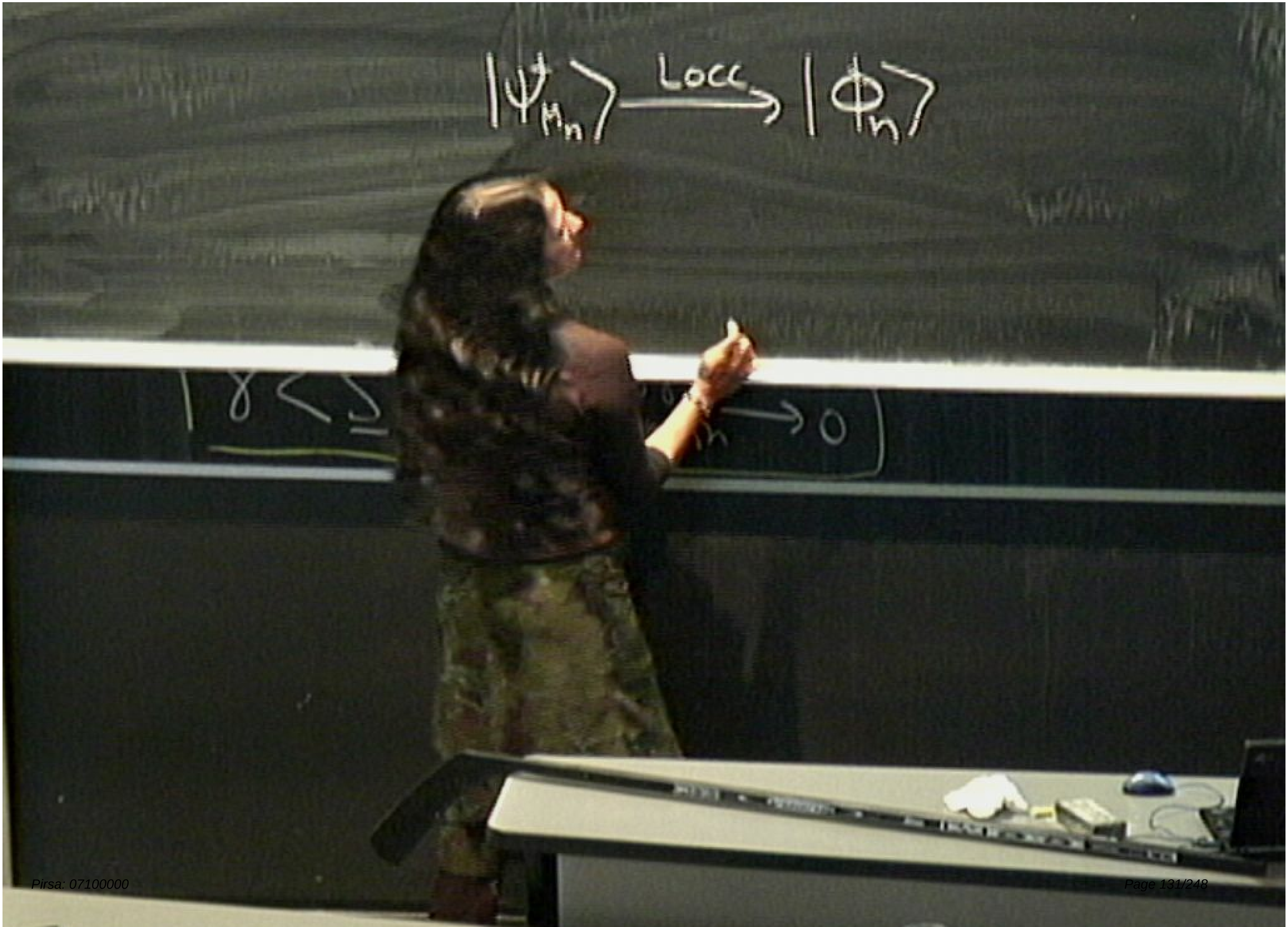
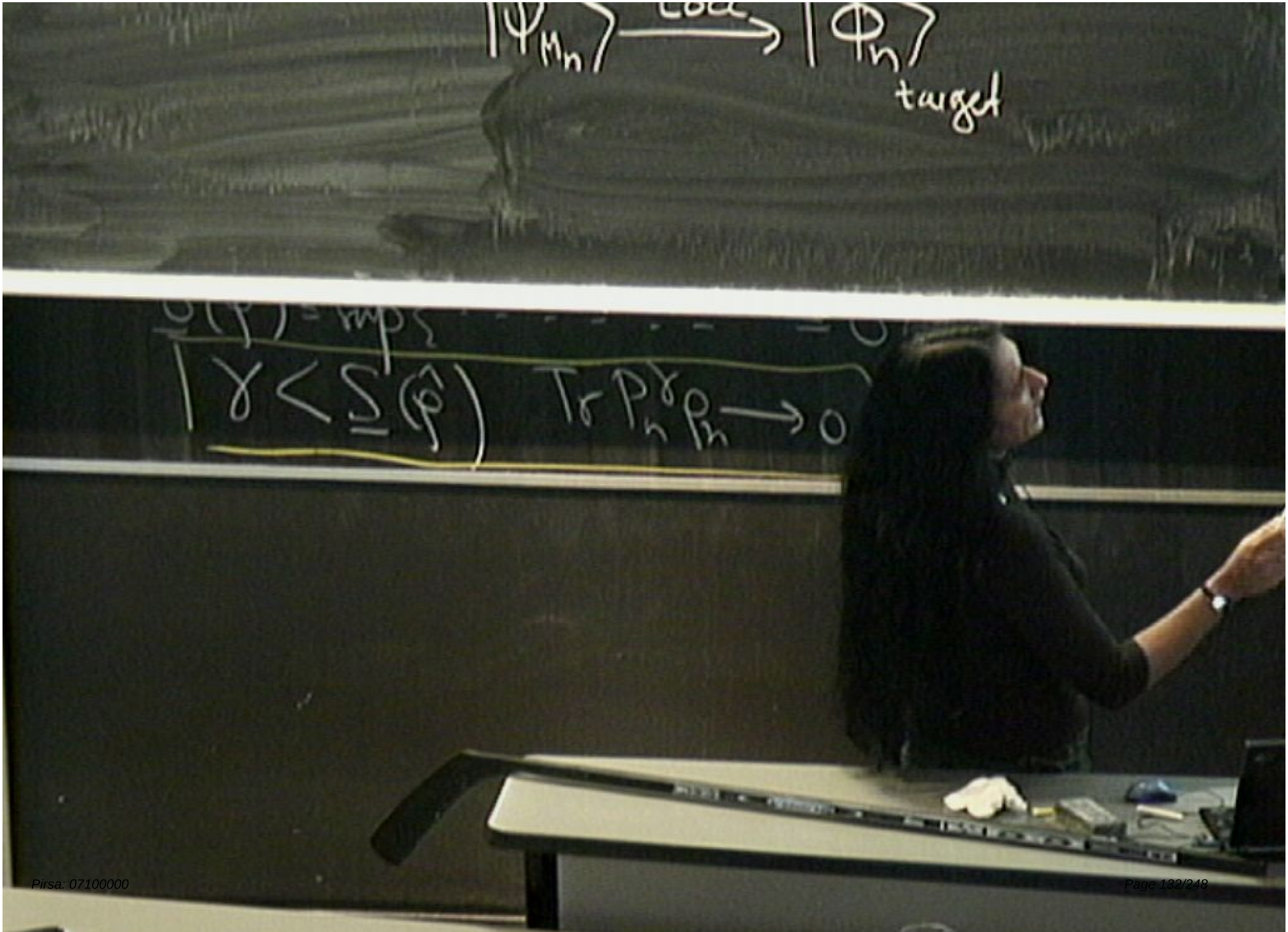

$|\psi_{M_n}\rangle \xrightarrow{LOCC} |\phi_n\rangle$



LOCC
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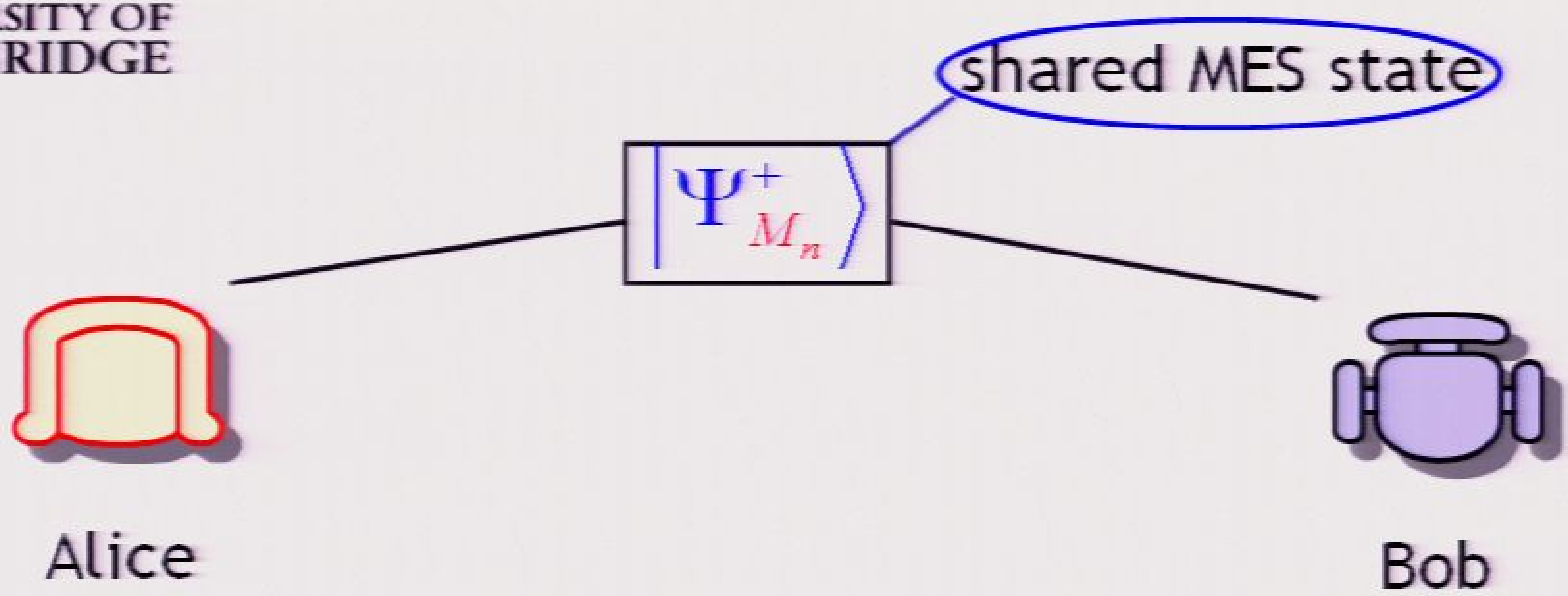
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Alice
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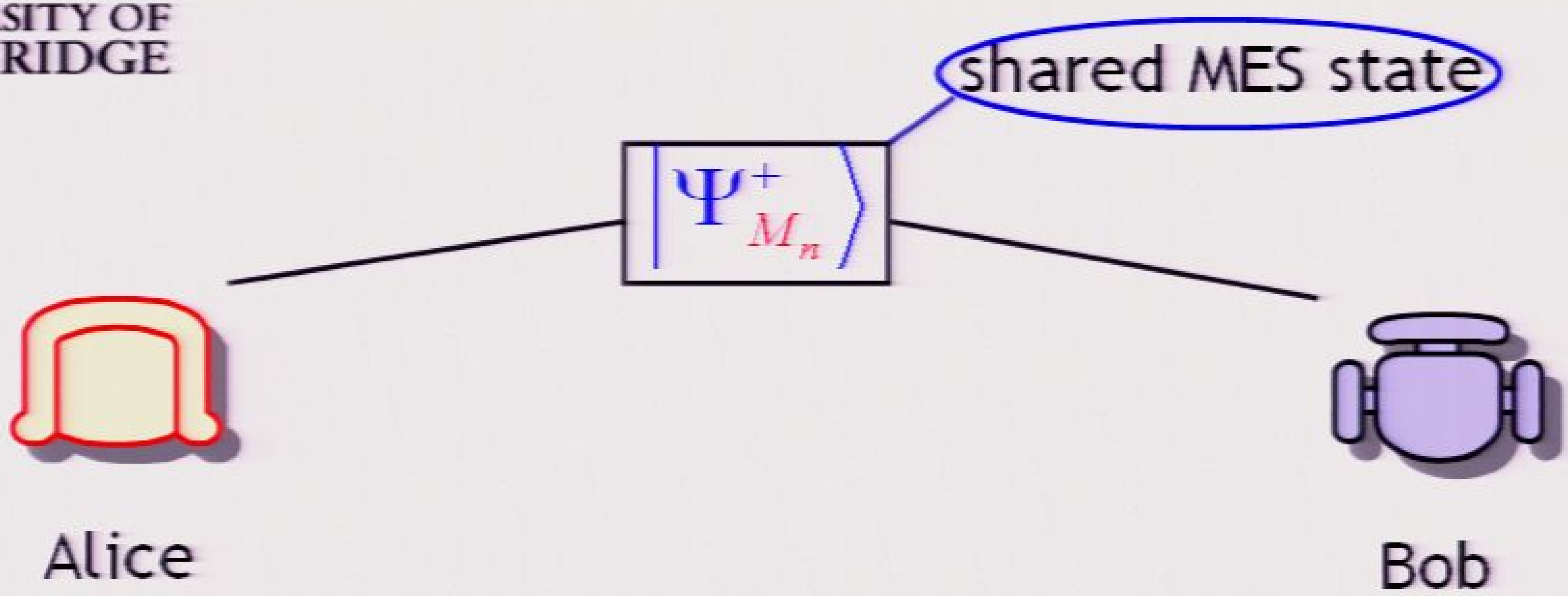

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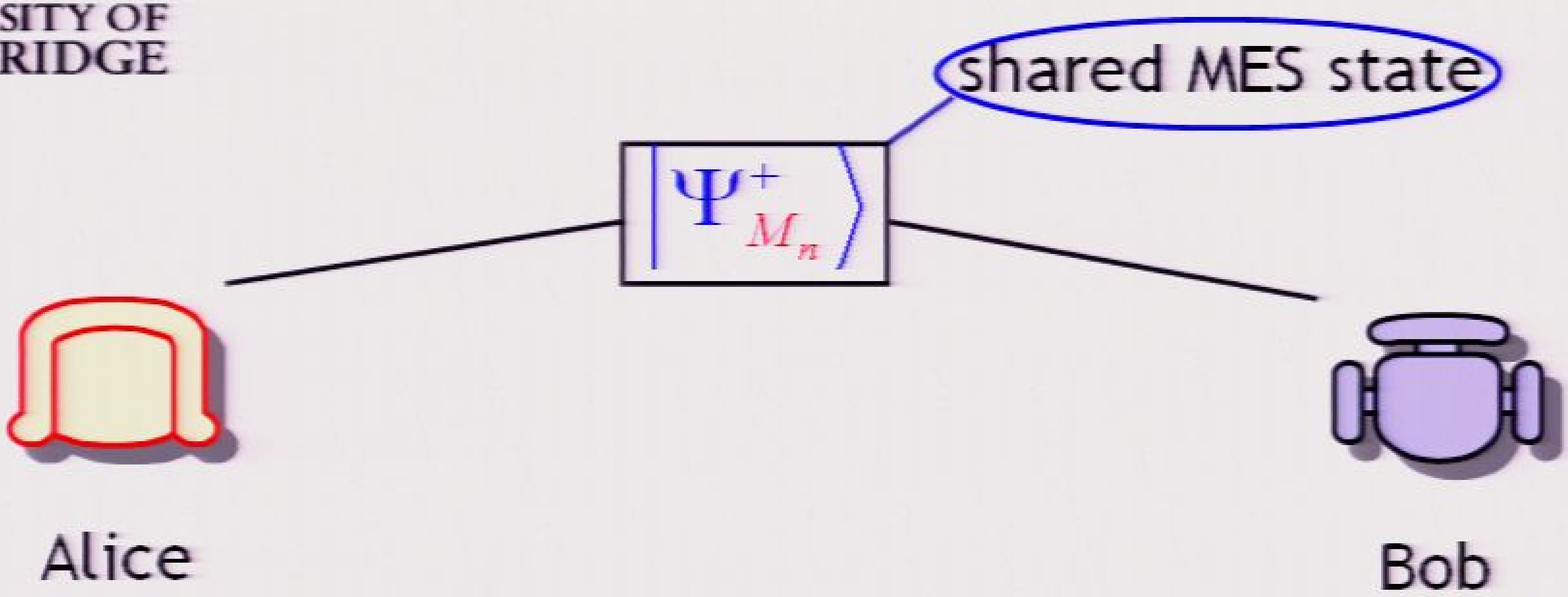

