Title: Effeciently generating generic entanglement

Date: Sep 07, 2007 09:30 AM

URL: http://pirsa.org/07090006

Abstract:

Waterloo, 7 Sep 2007

'Budding minds'
Perimeter Institute

Efficient Generation of Generic Entanglement

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www.imperial.ac.uk/quantuminformation

'Efficient Generation of Generic Entanglement'

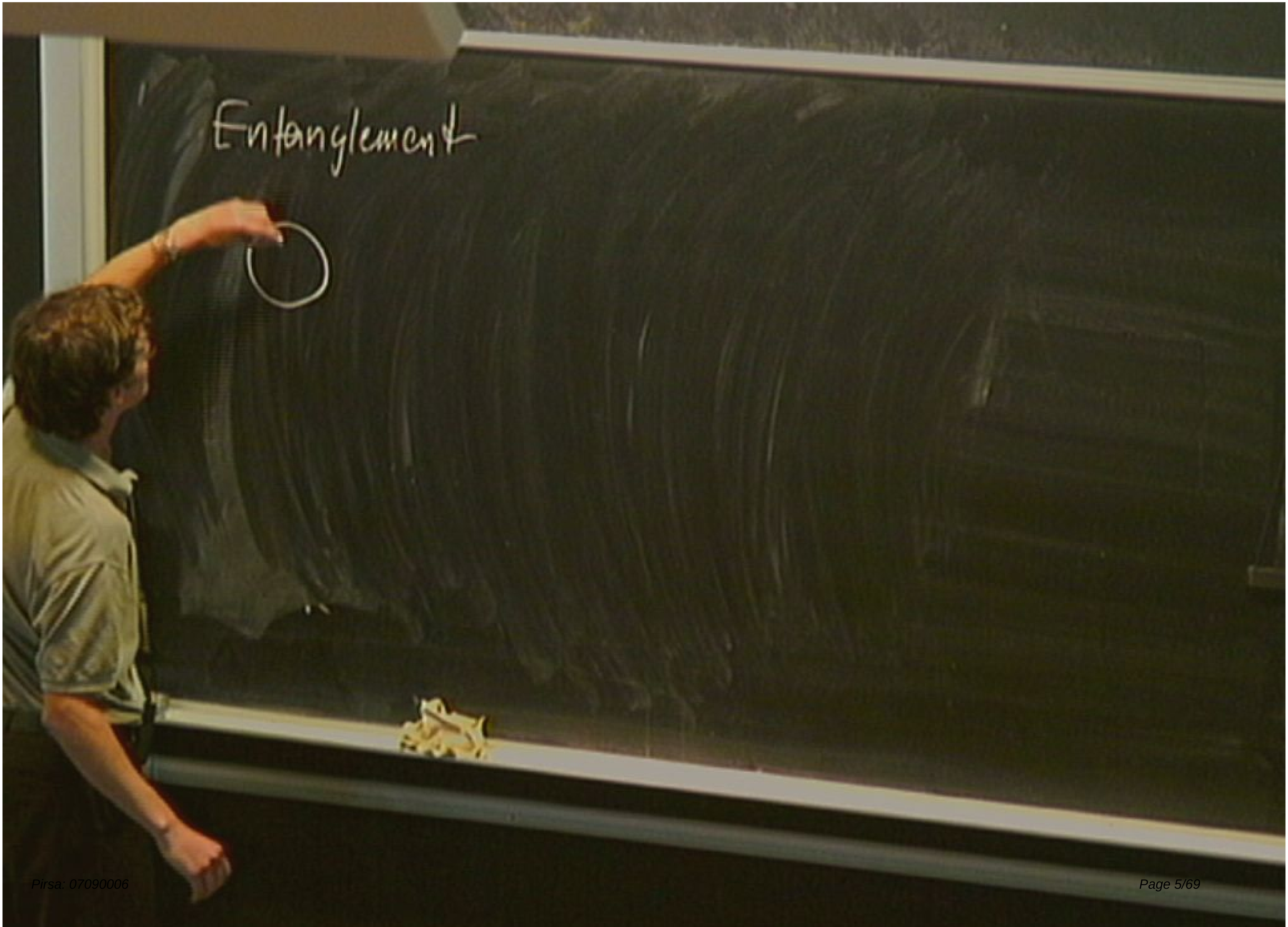
- By *Entanglement* we mean, unless otherwise stated, that taken between two parties sharing a pure state.

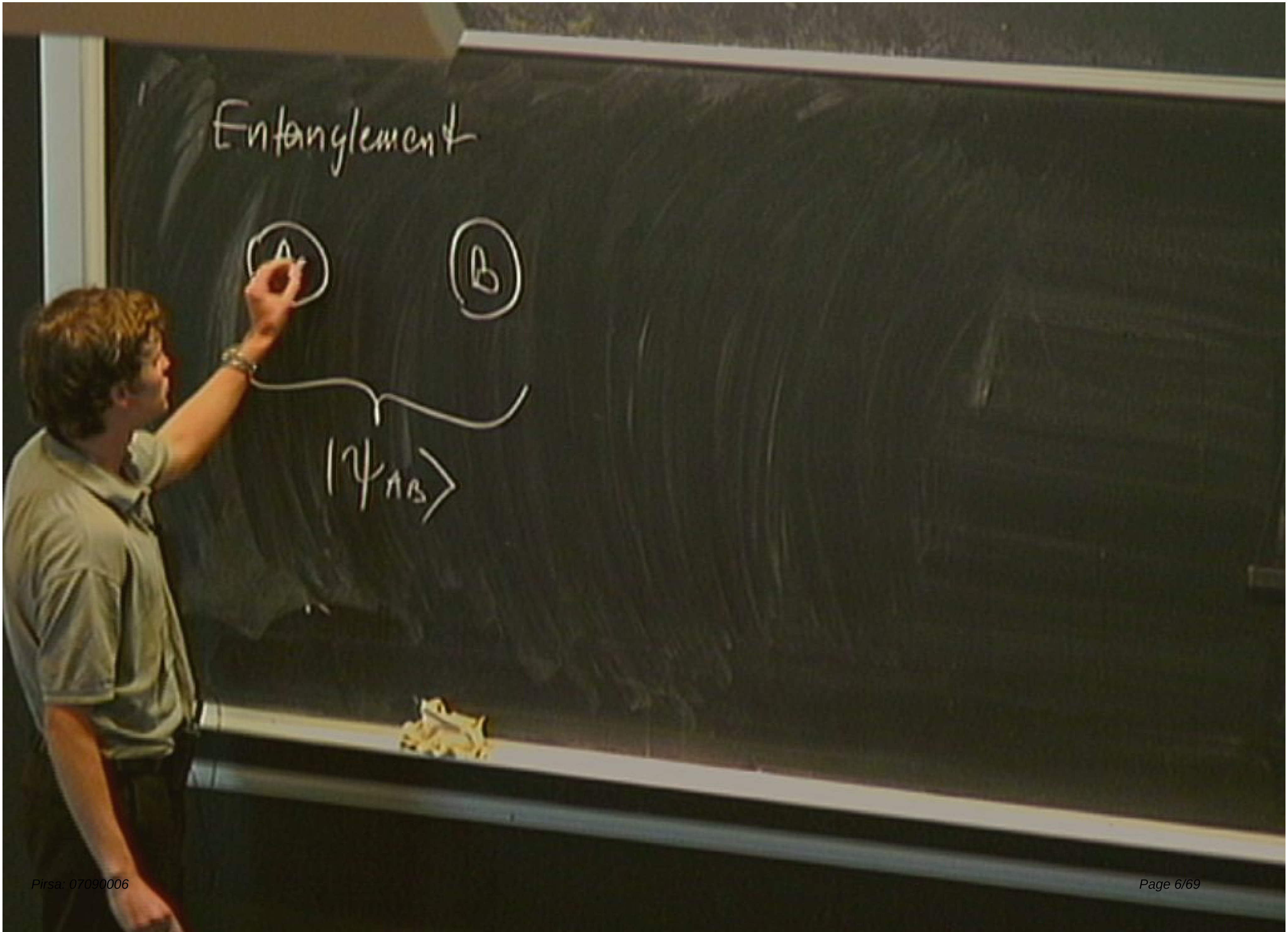
- By *Generic Entanglement* we mean the entanglement average over pure states picked from the uniform (Haar) distribution.

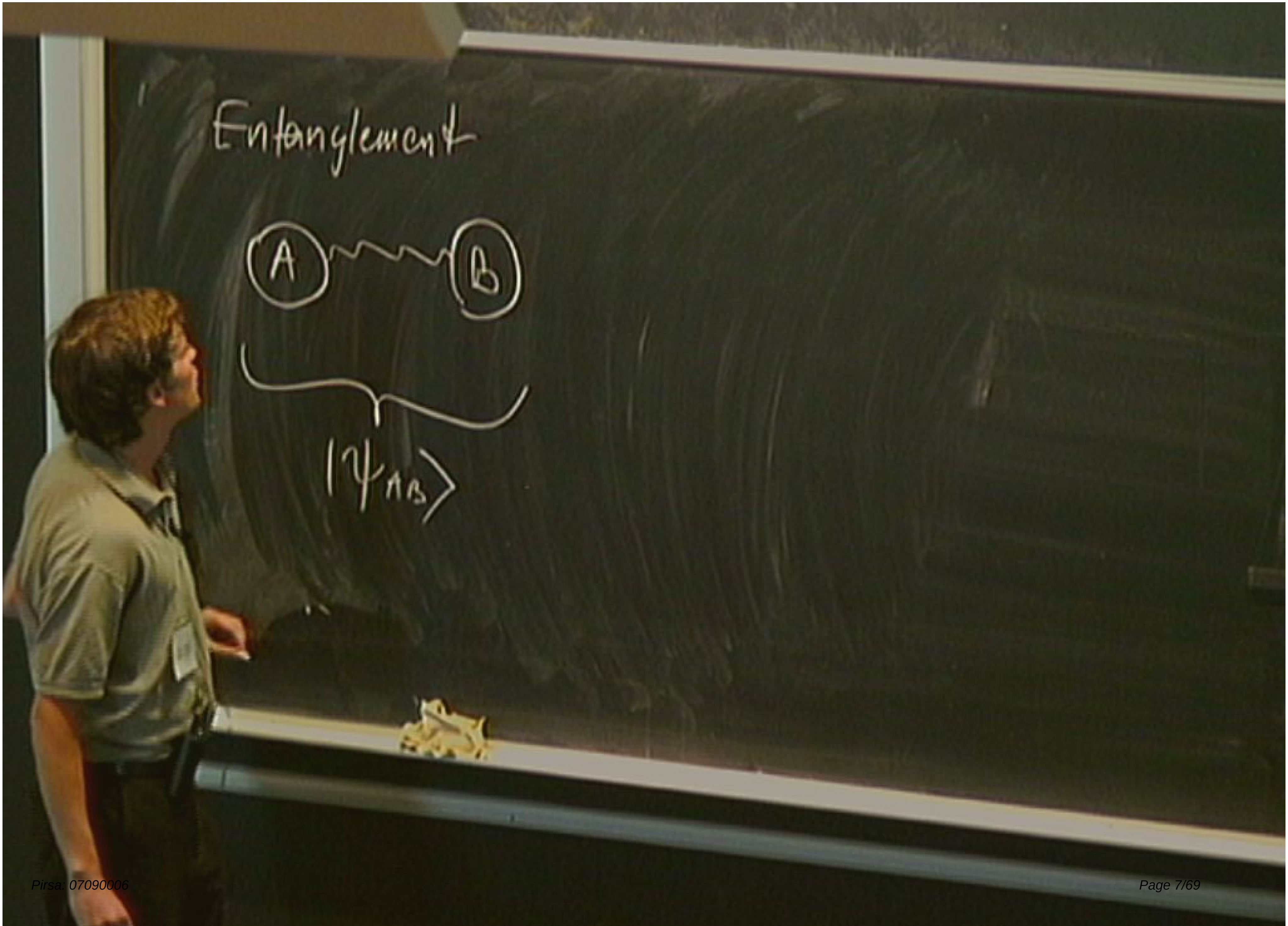
  [Hayden, Leung, Winter, Comm. Math.Phys. 2006]

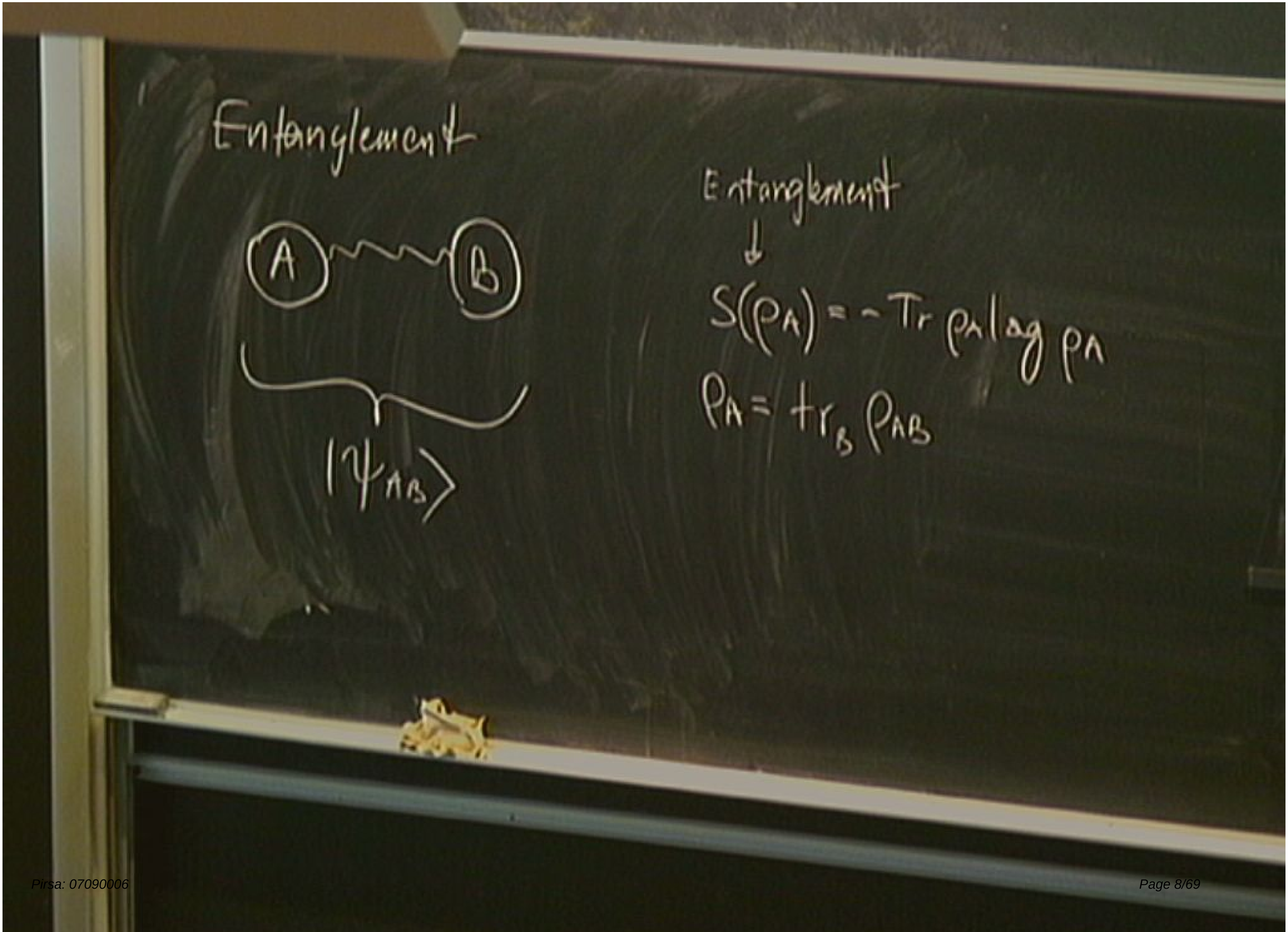
- By *Generating,* we mean that we have a random process yielding that average.

- By *Efficiently* we mean that the number of elementary(2-qubit) gates necessary grows as *poly(N)* where *N* is number of qubits carrying the state.


