Title: 3d quantum gravity and deformed special relativity

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Abstract: In 3d quantum gravity, Planck's constant, the Planck length and the cosmological constant control the lack of (co)-commutativity of quantities like angular momenta, momenta and postion coordinates. I will explain this statement, using the quantum groups which arise in the 3d quantum gravity but avoiding technical details. The non-commutative structures in 3d quantum gravity are quite different from those in the deformed version of special relativity desribed by the kappa-Poincare group, but can be related to the latter by an operation called semi-dualisation. I will explain this operation, and make comments on its possible physical significance. The talk is based on joint work with Shahn Majid.

2+1 dimensional gravity and deformed special relativity

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Motivation

- 2+1 gravity can be quantised!
- Non-commutative geometry?
- Deformed special relativity as a remnant of quantum gravity?

Physical constants in 3d quantum gravity

c, $\quad G$ (inverse mass), $\quad \Lambda_c$ (length^{-2}), $\quad \hbar$

Can form two length parameters

$$\ell_P = \hbar G \qquad \ell_C = \frac{1}{\sqrt{|\Lambda_c|}}$$

Overview

1. The Poincaré group and the free relativistic particle
2. Three-dimensional gravity as Poincaré gauge theory
3. Quantisation and quantum groups
4. The relation with the κ-Poincaré algebra

Previous work with Catherine Meusburger +
current work with Shahn Majid

Special relativity in 3d

- Minkowski space, metric $\eta_{ab} = (+, -, -)$, $a = 0, 1, 2$.
- Lorentz goup: $SO(2,1) = SL(2,\mathbb{R})/\mathbb{Z}_2 = SU(1,1)/\mathbb{Z}_2$
- Lie algebra $su(1,1) = so(2,1)$: $[J_a, J_b] = \epsilon_{abc} J^c$

J_3 generates rotatation, J_1, J_2 generate boosts.

The Poincaré group and its Lie algebra

The Poincaré group $P_3 = SU(1,1) \ltimes \mathbb{R}^3$:

$$(v_1, \boldsymbol{x}_1)(v_2, \boldsymbol{x}_2) = (v_1 v_2, \boldsymbol{x}_1 + \mathsf{Ad}(v_1)\boldsymbol{x}_2)$$

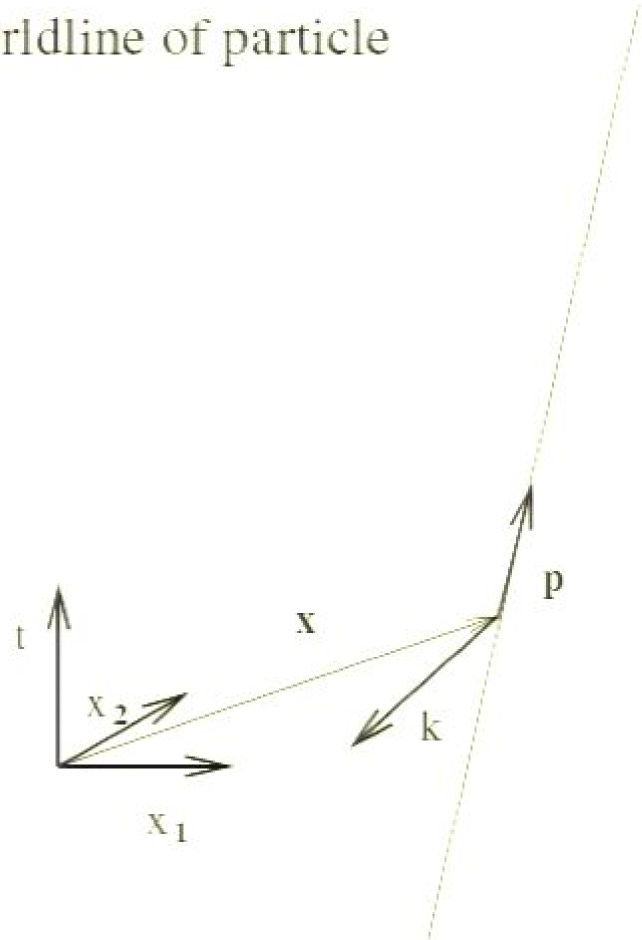
Lie algebra is

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c, \quad [P_a, P_b] = 0$$

with invariant inner product

$$\langle J_a, P_b \rangle = \eta_{ab}$$

Worldline of particle



$k = x \wedge p + s\,\hat{p}$

$p \cdot p = m^2$

$k \cdot p = m\,s$

Phase space of point particle=(Co)adjoint orbit

With $g = (v, \boldsymbol{x}) \in P_3$, and $P_0^* = J_0, J_0^* = P_0$, have orbit parametrisation

$$g(mJ_0 + sP_0)g^{-1} = p_a J^a + k_a P^a$$

Canonical Poisson brackets (dimensionful!)