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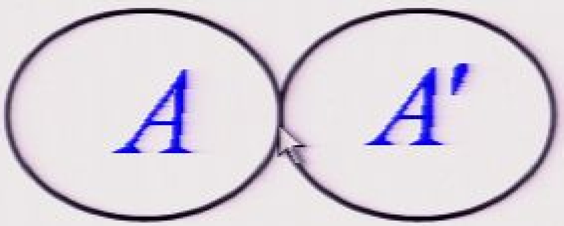

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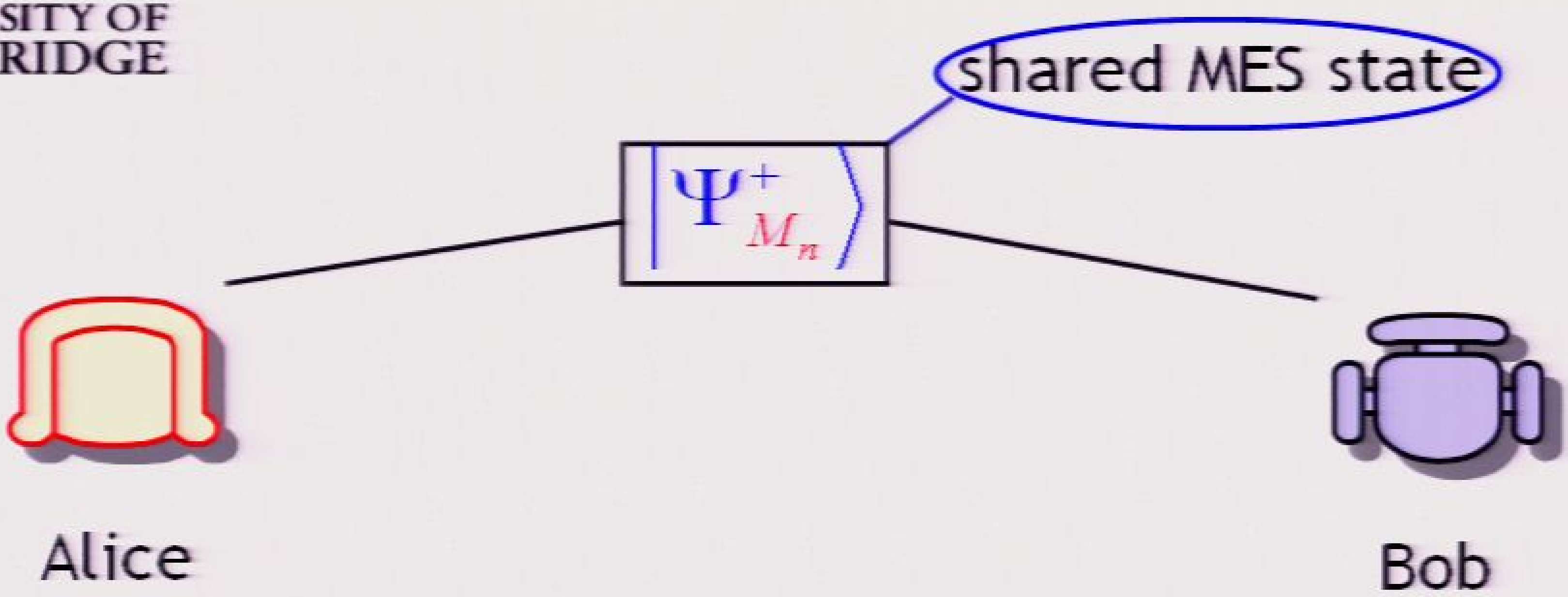

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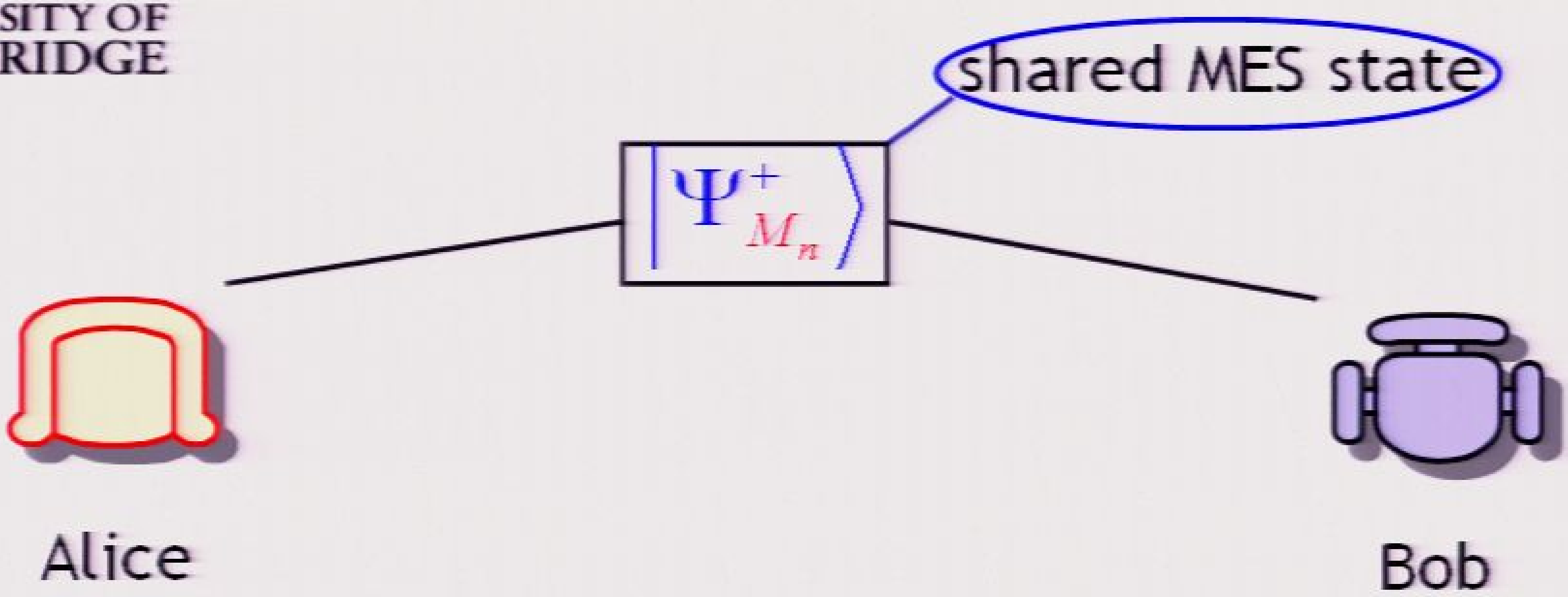

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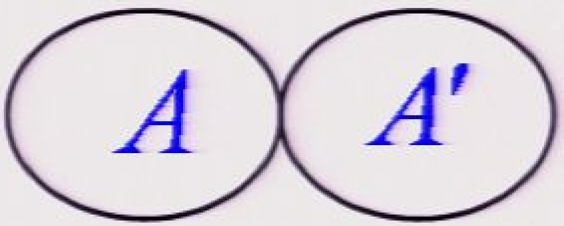

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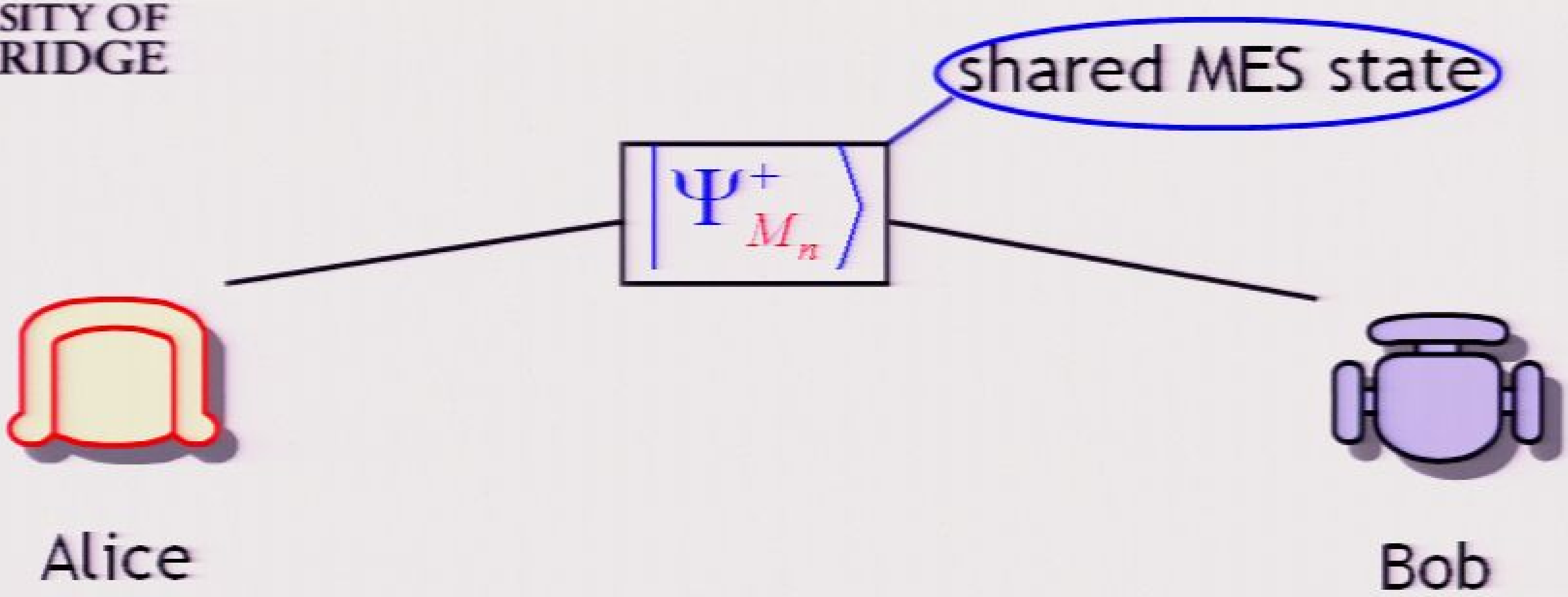

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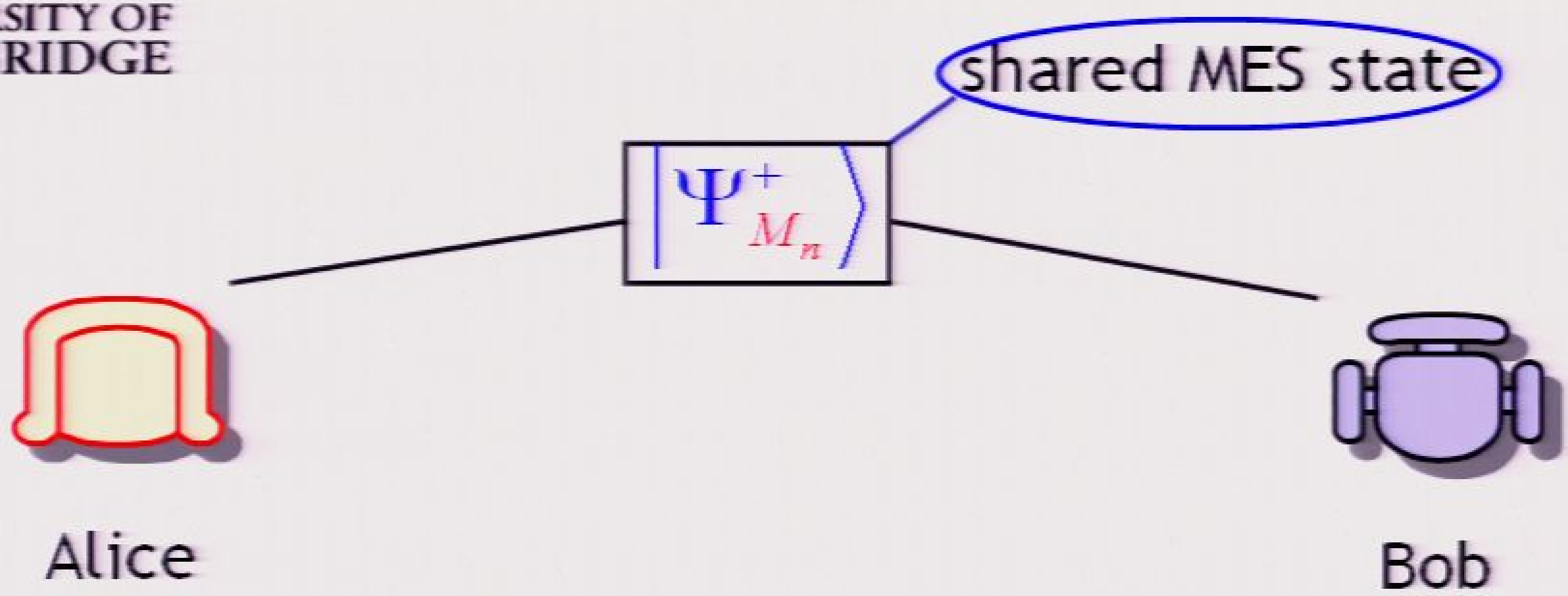

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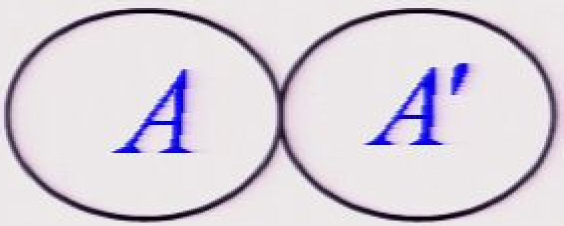

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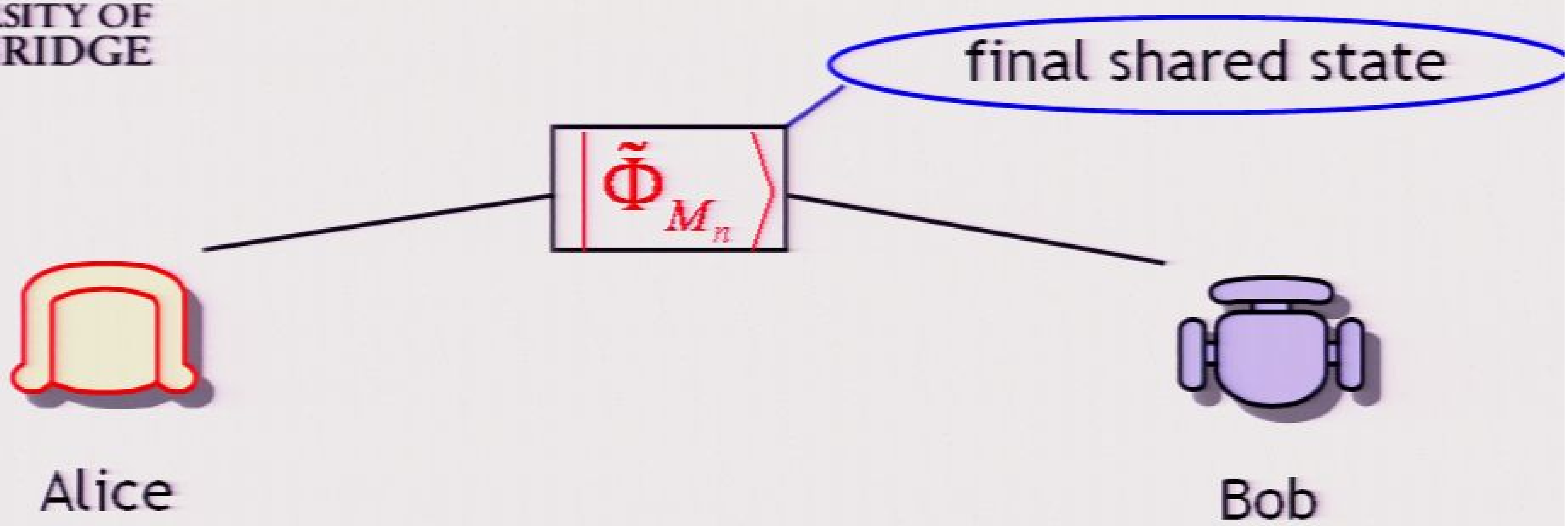
final shared state
$|\tilde{\Phi}_{M_n}\rangle$
Alice
Bob

