Title: Heterotic Twistor-String Theory

Date: Sep 25, 2007 11:00 AM

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Abstract: I'll discuss a reformulation of twistor-string theory as a heterotic string. This clarifies why conformal supergravity arises and provides a link between the Berkovits and Witten pictures. The talk is based on
arXiv:0708:2276 with Lionel Mason.

Outline

Twistor background
- Brief review of twistor correspondence
- Successes & failures

(0,2) Basics
- Fields & action
- Vertex operators
- Anomalies

Heterotic String Theory
- Coupling to YM
- Amplitudes

Beyond perturbation theory
- Generalized geometry

Relation to other twistor-string models
- Berkovits
- Witten

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The twistor programme

The twistor correspondence relates *points* of complexified spacetime M to holomorphic *lines* in twistor space $\mathbb{PT}$. In flat space

- $M = \mathbb{C}^4$ with coordinates $x^\mu = x^{a\dot{a}}$
- $\mathbb{T} = \mathbb{C}^4$ with coordinates $Z^\alpha = (\omega^a, \pi_{\dot{a}})$
- Incidence relation $\omega^a = \mathrm{i}x^{a\dot{a}}\pi_{\dot{a}}$

$(\omega^a, \pi_{\dot{a}})$ only defined upto overall scaling, so

$$\begin{aligned} x \in M \quad &\iff \quad L_x \simeq \mathbb{P}^1 \subset \mathbb{PT} \simeq \mathbb{P}^3 \\ x - y \ \ \text{null} \quad &\iff \quad L_x,\ L_y \ \text{intersect} \\ [\omega^a, \pi_{\dot{a}}] \in \mathbb{PT} \quad &\iff \quad \{x^{a\dot{a}} + \lambda^a \pi^{\dot{a}}\} \subset M \end{aligned}$$

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Penrose's twistor programme seeks to reformulate fundamental physics in twistor space.

Early successes

Penrose transform provides beautiful description of classical fields in terms of free data on twistor space

$$H^1(\mathbb{PT}', \mathcal{O}(-2h-2)) \simeq \left\{ \begin{matrix} \text{soln of wave eqn of massless} \\ \text{free field, helicity } h \end{matrix} \right\}$$

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The Penrose transform is essentially a contour integral.