$$\{k_a, k_b\} = -\epsilon_{abc} k^c, \quad \{k_a, p_b\} = -\epsilon_{abc} p^c, \quad \{p_a, p_b\} = 0$$

Re-write point particle action as

$$I_{\text{Point Particle}} = \int d\tau \, p_a \dot{x}^a + s\langle P_0, v^{-1}\dot{v}\rangle = \int d\tau \, \langle mJ_0 + sP_0, g^{-1}\dot{g}\rangle$$

2+1 gravity as a Poincaré gauge theory

Combine dreibein e_a and spin connection $\omega = \omega_a J^a$

$$A = e_a P^a + \omega_a J^a$$

Then

$$I_{\text{Einstein-Hilbert}} = \frac{1}{8\pi G} \int_{M_3} \langle A \wedge dA \rangle + \frac{1}{3} \langle [A, A], \wedge A \rangle$$

Equation of motion

$$F_A = 0 \Leftrightarrow R_\omega = 0 \quad \text{and} \quad T = 0$$

Introducing point particles

$$M_3 = S_{g.n} \times \mathbb{R}$$

Mark points on $S_{g.n}$ with coadjoint orbits and couple minimally:

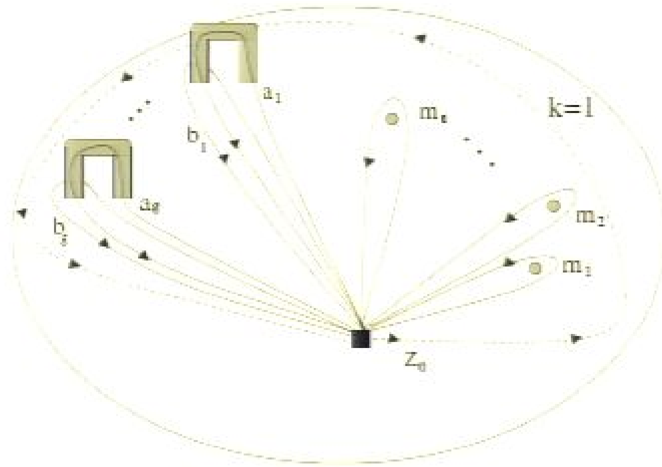
$$I_{\text{Point Particle}} = \int d\tau \, \langle mJ_3 + sP_3, g^{-1}\left(\frac{d}{d\tau} + A\right) g\rangle$$

New equation of motion ($\mu = 8\pi mG$)

$$F_A = -g(\mu J_0 + sP_0)g^{-1} \, dz \wedge d\bar{z} \, \delta^2(z - z_i)$$

Holonomy around puncture $\in \mathcal{C}_{\mu.s} := \{ge^{-\mu J_3 - sP_3}g^{-1} | g \in P_3\}$

Holonomies and phase space



Extended phase space: $\tilde{\mathcal{P}} = P_3^{2g} \times \mathcal{C}_{\mu_n s_n} \times \ldots \mathcal{C}_{\mu_1 s_1}$

Phase space:

$$\begin{aligned} \mathcal{P} \quad = & \{(A_g, B_g, \ldots, A_1, B_1, M_n, \ldots M_1) \in \tilde{\mathcal{P}} | \\ & [A_g, B_g^{-1}] \ldots [A_1, B_1^{-1}] M_n \ldots M_1 = 1\} / \text{conjugation}. \end{aligned}$$

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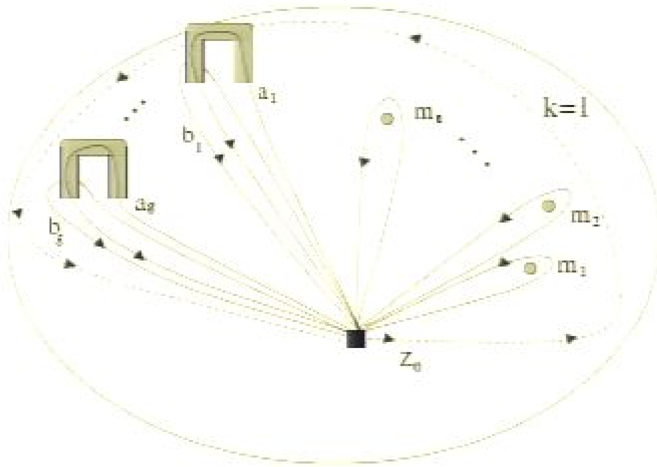

