Title: sN formalism for curvature perturbations

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Abstract:

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δN formalism for curvature perturbations

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Comments (mumbles…) to Paul Steinhardt

- In late 80's, people talked about crisis of Big Bang theory, because there were things which could not explained by BB theory.
- Later it was understood that "BB theory" was not there to explain "everything".
- Inflation came to rescue BB theory, giving the initial condition for the Hot Big Bang universe.

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- In late 80's, people talked about crisis of Big Bang theory, because there were things which could not explained by BB theory.
- Later it was understood that "BB theory" was not there to explain "everything".
- Inflation came to rescue BB theory, giving the initial condition for the Hot Big Bang universe.
- Now, people blame inflation because it cannot explain everything....

1. Introduction

2. Linear perturbation theory
   - metric perturbation & time slicing
   - δN formalism

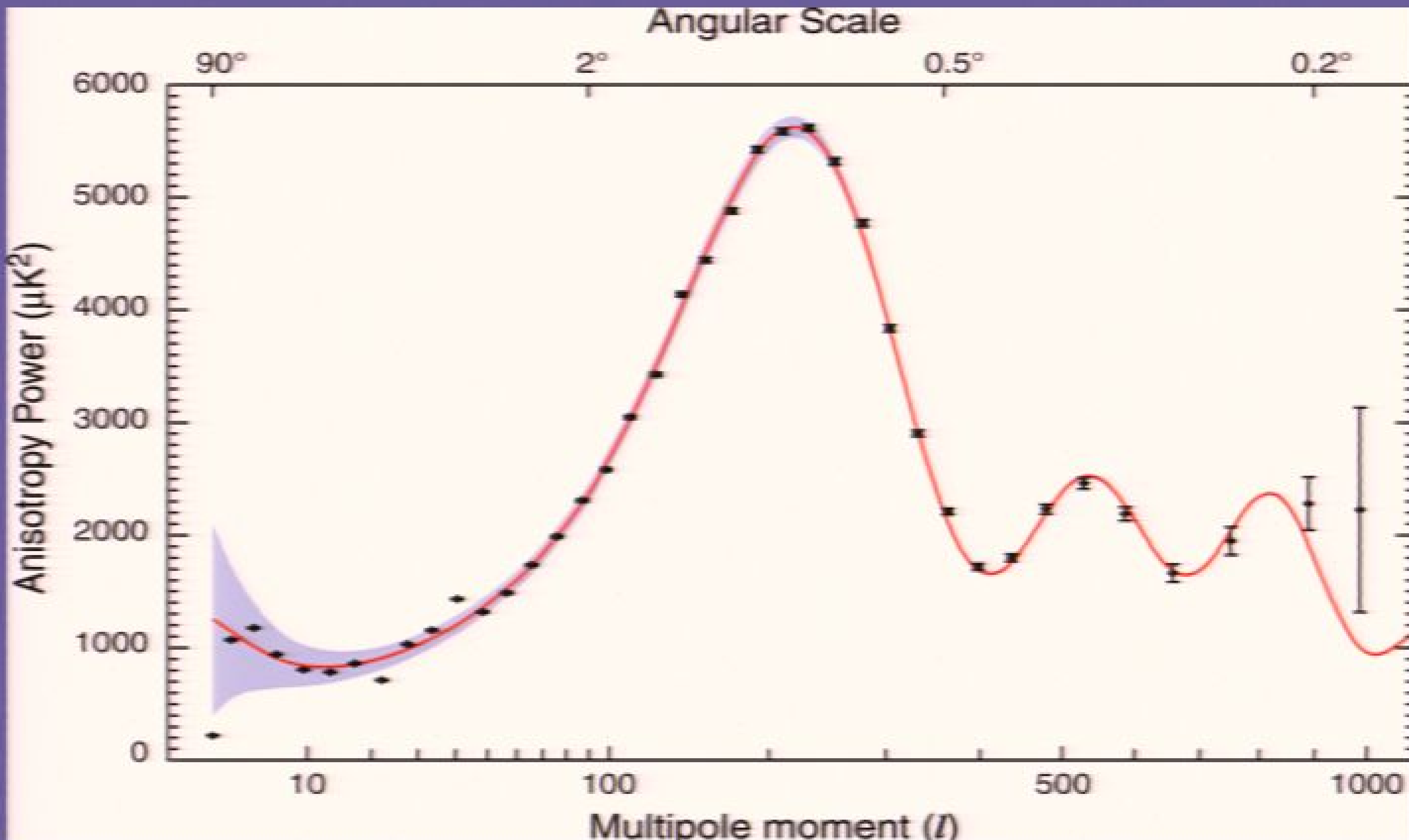
3. Nonlinear extension on superhorizon scales
   - gradient expansion, conservation law
   - local Friedmann equation

4. Nonlinear ΔN formula
   - ΔN for slowroll inflation
   - diagrammatic method for ΔN
   - IR divergence issue

5. Summary

1. Introduction

- Standard (single-field, slowroll) inflation predicts scale-invariant Gaussian curvature perturbations.



- CMB (WMAP) is consistent with the prediction.
- Linear perturbation theory seems to be valid.

- So, why bother doing more on theoretical models?

  Because observational data does not exclude other models.

  Tensor perturbations have not been detected yet.

  $T/S \sim 0.2 - 0.3$? or smaller?

- Inflation may not be so simple.

  multi-field, non-slowroll, extra-dim's, string theory...

- future CMB experiments may detect non-Gaussianity

  $\Psi = \Psi_{gauss} + f_{NL} \Psi^2_{gauss} + \cdots$; $|f_{NL}| \gtrsim 5$?

- Pre-bigbang, ekpyrotic, bouncing,....?

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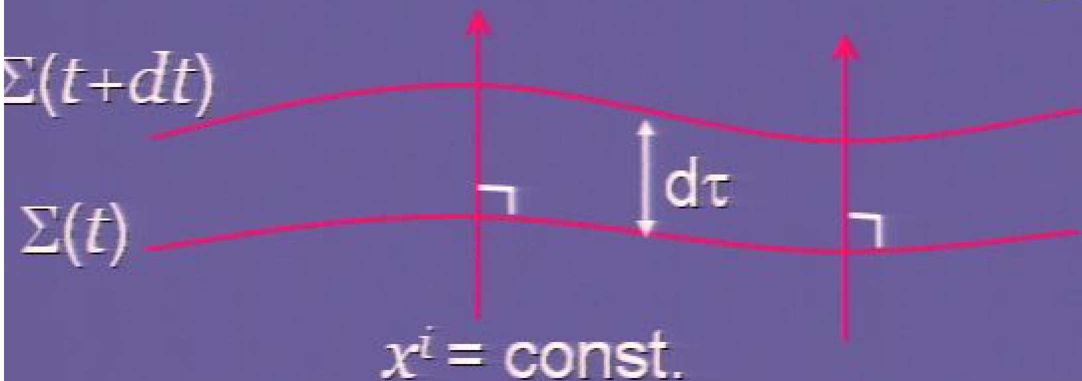
Re-consider the dynamics on super-horizon scales

2. Linear perturbation theory

Bardeen '80, Mukhanov '81, Kodama & MS '84, ...