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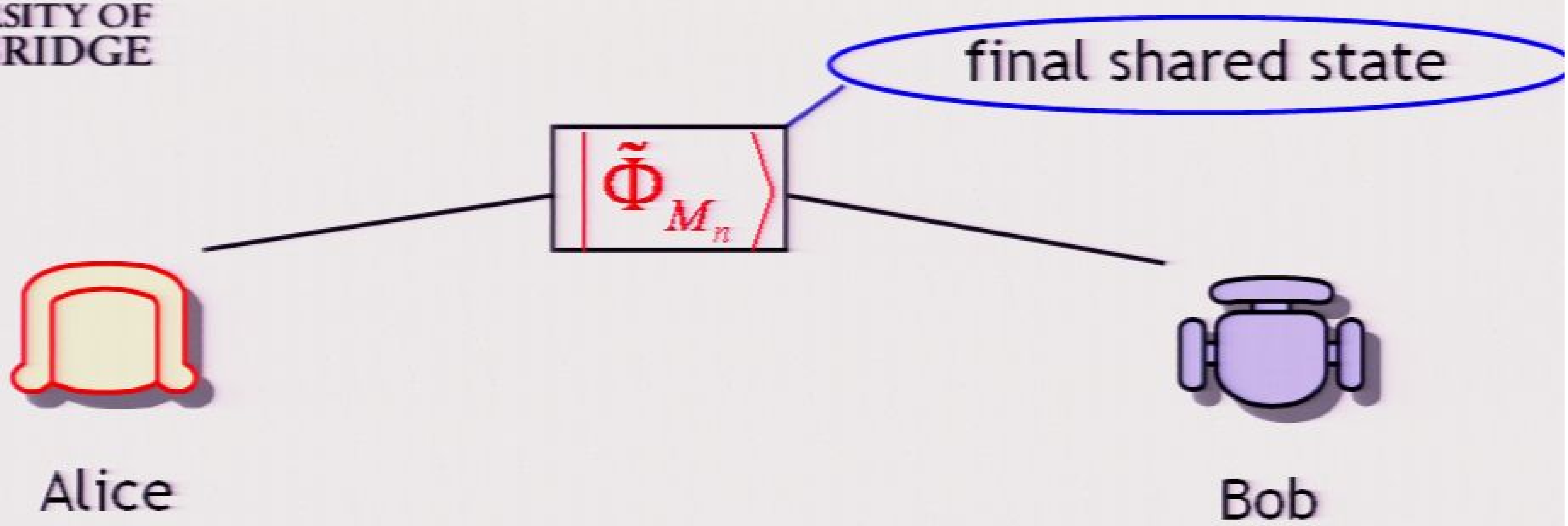

- If $M_n \geq N_n$ the teleportation can be done perfectly and the final shared state is the desired target state:

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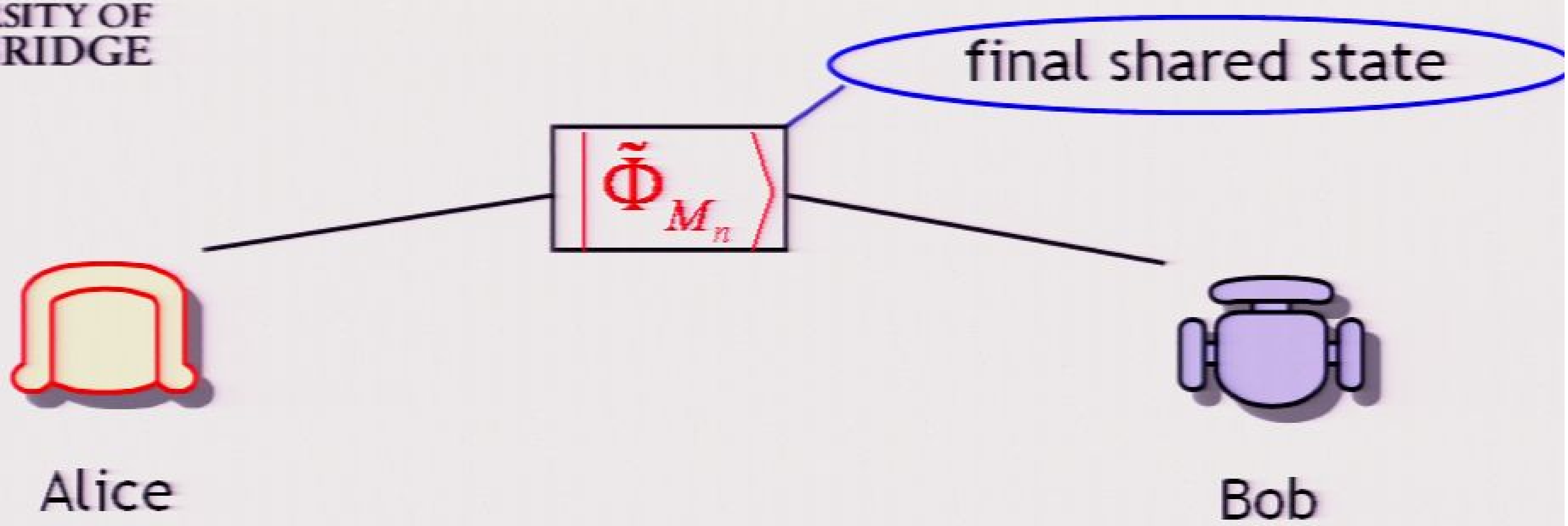

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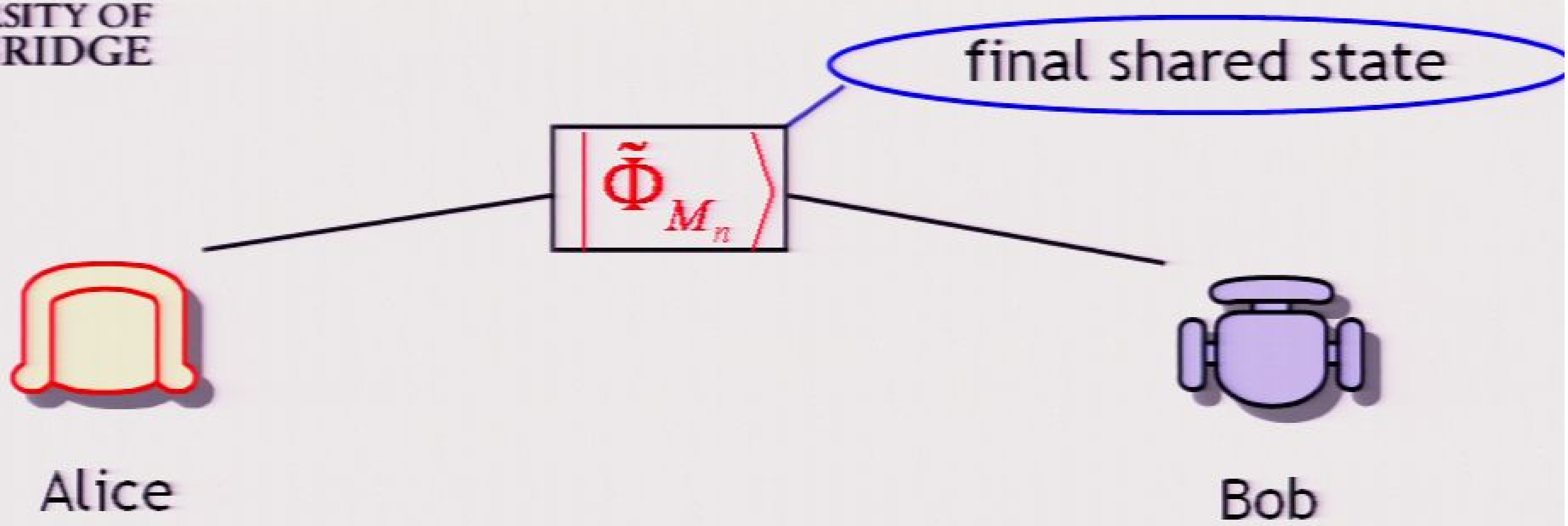

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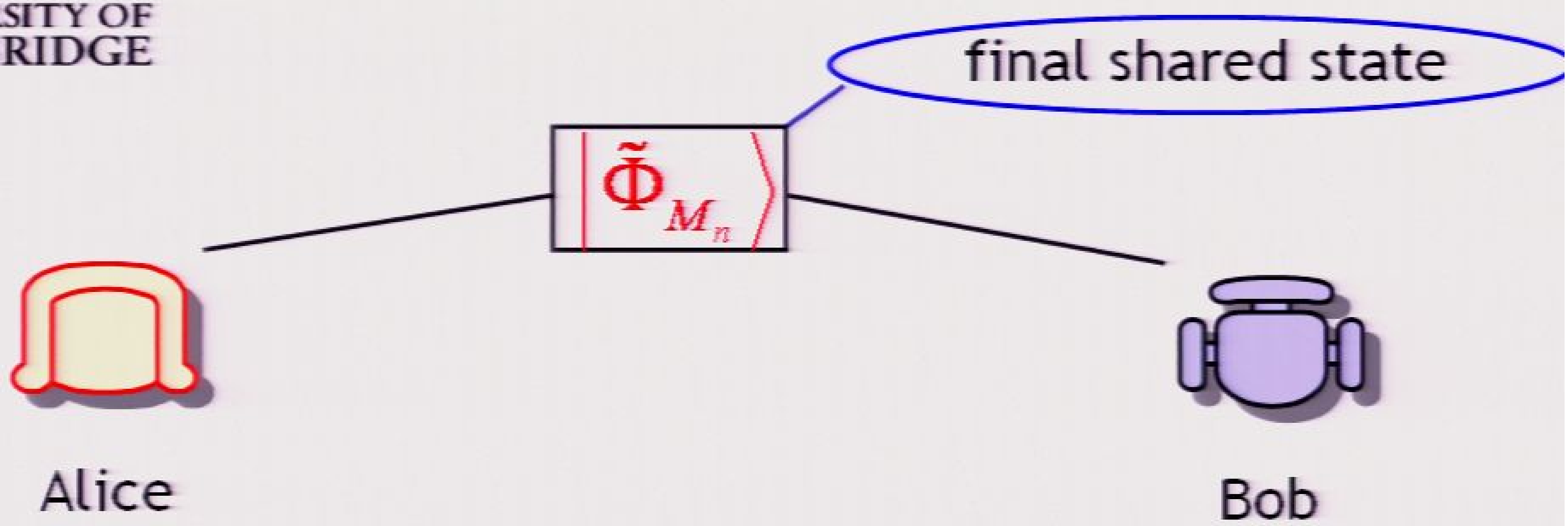

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- In this case the fidelity : $F_n = 1$

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- This is the "quantum scissors effect": if the quantum state to be teleported lives in a space of a dimension higher than the rank of the MES shared between the 2 parties, then the higher dimensional terms in the expansion of the state are "cut-off".

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- Hence, for $M_n < N_n$ the final shared state between Alice and Bob after the teleportation can be expressed as

$$\left|\tilde{\Phi}_{M_n}\right\rangle\left\langle\tilde{\Phi}_{M_n}\right| + \sigma_n^{AB}$$

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Fidelity for $M_n < N_n$

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$$F_n = F\left(\left|\tilde{\Phi}_{M_n}\right\rangle\left\langle\tilde{\Phi}_{M_n}\right| + \sigma_n^{AB}, \left|\Phi_n\right\rangle\left\langle\Phi_n\right|\right)$$

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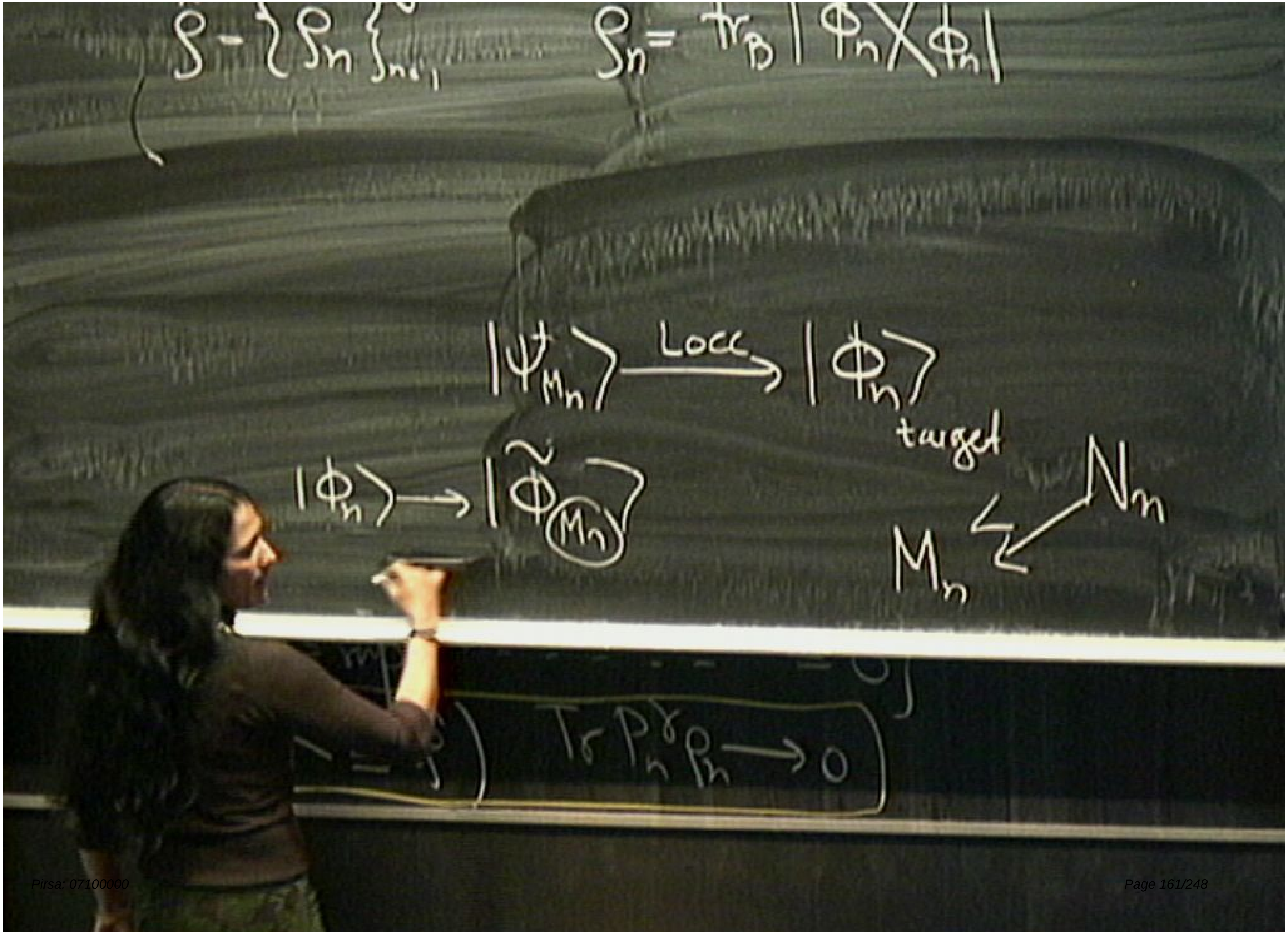
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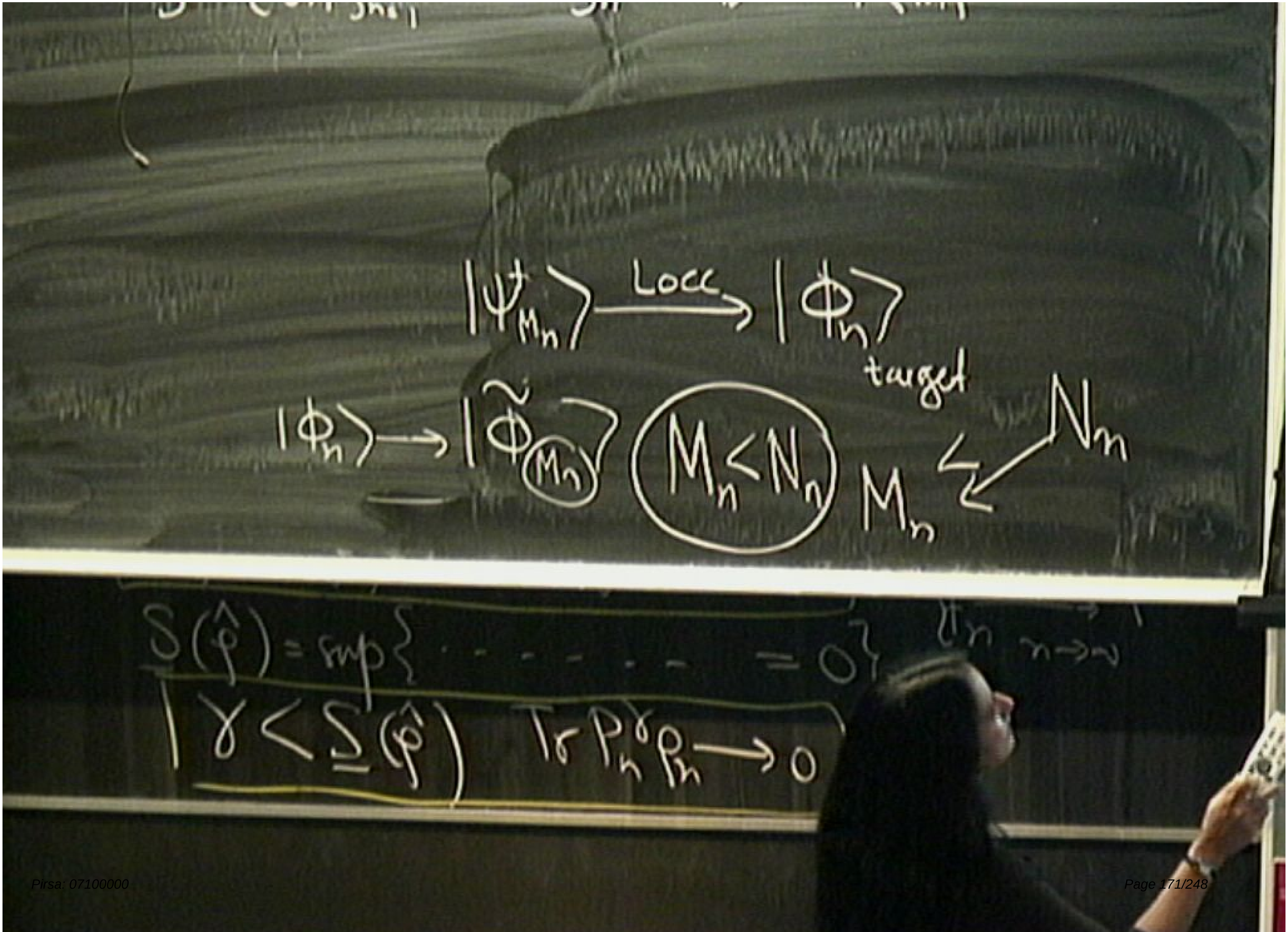
- We saw that $$\boxed{F_n \geq \mathrm{Tr}\left(Q^A_{M_n} \rho^A_n\right)}$$