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P_3 as Poisson-Lie group

- P_3 is Poisson-Lie group: $(v, \boldsymbol{x}) \in P_3$ then

$$\{x_a, x_b\} = G\epsilon_{abc}x^c, \quad \{x_a, f(v)\} = \{f(v), g(v)\} = 0.$$

- Dual Poisson-Lie group $P_3^* = SU(1,1) \times \mathbb{R}^3$. Write elements $(u, -\boldsymbol{j})$, with $u = \exp(-8\pi G p_a J_a)$:

$$\{j_a, j_b\} = -\epsilon_{abc}j^c, \quad \{j_a, p_b\} = -\epsilon_{abc}p^c, \quad \{p_a, p_b\} = 0.$$

Conjugacy class as particle phase space

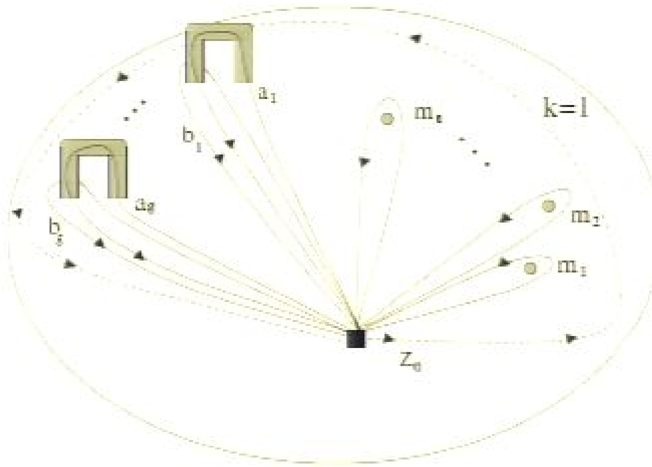
Symplectic leaves of P_3^* are P_3-conjugacy classes $\mathcal{C}_{\mu s}$:

$$(u, -\mathrm{Ad}(u)\boldsymbol{j}) = (v, \boldsymbol{x})e^{-\mu J_3 - sP_3}(v, \boldsymbol{x})^{-1}$$

$$\Rightarrow u = ve^{-\mu J_3}v^{-1} = e^{-8\pi G p_a J^a}$$

$$\boldsymbol{j} = (1 - \mathrm{Ad}(u^{-1}))\boldsymbol{x} + s\hat{p}_a P_a \approx [\boldsymbol{x}, p_a J^a] + s\hat{p}_a P_a$$

Holonomies and phase space



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Symplectic geometry of phase space

- Momentum space is the group manifold of $SU(1,1)$, or anti-de Sitter space: $\{(\boldsymbol{\pi}, \pi_3) \in \mathbb{R}^4 | \pi_0^2 - \pi_1^2 - \pi_2^2 + \pi_3^2 = \frac{1}{G^2}\}$
- Position coordinates x_a act by infinitesimal $SU(1,1)$ left-action
- Angular momentum coordinates j_a act by infinitesimal $SU(1,1)$ conjugation

LESSON 1

- Momentum space has curvature radius $\propto \frac{1}{G}$
- Position coordinates do not Poisson commute $\propto G$
- Angular momentum Poisson algebra is unchanged - but relation to position and momentum is changed

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Introducing the cosmological constant

Cosmological constant	Euclidean signature	Minkowskian signature
$\Lambda_c = 0$	E_3	P_3
$\Lambda_c > 0$	$SO(4) \simeq \frac{SU(2)\times SU(2)}{\mathbb{Z}_2}$	$SO(3,1) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$
$\Lambda_c < 0$	$SO(3,1) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$	$SO(2,2) \simeq \frac{SU(1,1)\times SU(1,1)}{\mathbb{Z}_2}$

The Lie algebra

Let $\Lambda = \begin{cases} \Lambda_c & \text{for Euclidean signature} \\ -\Lambda_c & \text{for Minkowskian signature} \end{cases}$