- metric on a spatially flat background (g_{0j}=0 for simplicity)

$$ds^2 = -(1+2A)dt^2 + a^2(t)\left[(1+2\mathcal{R})\delta_{ij} + H_{ij}\right]dx^i dx^j$$



$$\left(H_{ij}\right)_{\text{scalar}} = \partial_i \partial_j E$$

$$\left(H_{ij}\right)_{\text{tensor}} = \text{transverse-traceless}$$

- propertime along x^i = const.: $d\tau = (1+A)dt$
- curvature perturbation on $\Sigma(t)$: $\mathcal{R} \Longleftrightarrow \overset{(3)}{R} = -\dfrac{4}{a^2}\overset{(3)}{\Delta}\mathcal{R}$
- expansion (Hubble parameter): $\tilde{H} = H(1-A) + \partial_t\left[\mathcal{R} + \dfrac{1}{3}\overset{(3)}{\Delta}E\right]$

Choice of time-slicing

- comoving slicing $\quad T^{\mu}{}_{i}=0 \quad \left(\phi=\phi(t) \text{ for a scalar field}\right)$

- uniform density slicing $\quad -T^{0}{}_{0} \equiv \rho=\rho(t)$

- uniform Hubble slicing

$$\tilde{H}=H(t) \Leftrightarrow -H\,A+\partial_t\left[\mathcal{R}+\frac{1}{3}\overset{(3)}{\Delta}E\right]=0$$

- flat slicing $\quad \overset{(3)}{R}=-\frac{4}{a^2}\overset{(3)}{\Delta}\mathcal{R}=0 \Leftrightarrow \mathcal{R}=0$

- Newton (shear-free) slicing

$$\partial_t\left[H_{ij}\right]^{\text{scalar}}_{\text{traceless}} \equiv \left[\partial_i\partial_j-\frac{1}{3}\delta_{ij}\overset{(3)}{\Delta}\right]\partial_t E=0 \Leftrightarrow \partial_t E=0 \Leftrightarrow E=0$$

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matter-based slices

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comoving = uniform ρ = uniform H on superhorizon scales

- δN formalism in linear theory

MS & Stewart '96

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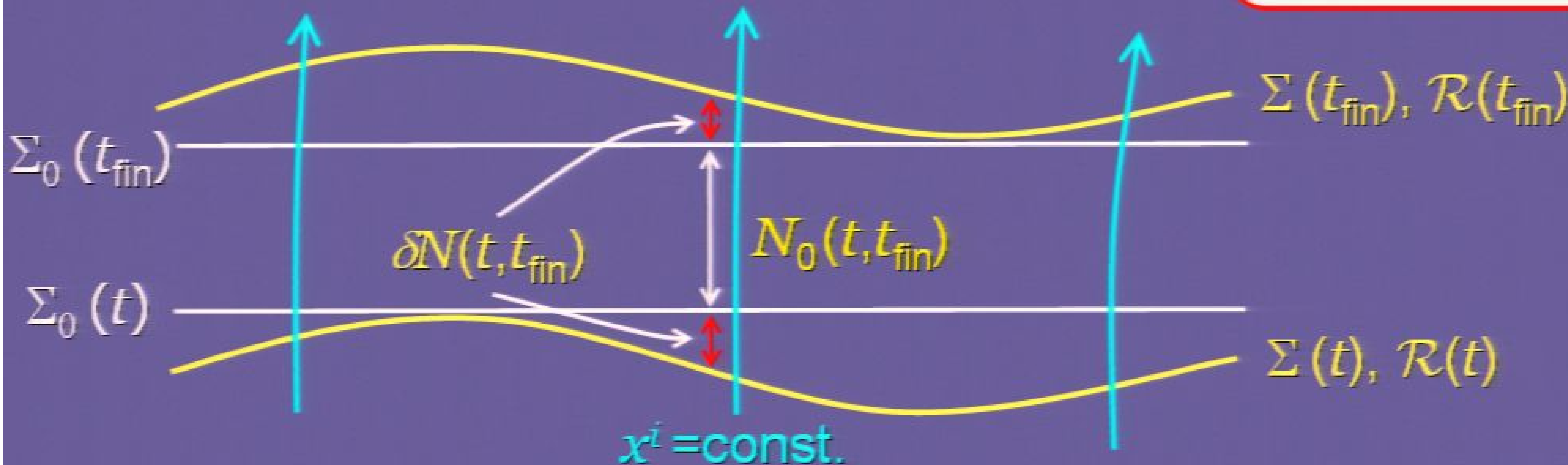
dep only on ini and fin t



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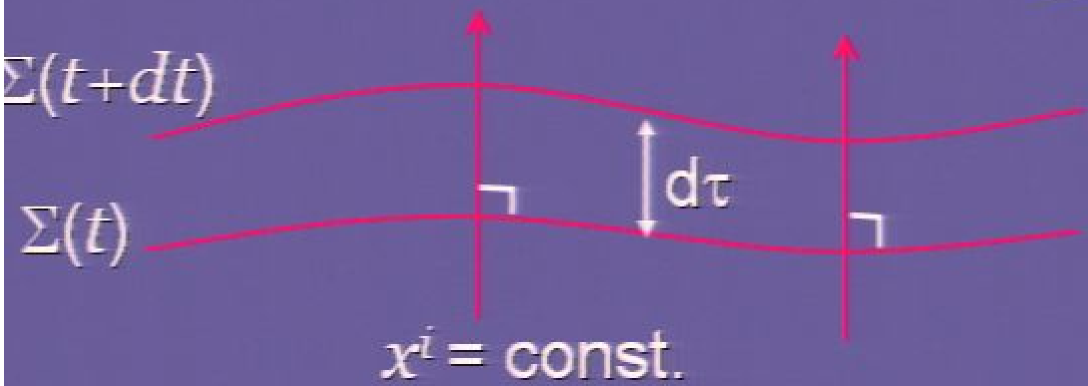

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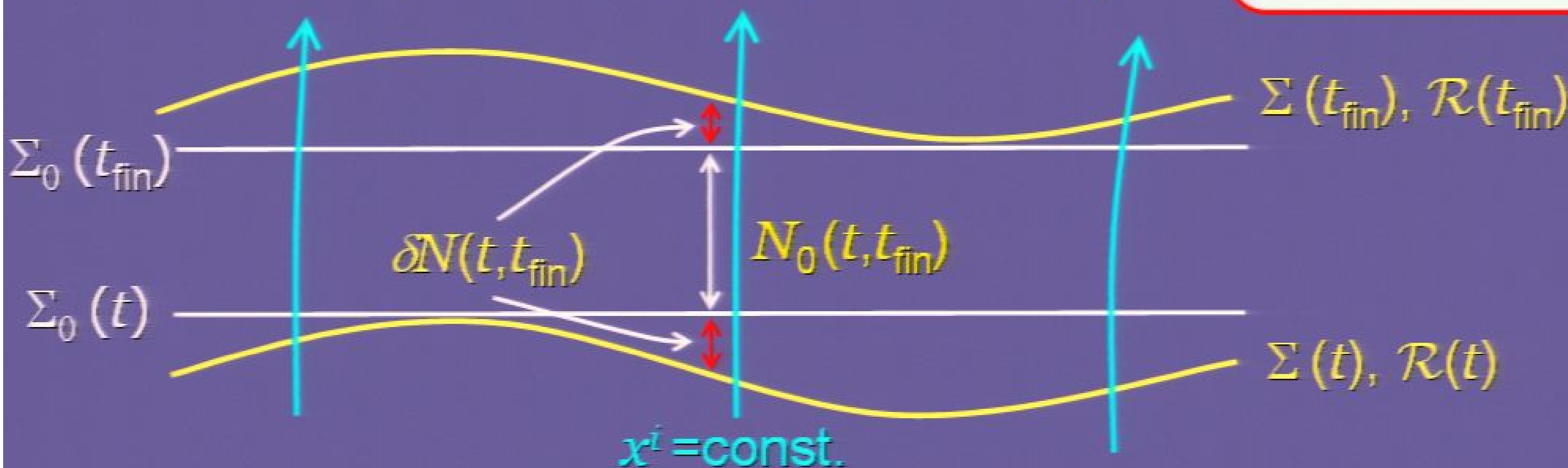