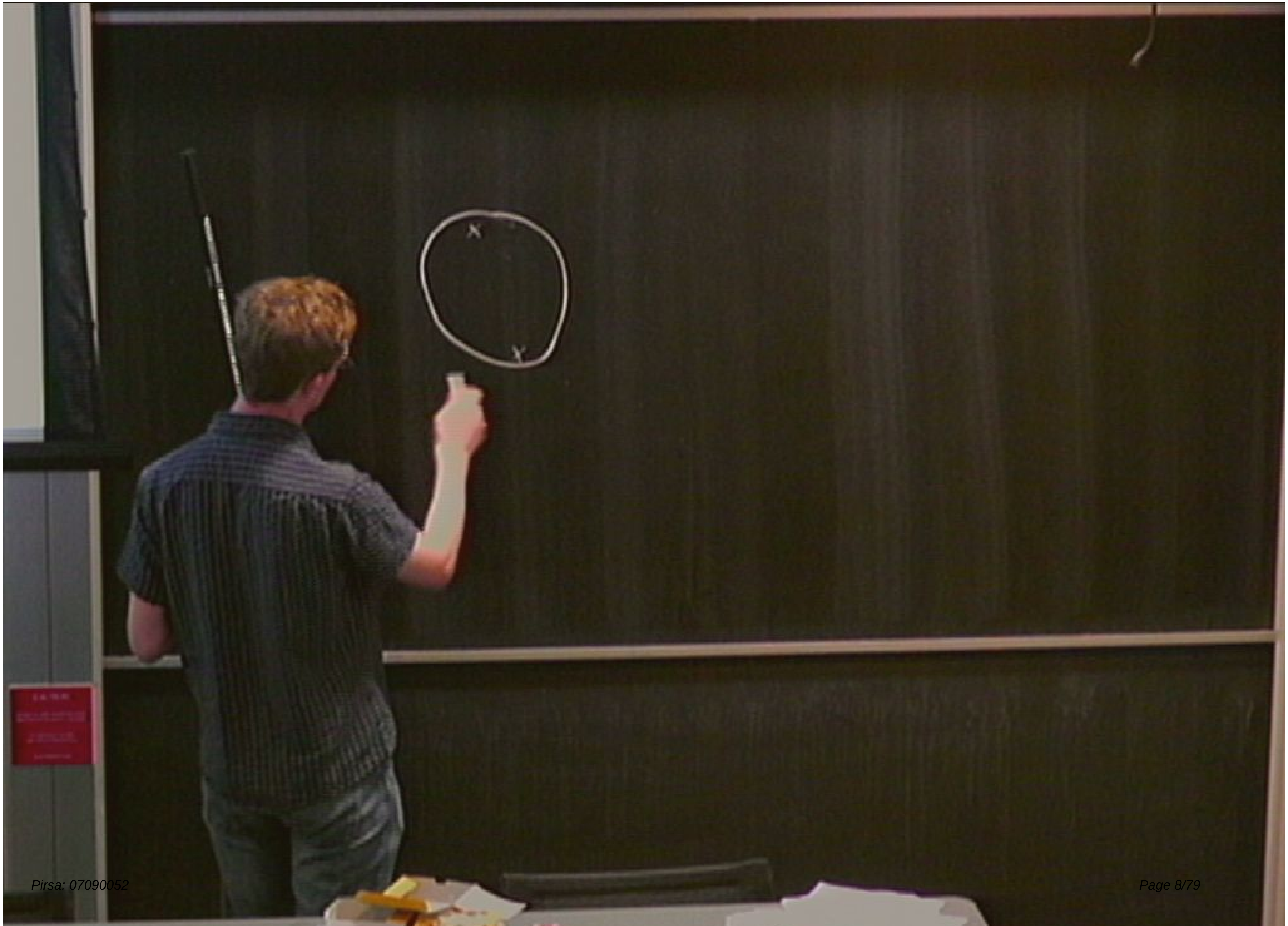
eg

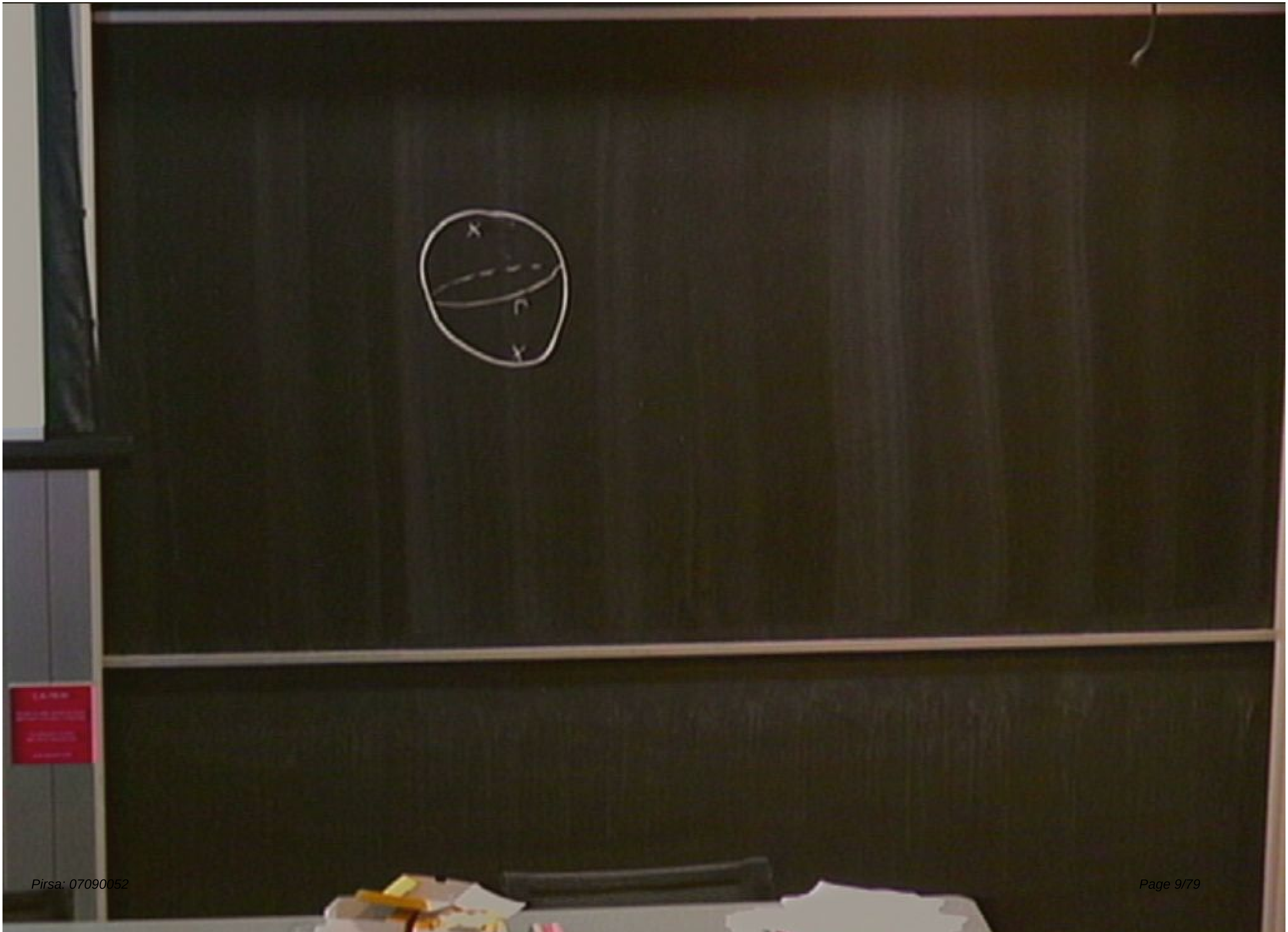
$$\phi(x) = \oint_{\Gamma \subset L_x} f_{-2}(Z)|_{L_x} \, \pi \cdot \mathrm{d}\pi$$

where the contour separates the two singularity regions of $f(Z)$ on L_x. Since $\partial/\partial x^{a\dot{a}}$ acts on $f(Z)|_{L_x}$ as $\mathrm{i}\pi_{\dot{a}}\partial/\partial\omega^a$, we find

$$\Box\phi(x) = -\oint \pi^{\dot{a}}\pi_{\dot{a}} \frac{\partial^2 f_{-2}}{\partial\omega^a\partial\omega_a} \pi \cdot \mathrm{d}\pi = 0$$

so ϕ is automatically on-shell.

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Beyond free theories

The nonlinear graviton & Ward transform extend this to asd solns of Einstein and YM equations. Leads to

- Atiyah-Ward and ADHM construction of YM instantons
- Twistor spaces for conformally flat spaces (*e.g.* Minkowski, AdS, FRW,...), ALE gravitational instantons, pp-waves, *etc.*
- In higher dimensions, $HK \to Z \to QK$ (*relevant e.g. for moduli spaces of 4d extremal black holes in Type II sugras*)
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However, there has never been a systematic way to encode non-asd interactions in twistor space, even perturbatively.

The twistor-string hope

Witten: cohomology classes in Penrose transform $\Leftrightarrow$ states in topological string theories. Once external states are known, worldsheet topology takes care of interactions.

- *For twistor theorists:* twistor-string theory offers a new approach to the main problem of twistor theory
- *For string theorists:* Penrose transform gives new way to connect essentially stringy behaviour with 4d physics, with no α' corrections or KK modes
- *For field theorists:* insight from twistor-string offers new picture of spacetime field theories

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Two original models of twistor-strings: open B-model + D-instantons (Witten) and open string $\beta\gamma$-system (Berkovits). Genus zero, leading-trace amplitudes of either equivalent to spacetime $\mathcal{N} = 4$ SYM at tree level. However...

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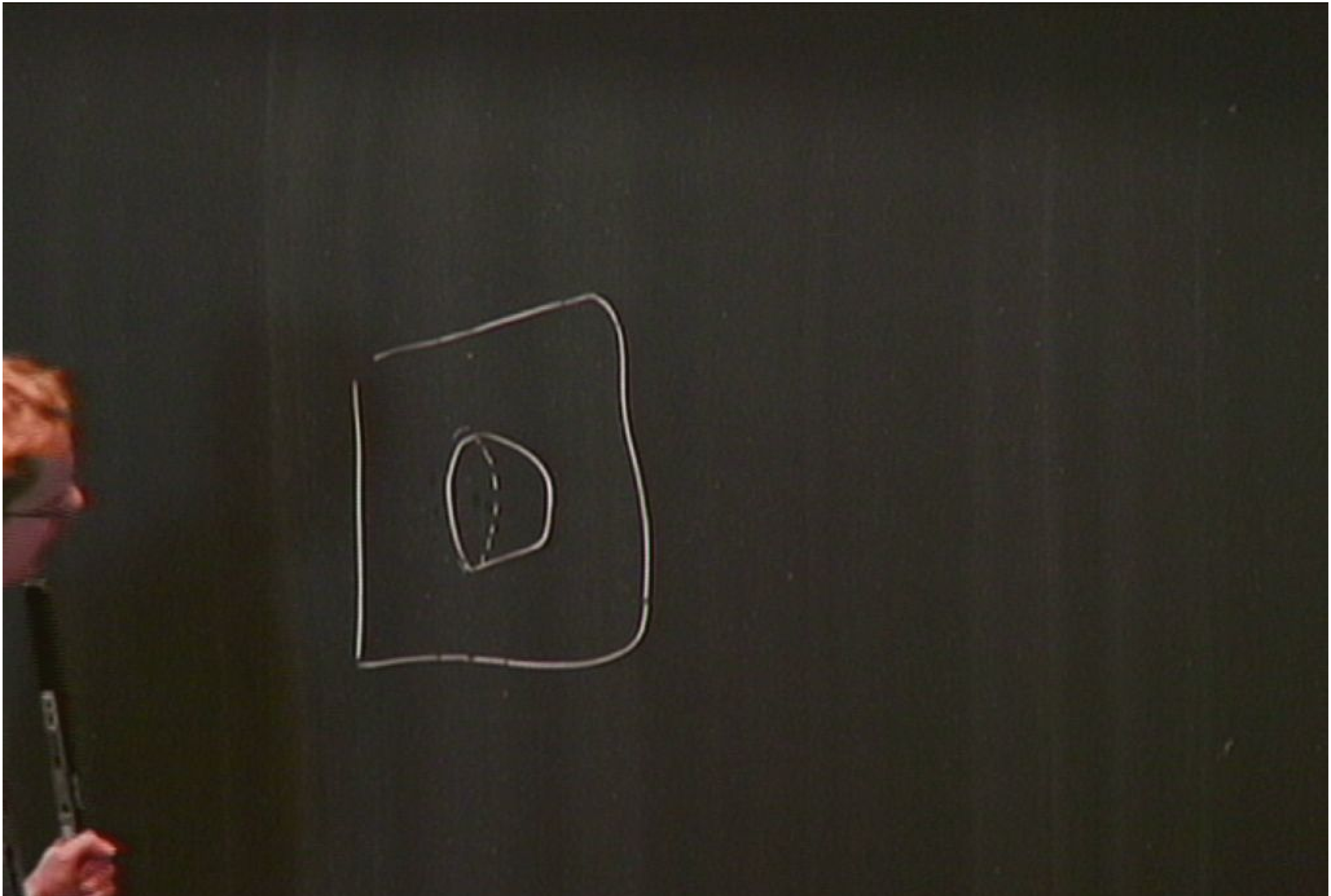