Entanglement


Entanglement
A
B
|ψ_AB⟩


Entanglement
A
B
$|\psi_{AB}\rangle$


Entanglement
A
B
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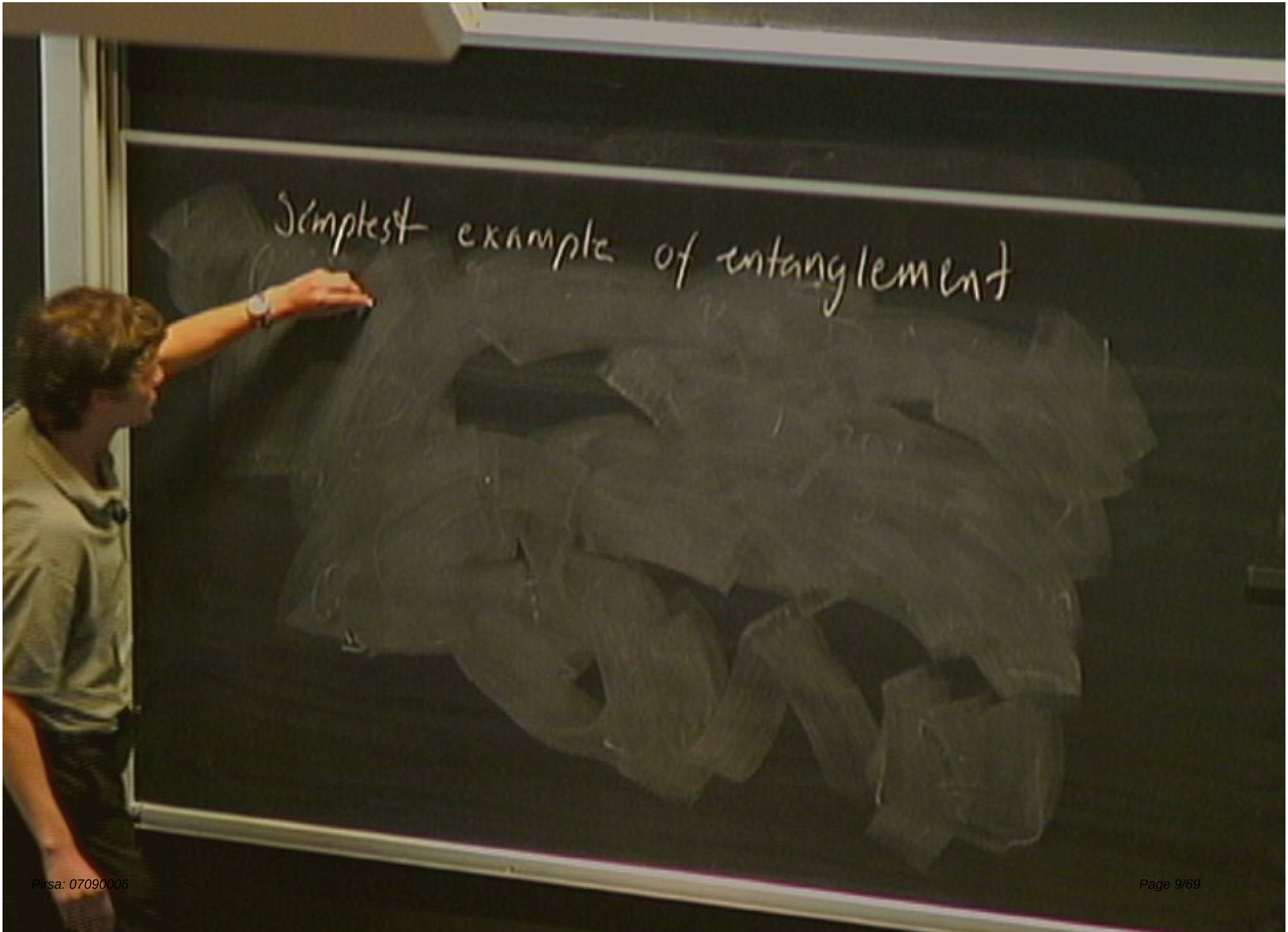
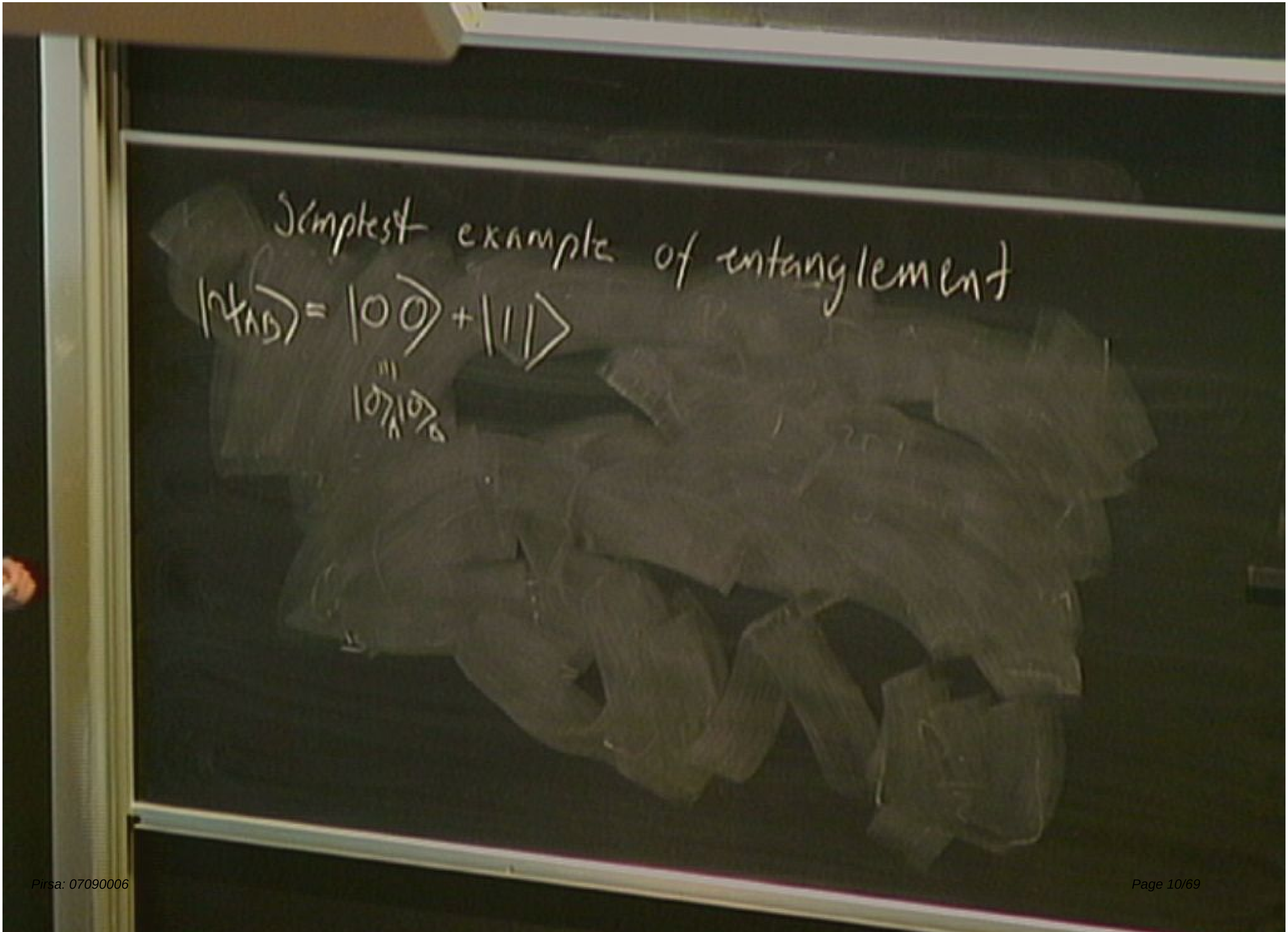
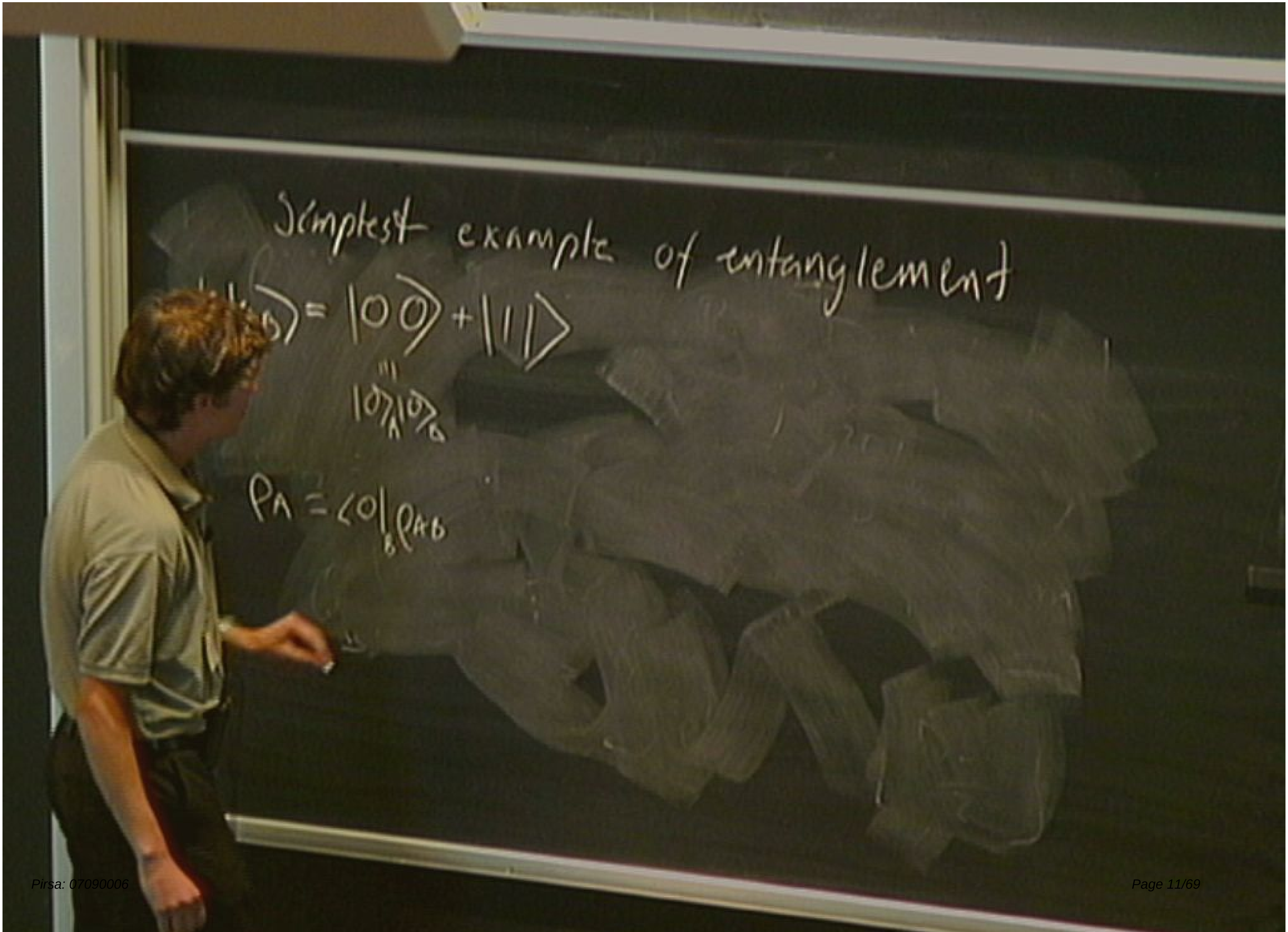
Entanglement
↓
$S(\rho_A) = -Tr\, \rho_A \log \rho_A$
$\rho_A = tr_B\, \rho_{AB}$


Simplest example of entanglement


Simplest example of entanglement
|ψ_AB⟩ = |00⟩ + |11⟩
=
|0⟩_A|0⟩_B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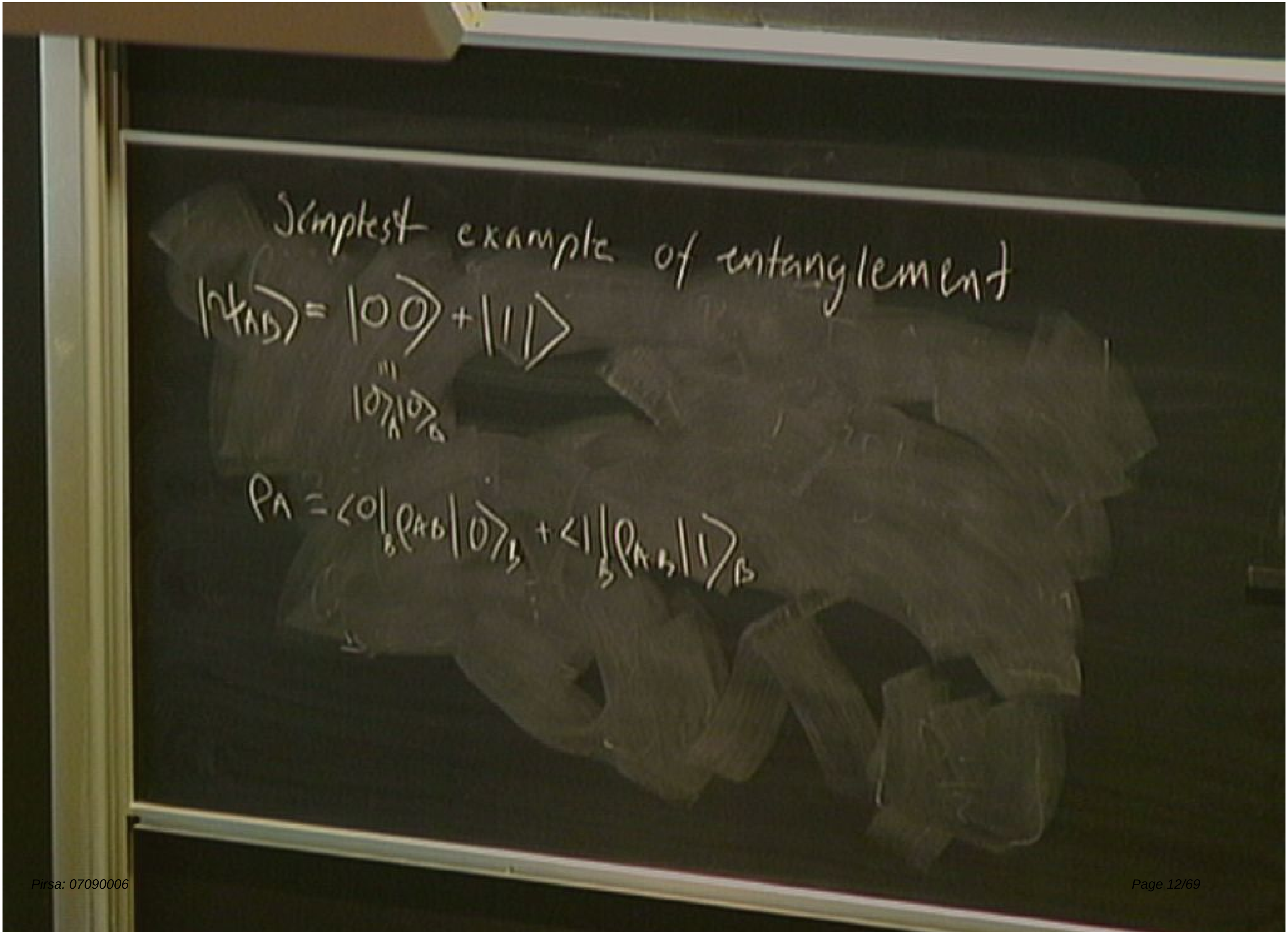
ρA = ⟨0|B ρAB


Simplest example of entanglement
$|\psi_{AB}\rangle = |00\rangle + |11\rangle$
$|0\rangle_A |0\rangle_B$
$\rho_A = \langle 0|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B$

