Title: Intro to Supersymmetry 4

Date: Sep 27, 2007 12:30 PM

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Abstract:

$\Rightarrow$ have explicite realization of SUSY algebra

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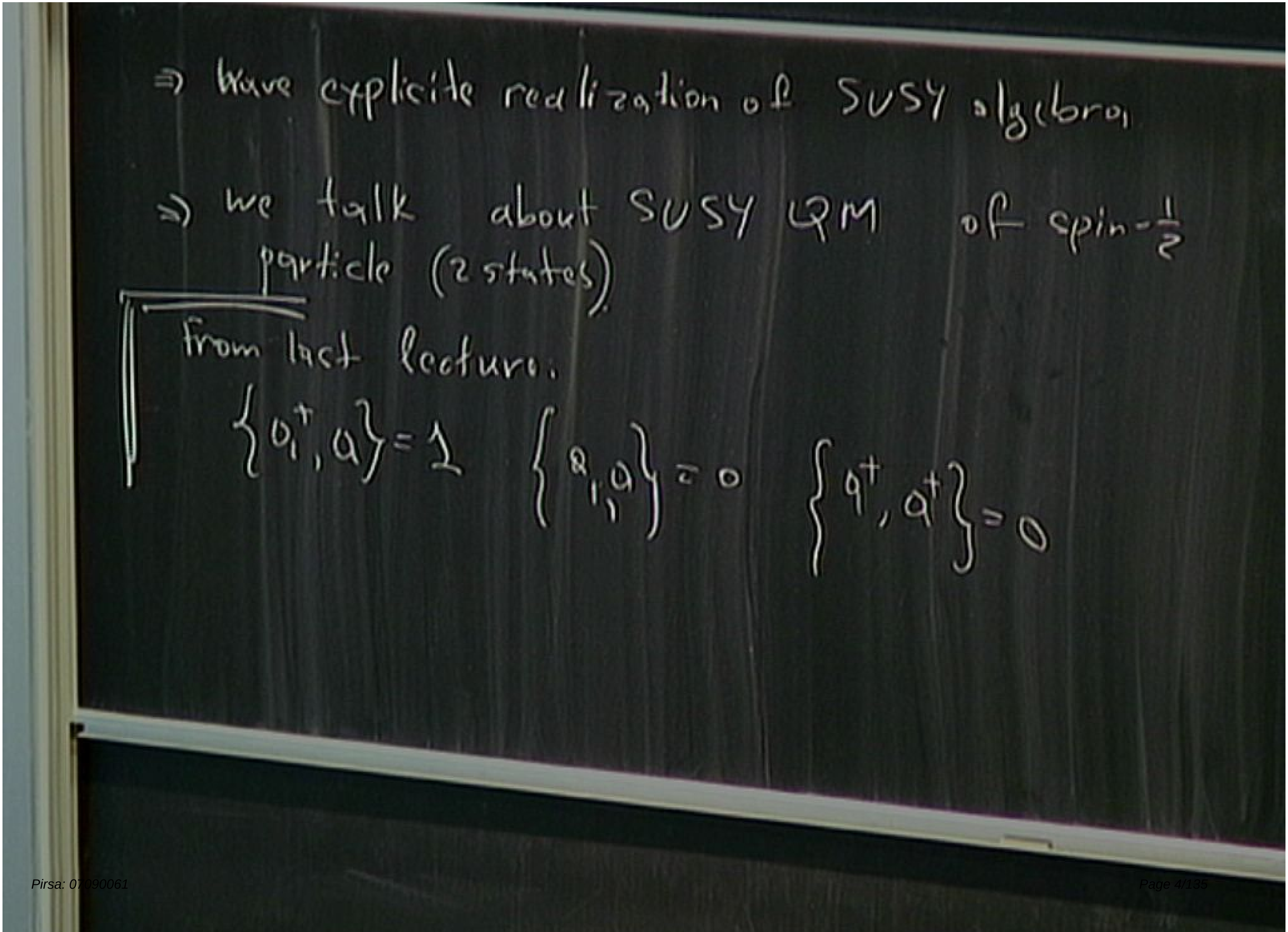
$\Rightarrow$ we talk about SUSY QM of spin-$\frac{1}{2}$ particle (2 states)



⇒ have explicite realization of SUSY algebra

⇒ we talk about SUSY QM of spin-$\frac{1}{2}$ particle (2 states)

from last lecture:

$$\{a_1^+, a\} = 1 \qquad \{a_1, a_1\} = 0 \qquad \{a^+, a^+\} = 0$$

⇒ have explicite realization of SUSY algebra

⇒ we talk about SUSY QM of spin-$\frac{1}{2}$ particle (2 states)

from last lecture:

$$\{a^+, a\} = 1 \qquad \{a, a\} = 0 \qquad \{a^+, a^+\} = 0$$

$$a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$|\Omega\rangle = \begin{pmatrix} \omega_+(\varphi) \\ \\ \omega_-(\varphi \end{pmatrix}$$

$\uparrow$

wave
function

$$\underset{\substack{\uparrow \\ \text{wave} \\ \text{function}}}{|\Omega\rangle} = \begin{pmatrix} \omega_+(\varphi) \\ \\ \omega_-(\varphi \end{pmatrix}$$

$$\hat{x} \Longleftrightarrow \hat{\varphi}$$

$$\hat{p} = -i\hbar \frac{\partial}{\partial x}$$

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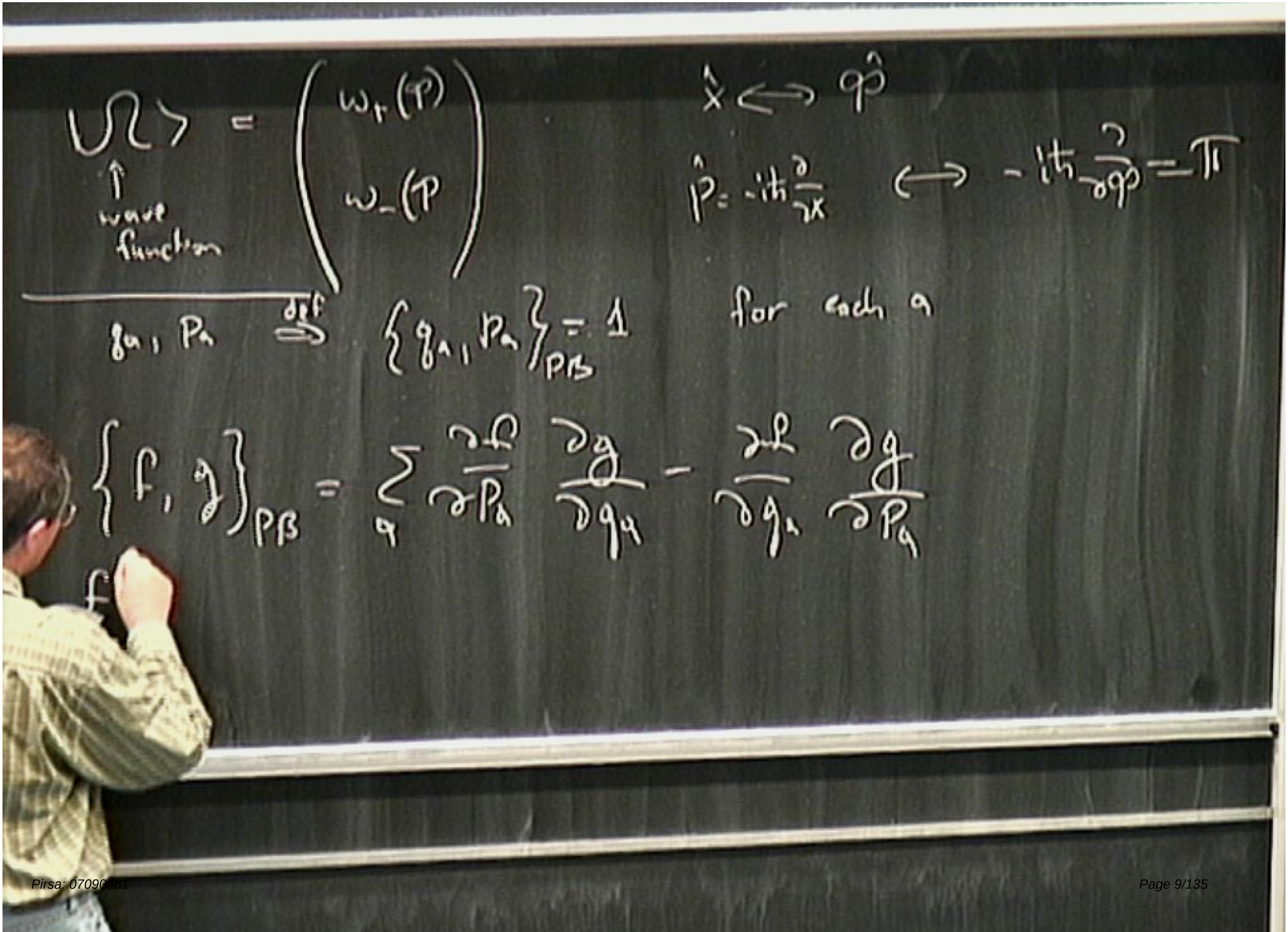
$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \longleftrightarrow -i\hbar\frac{\partial}{\partial\varphi} = \hat{\pi}$$



$$|\Omega\rangle = \begin{pmatrix} \omega_+(\varphi) \\ \omega_-(\varphi \end{pmatrix}$$

$\uparrow$ wave function

$\hat{x} \longleftrightarrow \hat{\varphi}$

$\hat{p} = -i\hbar\frac{\partial}{\partial x} \longleftrightarrow -i\hbar\frac{\partial}{\partial\hat{\varphi}} = \Pi$

$q_a, p_a \overset{\text{def}}{\Longleftrightarrow} \{q_a, p_a\}_{PB} = 1$ for each a

$$\{f, g\}_{PB} = \sum_a \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a} - \frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a}$$

f

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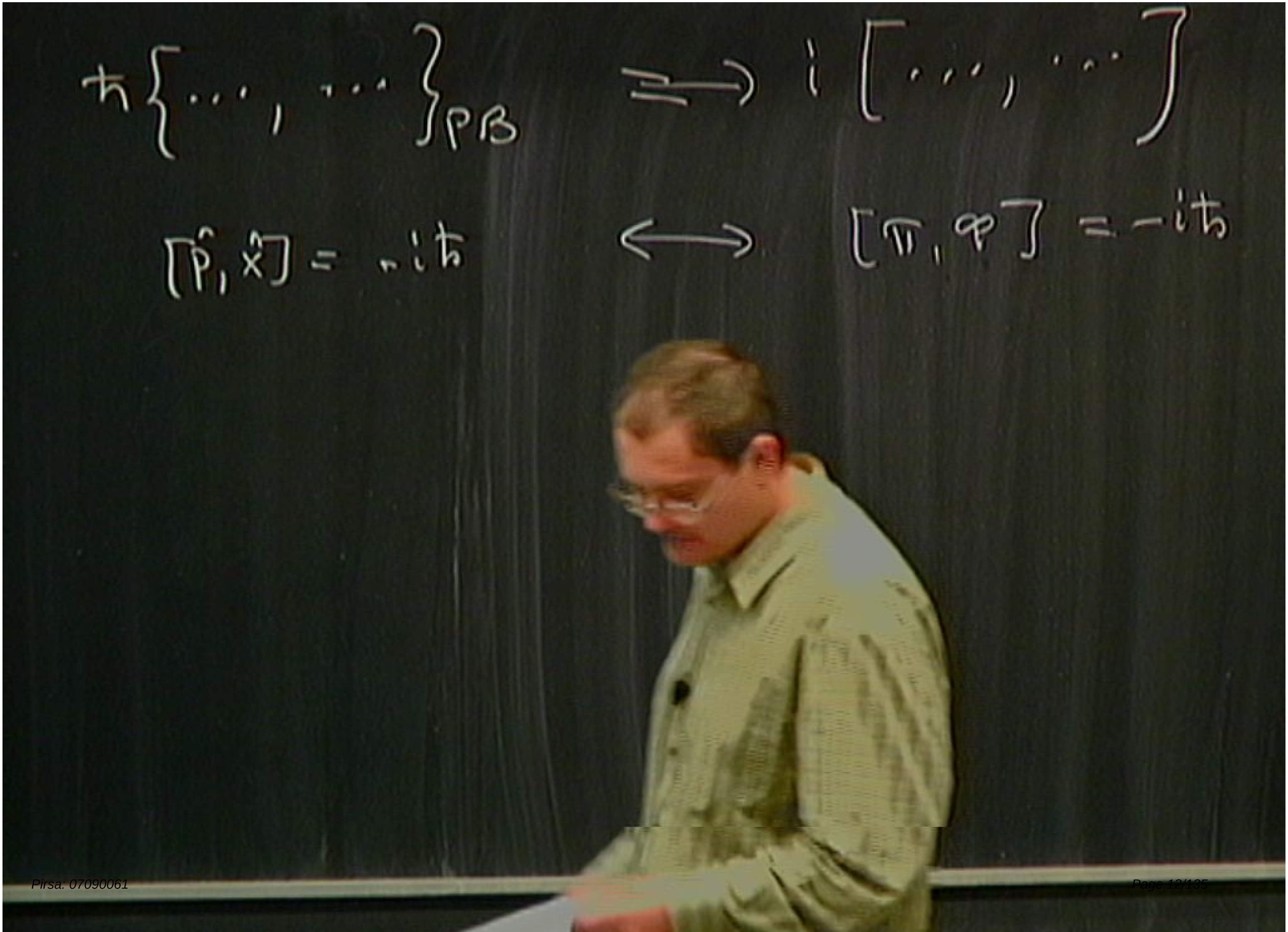
$f[q_a, P_a]$

$g[q_a, P_a]$

$$\hbar\{\ldots,\ldots\}_{PB} \Longrightarrow i[\ldots,\ldots]$$

$$[\hat{p},\hat{x}] = -i\hbar \longleftrightarrow [\pi,\varphi]$$



$$\hbar\left\{\ldots,\ \ldots\right\}_{PB} \Longrightarrow i\left[\ldots,\ \ldots\right]$$

$$[\hat{p}, \hat{x}] = -i\hbar \longleftrightarrow [\pi, \varphi] = -i\hbar$$

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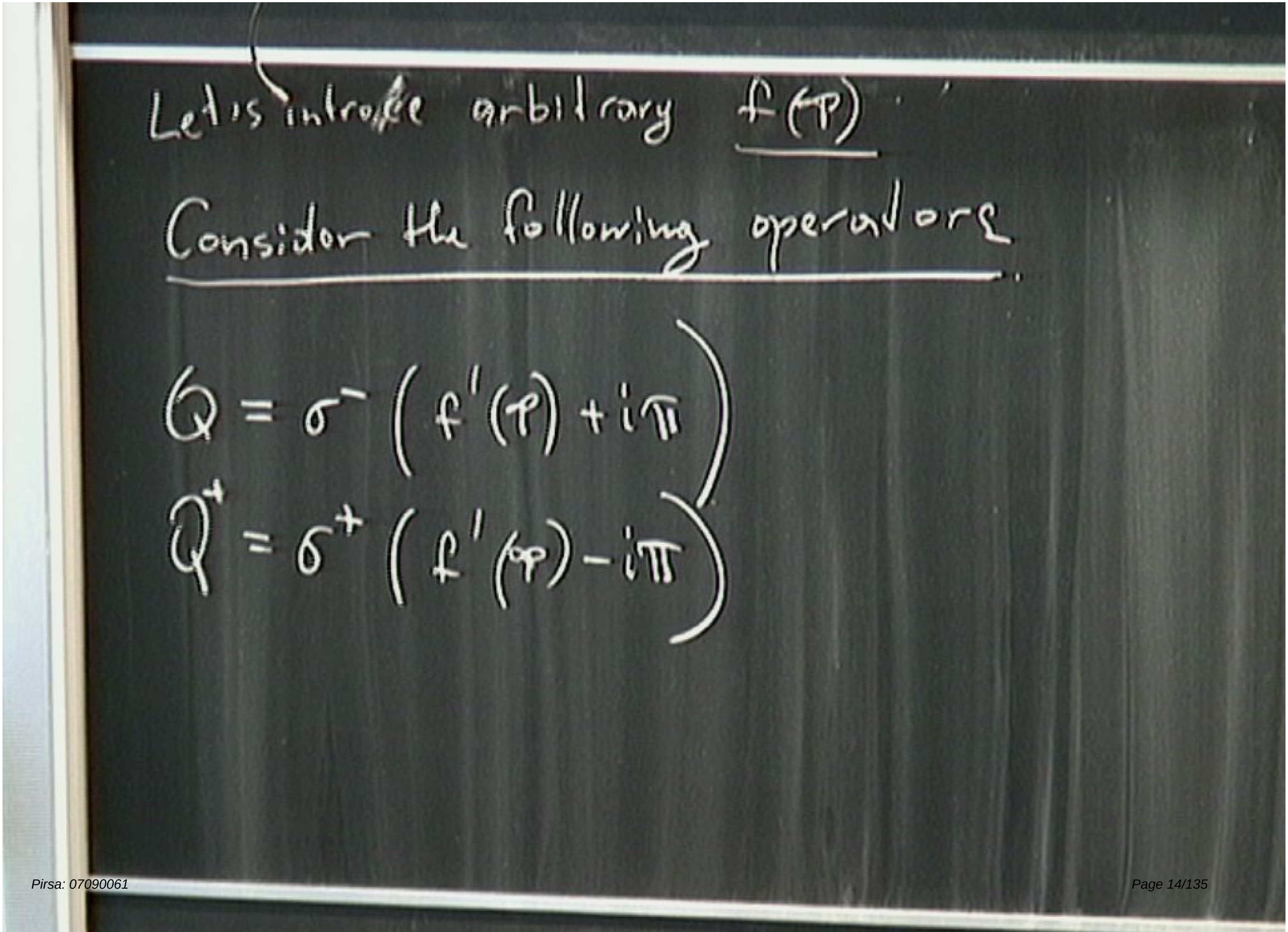
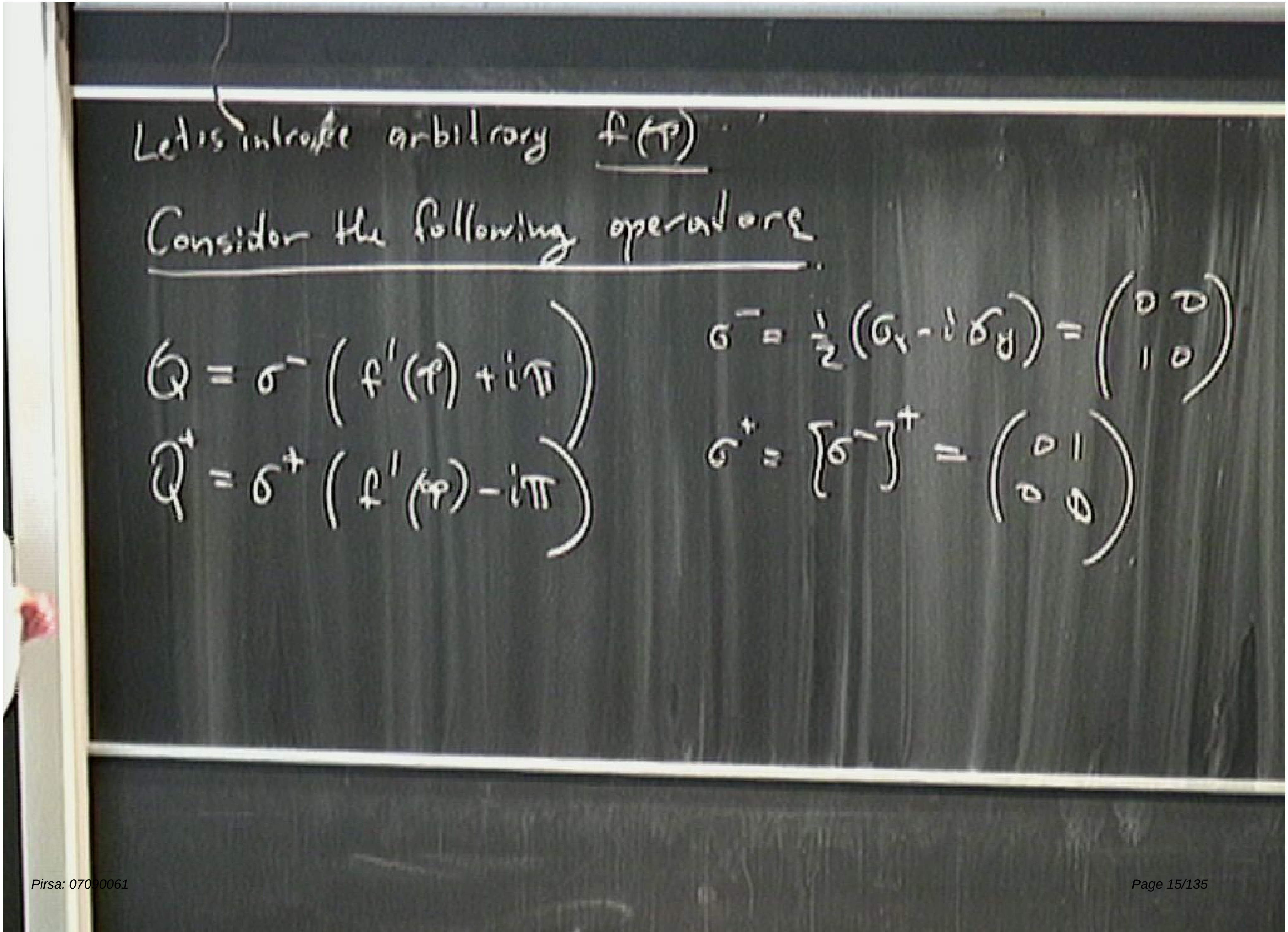


Let us introduce arbitrary $\underline{f(\varphi)}$

<u>Consider the following operators</u>

$$Q = \sigma^- \left(f'(\varphi) + i\pi \right)$$

$$Q^+ = \sigma^+ \left(f'(\varphi) - i\pi \right)$$



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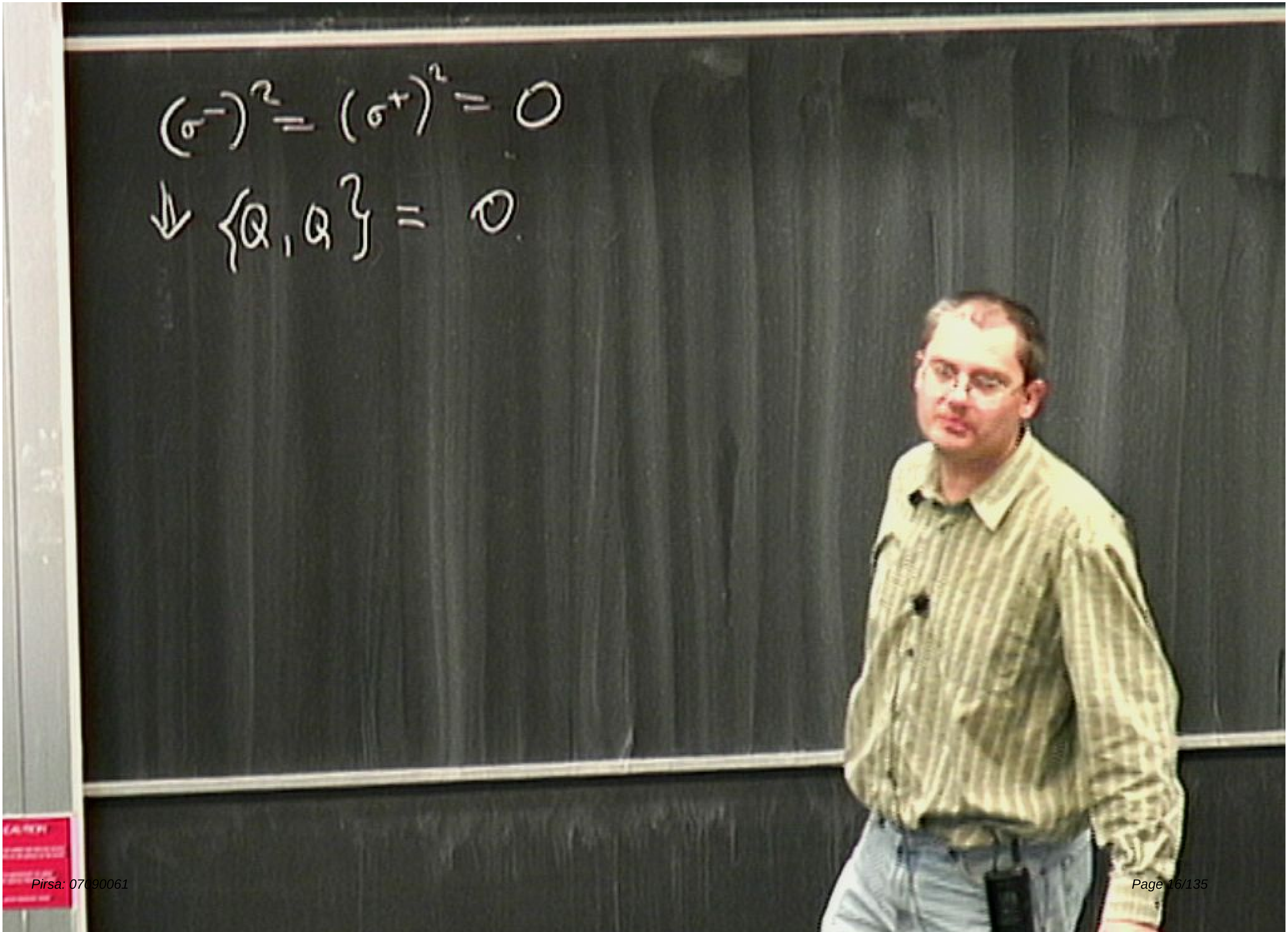
$$Q^+ = \sigma^+ \left(f'(\varphi) - i\pi \right)$$

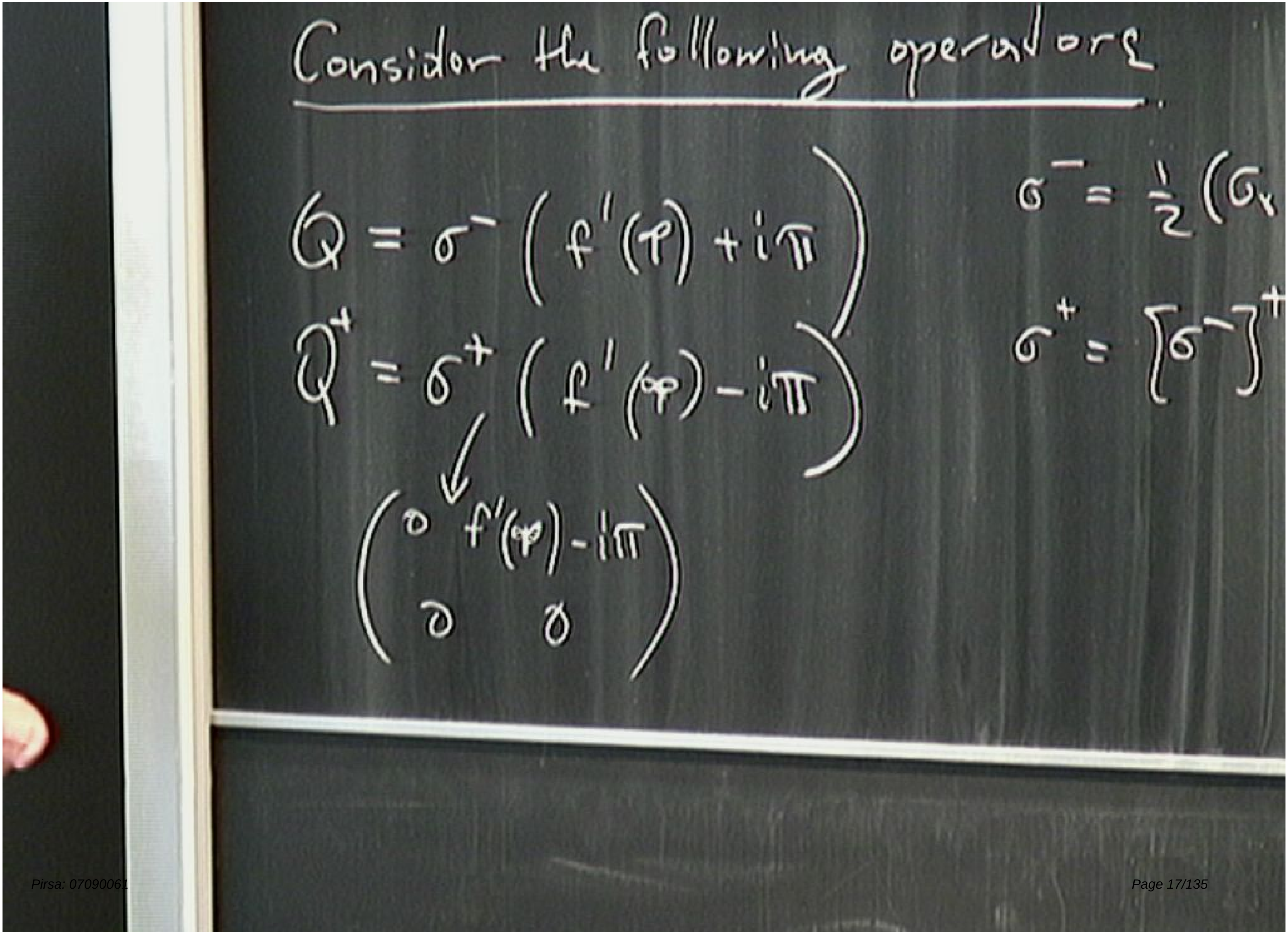
$$\sigma^- = \frac{1}{2}(\sigma_x - i\sigma_y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^+ = [\sigma^-]^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



$$(\sigma^-)^2 = (\sigma^+)^2 = 0$$

$$\Downarrow \{Q, Q\} = 0$$



Consider the following operators

$$Q = \sigma^- \left(f'(\varphi) + i\pi \right)$$

$$Q^\dagger = \sigma^+ \left(f'(\varphi) - i\pi \right)$$

$$\swarrow$$

$$\begin{pmatrix} 0 & f'(\varphi) - i\pi \\ 0 & 0 \end{pmatrix}$$

$\sigma^- = \frac{1}{2}(\sigma_x$

$\sigma^+ = [\sigma^-]^\dagger$

$$(\sigma^-)^2 = (\sigma^+)^2 = 0$$

$$\Downarrow \{Q, Q\} = 0 \qquad \{Q^\dagger, Q^\dagger\} = 0$$

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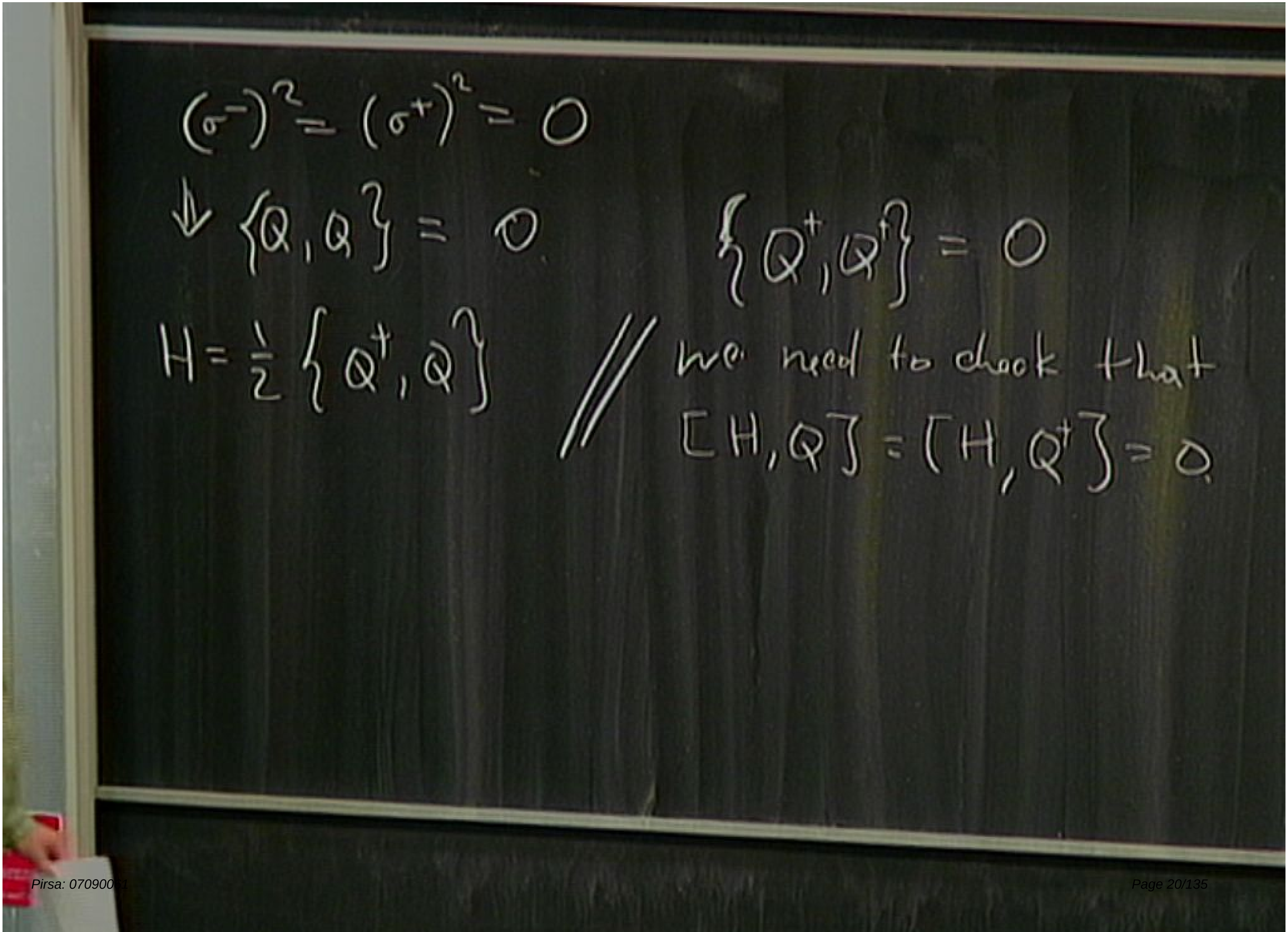
$$\Downarrow \quad \{Q, Q\} = 0 \qquad \{Q^\dagger, Q^\dagger\} = 0$$

$$H = \frac{1}{2}\{Q^\dagger, Q\}$$



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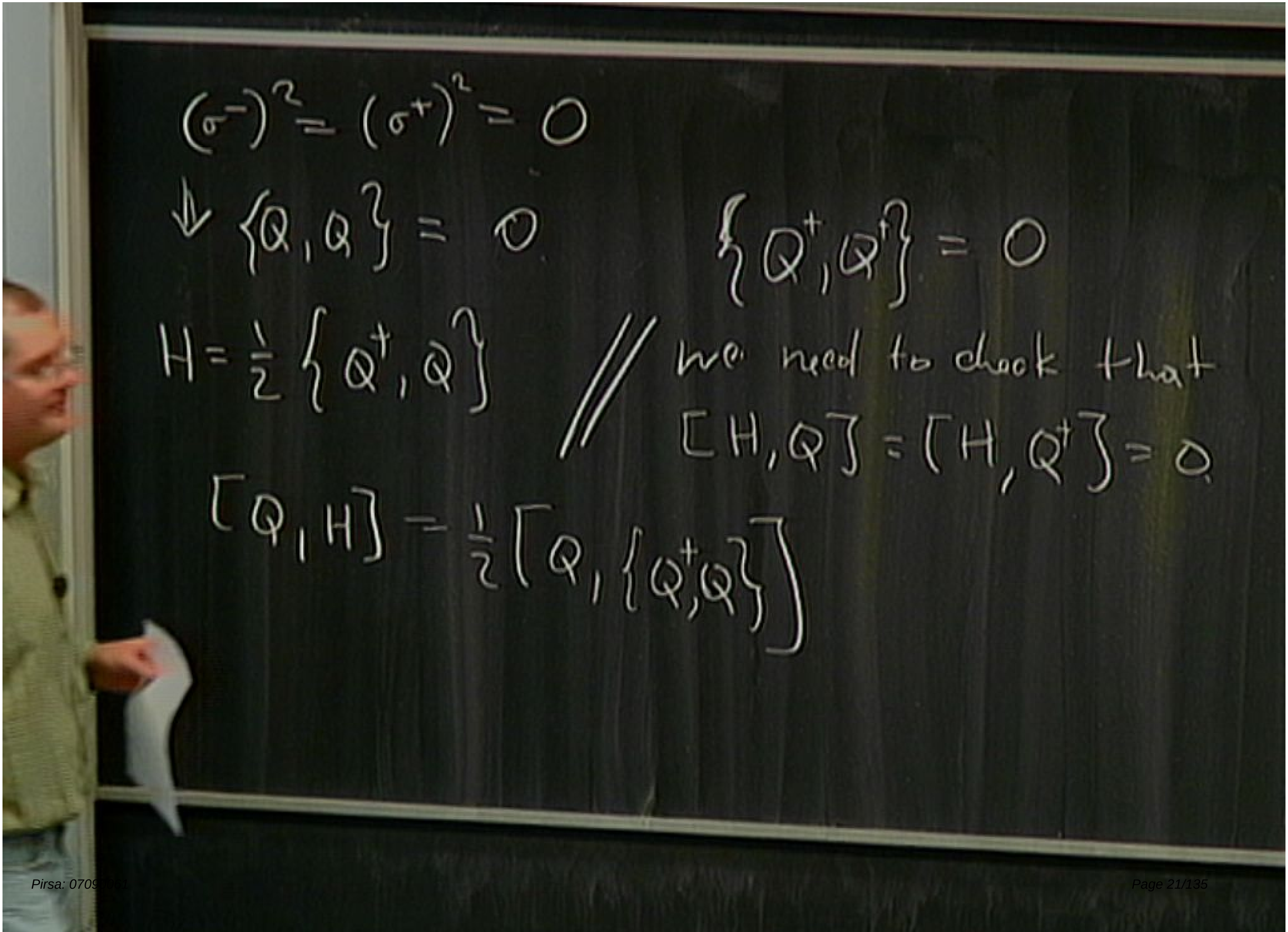
$$\Downarrow \quad \{Q, Q\} = 0 \qquad \{Q^+, Q^+\} = 0$$

$$H = \frac{1}{2}\{Q^+, Q\}$$

// we need to check that $[H, Q] = [H, Q^+] = 0.$



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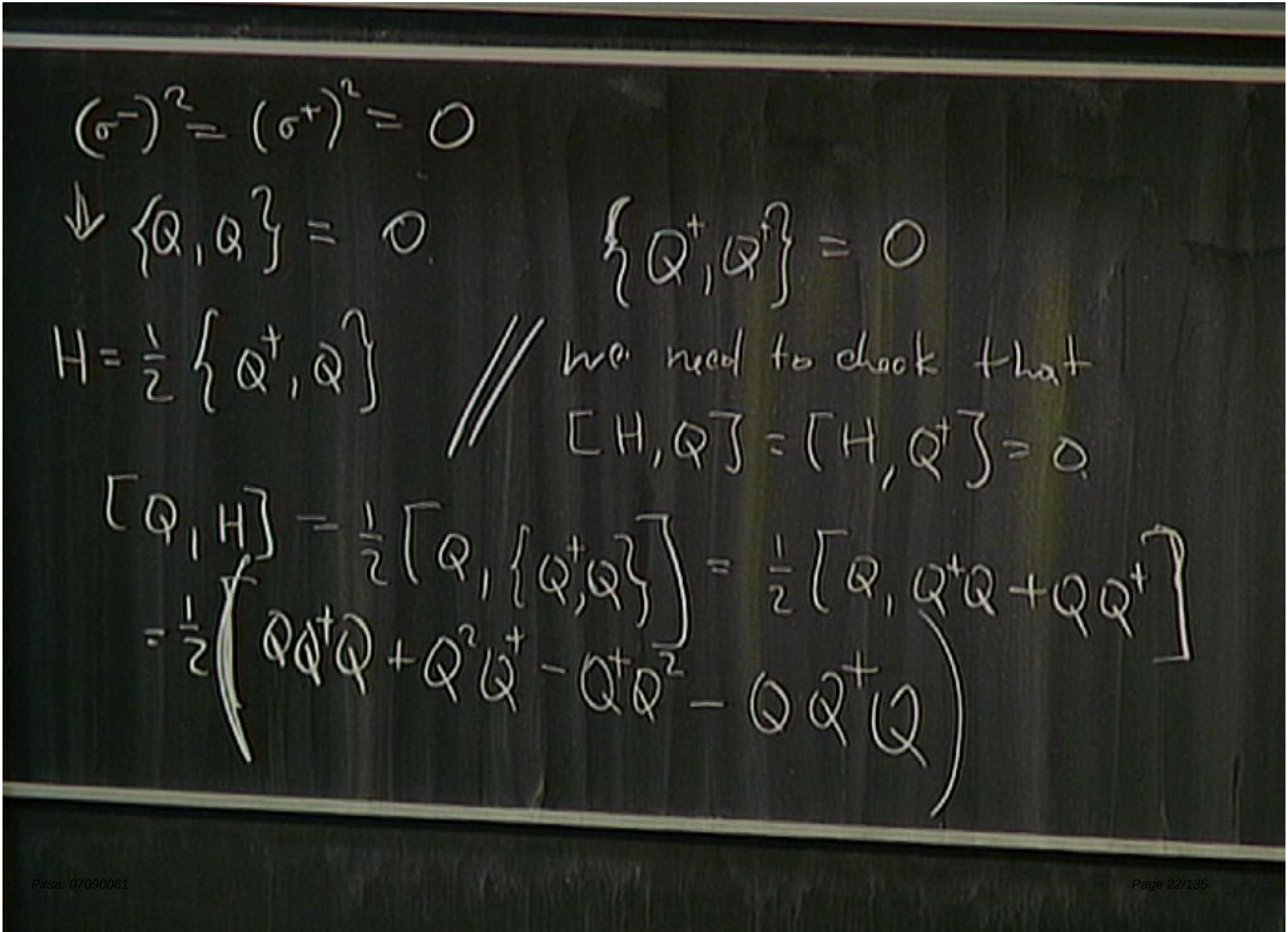
$$[Q,H] = \frac{1}{2}[Q,\{Q^\dagger,Q\}]$$



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// we need to check that $[H,Q] = [H,Q^+] = 0.$

$$[Q,H] = \frac{1}{2}[Q,\{Q^+,Q\}] = \frac{1}{2}[Q, Q^+Q + QQ^+]$$

$$= \frac{1}{2}\left(QQ^+Q + Q^2Q^+ - Q^+Q^2 - QQ^+Q\right)$$

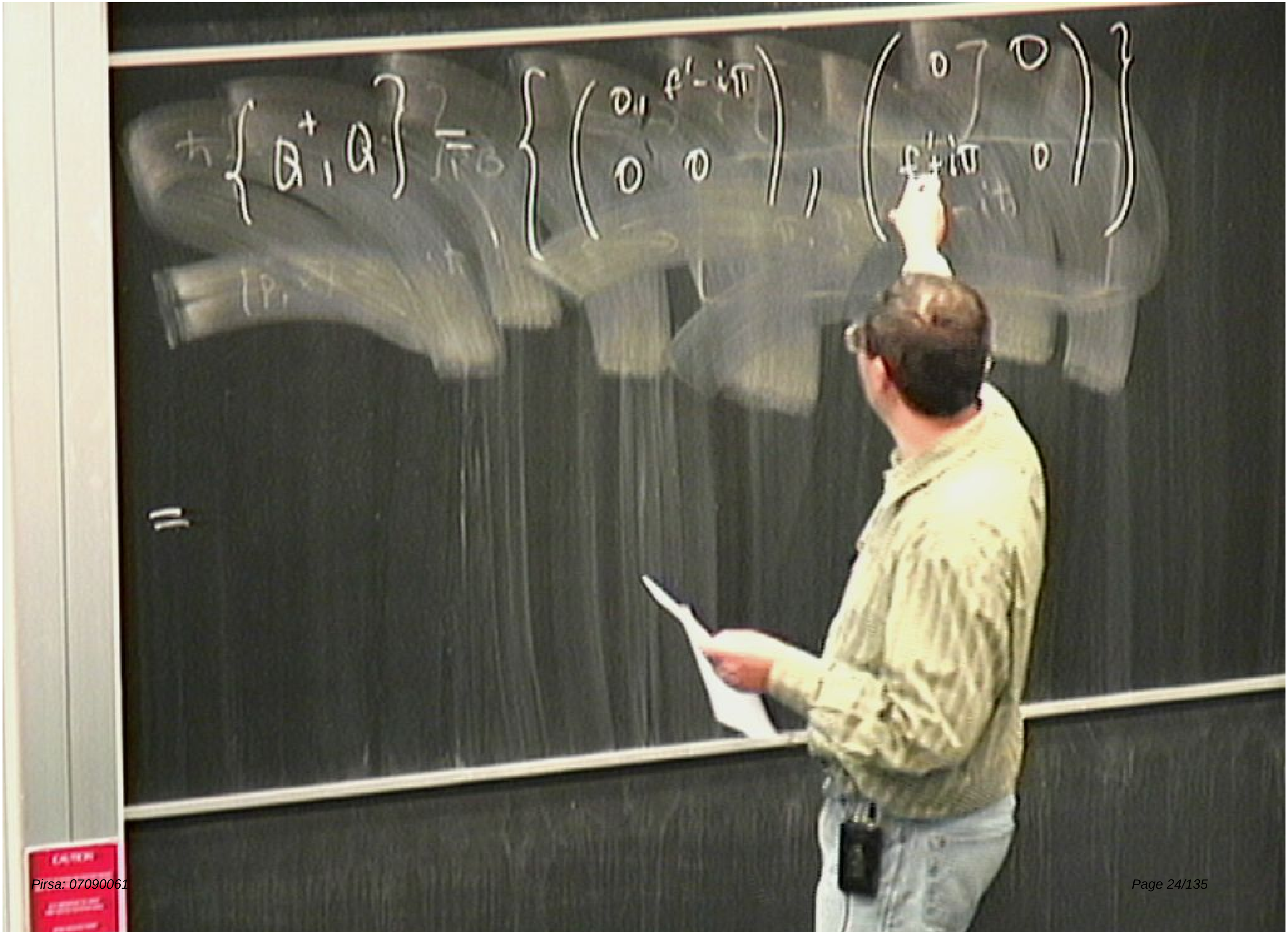
$(\sigma^-)^2 = (\sigma^+)^2 = 0$

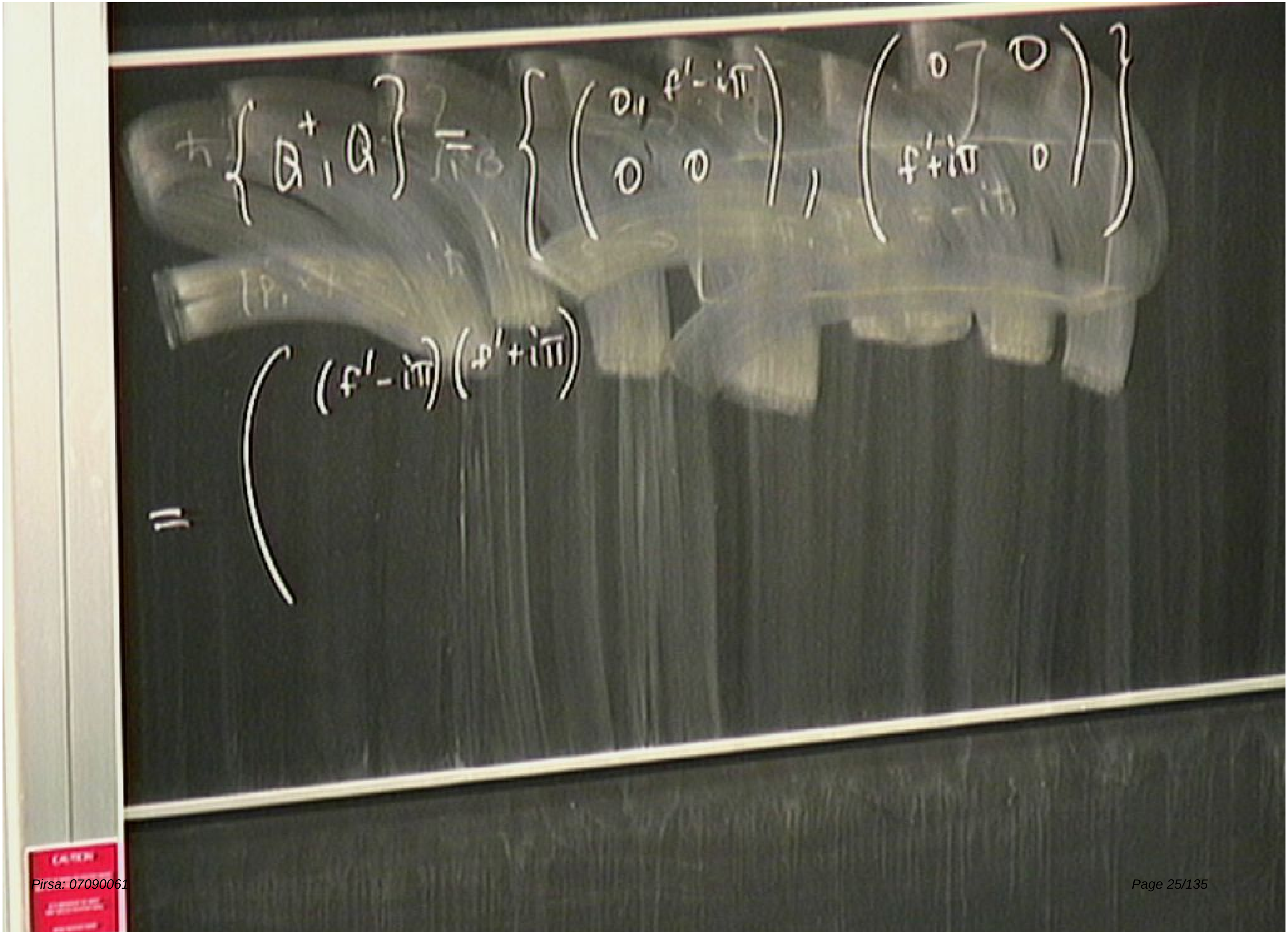
$\Downarrow$ $\{Q, Q\} = 0$ ✓ $\{Q^+, Q^+\} = 0$ ✓

$H = \frac{1}{2}\{Q^+, Q\}$ // we need to check that $[H, Q] = [H, Q^+] = 0$.