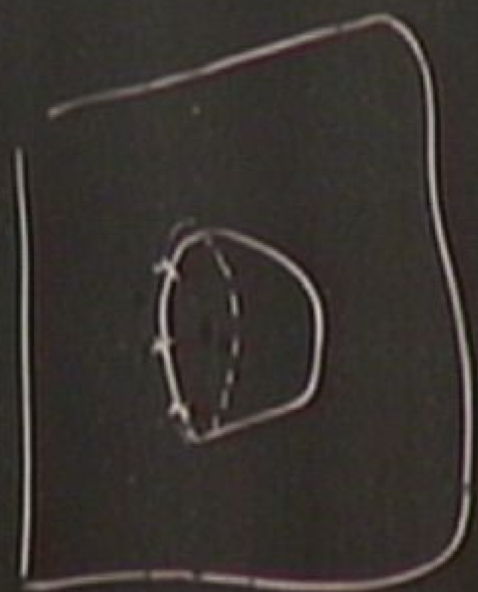

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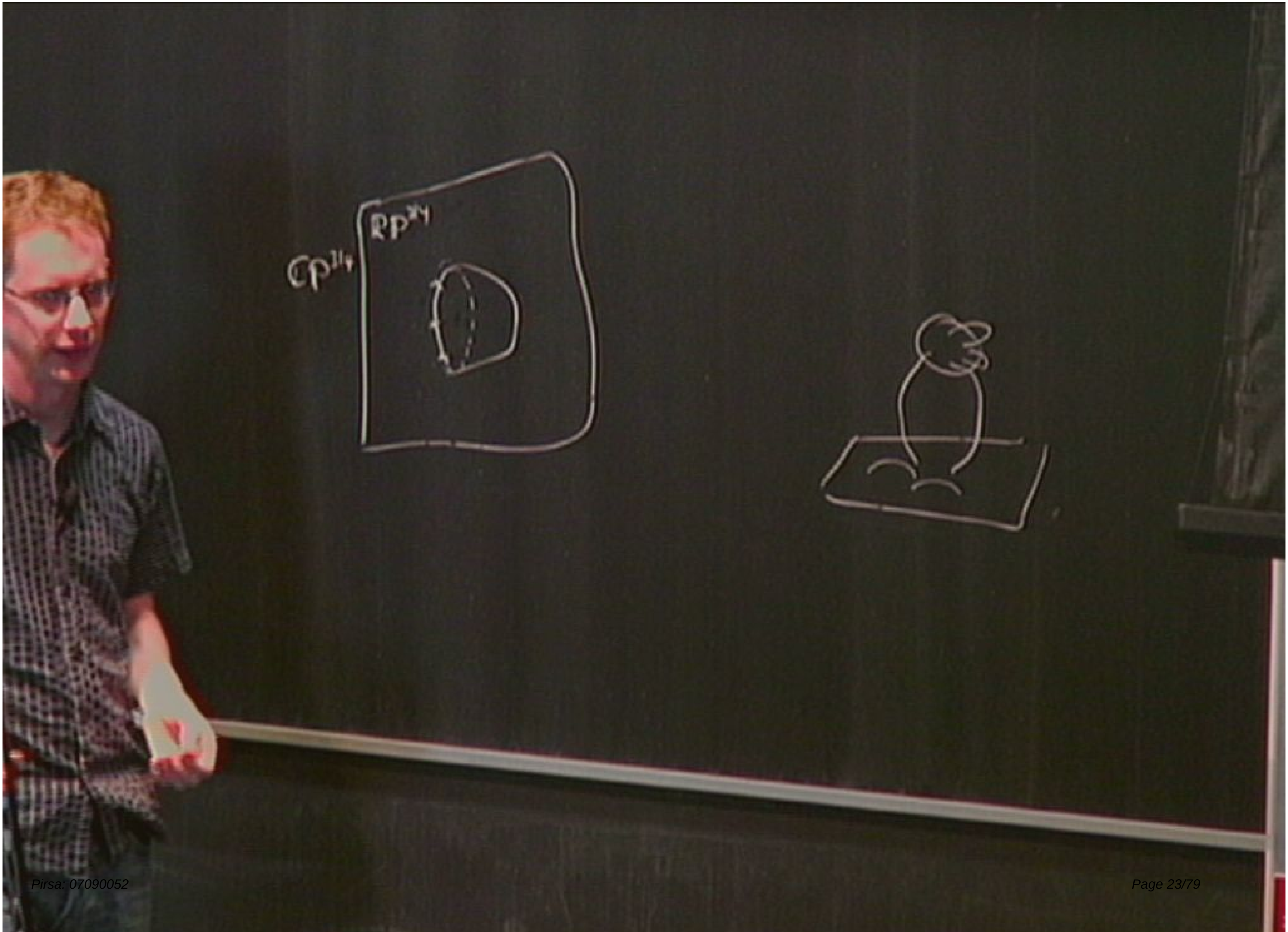
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Twisted (0,2) models

A theory of smooth maps $\Phi : \Sigma \to X$ from a closed, compact Riemann surface Σ to a complex manifold X.

Fields are worldsheet scalars $(\phi^i, \phi^{\bar{\jmath}})$ and

$$\bar{\rho}^{\bar{\jmath}} \in \Gamma(\Sigma, \phi^* \overline{T}_X) \qquad \rho^i \in \Gamma(\Sigma, \overline{K}_\Sigma \otimes \phi^* T_X)$$

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Susy transformations are

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$\overline{Q}$ acts on functions of $\phi, \bar{\phi}$ as the $\overline{\partial}$ operator on $\mathrm{Maps}(\Sigma, X)$

Action

The basic action is

$$S_0 = \mathrm{t}\int_\Sigma g\left(\overline{\partial}\phi, \partial\bar{\phi}\right) - g(\rho, \nabla\bar{\rho}) + \int_\Sigma \phi^*\omega$$
$$= \mathrm{t}\left\{\overline{Q}, \int_\Sigma g(\rho, \partial\bar{\phi})\right\} + \int_\Sigma \phi^*\omega$$

for $\mathrm{t} \in \mathbb{R}^+$ and g a Hermitian (*not pseudo-Hermitian*) metric on X with $\omega(X, Y) = g(X, JY)$