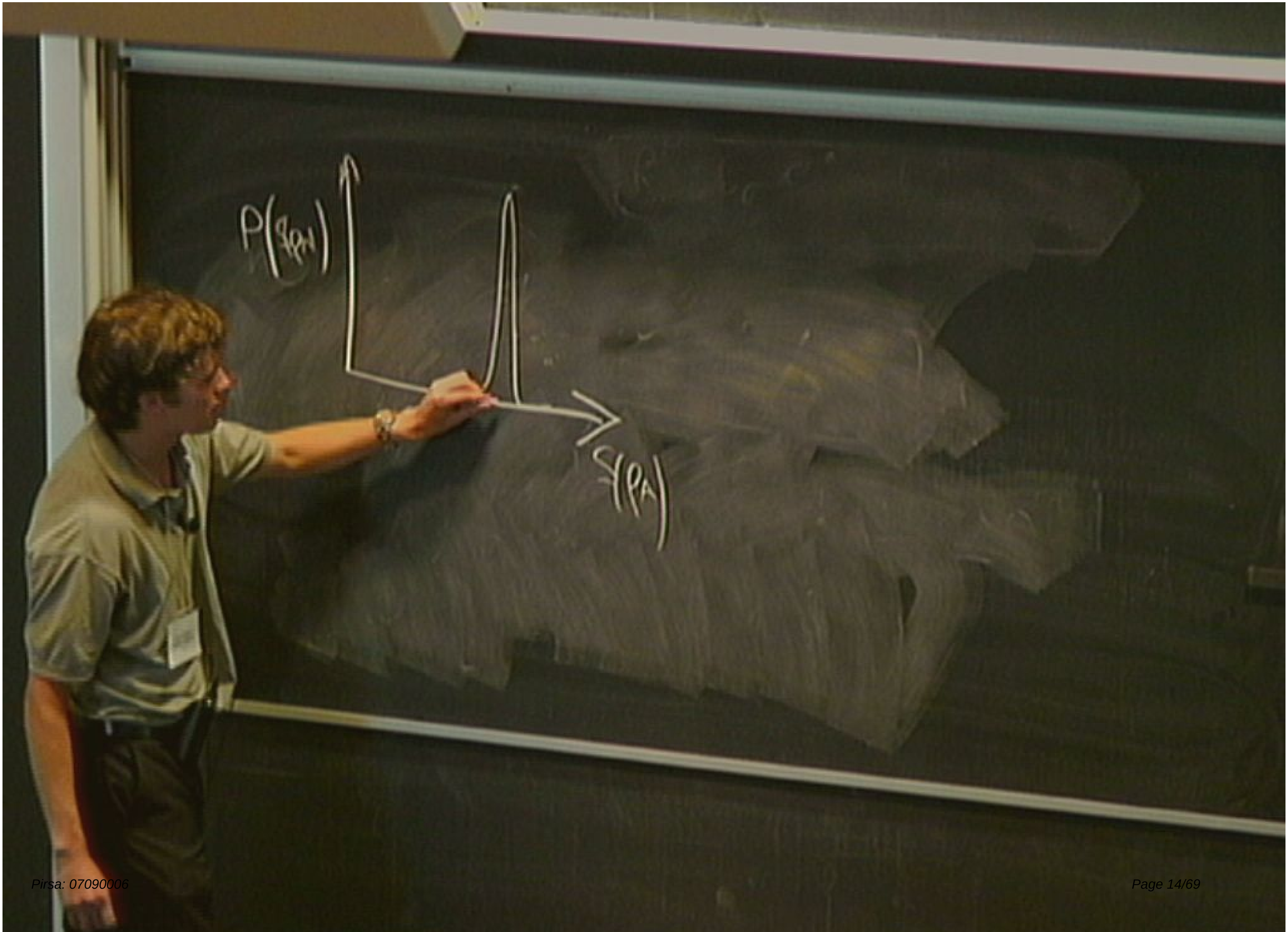
Simplest example of entanglement

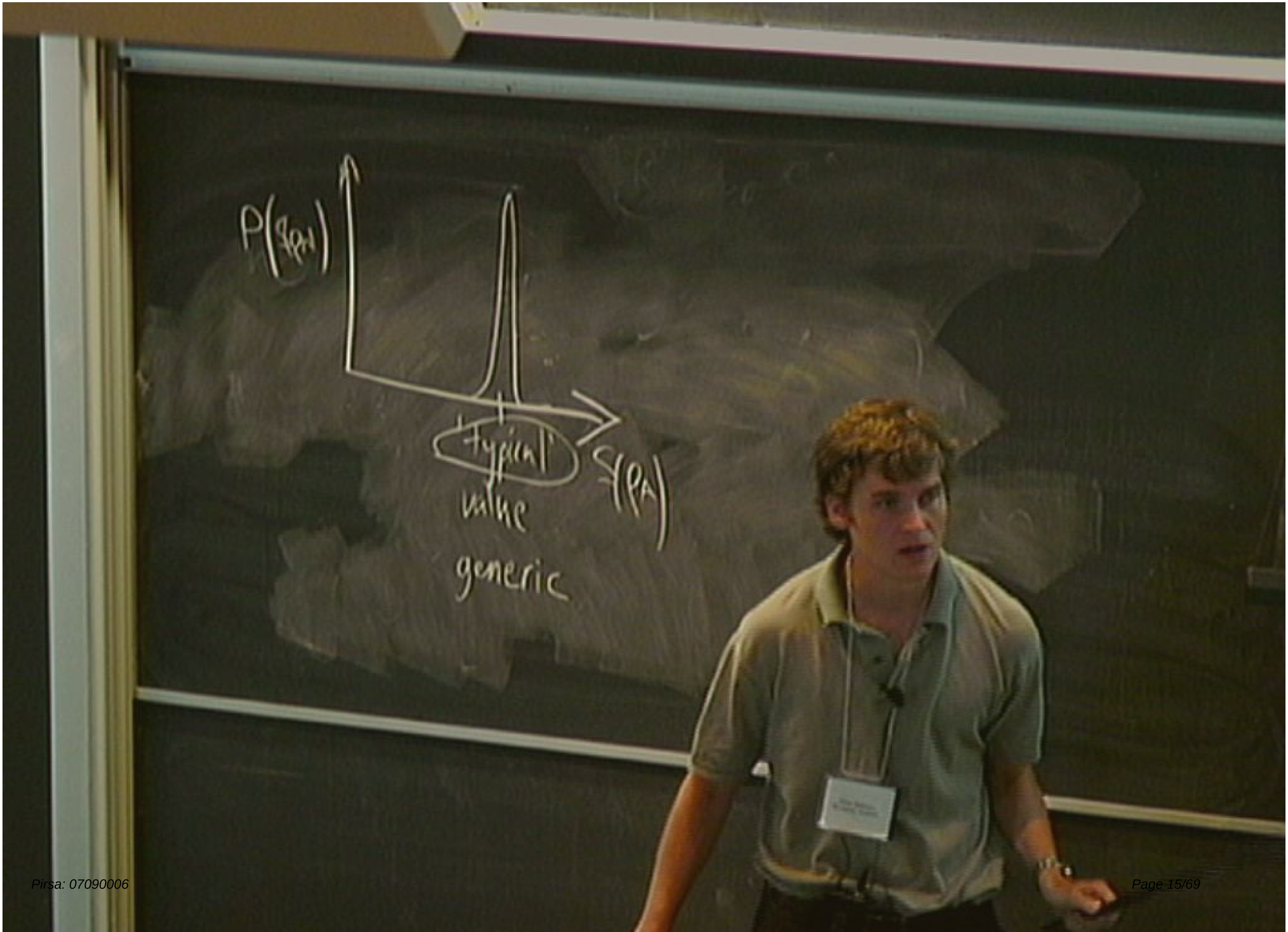
$$|\psi_{AB}\rangle = |00\rangle + |11\rangle$$

$$|00\rangle = |0\rangle_A |0\rangle_B$$

$$\rho_A = \langle 0|_B \rho_{AB} |0\rangle_B + \langle 1|_B \rho_{AB} |1\rangle_B$$

$$= \frac{1}{2}|0\rangle\langle 0|_A + \frac{1}{2}|1\rangle\langle 1|_B$$




'typical'
value
generic

‘*Efficient Generation of Generic Entanglement*’

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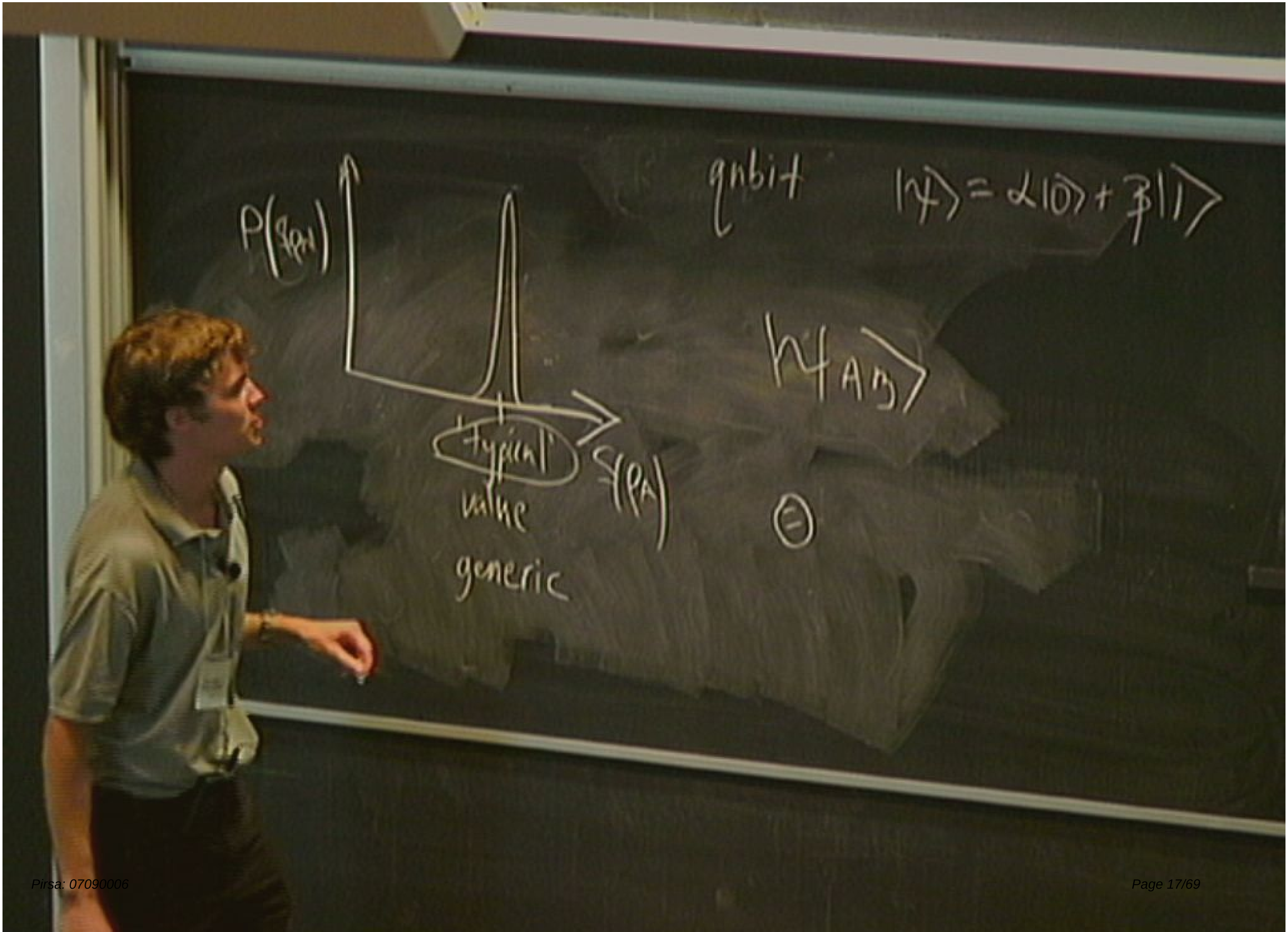
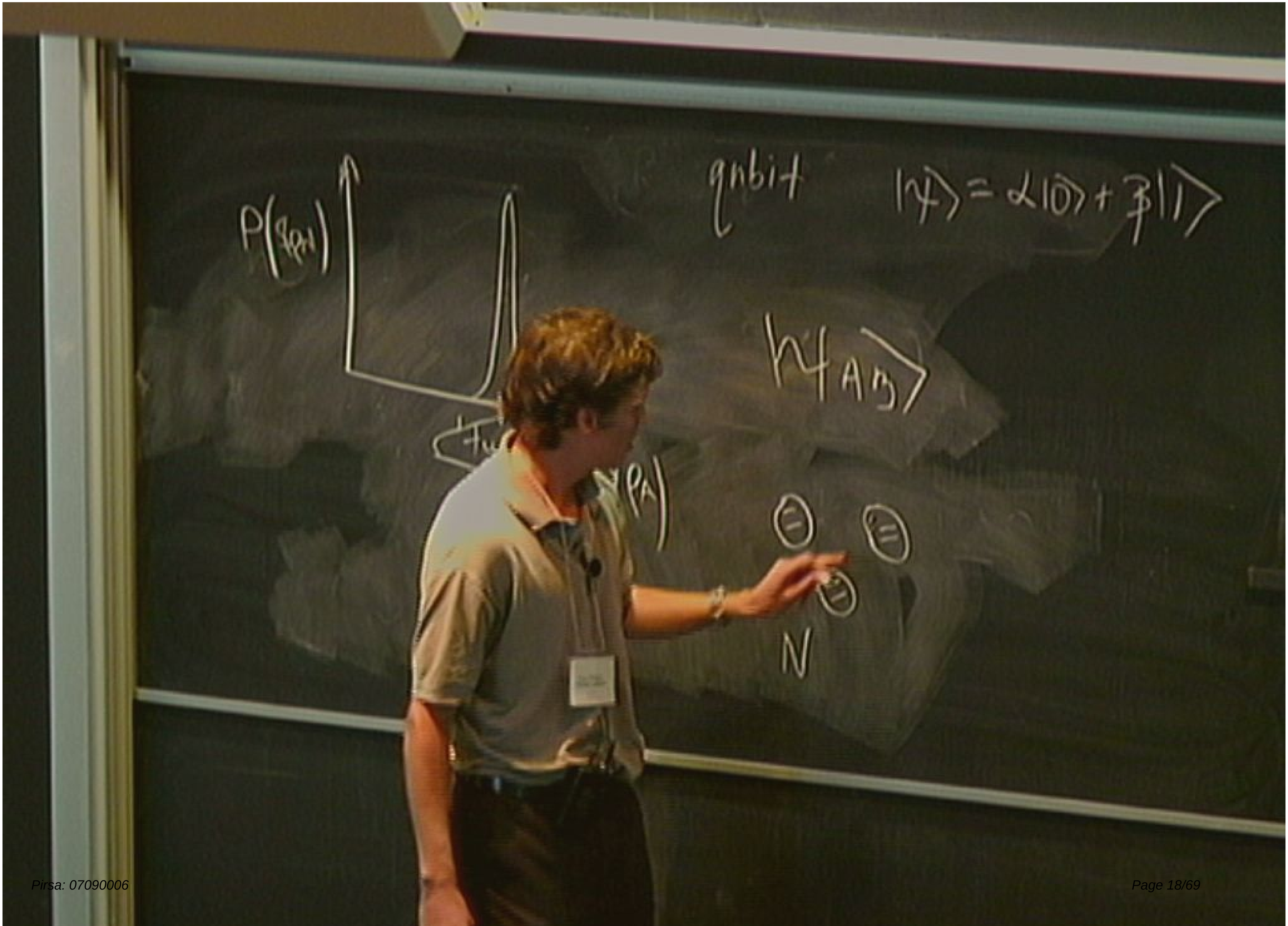
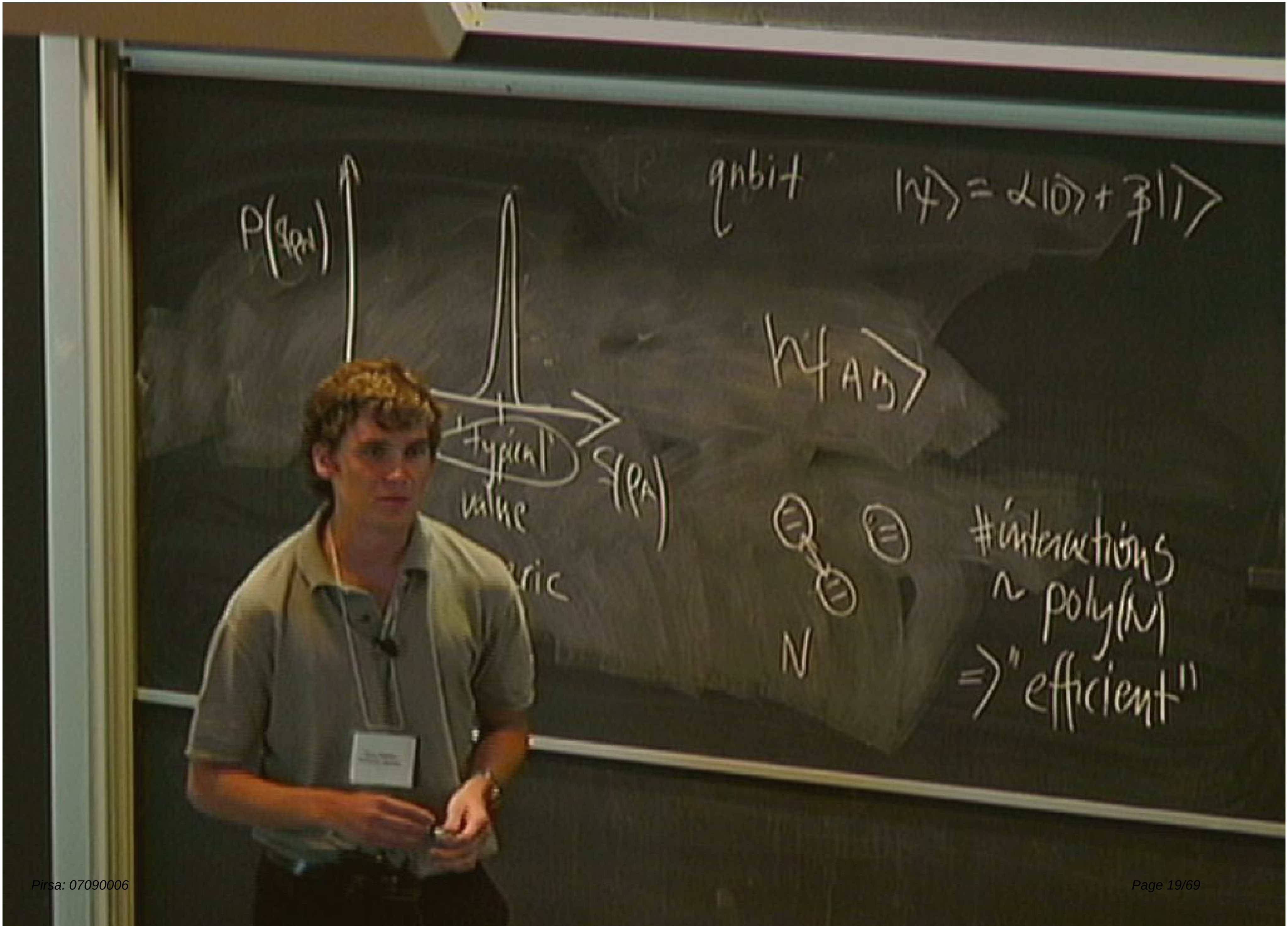

qubit
$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$
$P(S(\rho_A))$
'typical' value
generic
$S(\rho_A)$
$|\psi_{AB}\rangle$


qubit
|ψ⟩ = α|0⟩ + β|1⟩
|ψ_AB⟩
N


P(ρ_A)
'typical' value
S(ρ_A)
qubit
|ψ⟩ = α|0⟩ + β|1⟩
|ψ_AB⟩
N
#interactions
~ poly(N)
⇒ "efficient"

Talk Structure

This talk aims to explain key points of two papers relating to:
Efficient Generation of Generic Entanglement.
[Oliveira, Dahlsten and Plenio, q-ph/0605126, Phys. Rev. Lett. 2007]
[Dahlsten, Oliveira and Plenio, q-ph/0701125, J. Phys. A 2007]

- Introduction, aim of work
- **Result 1** (Theorem) : Generic entanglement is generated efficiently
- **Result 2** (Numerics): Generic entanglement is achieved at a particular instant.
- Conclusion

Motivation and Aim

- Restricting entanglement types to those that are typical/ generic could give a simplified entanglement theory.
  [Hayden, Leung, Winter, Comm. Math.Phys. 2006]

- 'Typical' has been defined relative to a flat distribution on pure states, the unitarily invariant measure, where $P(\psi) = P(U\psi)$.

- However *exp(N=system size)* two-qubit gates are necessary to get that flat distribution on states, so it seems unphysical.

- **Aim:** to prove that in spite of this, the **average entanglement associated with the unitarily invariant measure is physical** in that it appears after *poly(N)* gates.