Lie algebra has Cartan decomposition

$$[J_a, J_b] = \epsilon_{abc} J^c, \quad [J_a, P_b] = \epsilon_{abc} P^c \quad [P_a, P_b] = \Lambda \epsilon_{abc} J^c$$

and invariant pairing

$$\langle P_a, J^b \rangle = \delta_a{}^b.$$

Iwasawa decomposition $\tilde{P}_a = P_a + \epsilon_{abc} n^b J^c$, with $\boldsymbol{n}^2 = -\Lambda$:

$$[\tilde{P}_a, \tilde{P}_b] = n_a \tilde{P}_b - n_b \tilde{P}_a.$$

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$$[\tilde{P}_a, \tilde{P}_b] = n_a\tilde{P}_b - n_b\tilde{P}_a.$$

LESSON 2

- If $\Lambda \neq 0$ position space has curvature radius $\propto \ell_c$
- If $\Lambda \neq 0$ momenta don't Poisson commute $\propto \frac{1}{\ell_c}$

Quantisation

Recall: quantisation of free point-particle (co-adjoint orbit) brackets

$$\{k_a, k_b\} = -\epsilon_{abc}k^c, \quad \{k_a, p_b\} = -\epsilon_{abc}p^c, \quad \{p_a, p_b\} = 0$$

is the universal enveloping algebra $U(su(1,1)) \ltimes \mathbb{R}^3)$

$$[J_a, J_b] = \hbar\epsilon_{abc}J^c, \qquad [J_a, P_b] = \hbar\epsilon_{abc}P^c, \qquad [P_a, P_b] = 0$$

Can view this as

$$U(su(1,1)) \ltimes \mathbb{C}((\mathbb{R}^*)^3)$$

On the algebra of momenta

- The group of space-time translations $\boldsymbol{x} = x^a P_a$ is $\mathbb{R}^3$
- The group algebra of $\mathbb{R}^3$ is $\mathbb{C}(\mathbb{R}^3)$ with convolution product:

$$\delta_{\boldsymbol{x}} * \delta_{\boldsymbol{y}} = \delta_{\boldsymbol{x}+\boldsymbol{y}}$$

- Fourier transform to get $\mathbb{C}((\mathbb{R}^*)^3)$, with point-wise multiplication:

$$e^{i\boldsymbol{x}\boldsymbol{p}} e^{i\boldsymbol{y}\boldsymbol{p}} = e^{i(\boldsymbol{x}+\boldsymbol{y})\boldsymbol{p}}$$

- Momenta $p_a \in \mathbb{C}((\mathbb{R}^*)^3)$ are coordinate functions

The Poincaré group revisited

Can view Poincaré group algebra as transformation group algebra

$$SU(1,1) \ltimes \mathbb{C}((\mathbb{R}^*)^3)$$

with multiplication

$$(v_1 \otimes f_1 \bullet v_2 \otimes f_2)(\boldsymbol{p}) = v_1 v_2 \otimes f_1(\boldsymbol{p}) f_2(\mathrm{Ad}^{-1}(v_1)(\boldsymbol{p}))$$

Infinitesimal version is

$$U(su(1,1)) \ltimes \mathbb{C}((\mathbb{R}^*)^3))$$

A remark on the co-algebra of momenta

- Co-product for Hopf algebras $\Delta : A \to A \otimes A$ defines action on tensor products of representations
- For any manifold M, have $\mathbb{C}(M) \otimes \mathbb{C}(M) \simeq \mathbb{C}(M \times M)$
- For any group G, $\Delta : \mathbb{C}(G) \to \mathbb{C}(G \times G)$, $\Delta f(g, h) = f(gh)$ is co-product.
- For $G = (\mathbb{R}^*)^3$ get $\Delta p_a(\boldsymbol{p}, \boldsymbol{q}) = p_a + q_a$, so

$$\Delta(p_a) = p_a \otimes 1 + 1 \otimes p_a$$

The Lorentz double

Quantisation of "puncture brackets"

$$\{j_a, j_b\} = -\epsilon_{abc} j^c, \quad \{j_a, p_b\} = -\epsilon_{abc} p^c, \quad \{p_a, p_b\} = 0.$$

$(u = \exp(-8\pi G p_a J^a))$ is quantum double

$$U(su(1,1)) \ltimes \mathbb{C}(SU(1,1)) = D(U(su(1,1))$$

Co-multiplication $\Delta : \mathbb{C}(SU(1,1)) \to \mathbb{C}(SU(1,1) \times SU(1,1))$

$\Rightarrow$ non-commutative momentum addition.