$[Q, H] = \frac{1}{2}\left[Q, \{Q^+, Q\}\right] = \frac{1}{2}\left[Q, Q^+Q + QQ^+\right]$

$= \frac{1}{2}\left(QQ^+Q + \cancel{Q^2Q^+} - \cancel{Q^+Q^2} - QQ^+Q\right)$


ħ{Q†, Q} = {(0 f'−iπ; 0 0), (0 0; f'+iπ 0)}
=



$$\{Q^{+}, Q\} = \left\{ \begin{pmatrix} 0 & f' - i\pi \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ f' + i\pi & 0 \end{pmatrix} \right\}$$

$$= \begin{pmatrix} (f' - i\pi)(f' + i\pi) & \\ & \end{pmatrix}$$

$$\{Q^{+}, Q\} = \left\{ \begin{pmatrix} 0 & f'-i\pi \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ f'+i\pi & 0 \end{pmatrix} \right\}$$

$$\begin{pmatrix} (f'-i\pi)(f'+i\pi) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & (f'+i\pi)(f'-i\pi) \end{pmatrix}$$

$$(\phi' - i\pi)(\phi' + i\pi) =$$

$$= (\phi')^2 + \pi^2 - i[\pi, \phi']$$

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$$[\pi, \varphi] = -i\hbar$$

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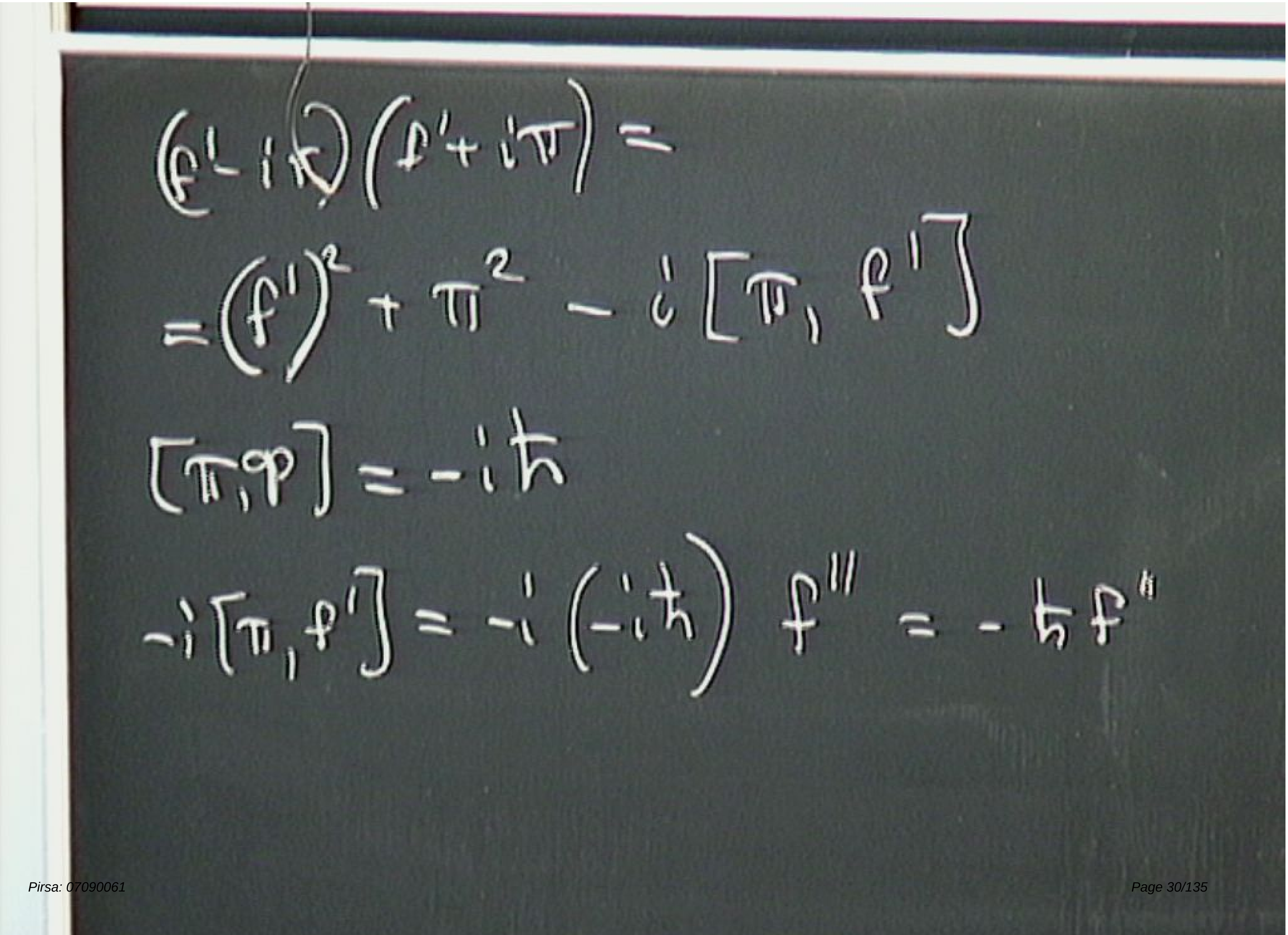
$$[\pi, \varphi] = -i\hbar$$

$$-i[\pi, \varphi'] = -i(-i\hbar)\,\varphi''$$



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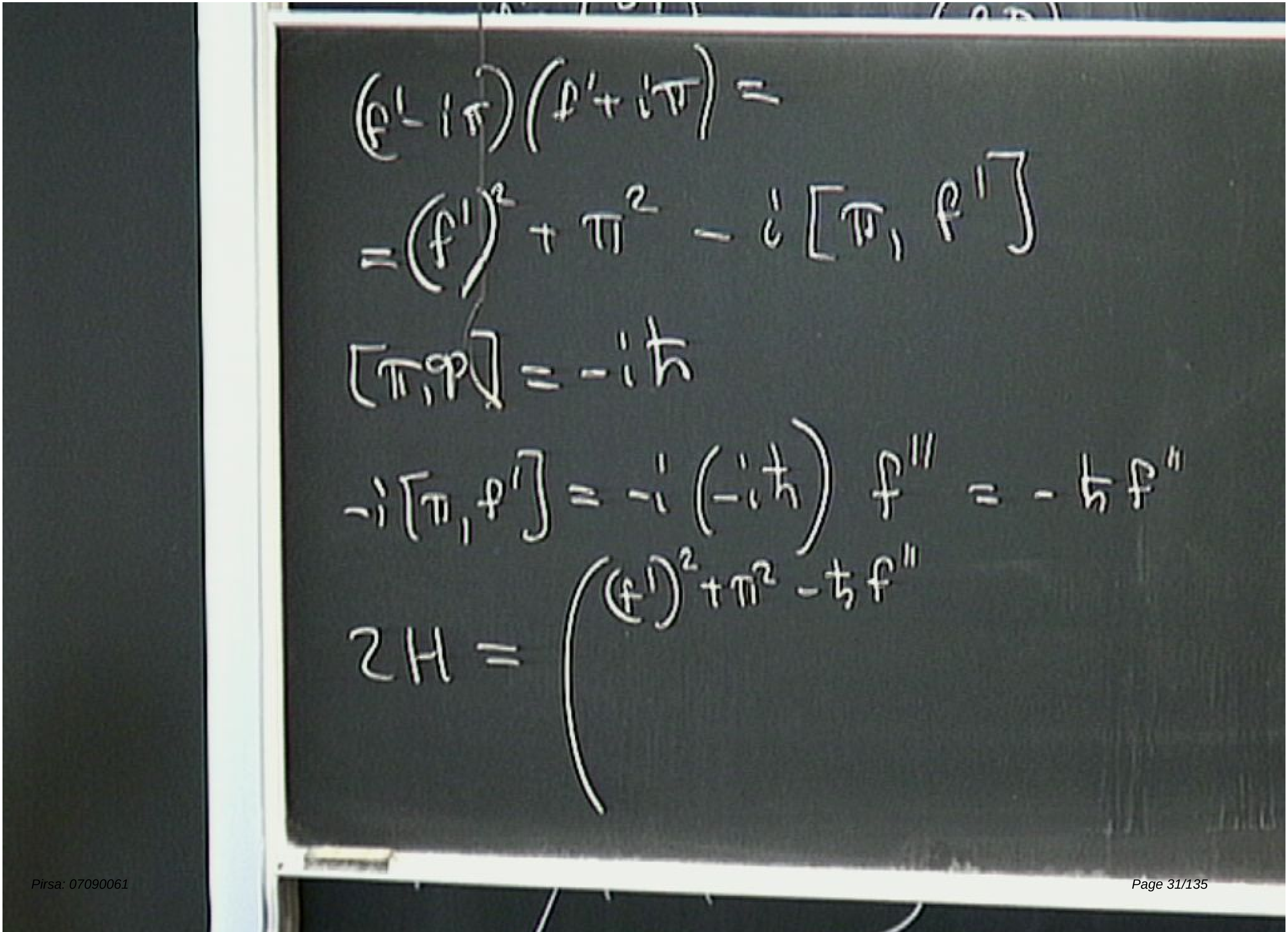

$$(f' - i\pi)(f' + i\pi) =$$

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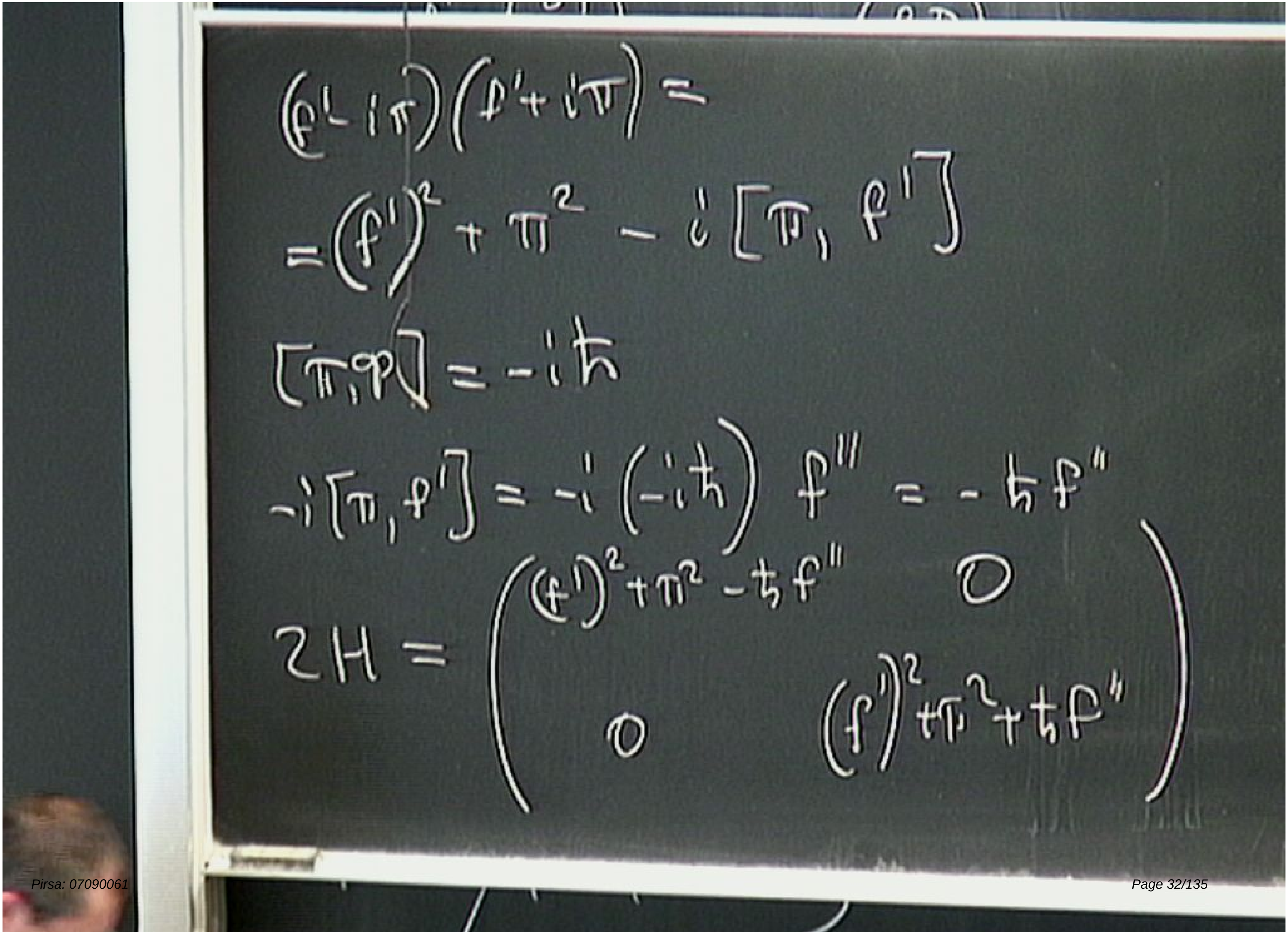
$$2H = \Big((f')^2 + \pi^2 - \hbar f''$$



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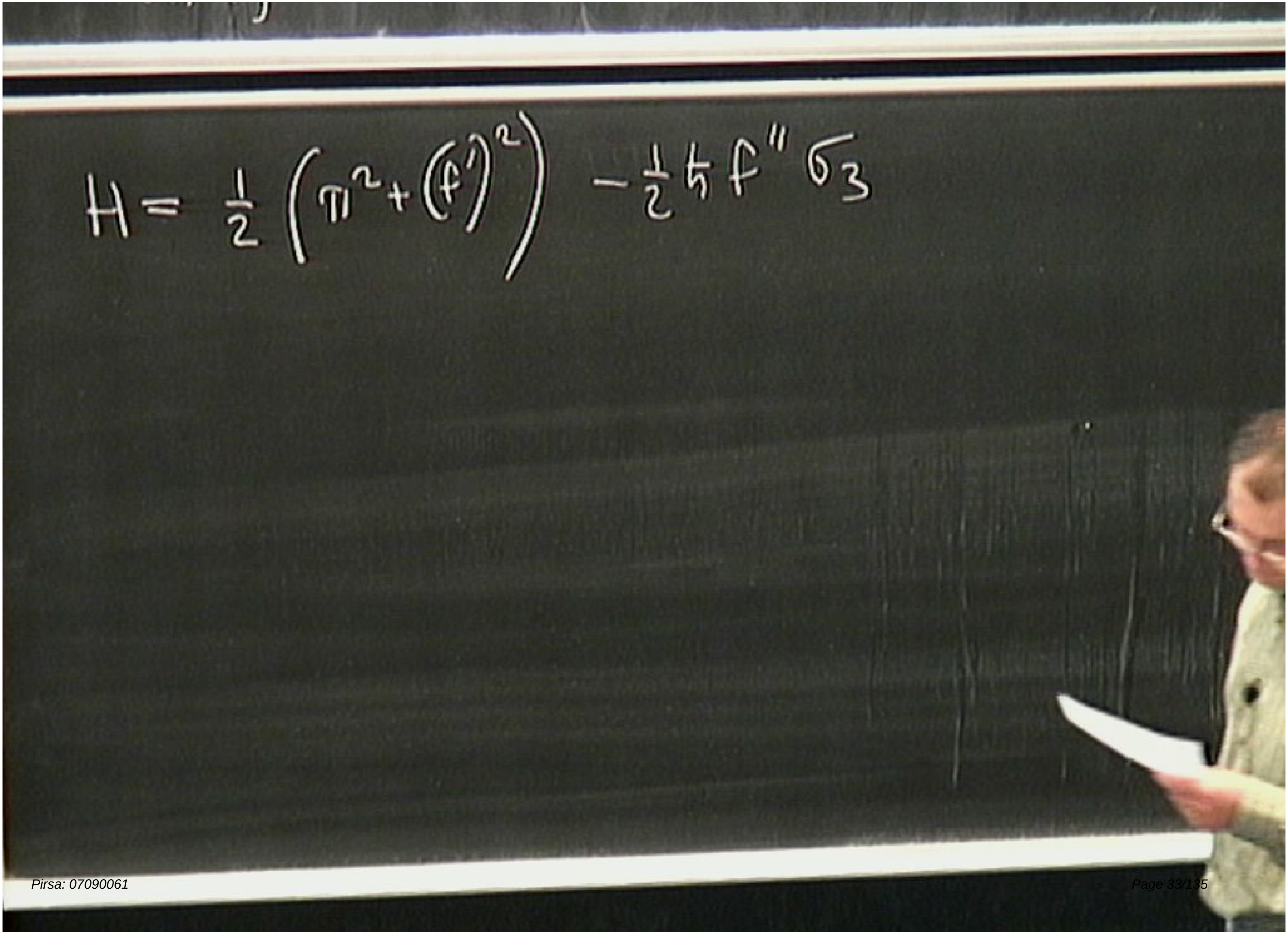
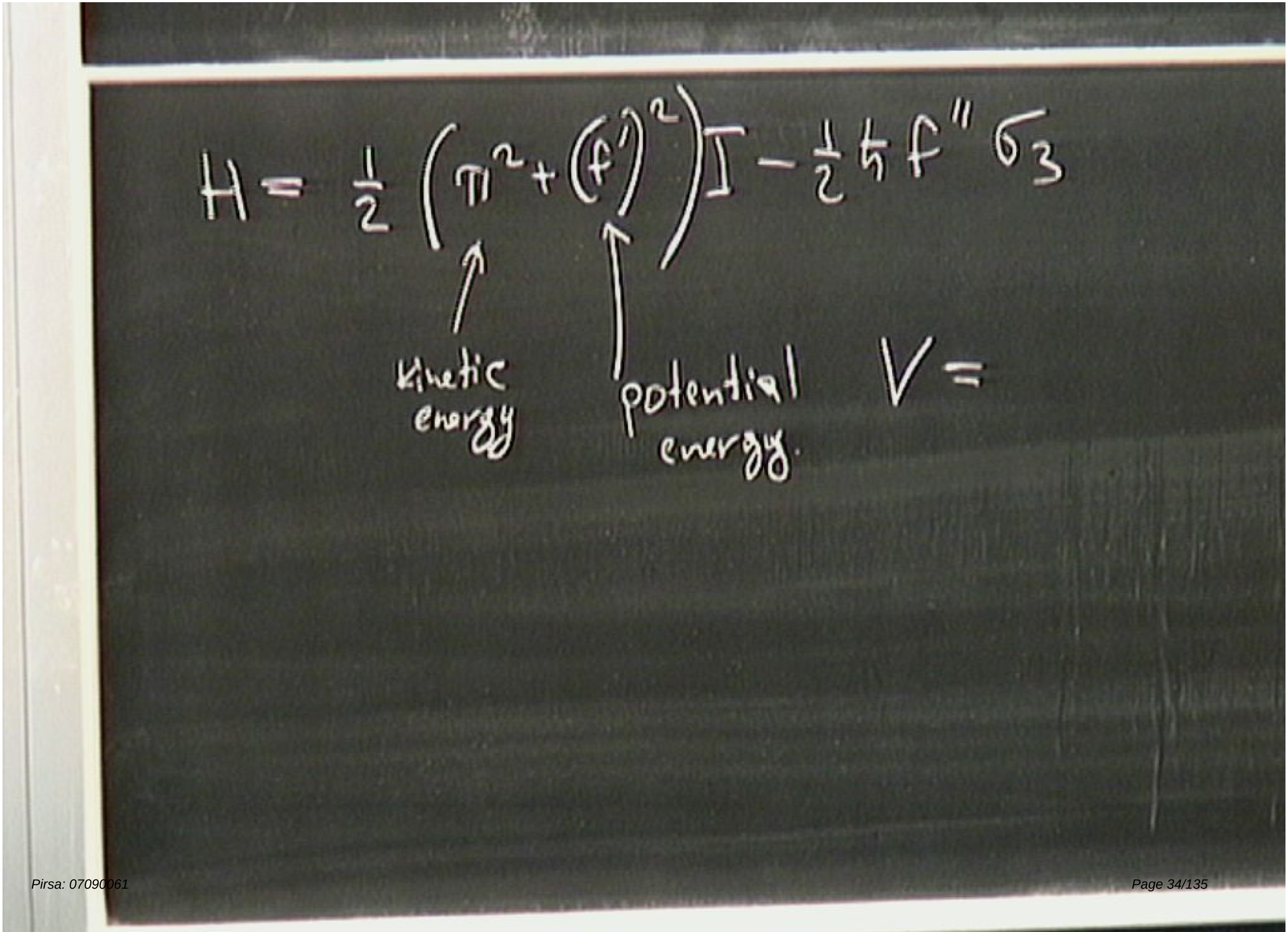
$$2H = \begin{pmatrix} (f')^2 + \pi^2 - \hbar f'' & 0 \\ 0 & (f')^2 + \pi^2 + \hbar f'' \end{pmatrix}$$



$$H = \frac{1}{2}\left(\pi^2 + (f')^2\right) - \frac{1}{2}\hbar f'' \sigma_3$$



$$H = \frac{1}{2}\left(\pi^2 + (f')^2\right) I - \frac{1}{2}\hbar f'' \sigma_3$$

kinetic energy

potential energy.

$V =$

$$H = \frac{1}{2}\Big(\underset{\substack{\uparrow \\ \text{kinetic} \\ \text{energy}}}{\pi}^2 + \big(\underset{\substack{\uparrow \\ \text{potential} \\ \text{energy.}}}{f'}\big)^2\Big) I - \frac{1}{2}\hbar f'' \sigma_3$$

$$V = \left(\frac{df}{d\varphi}\right)^2$$

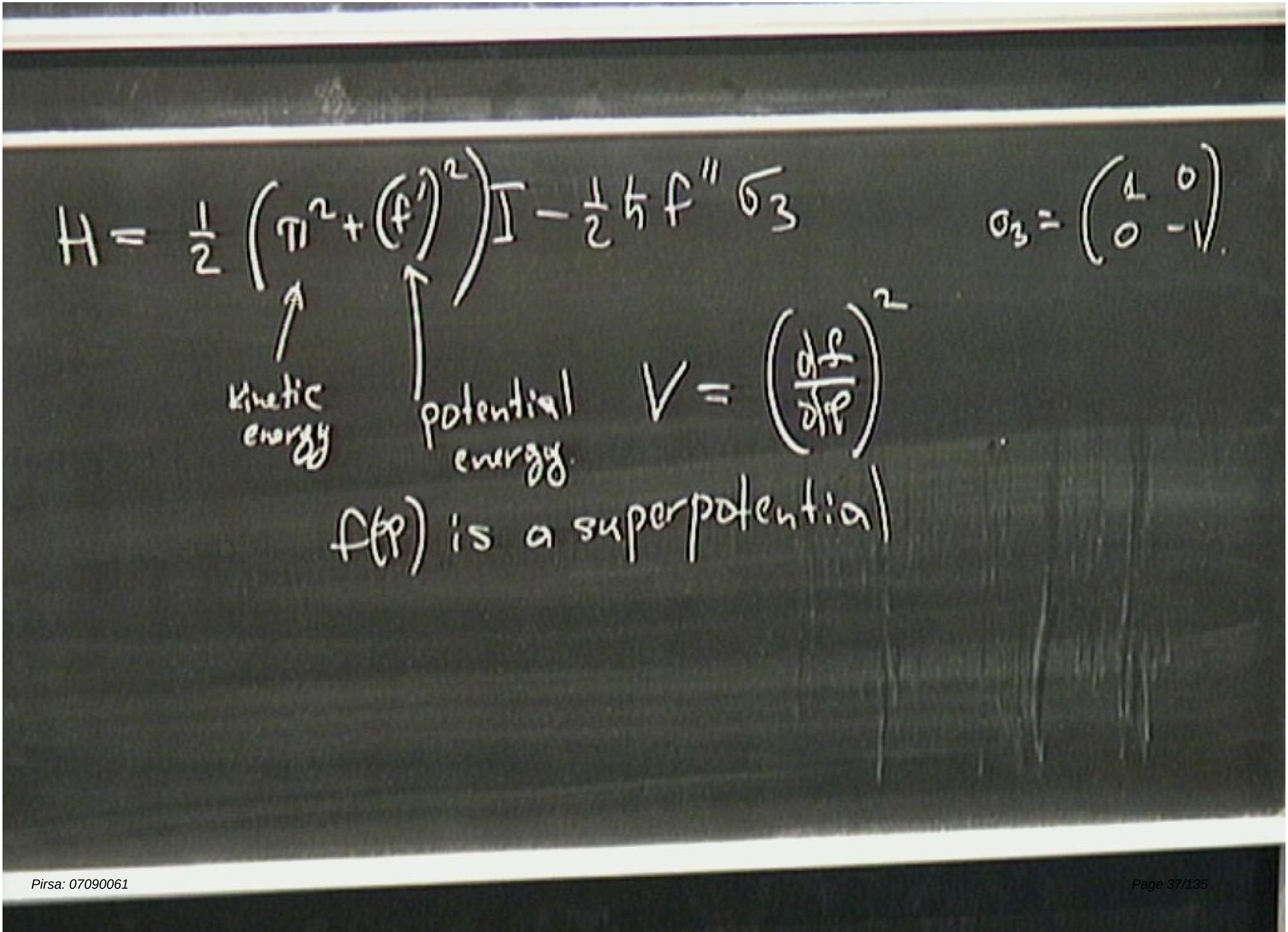
$$H = \frac{1}{2}\left(\pi^2 + (f')^2\right) I - \frac{1}{2}\hbar f'' \sigma_3$$

kinetic energy

potential energy

$$V = \left(\frac{df}{d\varphi}\right)^2$$

$f(\varphi)$ is a superpotential



$$H = \frac{1}{2}\Big(\underset{\substack{\uparrow\\ \text{kinetic}\\ \text{energy}}}{\pi^2} + \underset{\substack{\uparrow\\ \text{potential}\\ \text{energy.}}}{(f')^2}\Big) I - \frac{1}{2}\hbar f'' \sigma_3 \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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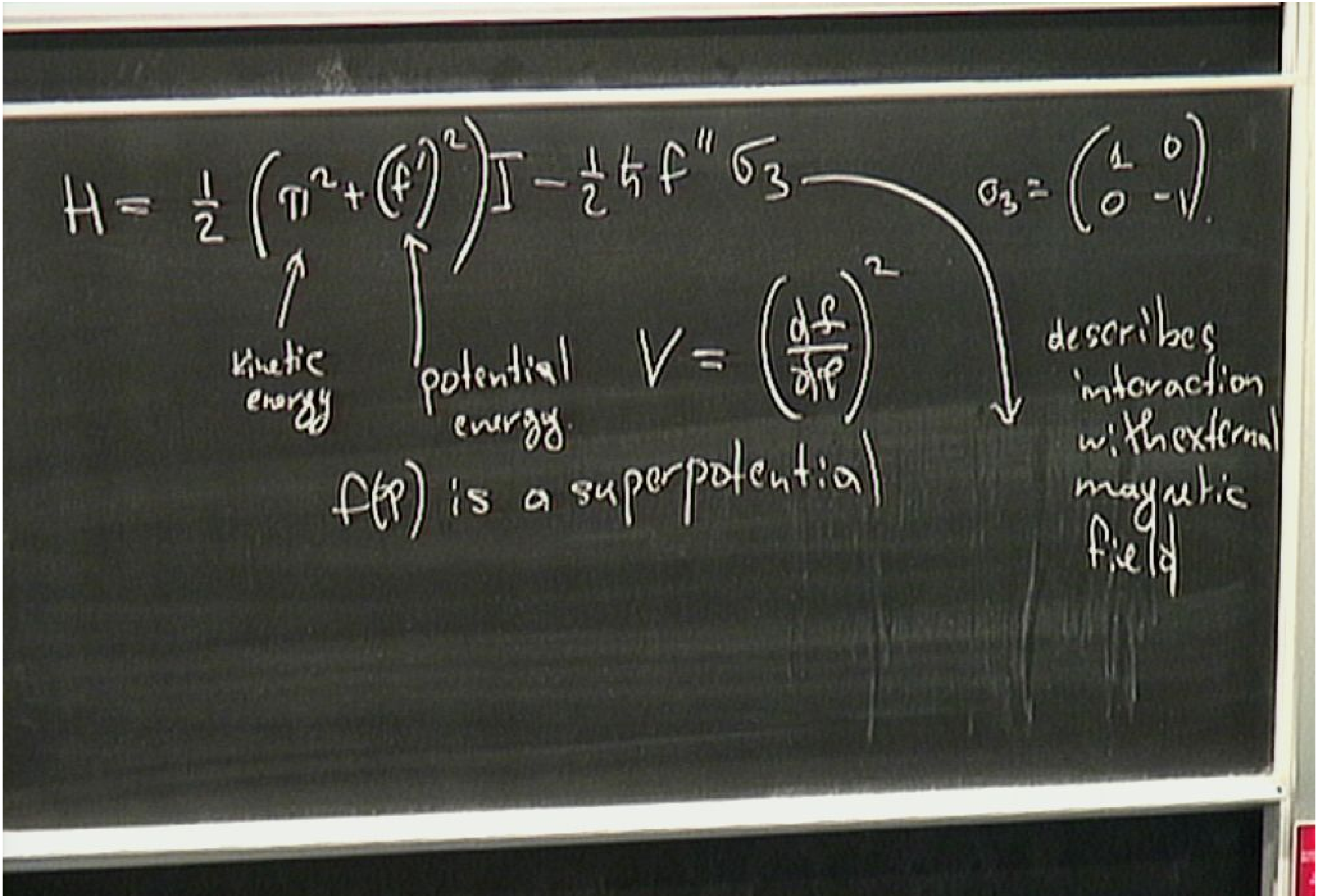

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kinetic energy

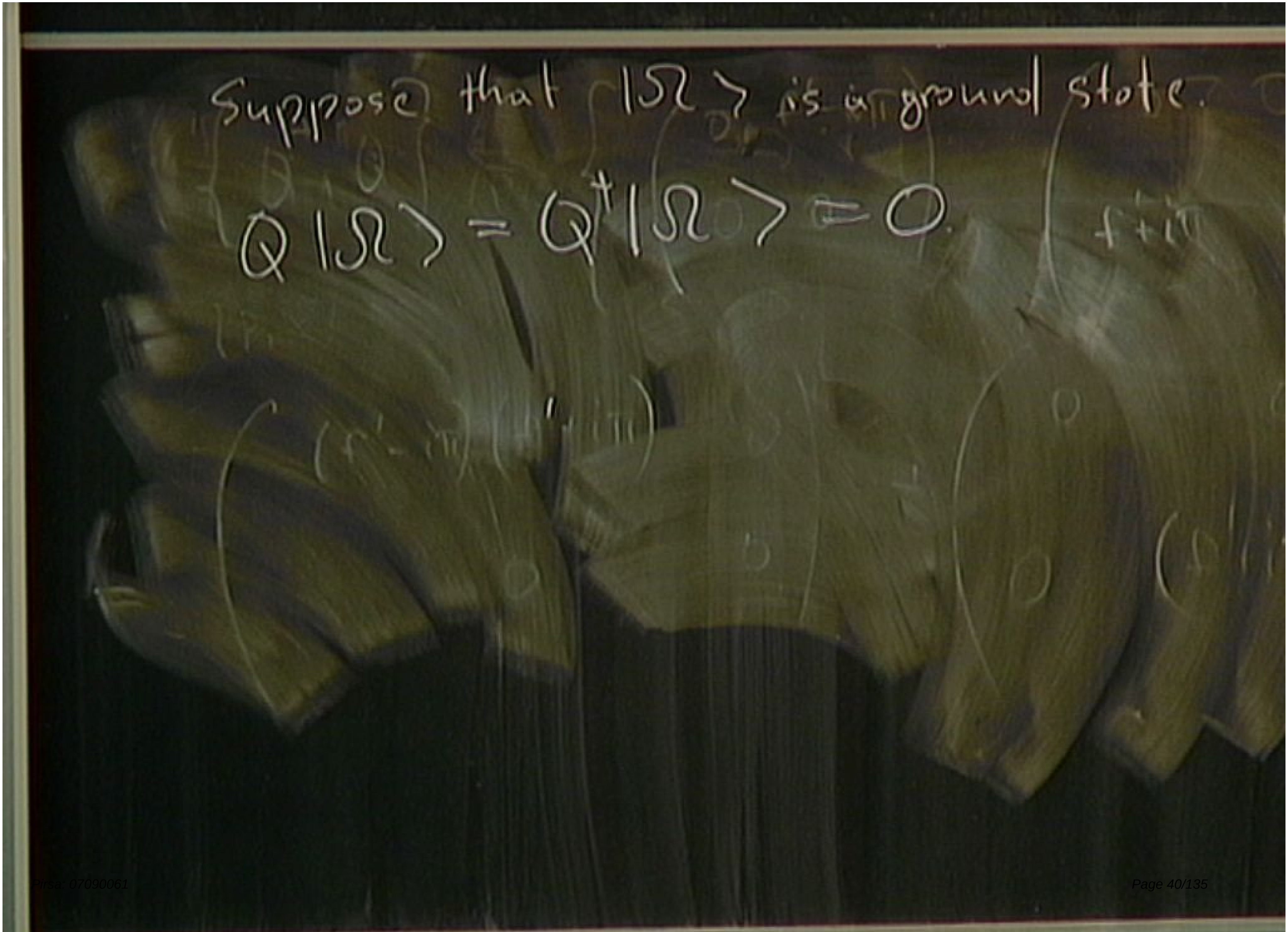
potential energy.

$$V = \left(\frac{df}{d\varphi}\right)^2$$

f(φ) is a superpotential

describes interaction with the external magnetic field

Suppose that $|\Omega\rangle$ is a ground state.



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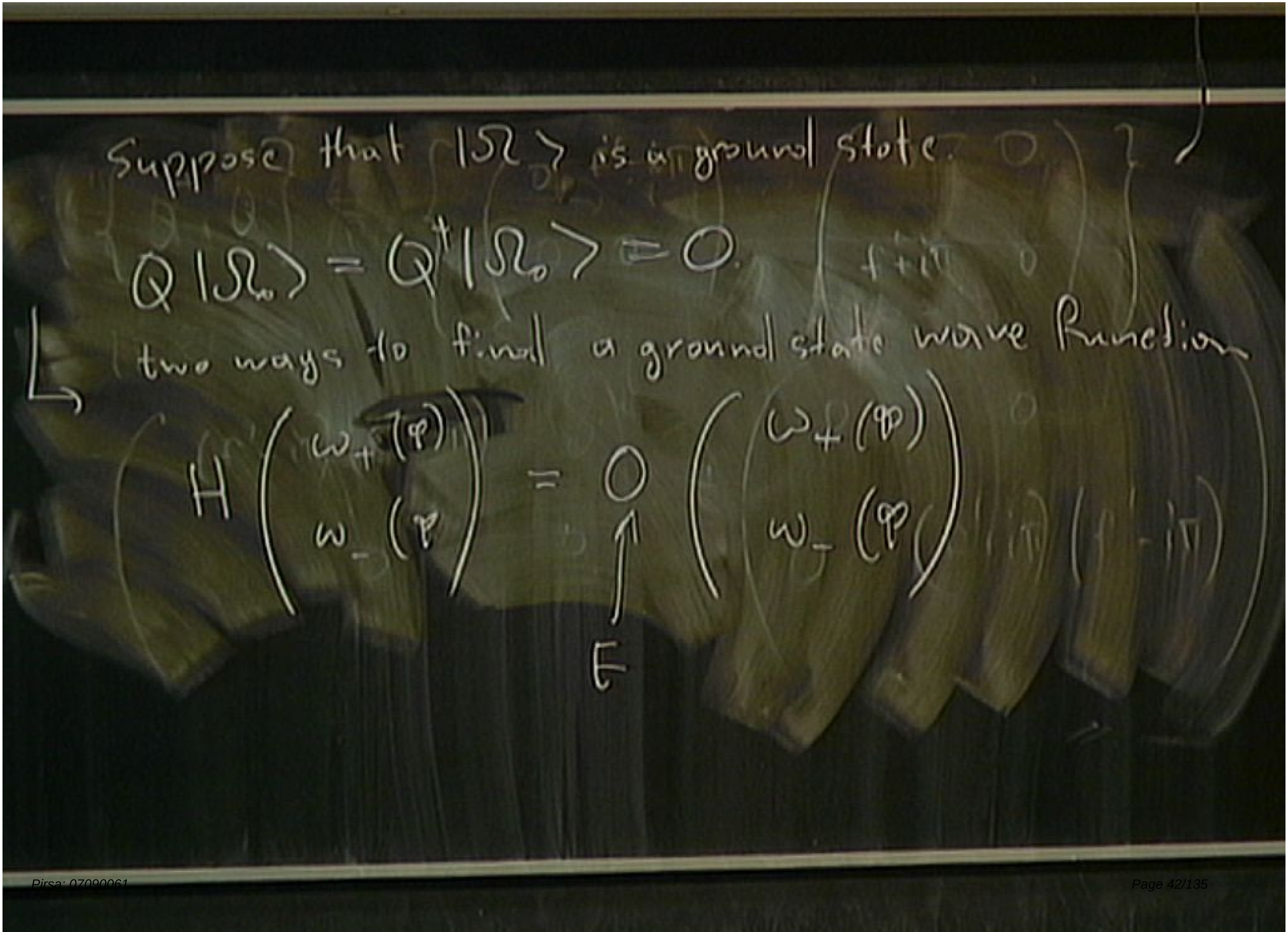
$$Q|\Omega\rangle = Q^\dagger|\Omega\rangle = 0.$$

Suppose that $|\Omega\rangle$ is a ground state.

$$Q|\Omega_0\rangle = Q^\dagger|\Omega_0\rangle = 0.$$

↳ two ways to find a ground state wave function

$$H\begin{pmatrix}\omega_+(\varphi)\\ \omega_-(\varphi)\end{pmatrix}$$



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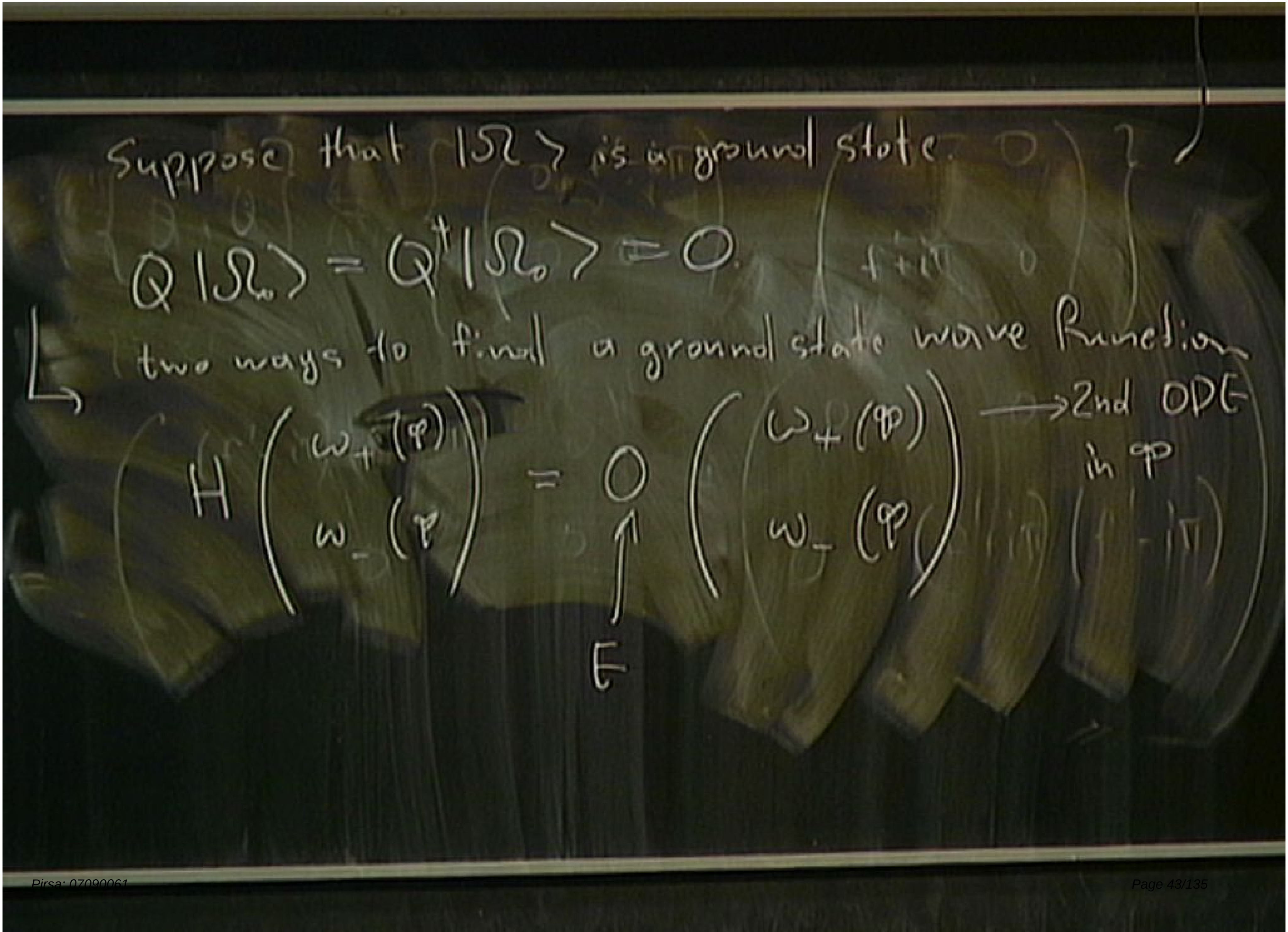
$$H\begin{pmatrix}\omega_+(\varphi)\\ \omega_-(\varphi)\end{pmatrix} = \underset{\substack{\uparrow\\ E}}{0}\begin{pmatrix}\omega_+(\varphi)\\ \omega_-(\varphi)\end{pmatrix}$$