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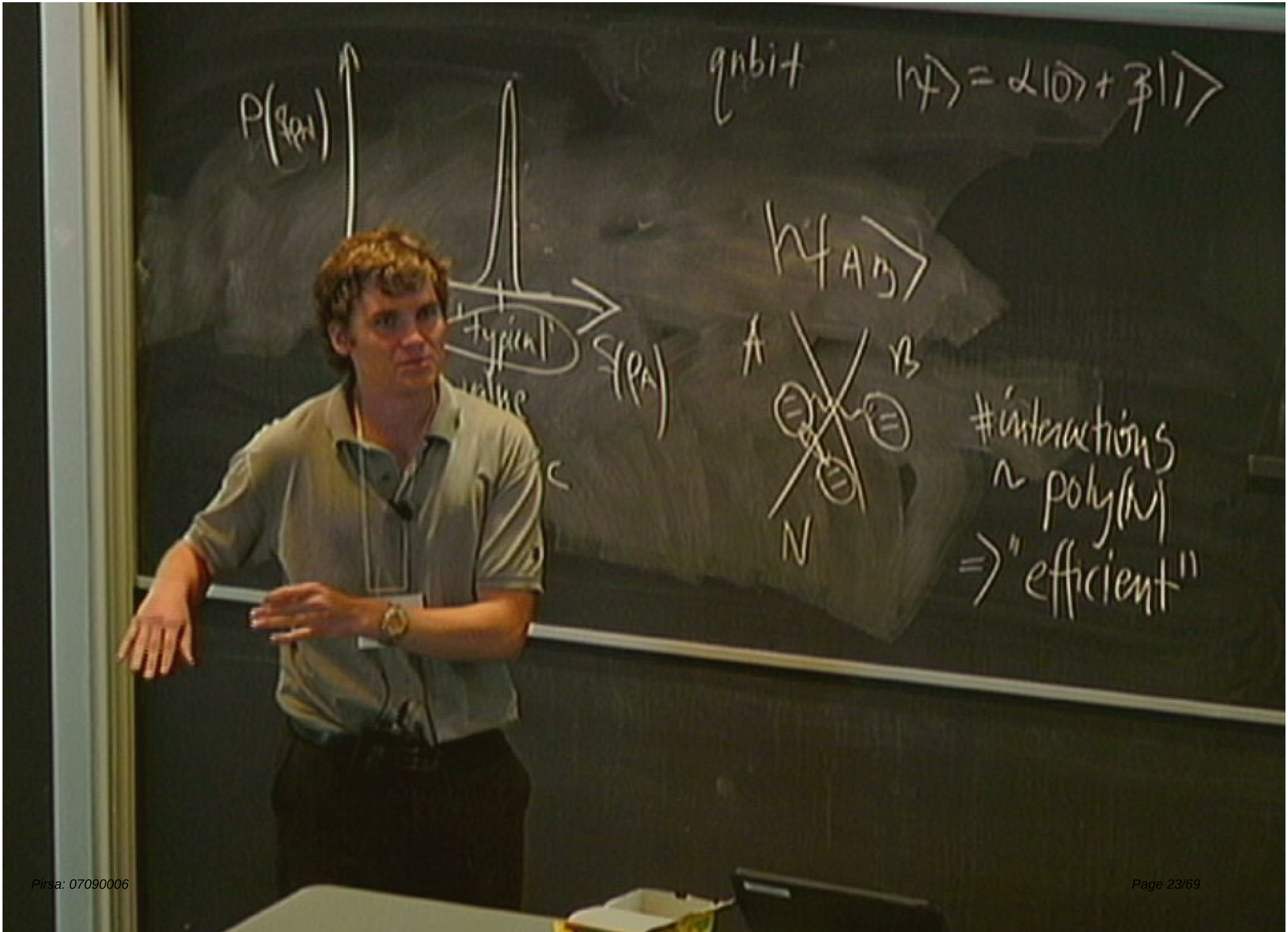

qubit
|ψ⟩ = α|0⟩ + β|1⟩
|ψAB⟩
A
B
N
"typical"
#interactions
~ poly(N)
=> "efficient"

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The random process

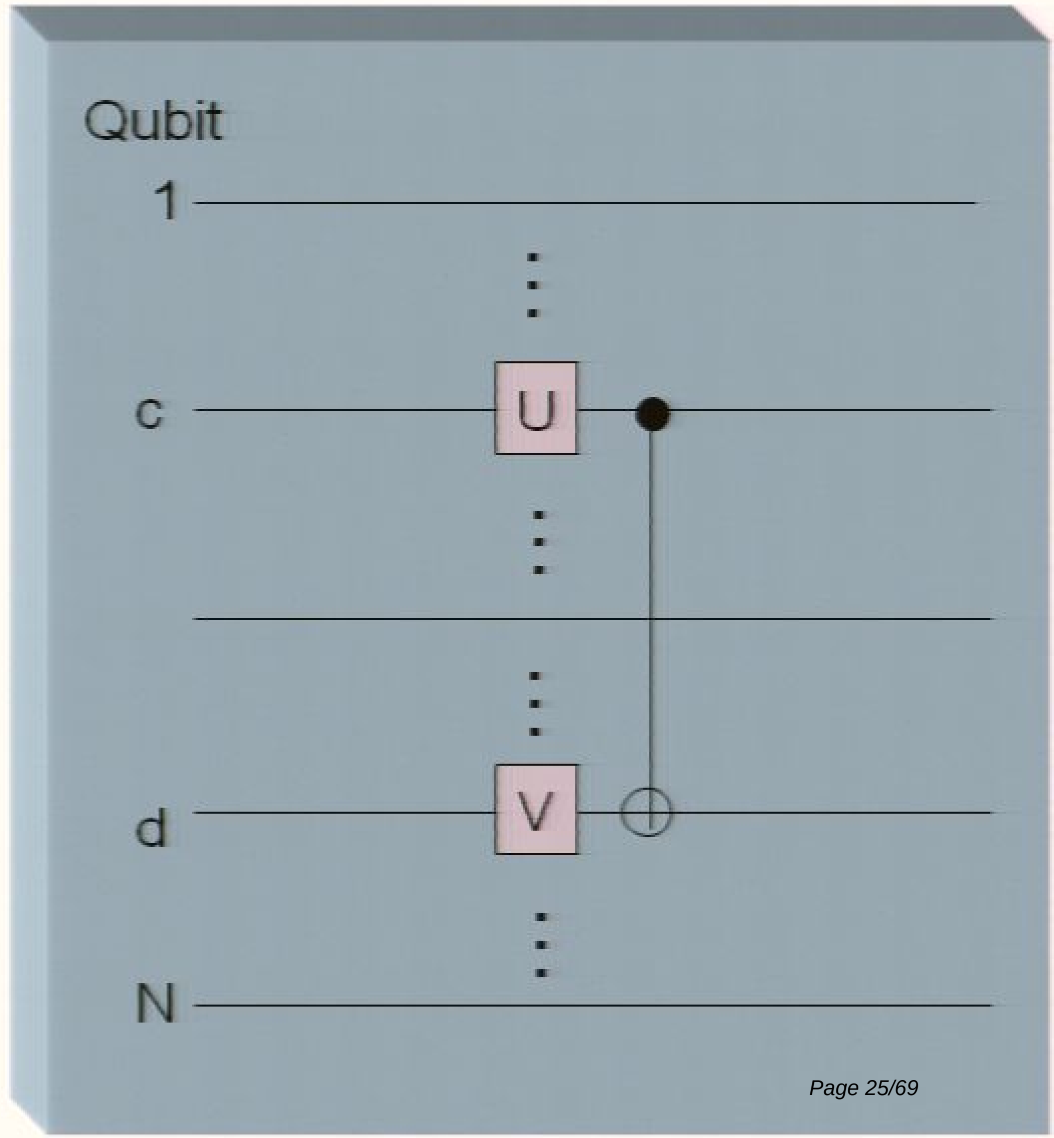
Consider random two-party interactions modelled as two-qubit gates:

1. Pick two single qubit unitaries, U and V, uniformly from the Bloch Sphere.

2. Choose a pair of qubits {c,d} without bias.

3. Apply U to c and V to d.

4. Apply a CNOT on c and d.



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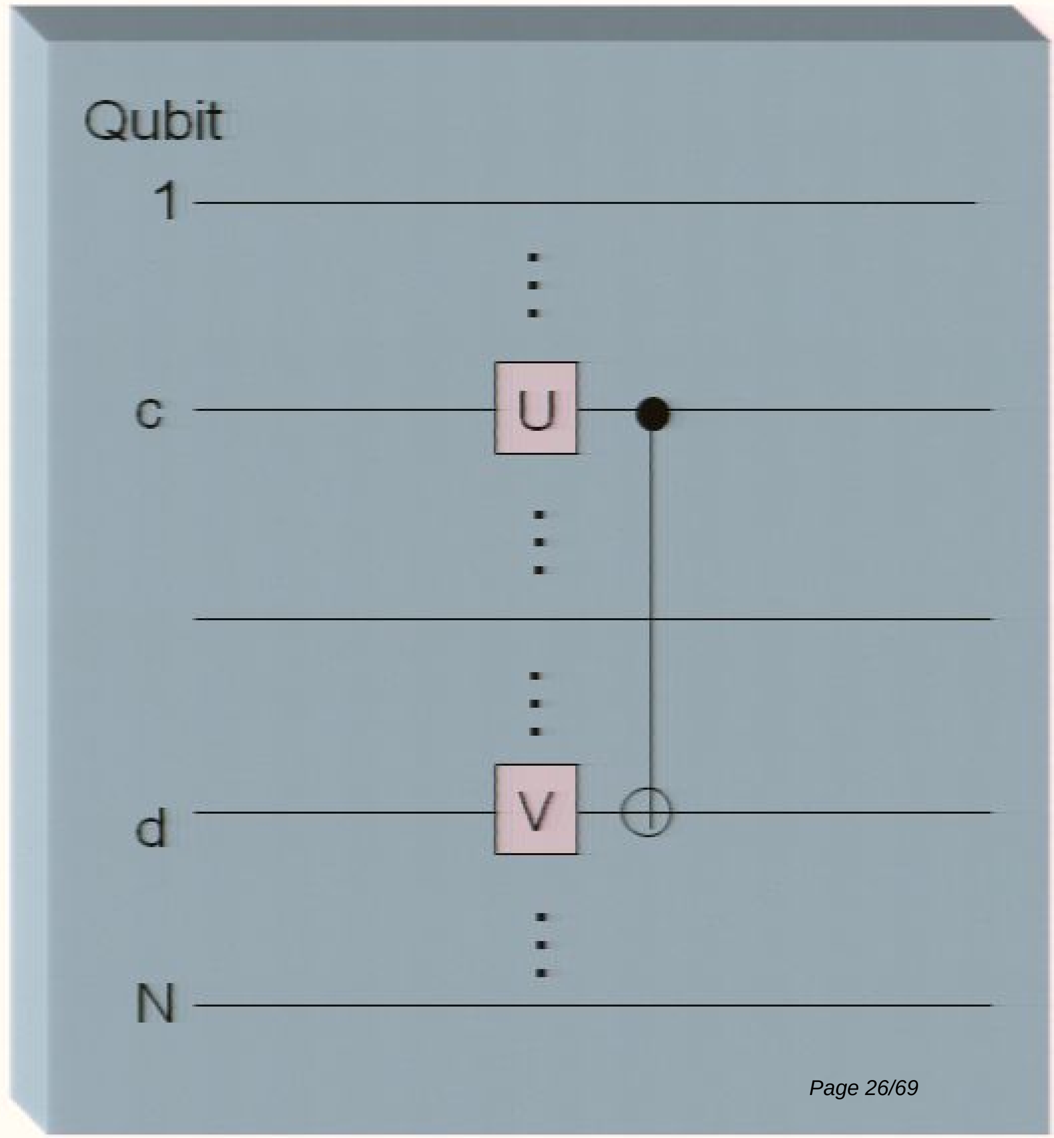
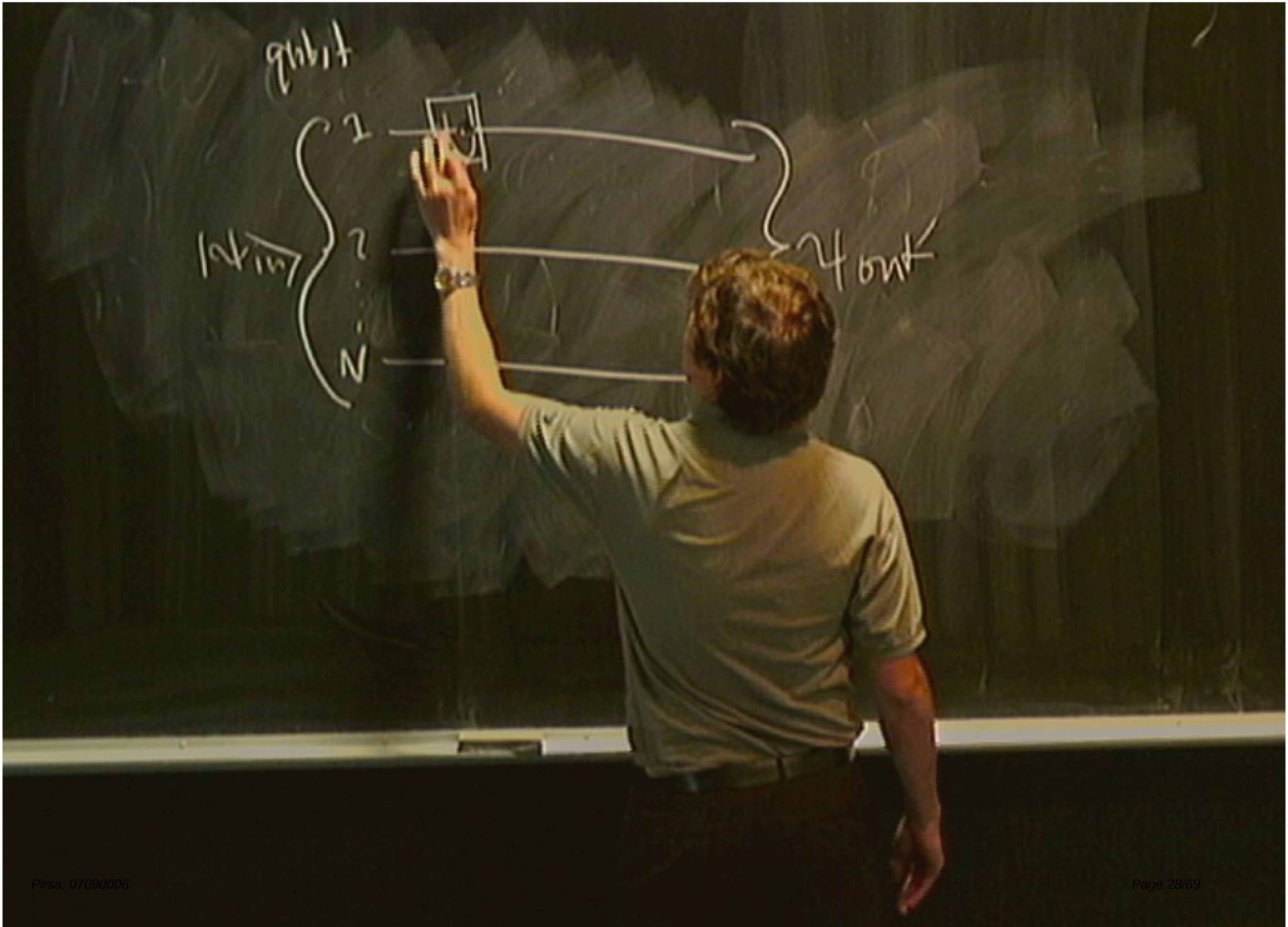
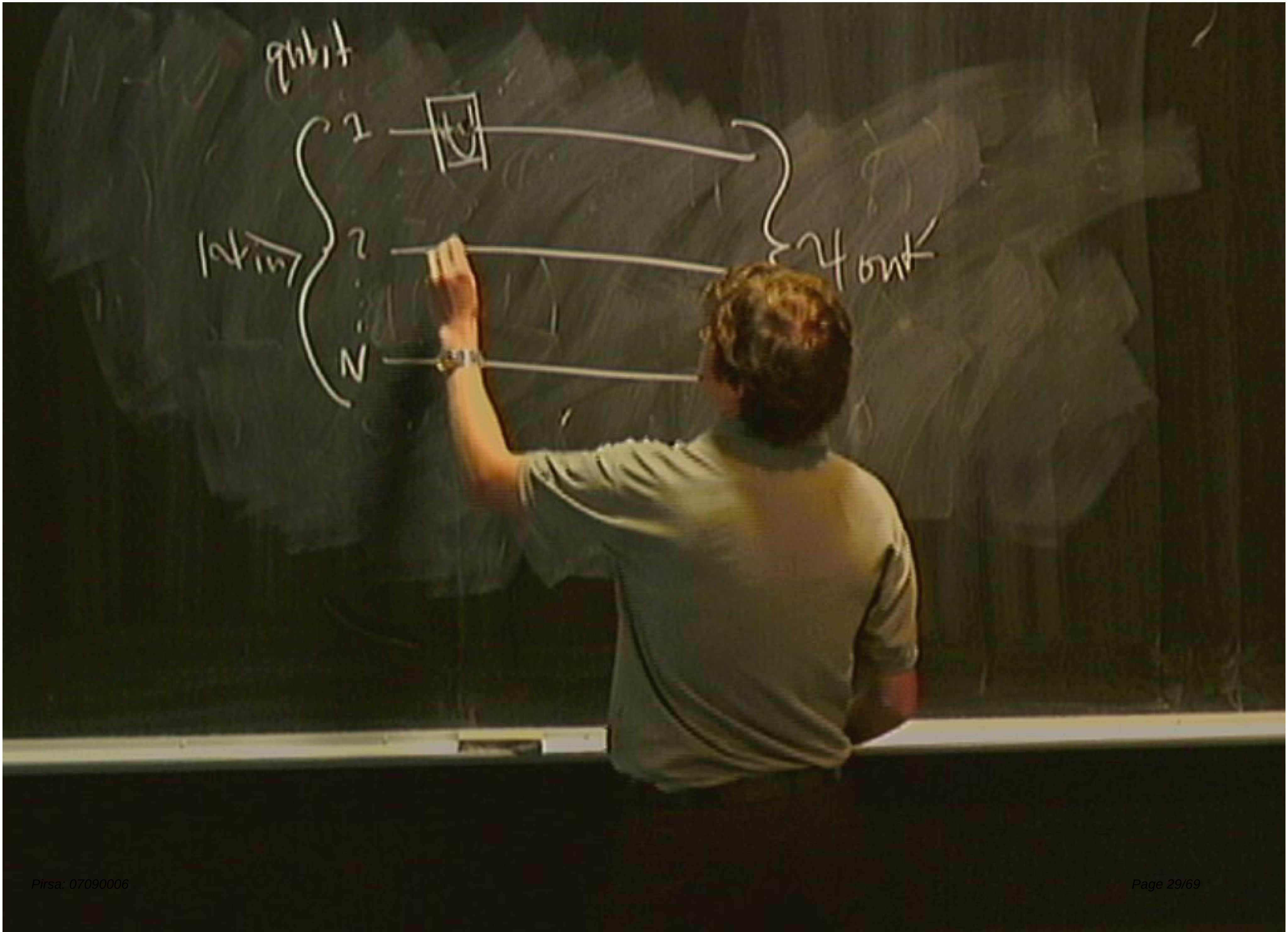