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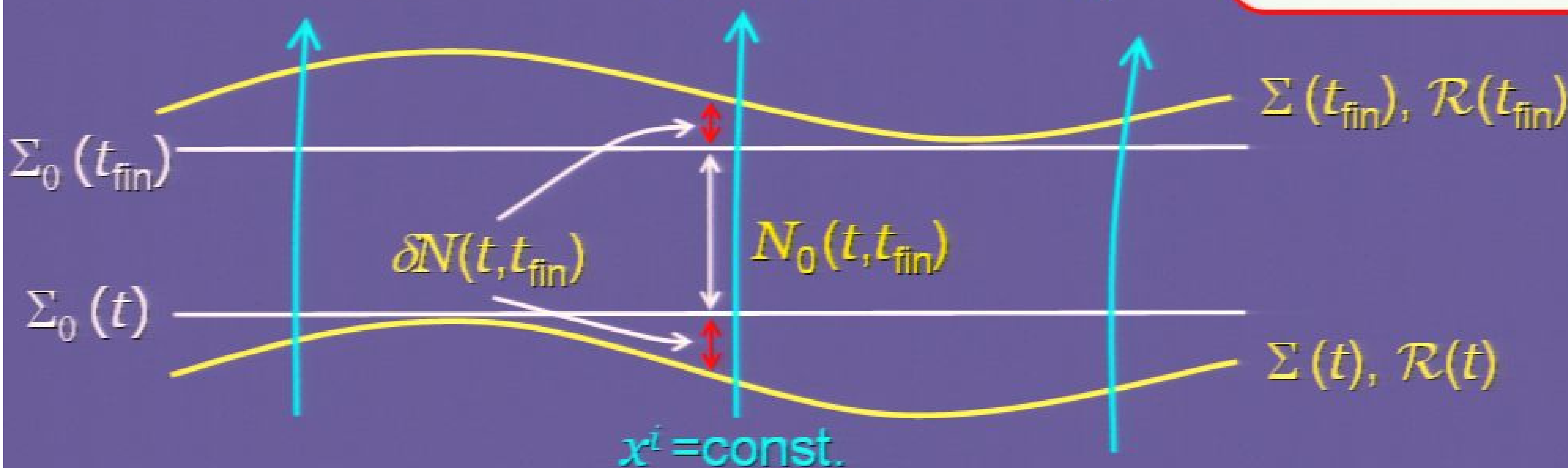

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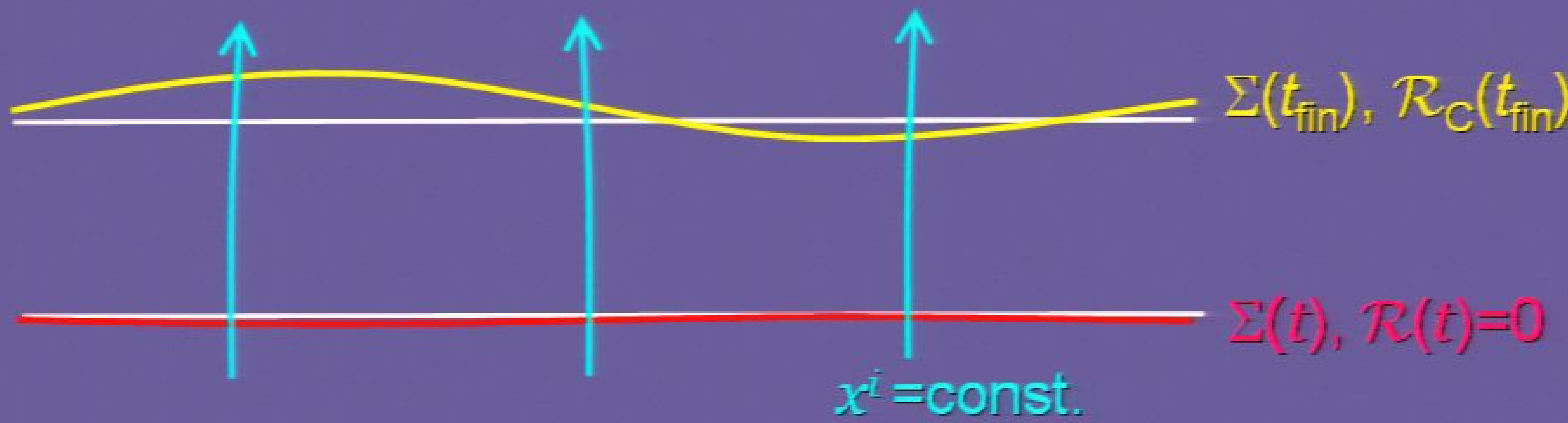

⇨ $\delta N(t;t_{\text{fin}}) = \mathcal{R}_C(t_{\text{fin}})$ on superhorizon scales

↖ curvature perturbation on comoving slice

(suffix 'C' for comoving)

The gauge-invariant variable 'ζ' used in the literature
is related to $\mathcal{R}_C$ as $\zeta = -\mathcal{R}_C$ or $\zeta = \mathcal{R}_C$ on superhorizon scales

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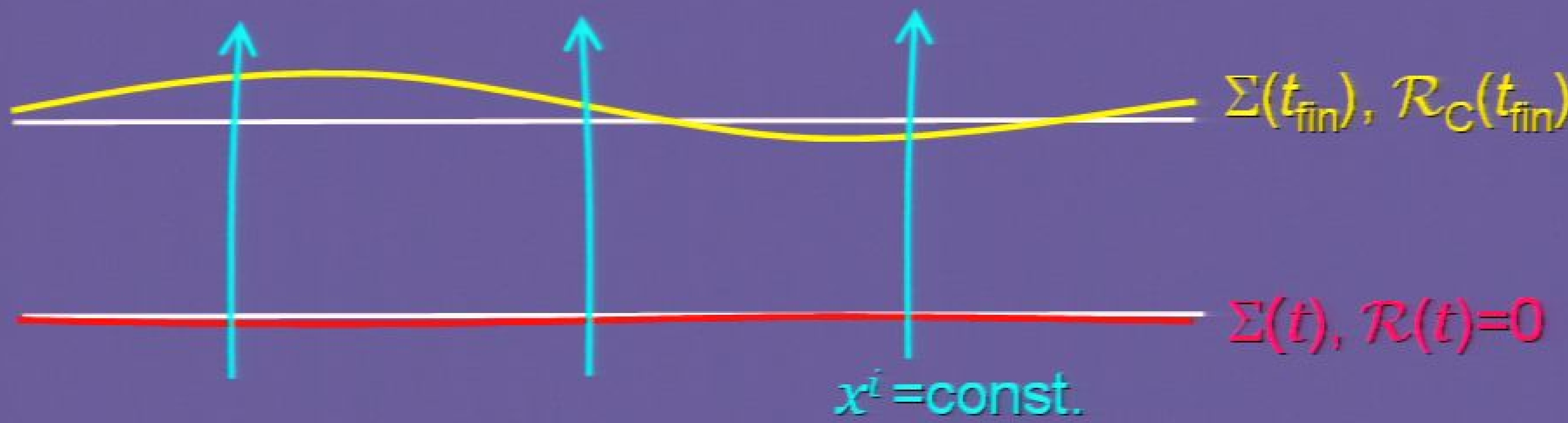

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By definition, $\delta N(t;\, t_{\rm fin})$ is t-independent

Example: slow-roll inflation

- single-field inflation, no extra degree of freedom

$\mathcal{R}_C$ becomes constant soon after horizon-crossing (t=t_h):