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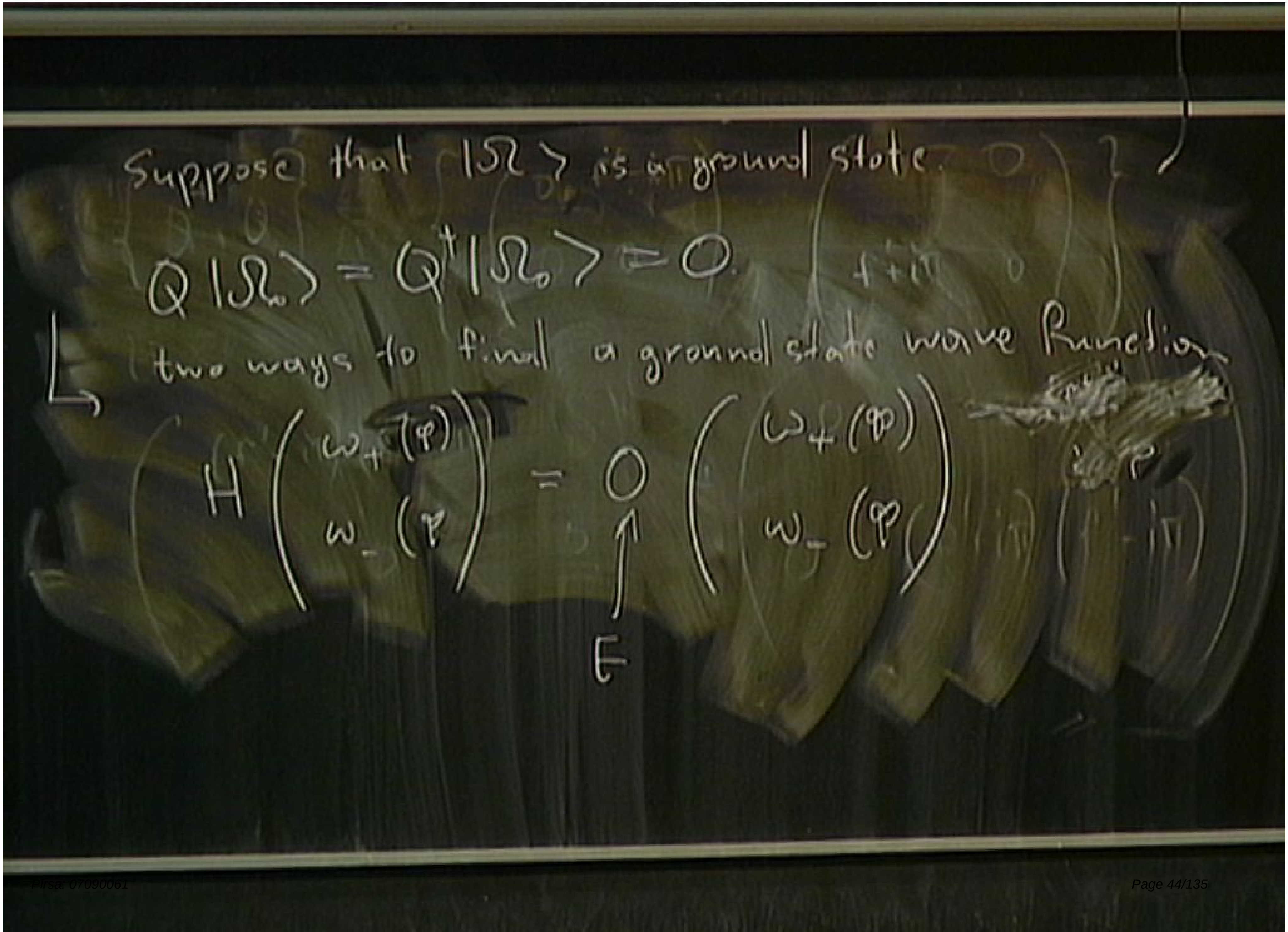
↳ two ways to find a ground state wave function

$$H\begin{pmatrix}\omega_+(\varphi)\\ \omega_-(\varphi)\end{pmatrix} = \underset{\underset{E}{\uparrow}}{0}\begin{pmatrix}\omega_+(\varphi)\\ \omega_-(\varphi)\end{pmatrix} \longrightarrow \text{2nd ODE in } \varphi$$



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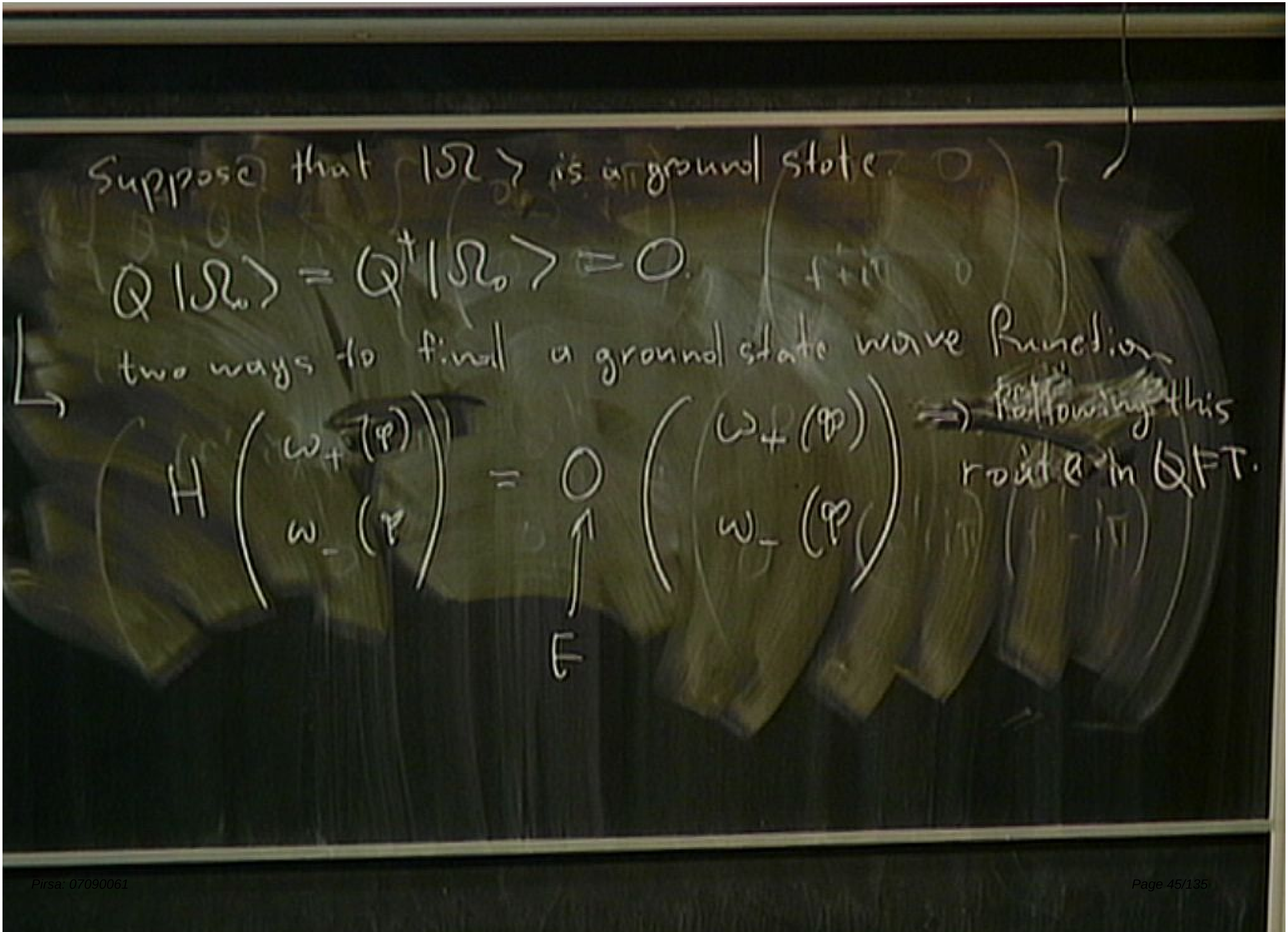

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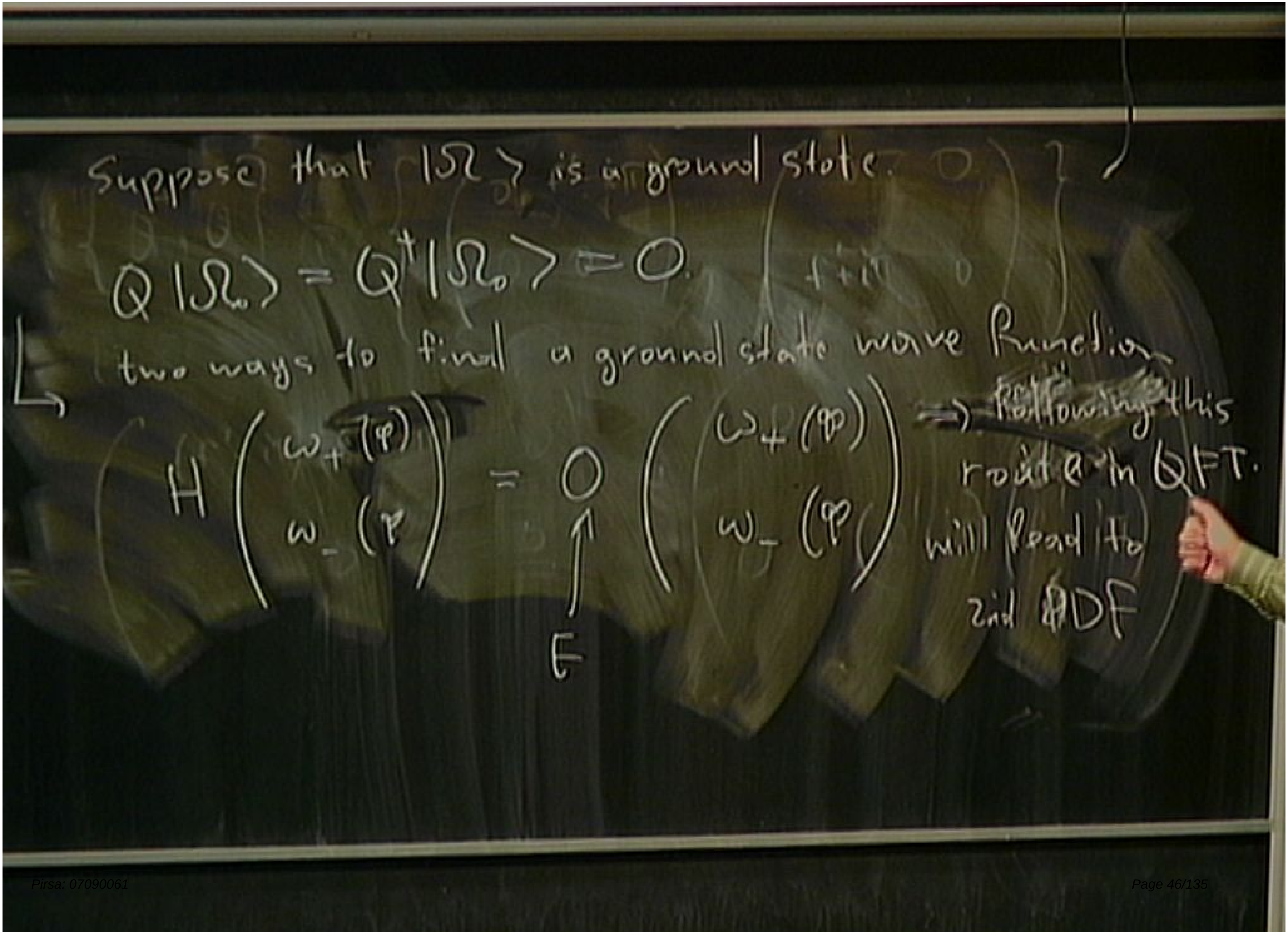
following this route in QFT.



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↳ two ways to find a ground state wave function

$$H\begin{pmatrix}\omega_+(\varphi)\\ \omega_-(\varphi)\end{pmatrix} = \underset{\overset{\uparrow}{E}}{0}\begin{pmatrix}\omega_+(\varphi)\\ \omega_-(\varphi)\end{pmatrix} \Rightarrow$$

Following this route in QFT will lead to 2nd PDF

$$H\Big(\omega_-(\varphi \Big) = 0 \qquad \Big(\omega_-$$

$$\uparrow$$

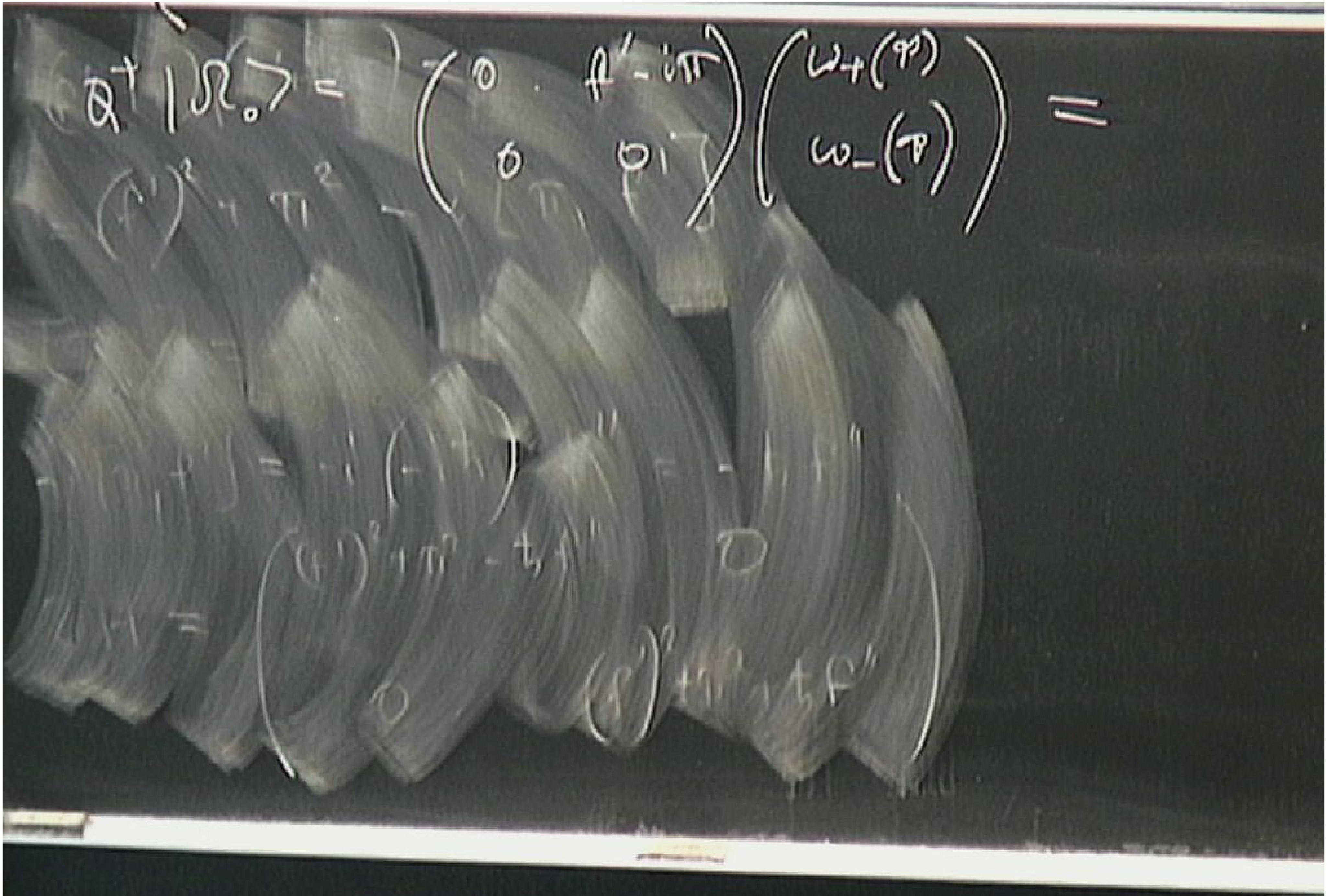
$$\pi^2 = \left(-i\hbar \frac{\partial}{\partial\varphi}\right)^2 = -\hbar^2 \frac{\partial^2}{\partial\varphi^2} \qquad E$$

$$Q^\dagger|\Omega_0\rangle = \begin{pmatrix} 0 & f' - i\pi \\ 0 & \rho' \end{pmatrix}$$

$(f')^2 + \pi^2 - i[\pi, f']$

$[\pi, f'] = -i(-\hbar \ldots) = -\hbar f''$

$$\Delta H = \begin{pmatrix} (f')^2 + \pi^2 - \hbar f'' & 0 \\ 0 & (f')^2 + \pi^2 + \hbar f'' \end{pmatrix}$$



$$Q^\dagger|\Omega_0\rangle = \begin{pmatrix} 0 & f' - i\pi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_+(\varphi) \\ \omega_-(\varphi) \end{pmatrix} =$$

$$(f')^2 + \pi^2 - i[\pi, f']$$

$$[\pi, f'] = -i(-\hbar f'')$$

$$\{Q, Q^\dagger\} = \begin{pmatrix} (f')^2 + \pi^2 - \hbar f'' & 0 \\ 0 & (f')^2 + \pi^2 + \hbar f'' \end{pmatrix}$$

$$Q^\dagger|\Omega_0\rangle = \begin{pmatrix} 0 & f' - i\pi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \omega_+(\varphi) \\ \omega_-(\varphi) \end{pmatrix} =$$

$$= \begin{pmatrix} (f' - i\pi)\,\omega_-(\varphi) \\ 0 \end{pmatrix}$$

$$Q^{\dagger}|\Omega_0\rangle = \begin{pmatrix} 0 & f' - i\pi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \omega_+(\varphi) \\ \omega_-(\varphi) \end{pmatrix} =$$

$$= \begin{pmatrix} (f' - i\pi)\,\omega_-(\varphi) \\ 0 \end{pmatrix} = 0.$$

$$Q^{\dagger}|\Omega_0\rangle = 0$$

$$\Downarrow$$

$$\left[\begin{array}{l} (f' - i\pi)\,\omega_-(\varphi) = 0 \\ \omega_+(\varphi) \text{ is anything} \end{array} \right]$$

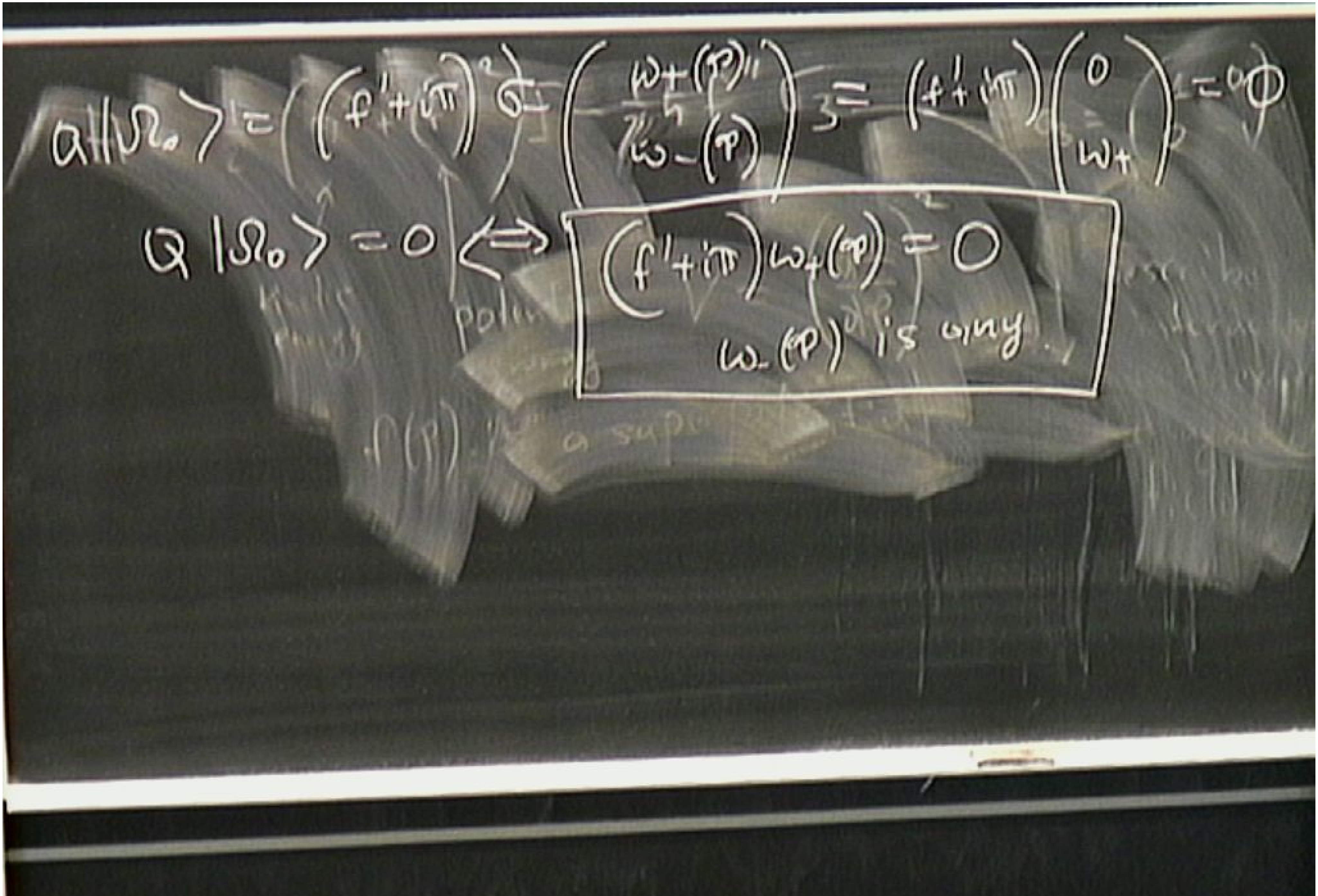
$$a\|\Omega_0\rangle' = \left((f' + i\pi)^2 \sigma_3 \right) \left(\frac{\omega_+(p)'}{\omega_-(p)} \right)_3 =$$

kinetic energy

potential energy

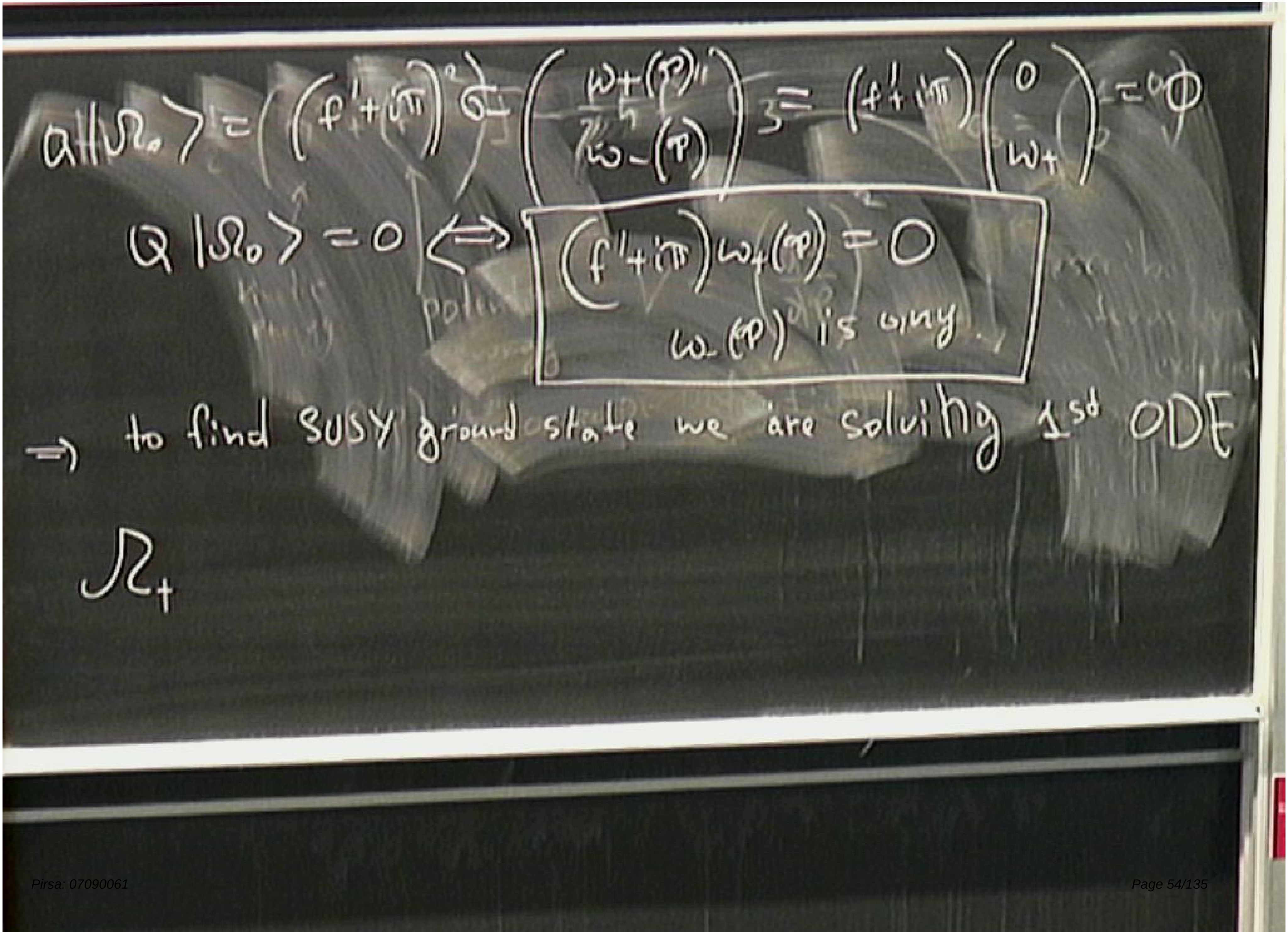
$$V = \left(\frac{df}{dp} \right)^2$$

$f(p)$ is a superpotential



$$Q|\Omega_0\rangle = \left((f'+i\pi)^2\,\sigma_-\right)\begin{pmatrix}\omega_+(p) \\ \omega_-(p)\end{pmatrix} = (f'+i\pi)\begin{pmatrix}0\\ \omega_+\end{pmatrix} = 0\,\phi$$

$$Q\,|\Omega_0\rangle = 0 \Longleftrightarrow \boxed{\begin{array}{c}(f'+i\pi)\,\omega_+(p) = 0 \\ \omega_-(p) \text{ is any.}\end{array}}$$



$$a|\Omega_0\rangle = \left((f'+i\pi)^2 \sigma_- \right)\begin{pmatrix} \omega_+(\mathbf{p})'' \\ \omega_-(\mathbf{p}) \end{pmatrix} = (f'+i\pi)\begin{pmatrix} 0 \\ \omega_+ \end{pmatrix} = 0\phi$$

$$Q|\Omega_0\rangle = 0 \iff \boxed{\begin{array}{c}(f'+i\pi)\,\omega_+(\mathbf{p}) = 0 \\ \omega_-(\mathbf{p}) \text{ is any}\end{array}}$$

⇒ to find SUSY ground state we are solving 1st ODE

Ω_+

$$(\varphi)\Big) = 0.$$

$$Q^{\dagger}|\Omega_0\rangle = 0$$

$$\Downarrow$$

$$\left[\begin{array}{l}(f' - i\pi)\,\omega_-(\varphi) = 0\\ \omega_+(\varphi) \text{ is anything}\end{array}\right]$$

$$\left(p' - i\bar{p}\right) w_-(\bar{p}) = 0$$

$$\left[p' - i\left(-i\hbar\frac{\partial}{\partial\bar{p}}\right)\right] w_- = 0$$

$$\left(p' - \hbar\frac{\partial}{\partial\bar{p}}\right) w_- = 0$$

$$\left(f' - i\pi\right) w_-(\varphi) = 0$$

$$\left[f' - i\left(-i\hbar\frac{\partial}{\partial\varphi}\right)\right] w_- = 0$$

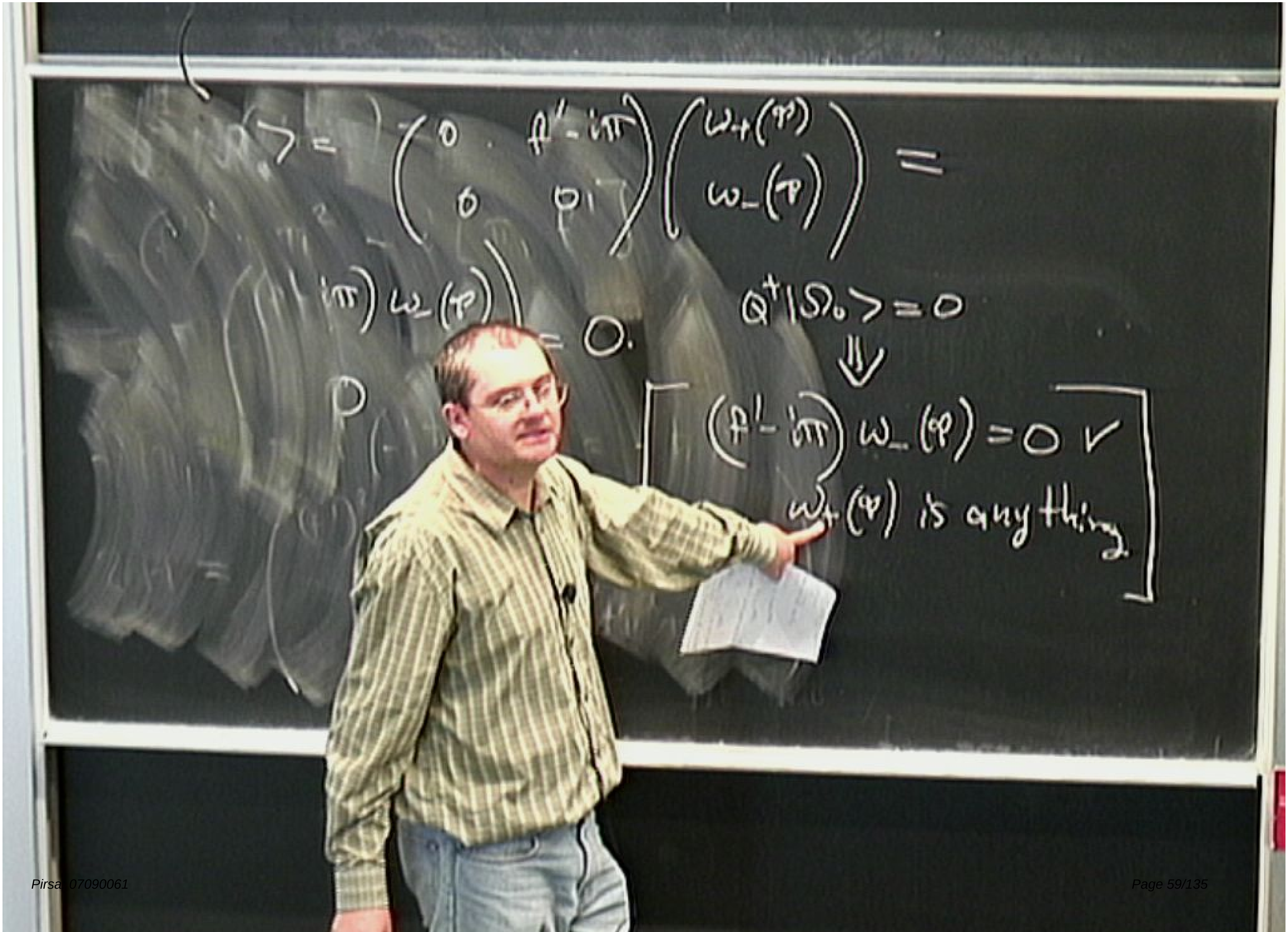
$$\left(f' - \hbar\frac{\partial}{\partial\varphi}\right) w_- = 0 \quad \Rightarrow \quad \hat{w}_-(\varphi) = A_-\, e^{f(\varphi)/\hbar}$$

$$\left(f' - i\pi\right)\omega_-(\varphi) = 0$$

$$\left[f' - i\left(-i\hbar\frac{\partial}{\partial\varphi}\right)\right]\omega_- = 0$$

$$\left(f' - \hbar\frac{\partial}{\partial\varphi}\right)\omega_- = 0 \Rightarrow \hat{\omega}_-(\varphi) = A_-\, e^{f(\varphi)/\hbar}$$

$$\left(f' + \hbar\frac{\partial}{\partial\varphi}\right)\omega_+ = 0 \Rightarrow \hat{\omega}_+(\varphi)$$



$$|\Omega_0\rangle = \begin{pmatrix} 0 & f'-i\pi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \omega_+(\varphi) \\ \omega_-(\varphi) \end{pmatrix} =$$

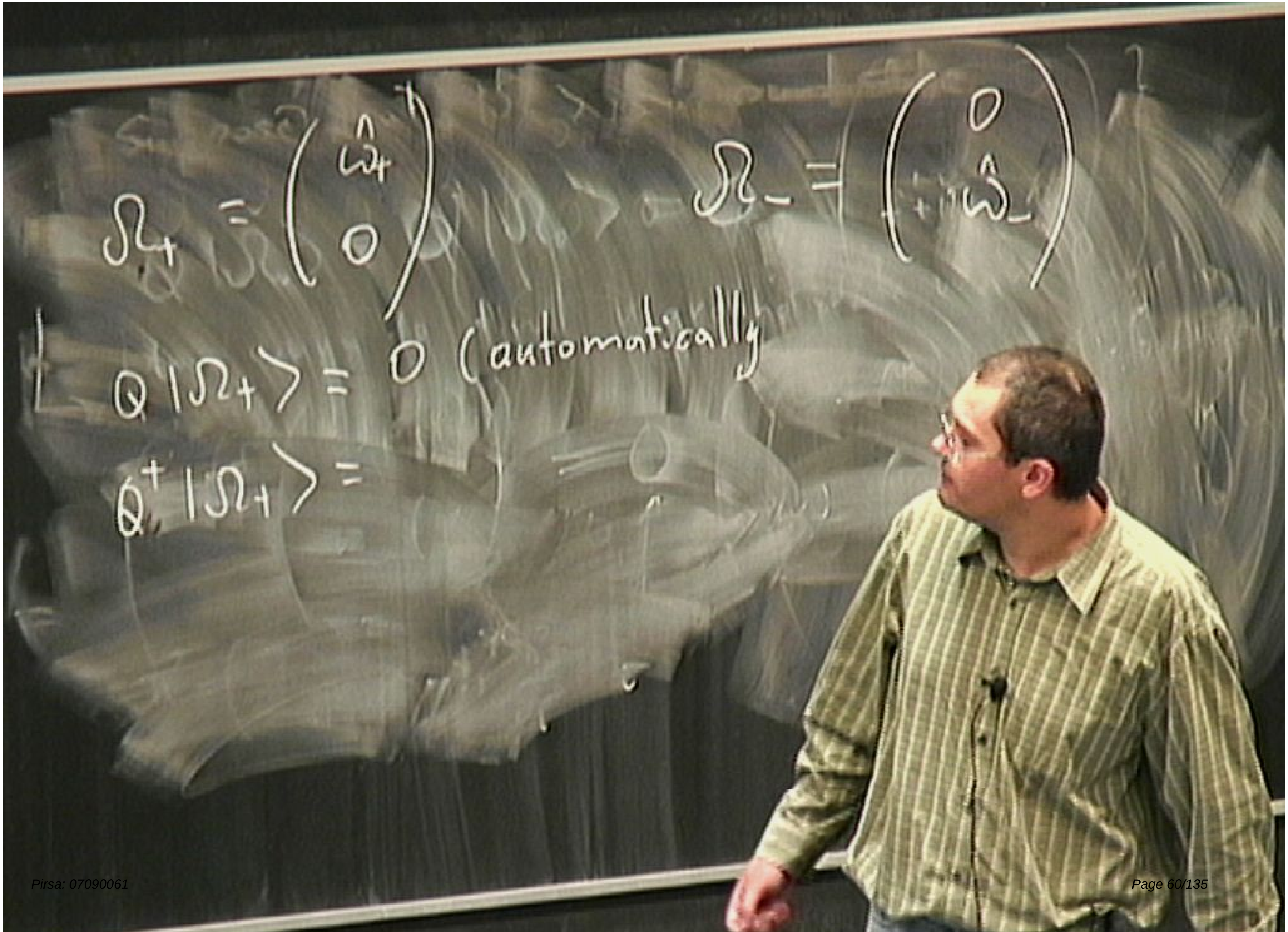
$$\left(\quad i\pi) \omega_-(\varphi) \right) = 0.$$

$$Q^+|\Omega_0\rangle = 0$$

$$\Downarrow$$

$$(f'-i\pi)\,\omega_-(\varphi) = 0 \checkmark$$

$\omega_+(\varphi)$ is anything



$$\Omega_+ = \begin{pmatrix} \hat{\omega}_+ \\ 0 \end{pmatrix} \qquad \Omega_- = \begin{pmatrix} 0 \\ \hat{\omega}_- \end{pmatrix}$$

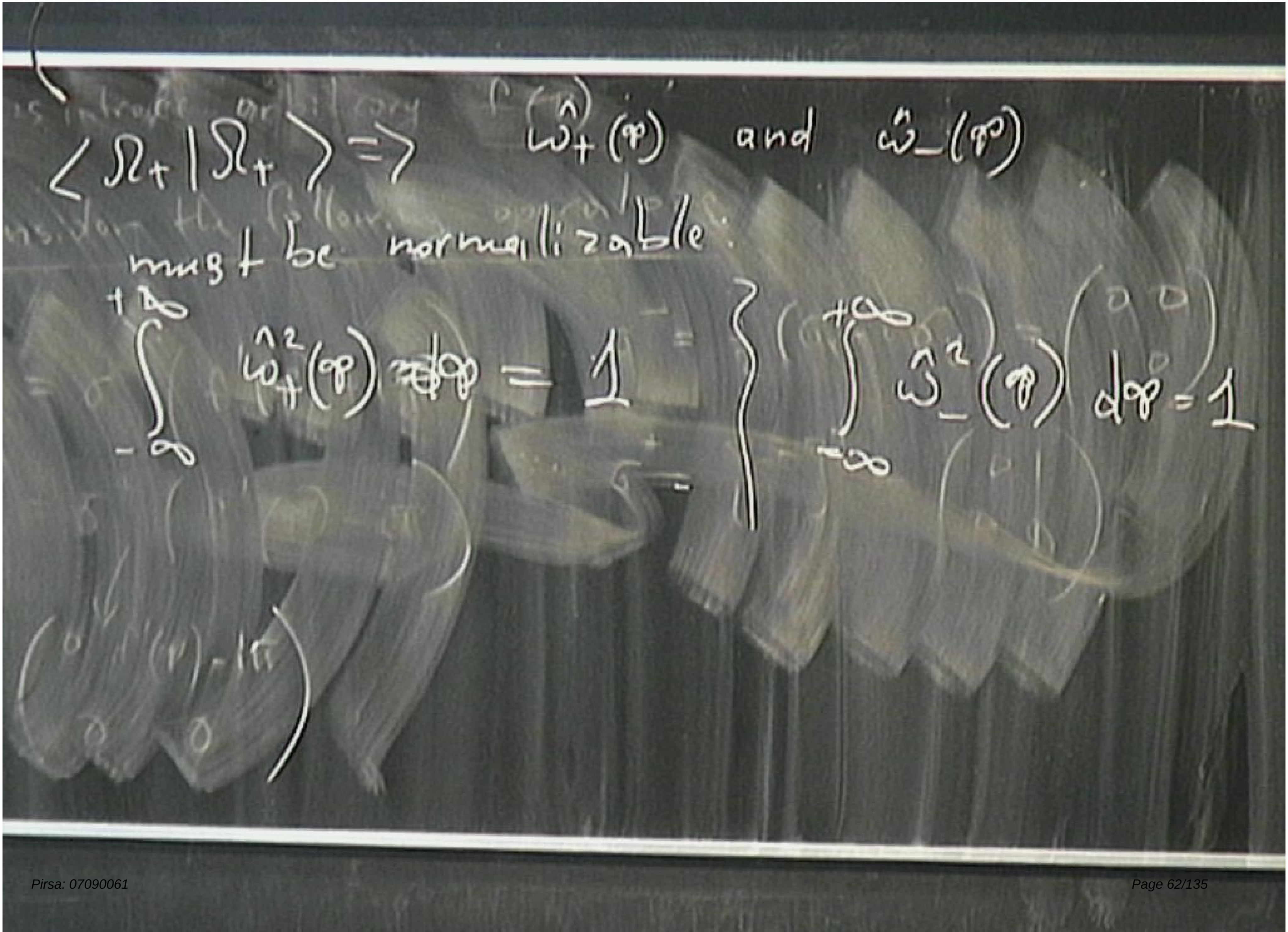
$Q|\Omega_+\rangle = 0$ (automatically

$Q^\dagger|\Omega_+\rangle =$

$$\Omega_+ = \begin{pmatrix} \hat{\omega}_+ \\ 0 \end{pmatrix} \qquad \Omega_- = \begin{pmatrix} 0 \\ \hat{\omega}_- \end{pmatrix}$$

$Q|\Omega_+\rangle = 0$ (automatically)

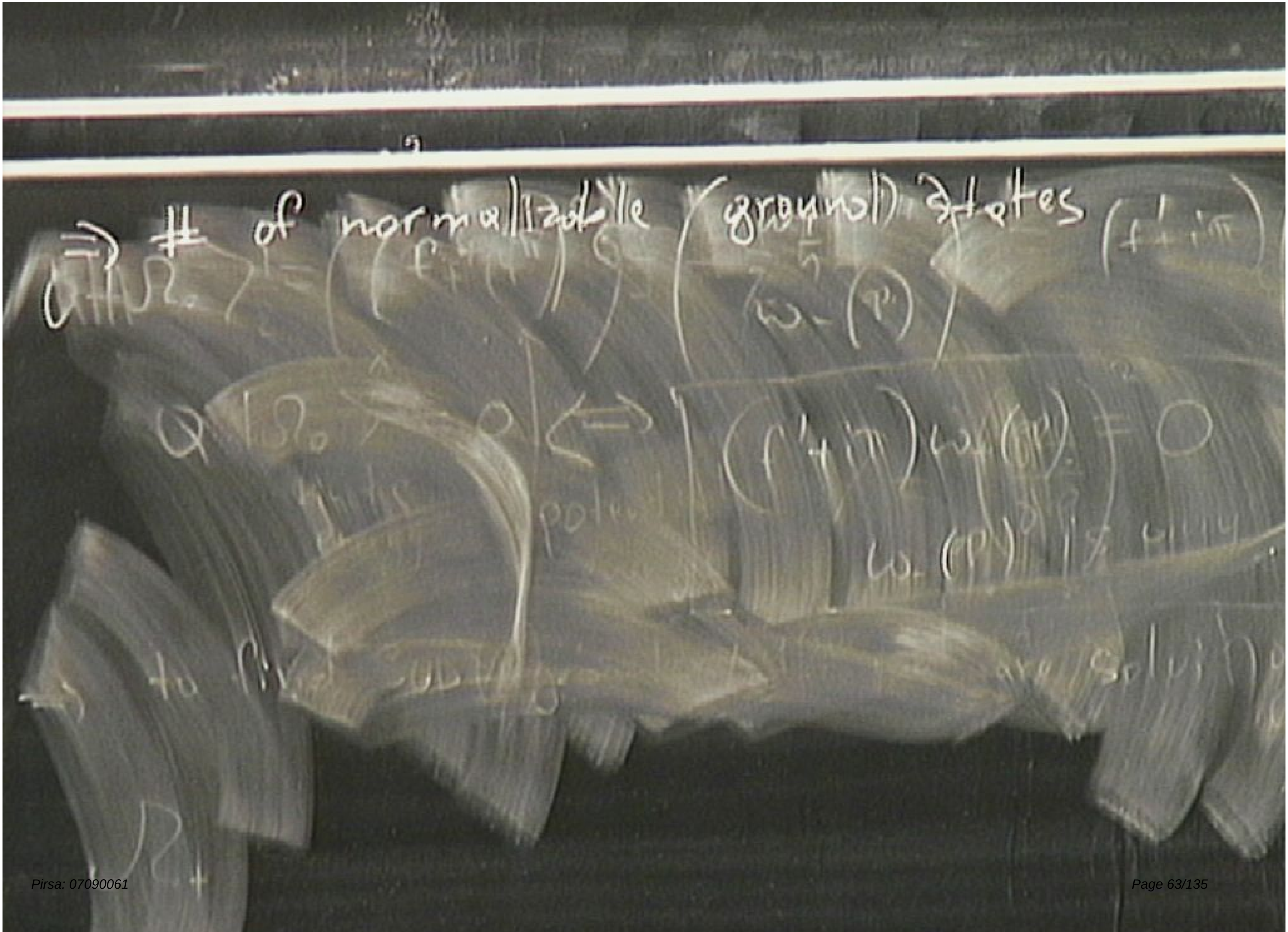
$Q^\dagger|\Omega_+\rangle = 0$ (by solution)