[redacted]


|ψ_{M_n}⟩ LOCC → |φ_n⟩
target
|φ_n⟩ → |φ̃_{(M_n)}⟩
M_n < N_n
M_n ← N_n
S(ρ̂) = sup{ - - - - - - = 0}
γ < S(ρ̂) Tr P_n^γ ρ_n → 0

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- We saw that $$F_n \geq \mathrm{Tr}\left(Q^A_{M_n}\rho^A_n\right)$$

Q1) How can we prove that $F_n \xrightarrow[n\to\infty]{} 1$?

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- (Q2) Why?
- (A2) Because $\mathrm{Tr}\left(P^\gamma_n \rho^A_n\right) \xrightarrow[n\to\infty]{} 1$
- (Q3) How can we choose M_n such that (a) holds ?

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Eigenvalues of ρ_n^A in decreasing order

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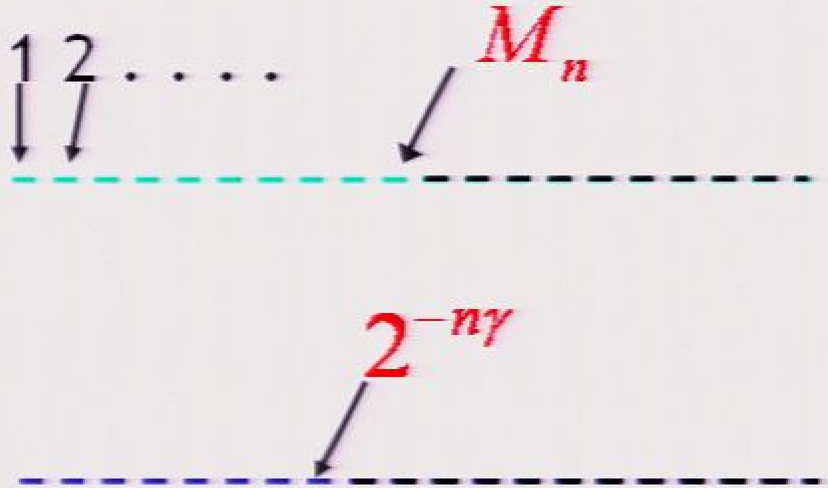
Eigenvalues of ρ_n^A *in decreasing order*

- labels 1 2 . . . . M_n

$Q_{M_n}^A$ *projects onto first* M_n *eigenvalues*

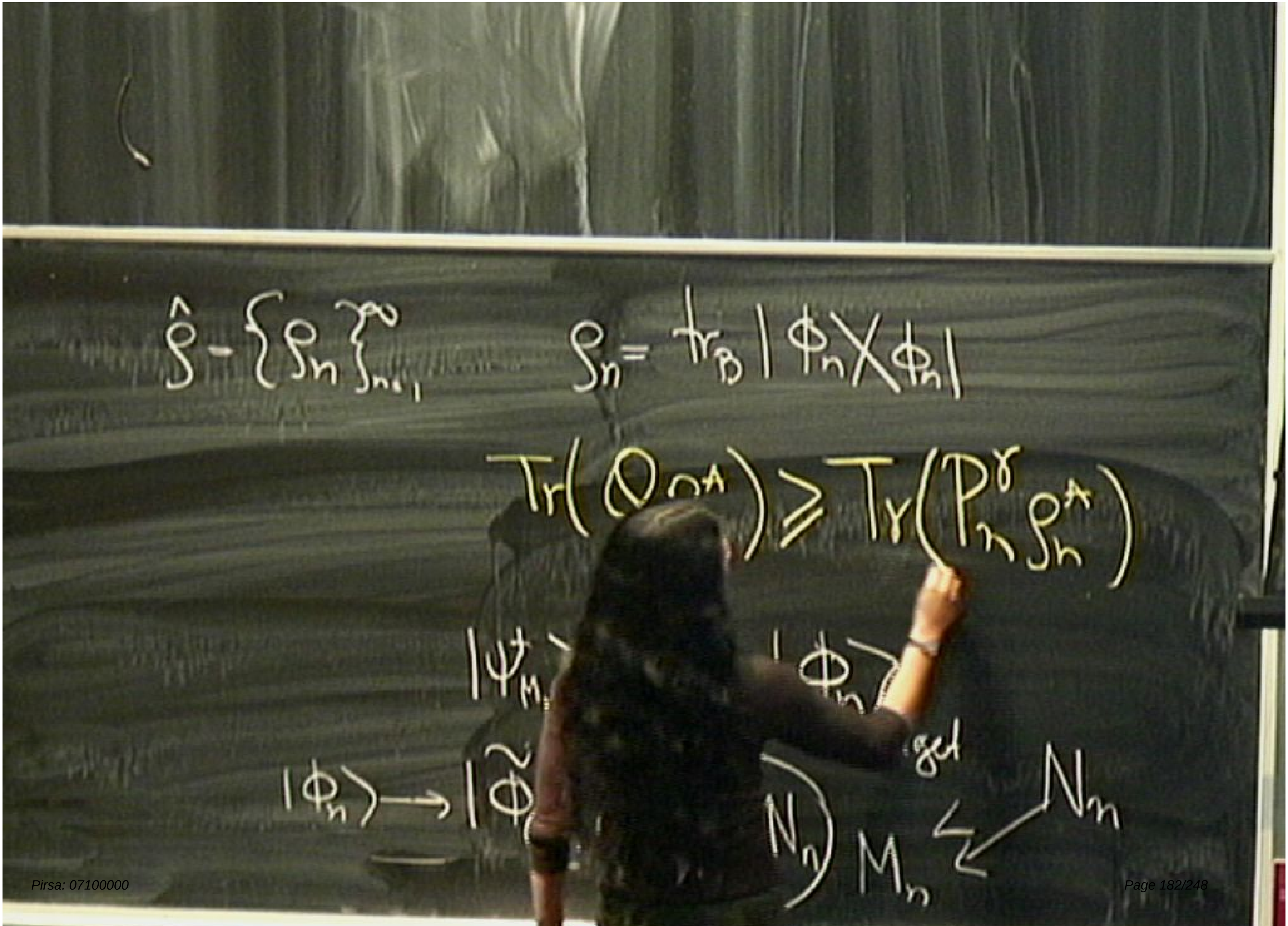
Eigenvalues of ρ_n^A *in decreasing order*

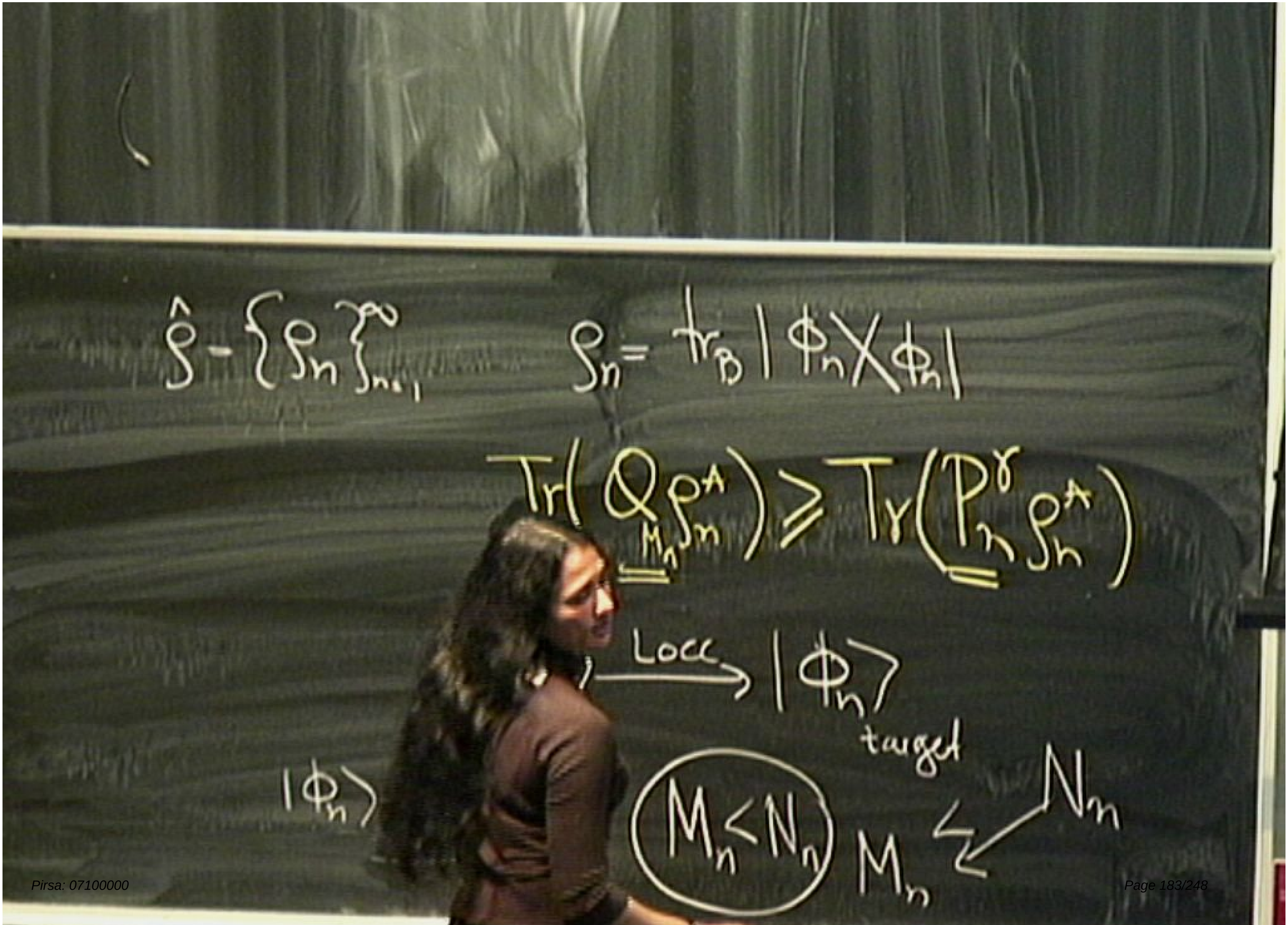
- labels
- values



$Q_{M_n}^A$ *projects onto first* M_n *eigenvalues*

P_n^γ *projects onto all eigenvalues* $\geq 2^{-n\gamma}$


$\hat{\rho} = \{\rho_n\}_{n=1}^{\infty}$
$\rho_n = tr_B |\phi_n\rangle\langle\phi_n|$
$Tr(Q\rho^A) \geq Tr(P_n^{\gamma}\rho_n^A)$
$|\phi_n\rangle \longrightarrow |\tilde{\phi}$
N_n
M_n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LOCC
$|\phi_n\rangle$
target
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N_n