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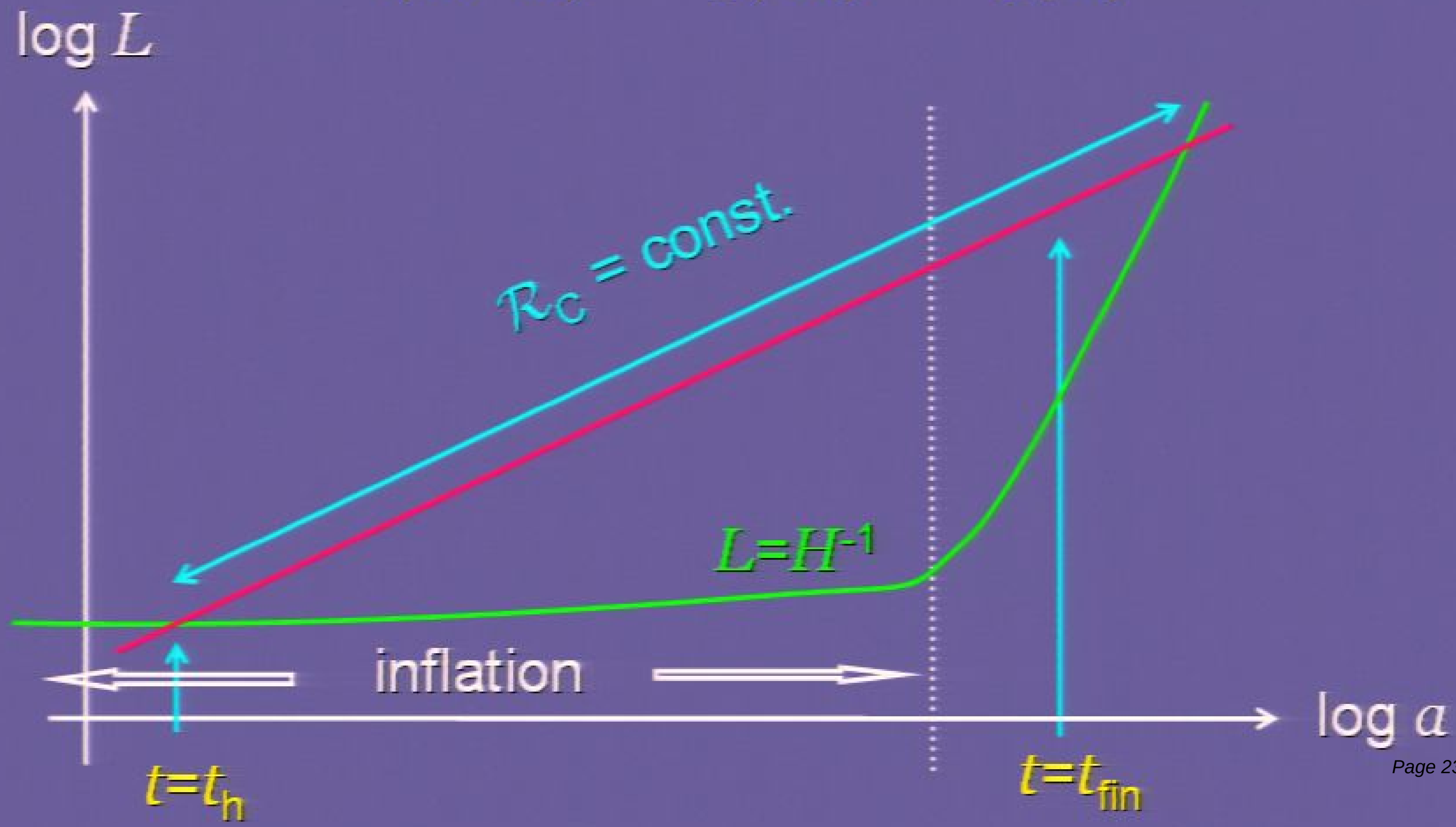

Also $\delta N = H(t_h)\,\delta t_{F\to C}$, where $\delta t_{F\to C}$ is the time difference between the comoving and flat slices at $t=t_h$.

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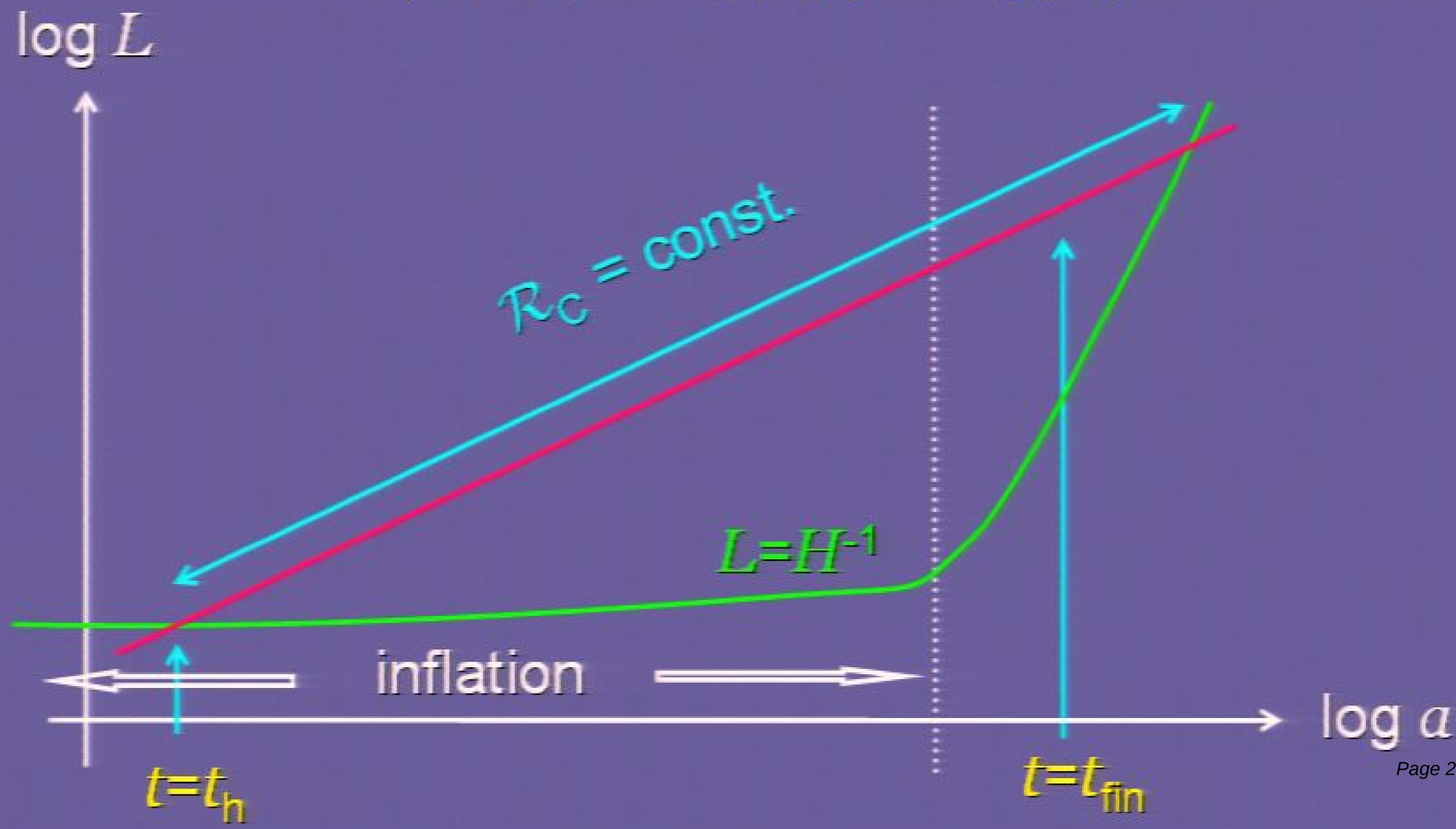