qubit
1
2
N
|ψ_in⟩
ψ_out


qubit
1
2
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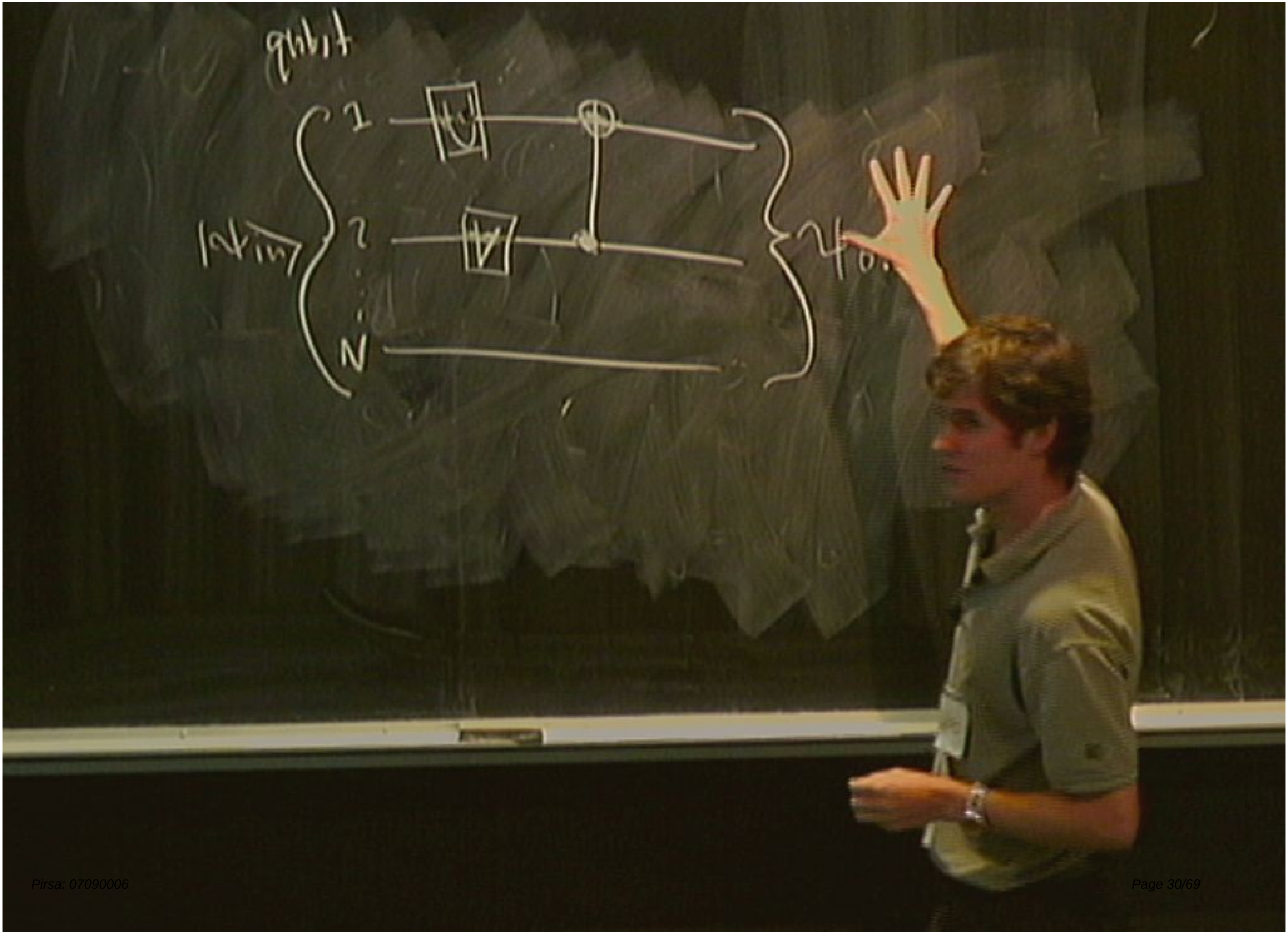
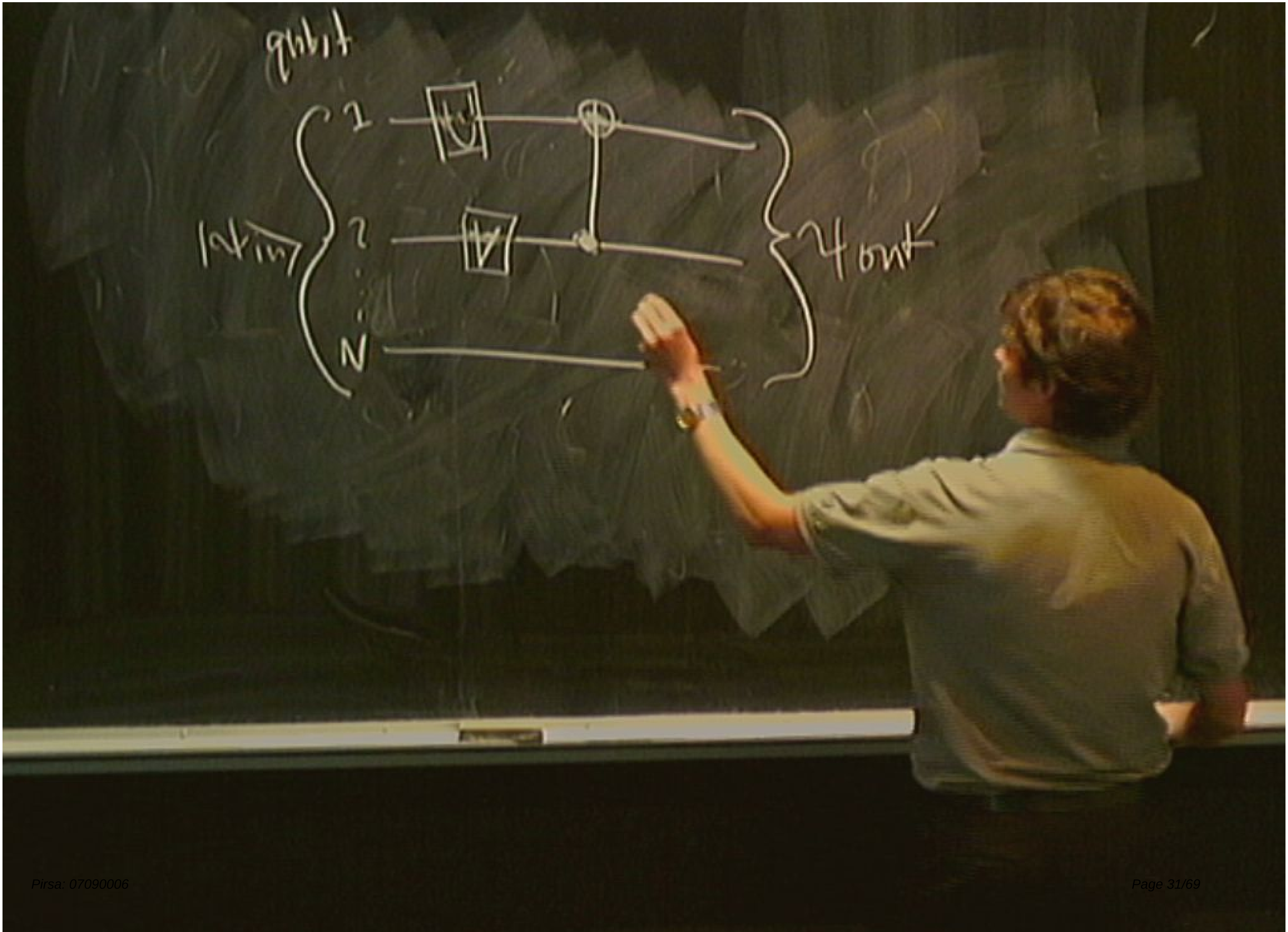

qubit
1
2
N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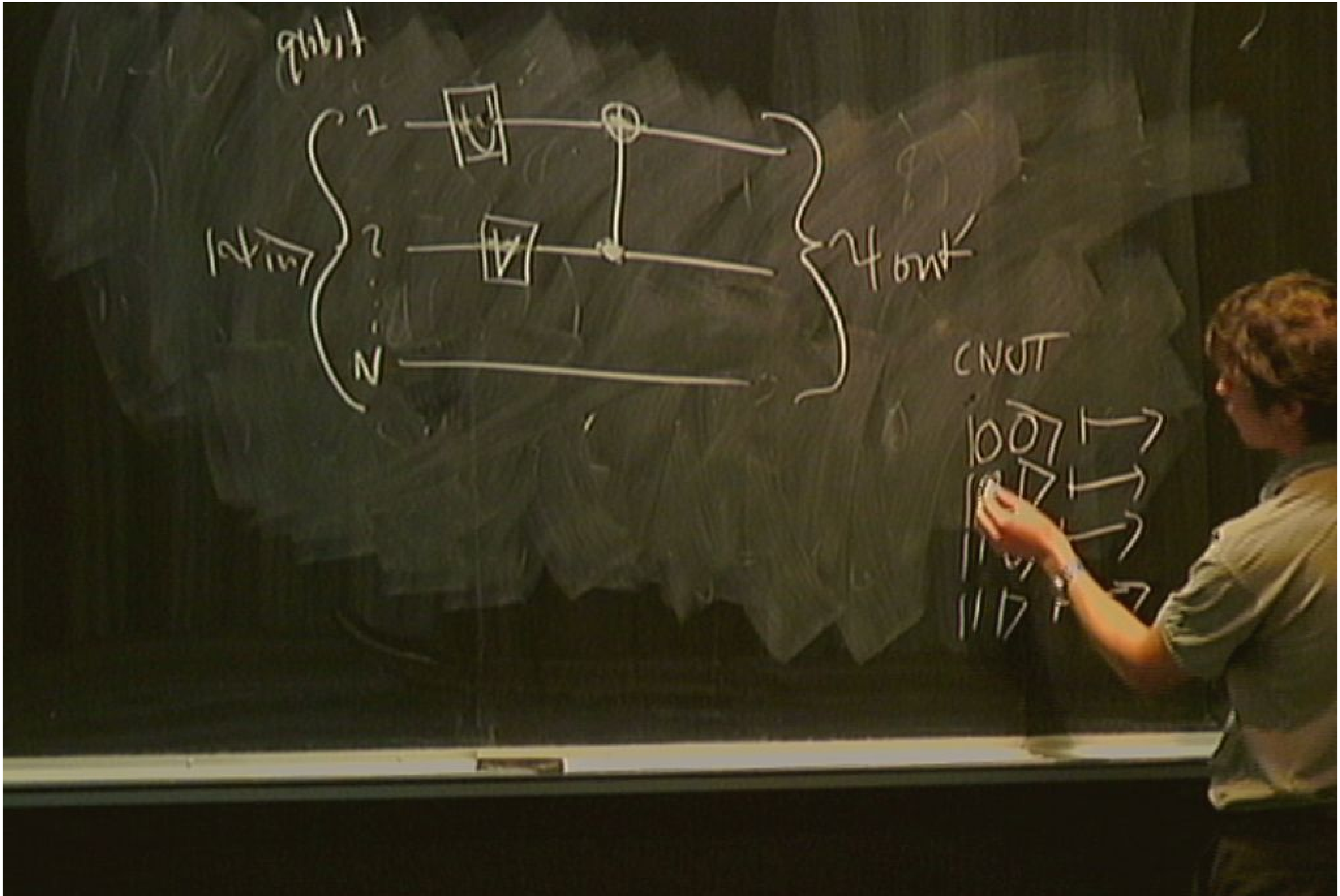
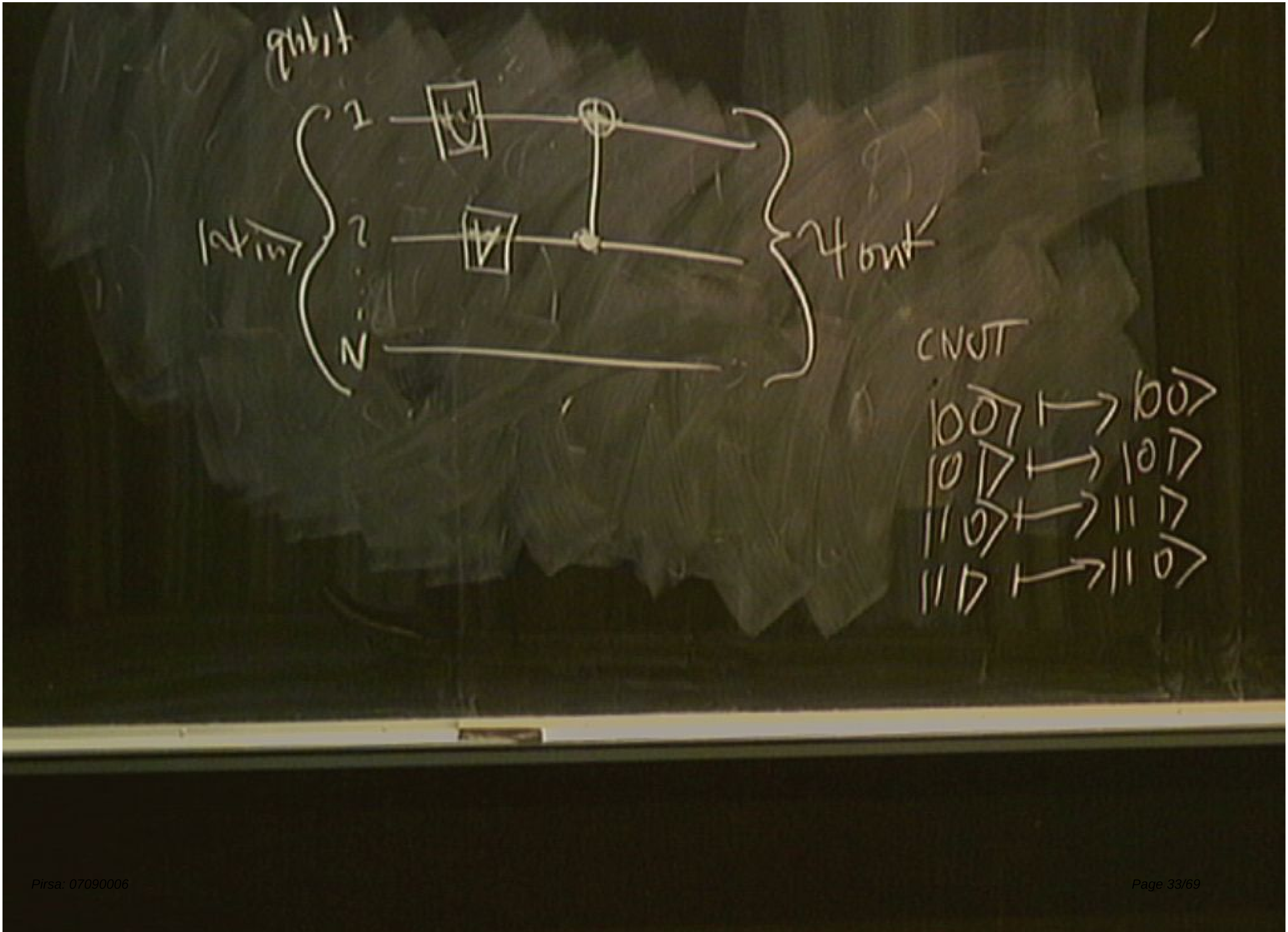