Eigenvalues of ρ_n^A *in decreasing order*

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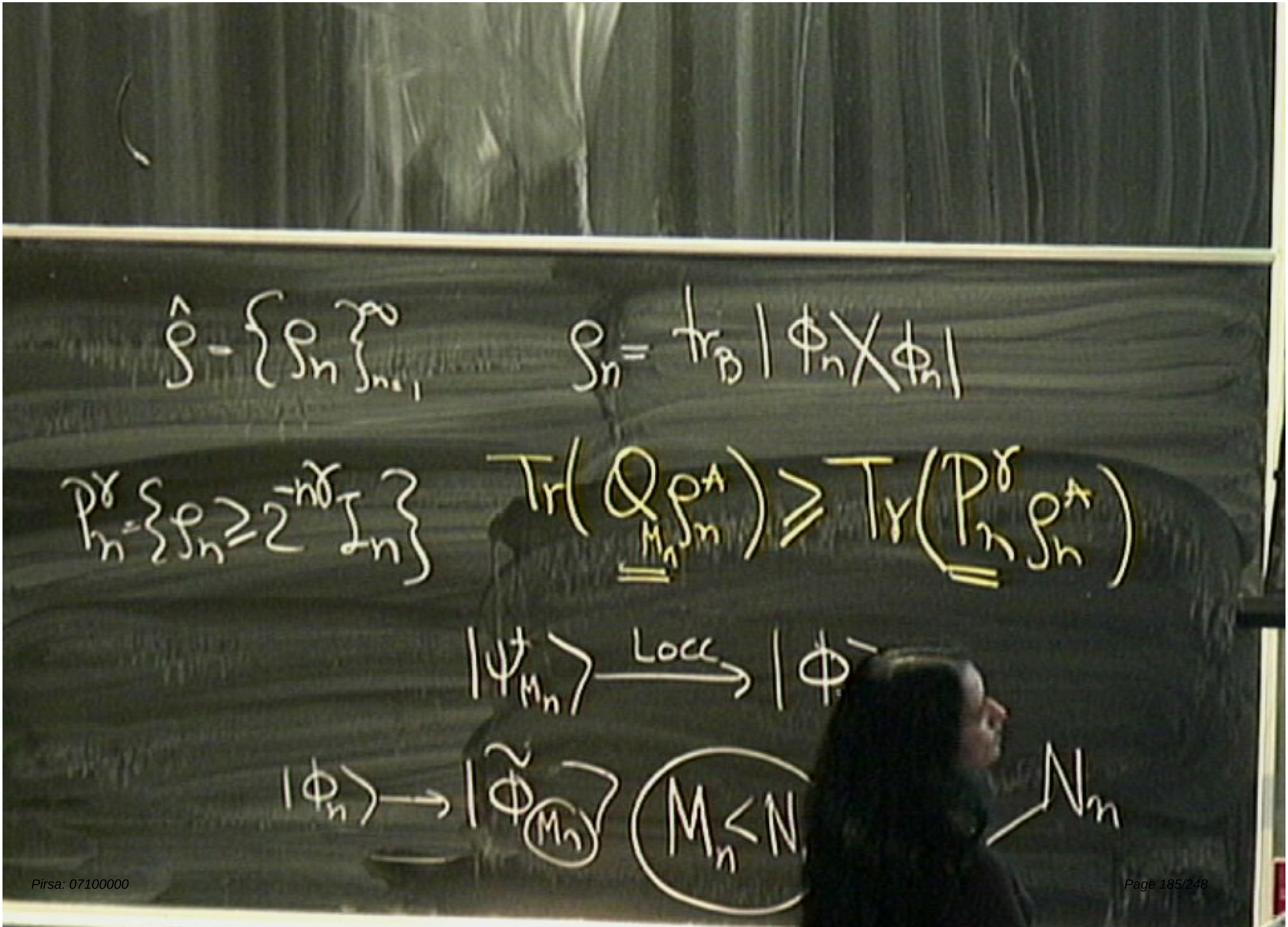
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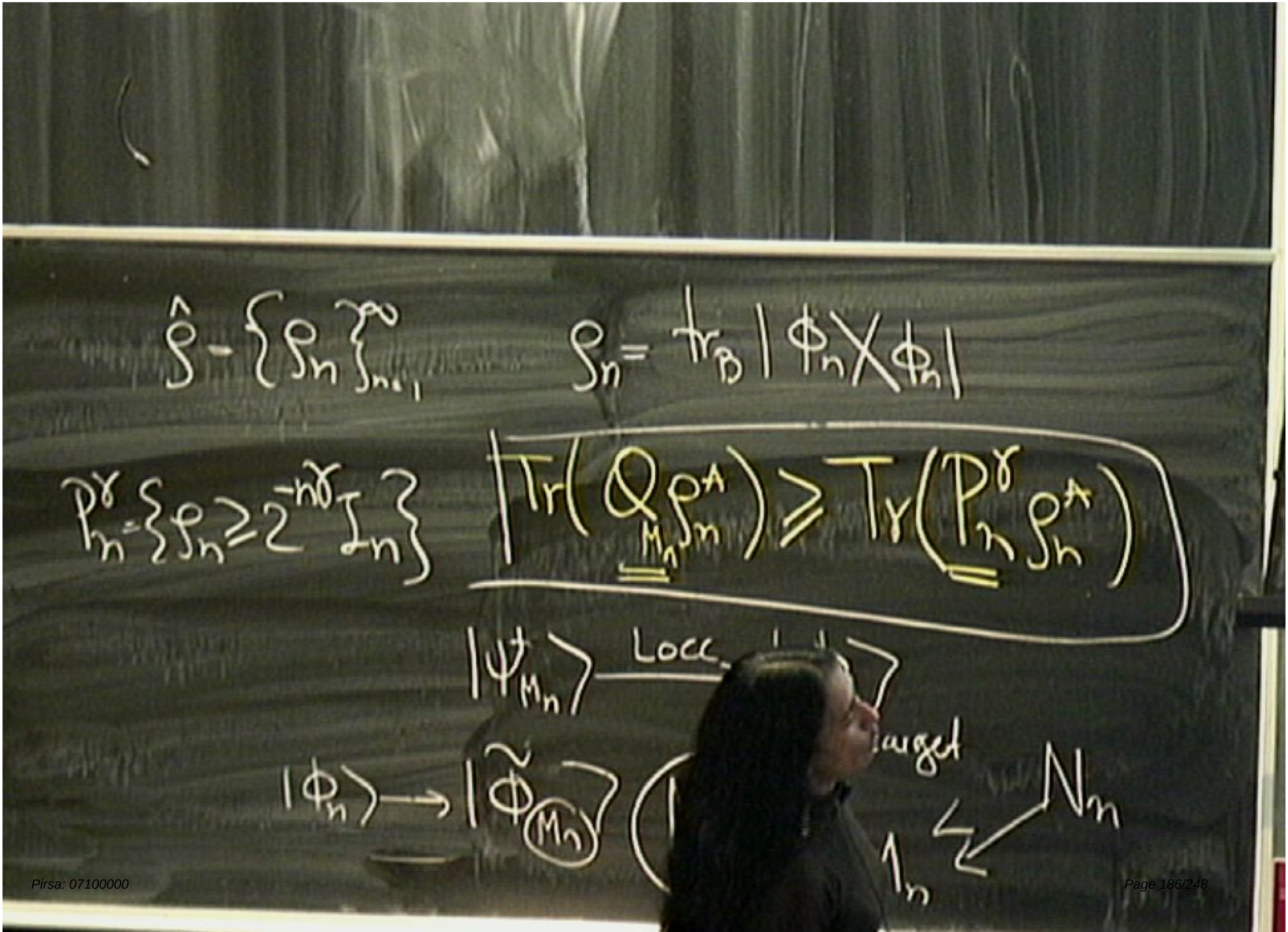
$|\phi_n\rangle \longrightarrow |\tilde{\phi}_{M_n}\rangle$
$M_n < N$
N_n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(there are $\leq 2^{n\gamma}$ such values $\mathrm{Tr}P_n^\gamma \leq 2^{n\gamma}$)

- If we choose $M_n \geq 2^{n\gamma}$ then $\mathrm{Tr}\left(Q_{M_n}^A \rho_n^A\right) \geq \mathrm{Tr}(P_n^\gamma \rho_n^A)$

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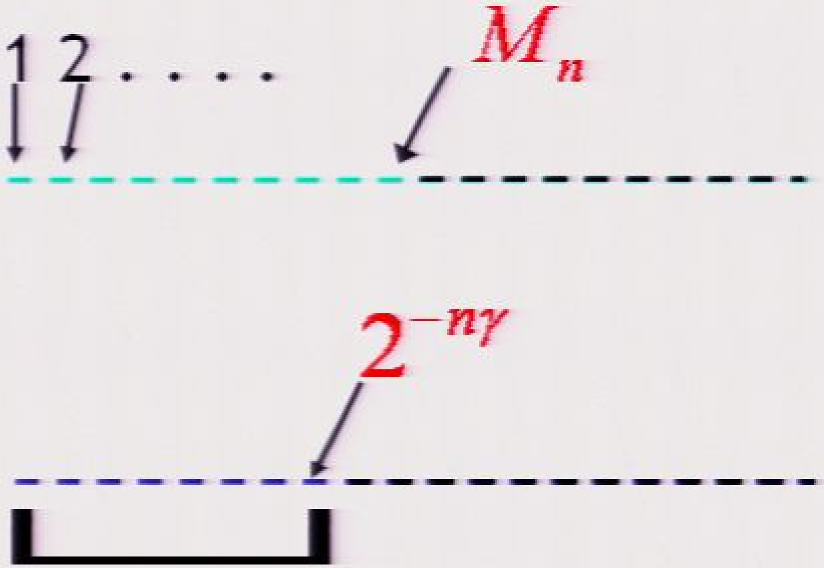
Eigenvalues of ρ_n^A *in decreasing order*



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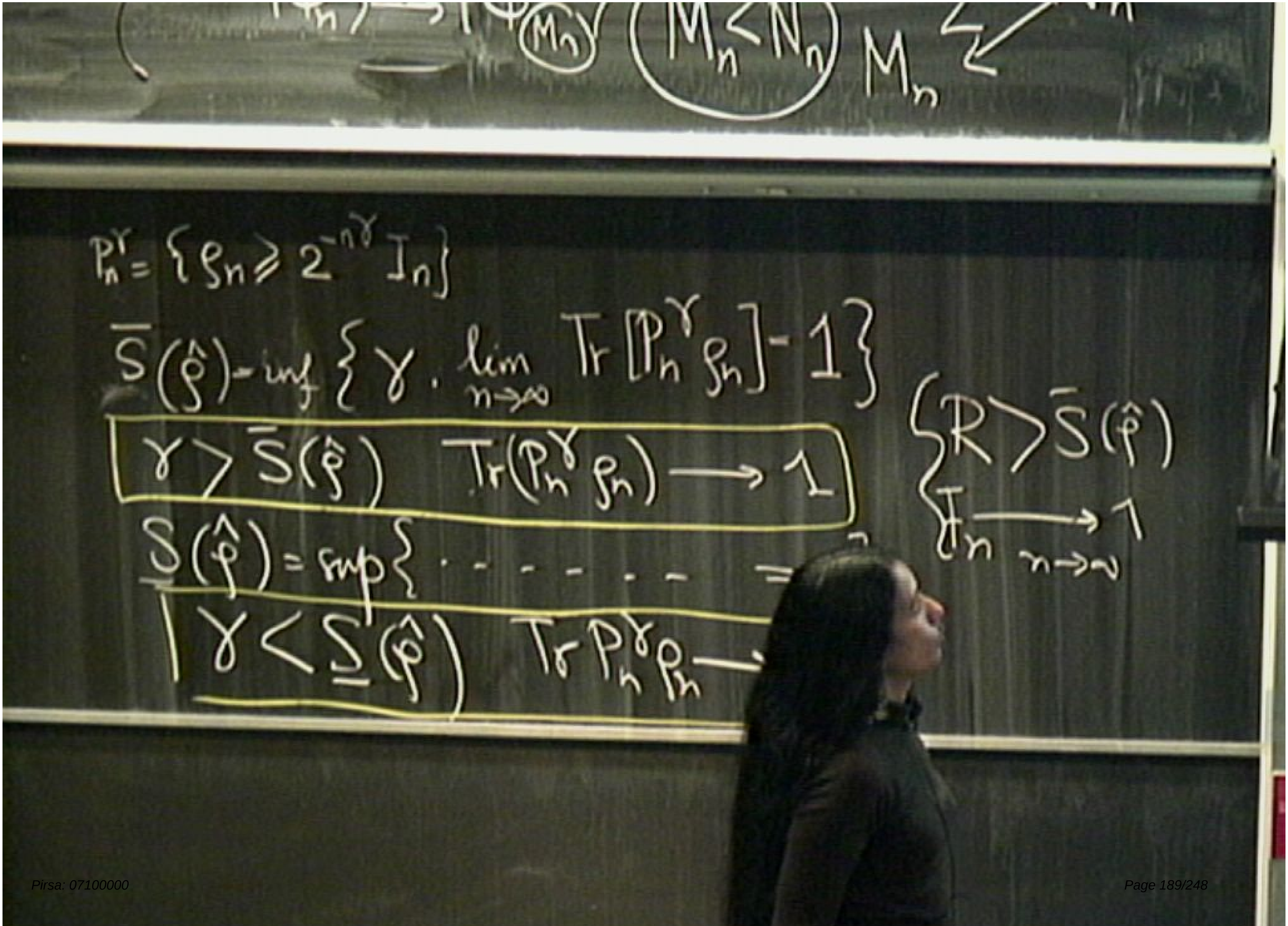
- &

$$F_n \geq \mathrm{Tr}\left(Q_{M_n}^A \rho_n^A\right) \geq \mathrm{Tr}\left(P_n^\gamma \rho_n^A\right) \xrightarrow[n\to\infty]{} 1 \quad \text{for} \quad \gamma > \overline{S}(\hat{\rho})$$



$P_n^\gamma = \{\rho_n \geq 2^{-n\gamma} I_n\}$
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$R > \bar{S}(\hat{\rho})$
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If the rank M_n of the initial shared MES $\left|\Psi^+_{M_n}\right\rangle$ is:

$M_n = \lceil 2^{n\gamma} \rceil$ with $\gamma > \overline{S}(\hat{\rho})$, then $F_n \xrightarrow[n\to\infty]{} 1$

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- Hence, a rate $R = \frac{1}{n}\log M_n > \gamma > \bar{S}(\hat{\rho})$ is achievable!

- *Weak converse*: A rate $R < \bar{S}(\hat{\rho})$ is not achievable

- Hence, entanglement cost:

$$E_C = \inf R = \bar{S}(\hat{\rho})$$

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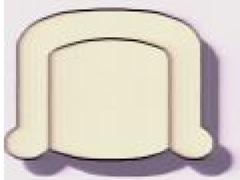
Schematic summary of protocol for entanglement dilution

Aim:

$$\left\{\left|\Psi^{+}_{M_n}\right\rangle\right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{\left|\Phi_n\right\rangle\right\}_{n=1}^{\infty}$$

where

$$\left|\Phi_n\right\rangle = \sum_{k=1}^{N_n} \sqrt{\lambda_{n,k}}\left|k_A^{(n)}\right\rangle\left|k_B^{(n)}\right\rangle$$



(1) Locally prepares AA' in state $\left|\Phi_n\right\rangle$

(2) She teleports A' to Bob

a) If $M_n \geq N_n$ then $F_n = 1$

b) If $M_n < N_n$ then $F_n \xrightarrow[n\to\infty]{} 1$ if we choose M_n

such that $\frac{1}{n}\log M_n > \bar{S}(\hat{\rho})$

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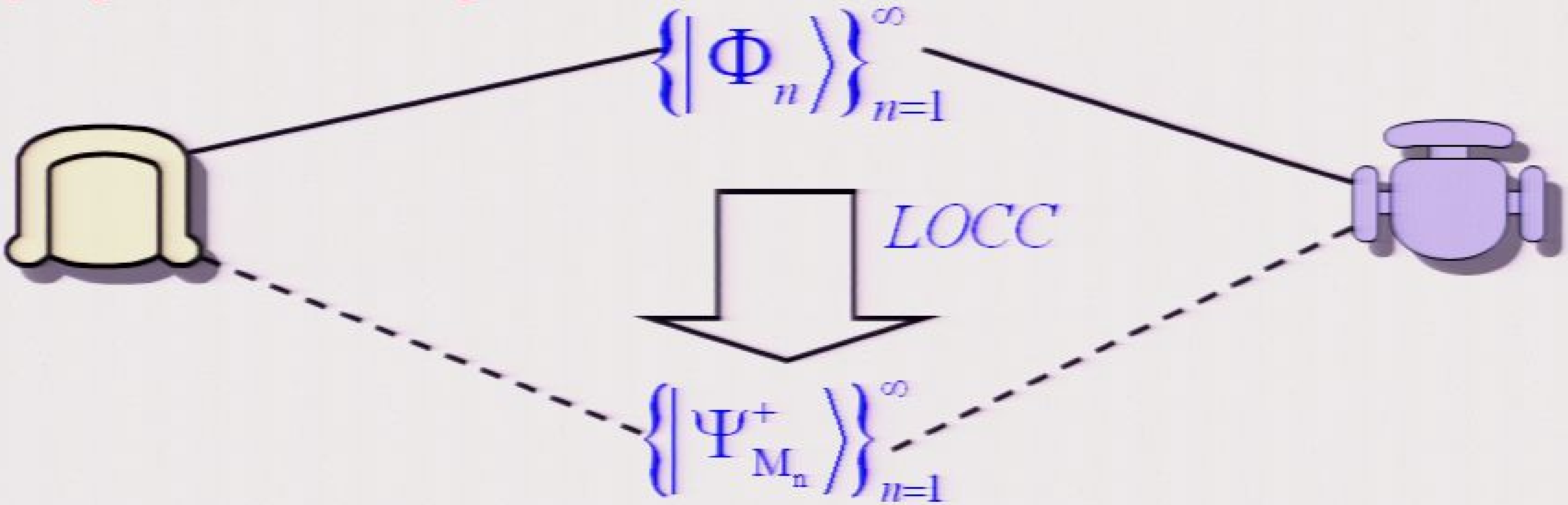
Asymptotic Entanglement Concentration of Pure States



$\left\{|\Phi_n\rangle\right\}_{n=1}^{\infty}$: partially entangled pure states

AIM:

$$\left\{|\Phi_n\rangle\right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{|\Psi^+_{M_n}\rangle\right\}_{n=1}^{\infty}$$

- If the fidelity of this LOCC transformation: $F_n \xrightarrow[n\to\infty]{} 1$

then, any $R \leq \frac{1}{n}\log M_n$ is an achievable rate:

Distillable entanglement: $$E_D = \sup R$$

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- THEOREM (Hayashi): For the entanglement concentration protocol $\left\{\left|\Phi_n\right\rangle\right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{\left|\Psi^+_{M_n}\right\rangle\right\}_{n=1}^{\infty}$

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$$E_D = \underline{S}(\hat{\rho})$$

where $\hat{\rho} = \left\{\rho_n^A\right\}_{n=1}^{\infty}$ with $\rho_n^A = \mathrm{Tr}_B |\Phi_n\rangle\langle\Phi_n|$

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- Proof: Let initial shared state: $$|\Phi_n\rangle = \sum_k \sqrt{\lambda_{n,k}}\,|k_A^{(n)}\rangle|k_B^{(n)}\rangle$$