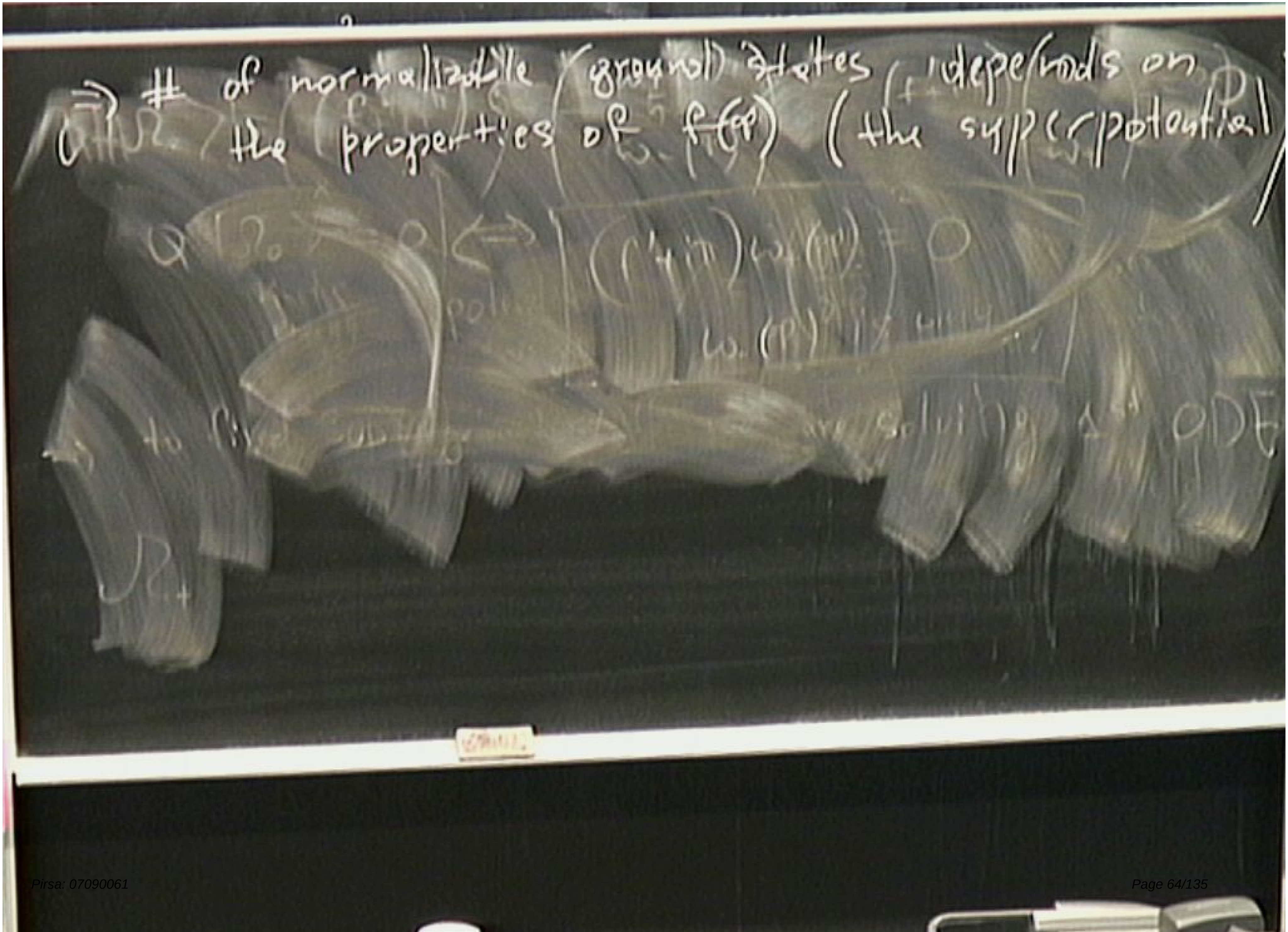
$\langle \Omega_+ | \Omega_+ \rangle \Rightarrow$ $\hat{\omega}_+(\varphi)$ and $\hat{\omega}_-(\varphi)$

must be normalizable:

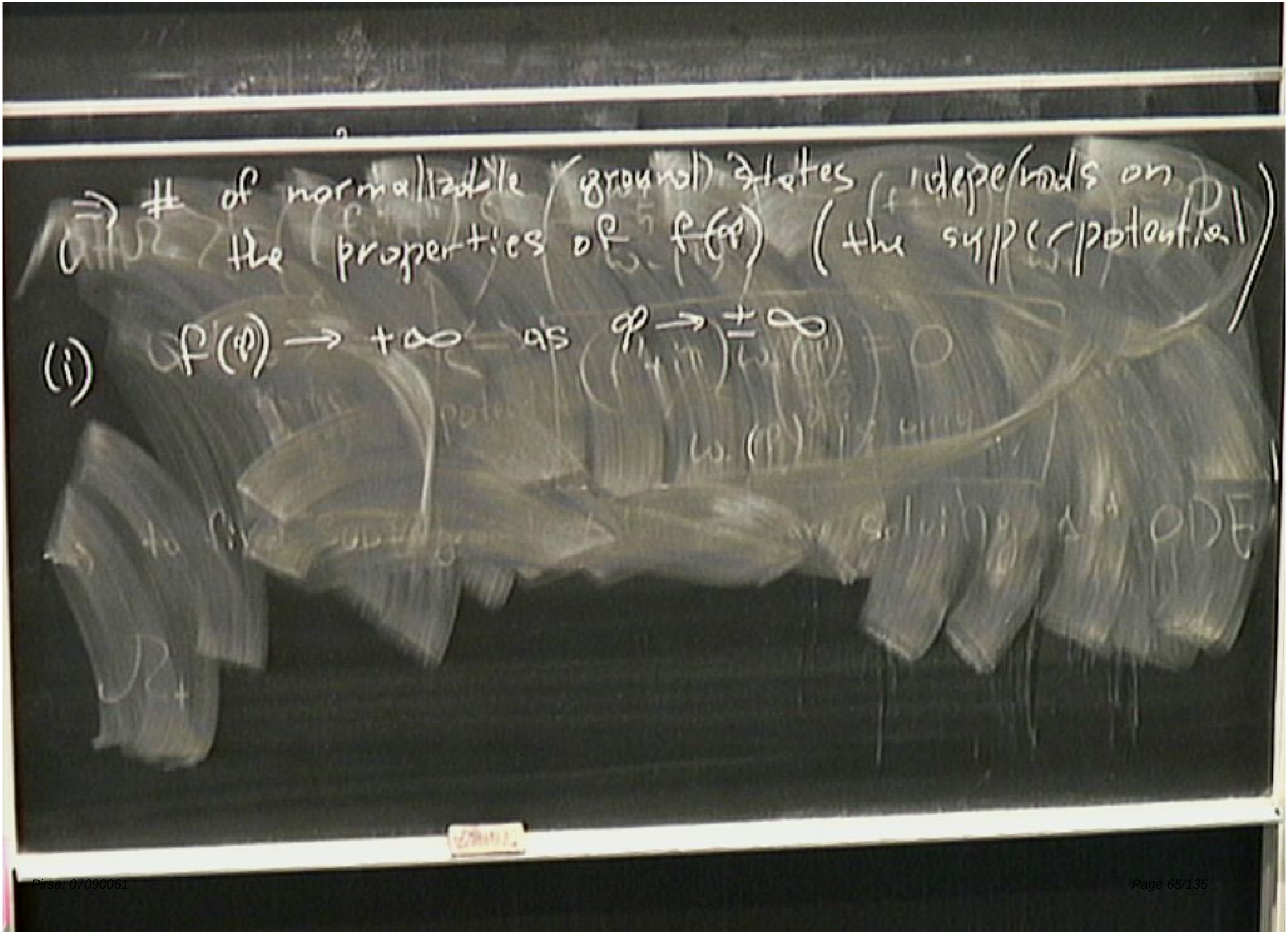
$$\int_{-\infty}^{+\infty} \hat{\omega}_+^2(\varphi)\, d\varphi = 1 \quad \Bigg\} \quad \int_{-\infty}^{+\infty} \hat{\omega}_-^2(\varphi)\, d\varphi = 1$$



⇒ # of normalizable (ground) states

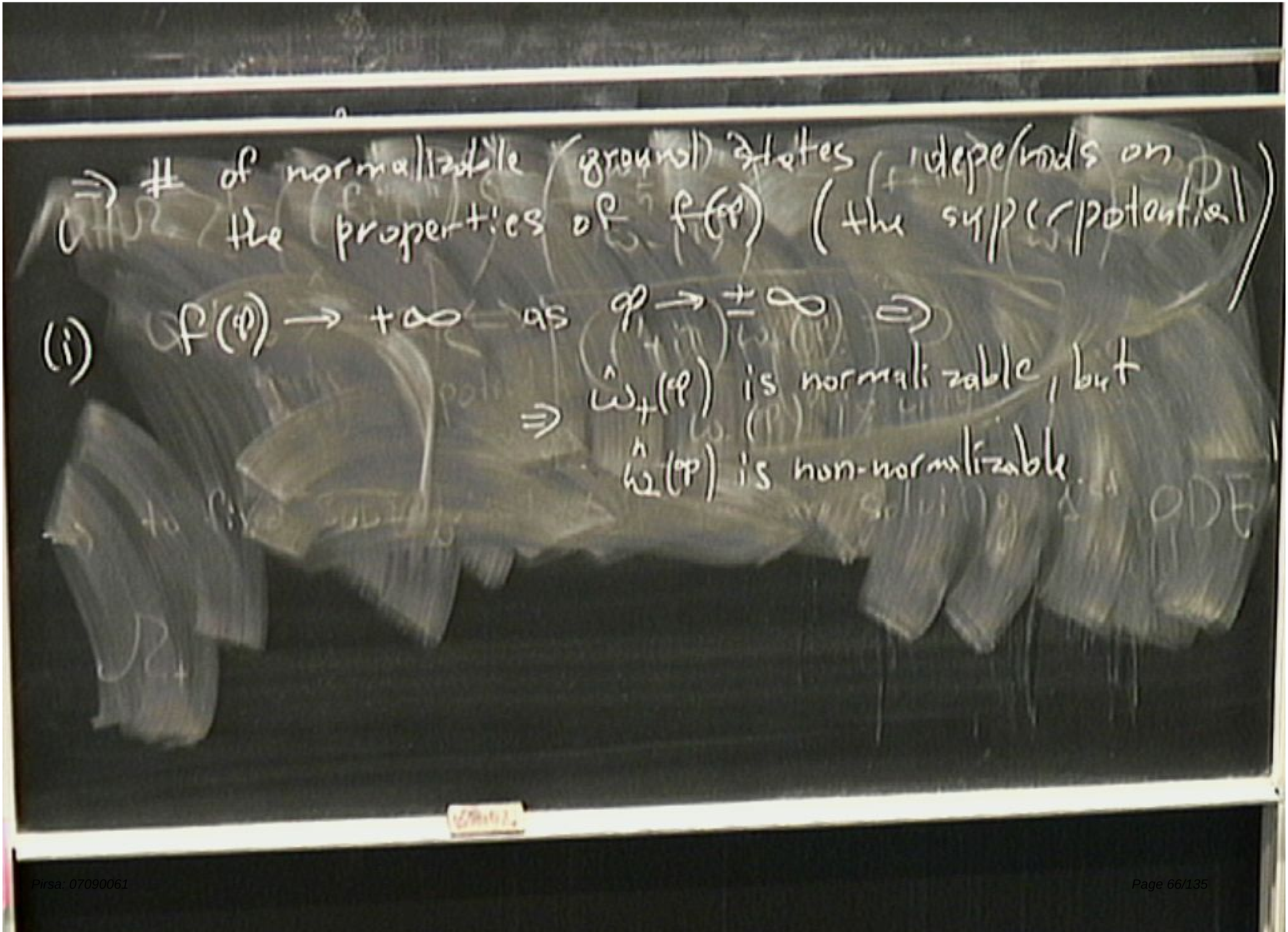


$\Rightarrow$ # of normalizable (ground) states depends on the properties of $f(\varphi)$ (the superpotential)



⇒ # of normalizable (ground) states depends on the properties of $f(\varphi)$ (the superpotential)

(i) $f(\varphi) \to +\infty$ as $\varphi \to \pm\infty$

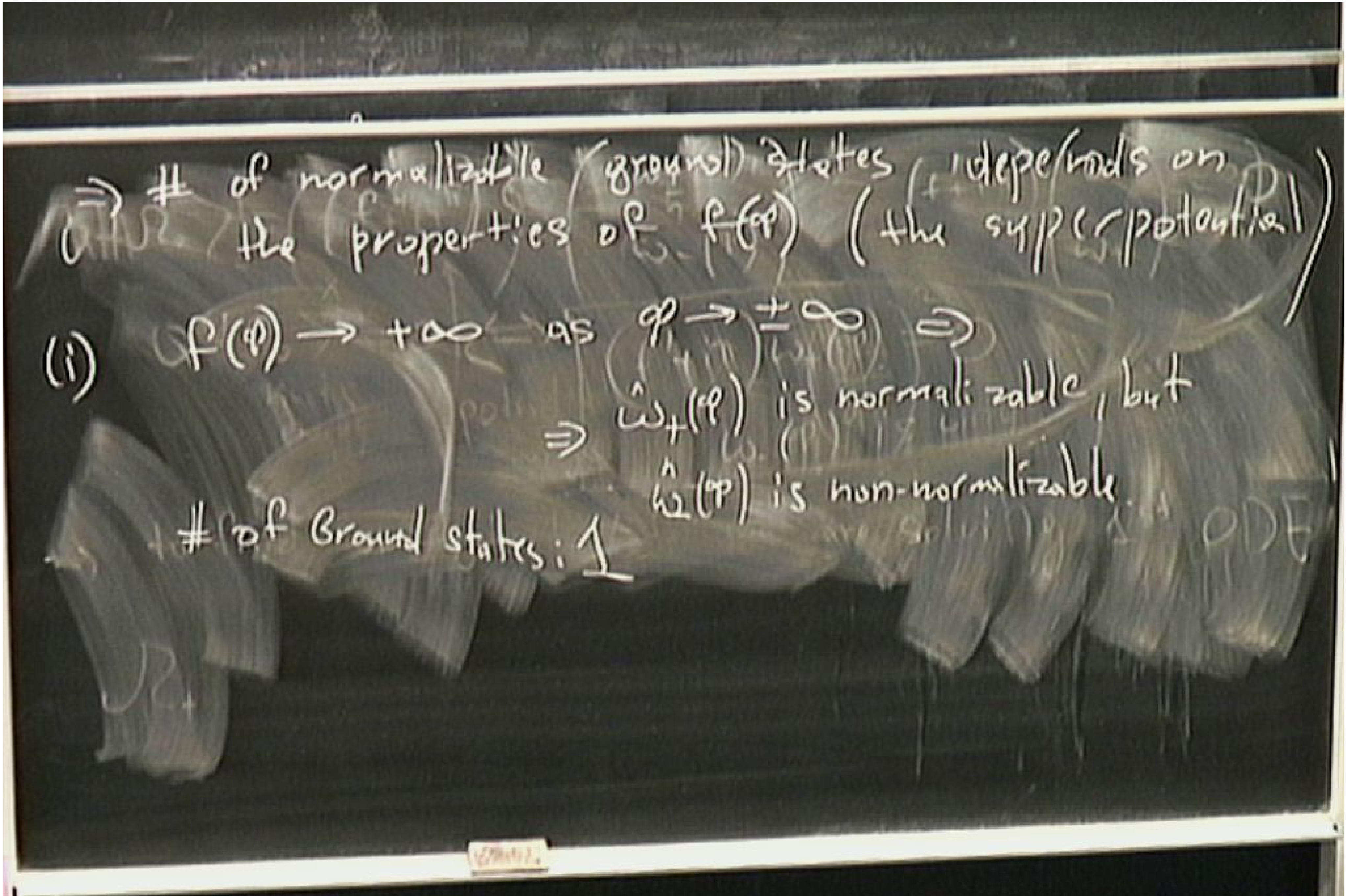


$\Rightarrow$ # of normalizable (ground) states (depends on the properties of $f(\varphi)$ (the superpotential))

(i) $f(\varphi) \to +\infty$ as $\varphi \to \pm\infty$ $\Rightarrow$

$\Rightarrow$ $\hat{\omega}_+(\varphi)$ is normalizable, but

$\hat{\omega}_-(\varphi)$ is non-normalizable



⇒ # of normalizable (ground) states depends on the properties of $f(\varphi)$ (the superpotential)

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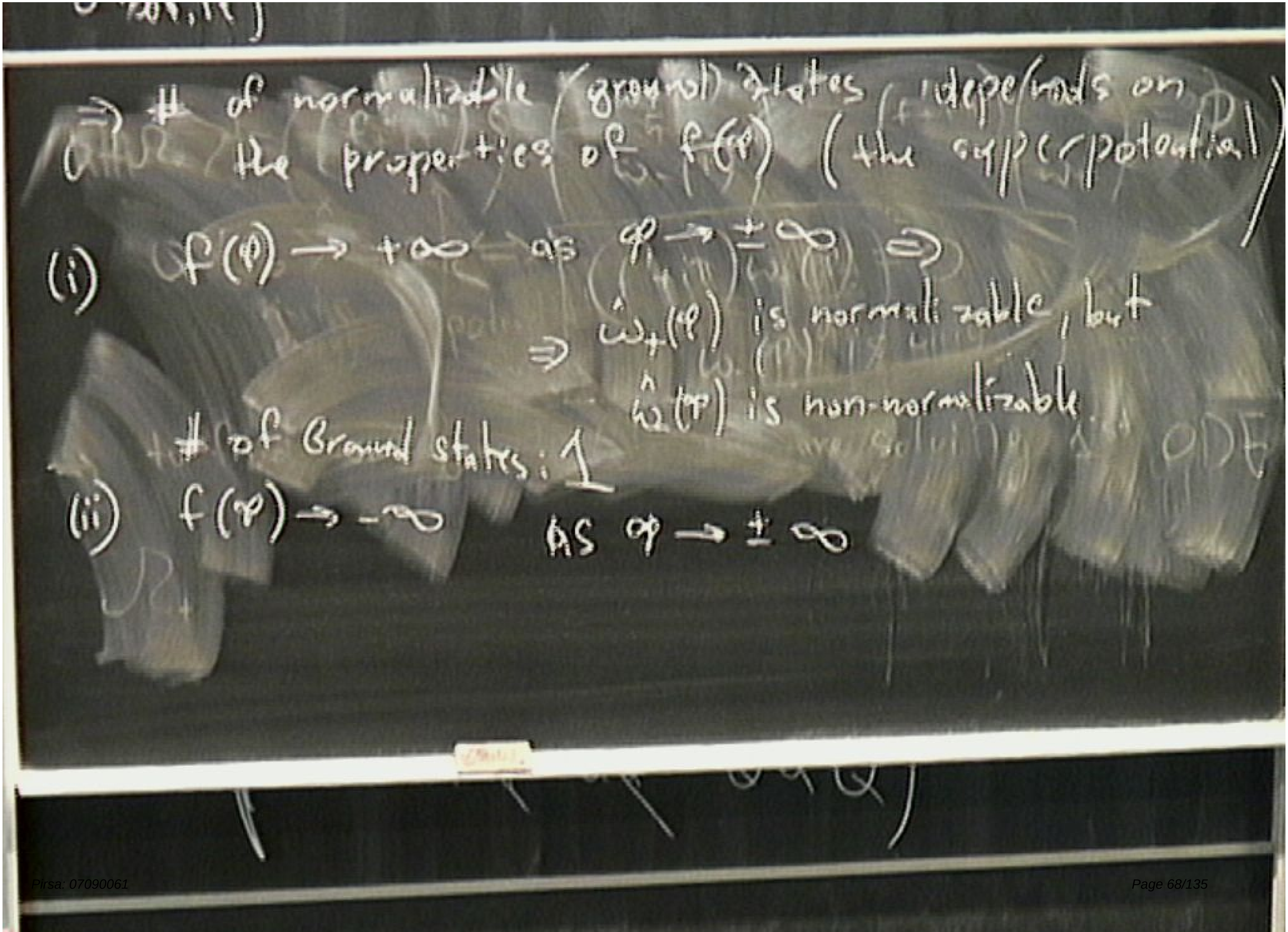
$\hat{\omega}_-(\varphi)$ is non-normalizable

# of Ground states: $\underline{1}$



$\Rightarrow$ # of normalizable (ground) states depends on the properties of $f(\varphi)$ (the superpotential)

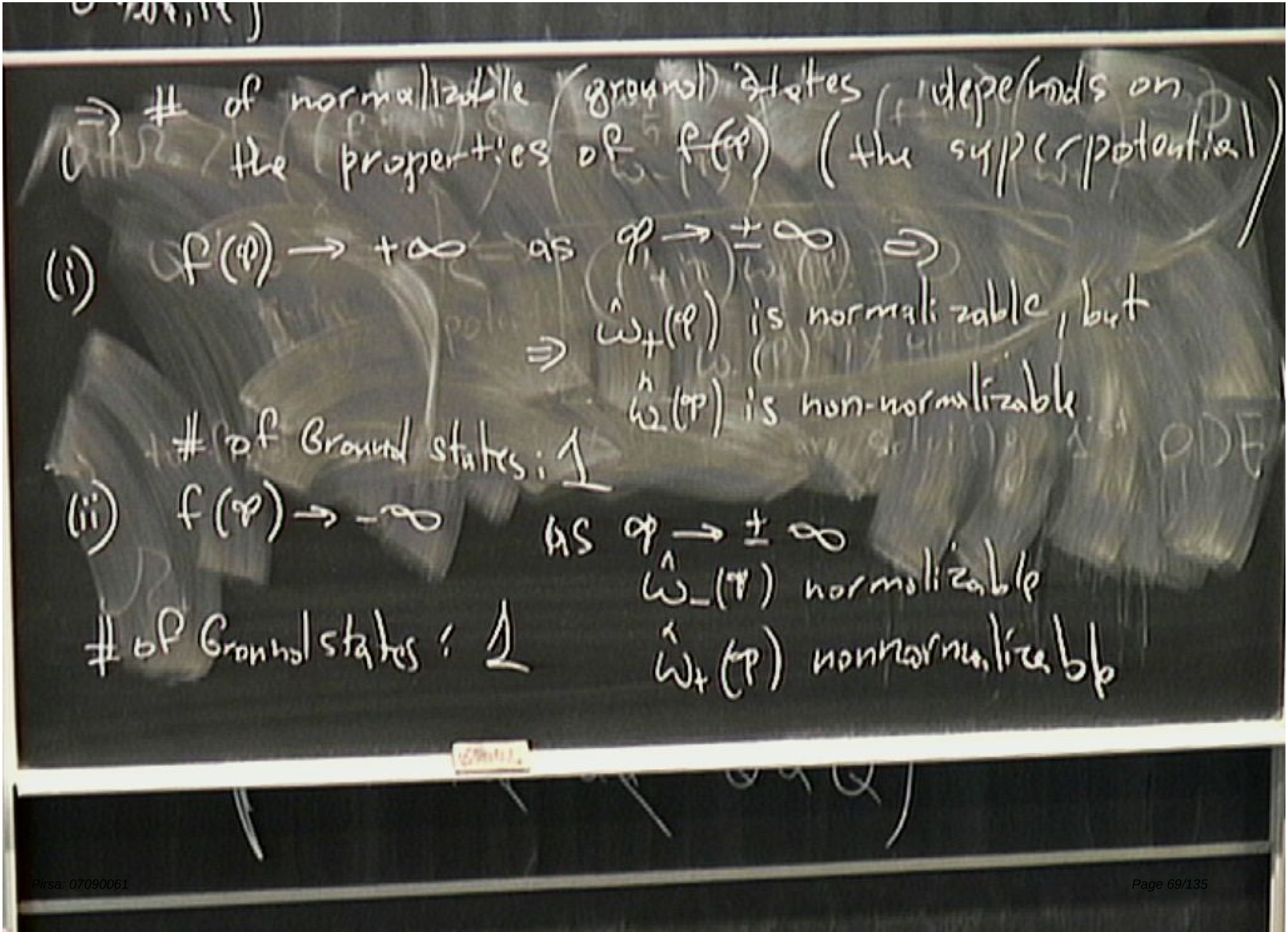
(i) $f(\varphi) \rightarrow +\infty$ as $\varphi \rightarrow \pm\infty$ $\Rightarrow$

$\Rightarrow$ $\hat{\omega}_+(\varphi)$ is normalizable, but

$\hat{\omega}_-(\varphi)$ is non-normalizable

# of Ground states: $\underline{1}$

(ii) $f(\varphi) \rightarrow -\infty$ as $\varphi \rightarrow \pm\infty$



⇒ # of normalizable (ground) states depends on the properties of $f(\varphi)$ (the superpotential)

(i) $f(\varphi) \to +\infty$ as $\varphi \to \pm\infty$ ⇒

⇒ $\hat{\omega}_+(\varphi)$ is normalizable, but

$\hat{\omega}_-(\varphi)$ is non-normalizable

# of Ground states: 1

(ii) $f(\varphi) \to -\infty$ as $\varphi \to \pm\infty$

$\hat{\omega}_-(\varphi)$ normalizable

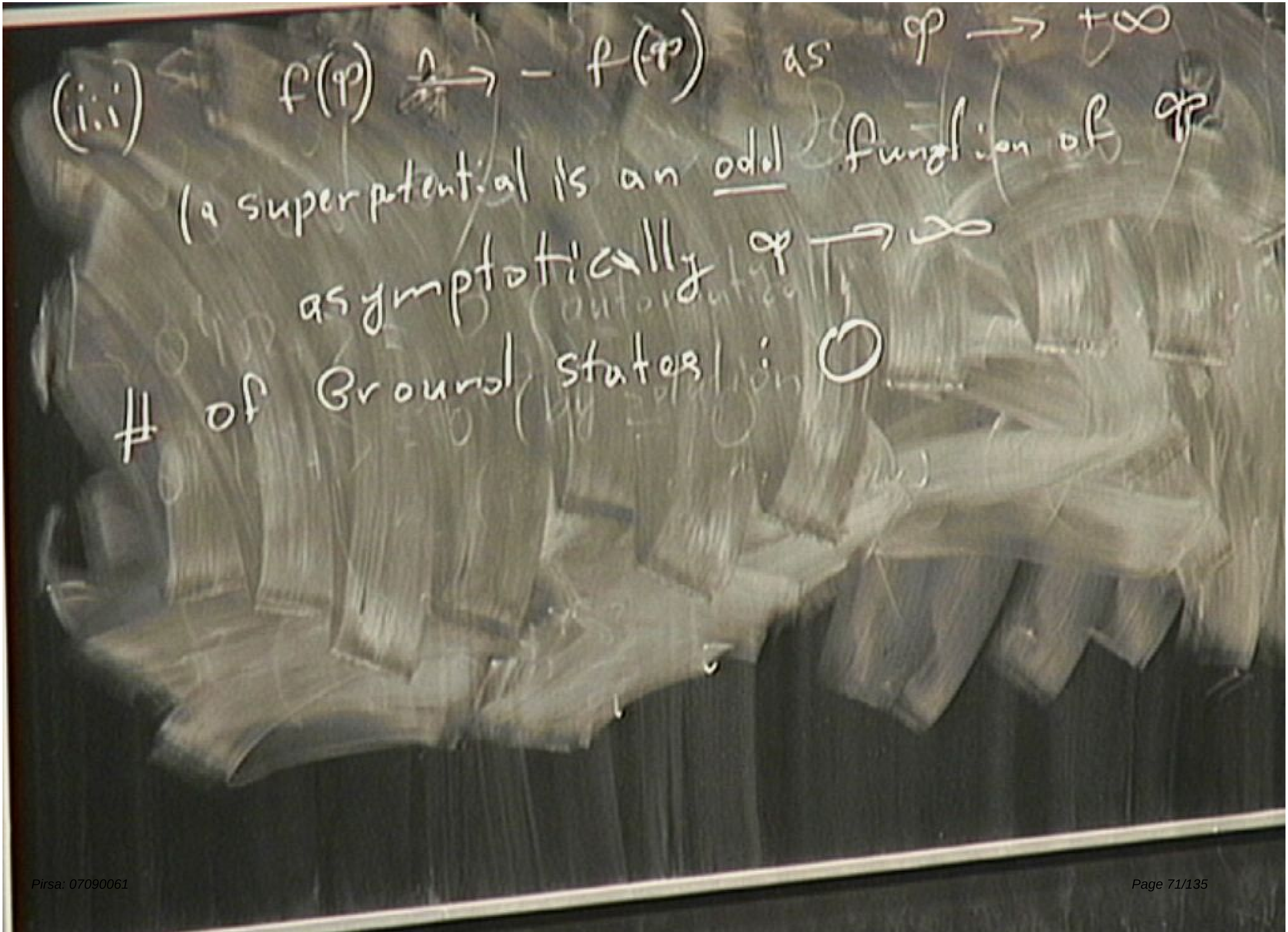
$\hat{\omega}_+(\varphi)$ nonnormalizable

# of Ground states: 1

(iii) $f(\varphi) \rightarrow -f(-\varphi)$ as $\varphi \rightarrow +\infty$

(a superpotential is an odd function of φ

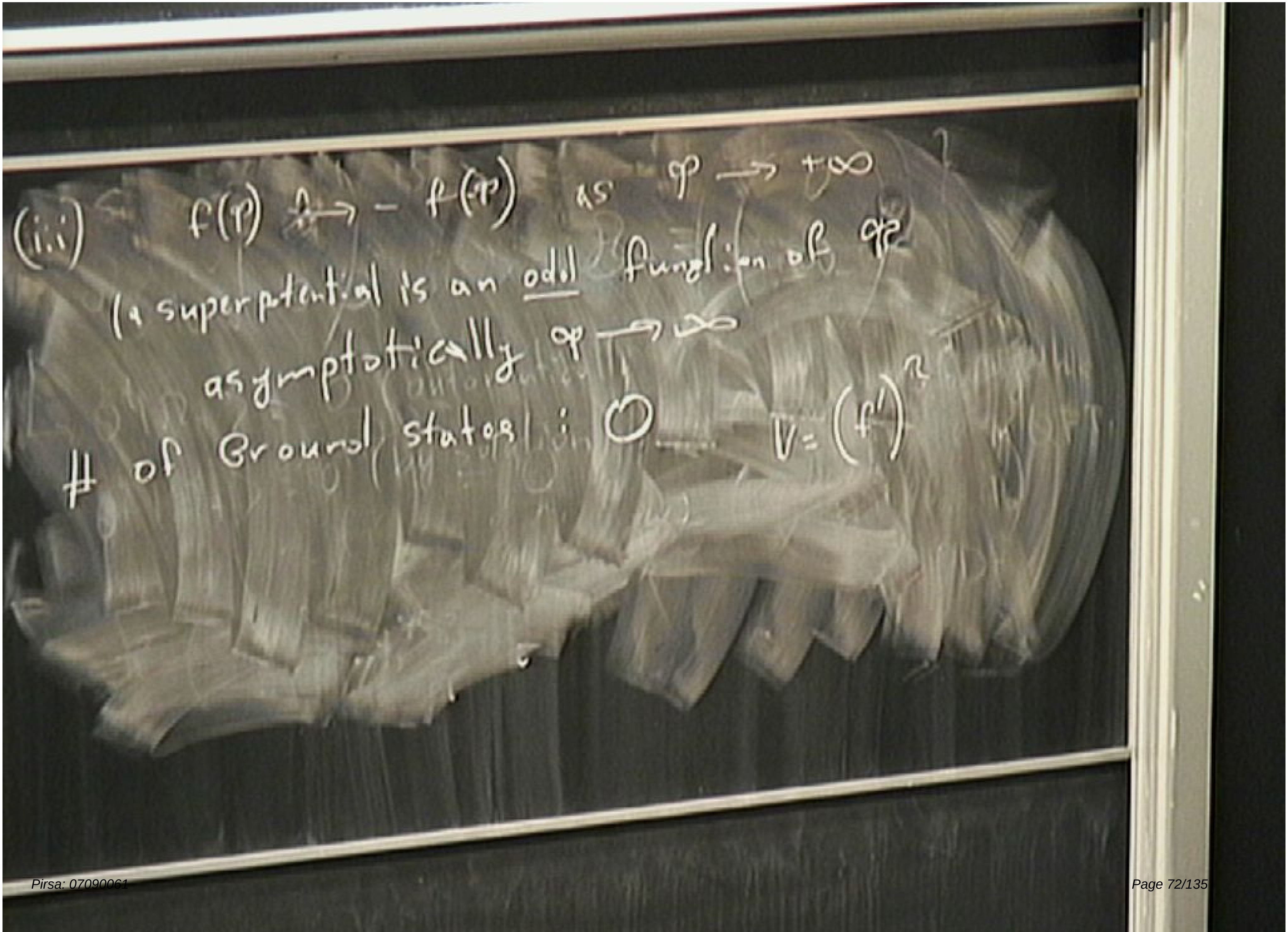
asymptotically $\varphi \rightarrow \infty$



$(\mathrm{i.i}) \Leftarrow \quad f(\varphi) \to -f(-\varphi)$ as $\varphi \to +\infty$

(a superpotential is an odd function of φ asymptotically $\varphi \to \infty$)

# of Ground states : 0

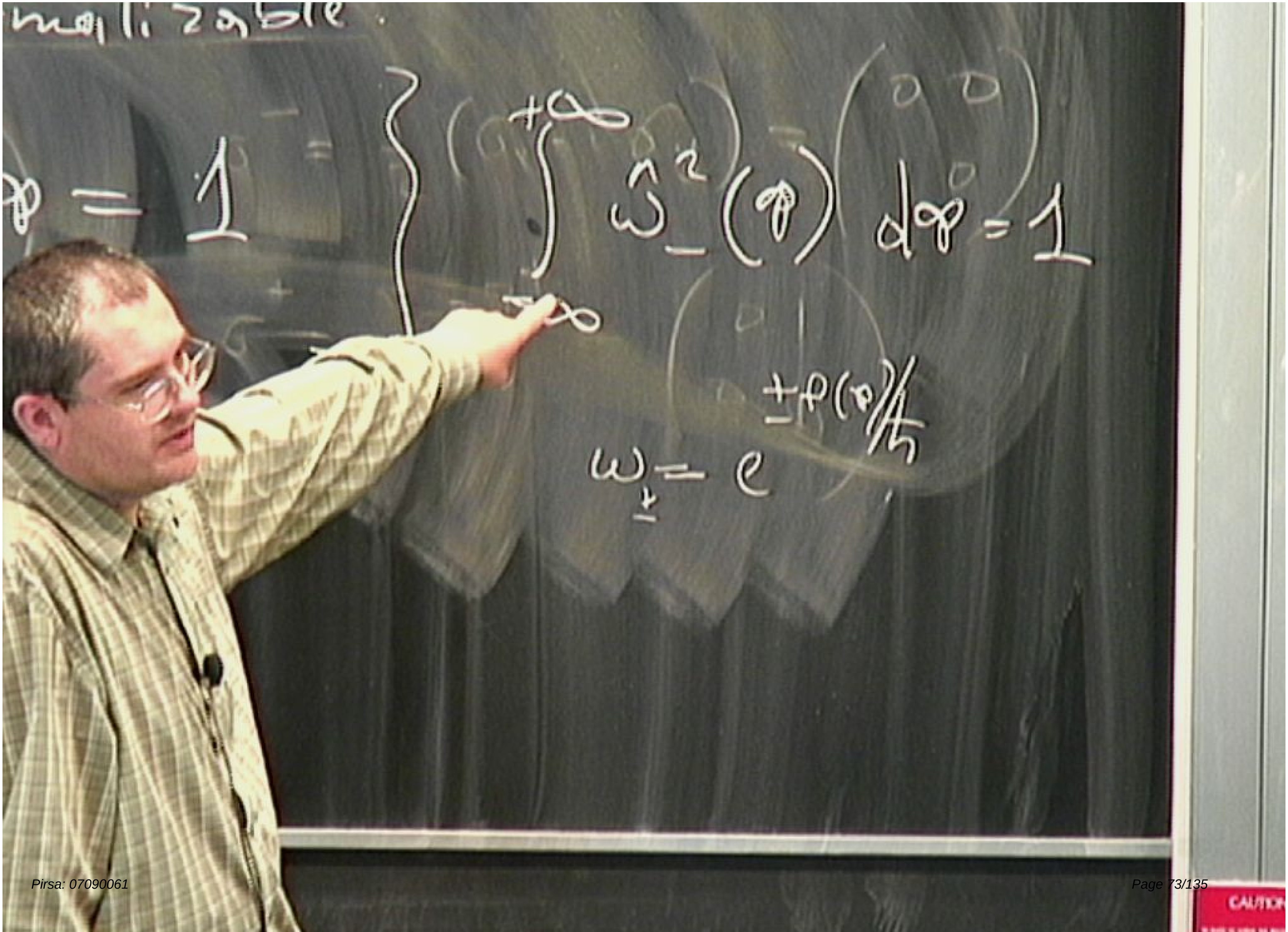


(ii) $\Leftarrow$ $f(\varphi) \to -f(-\varphi)$ as $\varphi \to \pm\infty$

(a superpotential is an $\underline{\text{odd}}$ function of φ

asymptotically $\varphi \to \infty$

$\#$ of Ground states : 0 $\qquad V = (f')^2$

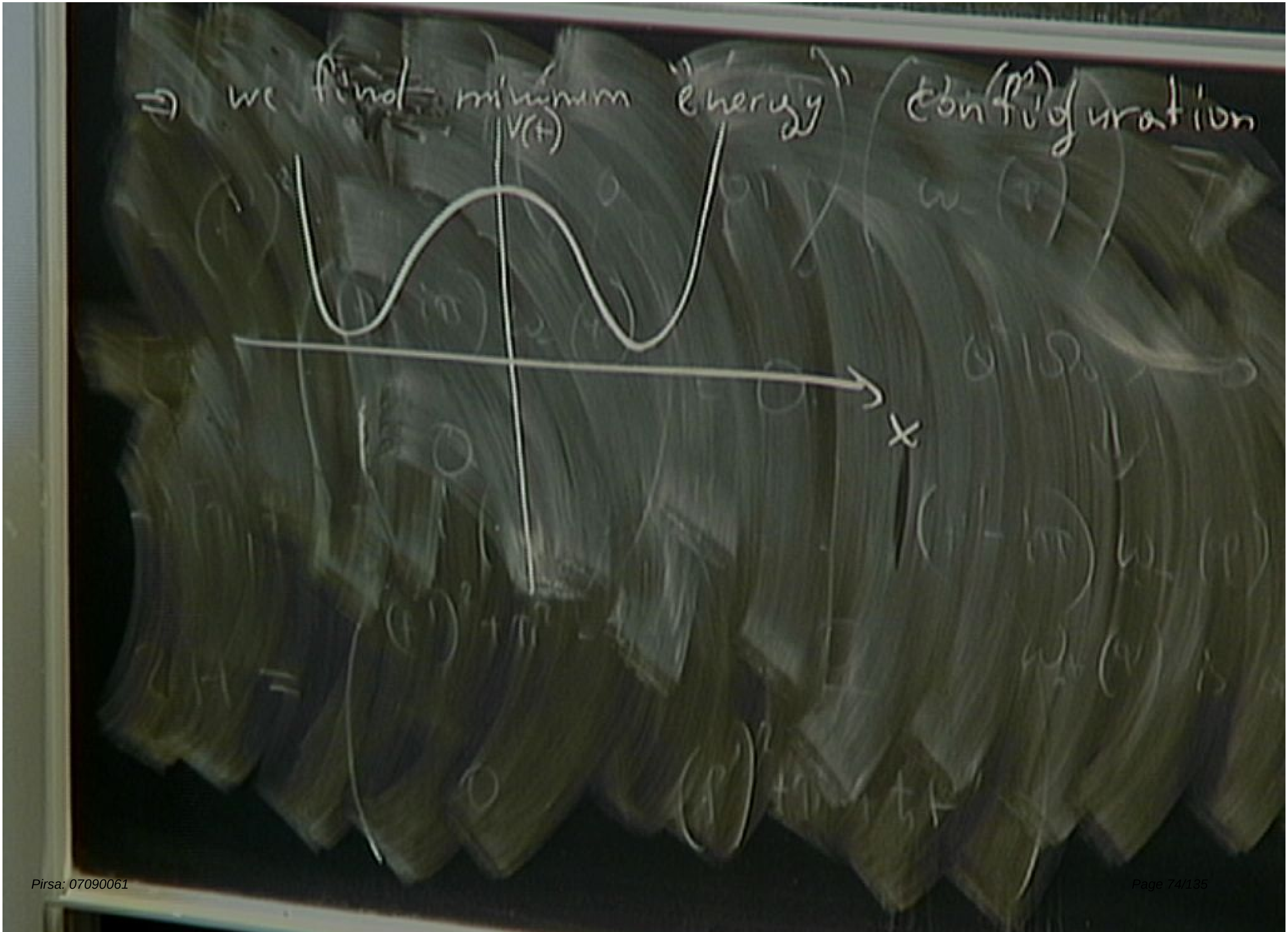


rmalizable.