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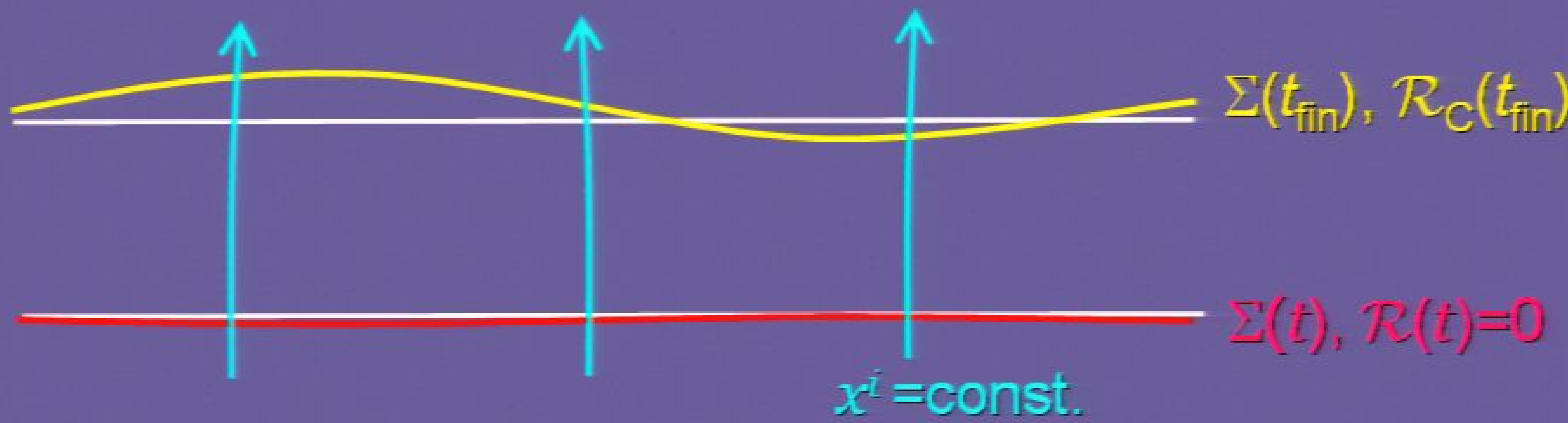

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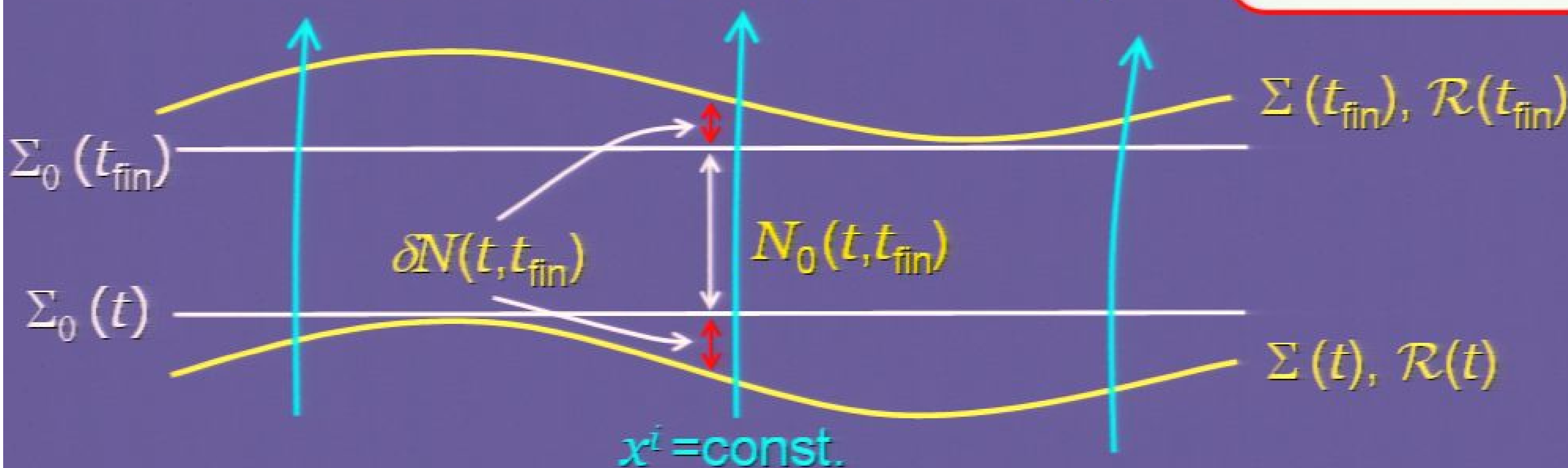

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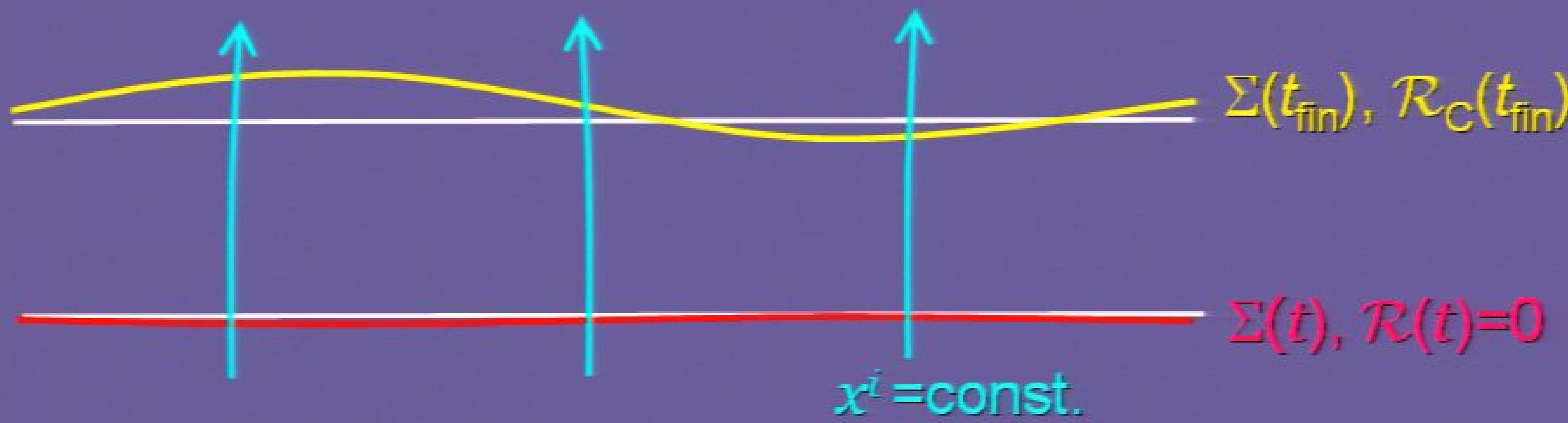