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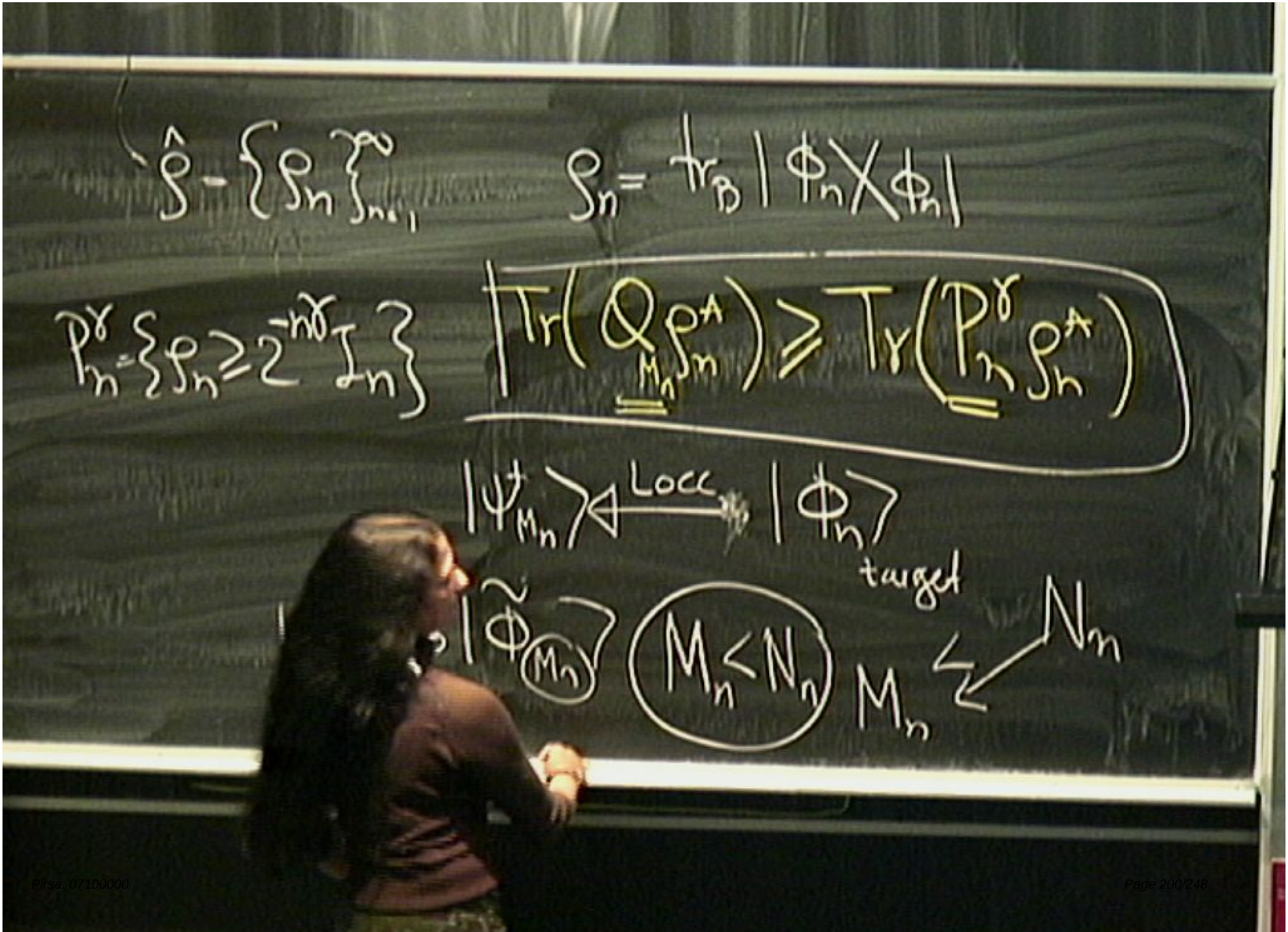
$\hat{\rho} - \{\rho_n\}_{n=1}^{\infty}$
$\rho_n = tr_B |\phi_n\rangle\langle\phi_n|$
$P_n^{\gamma} = \{\rho_n \geq 2^{-n\gamma} I_n\}$
$Tr(Q_{M_n} \rho_n^A) \geq Tr(P_n^{\gamma} \rho_n^A)$
$|\psi^+_{M_n}\rangle$
LOCC
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target
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$M_n < N_n$
N_n
M_n

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- Proof: Let initial shared state: $$|\Phi_n\rangle = \sum_k \sqrt{\lambda_{n,k}}\,|k_A^{(n)}\rangle|k_B^{(n)}\rangle$$

- Consider projection operators $$P_n^\gamma := \left\{\rho_n^A \geq 2^{-n\gamma} I_n^A\right\}$$

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- Hence for $\gamma < \underline{S}(\hat{\rho})$:

$$\mathrm{Tr}\,(P_n^\gamma \rho_n^A) \xrightarrow[n\to\infty]{} 0$$
$$\mathrm{Tr}\,(\overline{P}_n^\gamma \rho_n^A) \xrightarrow[n\to\infty]{} 1$$

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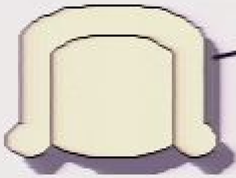
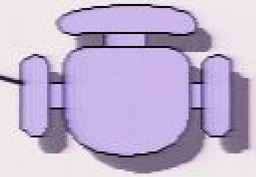
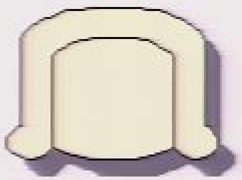


$|\Phi_n\rangle$



initial shared state

PROTOCOL:



(1) Does a von Neumann measurement corrs. to $P_n^\gamma, \overline{P}_n^\gamma$ on her part of shared state $|\Phi_n\rangle$

If outcome corrs. to P_n^γ

Failure!

Protocol aborted!

Probability= $\mathrm{Tr}\,(P_n^\gamma \rho_n^A)$

If outcome corrs. to $\overline{P}_n^\gamma$

Success!

Probability= $\mathrm{Tr}\,(\overline{P}_n^\gamma \rho_n^A)$

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If the outcome corrs. to $\bar{P}_n^\gamma$: post-measurement state:

$$|\Phi_n\rangle_{AB} \propto (\bar{P}_n^\gamma \otimes I_n^B)|\Phi_n\rangle_{AB}$$

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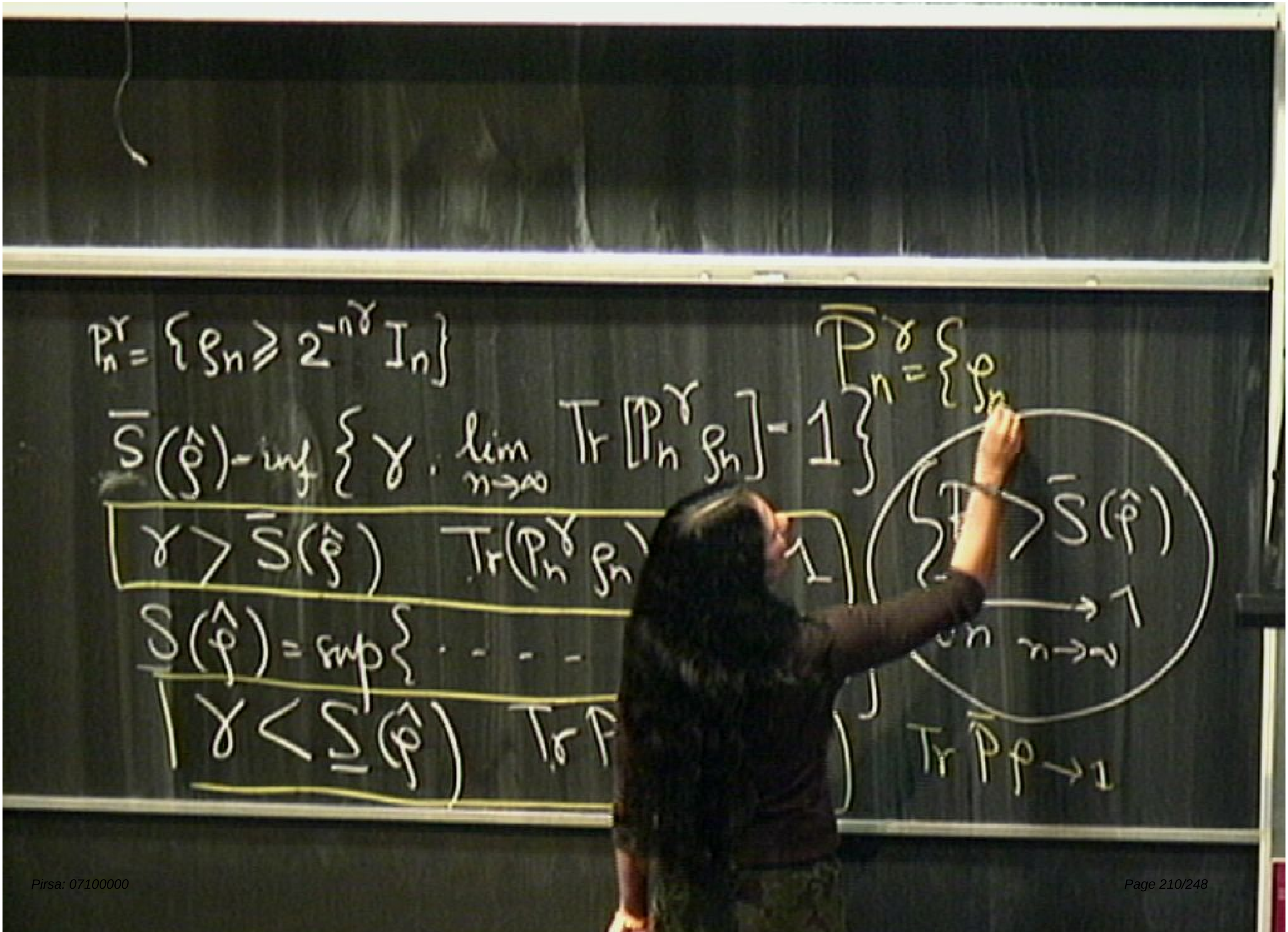
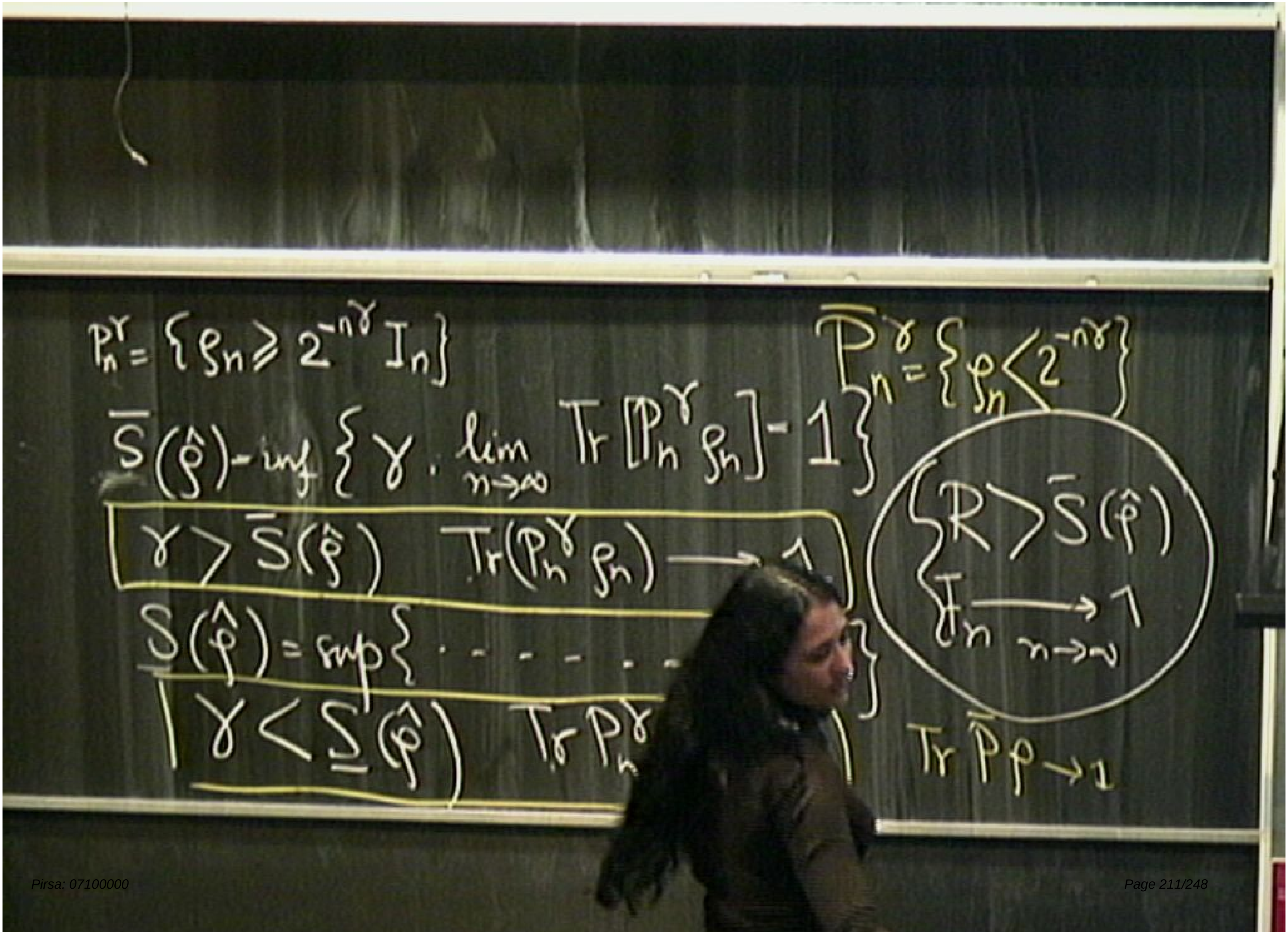

P_n^γ = { ρ_n ≥ 2^{-nγ} I_n }
S̄(ρ̂) = inf { γ : lim_{n→∞} Tr[P_n^γ ρ_n] = 1 }
γ > S̄(ρ̂) Tr(P_n^γ ρ_n)
S(ρ̂) = sup { . . . .
γ < S(ρ̂) Tr P
P̄_n^γ = { ρ_n
S̄(ρ̂)
n→∞
Tr P̄ ρ → 1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$\{R > \bar{S}(\hat{\rho})$
$F_n \xrightarrow[n\to\infty]{} 1$
$\mathrm{Tr}\, \bar{P} \rho \to 1$

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$$|\Phi_n\rangle_{AB} \propto (\bar{P}_n^\gamma \otimes I_n^B)|\Phi_n\rangle_{AB} \propto \sum_{k:\lambda_{n,k}<2^{-n\gamma}} \sqrt{\lambda_{n,k}}\, |k_A^{(n)}\rangle |k_B^{(n)}\rangle$$

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We need: $M_n \leq 2^{n\gamma} \mathrm{Tr}(P_n^\gamma \rho_n^A)$; Let $M_n = \lfloor 2^{n\gamma} \mathrm{Tr}(\bar{P}_n^\gamma \rho_n^A) \rfloor$

If $\gamma < \underline{S}(\hat{\rho})$ where $\hat{\rho} = \{\rho_n^A\}_{n=1}^\infty$ then

Probability of failure: $\mathrm{Tr}\,(P_n^\gamma \rho_n^A) \xrightarrow[n\to\infty]{} 0$

Probability of success: $\mathrm{Tr}\,(\bar{P}_n^\gamma \rho_n^A) \xrightarrow[n\to\infty]{} 1$

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Achievable rate: $R \le \frac{1}{n}\log M_n < \underline{S}(\hat{\rho})$

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We need: $M_n \le 2^{n\gamma}\,\mathrm{Tr}(\mathrm{P}_n^\gamma \rho_n^A)$; Let $M_n = \lfloor 2^{n\gamma}\,\mathrm{Tr}(\bar{P}_n^\gamma \rho_n^A) \rfloor$

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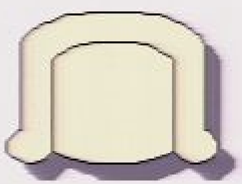
Distillable Entanglement: $E_D = \underline{S}(\hat{\rho})$

chematic summary: protocol for entanglement concentration

Aim : $\left\{ |\Phi_n\rangle_{AB} \right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{ |\Psi^+_{M_n}\rangle_{AB} \right\}_{n=1}^{\infty}$

PROTOCOL:



(1) Does a von Neumann measurement corrs. to $P_n^{\gamma}, \overline{P}_n^{\gamma}$ on her part of shared state $|\Phi_n\rangle$

If outcome corrs. to P_n^{γ}

Failure!

If outcome corrs. to $\overline{P}_n^{\gamma}$

Success!