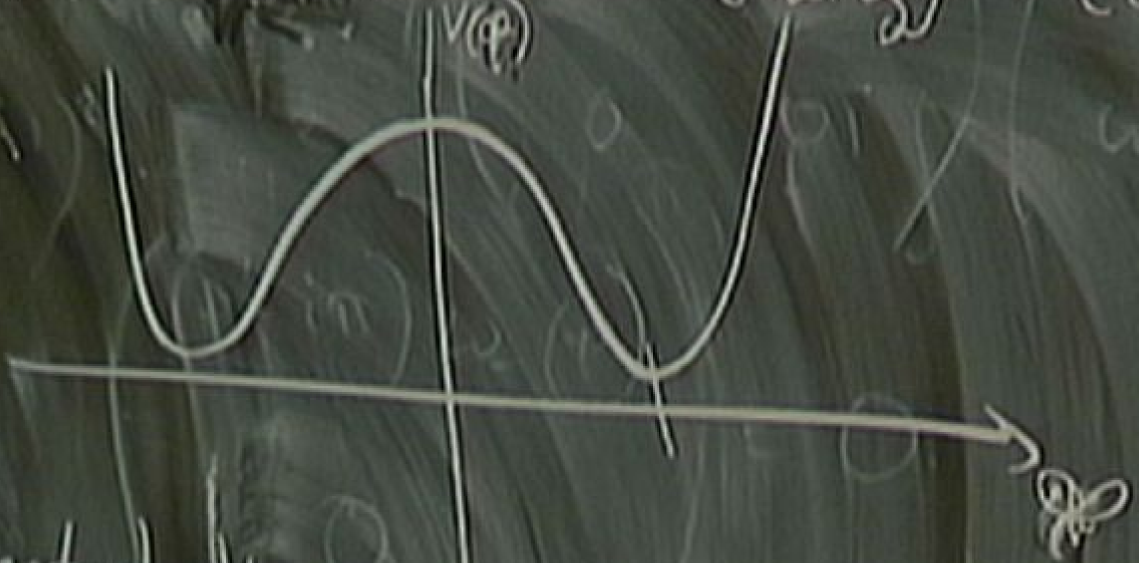
$$\varphi = 1 \quad \Big\} \quad \int_{-\infty}^{+\infty} \hat{\omega}_-^2(\varphi)\, d\varphi = 1$$

$$\omega_\pm = e^{\pm \varphi(\varphi)/\hbar}$$

⇒ we find minimum "energy" configuration

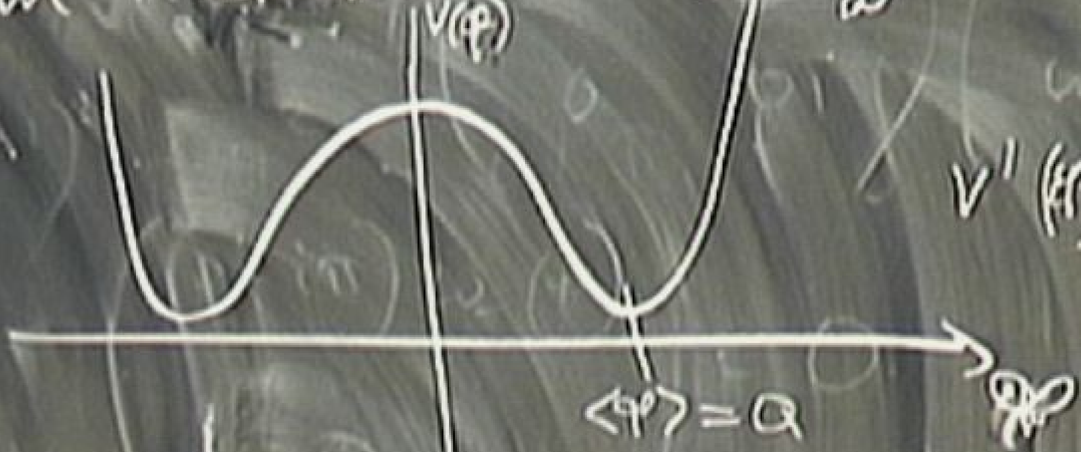


⇒ we find minimum "energy" configuration



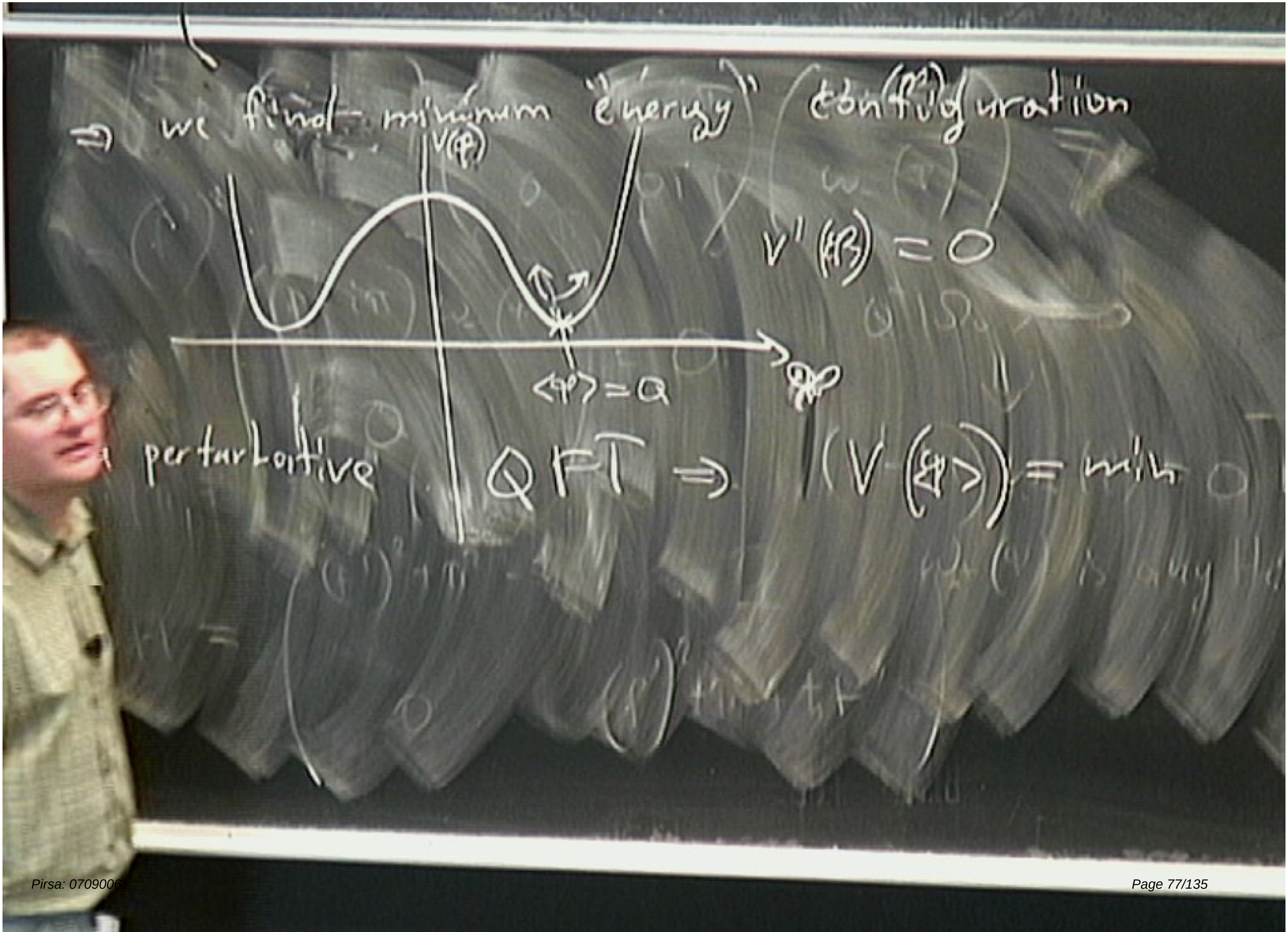
a perturbative QFT ⇒

⇒ we find minimum "energy" configuration



$V'(\varphi) = 0$

a perturbative QFT ⇒ $V(\langle\varphi\rangle) = \min$

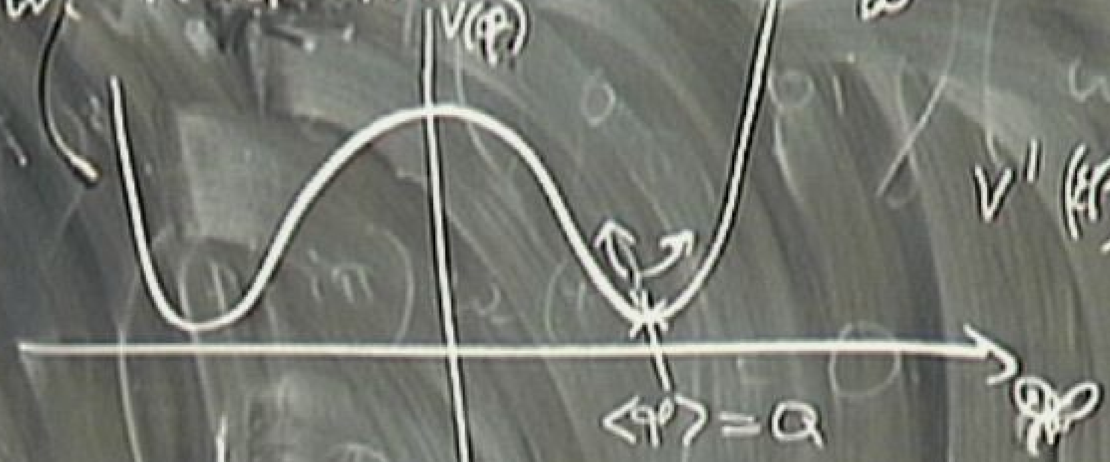


⇒ we find minimum "energy" configuration

$V'(\varphi) = 0$

⇒ perturbative QFT ⇒ $V(\langle\varphi\rangle) = \min$

$\Rightarrow$ we find minimum "energy" configuration

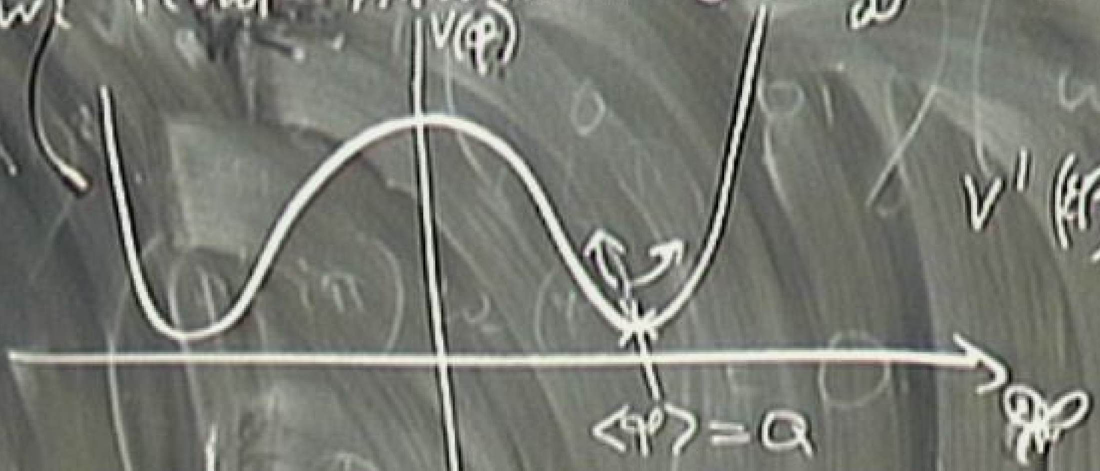


$$V'(\varphi) = 0$$

a perturbative QFT $\Rightarrow$ $V(\langle\varphi\rangle) = \min$

$\Rightarrow$ we study perturbatively small fluctuations.

⇒ we find minimum "energy" configuration

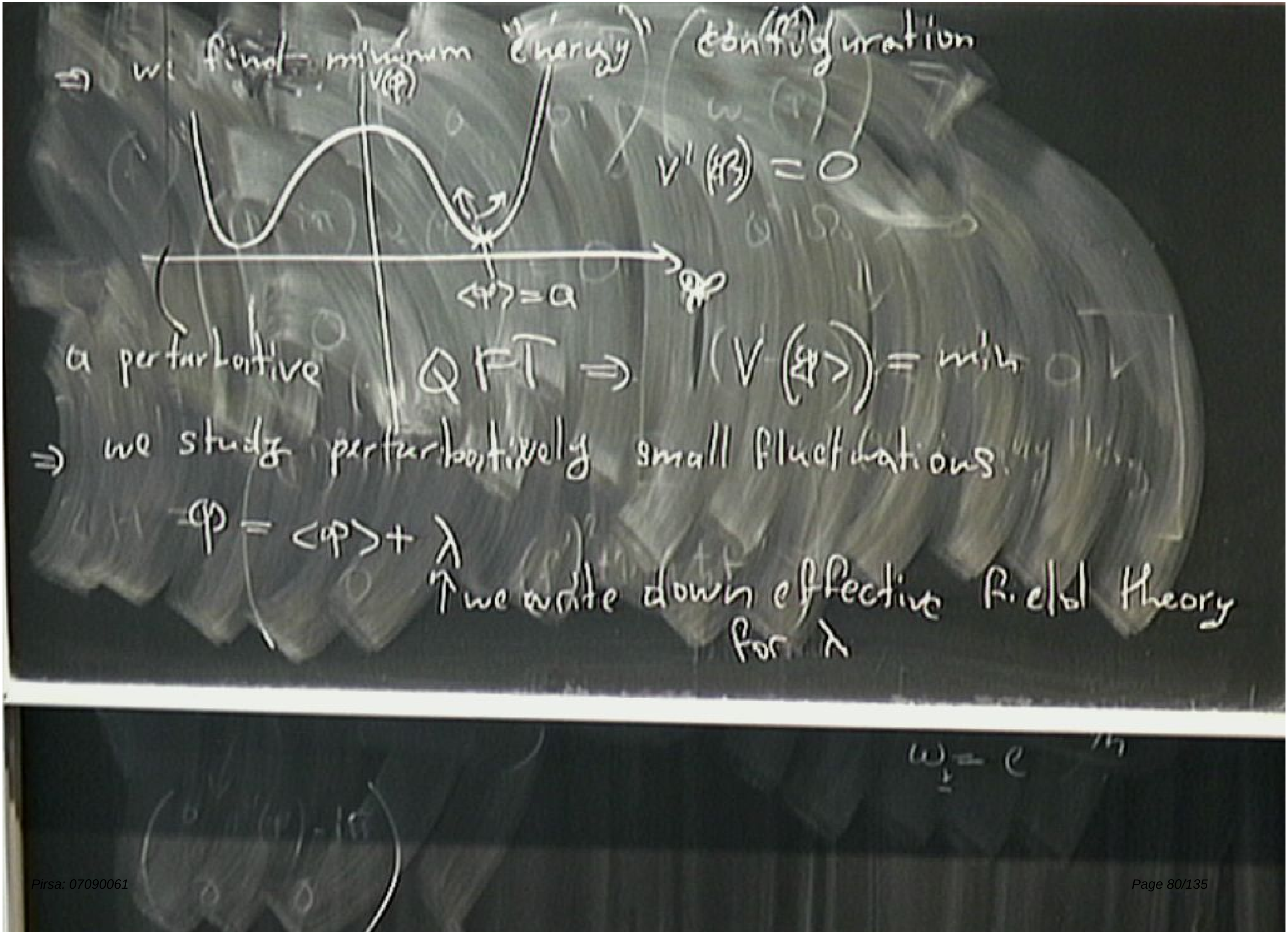


$V'(\varphi) = 0$

a perturbative QFT ⇒ $V(\langle\varphi\rangle) = \min$

⇒ we study perturbatively small fluctuations

$$\varphi = \langle\varphi\rangle + \lambda$$



⇒ we find minimum "energy" configuration

$V'(\varphi) = 0$

a perturbative QFT ⇒ $V(\langle\varphi\rangle) = \min$

⇒ we study perturbatively small fluctuations

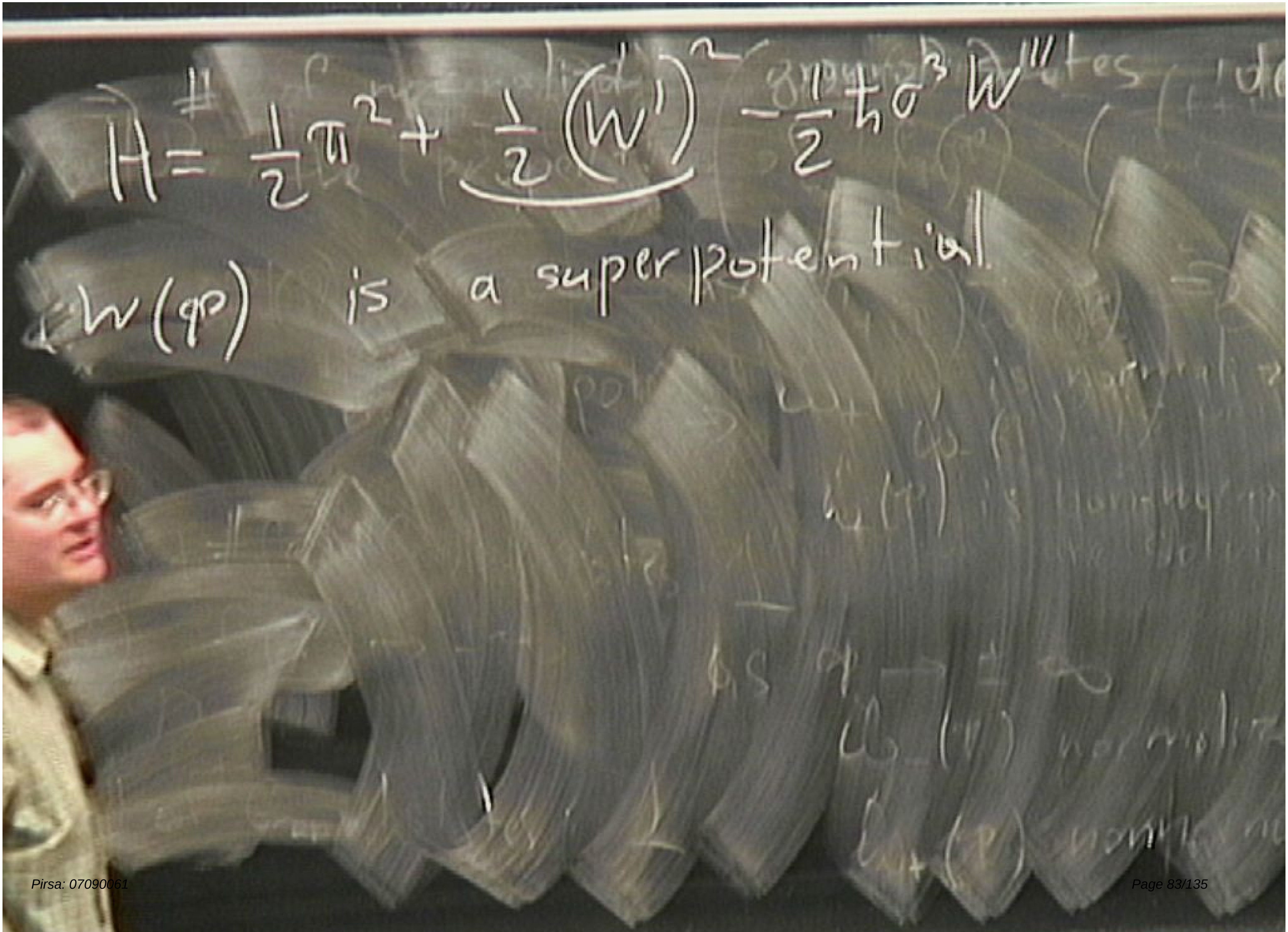
$\varphi = \langle\varphi\rangle + \lambda$

↑ we write down effective field theory for λ

$$H = \frac{1}{2}\pi^2 + \frac{1}{2}(W')^2 - \frac{1}{2}\hbar\sigma^3 W''$$

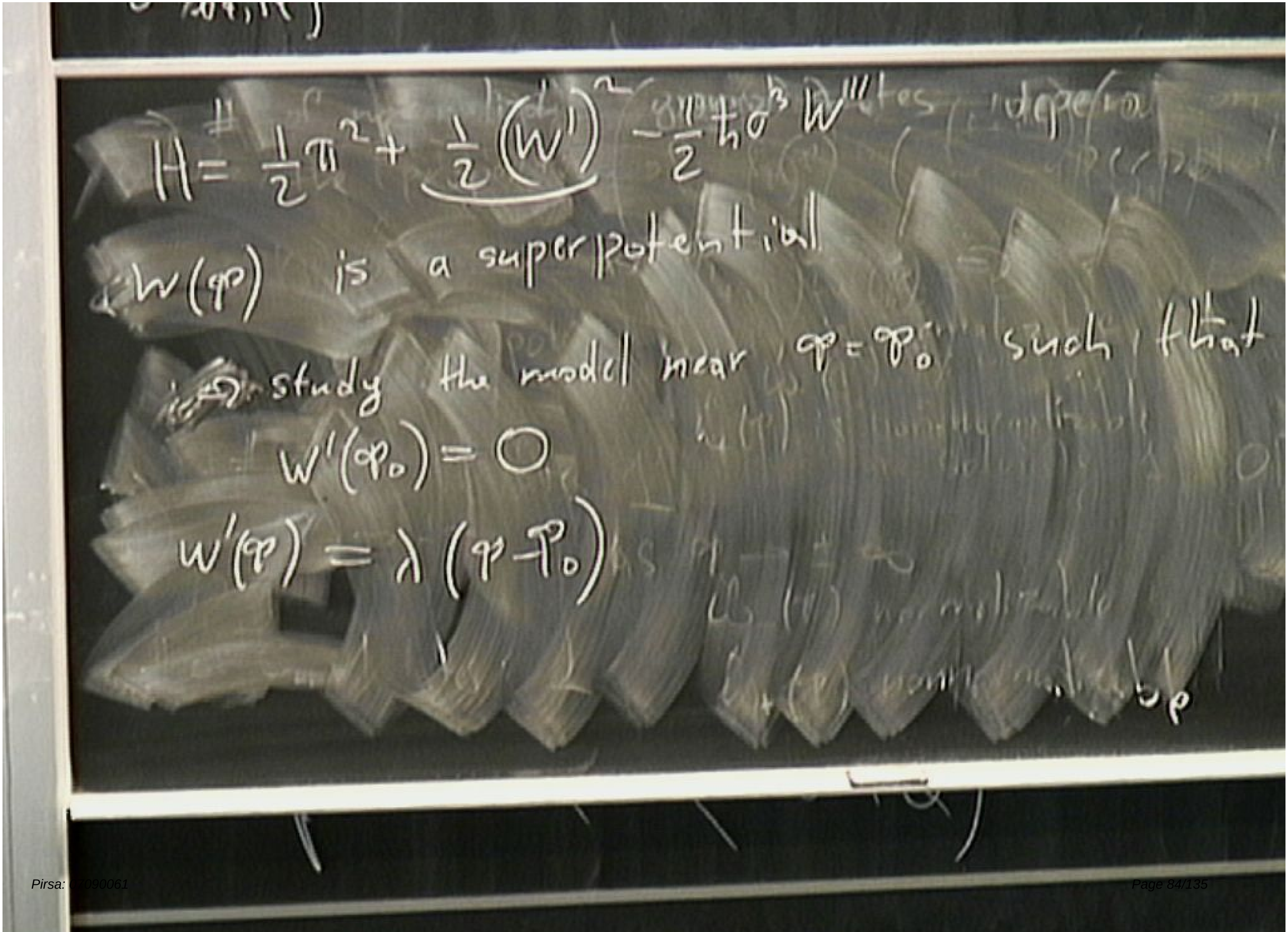
$$\overset{\#}{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(W')^2 - \frac{1}{2}\hbar\sigma^3 W''$$

$W(\varphi)$ is a superpotential



$$H = \frac{1}{2}\pi^2 + \underline{\frac{1}{2}(W')^2} - \frac{1}{2}\hbar\sigma^3 W''$$

$W(\varphi)$ is a superpotential



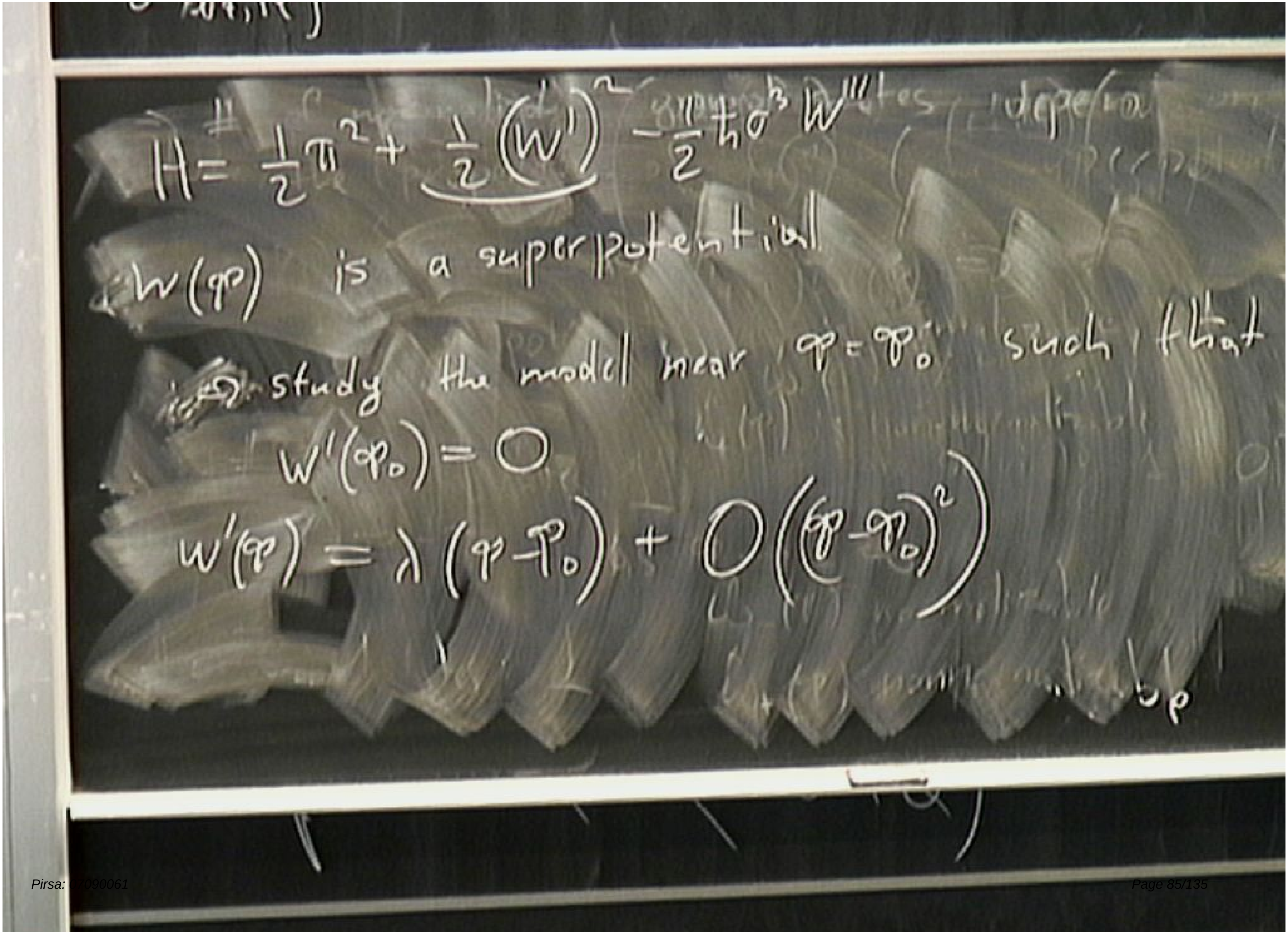
$$H = \frac{1}{2}\pi^2 + \underline{\frac{1}{2}(W')^2} - \frac{1}{2}\hbar\sigma^3 W''$$

$W(\varphi)$ is a superpotential

study the model near $\varphi = \varphi_0$ such that

$$W'(\varphi_0) = 0$$

$$W'(\varphi) = \lambda(\varphi - \varphi_0)$$



$$H = \frac{1}{2}\pi^2 + \underline{\frac{1}{2}\left(W'\right)^2} - \frac{1}{2}\hbar\sigma^3 W''$$

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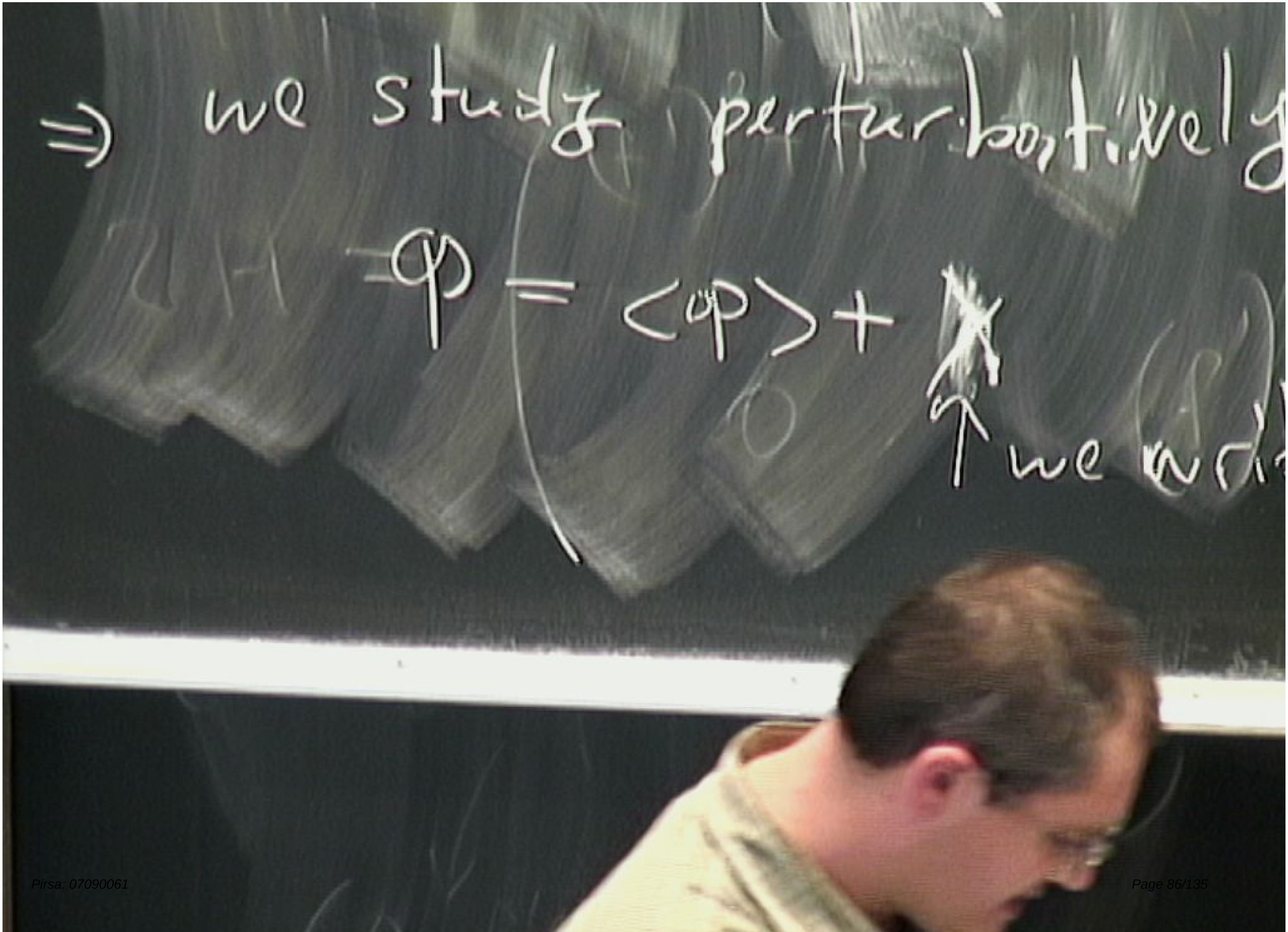
study the model near $\varphi = \varphi_0$ such that

$$W'(\varphi_0) = 0$$

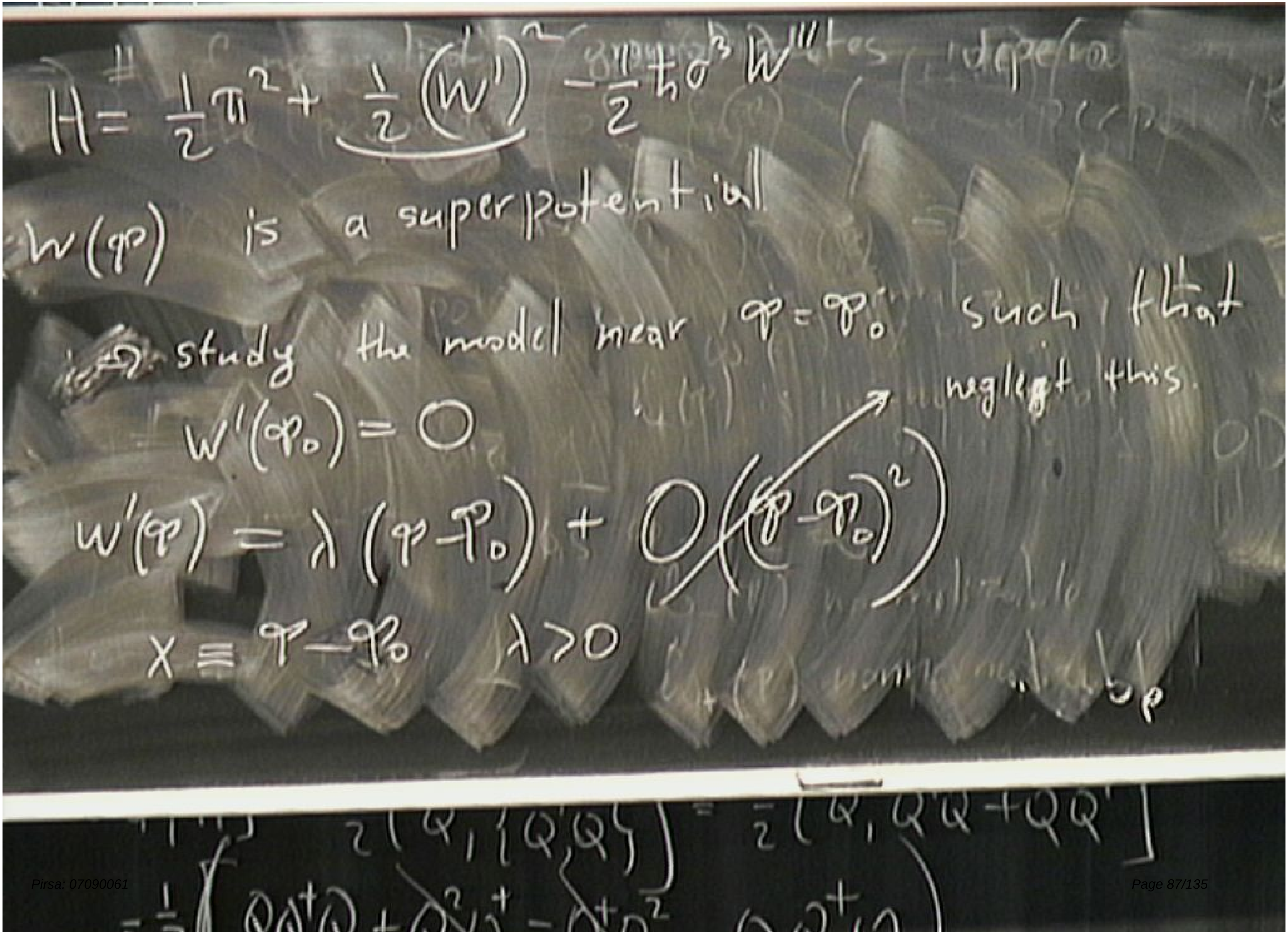
$$W'(\varphi) = \lambda\left(\varphi - \varphi_0\right) + O\left(\left(\varphi-\varphi_0\right)^2\right)$$



⇒ we study perturbatively

$\varphi = \langle \varphi \rangle +$

$\uparrow$ we wri



$$H = \frac{1}{2}\pi^2 + \underline{\frac{1}{2}(W')^2} - \frac{1}{2}\hbar\sigma^3 W''$$

$W(\varphi)$ is a superpotential

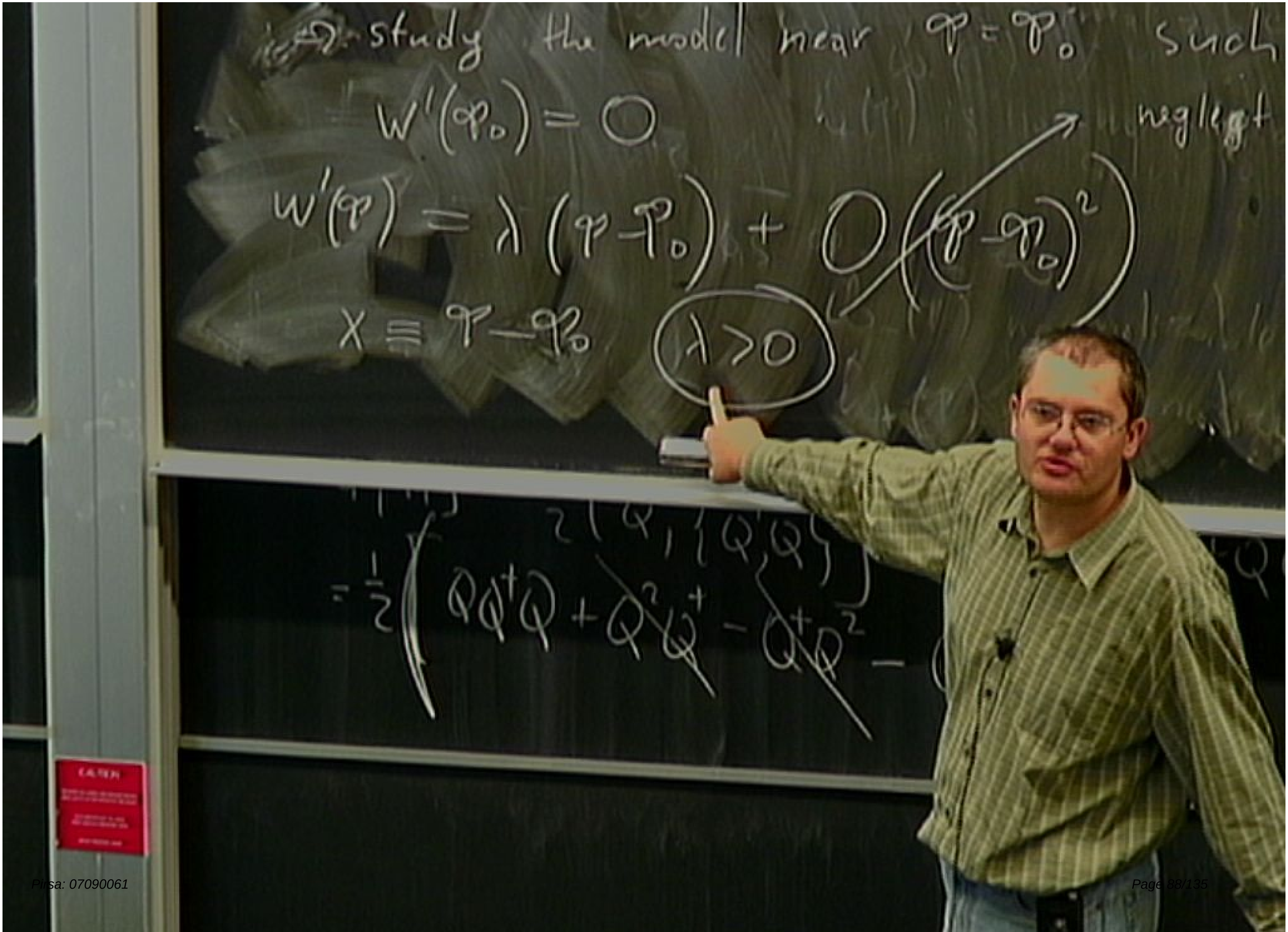
study the model near $\varphi = \varphi_0$ such that

$$W'(\varphi_0) = 0$$

$$W'(\varphi) = \lambda(\varphi - \varphi_0) + \cancel{O\left((\varphi-\varphi_0)^2\right)} \rightarrow \text{neglect this.}$$

$$X \equiv \varphi - \varphi_0 \qquad \lambda > 0$$

$$\frac{1}{2}(Q,\{Q,Q\}) = \frac{1}{2}[Q, Q'Q + QQ']$$



study the model near $\varphi = \varphi_0$ such

$W'(\varphi_0) = 0$

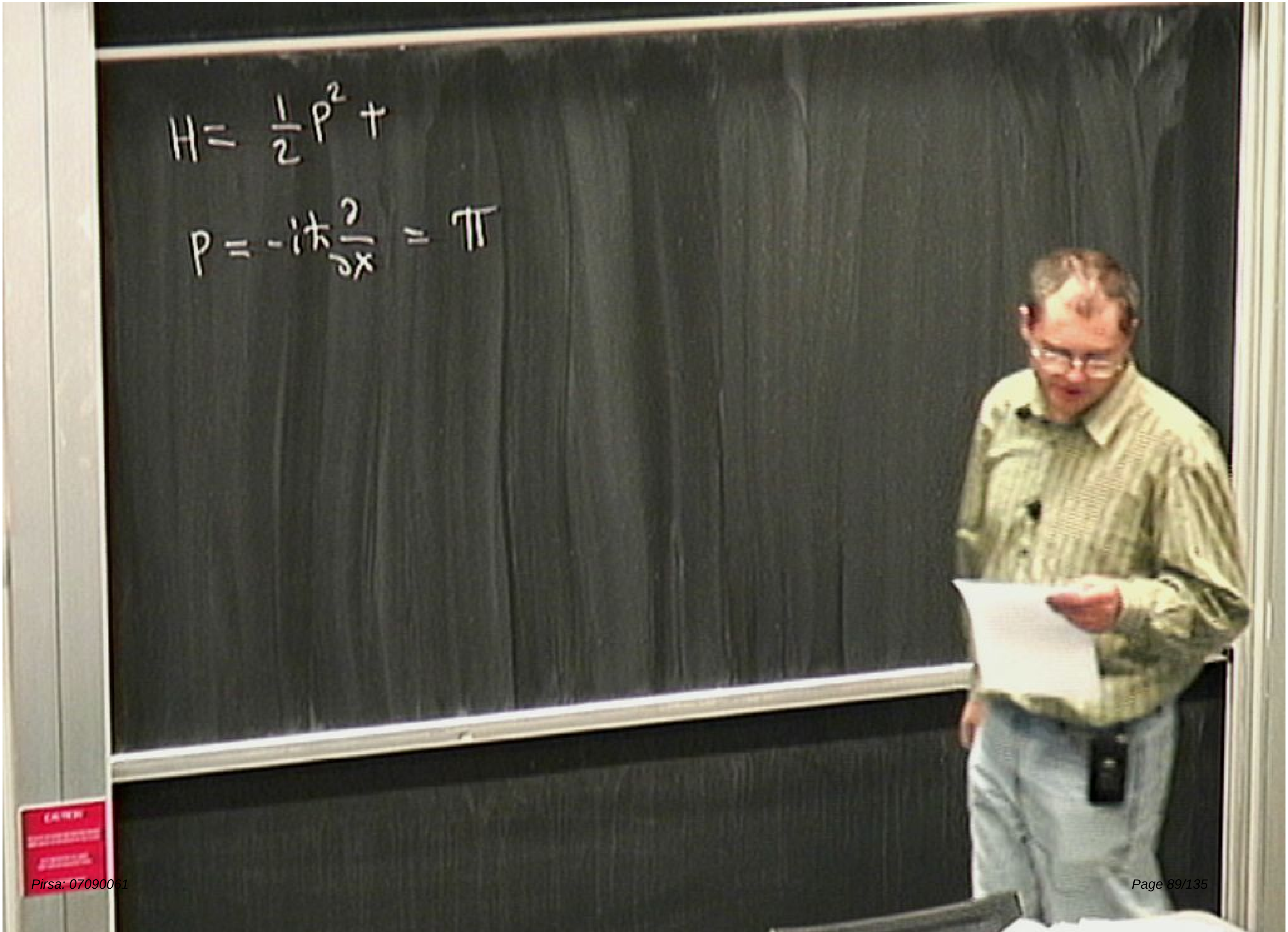
neglect

$W'(\varphi) = \lambda(\varphi - \varphi_0) + O\left(\cancel{(\varphi-\varphi_0)^2}\right)$

$X \equiv \varphi - \varphi_0$ $\boxed{\lambda > 0}$

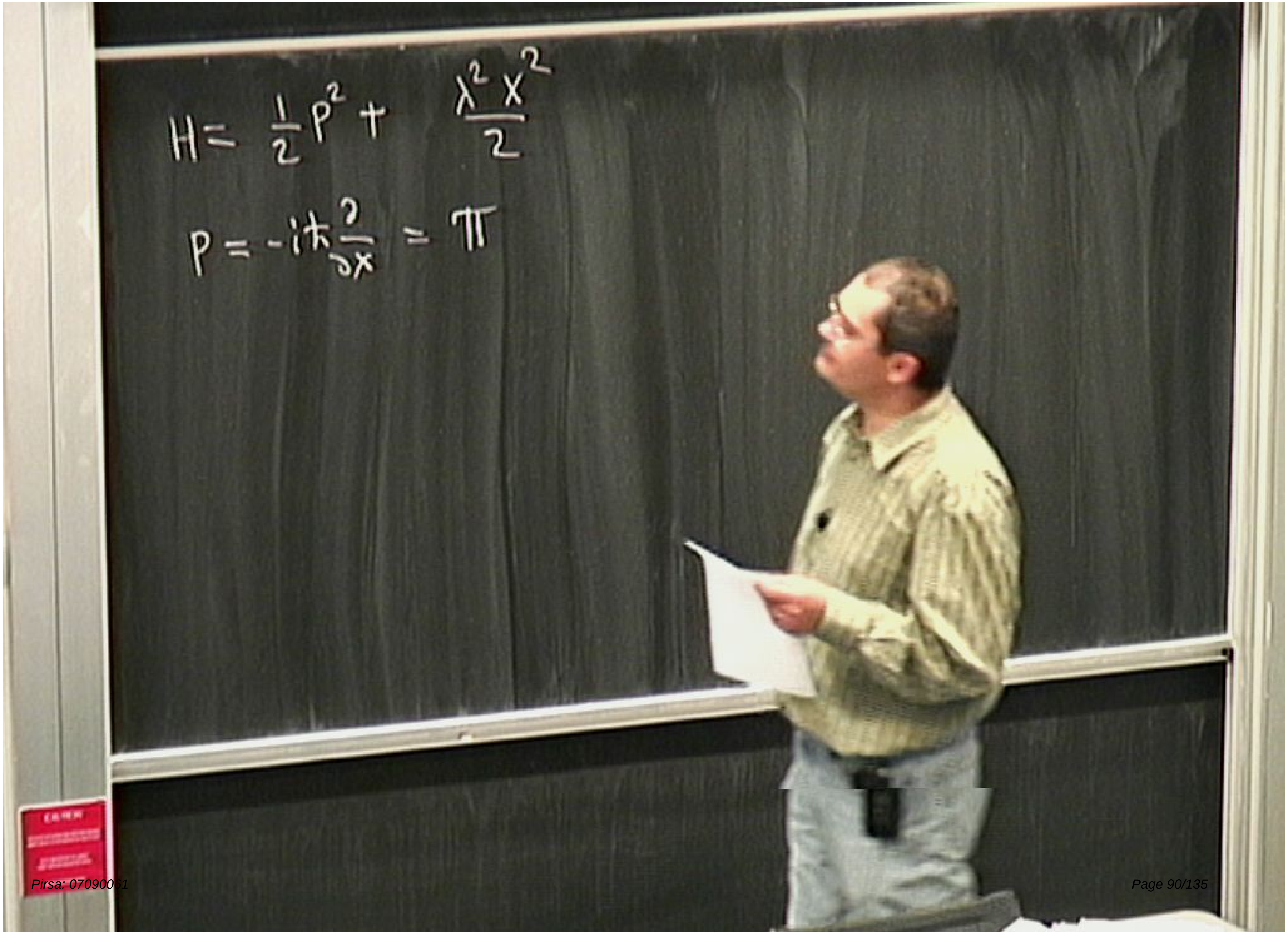
$2\{Q,\{Q,Q\}\}$

$= \frac{1}{2}\left(QQ^\dagger Q + \cancel{Q^2 Q^\dagger} - \cancel{Q^\dagger Q^2} - \right.$