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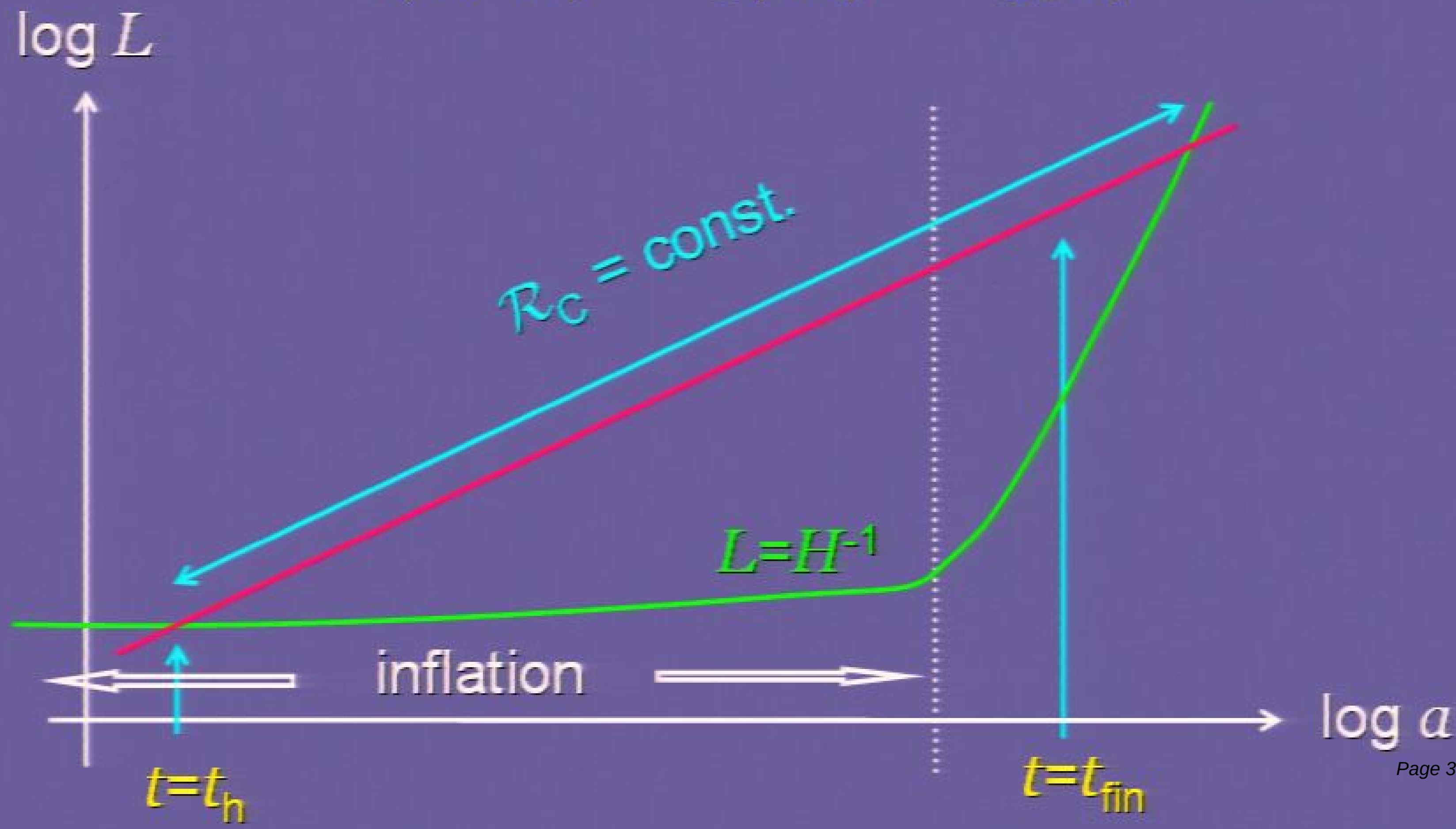

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… δN formula

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··· δN formula

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Only the knowledge of the background evolution is necessary to calculate $\mathcal{R}_C(t_{fin})$.

- δN for a multi-component scalar:
  (for slowroll inflation)

$$\mathcal{R}_{\mathrm{C}}(t_{\mathrm{fin}}) = \delta N = \sum_a \frac{\partial N}{\partial \phi^a} \delta\phi_{\mathrm{F}}^a(t_{\mathrm{h}})$$

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N.B. $\mathcal{R}_{\mathrm{C}}$ (=ζ) is no longer constant in time:

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$$\left\langle |\mathcal{R}_{\mathrm{c}}|^2 (t_{\mathrm{fin}}) \right\rangle = \|\nabla N\|^2 \|\delta\phi_{\mathrm{F}}\|^2 = \|\nabla N\|^2 \frac{H^2(t_{\mathrm{h}})}{(2\pi)^2}$$

$$\nabla_a N \equiv \frac{\partial N}{\partial \phi^a}$$

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··· time varying even on superhorizon scales

$$\left\langle |\mathcal{R}_c|^2 (t_{\rm fin}) \right\rangle = \|\nabla N\|^2 \|\delta\phi_F\|^2 = \|\nabla N\|^2 \frac{H^2(t_h)}{(2\pi)^2}$$

$$\nabla_a N \equiv \frac{\partial N}{\partial \phi^a}$$

Further extension to non-slowroll case is possible, if **general slow-roll condition** is satisfied at horizon-crossing.

Lee, MS, Stewart, Tanaka & Yokoyama '05

$$\frac{\dot{\phi}^2}{2H^2} = O(\xi), \frac{\ddot{\phi}}{H\dot{\phi}} = O(\xi), \frac{\dddot{\phi}}{H^2\dot{\phi}} = O(\xi), \ldots, \ \xi \ll 1$$

3. Nonlinear extension

- On superhorizon scales, gradient expansion is valid:

$$\left|\frac{\partial}{\partial x^i}Q\right| \ll \left|\frac{\partial}{\partial t}Q\right| \sim HQ; \;\; H \sim \sqrt{G\rho}$$

Belinski et al. '70, Tomita '72, Salopek & Bond '90, ...

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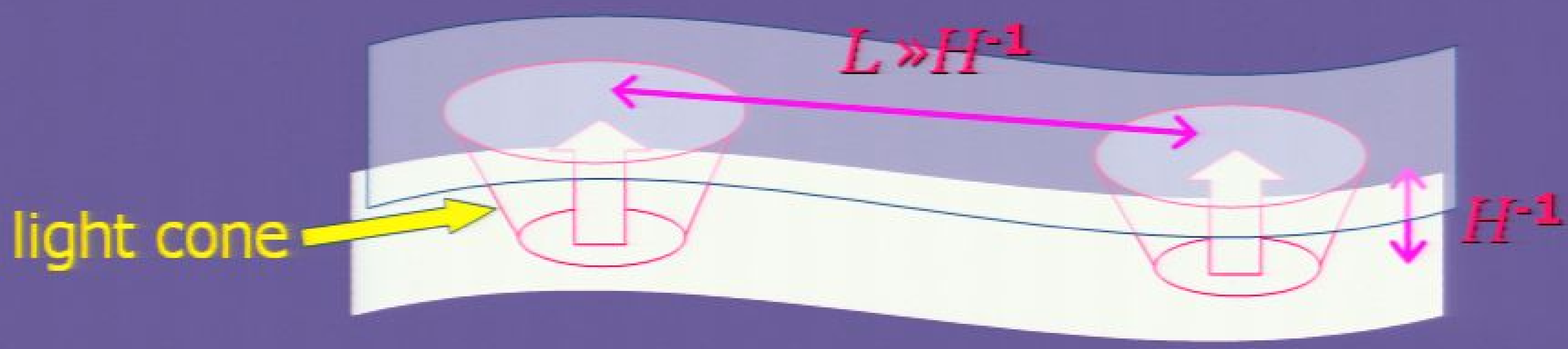

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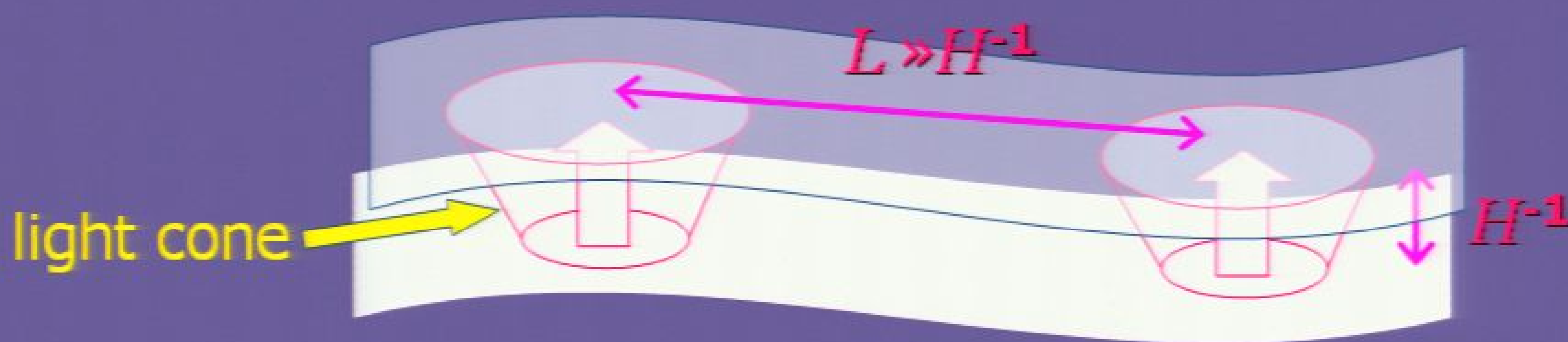