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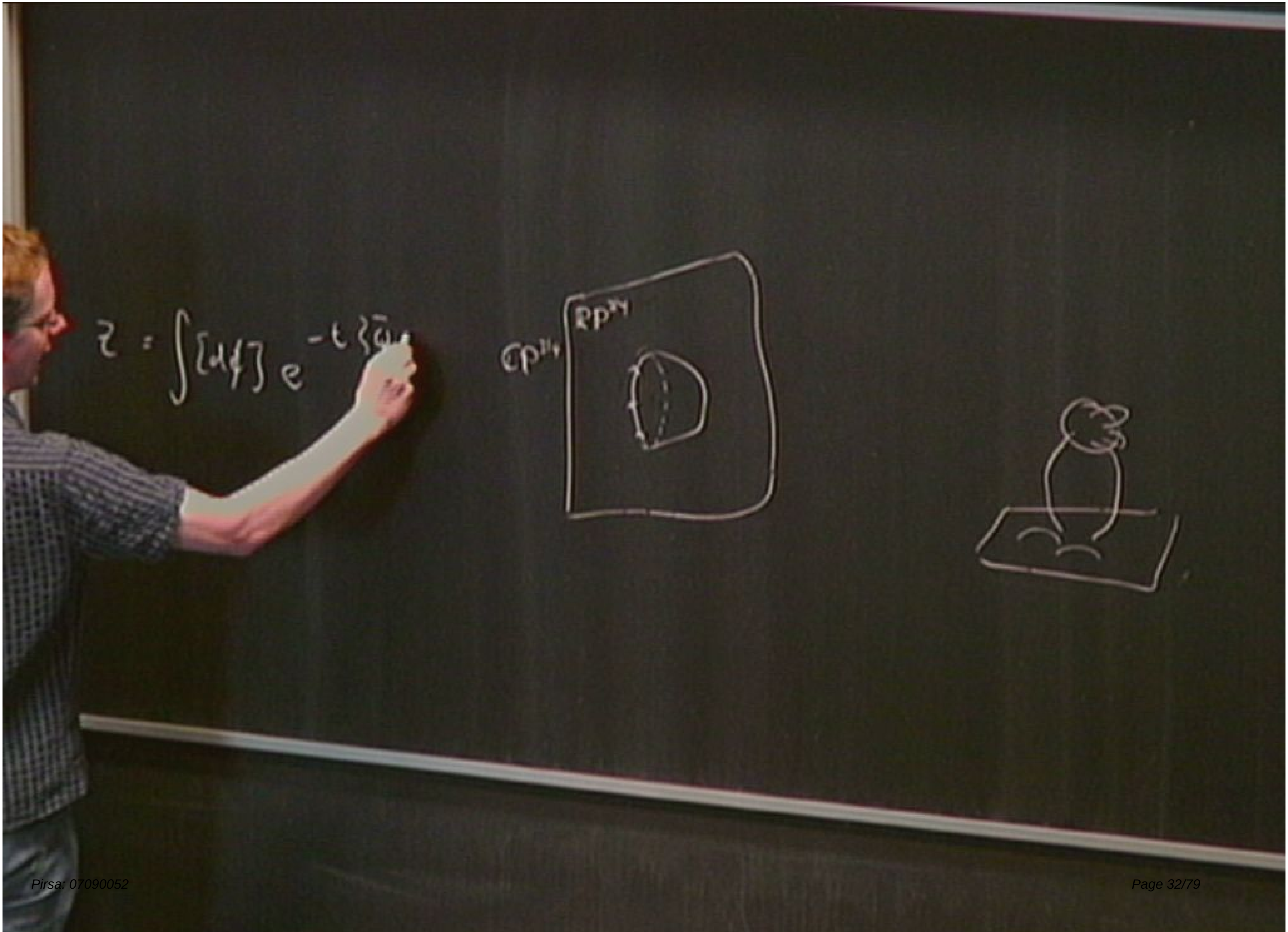
- Action is $\overline{Q}$-exact $\Rightarrow$ partition function independent of t, g
- $S_0 = -\mathrm{t}|\overline{\partial}\phi|^2 +$ fermions $\Rightarrow$ localize on holomorphic maps
- Manifestly invariant under $\overline{Q}$; also invariant under $\overline{Q}^\dagger$ if X is Kähler
- Can generalize by coupling to B-field: $\partial\overline{\partial}\omega = 0$ and ∇ has torsion determined by B


$\mathbb{C}P^{3|4}$
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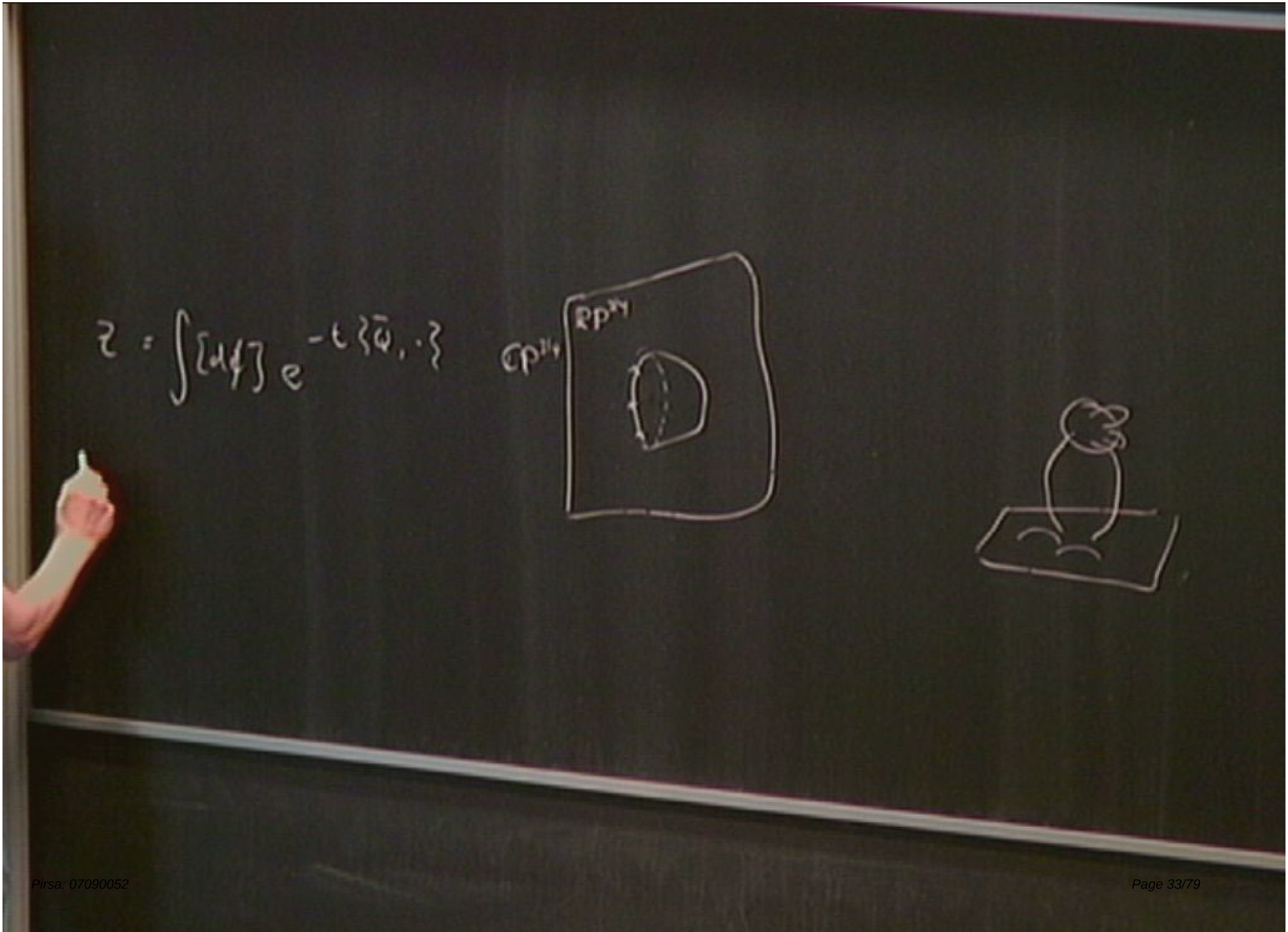
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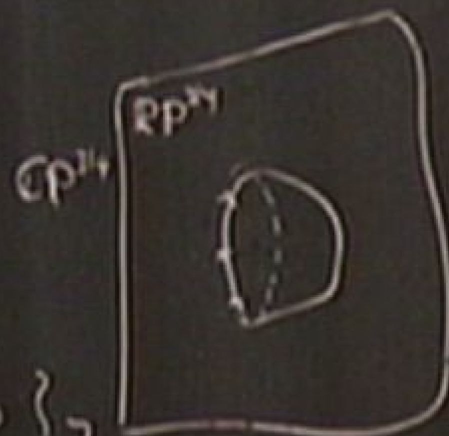