LESSON 3

- $[J_a, J_b] = \hbar \ \epsilon_{abc} J^c$:

  Angular momentum coordinates don't commute $\propto \hbar$

- $\Delta(p_a) = 1 \otimes p_a + p_a \otimes 1 + G \ \epsilon_{abc} p^b \otimes p^c + \dots$

  Momenta don't co-commute $\propto G$

- $[X_a, X_b] = l_P \ \epsilon_{abc} X^c$

  Position coordinates don't commute $\propto l_P$

Quantum groups: $q = e^{-\hbar G\sqrt{-\Lambda}} = \text{``} e^{-\frac{\ell_P}{\ell_C}} \text{''}$

Cosmological constant	Euclidean signature	Minkowskian signature
$\lambda = 0$	$D(U(su(2))$	$D(U(su(1,1))$
$\lambda > 0$	$D(U_q(su(2)))$, q root of unity	$D(U_q(su(1,1)))$ $q \in \mathbb{R}$
$\lambda < 0$	$D(U_q(su(2)))$, $q \in \mathbb{R}$	$D(U_q(su(1,1)))$, $q \in U(1)$

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LESSON 4

	Commutator	Co-commutator
Angular momentum	$\hbar$	$\frac{G}{\ell_c}$
Momentum	$\frac{\hbar}{\ell_c}$	G
Position	$\hbar G$	$\frac{1}{\ell_c}$

κ-Poincaré

Idea: curved momentum space on which Lorentz group acts, e.g.

$$\mathrm{dS} = \{(\boldsymbol{\pi}, \pi_3) \in \mathbb{R}^4 | -\pi_0^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = \kappa^2\}$$

Define transformation group algebra $SU(1,1) \ltimes \mathbb{C}(\mathrm{dS})$:

$$(v_1 \otimes f_1 \bullet v_2 \otimes f_2)(\pi) = v_1 v_2 \otimes f_1(\pi) f_2(v_1^{-1}.\pi), \quad v_i \in SU(1,1), \pi \in \mathrm{dS}$$

Infinitesimal version:

$$U(su(1,1)) \ltimes \mathbb{C}(\mathrm{dS}).$$

Iwasawa decomposition of $SO(3,1)$

Need (local) factorisation trick:

$$g \in SL(2,\mathbb{C}) \Rightarrow g = u \cdot s = r \cdot v, \qquad u, v \in SU(1,1), \quad r, s \in AN.$$

where $AN \simeq \mathbb{R} \ltimes \mathbb{R}^2$ is group of matrices of form

$$t = \begin{pmatrix} e^{-\frac{p_0}{\kappa}} & \frac{p_1}{\kappa} + i\frac{p_2}{\kappa} \\ 0 & e^{\frac{p_0}{\kappa}} \end{pmatrix}.$$

Factorisation of $g \in SL(2,\mathbb{C})$ gives rise to dressing action

$$u = L_r(v), \qquad s = R_v(r)$$

(Half of) de Sitter space as a group

Realise dS as subspace of Hermitian 2×2 matrices:

$$\mathrm{dS} = \{\pi_0 + \pi_1\sigma_1 + \pi_2\sigma_2 + \pi_3\sigma_3 | -\pi_0^2 + \pi_1^2 + \pi_2^2 + \pi_3^2 = \kappa^2\}$$

Can define map

$$S : AN \to \mathrm{dS}, \quad t \mapsto \kappa t \sigma_3 t^\dagger$$

whose image is "half of dS".

Bicross product structure of κ-Poincaré (Majid)

Define $P_\kappa = SU(1,1) \bowtie \mathbb{C}(AN)$ with multiplication

$$(v_1 \otimes f_1 \bullet v_2 \otimes f_2)(t) = v_1 v_2 \otimes f_1(t) f_2(R_{v_1}(t))$$

and co-product

$$\Delta(v \otimes f)(t_1, t_2) = L_{t_2}(v) \otimes v \otimes f(t_1 t_2).$$

Non-commutative momentum addition

$$\Delta(p_i) = p_i \otimes 1 + e^{-\frac{p_0}{\kappa}} \otimes p_i$$

and "twisted" angular momentum addition.