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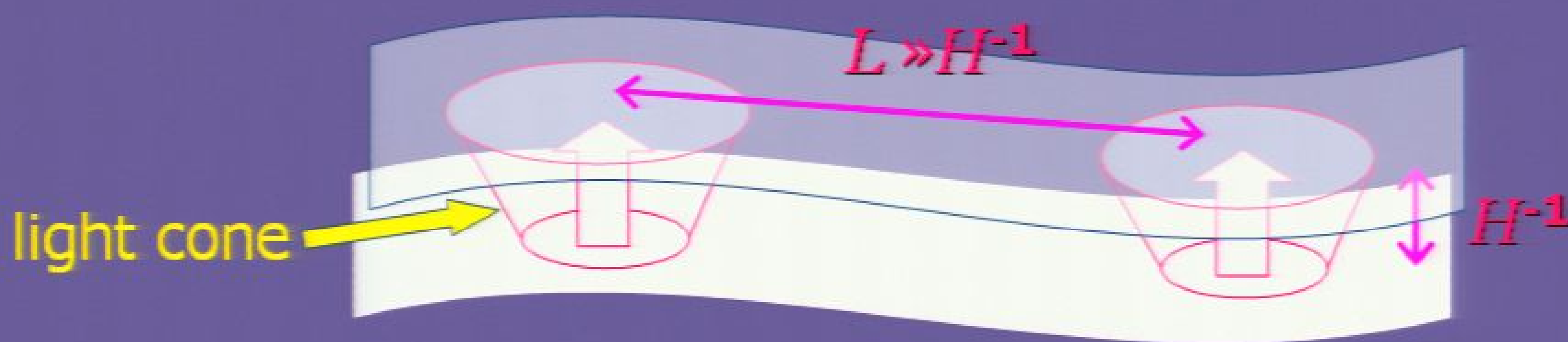

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Field equations reduce to ODE's

metric on superhorizon scales

- gradient expansion:

$$\partial_i \to \varepsilon \partial_i\,, \quad \varepsilon = \text{expansion parameter}$$

- metric:

$$ds^2 = -N^2 dt^2 + e^{2\alpha}\,\tilde{\gamma}_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

$$\det \tilde{\gamma}_{ij} = 1, \quad \beta^i = O(\varepsilon)$$

↑ contains GW (~ tensor) modes

↳ the only non-trivial assumption

$$\alpha\left(t, x^i\right) = \ln a(t) + \psi\left(t, x^i\right);\ \psi \sim \text{curvature perturbation}$$

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e.g., choose $\psi(t_*, 0) = 0$

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$$\tilde{H}^2(t,x^i) = \frac{8\pi G}{3}\rho(t,x^i) + O(\varepsilon^2) \qquad \tilde{H} = \frac{\partial_t \alpha}{N}$$

x^i : comoving (Lagrangean) coordinates.

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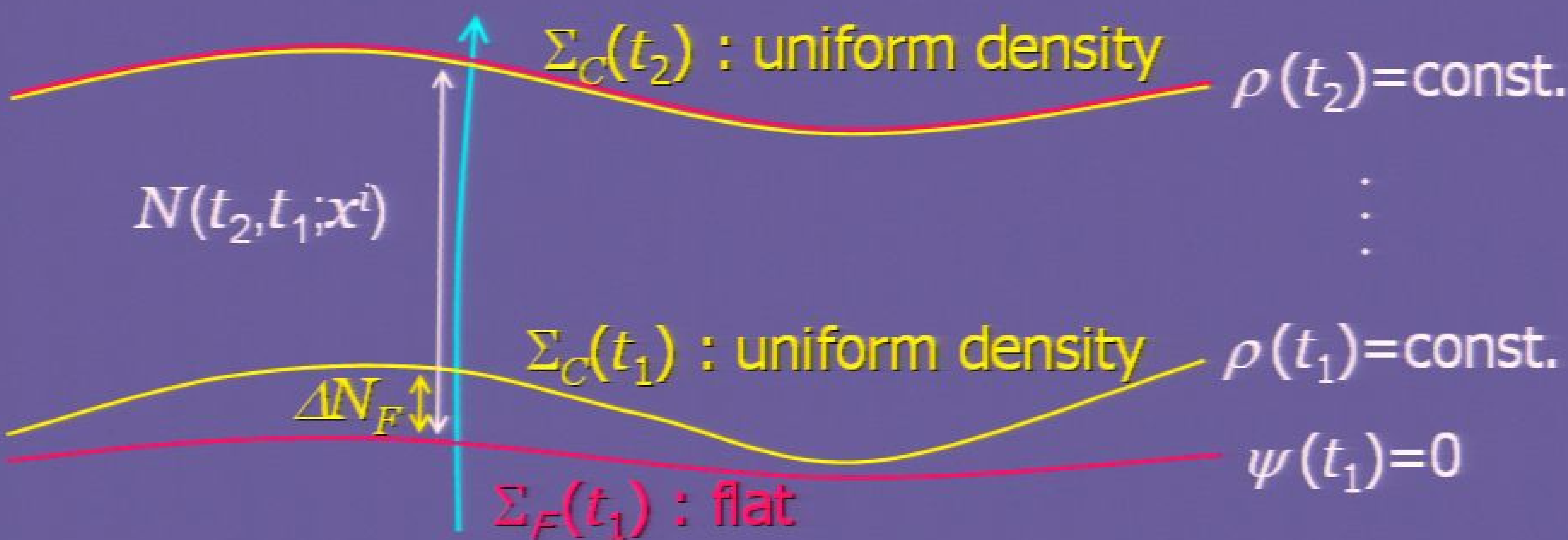
• ΔN - formula

Lyth & Wands '03, Malik, Lyth & MS '04,
Lyth & Rodriguez '05, Langlois & Vernizzi '05

Let us take slicing such that $\Sigma(t)$ is flat at $t = t_1$ [$\Sigma_F(t_1)$]
and uniform density/uniform H/comoving at $t = t_2$ [$\Sigma_C(t_1)$] :

('flat' slice: $\Sigma(t)$ on which $\psi = 0 \leftrightarrow e^{\alpha} = a(t)$)



$$N(t_2,t_1;x^i) = N_0(t_2,t_1) + \Delta N_F$$

$$N_0(t_2,t_1) = \ln\left(\frac{a(t_2)}{a(t_1)}\right) \text{ between } \Sigma_C(t_1) \text{ and } \Sigma_C(t_2)$$

Then

$$\Delta N_F = \psi(t_2, x^i) - \psi(t_1, x^i) = \psi_C(t_2, x^i)$$

suffix C for comoving/uniform ρ/uniform H

where ΔN_F is equal to e-folding number from $\Sigma_F(t_1)$ to $\Sigma_C(t_1)$:

$$\Delta N_F = -\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_C(t_2)} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt + \frac{1}{3}\int_{\Sigma_C(t_1)}^{\Sigma_C(t_2)} \frac{\partial_t \rho}{\rho + P} dt$$

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For slow-roll inflation in linear theory, this reduces to

$$\psi_C(t_2) \equiv \mathcal{R}_C(t_2) = \delta N(t_1; t_2) = H(t_1)\delta t_{F\to C} = \left[\sum_a \frac{\partial N}{\partial \phi^a}\delta\phi^a_F\right](t_1)$$

- Conserved nonlinear curvature perturbation

Lyth & Wands '03, Rigopoulos & Shellard '03, ...