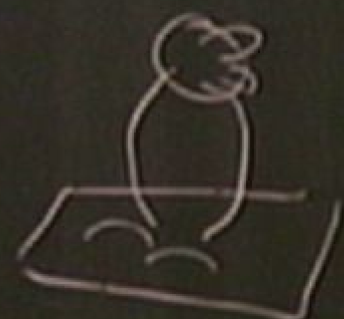

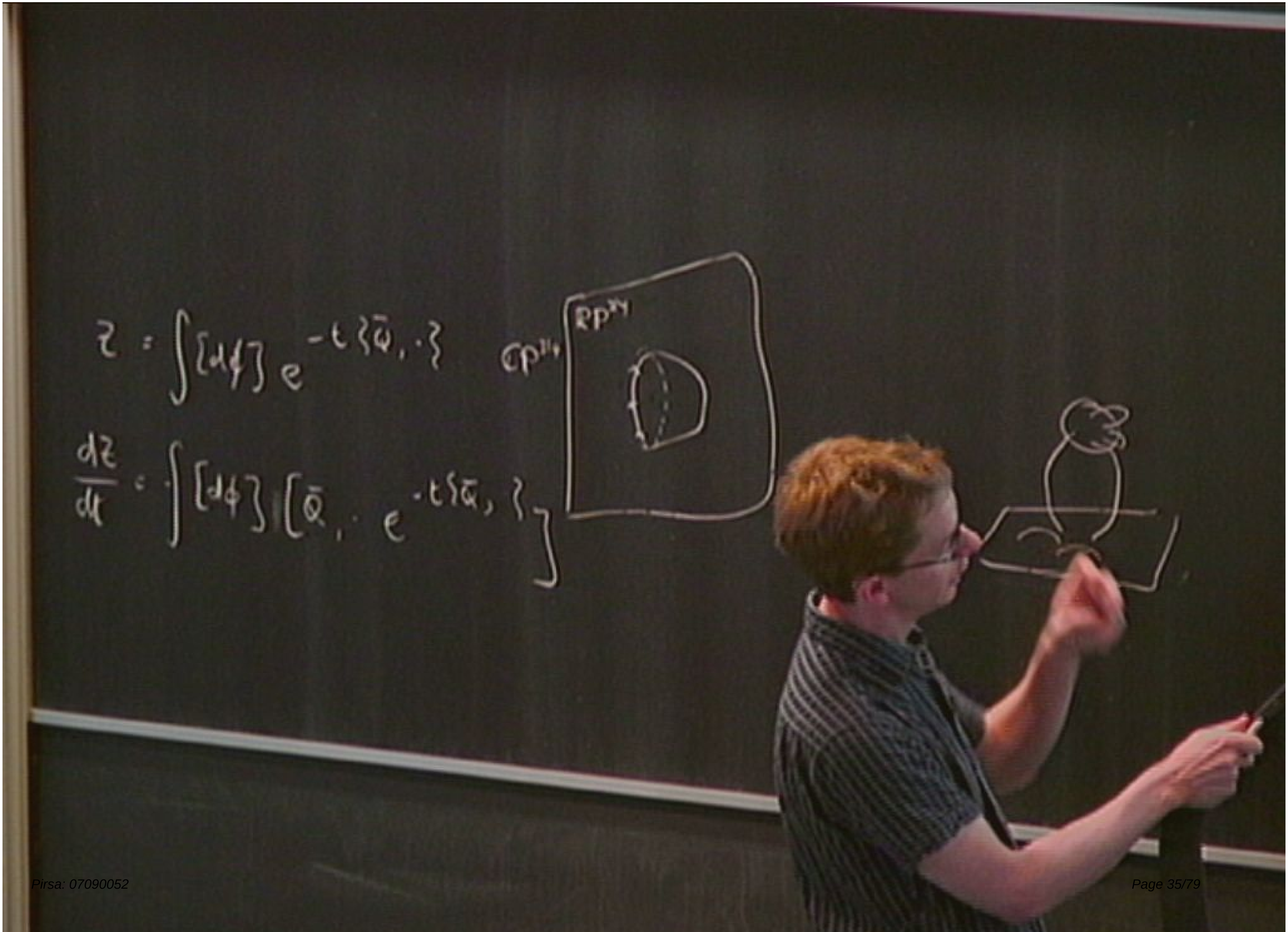

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Coupling to a bundle

We can also couple in a holomorphic bundle $\mathcal{V} \to X$ by introducing

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and action

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Total action $S_0 + S_1$ is twisted version of heterotic string on general background

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We could choose $X = \mathbb{P}^{3|4}$, but

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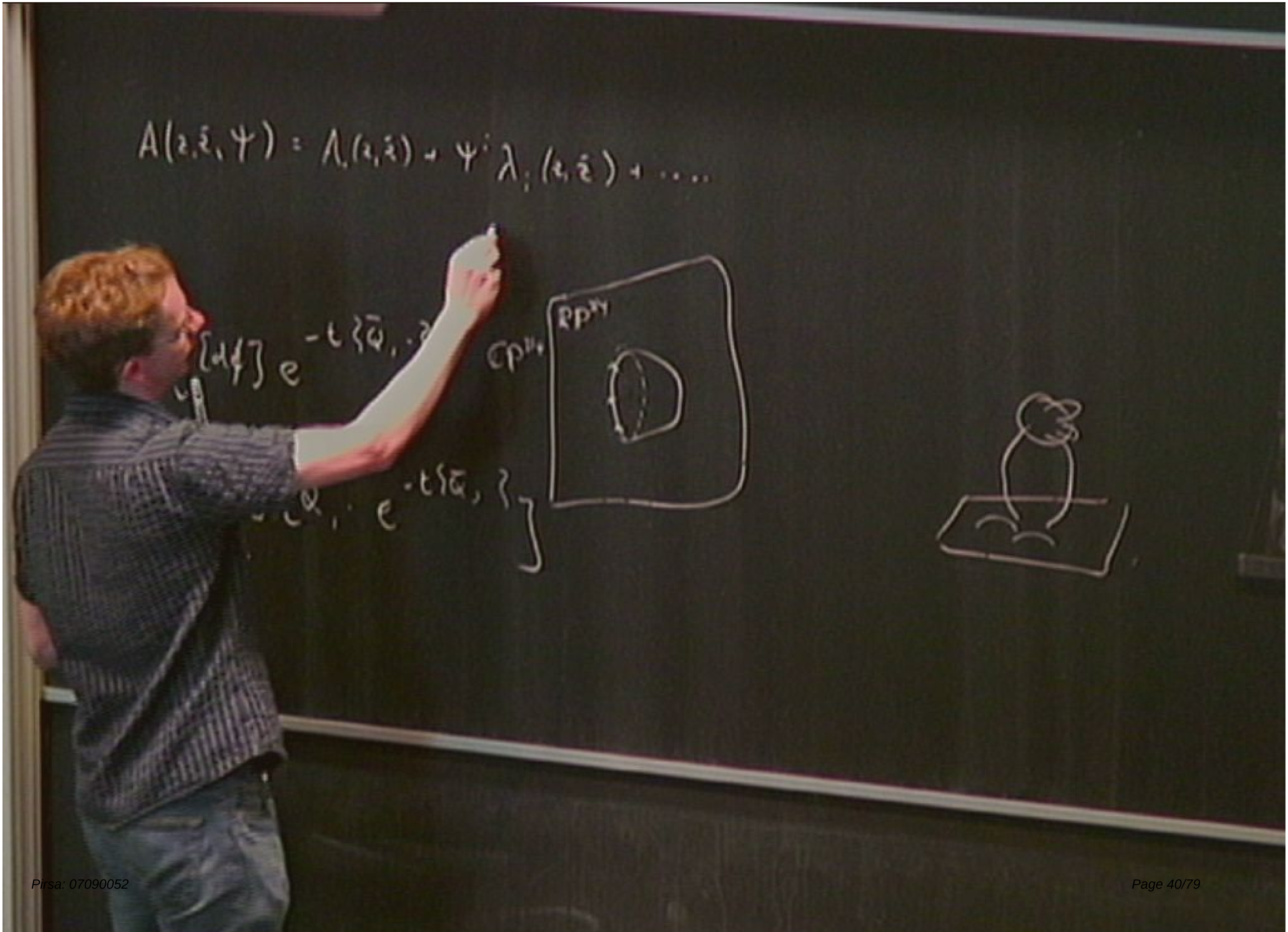
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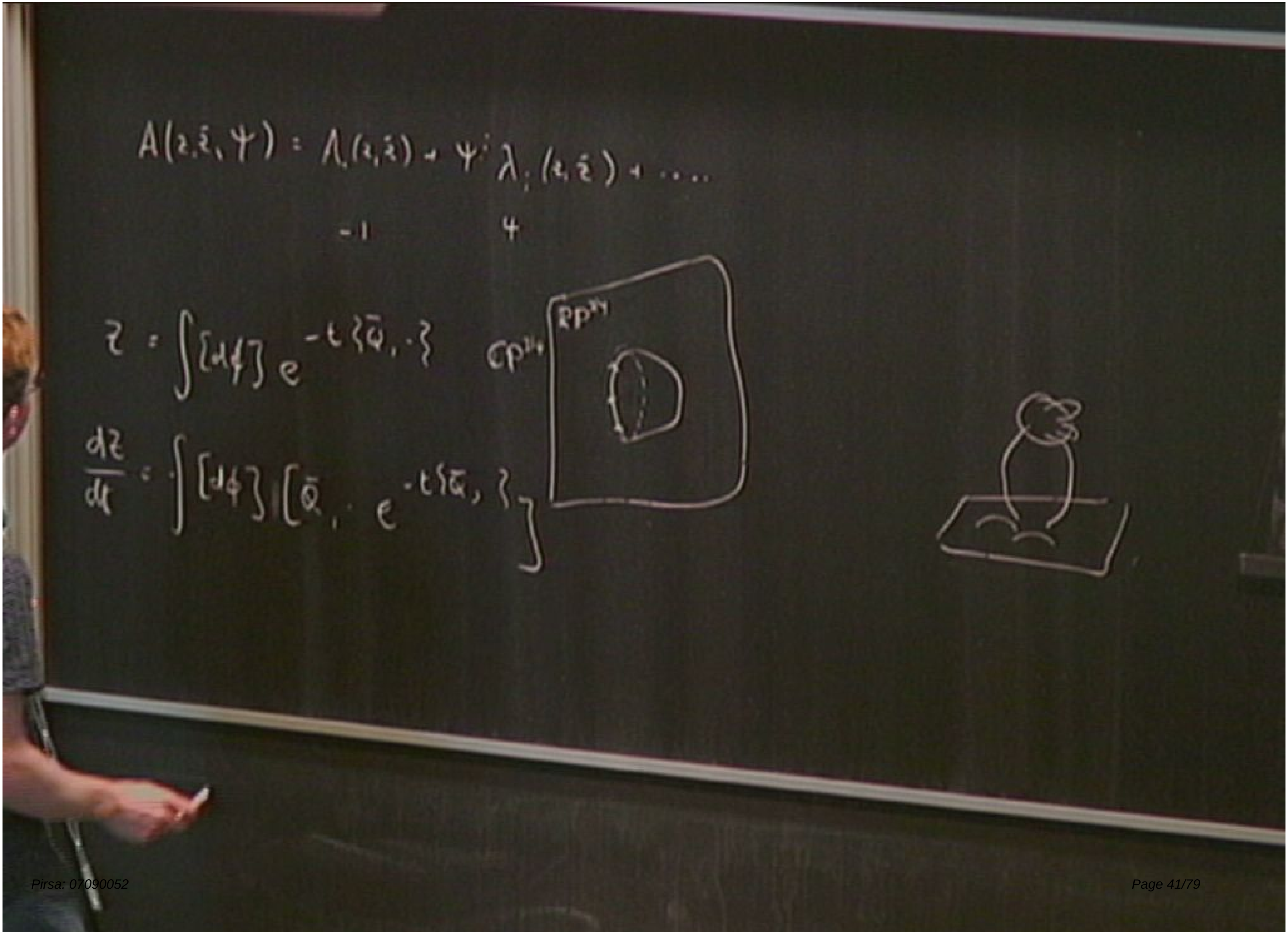
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-1
4
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$$A(z,\bar z,\psi,\bar\psi) = \Lambda(z,\bar z) + \psi^i \lambda_i(z,\bar z) + \dots$$