qubit
1
2
N
CNOT


qubit
1
2
N
|ψin⟩
ψout
CNOT
|00⟩ ↦ |00⟩
|01⟩ ↦ |01⟩
|10⟩ ↦ |11⟩
|11⟩ ↦ |10⟩

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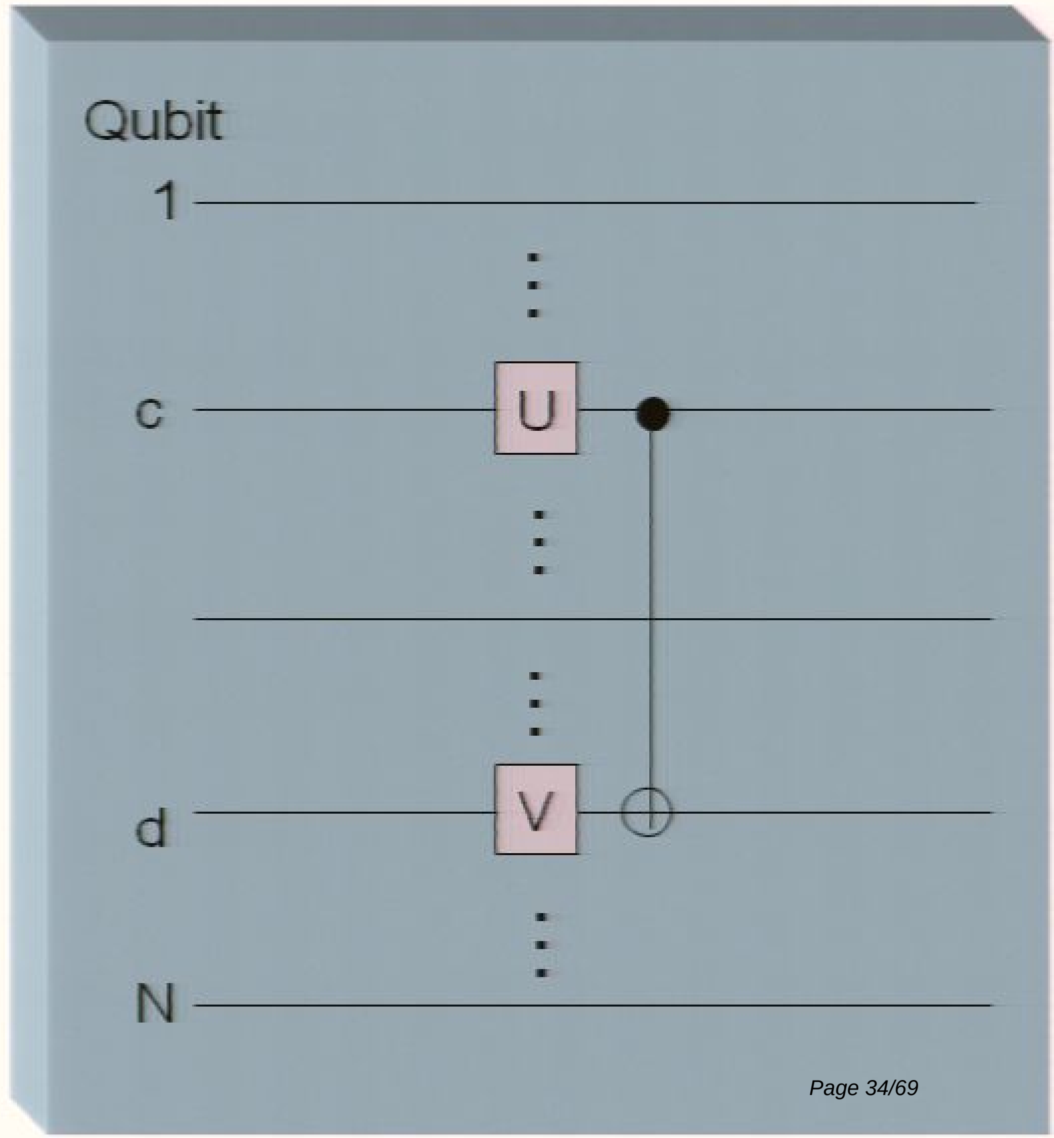

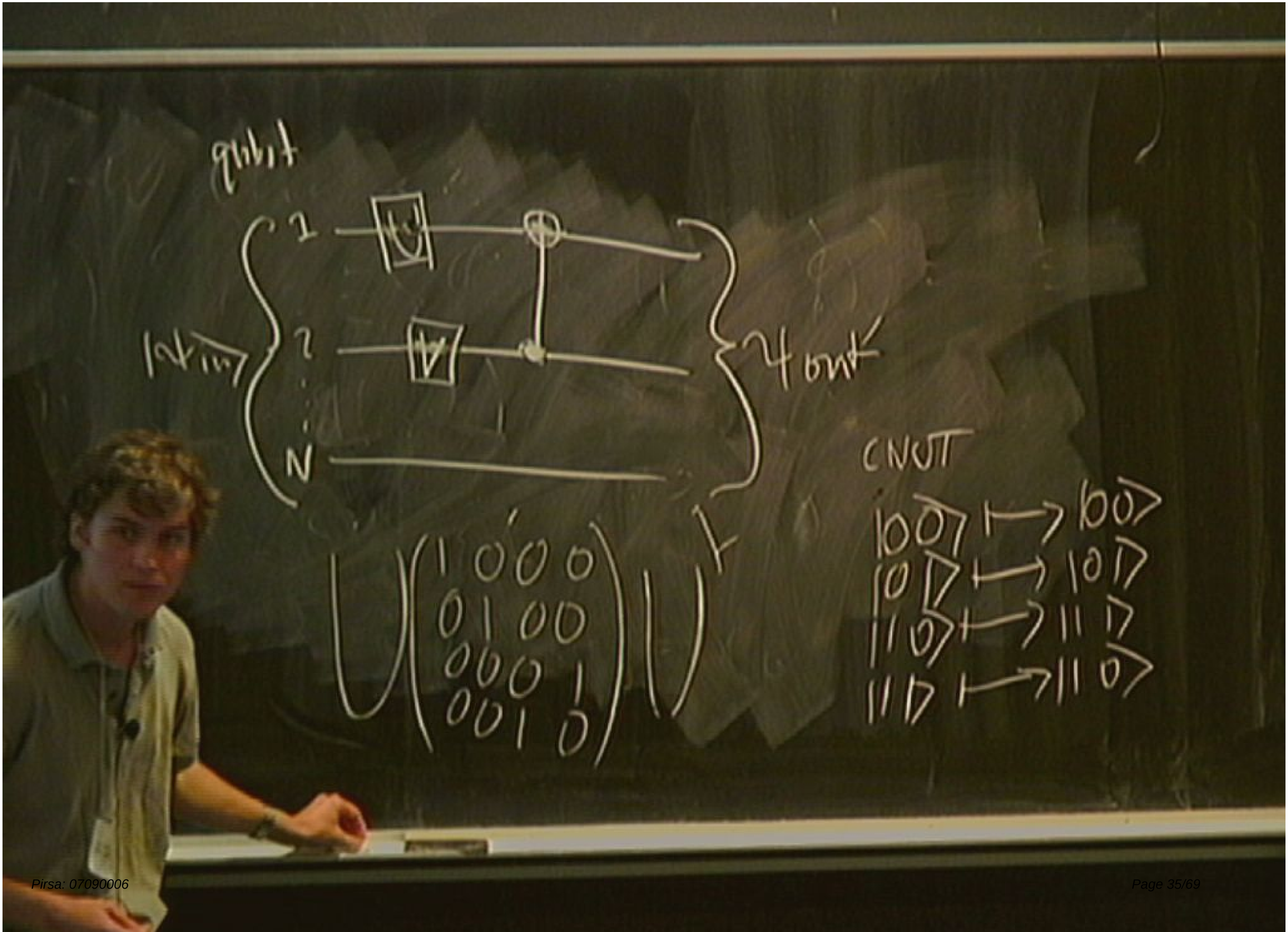
qubit
1
2
N
|ψ in⟩
|ψ out⟩
CNOT
|00⟩ ↦ |00⟩
|01⟩ ↦ |01⟩
|10⟩ ↦ |11⟩
|11⟩ ↦ |10⟩
U
1 0 0 0
0 1 0 0
0 0 0 1
0 0 1 0

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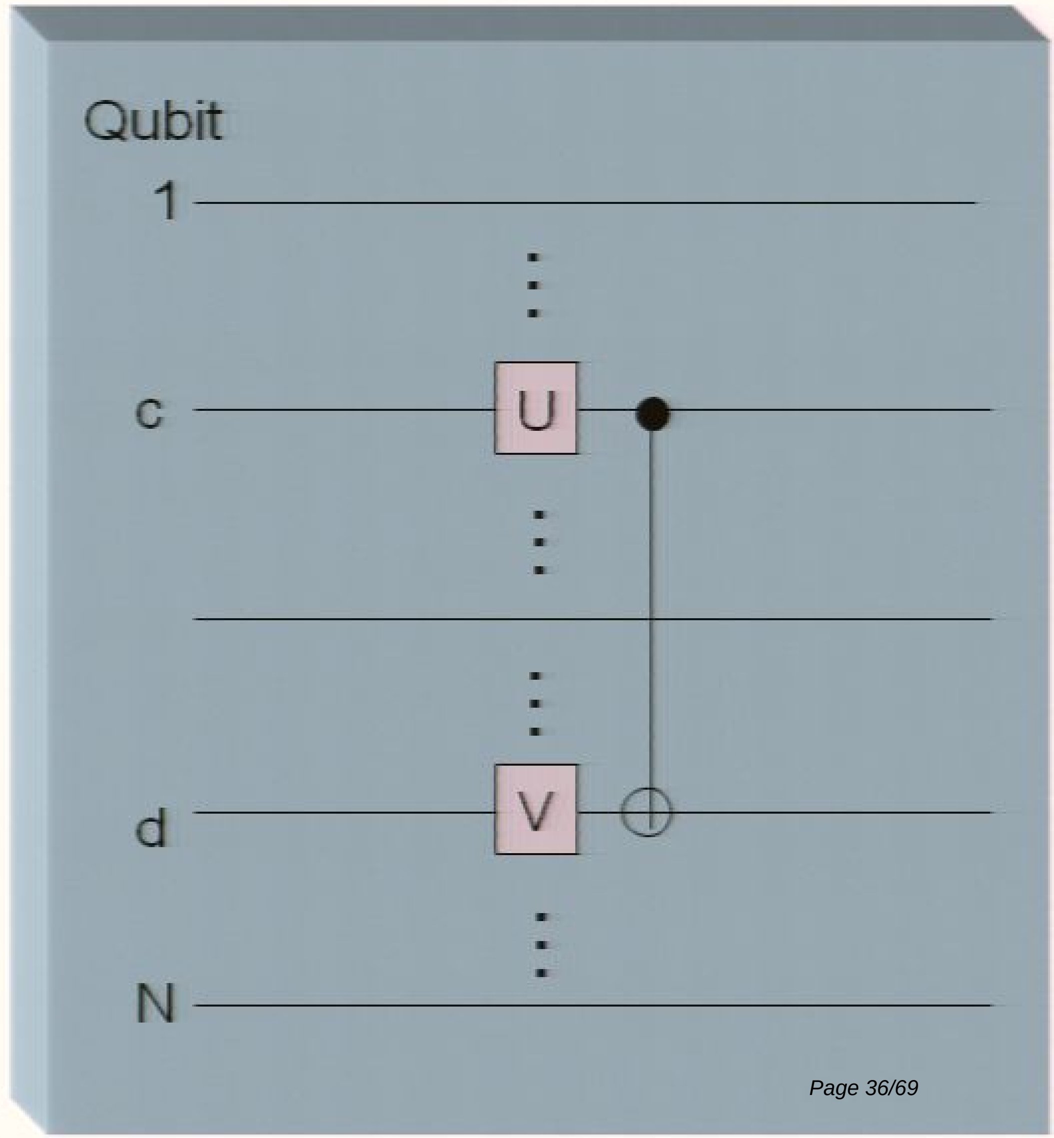
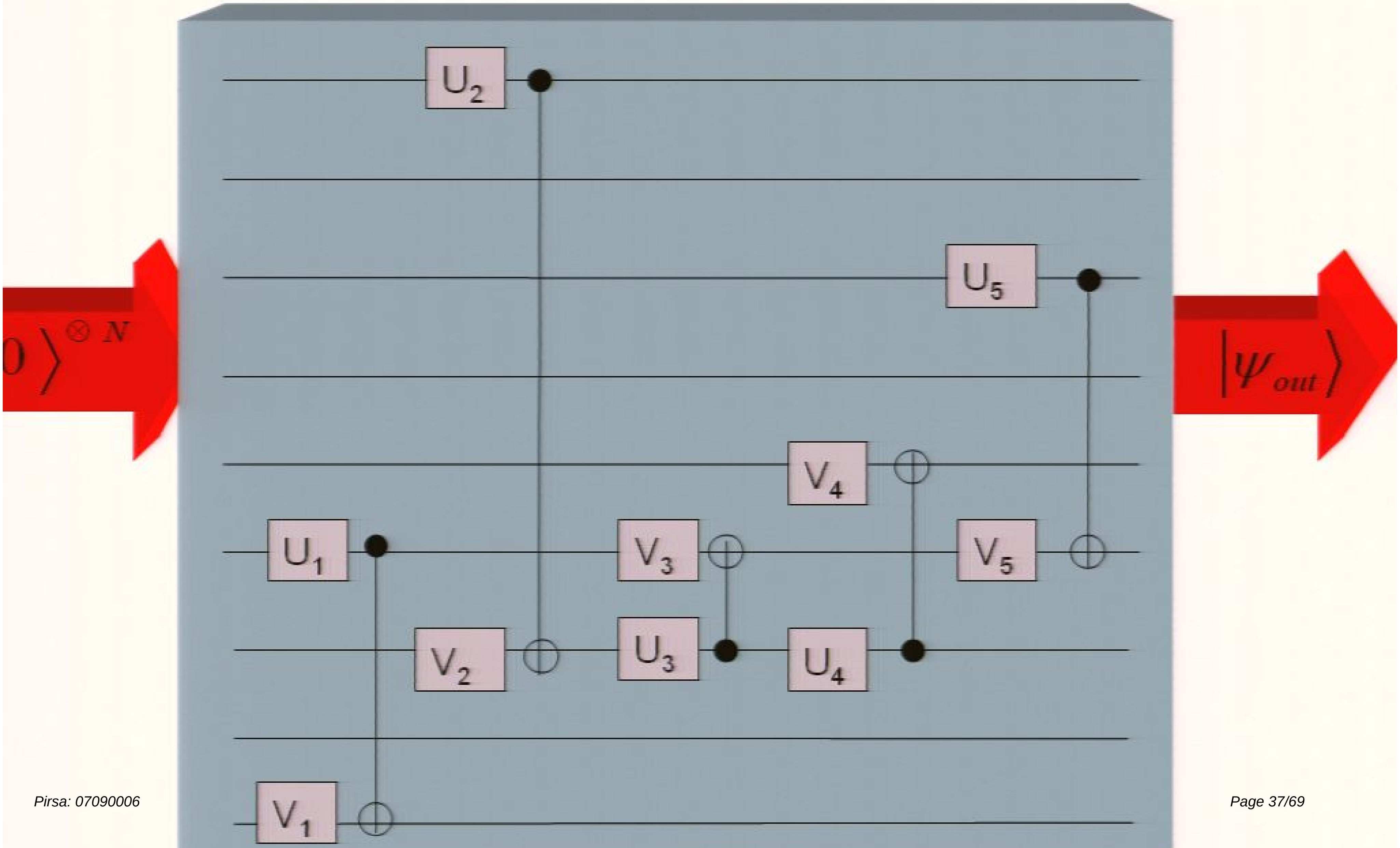


Random circuit example



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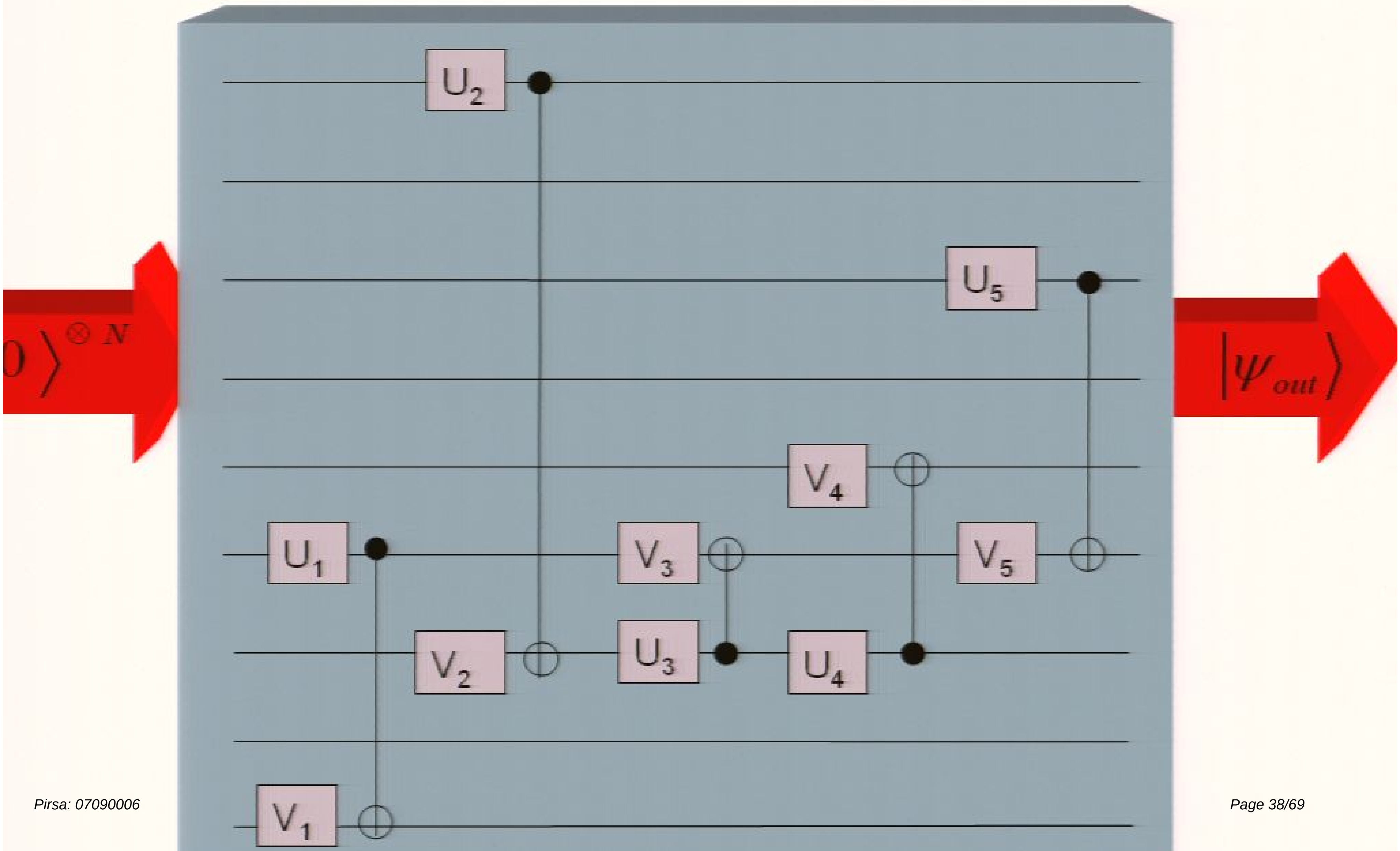