For adiabatic case (P=$P(\rho)$,or single-field slow-roll inflation),

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$$\Rightarrow \quad \varsigma_{NL}(x^i)\equiv\psi(t,x^i)+\frac{1}{3}\int_{\rho(t)}^{\rho(t,x^i)}\frac{d\rho}{\rho+P(\rho)}$$

···slice-independent

Lyth, Malik & MS '04

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$$\Rightarrow \quad \varsigma_{NL}(x^i)\equiv\psi(t,x^i)+\frac{1}{3}\int_{\rho(t)}^{\rho(t,x^i)}\frac{d\rho}{\rho+P(\rho)}$$

···slice-independent

Lyth, Malik & MS '04

non-linear generalization of 'gauge'-invariant quantity ζ or $\mathcal{R}_c$

- ψ and ρ can be evaluated on any time slice
- applicable to each decoupled matter component

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MS & Tanaka '98, Lyth & Rodriguez '05

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where $\delta\phi = \delta\phi_F$ (on flat slice) at horizon-crossing.

($\delta\phi_F$ may contain non-gaussianity from subhorizon interactions)

cf. Maldacena '03, Weinberg '05, ...

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Then

$$\Delta N_F = \psi(t_2, x^i) - \psi(t_1, x^i) = \psi_C(t_2, x^i)$$

suffix C for comoving/uniform ρ/uniform H

where ΔN_F is equal to e-folding number from $\Sigma_F(t_1)$ to $\Sigma_C(t_1)$:

$$\Delta N_F = -\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_C(t_2)} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt + \frac{1}{3}\int_{\Sigma_C(t_1)}^{\Sigma_C(t_2)} \frac{\partial_t \rho}{\rho + P} dt$$

$$= -\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_C(t_1)} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt$$

For slow-roll inflation in linear theory, this reduces to

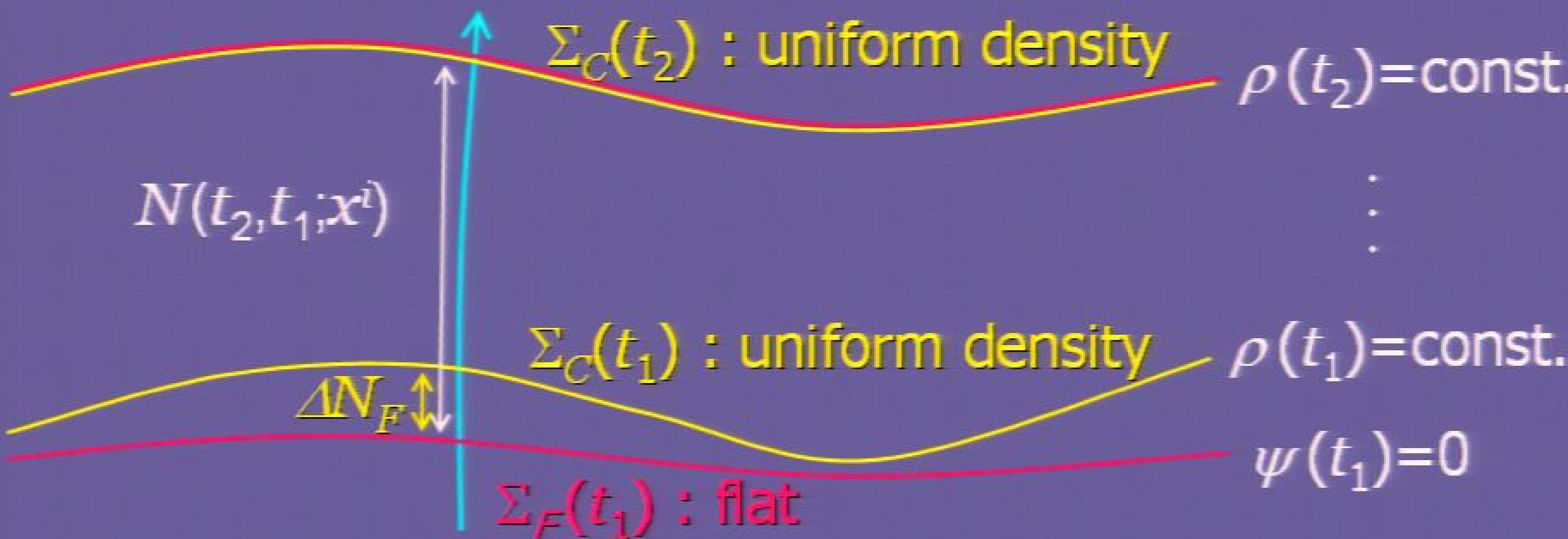
$$\psi_C(t_2) \equiv \mathcal{R}_C(t_2) = \delta N(t_1; t_2) = H(t_1)\delta t_{F\to C} = \left[\sum_a \frac{\partial N}{\partial \phi^a}\delta\phi^a_F\right](t_1)$$

- ΔN - formula

Lyth & Wands '03, Malik, Lyth & MS '04, Lyth & Rodriguez '05, Langlois & Vernizzi '05

Let us take slicing such that $\Sigma(t)$ is flat at $t = t_1$ [$\Sigma_F(t_1)$] and uniform density/uniform H/comoving at $t = t_2$ [$\Sigma_C(t_1)$] :

('flat' slice: $\Sigma(t)$ on which $\psi = 0 \leftrightarrow e^{\alpha} = a(t)$)



$$N(t_2,t_1;x^i) = N_0(t_2,t_1) + \Delta N_F$$

$$N_0(t_2,t_1) = \ln\left(\frac{a(t_2)}{a(t_1)}\right)$$ between $\Sigma_C(t_1)$ and $\Sigma_C(t_2)$

4. Nonlinear ΔN formula

- energy conservation:

  (applicable to each independent matter component)

$$\frac{\partial_t \rho}{3(\rho+p)} + O(\varepsilon^2) = -\partial_t \alpha = -\left(\frac{\dot{a}}{a} + \partial_t \psi\right) = -\tilde{H} N + O(\varepsilon^2)$$

- e-folding number:

$$N(t_2, t_1; x^i) \equiv \int_{t_1}^{t_2} \tilde{H} N dt = -\frac{1}{3}\int_{t_1}^{t_2} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt$$

where x^i=const. is a comoving worldline.

This definition applies to any choice of time-slicing.

$$\Rightarrow \quad \psi(t_2, x^i) - \psi(t_1, x^i) = \Delta N(t_2, t_1; x^i)$$

where

$$\Delta N(t_2, t_1; x^i) \equiv N(t_2, t_1; x^i) - \ln\left(\frac{a(t_2)}{a(t_1)}\right)$$

Conserved nonlinear curvature perturbation

Lyth & Wands '03, Rigopoulos & Shellard '03, …

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- $\varsigma_{NL}(x^i)$ …sl

8. Summary

- Superhorizon scale perturbations can never affect local (horizon-size) dynamics, hence never cause backreaction.

  nonlinearity on superhorizon scales are always local.

  However, nonlocal nonlinearity (non-Gaussianity) may appear due to quantum interactions on subhorizon scales.

  cf. Weinberg '06

- There exists a nonlinear generalization of δN formula which is useful in evaluating non-Gaussianity from inflation.

  diagrammatic method can by systematically applied.

  IR divergence from loop diagrams needs further consideration.

スライド ショーの最後です。クリックすると終了します。

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