$-1 \qquad 4$

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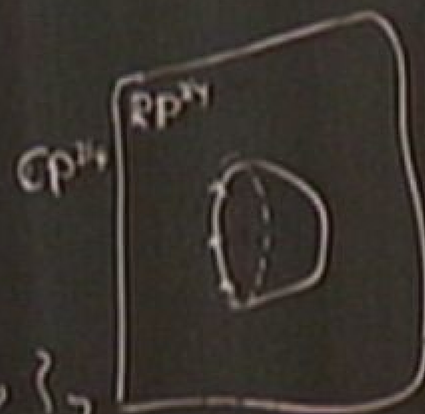

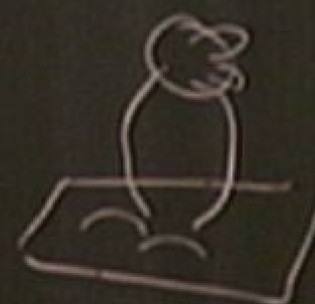

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Instead, we'll choose $X = \mathbb{P}^3$ and include the bundle $\mathcal{V} = \mathcal{O}(1)^{\oplus 4}$. The advantages are

- ψ is a worldsheet scalar, as it would be with $\mathbb{P}^{3|4}$ target, but $\overline{\psi}$ is a 1-form – naturally on different footing
- First-order action for worldsheet fermions
- Worldsheet superpartners are auxiliary

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- Not clear how to promote to string theory

Instead, we'll choose $X = \mathbb{P}^3$ and include the bundle $\mathcal{V} = \mathcal{O}(1)^{\oplus 4}$. The advantages are

- ψ is a worldsheet scalar, as it would be with $\mathbb{P}^{3|4}$ target, but $\overline{\psi}$ is a 1-form – naturally on different footing
- First-order action for worldsheet fermions
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Non-zero modes equivalent in either picture. Zero-modes more difficult to handle on $\mathbb{P}^{3|4}$, but equivalent after appropriate choice of contour.

Sheaves of chiral algebras

The antiholomorphic stress tensor $T_{\bar{z}\bar{z}} = \{\overline{Q}, \overline{G}_{\bar{z}\bar{z}}\}$, so all the antiholomorphic Virasoro generators $\bar{L}_n$ are $\overline{Q}$-exact.

$[\bar{L}_0, \mathcal{O}] = \bar{h}\mathcal{O}$, but since $\bar{L}_0 = \{\overline{Q}, \overline{G}_0\}$ we find

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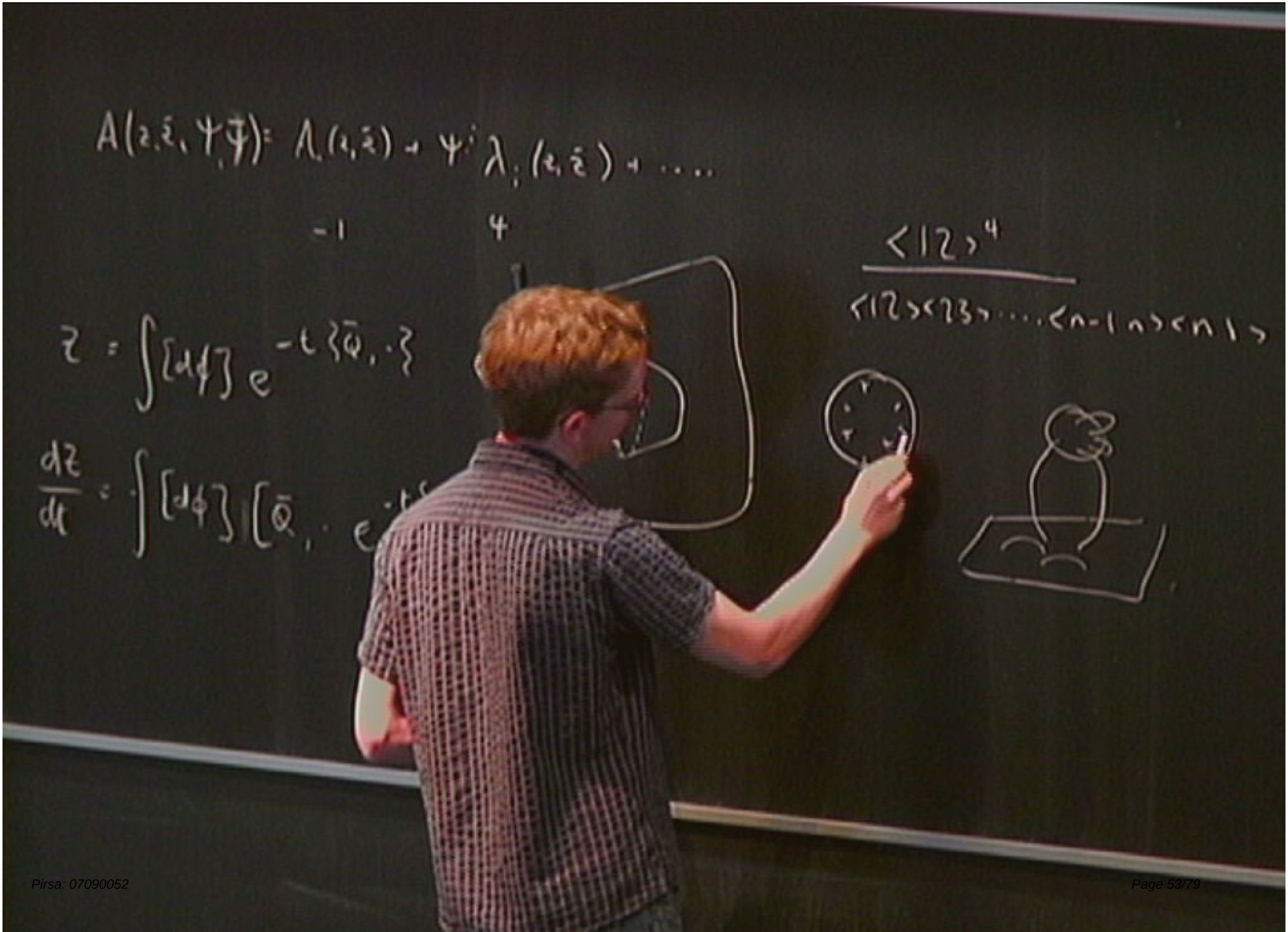
In the A- or B-model, we'd similarly find $h = 0$, but in a (0,2) model there is no holomorphic susy and all $h \geq 0$ are allowed. Vertex operators form "sheaf of chiral algebras" over target.