The idea of semi-dualisation

Avoid global issues by taking infinitesimal version

$$P_\kappa = U(su(1,1)) \triangleright\!\!\blacktriangleleft \mathbb{C}(AN)$$

Note $U(sl(2,\mathbb{C})) \simeq U(su(1,1)) \bowtie U(an)$ and duality of Hopf algebras

$$U(an) - \mathbb{C}(AN).$$

Then define **semidualisation** (Majid)

$$U(sl(2,\mathbb{C})) \stackrel{S}{-} P_\kappa$$

Relation with 2+1 gravity?

A purely quantum phenomenon ($q \in \mathbb{R}, q \neq 1$):

$$\mathbb{C}_q(SU(1,1)) \simeq U_q(an)$$

and its classical limit:

$$U_q(su(1,1)) \bowtie \mathbb{C}_q(SU(1,1)) \overset{q \neq 1}{\simeq} U_q(sl(2,\mathbb{C})) \overset{S}{\longmapsto} U_q(su(1,1)) \blacktriangleright\!\!\lhd \mathbb{C}_q(AN)$$

$$\downarrow q \to 1 \qquad\qquad\qquad \downarrow q \to 1$$

$$D(U(su(1,1))) \qquad\qquad\qquad P_\kappa$$

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LESSON 5

- In 2+1 gravity momentum space is EITHER Euclidean and postively curved (three-sphere) OR Lorentzian and negatively curved (anti-de-Sitter). Position algebra is
$[X_a, X_b] = \ell_P \, \epsilon_{abc} X^c$.

- In standard bicrossproduct construction of κ-Poincaré, momentum space is Lorentzian and positively curved (de-Sitter). Position algebra is $[X_0, X_i] = \ell_P X_i$.

- Lorentz double and κ-Poincaré are different Hopf algebras arising as $q \to 0$ limits of semidual Hopf-algebras .

Relation with 2+1 gravity?

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$$\mathbb{C}_q(SU(1,1)) \simeq U_q(an)$$

and its classical limit:

$$U_q(su(1,1)) \bowtie \mathbb{C}_q(SU(1,1)) \quad \overset{q \neq 1}{\simeq} \quad U_q(sl(2,\mathbb{C})) \quad \overset{S}{\longmapsto} \quad U_q(su(1,1)) \blacktriangleright\!\!\!\triangleleft \mathbb{C}_q(AN)$$

$$\downarrow q \to 1 \qquad\qquad\qquad\qquad \downarrow q \to 1$$

$$D(U(su(1,1))) \qquad\qquad\qquad\qquad P_\kappa$$

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LESSON 5

- In 2+1 gravity momentum space is EITHER Euclidean and postively curved (three-sphere) OR Lorentzian and negatively curved (anti-de-Sitter). Position algebra is $[X_a, X_b] = \ell_P \, \epsilon_{abc} X^c$.

- In standard bicrossproduct construction of κ-Poincaré, momentum space is Lorentzian and positively curved (de-Sitter). Position algebra is $[X_0, X_i] = \ell_P X_i$.

- Lorentz double and κ-Poincaré are different Hopf algebras arising as $q \to 0$ limits of semidual Hopf-algebras .

Outlook

Workshop on Non-commutative Deformations of Special Relativity

July 2008 at ICMS, Edinburgh

Organised by Giovanni Amelino-Camelia, Shahn Majid, Jerzy Kowalski-Glikman, Bernd Schroers

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The idea of semi-dualisation

Avoid global issues by taking infinitesimal version

$$P_\kappa = U(su(1,1)) \blacktriangleright\!\!\triangleleft \mathbb{C}(AN)$$

Note $U(sl(2,\mathbb{C})) \simeq U(su(1,1)) \bowtie U(an)$ and duality of Hopf algebras

$$U(an) - \mathbb{C}(AN).$$

Then define **semidualisation** (Majid)

$$U(sl(2,\mathbb{C})) \stackrel{S}{-} P_\kappa$$

Bicross product structure of κ-Poincaré (Majid)

Define $P_\kappa = SU(1,1) \bowtie \mathbb{C}(AN)$ with multiplication

$$(v_1 \otimes f_1 \bullet v_2 \otimes f_2)(t) = v_1 v_2 \otimes f_1(t) f_2(R_{v_1}(t))$$

and co-product

$$\Delta(v \otimes f)(t_1, t_2) = L_{t_2}(v) \otimes v \otimes f(t_1 t_2).$$

Non-commutative momentum addition

$$\Delta(p_i) = p_i \otimes 1 + e^{-\frac{p_0}{\kappa}} \otimes p_i$$

and "twisted" angular momentum addition.