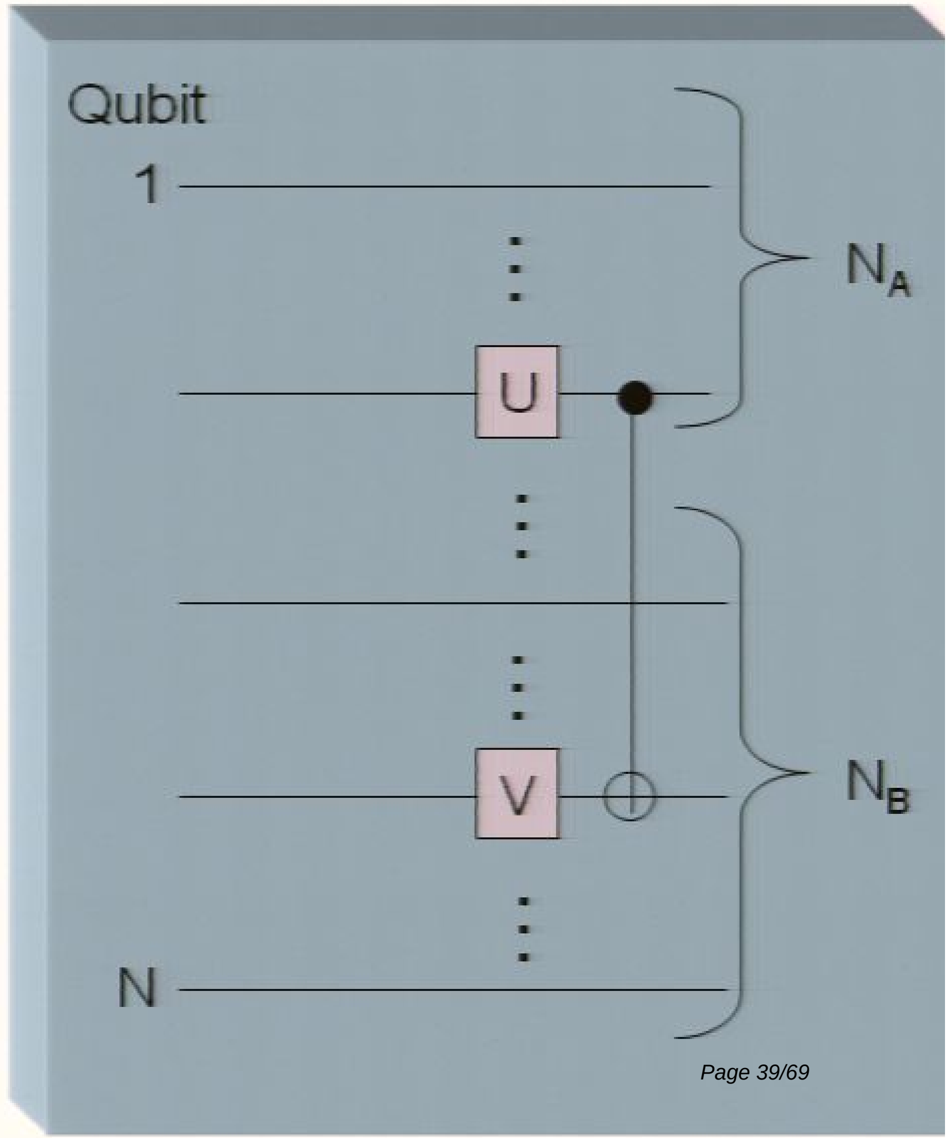
Entanglement after infinite time

- After infinite time(steps), the entanglement E is expected to be nearly maximal.

$$\langle E(\psi)\rangle \geq \min(N_A, N_B) - \frac{2^{-|N_B - N_A|}}{\ln 2}$$

[Lubkin,,J. Math. Phys.1978][Lloyd, Pagels, Ann. of Phys. 1988][Page, PRL, 1993][Foong, Kanno PRL, 1994] [Hayden, Leung, Winter, Comm. Math.Phys. 2006][Emerson, Livine, Lloyd, PRA 2005]

- But this average is only physical if it is reached in *poly(N)* steps.



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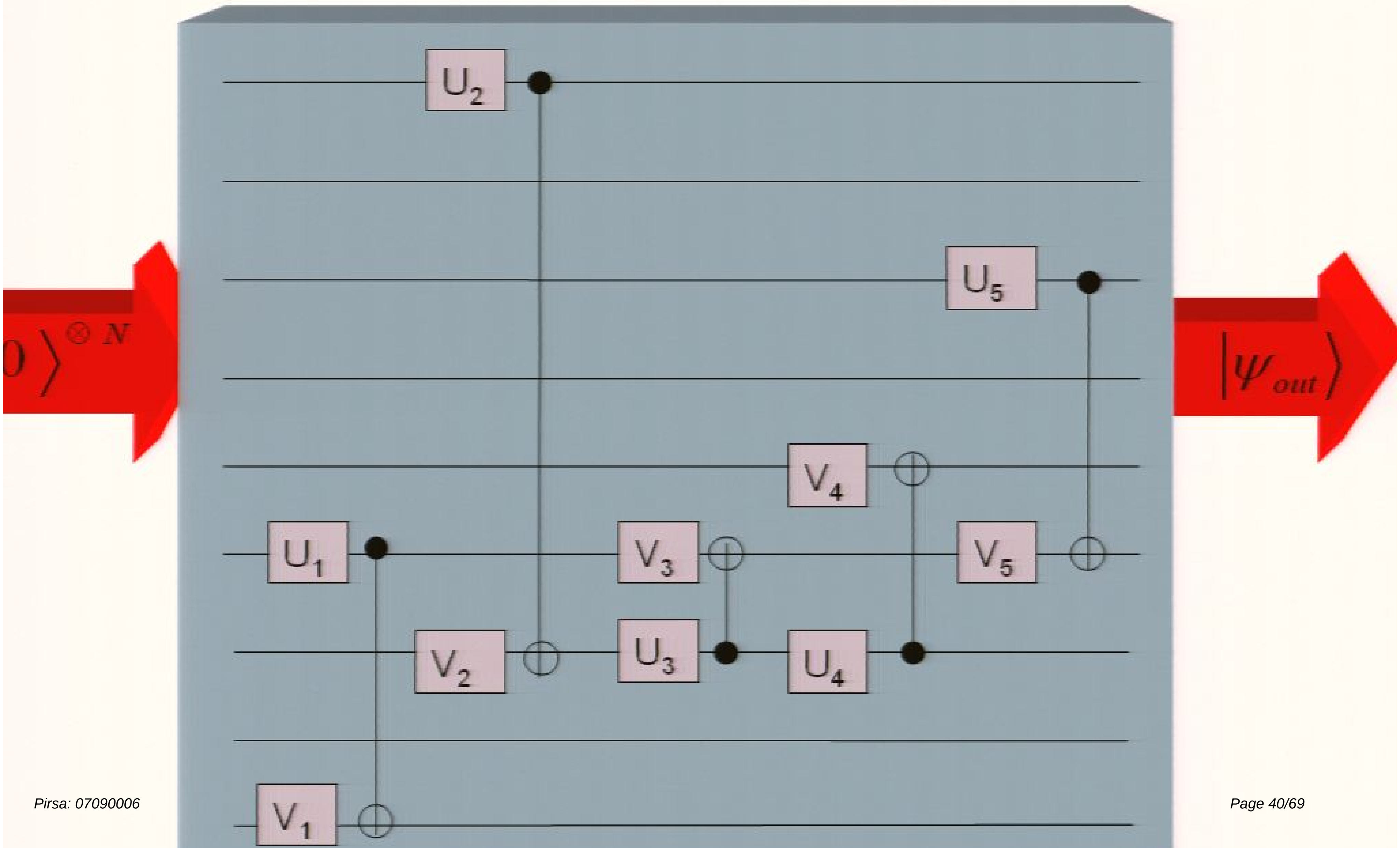

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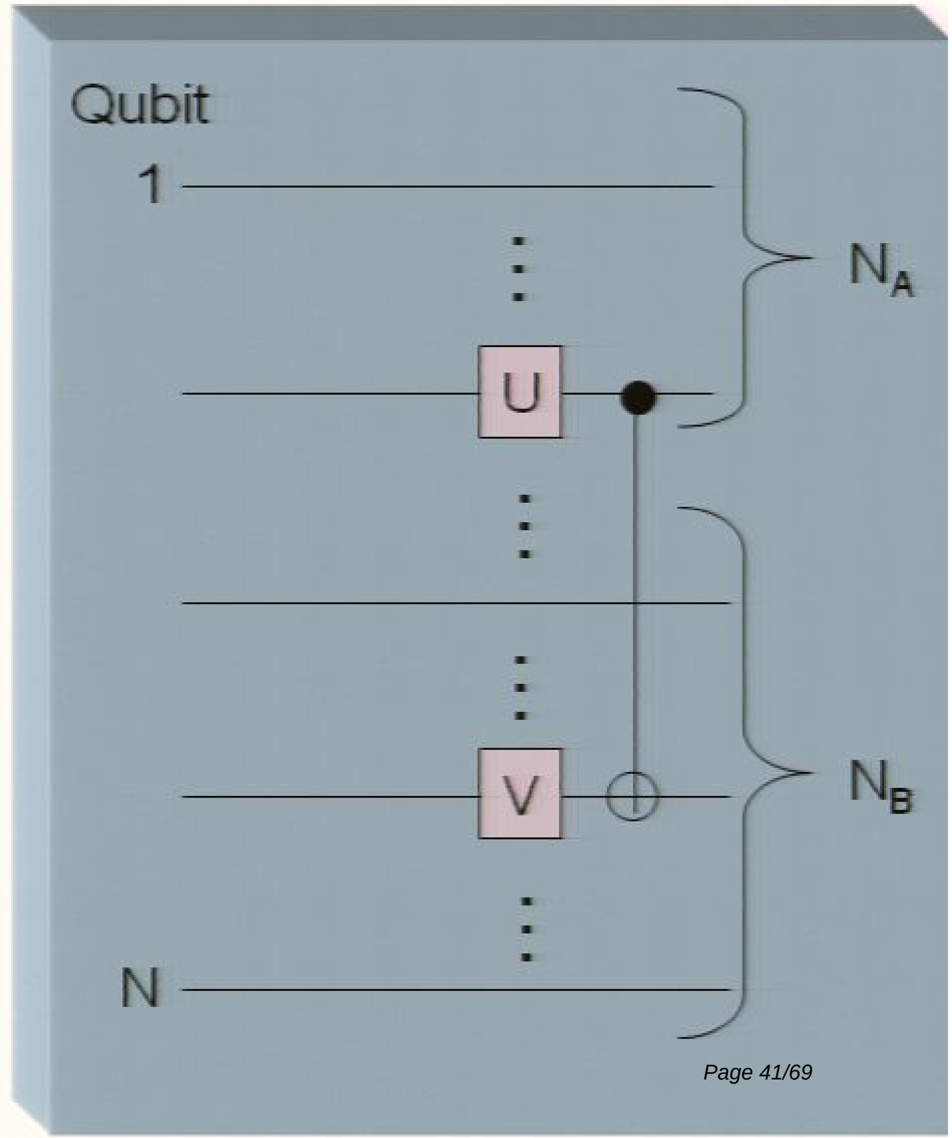
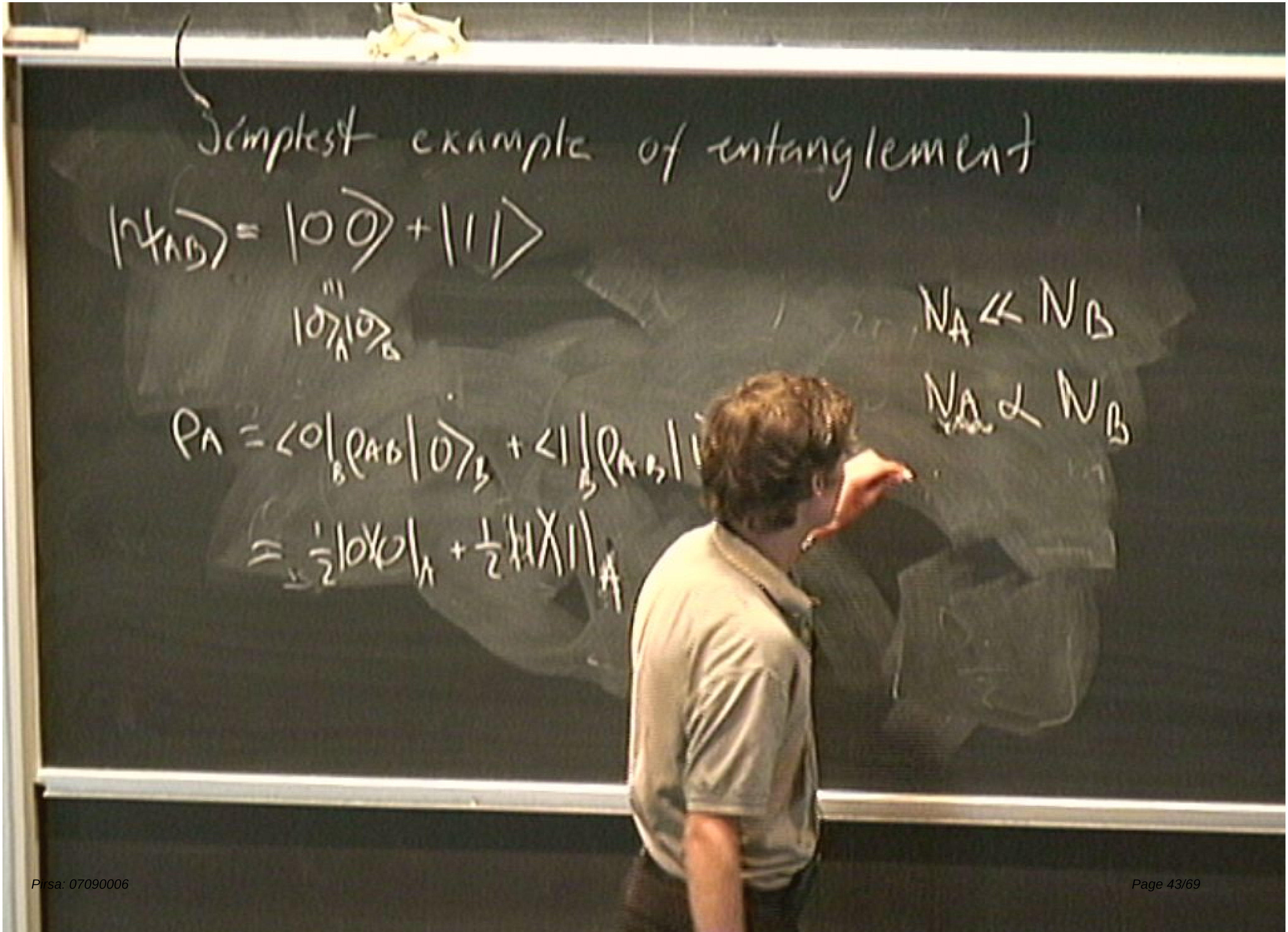
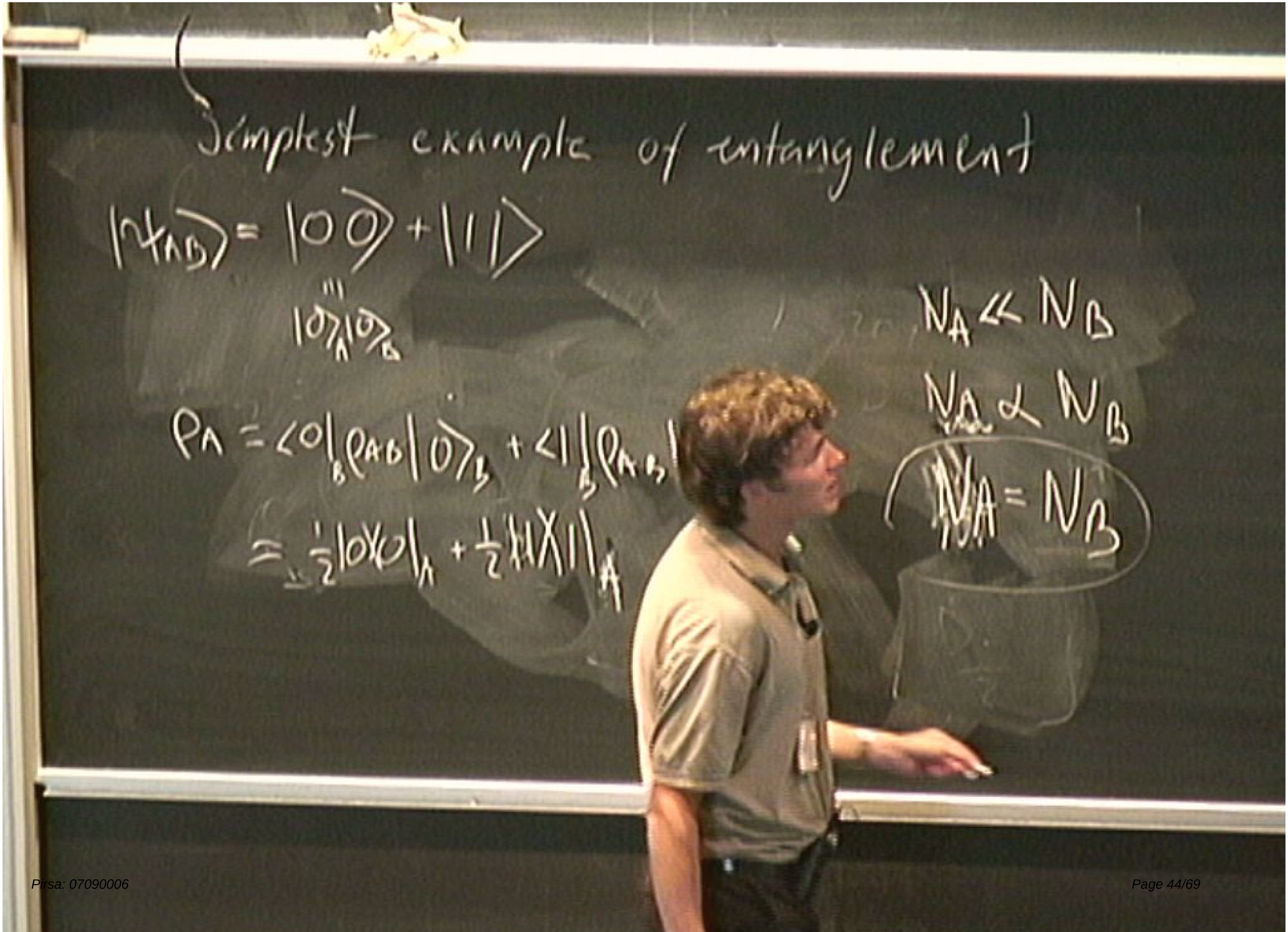