$|\Phi_n\rangle \longrightarrow |\Psi_n\rangle$

LOCC (ii)

$|\Psi^+_{M_n}\rangle$

- If $\gamma < \underline{S}(\hat{\rho})$:

(i) Prob.of success $\xrightarrow[n\to\infty]{} 1$ &

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Summary

Entangement Dilution

$$\left\{\left|\Psi^{+}_{M_n}\right\rangle_{AB}\right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{\left|\Phi_n\right\rangle_{AB}\right\}_{n=1}^{\infty}$$

Entanglement cost $\boxed{E_C = \overline{S}(\hat{\rho})}$

where $\hat{\rho} = \left\{\rho^{A}_{\Phi_n}\right\}_{n=1}^{\infty}$ with $\rho^{A}_{\Phi_n} = \mathrm{Tr}_B \left|\Phi_n\right\rangle\left\langle\Phi_n\right|$

Entanglement Concentration $\left\{\left|\Phi_n\right\rangle_{AB}\right\}_{n=1}^{\infty} \xrightarrow{LOCC} \left\{\left|\Psi^{+}_{M_n}\right\rangle_{AB}\right\}_{n=1}^{\infty}$

Distillable entanglement $\boxed{E_D = \underline{S}(\hat{\rho})}$

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- Any sequence of bipartite pure states $\left\{ |\Phi_n\rangle_{AB} \right\}_{n=1}^{\infty}$ for which

$$\underline{S}(\hat{\rho}) = \lim_{n\to\infty} \frac{1}{n} S(\rho_n) = \overline{S}(\hat{\rho})$$

: information stable on its subsystems

asymptotic entanglement measure:

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here $\rho_n = \rho_n^A = \mathrm{Tr}_B|\Phi_n\rangle\langle\Phi_n|$

- e.g. if $\left\{|\Phi_n\rangle_{AB}\right\}_{n=1}^{\infty} = \left\{|\varphi\rangle_{AB}^{\otimes n}\right\}_{n=1}^{\infty}$ sequences of memoryless states

then $\hat{\rho} = \left\{\rho^{\otimes n}\right\}_{n=1}^{\infty}$ with $\rho = \mathrm{Tr}_B|\varphi\rangle\langle\varphi|$

and

$$\underline{S}(\hat{\rho}) = \overline{S}(\hat{\rho}) = S(\rho)$$

- Hence, $E_C = E_D = S(\rho)$:unique entanglement measure

- However, $\exists$sequences $\left\{|\Phi_n\rangle_{AB}\right\}_{n=1}^{\infty}$ of bipartite pure states for which the corrs. sequence of subsystem states are *not information stable*: $\underline{S}(\hat{\rho}) \neq \overline{S}(\hat{\rho})$

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- e.g. sequences of states for which the subsystem states are:

$$\rho_n = t\sigma^{\otimes n} + (1-t)\omega^{\otimes n} \quad ; t \in (0,1)$$

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- For such sequences $E_D = S(\sigma) < S(\omega) = E_C$

- Hence, the asymptotic entanglement measure is unique only for information stable sequences !

SUMMARY

- For an arbitrary sequence of pure bipartite states $\left\{|\Phi_n\rangle_{AB}\right\}_{n=1}^{\infty}$

  entanglement cost $\boxed{E_C = \overline{S}(\hat{\rho})}$; $\hat{\rho} = \left\{\mathrm{Tr}_B |\Phi_n\rangle\langle\Phi_n|\right\}_{n=1}^{\infty}$

  distillable entanglement $\boxed{E_D = \underline{S}(\hat{\rho})}$

- $\boxed{E_C = E_D}$ only for sequences of states which are

  information stable, i.e., for which $\boxed{\underline{S}(\hat{\rho}) = \overline{S}(\hat{\rho})}$

- only such sequences have a unique asymptotic entanglement measure.

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NOTE: The quantities $\underline{S}(\hat{\rho})$, $\overline{S}(\hat{\rho})$

are obtainable from 2 fundamental quantities: the *spectral divergence rates:*

$$\overline{D}(\hat{\rho}\,\|\,\hat{\omega}) := \inf\left\{\gamma : \limsup_{n\to\infty} \mathrm{Tr}\left[\{\Pi_n(\gamma)\geq 0\}\Pi_n(\gamma)\right]=0\right\}$$

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- From $\overline{D}(\hat{\rho} \| \hat{\omega})$ and $\underline{D}(\hat{\rho} \| \hat{\omega})$ we obtain

$$\boxed{\underline{S}(\hat{\rho}) = -\overline{D}(\hat{\rho} \| \hat{I})}$$ and $$\boxed{\overline{S}(\hat{\rho}) = -\underline{D}(\hat{\rho} \| \hat{I})}$$

by substituting $\hat{\omega} = \hat{I} = \{I_n\}_{n=1}^{\infty}$

- The spectral divergences rates can be viewed as generalizations of the quantum relative entropy:

$$\boxed{S(\rho \| \omega) = \mathrm{Tr}\, \rho \log \rho - \mathrm{Tr}\, \rho \log \omega}$$

since

$$\boxed{\begin{aligned} S(\rho) &= -S(\rho \| I) \\ S(A|B) &= -S(\rho^{AB} \| I^A \otimes \rho^B) \end{aligned}}$$

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OPEN PROBLEMS

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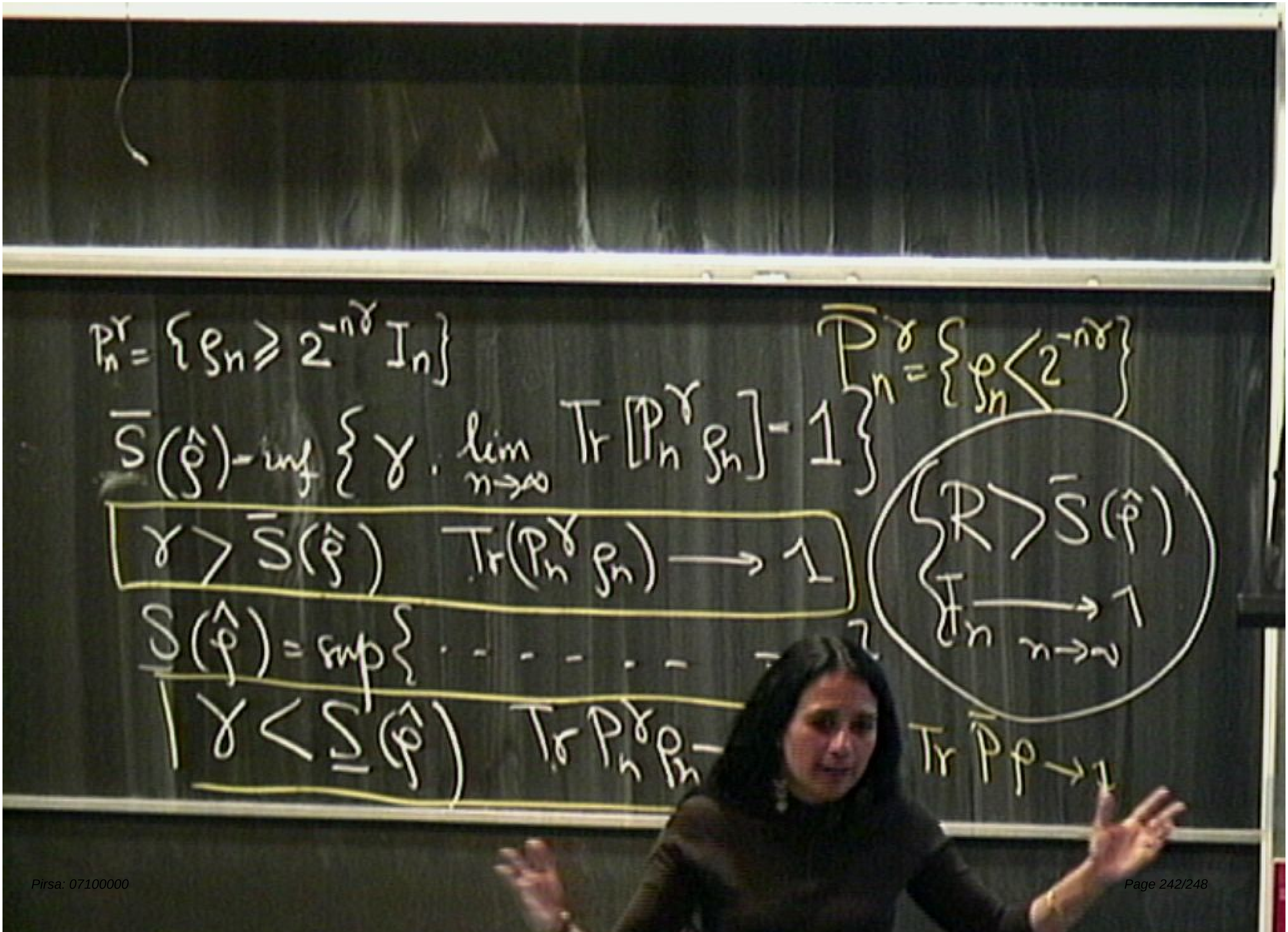
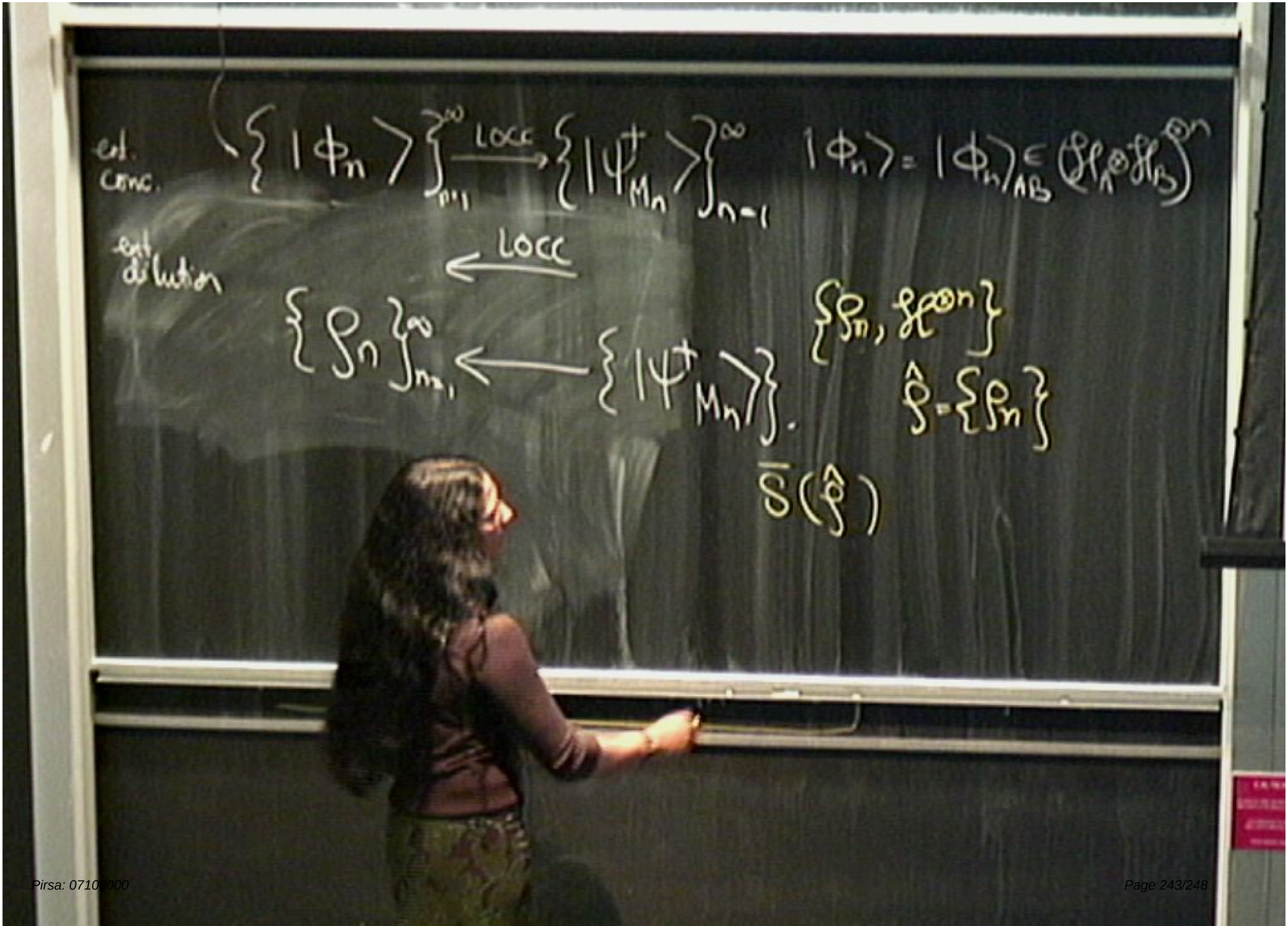
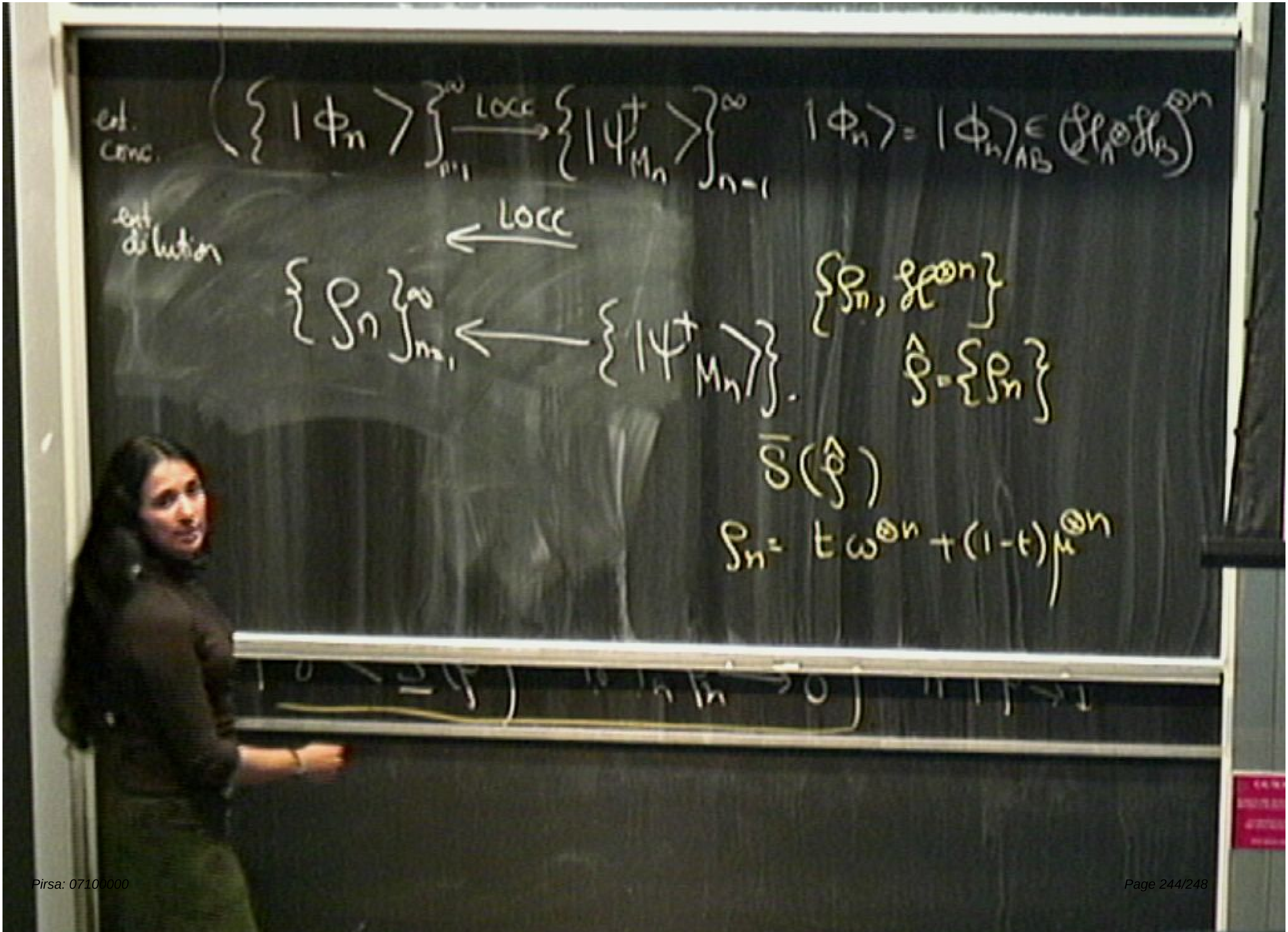