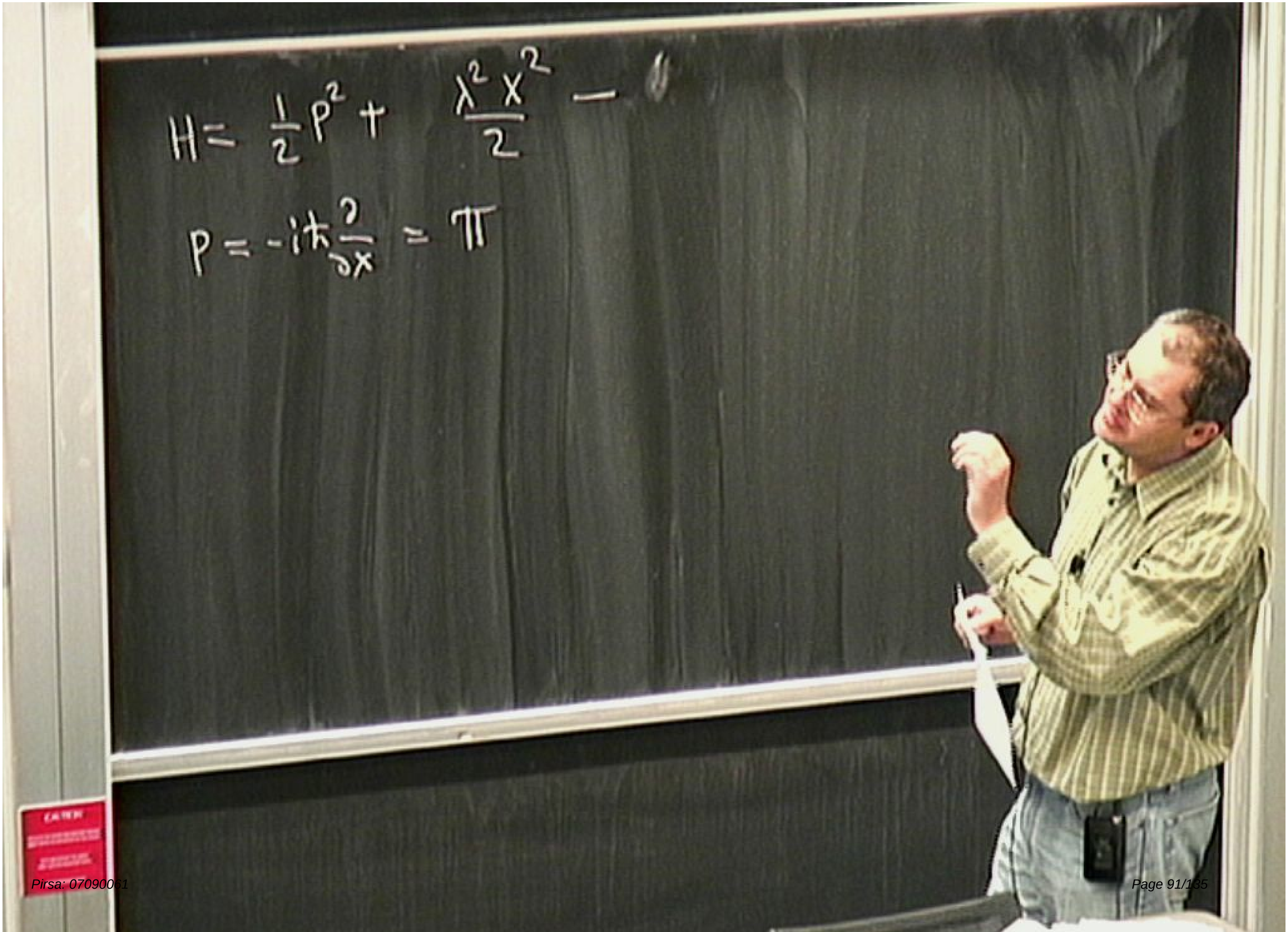
$$H = \frac{1}{2}p^2 +$$

$$p = -i\hbar\frac{\partial}{\partial x} = \pi$$



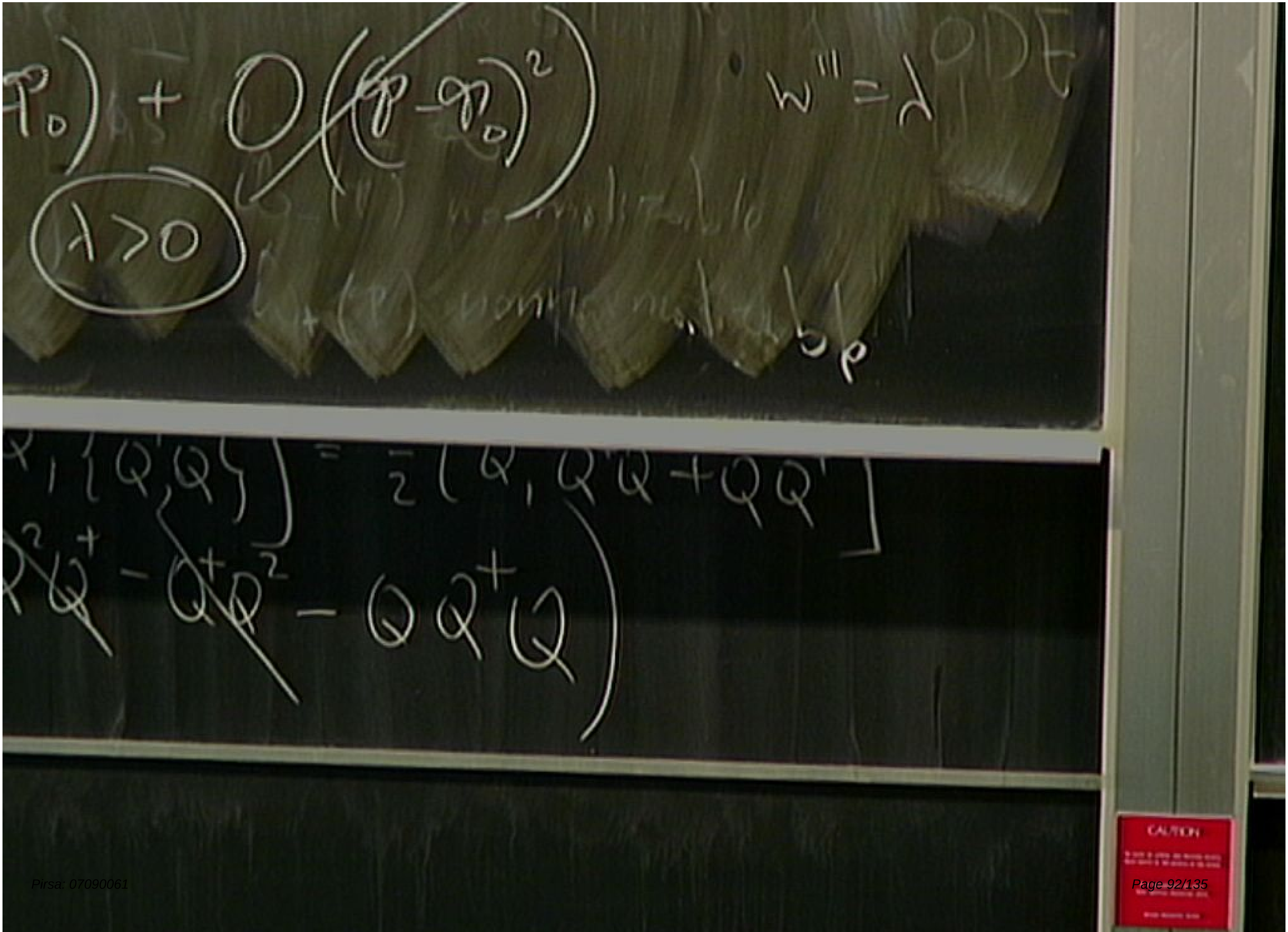
$$H = \frac{1}{2}P^2 + \frac{\lambda^2 X^2}{2}$$

$$P = -i\hbar\frac{\partial}{\partial x} = \Pi$$



$$H = \frac{1}{2}p^2 + \frac{\lambda^2 x^2}{2} -$$

$$p = -i\hbar\frac{\partial}{\partial x} = \pi$$



$$\varphi_0) + O((\varphi - \varphi_0)^2)$$

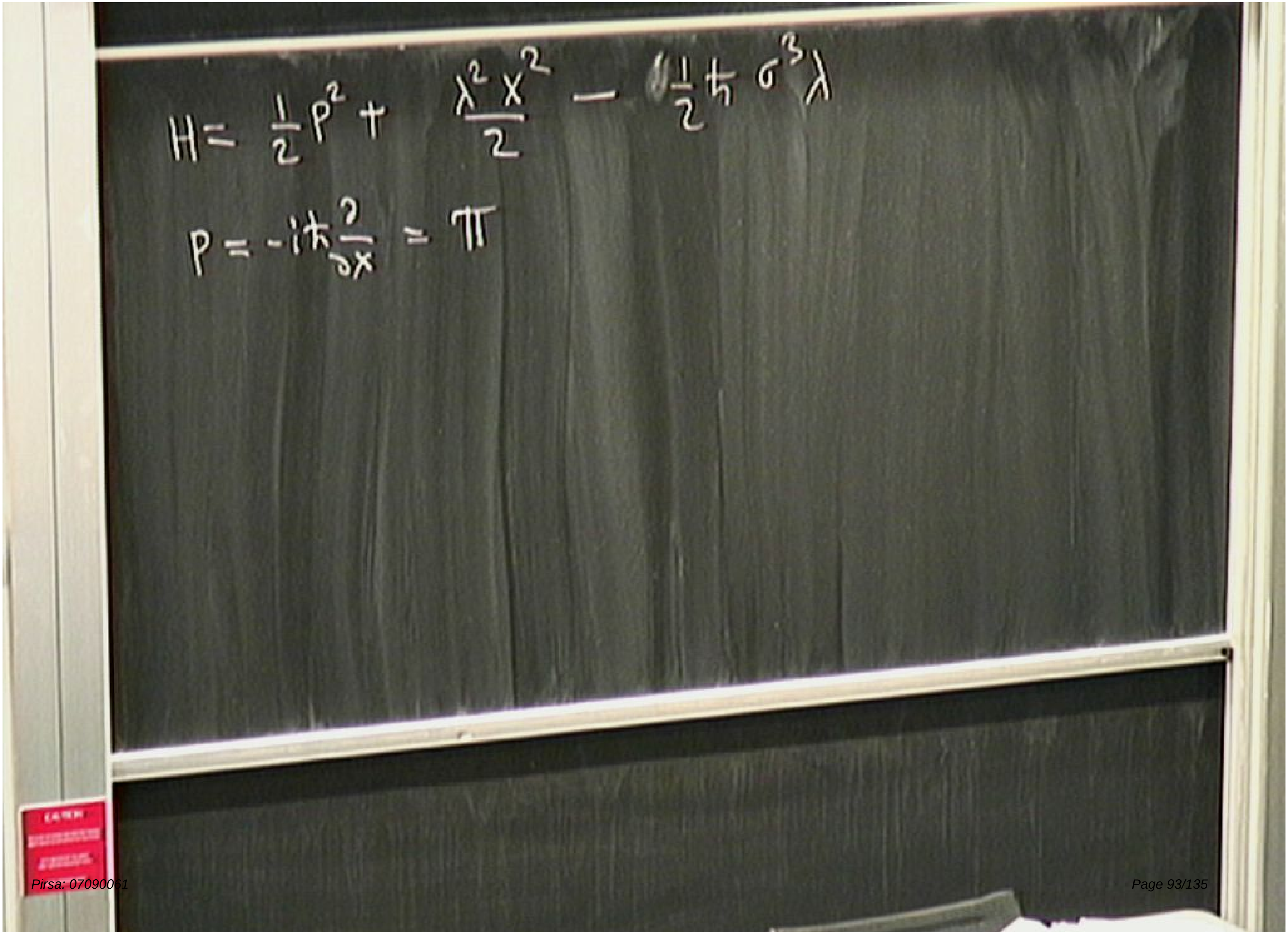
$$w'' = \lambda \quad \text{ODE}$$

$$\boxed{\lambda > 0}$$

$$b_p$$

$$Q, \{Q', Q\}) = \frac{-}{2}[Q, Q'Q + QQ']$$

$$Q^2 + \cancel{Q^{+}Q} - \cancel{Q^{+}Q^2} - QQ^{+}Q)$$



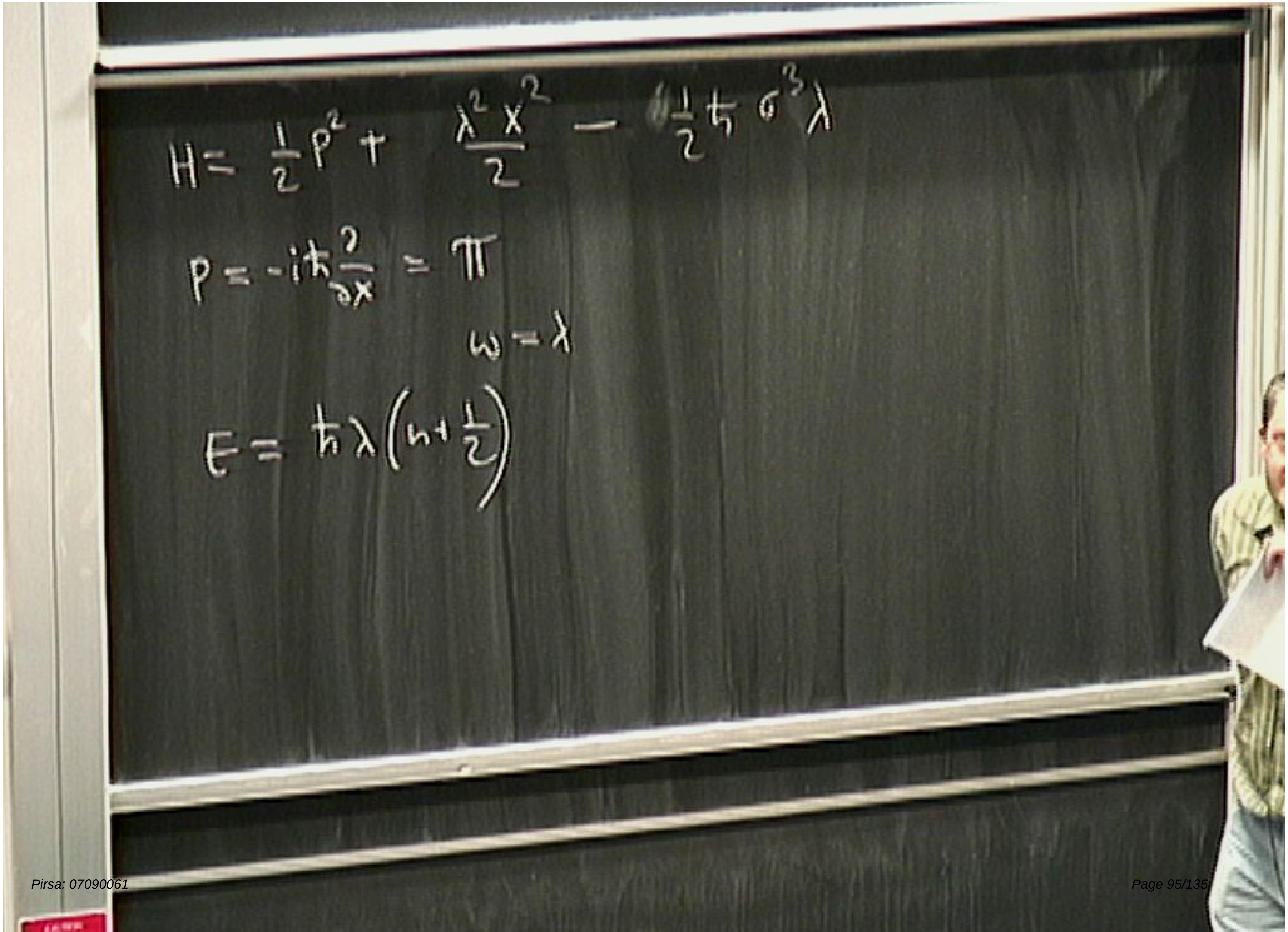
$$H = \frac{1}{2}p^2 + \frac{\lambda^2 x^2}{2} - \frac{1}{2}\hbar\,\sigma^3\lambda$$

$$p = -i\hbar\frac{\partial}{\partial x} = \pi$$

$$H = \frac{1}{2}p^2 + \frac{\lambda^2 x^2}{2} - \frac{1}{2}\hbar\,\sigma^3\lambda$$

$$p = -i\hbar\frac{\partial}{\partial x} = \pi$$

$$\omega = \lambda$$



$$H = \frac{1}{2}p^2 + \frac{\lambda^2 x^2}{2} - \frac{1}{2}\hbar\,\sigma^3\lambda$$

$$p = -i\hbar\frac{\partial}{\partial x} = \pi$$

$$\omega = \lambda$$

$$E = \hbar\lambda\left(n + \frac{1}{2}\right)$$

$$H = \frac{1}{2}p^2 + \frac{\lambda^2 x^2}{2} - \frac{1}{2}\hbar\,\sigma^3\lambda$$

$$p = -i\hbar\frac{\partial}{\partial x} = \pi$$

$$\omega = \lambda$$

$$E_n^{\pm} = \hbar\lambda\left(n + \frac{1}{2}\right) \pm \frac{1}{2}\hbar\lambda$$

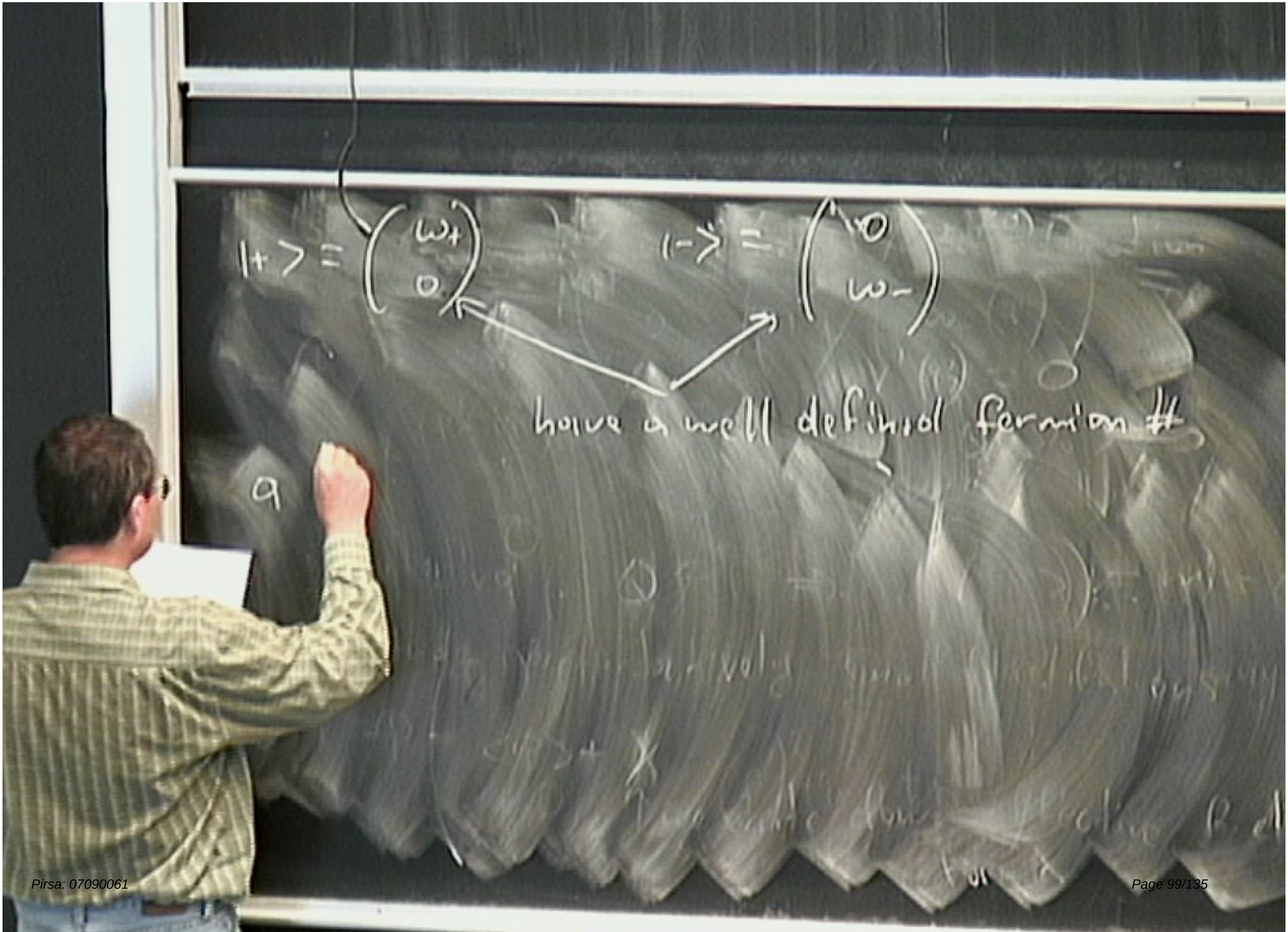
$$H = \frac{1}{2}p^2 + \frac{\lambda^2 x^2}{2} - \frac{1}{2}\hbar\sigma^3\lambda$$

$$p = -i\hbar\frac{\partial}{\partial x} = \pi$$

$$\omega = \lambda$$

$$E_n^{\pm} = \underbrace{\hbar\lambda\left(n+\frac{1}{2}\right)}_{\text{SHO}} \pm \underbrace{\frac{1}{2}\hbar\lambda}_{\substack{\text{SPIN interaction} \\ \text{with a constant magnetic} \\ \text{field}}}$$

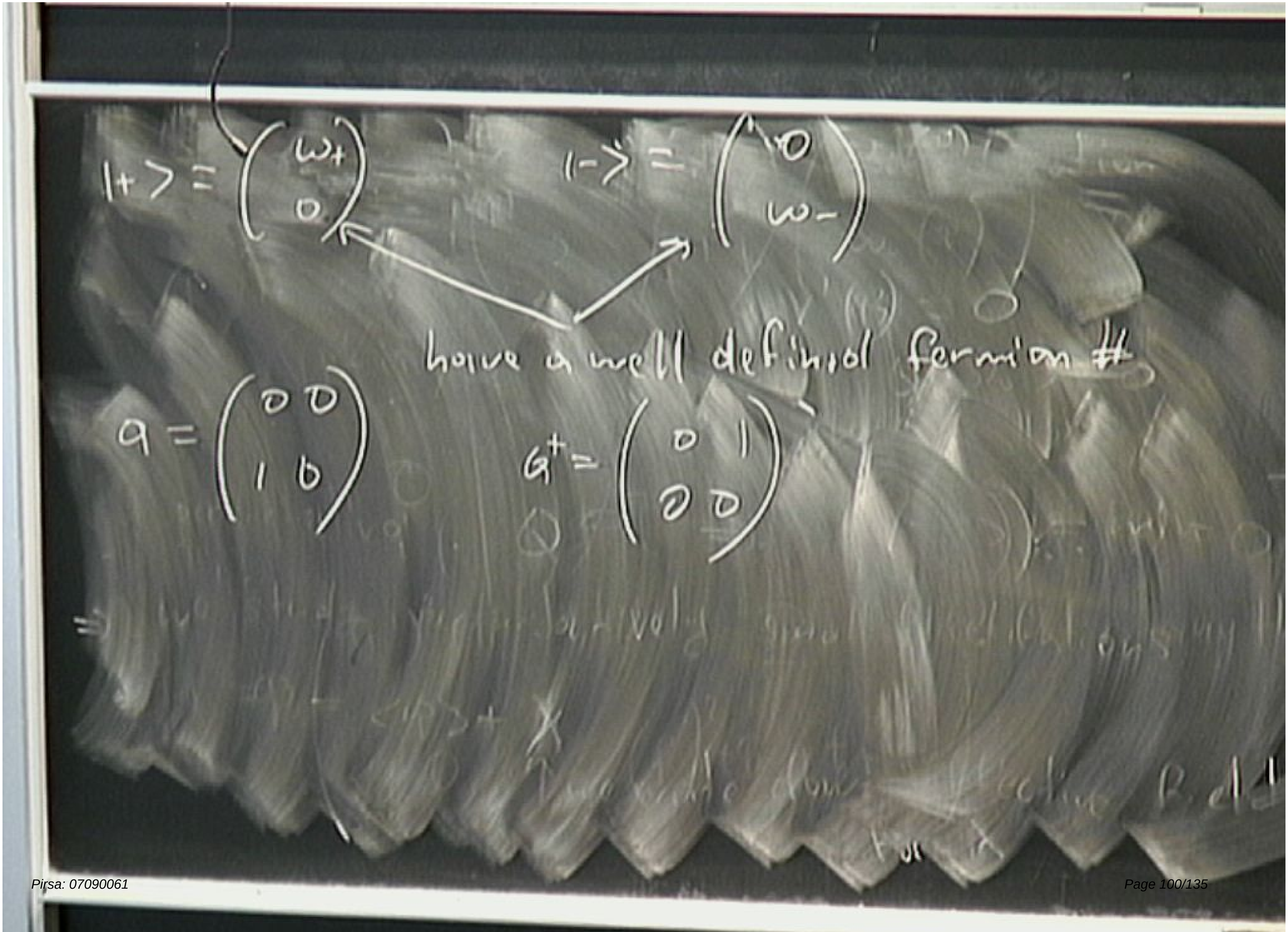
$$|+\rangle = \begin{pmatrix} \omega_+ \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ \omega_- \end{pmatrix}$$



$$|+\rangle = \begin{pmatrix}\omega_+\\ 0\end{pmatrix} \qquad |-\rangle = \begin{pmatrix}0\\ \omega_-\end{pmatrix}$$

have a well defined fermion #

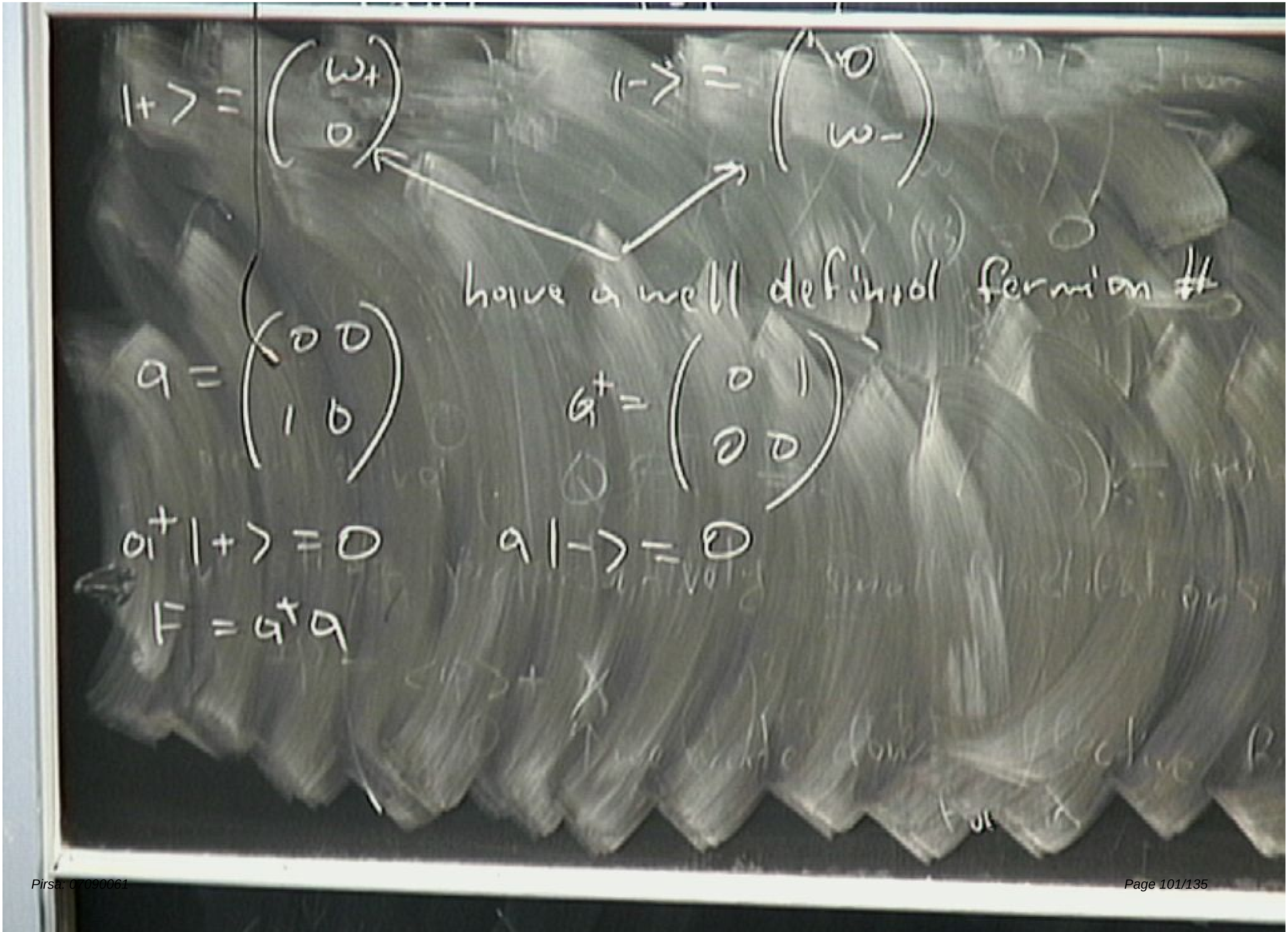
a



$$|+\rangle = \begin{pmatrix} \omega_+ \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ \omega_- \end{pmatrix}$$

have a well defined fermion #

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



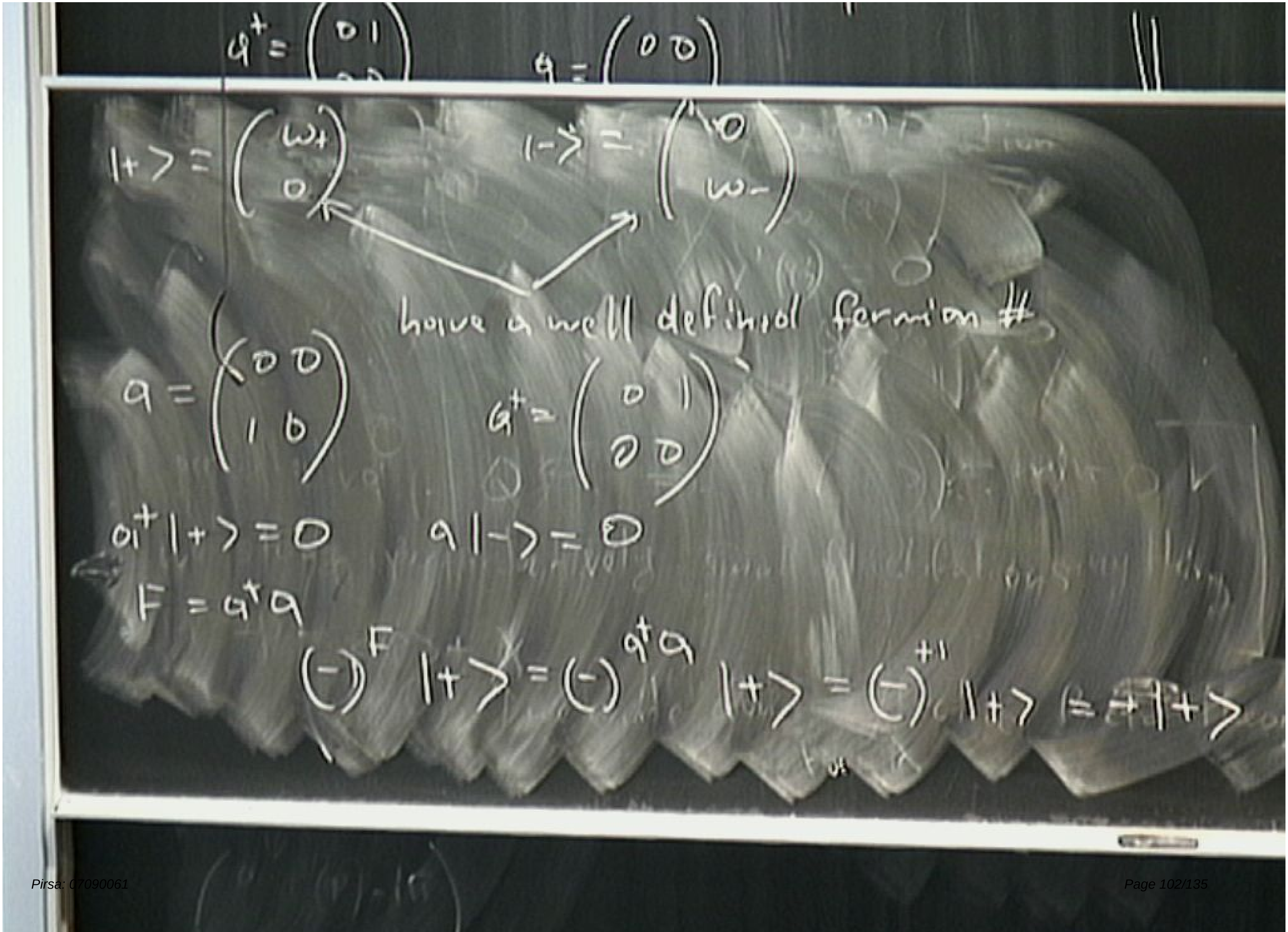
$$|+\rangle = \begin{pmatrix} \omega_+ \\ 0 \end{pmatrix} \qquad |-\rangle = \begin{pmatrix} 0 \\ \omega_- \end{pmatrix}$$

have a well defined fermion #

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$a^+|+\rangle = 0 \qquad a|-\rangle = 0$$

$$F = a^+ a$$



$a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $a = \begin{pmatrix} 0 & 0 \\ \end{pmatrix}$

$|+\rangle = \begin{pmatrix} \omega_+ \\ 0 \end{pmatrix}$ $|-\rangle = \begin{pmatrix} 0 \\ \omega_- \end{pmatrix}$

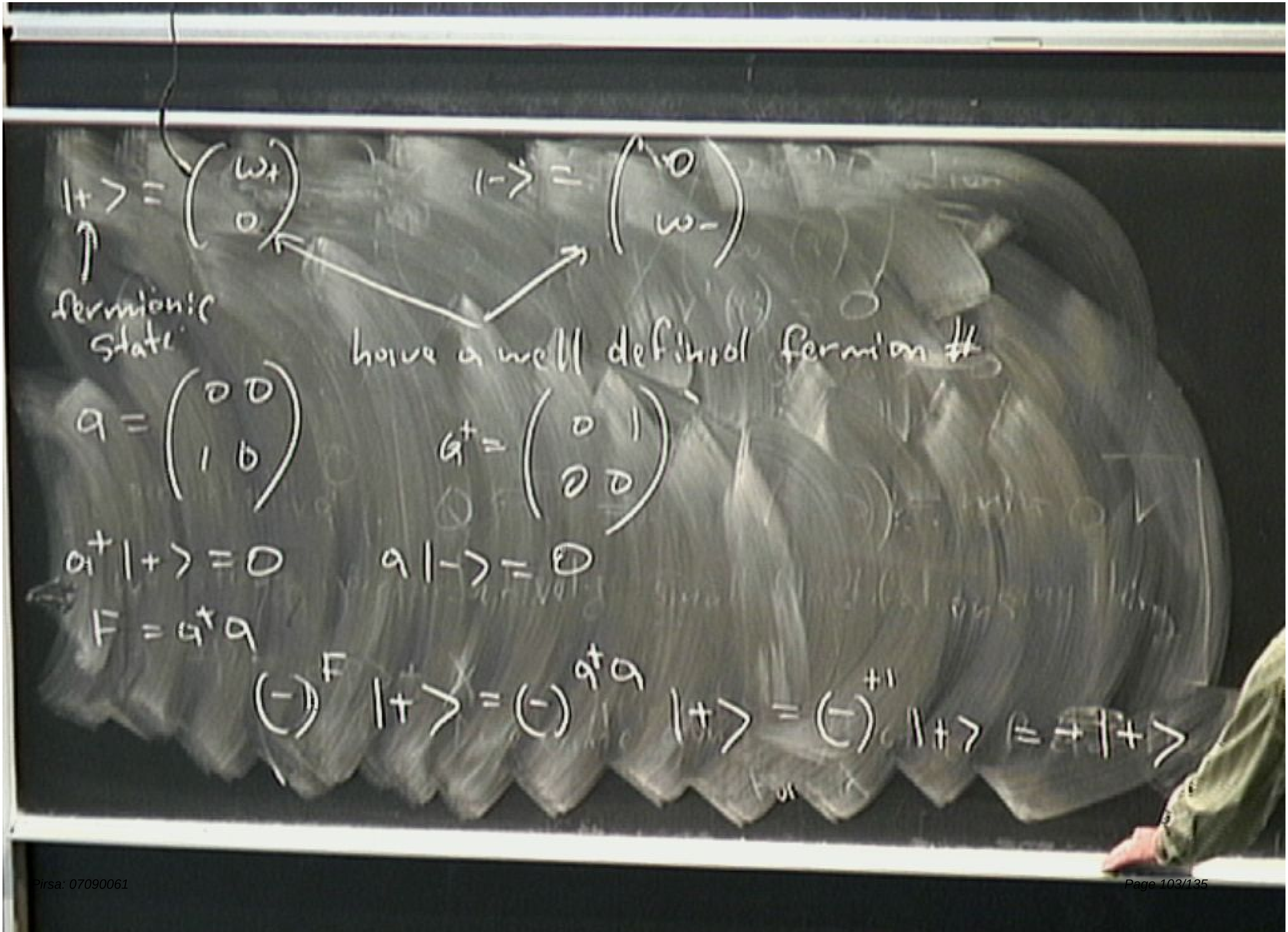
have a well defined fermion #

$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $a^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$a^+ |+\rangle = 0$ $a|-\rangle = 0$

$F = a^+ a$

$(-)^F |+\rangle = (-)^{a^+a} |+\rangle = (-)^{+1} |+\rangle = -|+\rangle$



$|+\rangle = \begin{pmatrix} \omega_+ \\ 0 \end{pmatrix}$ $\quad |-\rangle = \begin{pmatrix} 0 \\ \omega_- \end{pmatrix}$

fermionic state

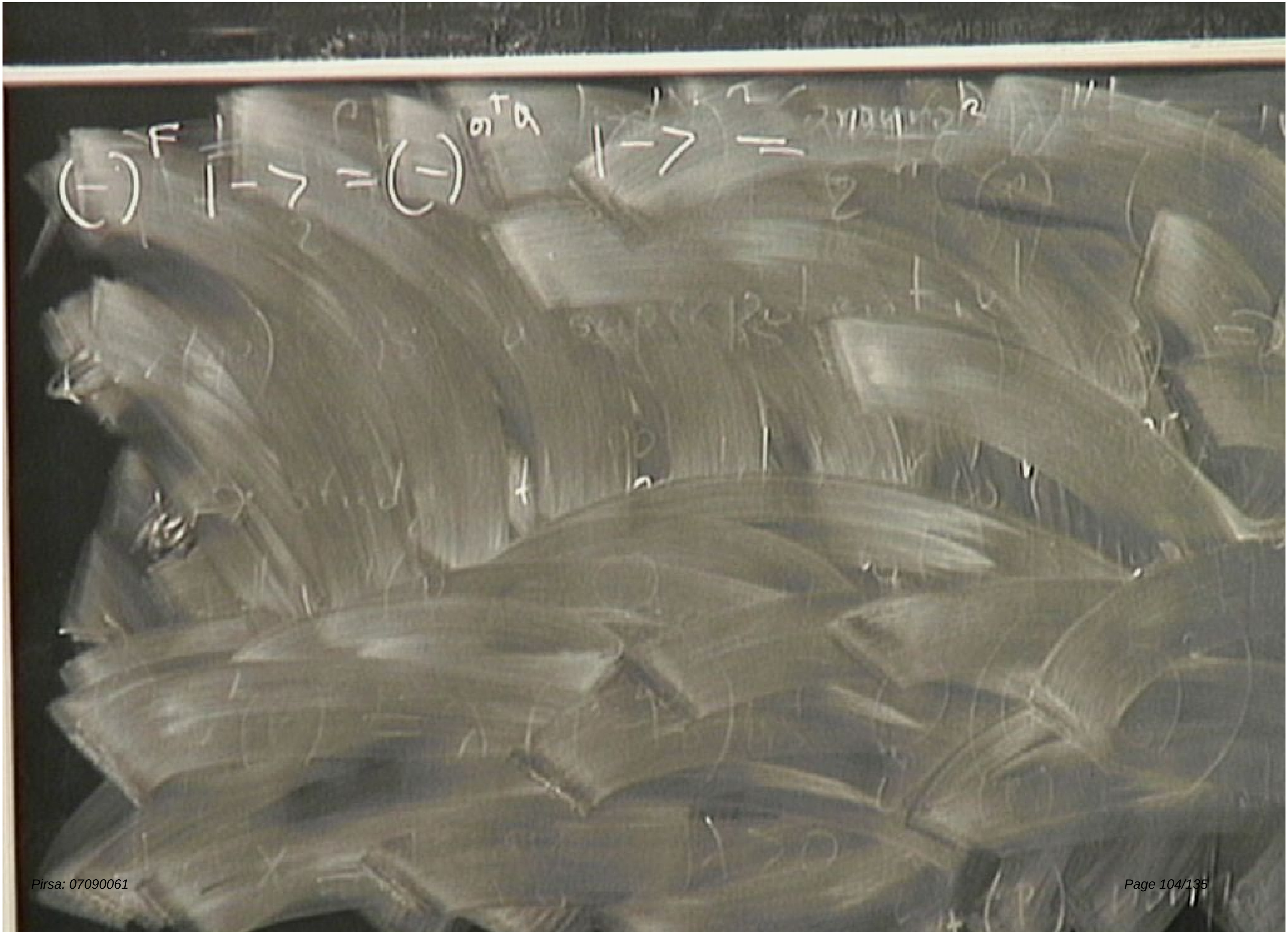
have a well defined fermion #

$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\quad a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

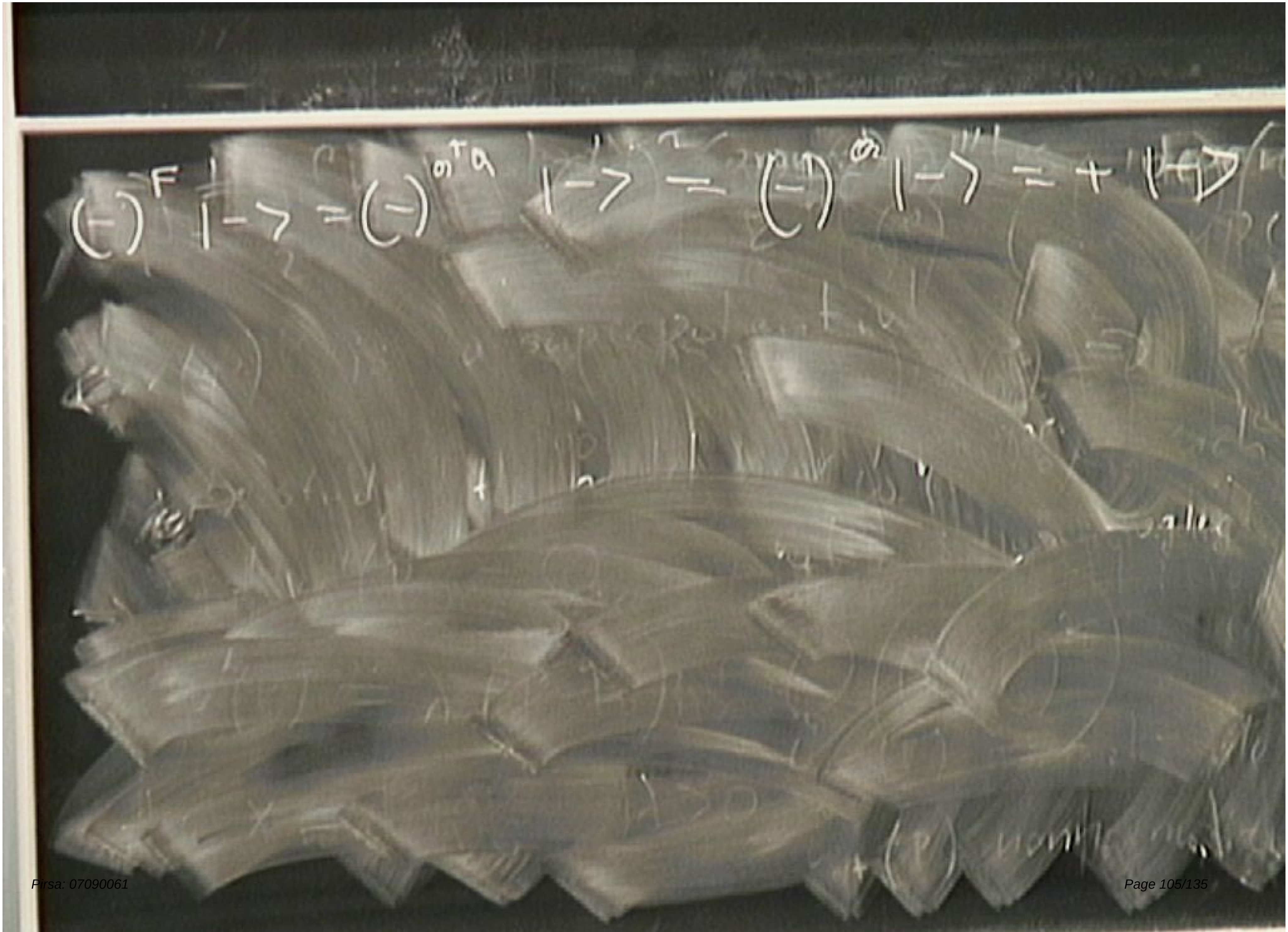
$a^\dagger |+\rangle = 0 \qquad a|-\rangle = 0$