(0,2) model is holomorphic (not topological) field theory... but twistor-string theory can't be a topological theory! (*cf. MHV amplitudes*)


A(z, z̄, ψ, ψ̄) = Λ(z, z̄) + ψ^i λ_i(z, z̄) + ...
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(0,2) moduli

Focus on operators with $(h, \bar{h}) = (1, 0)$ and ghost number $+1$ (*related to deformations of the (0,2) action via descent*).

$$\mathcal{O}_M := g_{i\bar{k}} M^i{}_{\bar{j}} \bar{\rho}^{\bar{j}} \partial \phi^{\bar{k}} \qquad \mathcal{O}_\mu := \mu^a{}_{\bar{j}} \bar{\rho}^{\bar{j}} \bar{\psi}_a$$

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- M, μ, b & β may depend on ψ as this has $h = 0$. They must be independent of $\bar{\psi}$, which has $h = 1$.
- Non-trivial in $\overline{Q}$-cohomology if $[M] \in H^{0,1}(\mathbb{PT}', T_{\mathbb{PT}'})$, plus supersymmetric extensions.
- $b \to b + \partial\chi$ changes vertex operator by total derivative (*upto* ρ *eom*) $\Rightarrow \mathcal{H} = \partial b$ nontrivial in $H^{0,1}(\mathbb{PT}', \Omega^2_{\mathrm{cl}})$, plus super extension

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Anomalies

Sigma model anomaly unless

$$\mathrm{ch}_2(T_X) - \mathrm{ch}_2(\mathcal{V}) = 0 \qquad \mathrm{c}_1(T_\Sigma)\,(\mathrm{c}_1(T_X) - \mathrm{c}_1(\mathcal{V})) = 0$$

Twistor-strings: $\mathrm{c}(T_{\mathbb{P}^3}) = \mathrm{c}(\mathcal{O}(1)^{\oplus 4}) \Rightarrow$ no sigma model anomaly

Anomalies in global symmetries

$$\mathrm{ind}(\bar{\partial}_{\phi^* T_{\mathbb{P}^3}}) = 4d + 3(1-g)$$
$$\mathrm{ind}(\bar{\partial}_{\phi^* \mathcal{O}(1)^{\oplus 4}}) = 4(d+1-g)$$

for a map of degree d, genus g.
Amplitudes with n_h external SYM states of helicity h supported on maps of degree

$$d = g - 1 + \sum_{h=-1}^{+1} \frac{h+1}{2} n_h$$

Coefficient of $(\psi)^{\mathrm{top}}$ is a section of canonical bundle of instanton moduli space

Perturbative corrections

There are also perturbative corrections to the theory. (0,2) susy ensures that $\Delta \overline{T}_{\bar{z}\bar{z}}$ and $\Delta T_{z\bar{z}}$ are $\overline{Q}$-exact, but there is no such statement for T_{zz}.
At one loop, correction to worldsheet action is

$$\Delta S^{1-\text{loop}} = \left\{ \overline{Q}, \int_\Sigma R_{i\bar{\jmath}} \rho^i \partial \phi^{\bar{\jmath}} + g^{i\bar{\jmath}} F_{i\bar{\jmath}}{}^a{}_b \bar{\psi}_a r^b \right\}$$

- On $\mathbb{P}^{3|4}$ we have $R = 0$ and no bundle
- For $\mathbb{P}^3$ and bundle $\mathcal{O}(1)^{\oplus 4}$ we have $R_{i\bar{\jmath}} = 4g_{i\bar{\jmath}}$ and $F_{i\bar{\jmath}}{}^a{}_b = \delta^a{}_b g_{i\bar{\jmath}}$ so the 1-loop correction is $\propto$ classical action.

The twistor model is a holomorphic CFT provided we study correlators of $\overline{Q}$-closed operators.