The idea of semi-dualisation

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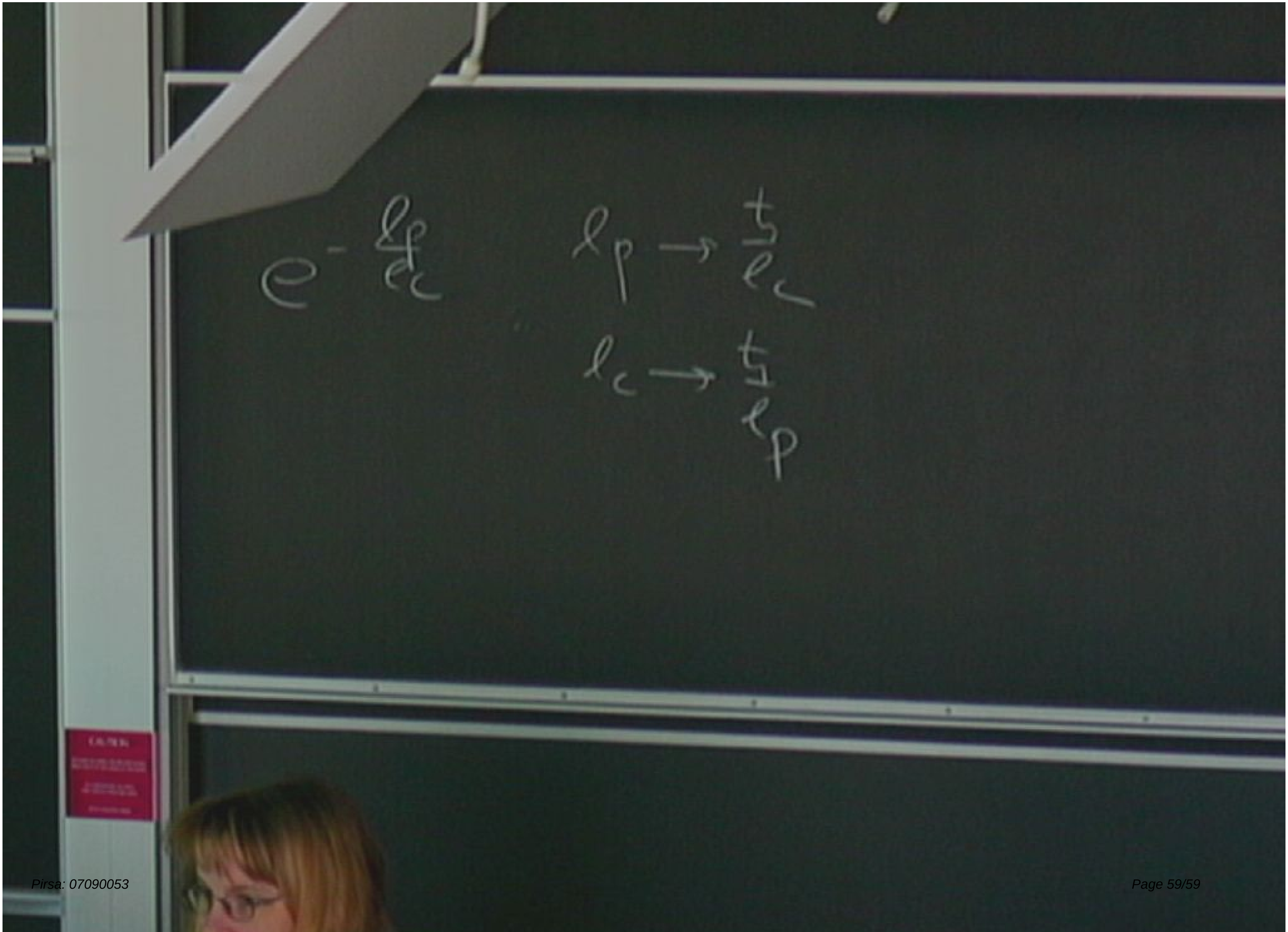
Then define **semidualisation** (Majid)

$$U(sl(2,\mathbb{C})) \xrightarrow{S} P_\kappa$$


$e^{-\frac{l_p}{l_c}}$
$l_p \rightarrow \frac{\hbar}{l_c}$
$l_c \rightarrow \frac{\hbar}{l_p}$