P_n^γ = { ρ_n ≥ 2^{-nγ} I_n }
P̄_n^γ = { ρ_n < 2^{-nγ} }
S̄(ρ̂) = inf { γ : lim_{n→∞} Tr [P_n^γ ρ_n] = 1 }
γ > S̄(ρ̂)   Tr(P_n^γ ρ_n) → 1
R > S̄(ρ̂)
S(ρ̂) = sup { . . . . . . }
γ < S(ρ̂)   Tr P_n^γ ρ_n
Tr P̄ ρ → 1


ent.
conc.
LOCC
ent.
dilution
LOCC


ent. conc.
LOCC
ent dilution
LOCC
$\{|\Phi_n\rangle\}_{n=1}^{\infty} \xrightarrow{LOCC} \{|\Psi^+_{M_n}\rangle\}_{n=1}^{\infty}$
$|\Phi_n\rangle = |\Phi_n\rangle_{AB} \in (\mathcal{H}_A \otimes \mathcal{H}_B)^{\otimes n}$
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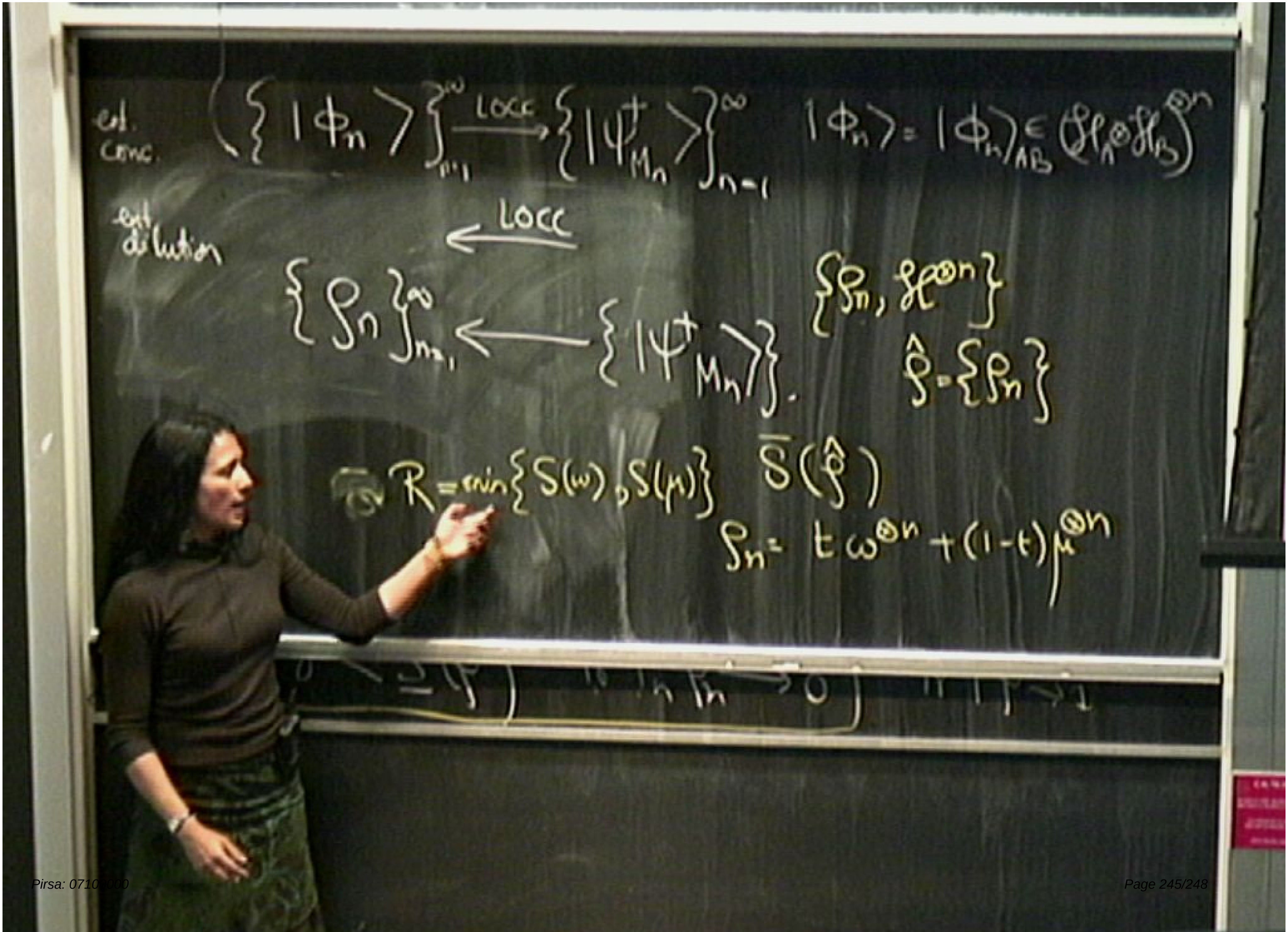
$\{\rho_n, \mathcal{H}^{\otimes n}\}$
$\hat{\rho} = \{\rho_n\}$
$\overline{S}(\hat{\rho})$
$\rho_n = t\,\omega^{\otimes n} + (1-t)\mu^{\otimes n}$


ent. conc.
LOCC
ent. dilution
LOCC
$\hat{\rho} = \{\rho_n\}$
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$\bar{S}(\hat{\rho})$
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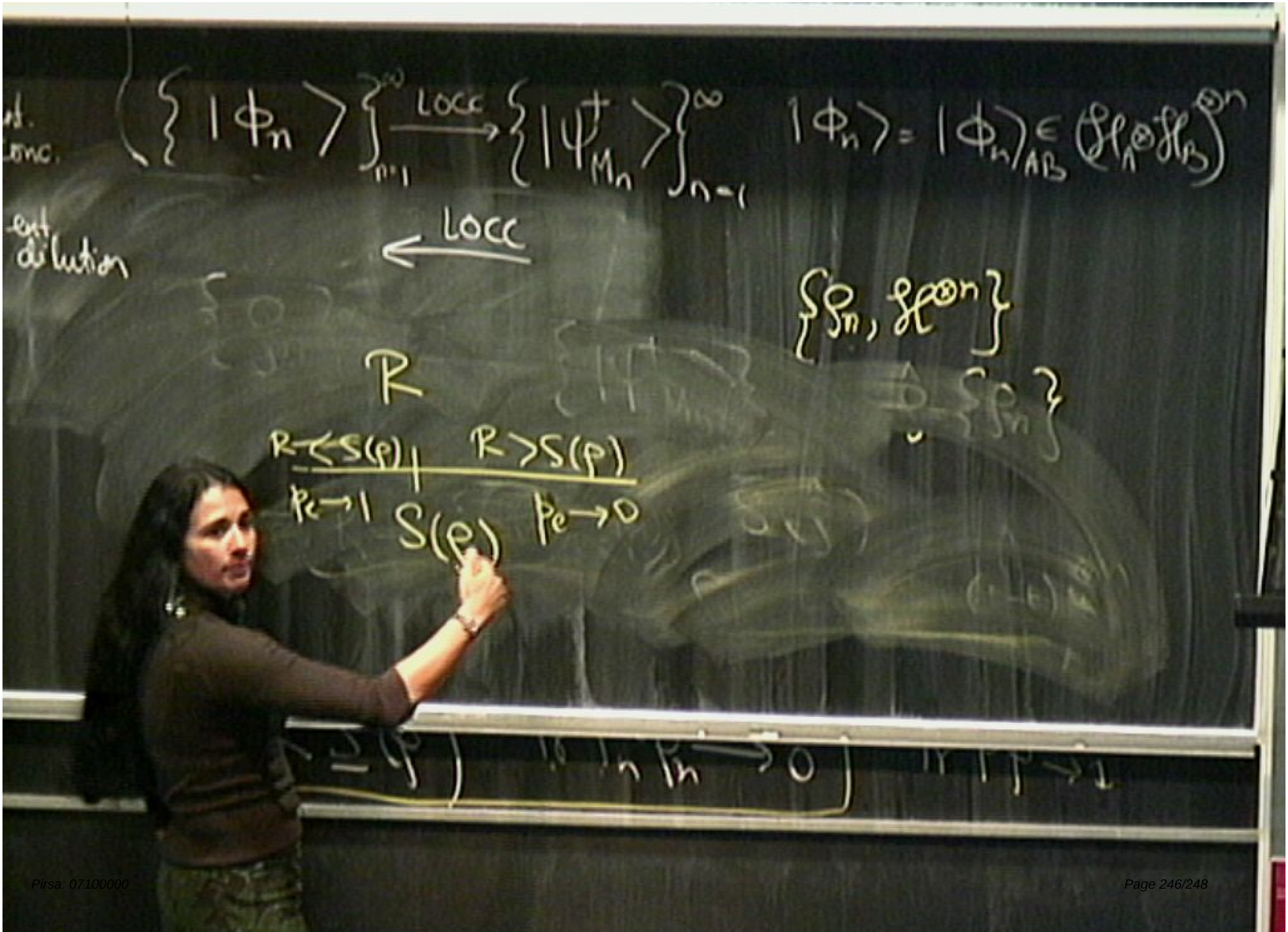
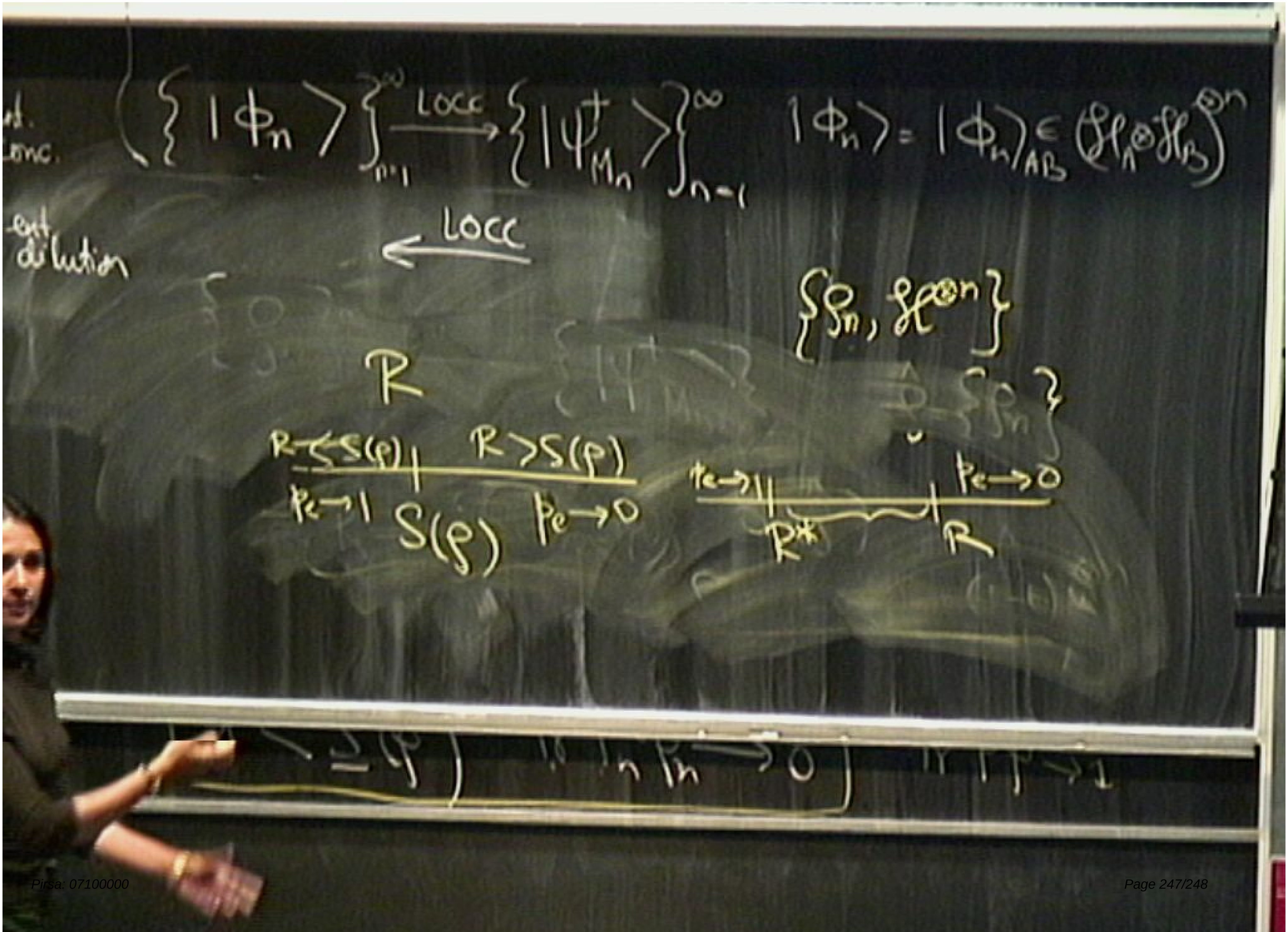

LOCC
LOCC
dilution
R
R<S(ρ)
R>S(ρ)
S(ρ)


LOCC
LOCC
ent.
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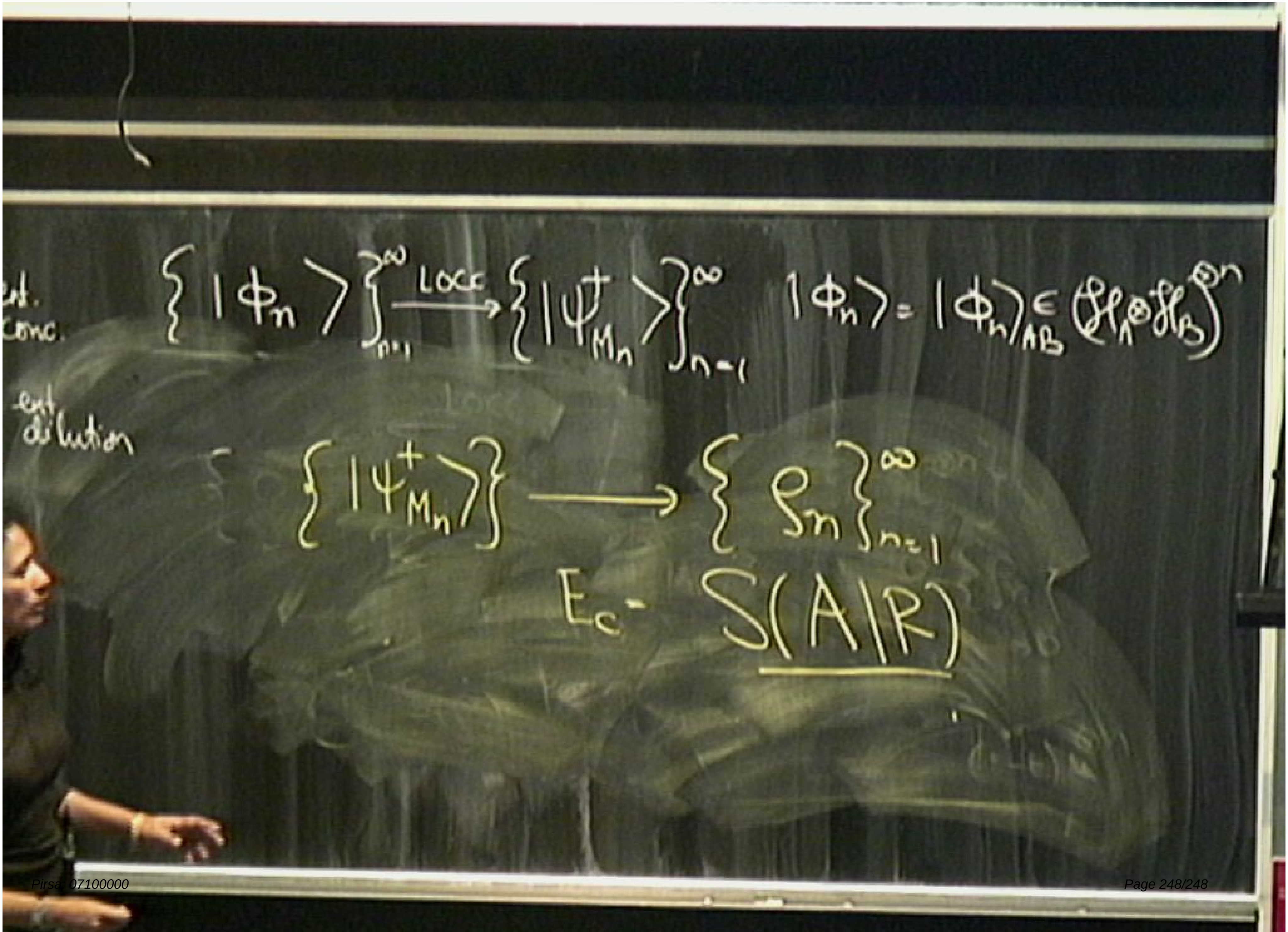
R
R < S(ρ)
R > S(ρ)
pe→1
S(ρ)
pe→0
pe→1
pe→0
R*
R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{|φ_n⟩}_{n=1}^∞ LOCC → {|ψ^+_{M_n}⟩}_{n=1}^∞ |φ_n⟩ = |φ_n⟩_{AB} ∈ (H_A ⊗ H_B)^{⊗n}
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