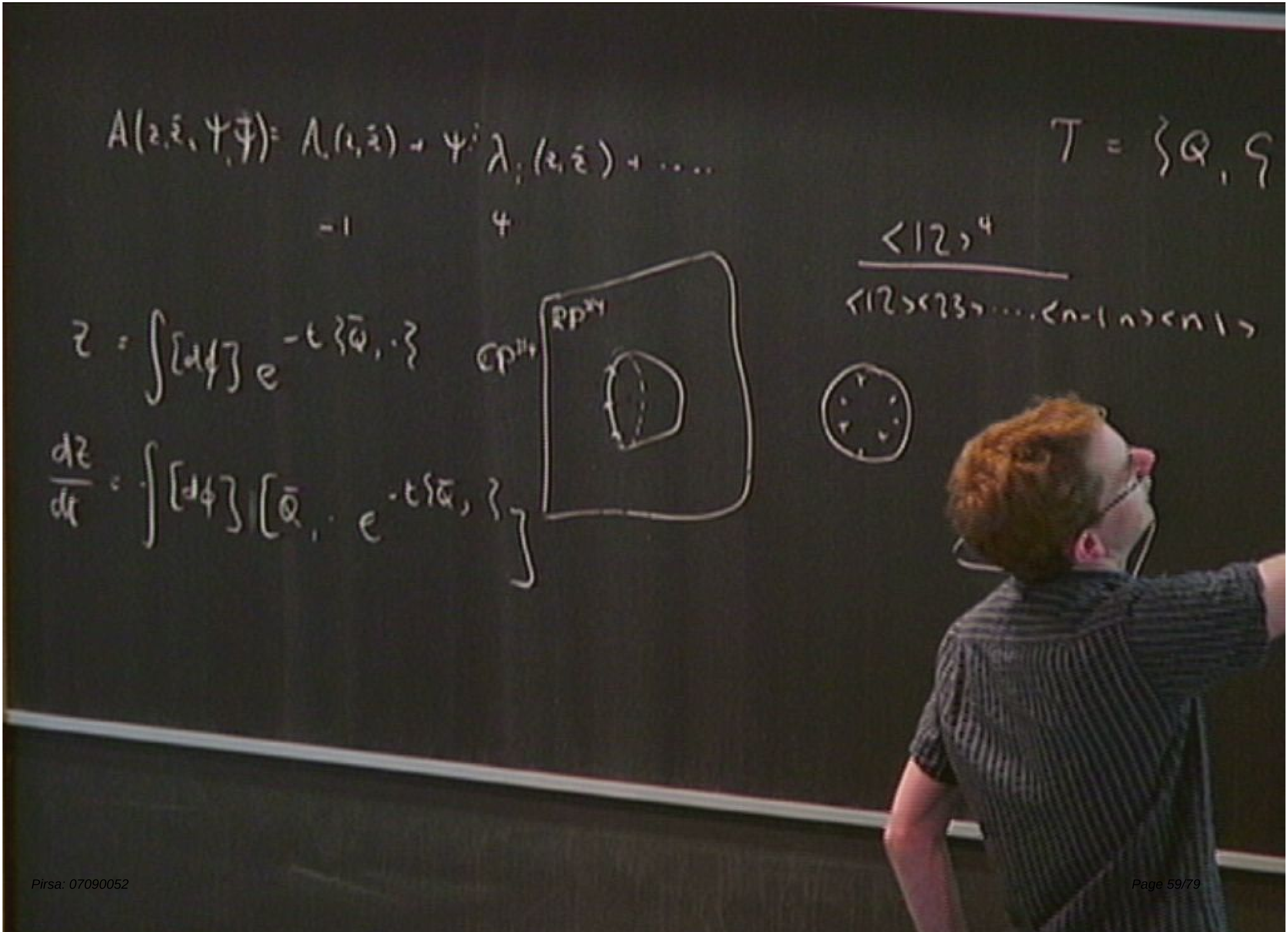
Holomorphic bc-system

Supercurrent $\overline{G}_{\bar{z}\bar{z}}$ plays role of $\bar{b}$-antighost

No left-moving susy, so need to include holomorphic bc-ghost system

$$S = \int_\Sigma b\bar{\partial}c \qquad b \in \Gamma(\Sigma, K_\Sigma \otimes K_\Sigma)\ ;\ c \in \Gamma(\Sigma, T_\Sigma)$$

- Provides holomorphic BRST operator Q
- $Q + \overline{Q}$ has complete descent chain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Physical string states $\Leftrightarrow$ (0,2) moduli $\Leftrightarrow$ $\mathcal{N} = 4$ conformal supergravity

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Yang-Mills current algebra

In order for $Q^2 = 0$ we need to include a holomorphic current algebra contributing central charge $c = 28$ $(= 26 + 2 \times (4 - 3))$, as in both Berkovits' and Witten's models (*see later* ...)

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e.g. Could include further fermions

$$\lambda^\alpha \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^* E) \qquad \bar{\lambda}_\alpha \in \Gamma(\Sigma, \sqrt{K_\Sigma} \otimes \phi^* E^\vee)$$

for some holomorphic bundle $E \to X$ (together with auxiliary superpartners).

- Conformal invariance requires $\mathrm{c}_1(E) = 0$
- Freedom from sigma model anomalies requires $\mathrm{ch}_2(E) = 0$

$\Rightarrow$ E corresponds to a zero-instanton spacetime bundle
Vertex operators $c\mathcal{A}_{\bar{\jmath}}{}^\alpha{}_\beta \bar{\lambda}_\alpha \lambda^\beta \Leftrightarrow$ External states in $\mathcal{N} = 4$ SYM

Yang-Mills instantons

Heterotic strings contain NS branes which couple magnetically to the NS B-field.

- Physical heterotic strings (10-manifold) $\rightarrow$ 5-branes
- Twisted heterotic strings (complex 3-fold) $\rightarrow$ 1-branes

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Modified Green-Schwarz condition

$$\mathrm{ch}_2(T_X) - \mathrm{ch}_2(\mathcal{V}) - \mathrm{ch}_2(E) + \sum_i [NS]_i = 0$$

$\Rightarrow$ instanton backgrounds allowed

e.g. 't Hooft $SU(2)$ k-instanton

$$A(x) = \mathrm{i}\,\mathrm{d}x^{\mu}\sigma_{\mu\nu}\partial^{\nu}\log\Phi\ , \qquad \Phi(x) = \sum_{i=0}^{k}\frac{\lambda_i}{(x-x_i)^2}$$

wrap NS branes on the $k+1$ lines in twistor space corresponding to the x_is

A puzzle

	Physical heterotic	Twistor-string
c	16	28
Field theory	$SO(32)$, $E_8 \times E_8$, $E_8 \times U(1)^{248}$, $U(1)^{496}$	$SU(2) \times U(1)$, $U(1)^4$
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- Change level of current algebra? $c_{\mathrm{Sug}} = k\dim(G)/(k + h(G))$ and no solns for $G = SU(2) \times U(1)$ or $U(1)^4$ with $k \in \mathbb{N}$
- Include additional fields contributing to c? Monster CFT?
- Promote to string theory by some other means than bc-system?

Clear that modular invariance is key test.

Amplitudes and contours

Choose basis of Beltrami differentials μ and compute

$$\left\langle \prod_{i=1}^{3g-3+n} (\mu^{(i)}, b)(\overline{\mu}^{(i)}, \overline{G}) \prod_{j=1}^{n} \mathcal{O}_j \right\rangle$$

where $\mathcal{O}_j$ are fixed vertex operators.

- bc-ghost number anomaly absorbed by (μ, b) and vertex operators
- $U(1)_R$ anomaly is $3(1-g)+4d$. Remaining anomaly of $4d = \mathrm{vdim}_{\mathbb{C}}\, \overline{\mathcal{M}}_{g,0}(\mathbb{P}^3, d)$

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Integrand is effectively a $(4d, 0)$ form on moduli space of stable maps $\Rightarrow$ contour integral (*in Dolbeault picture*)

- Absorb anomaly by inserting Poincaré dual into path integral, soaking up remaining $\bar{\rho}$ zero-modes
- Choice of contour $\Leftrightarrow$ choice of spacetime signature
- Leading-trace SYM amplitudes agree with Witten's & Berkovits' models. Sub-leading trace = cSUGRA (*unitarity*)

Instanton corrections and twistor actions

At degree d, the heterotic generating function for amplitudes in $\mathcal{N} = 4$ csugra + SYM is

$$\int_{\mathcal{M}_{g,d}} \mathrm{d}\mu \ \exp\left(\frac{-A(C)}{2\pi} + \mathrm{i}\int_C B\right) \frac{\det \overline{\partial}_{E\otimes S_-}}{\det' \overline{\partial}_{N_{C|\mathbb{PT}_s}}} \qquad (\star)$$

- $\mathcal{M}_{g,d}$ is contour in space of genus g, degree d curves, measure $\mathrm{d}\mu$ ($= \mathrm{d}^{4|8}x$ at $g = 0$, $d = 1$)
- $A(C) =$ area of curve C (*from the restriction of the Kähler form*)
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In compactifications on $CY \times \mathbb{R}^4$, $(\star)$ describes instanton corrections to $4d$ superpotential.
Here, the $d = 1$ contribution can be used together with the Chern-Simons ($d = 0$ term) as a twistor action.

Finite deformations

Vertex operator $\mathcal{O}_M = g_{i\bar{k}} M^i{}_{\bar{j}} \bar{\rho}^{\bar{j}} \partial \phi^{\bar{k}}$ contains graviton of $h = -2$ and generates infinitesimal deformation of complex structure.
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Answer: a *generalized* complex manifold (*ie* a smooth manifold with a map

$$\mathcal{J} : T \oplus T^\vee \to T \oplus T^\vee, \qquad \mathcal{J}^2 = -1$$

where holomorphic objects are in $L \subset T \oplus T^\vee$, and L is preserved by the twisted Courant bracket)

Could generalized geometry solve the googly problem?

Conclusions & Outlook

We've given a construction of twistor-string theory as a heterotic string.

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- Modular invariance & $c = 28$
- Amplitudes as contour integrals. *Derivation of RSV?*
- Understand finite deformations. *Generalized geometry?*
- Poincaré supergravity? *Complex Poisson structure on $\mathbb{PT}'$*
- A true open string version for pure $\mathcal{N} = 4$ SYM?