$F = a^\dagger a$

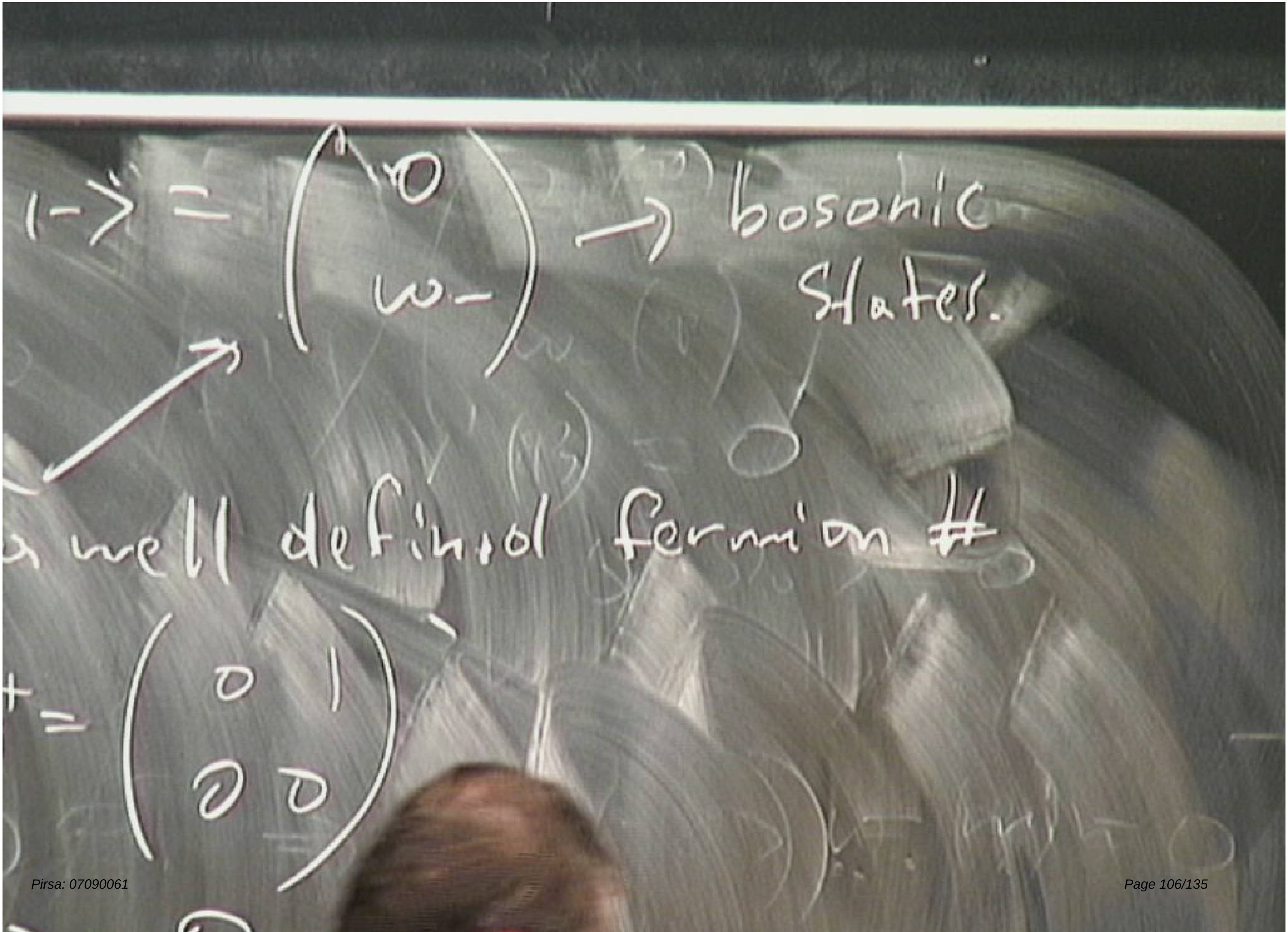
$$(-)^F |+\rangle = (-)^{a^\dagger a} |+\rangle = (-)^{+1} |+\rangle = -|+\rangle$$



$$(-)^F |-\rangle = (-)^{a^\dagger a} |-\rangle =$$



$$(-)^F |-\rangle = (-)^{a^\dagger a} |-\rangle = (-)^{0} |-\rangle = +|-\rangle$$



$$|-\rangle = \begin{pmatrix} 0 \\ \omega_- \end{pmatrix} \longrightarrow \text{bosonic states.}$$

well defined fermion #

$$^{+} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

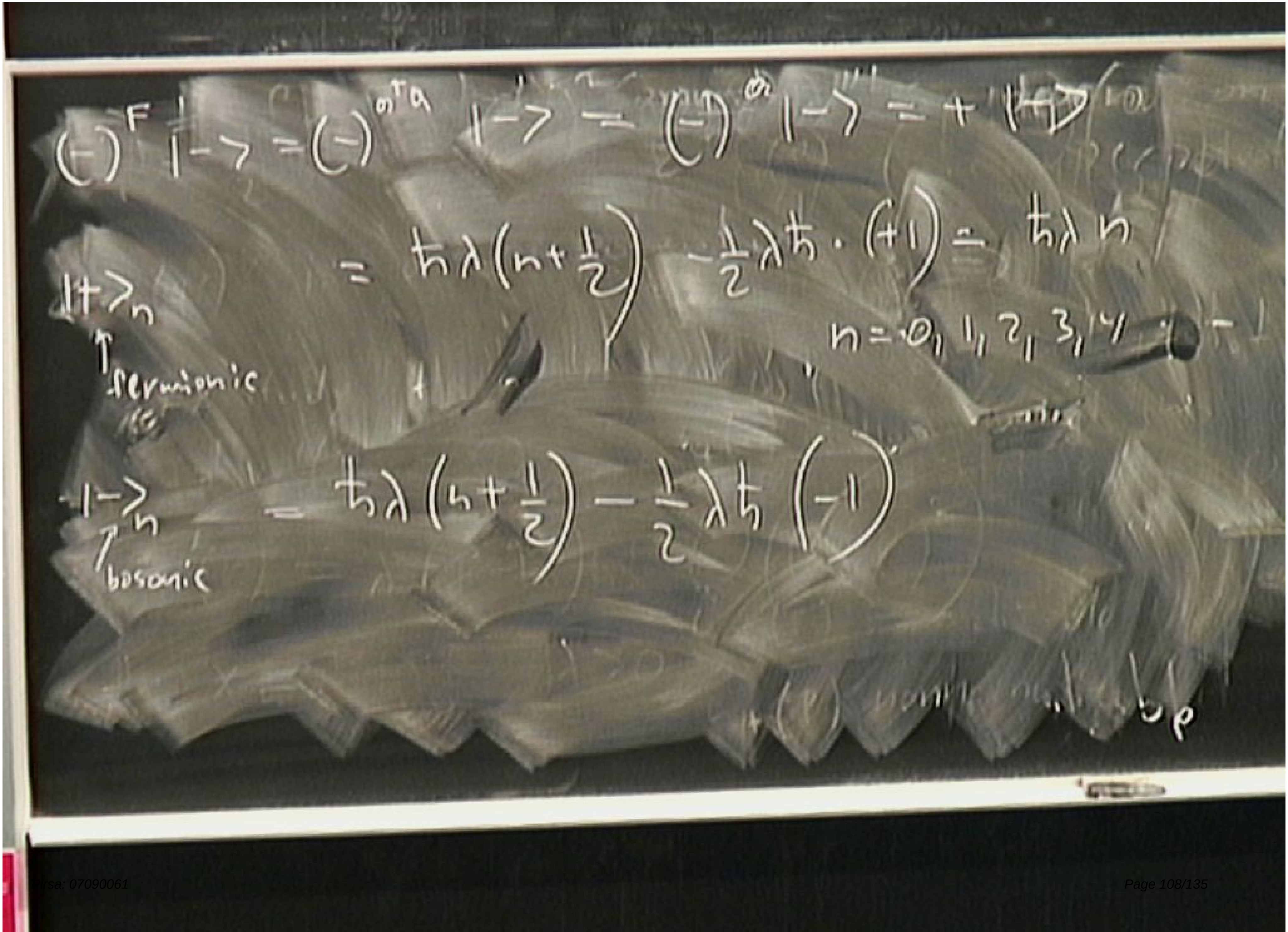
$$(-)^F |-\rangle = (-)^{a^\dagger a} |-\rangle = (-)^0 |-\rangle = +|-\rangle$$

$$= \hbar\lambda\left(n+\frac{1}{2}\right) - \frac{1}{2}\lambda\hbar .$$

$|-\rangle_n$

$\uparrow$

fermionic



$$(-)^F |-\rangle = (-)^{a^\dagger a} |-\rangle = (-1)^0 |-\rangle = + |-\rangle$$

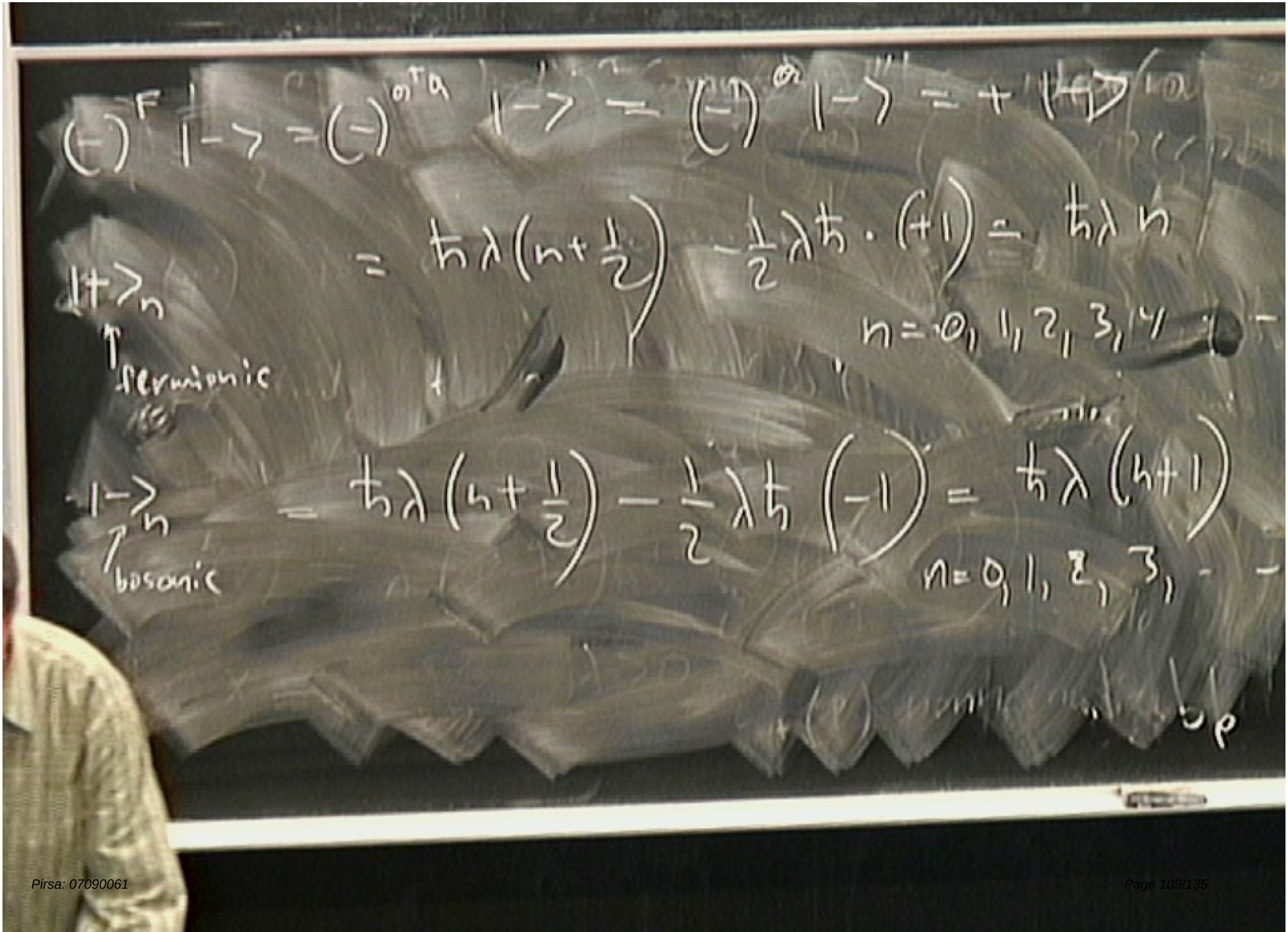
$|+\rangle_n$ ↑ fermionic

$$= \hbar\lambda\left(n+\tfrac{1}{2}\right) - \tfrac{1}{2}\lambda\hbar\cdot(+1) = \hbar\lambda n$$

$$n = 0, 1, 2, 3, 4, \ldots$$

$|-\rangle_n$ ↑ bosonic

$$= \hbar\lambda\left(n+\tfrac{1}{2}\right) - \tfrac{1}{2}\lambda\hbar(-1)$$



$$(-)^F |-\rangle = (-)^{a^\dagger a} |-\rangle = (-)^0 |-\rangle = +|-\rangle$$

$|+\rangle_n$ ↑ fermionic

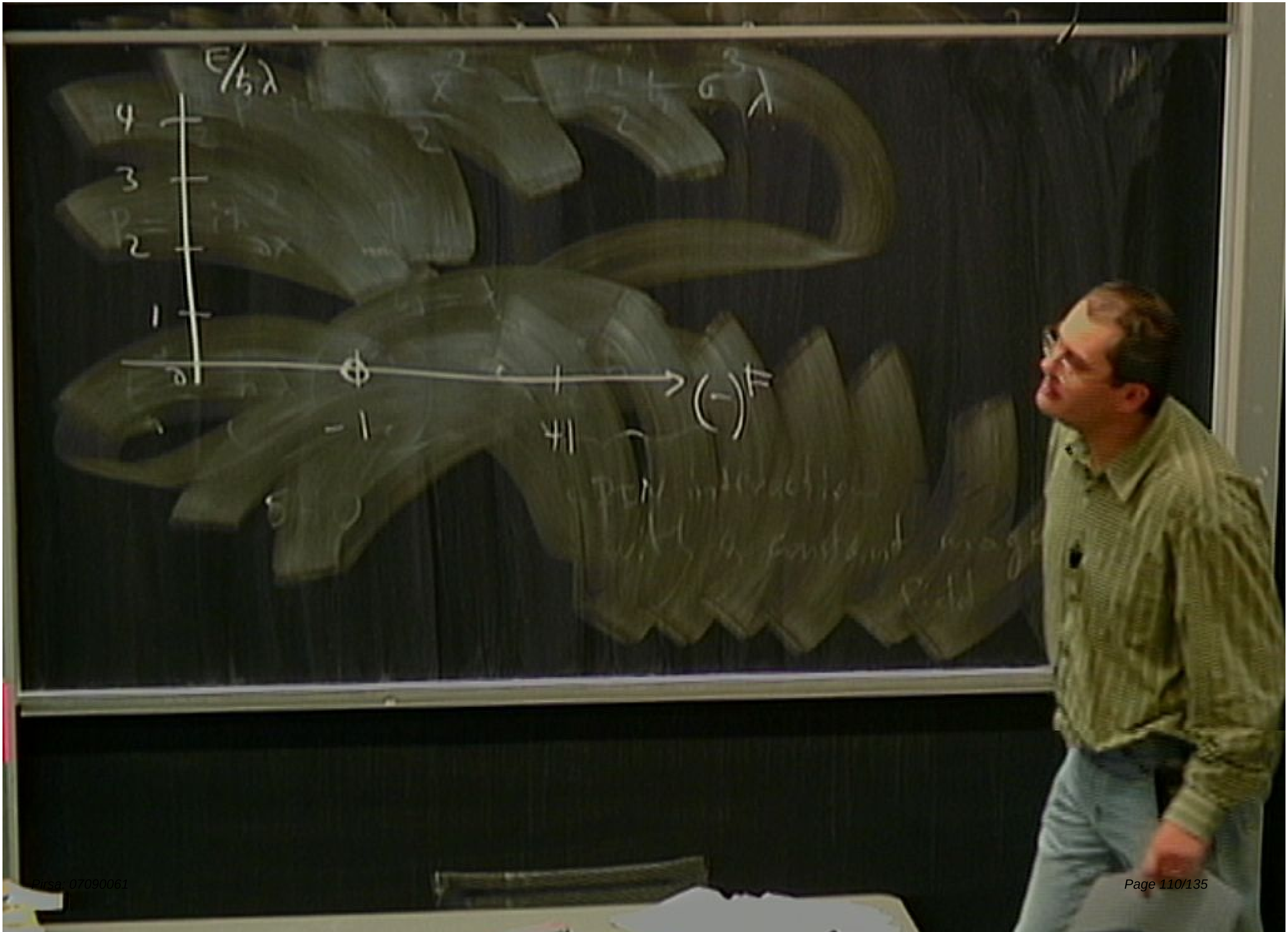
$$= \hbar\lambda\left(n+\frac{1}{2}\right) - \frac{1}{2}\lambda\hbar\cdot(+1) = \hbar\lambda n$$

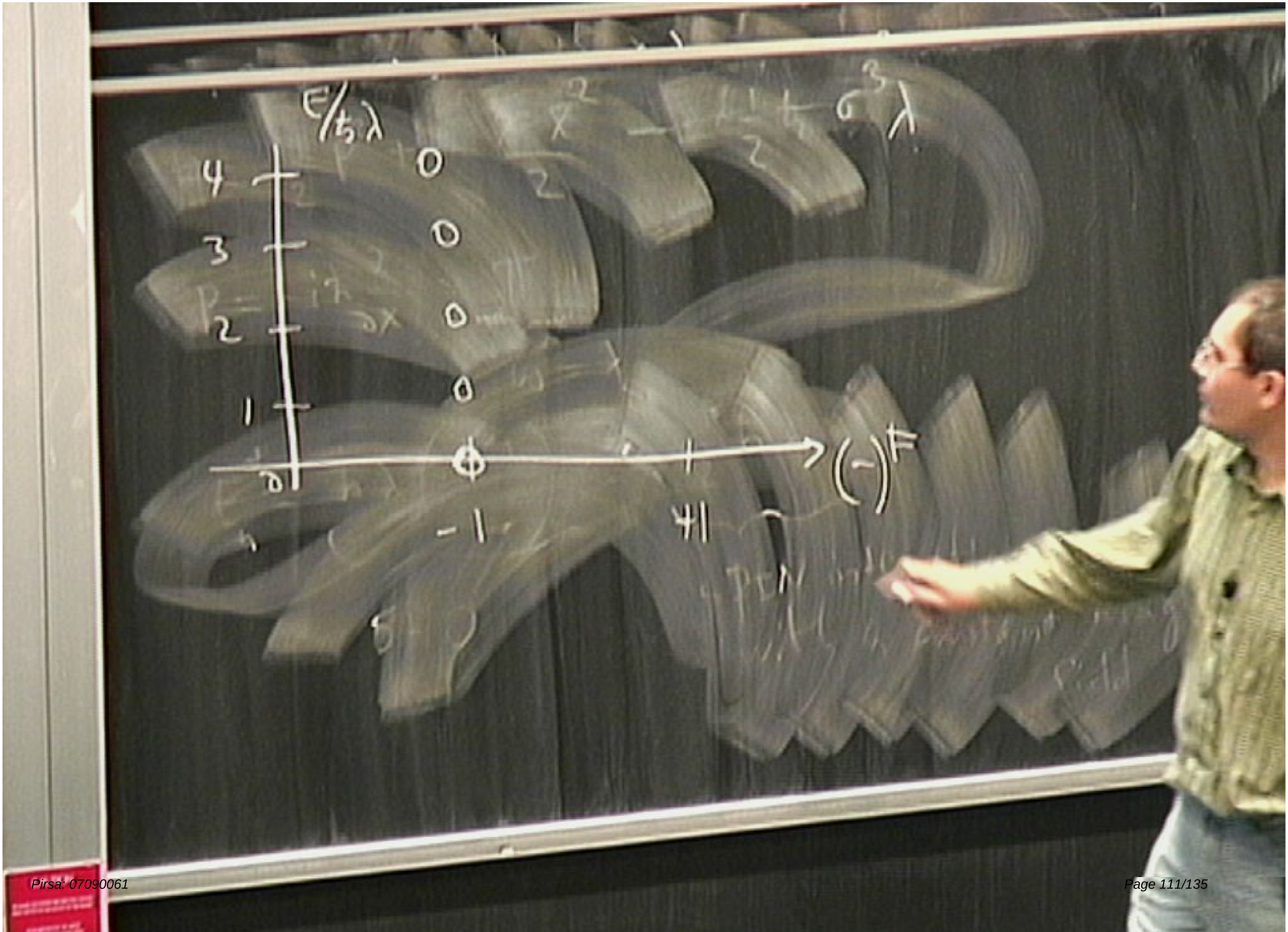
$n = 0, 1, 2, 3, 4, \ldots$

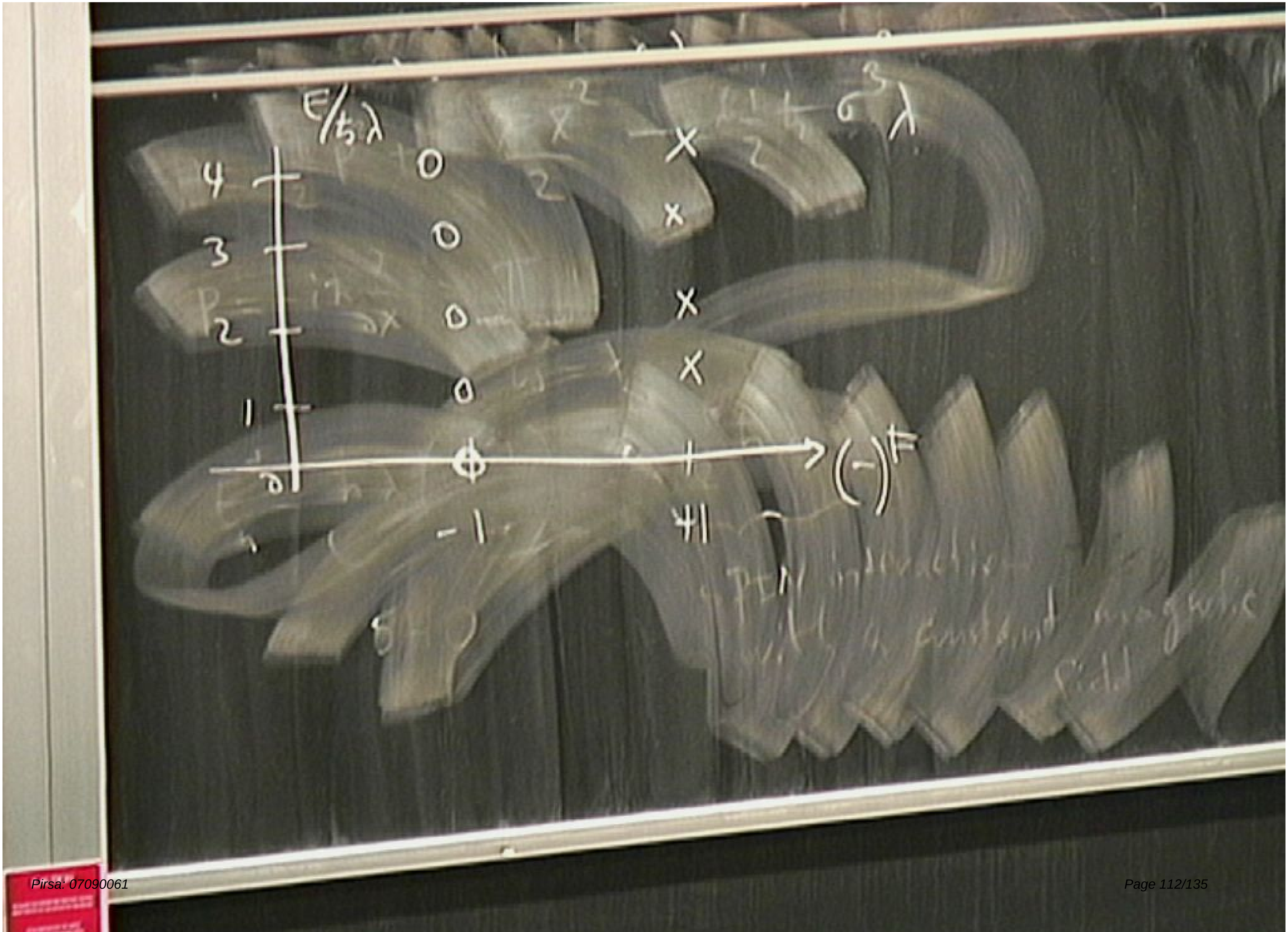
$|-\rangle_n$ ↑ bosonic

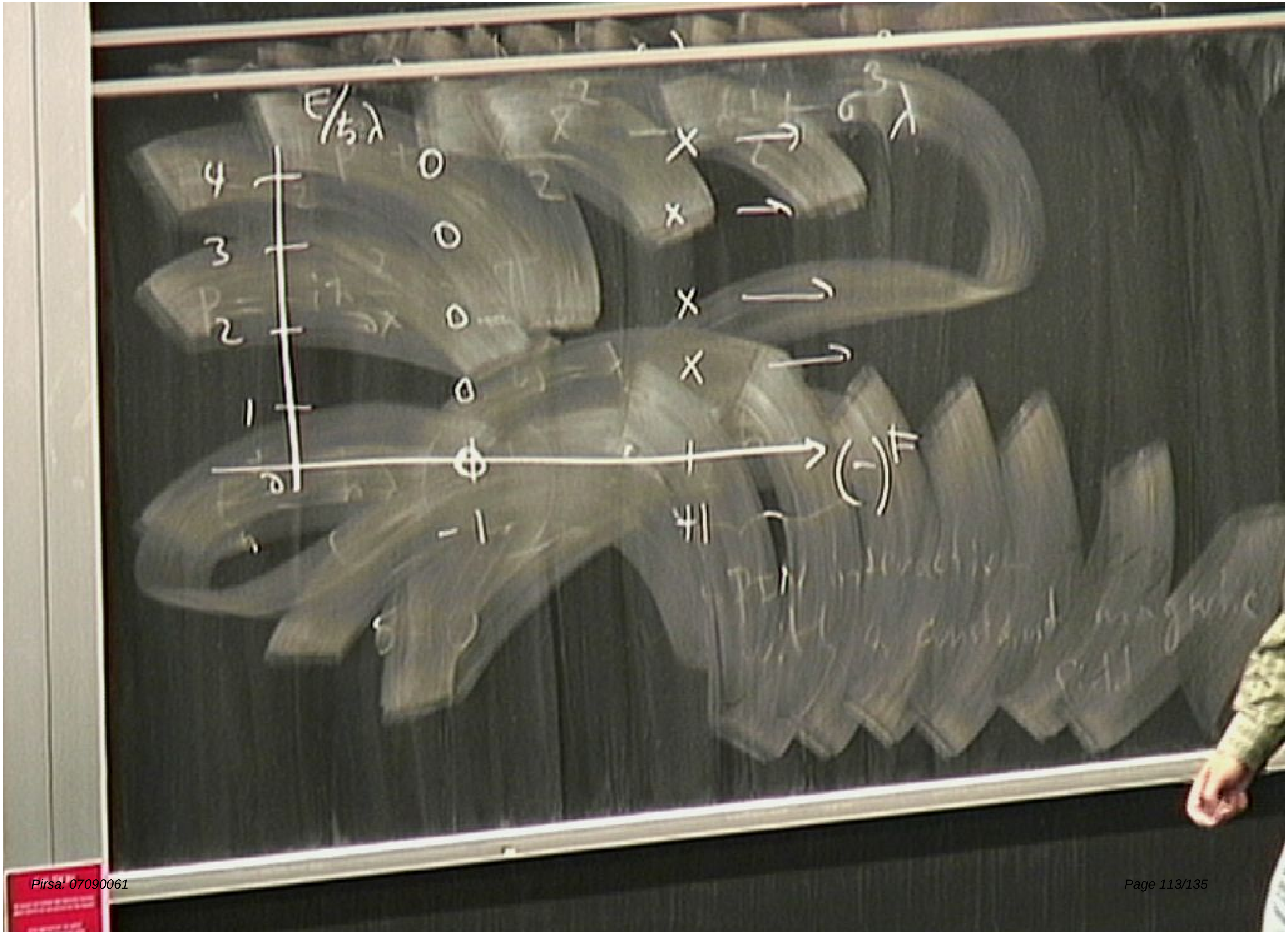
$$= \hbar\lambda\left(n+\frac{1}{2}\right) - \frac{1}{2}\lambda\hbar(-1) = \hbar\lambda(n+1)$$

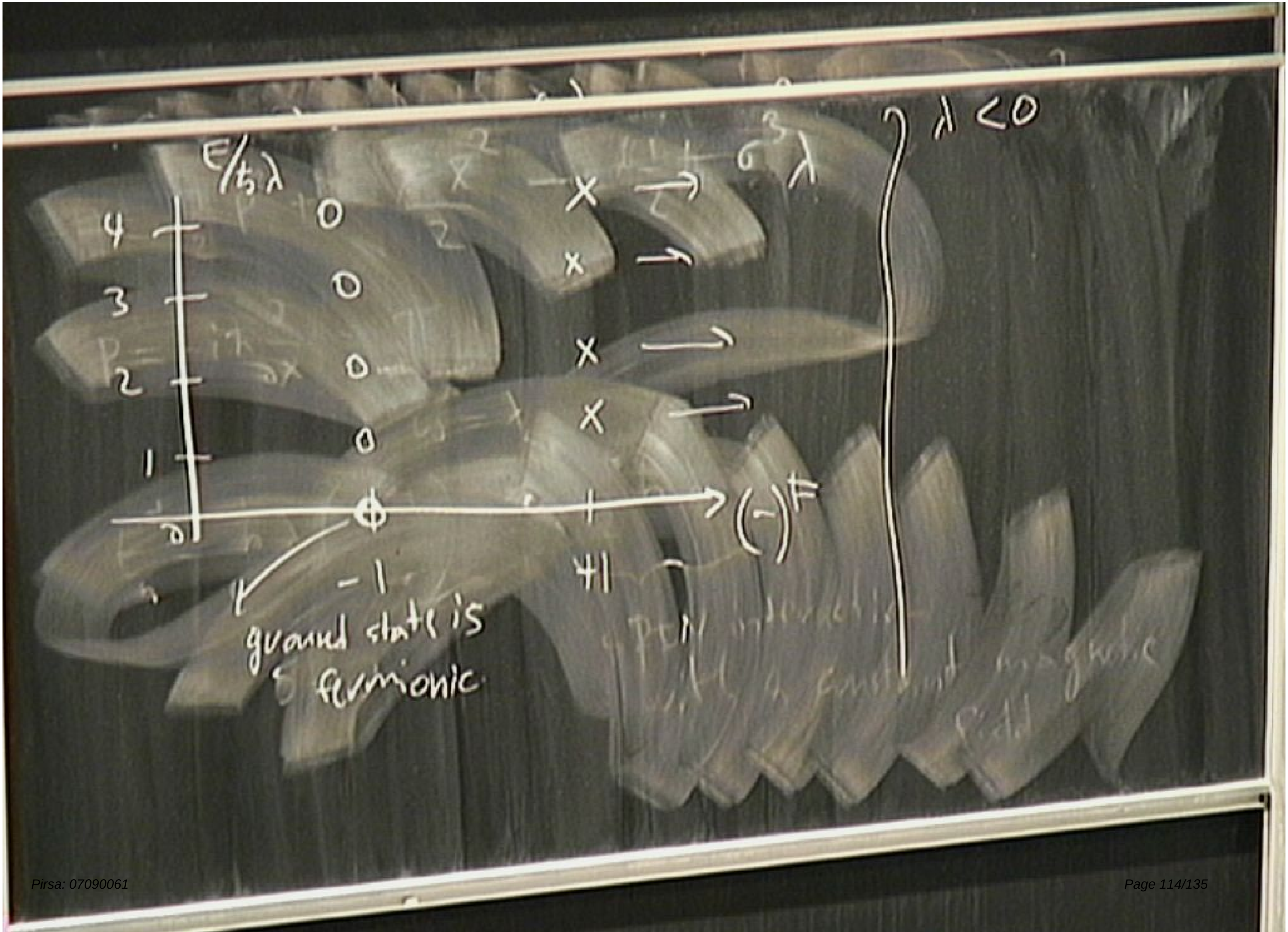
$n = 0, 1, 2, 3, \ldots$

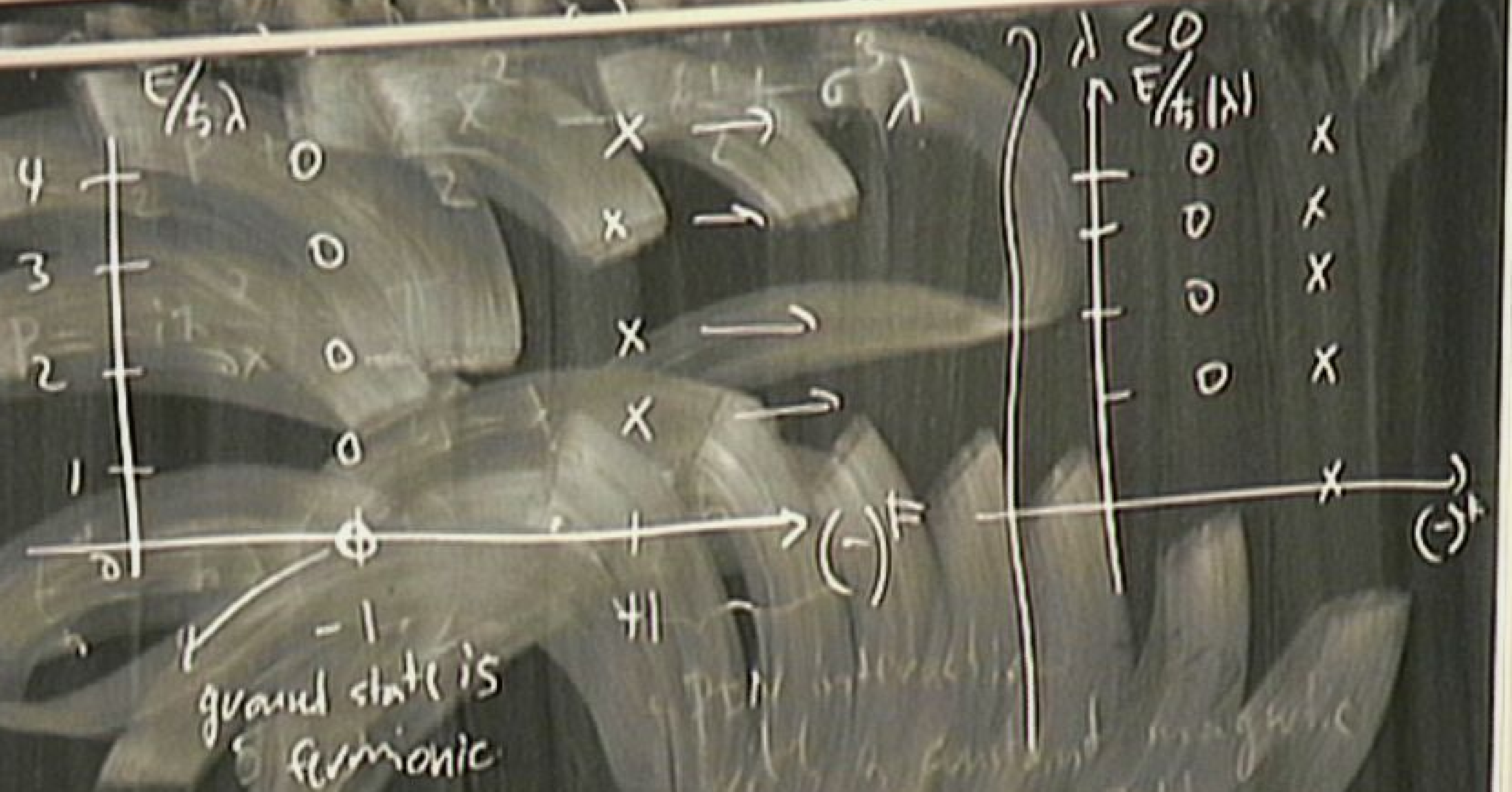

E/ħλ
4
3
2
1
0
Φ
-1
+1
(-)^F


E/ħλ
4
3
2
1
0
0
0
0
0
Φ
1
-1
+1
(-)^F

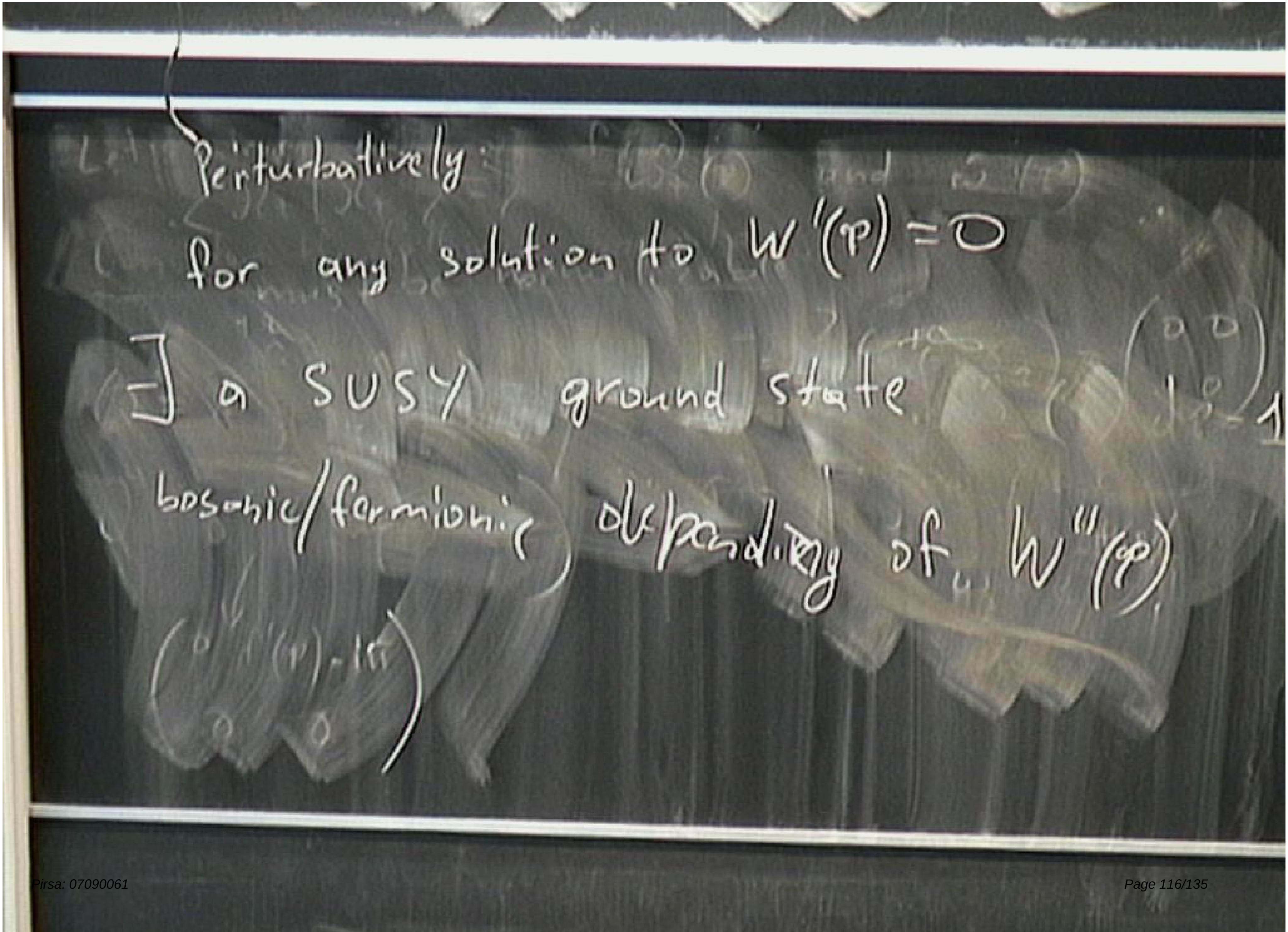

E/ħλ
4
3
2
1
0
0
0
0
0
ϕ
-1
×
×
×
×
+
±1
(-)^F


E/ħλ
4
3
2
1
0
0
0
0
0
Φ
-1
X
X
X
X
+1
+1
$(-)^F$


λ<0
E/ħλ
σ³λ
4
3
2
1
0
0
0
0
0
x
x
x
x
(-)^F
-1
+1
ground state is fermionic



ground state is fermionic



Perturbatively:

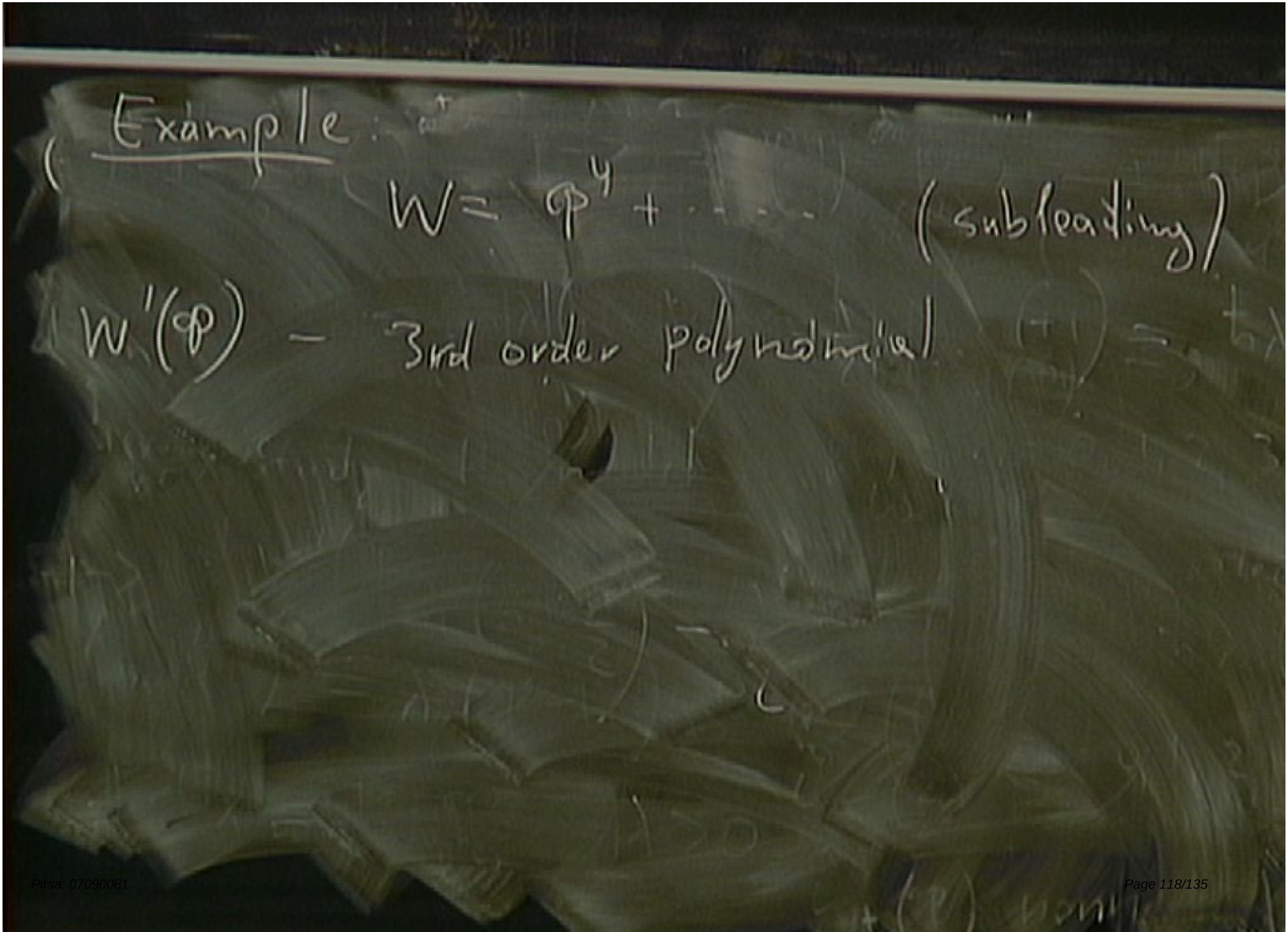
for any solution to $W'(\varphi)=0$

$\exists$ a SUSY ground state

bosonic/fermionic depending of $W''(\varphi)$

(<u>Example</u>:

$$W = \varphi^4 + \cdots \quad (\text{subleading})$$

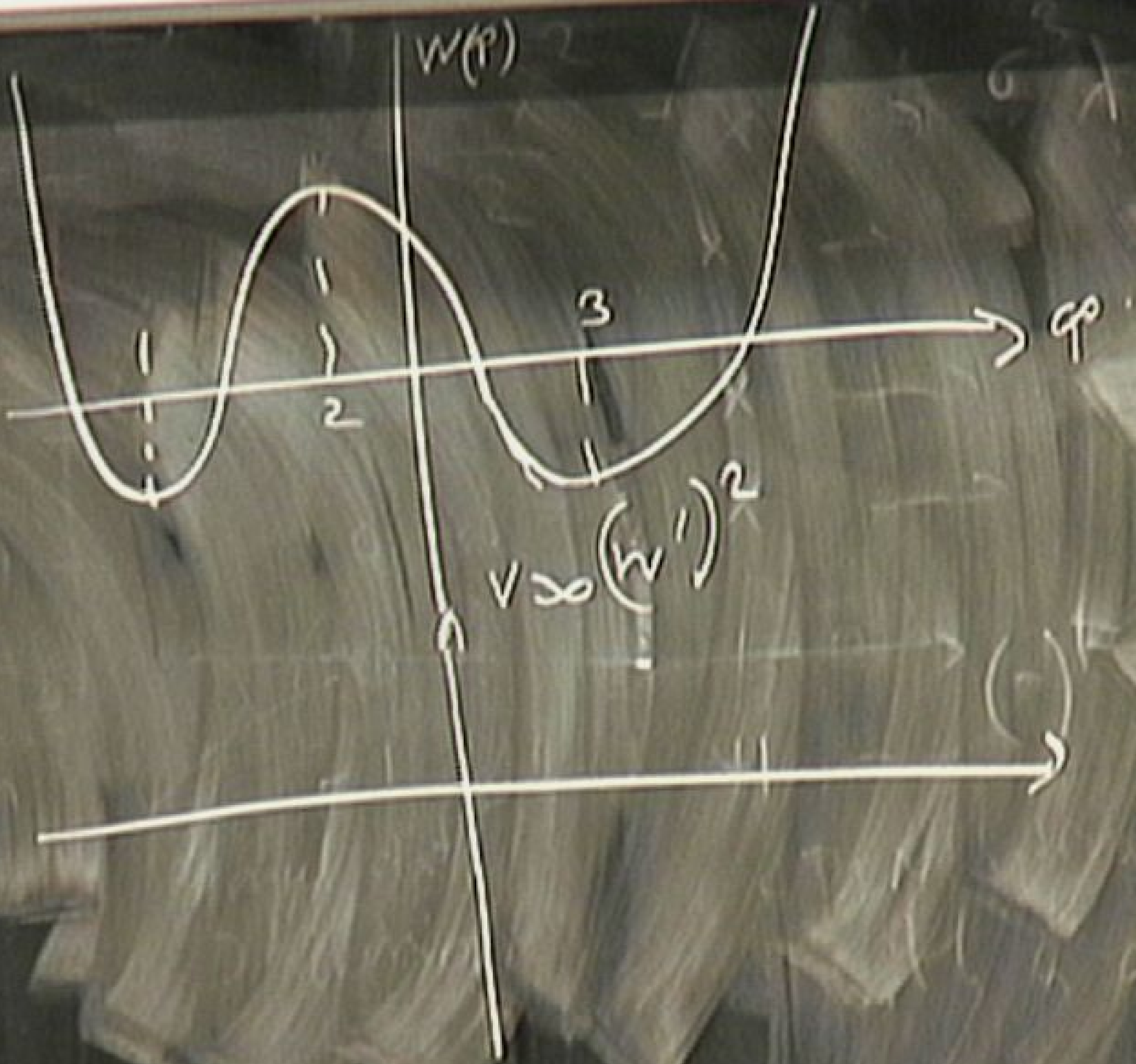


(Example:

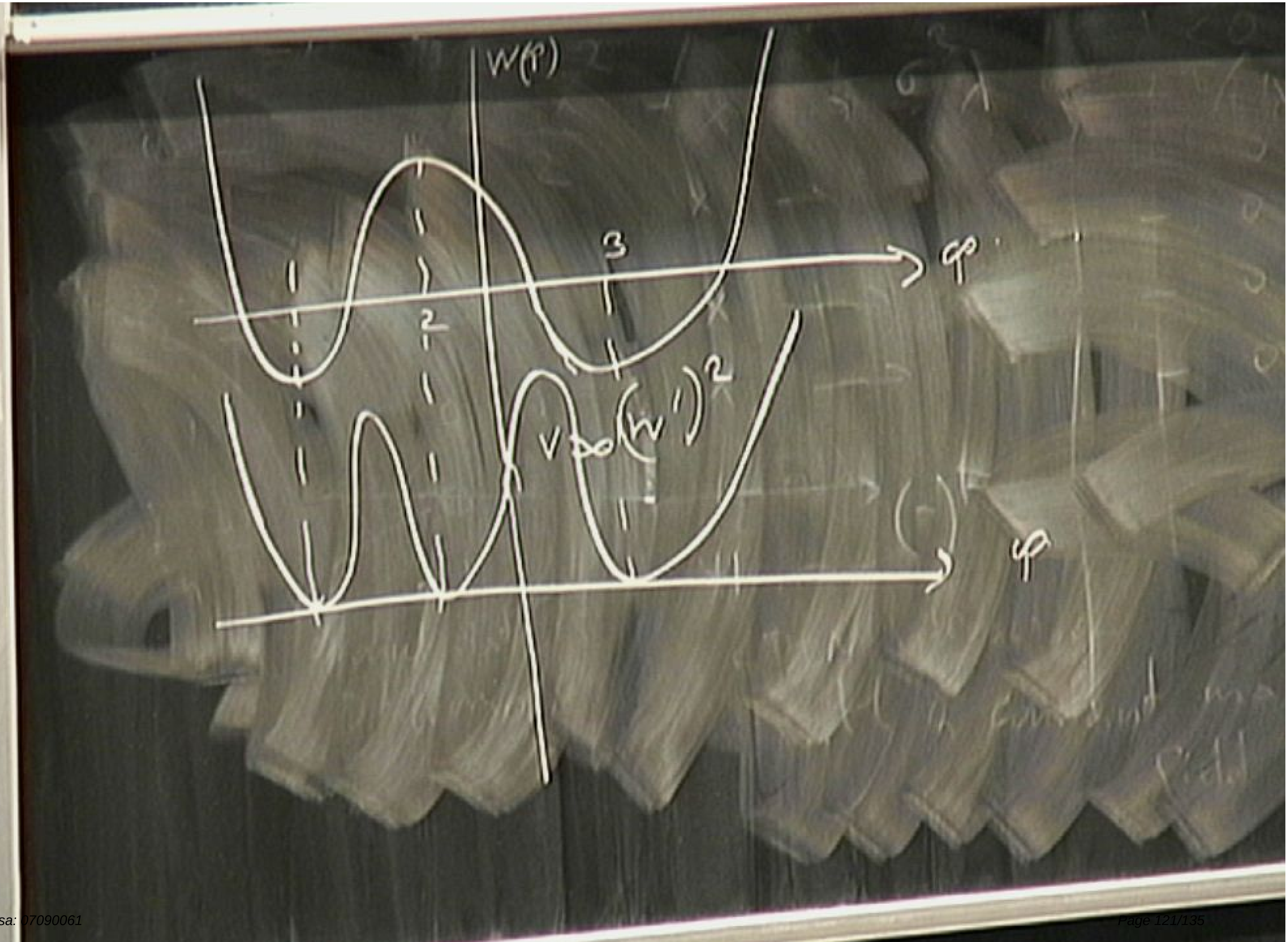
$$W = \varphi^4 + \ldots \quad (\text{subleading})$$

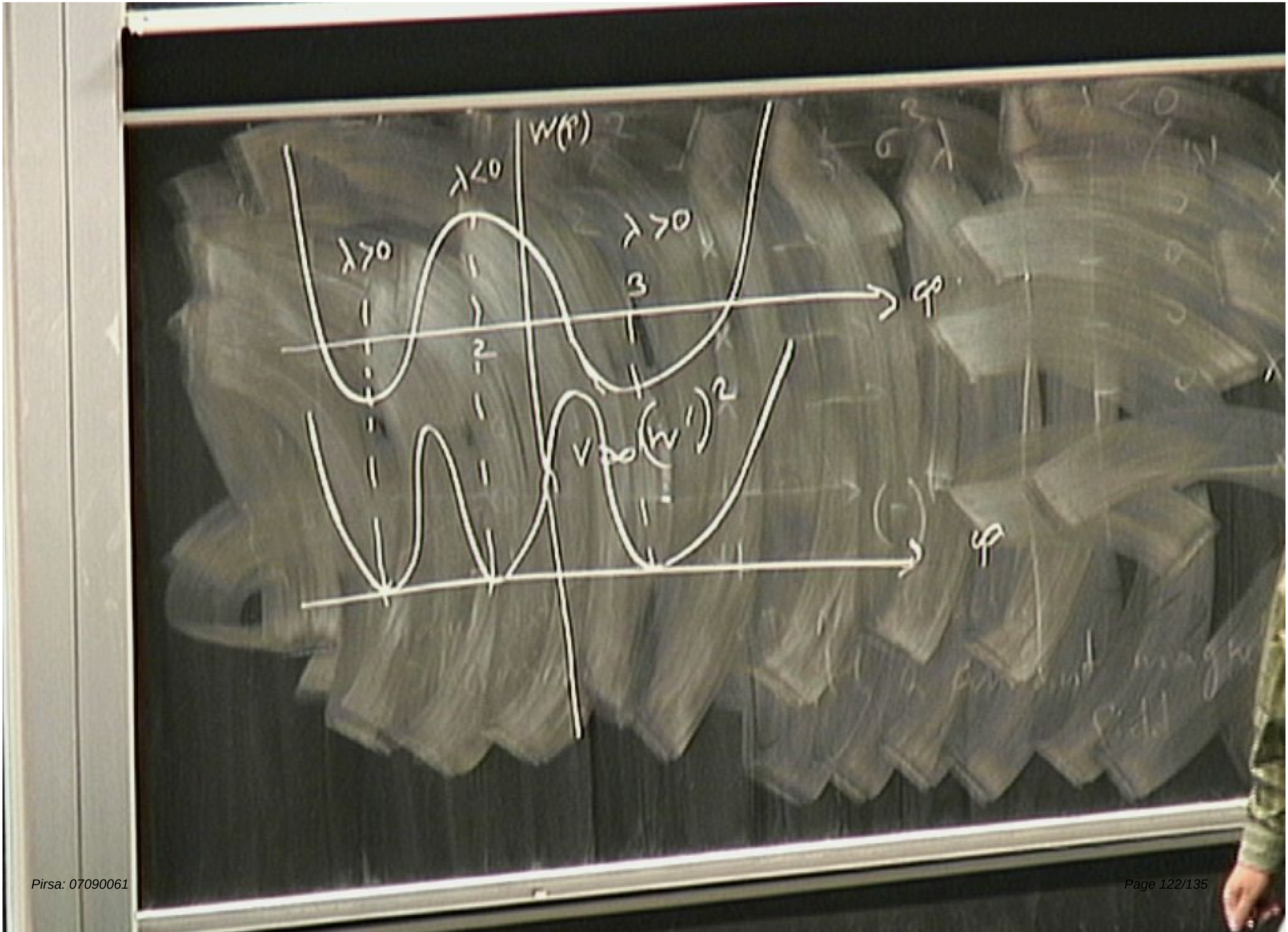
$W'(\varphi)$ – 3rd order polynomial


W(φ)
φ



$V \propto (W')^2$


W(φ)
1
2
3
φ
V ∝ (W')²
φ

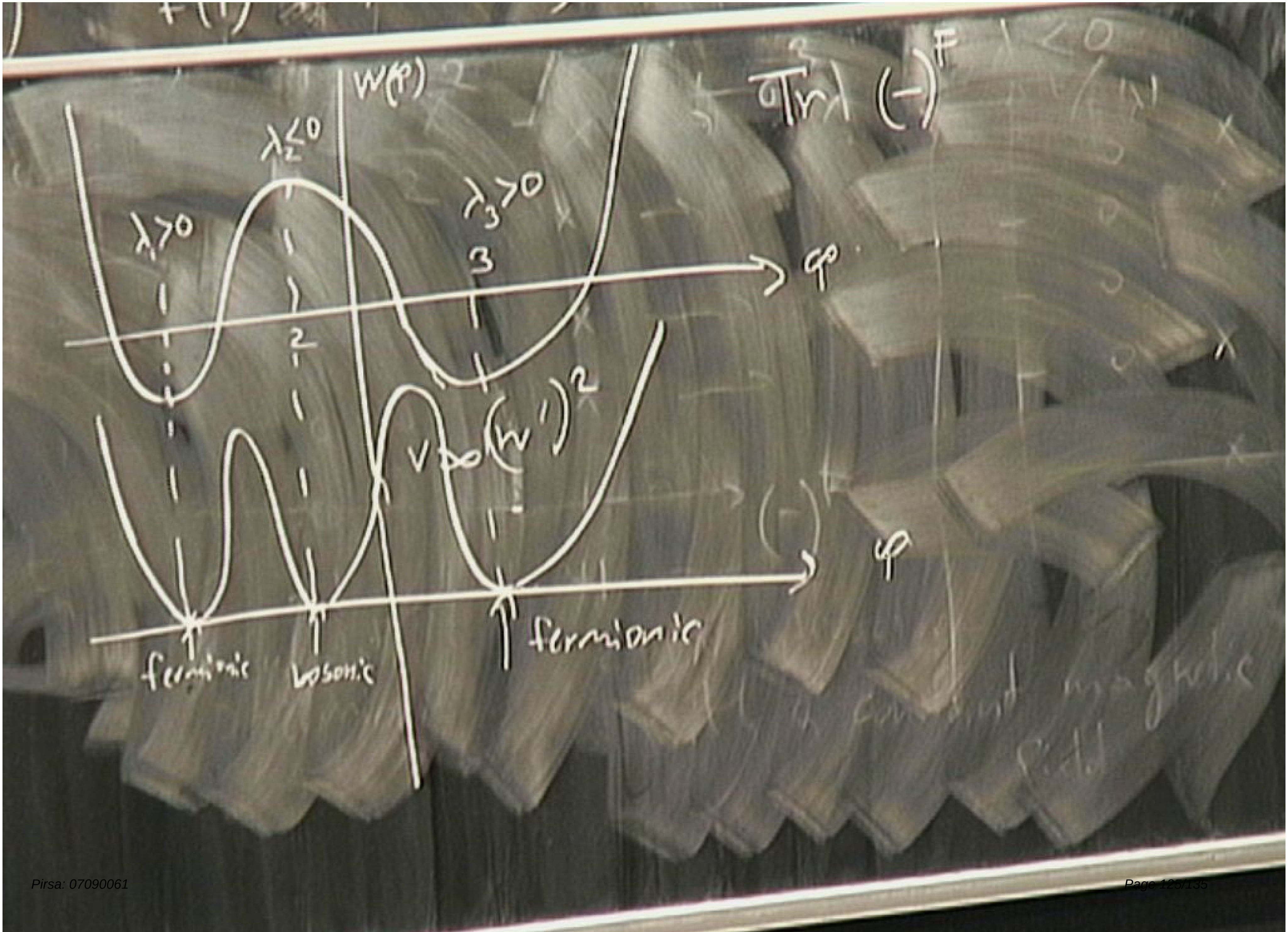

W(φ)
λ<0
λ>0
λ>0
1
2
3
φ
V ∝ (W')²
φ

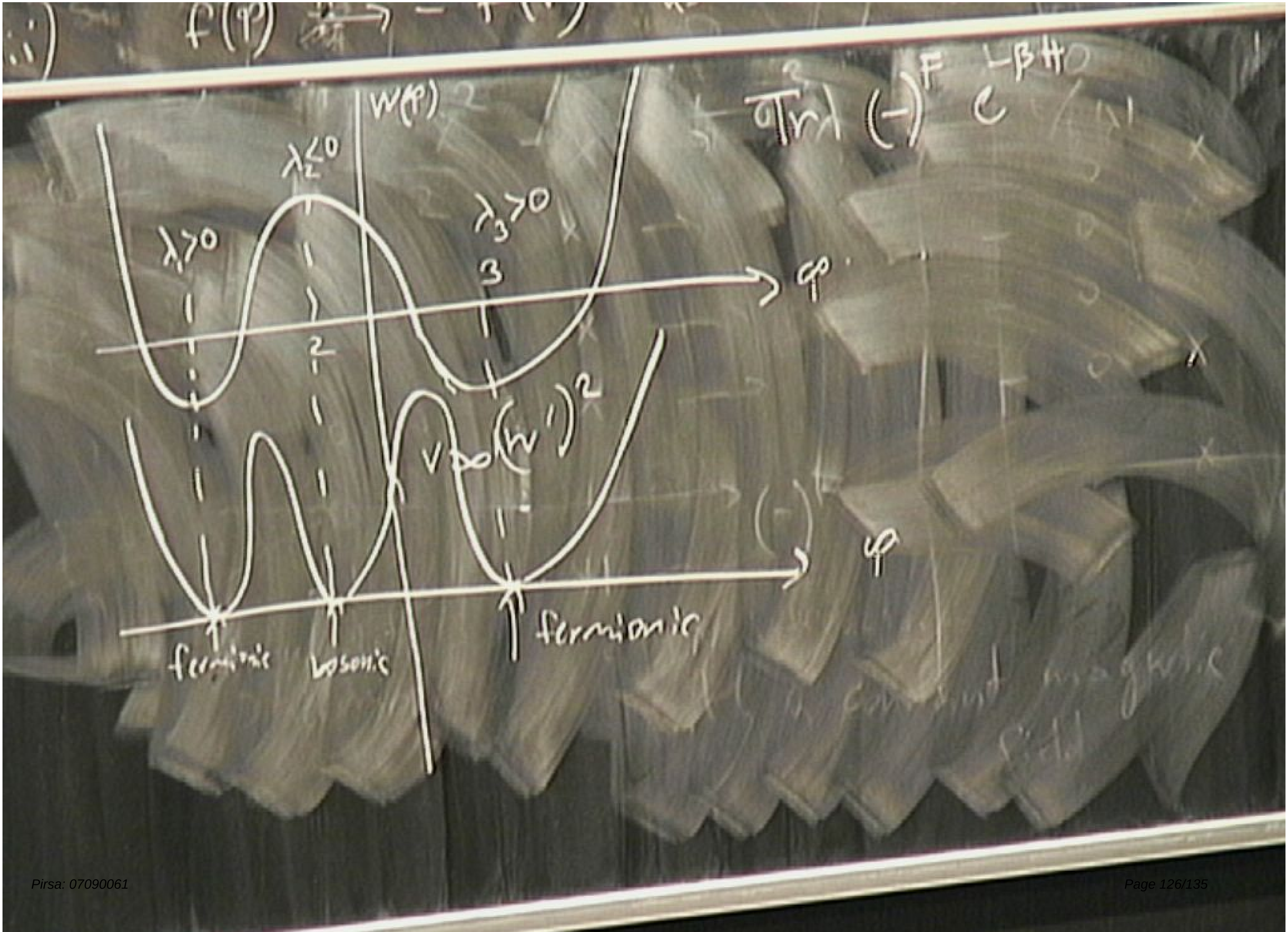
nonperturbatively :

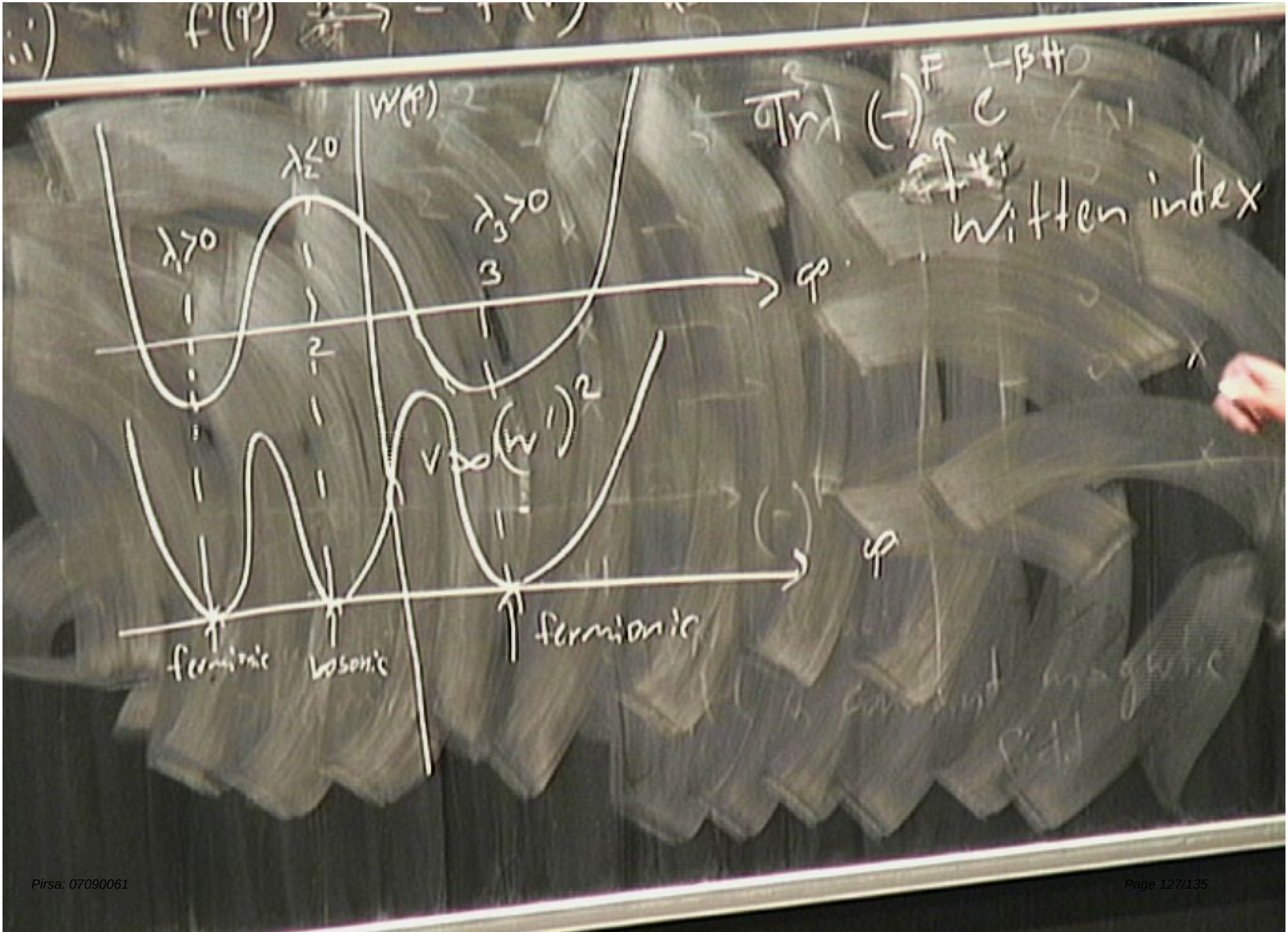
$W(\varphi) \to +\infty \Rightarrow \hat{\omega}_+(\varphi)$ is normalizable nonperturbatively :

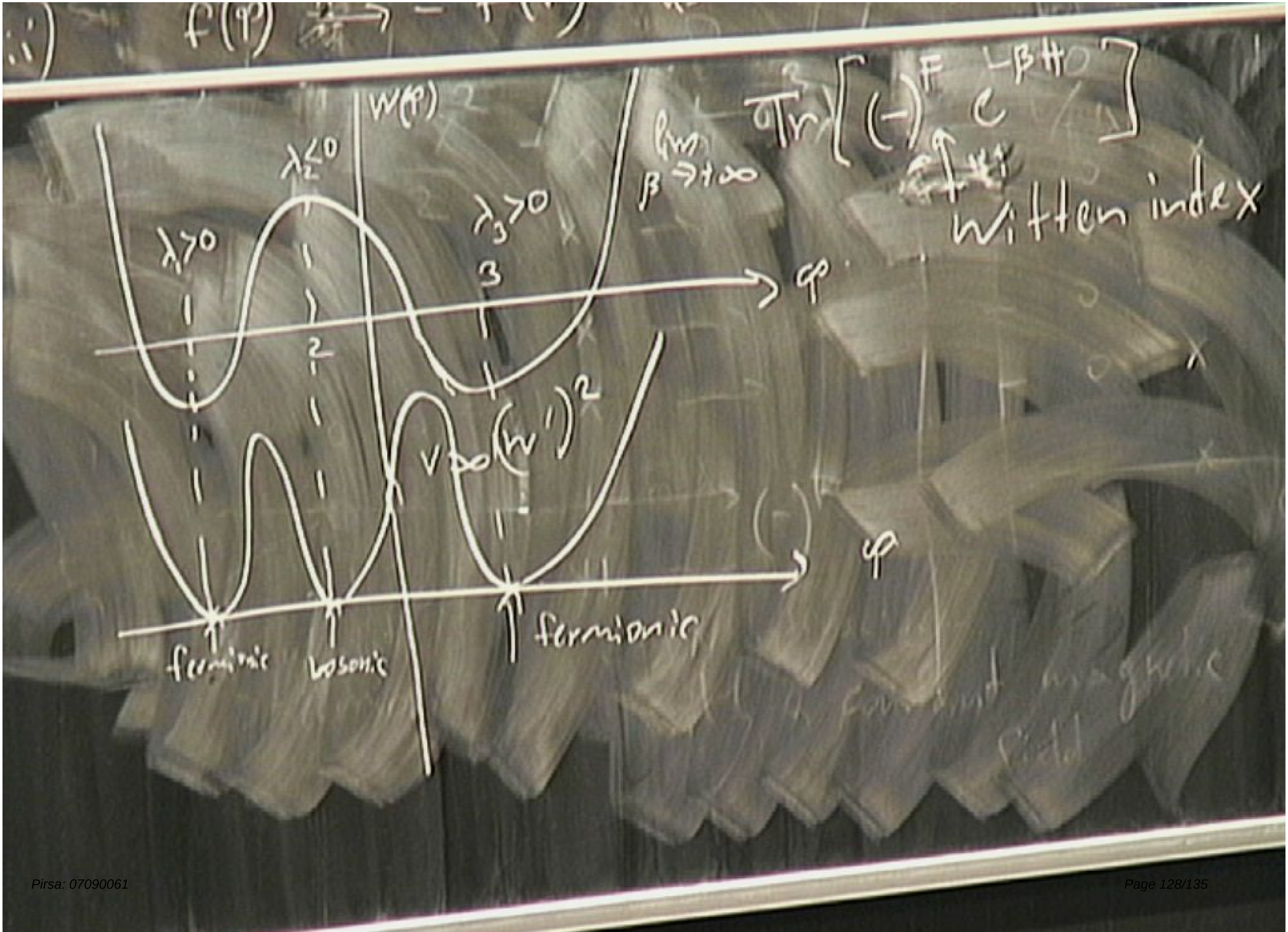
$W(\varphi) \to +\infty \Rightarrow \hat{\omega}_+(\varphi)$ is normalizable

nonperturbatively $\exists$ 1 (fermionic) ground state


W(φ)
λ₁>0
λ₂<0
λ₃>0
1
2
3
φ
Tr (−)^F
V ∝ (W')²
φ
fermionic
bosonic
fermionic

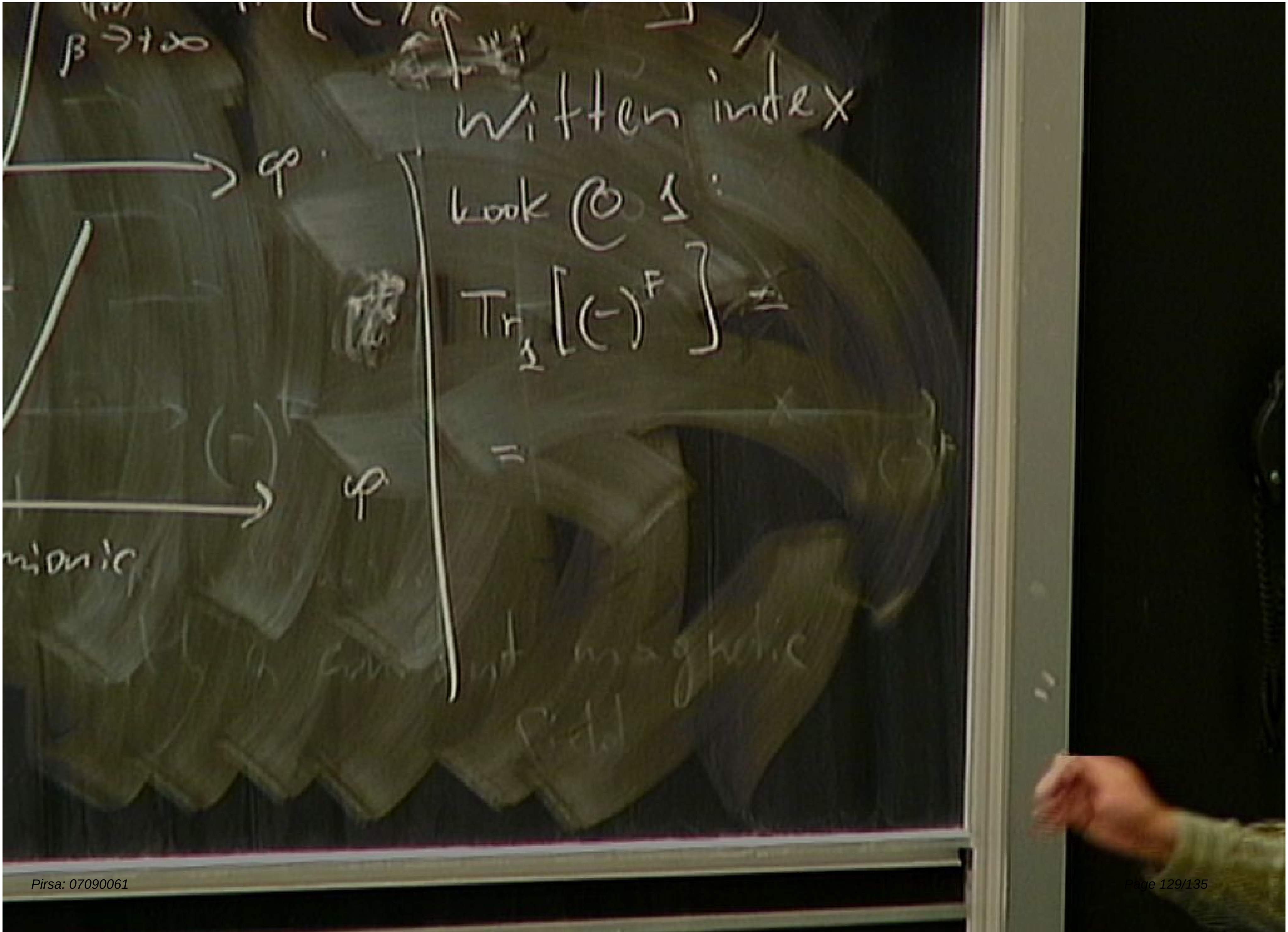

f(φ) → − f(φ)
Tr (−)^F e^{−βH}
W(φ)
λ₂<0
λ₃>0
λ₁>0
3
2
φ
V ∝ (W′)²
fermionic
bosonic
fermionic
φ


W(φ)
λ₂<0
λ₁>0
λ₃>0
1
2
3
φ
V ∝ (W')²
φ
fermionic
bosonic
fermionic
Tr (−)^F e^{−βH}
C
Witten index



$$\lim_{\beta\to+\infty} \mathrm{Tr}\left[(-)^F e^{-\beta H}\right]$$

Witten index



$\beta \to +\infty$

$\to \varphi$

Witten index

Look @ 1 :

$\mathrm{Tr}_1 [(-)^F] =$

$=$

$(-)$

$\to \varphi$

mionic

magnetic field

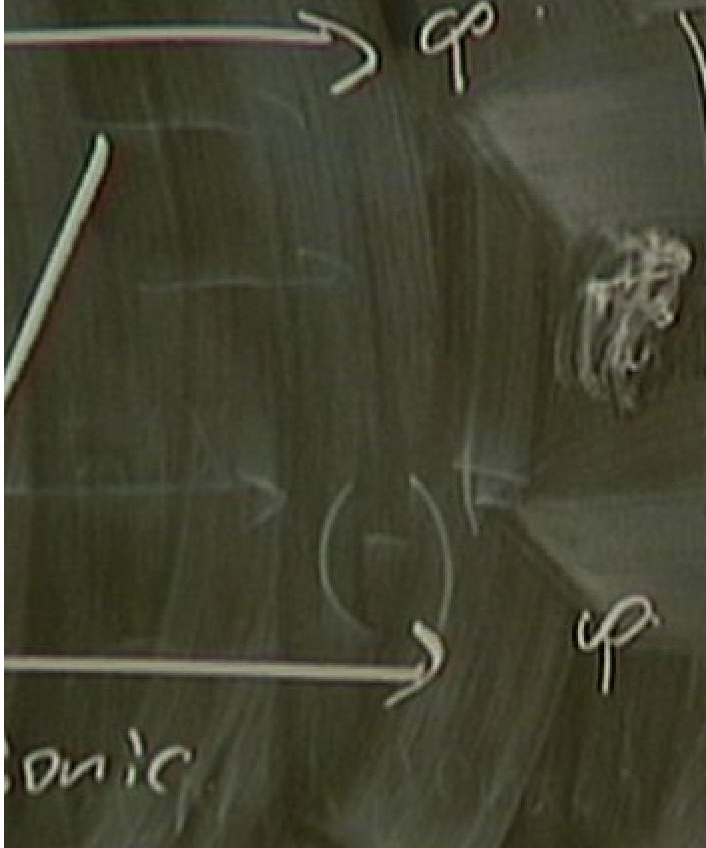
$\beta \to +\infty$

Witten index

look @ 1:

$$\mathrm{Tr}_1\left[(-)^F\right] = (-) + (-1+1) + (-1+1) + \dots$$

$\beta \to +\infty$



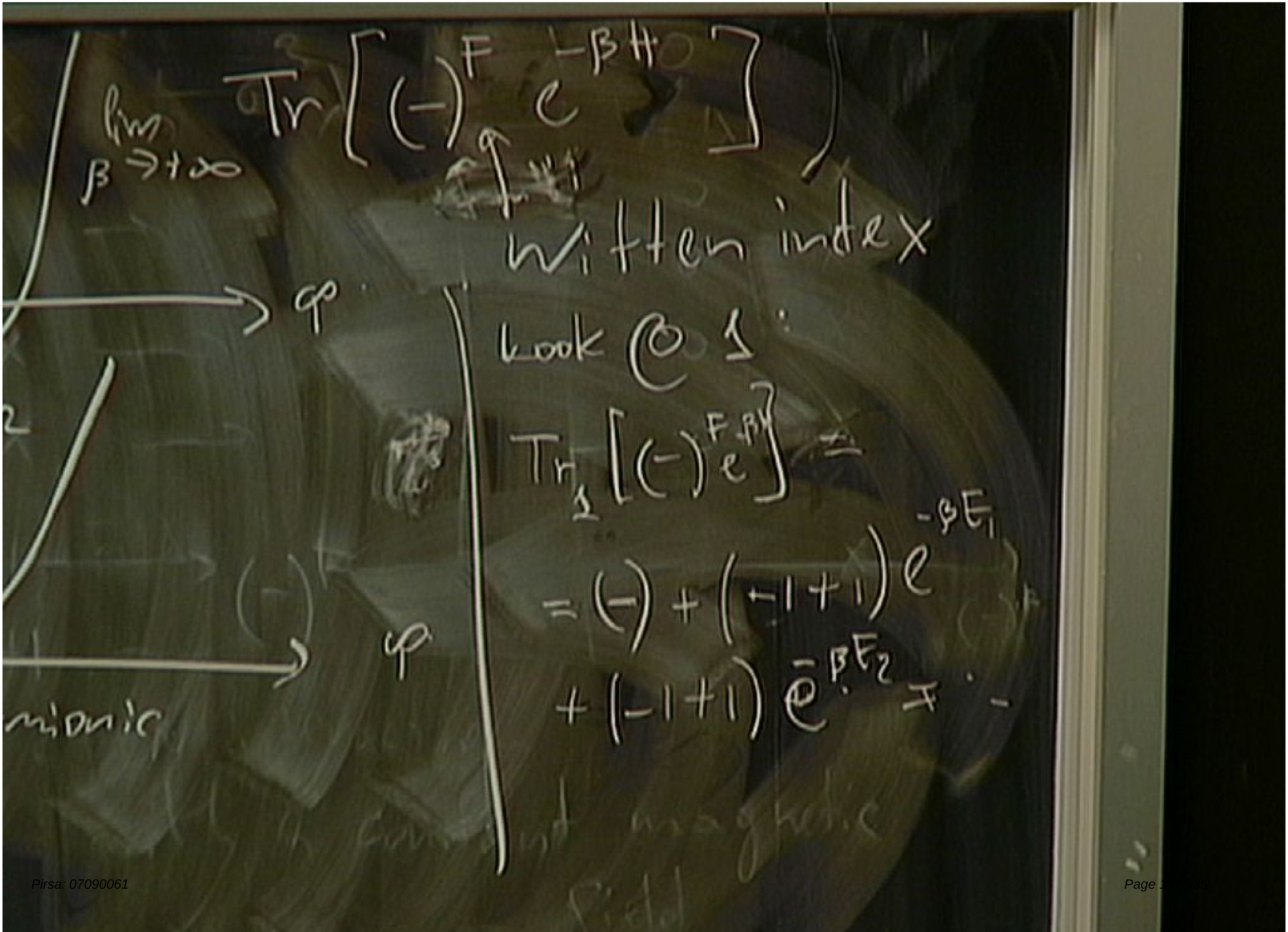
Witten index

Look @ 1 :

$$\mathrm{Tr}_1\left[(-)^F e^{-\beta H}\right] =$$

$$= (-) + (-1+1)e + (-1+1) + \cdots$$

constant magnetic field



$$\lim_{\beta \to +\infty} \mathrm{Tr}\left[(-)^F e^{-\beta H} \right] \Big)$$

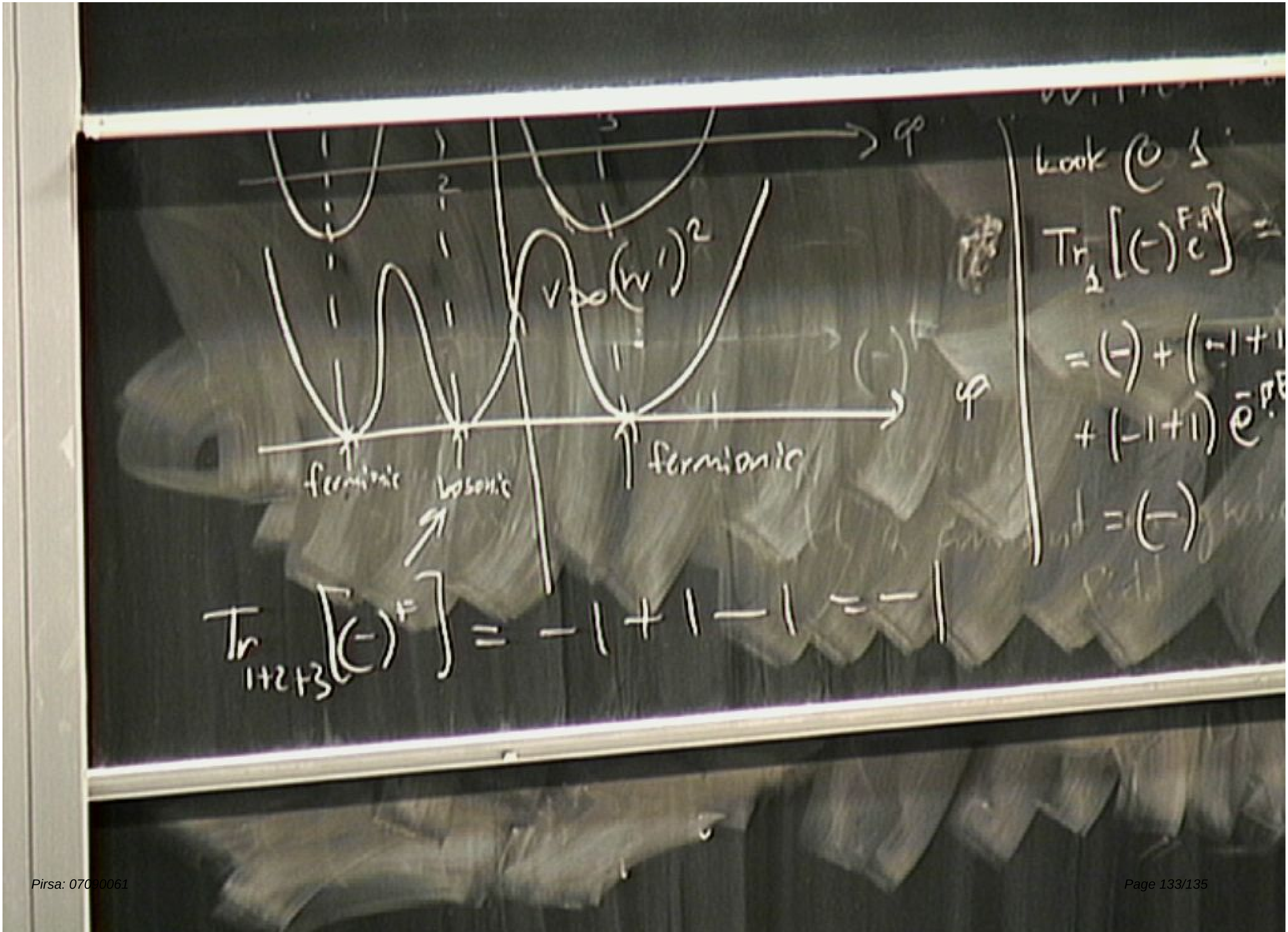
Witten index

Look @ 1 :

$$\mathrm{Tr}_1 \left[(-)^F e^{-\beta H} \right] =$$

$$= (-) + (-1+1) e^{-\beta E_1}$$

$$+ (-1+1) e^{-\beta E_2} + \cdots$$



Look @ 1

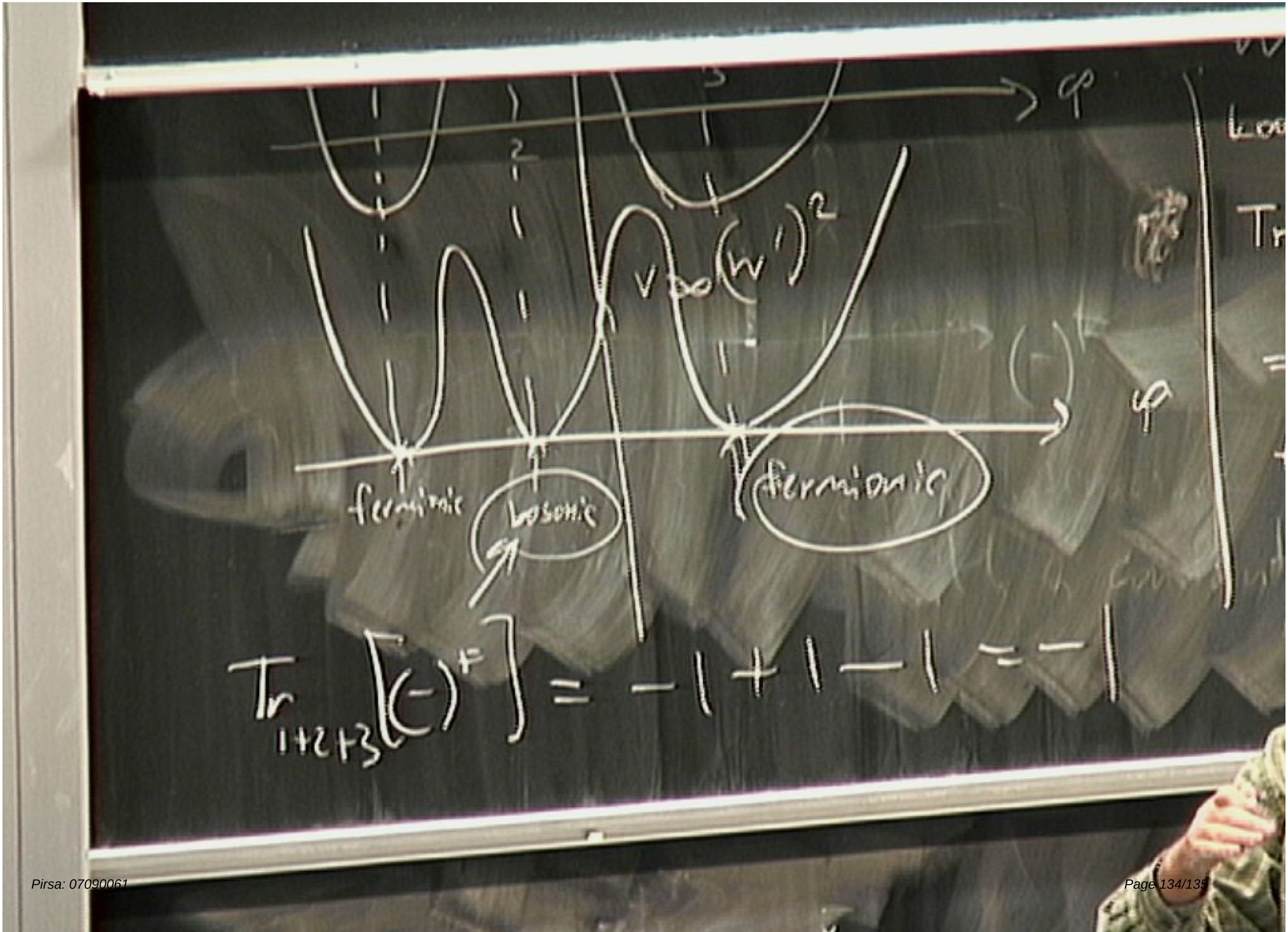
$\mathrm{Tr}_1 \left[(-)^{F+n}_{c} \right] =$

$= (-) + (-1+1$

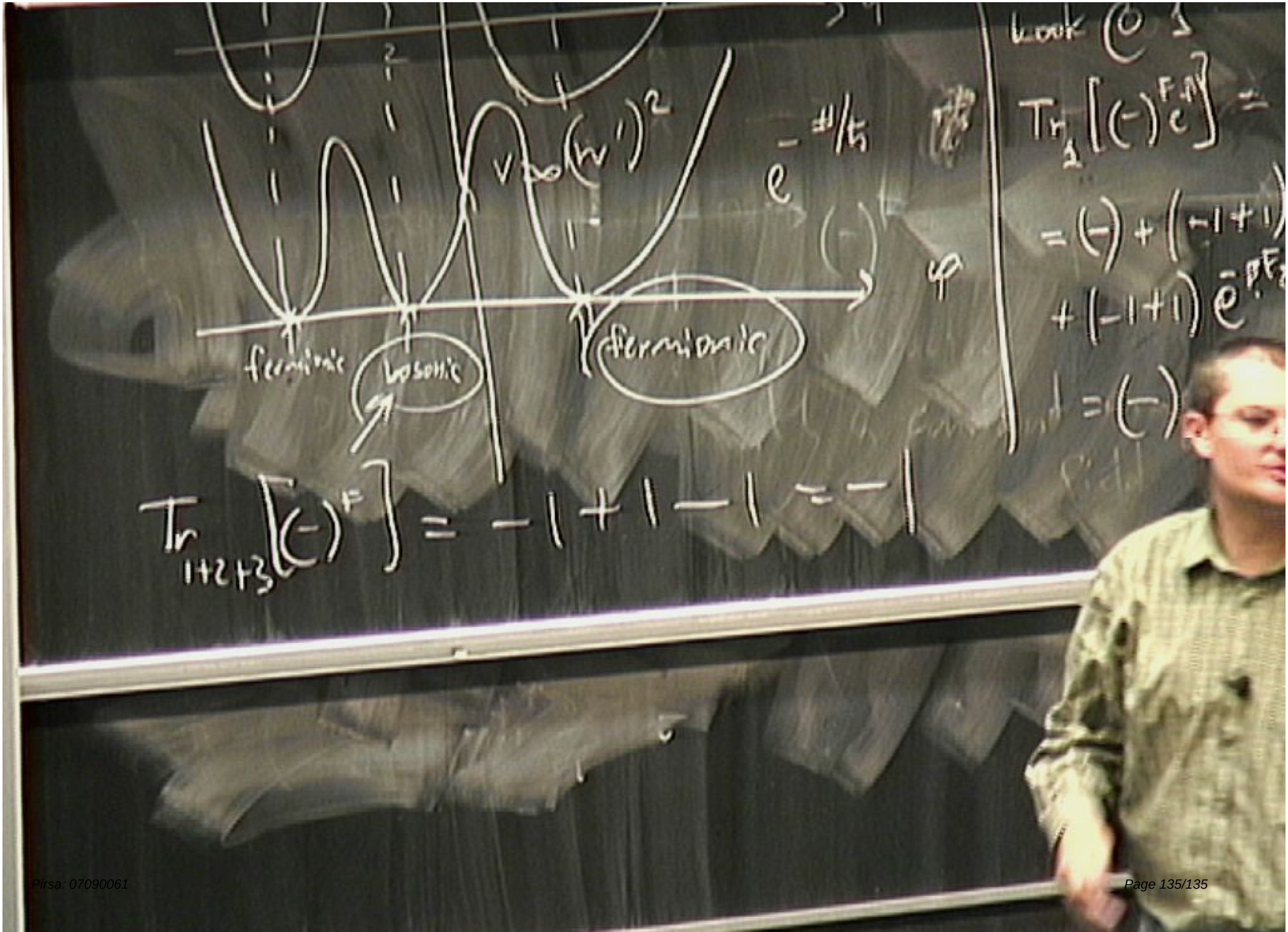
$+ (-1+1)\, e^{-\beta E}$

$= (-)$

$$\mathrm{Tr}_{1+2+3} \left[(-)^F \right] = -1 + 1 - 1 = -1$$



$$\mathrm{Tr}_{1+2+3}\left[(-)^F\right] = -1+1-1 = -1$$



look @ 1

$$\mathrm{Tr}_1\left[(-)^F e^{-\beta H}\right] =$$

$$= (-) + (-1+1)$$

$$+ (-1+1)\, e^{-\beta E_2}$$

$$1 = (-)$$

$$\mathrm{Tr}_{1+2+3}\left[(-)^F\right] = -1+1-1 = -1$$