Simplest example of entanglement
|ψ_AB⟩ = |00⟩ + |11⟩
|0⟩_A|0⟩_B
ρ_A = ⟨0|_B ρ_AB |0⟩_B + ⟨1|_B ρ_AB |1
= ½|0⟩⟨0|_A + ½|1⟩⟨1|_A
N_A ≪ N_B
N_A ∝ N_B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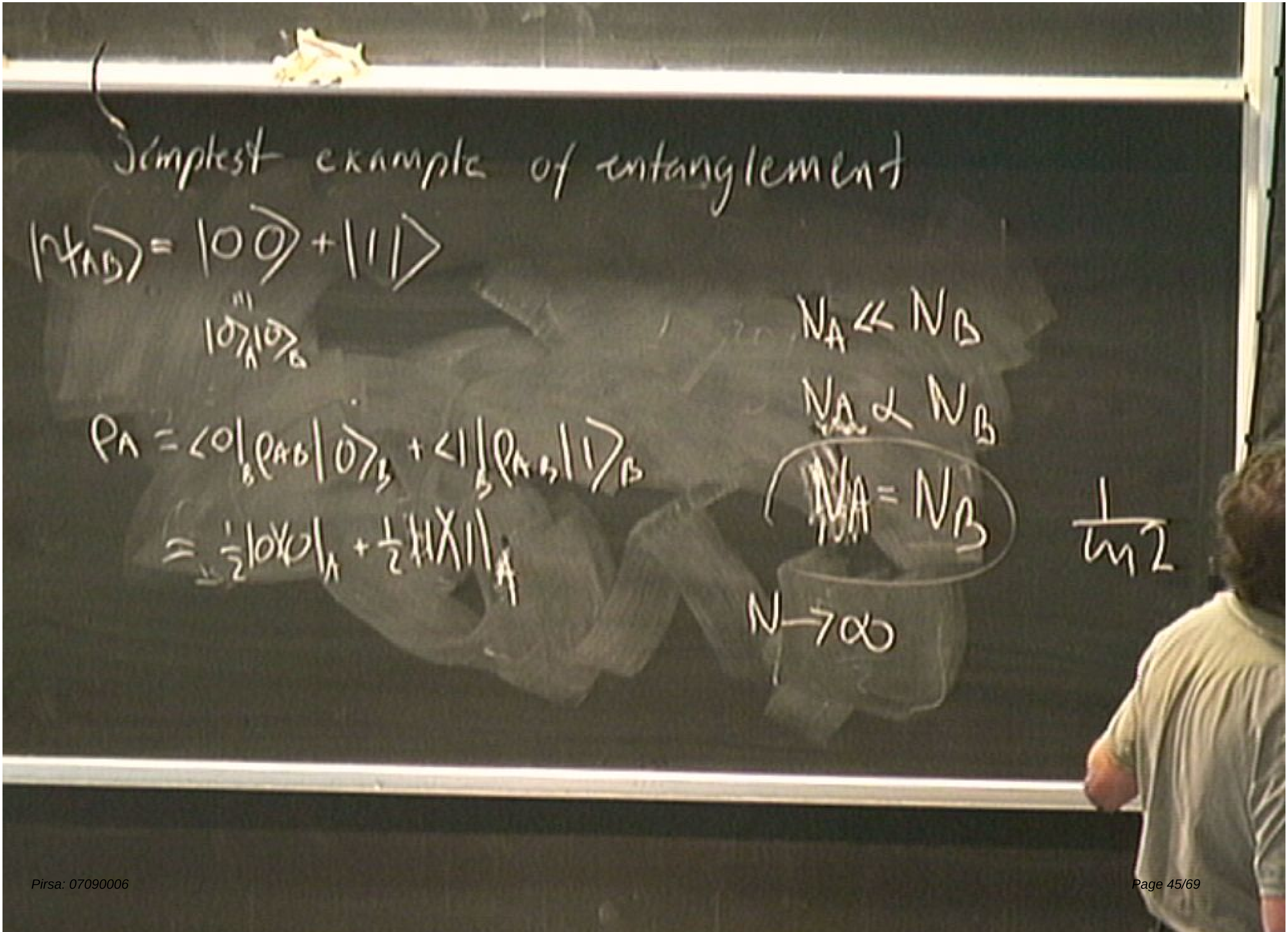
= ½|0⟩⟨0|A + ½|1⟩⟨1|A
NA ≪ NB
NA ∝ NB
NA = NB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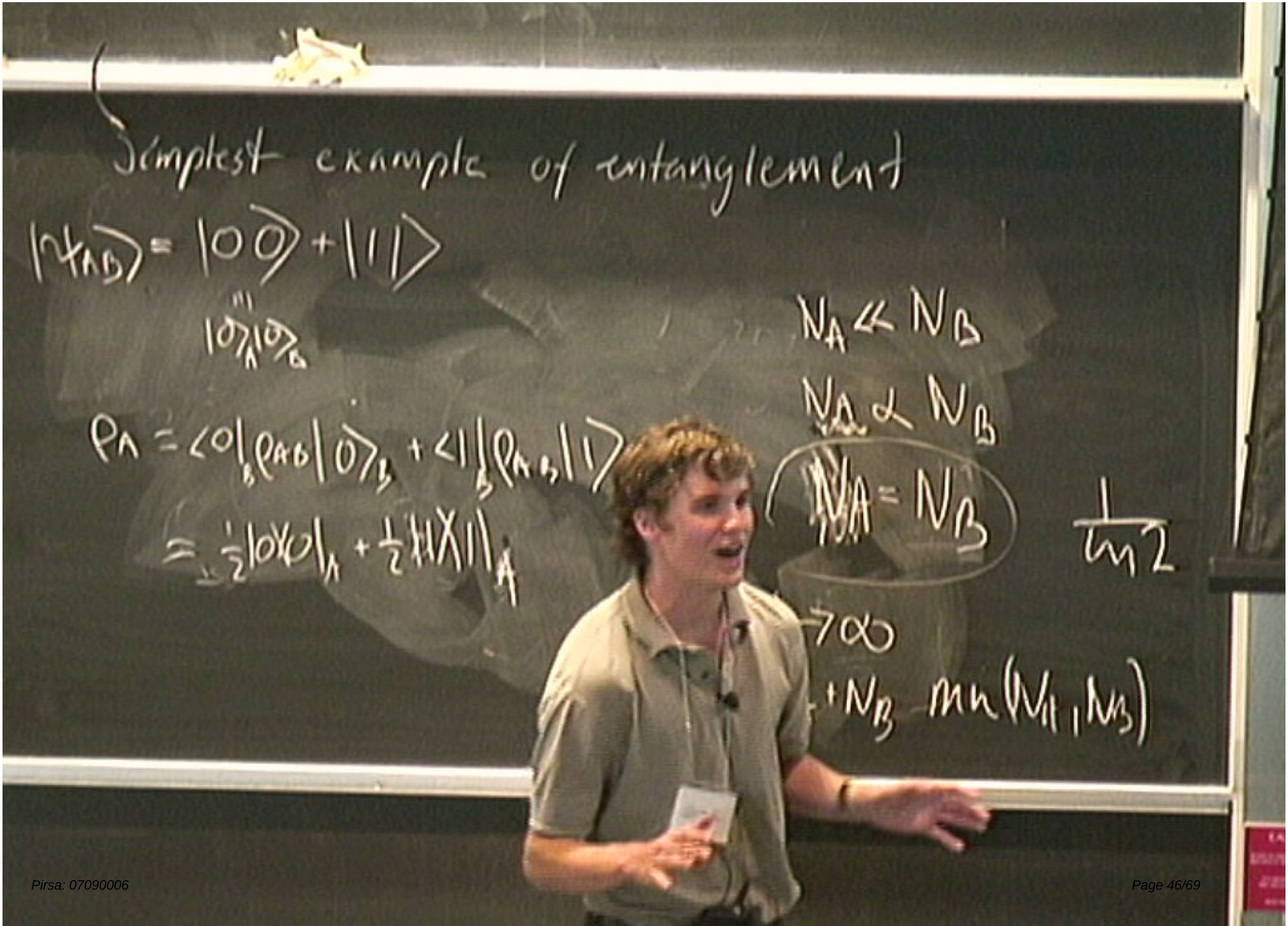
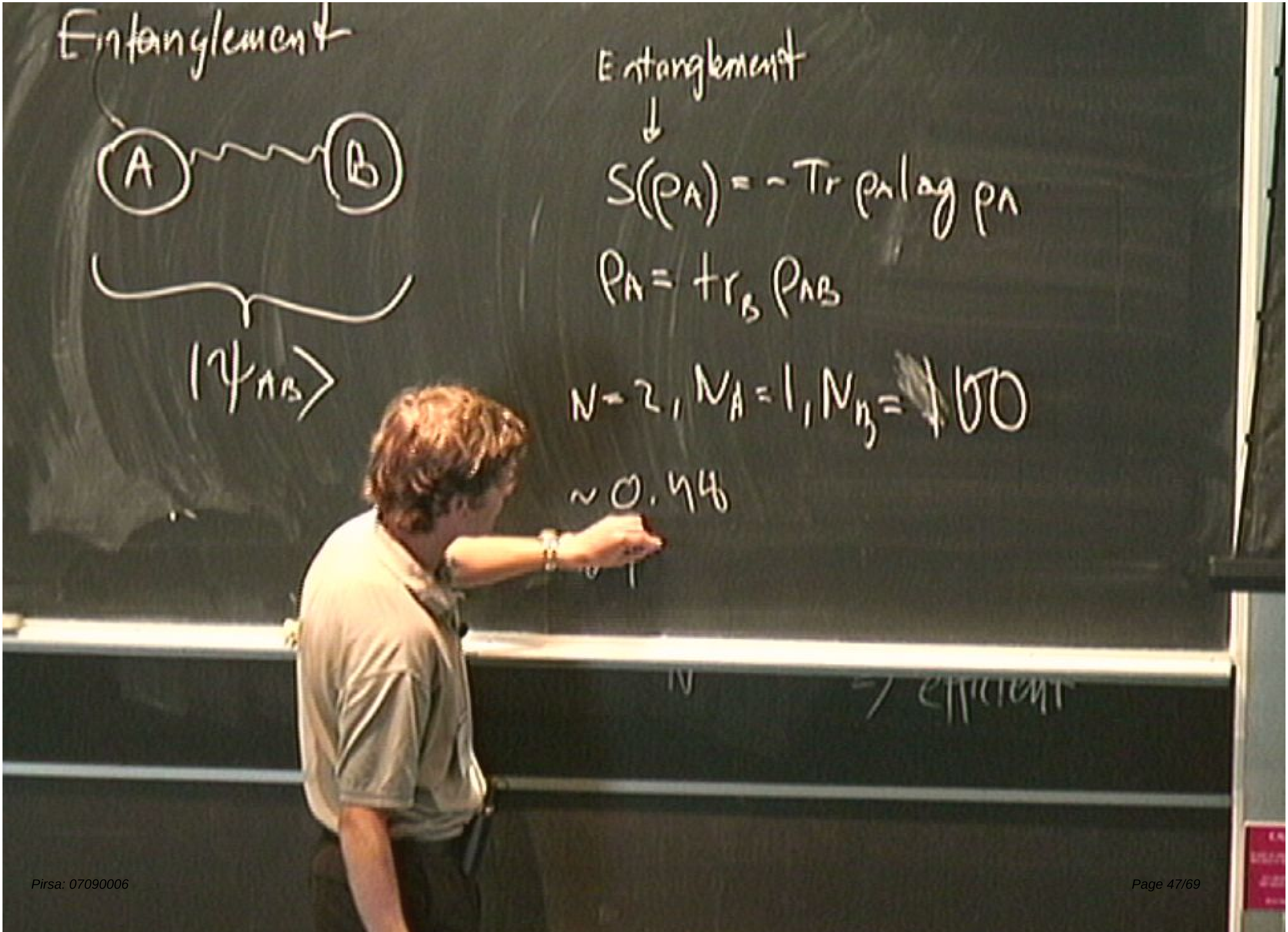
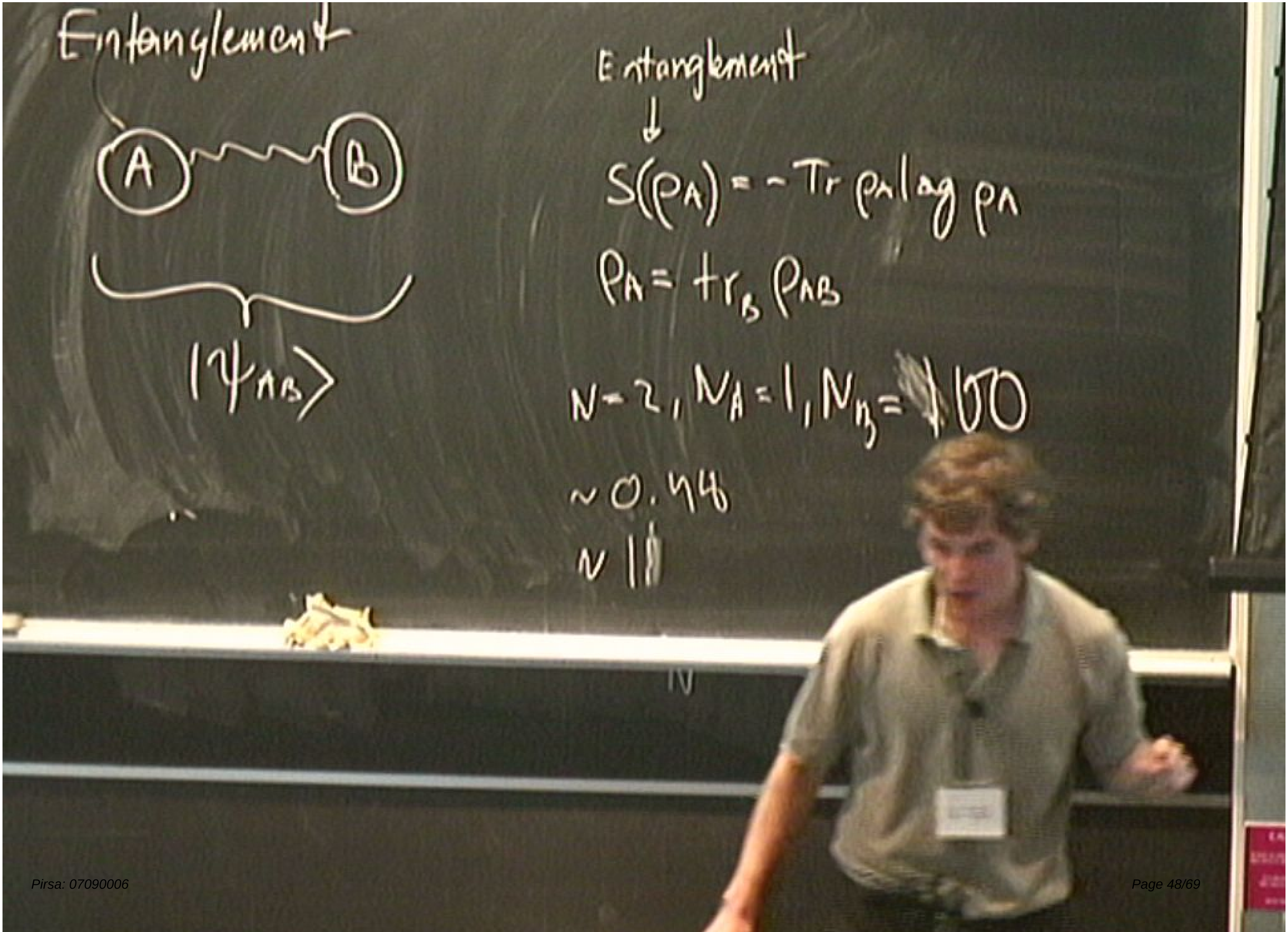
= ½|0⟩⟨0|A + ½|1⟩⟨1|A
NA ≪ NB
NA < NB
NA = NB
N → ∞
1/ln 2

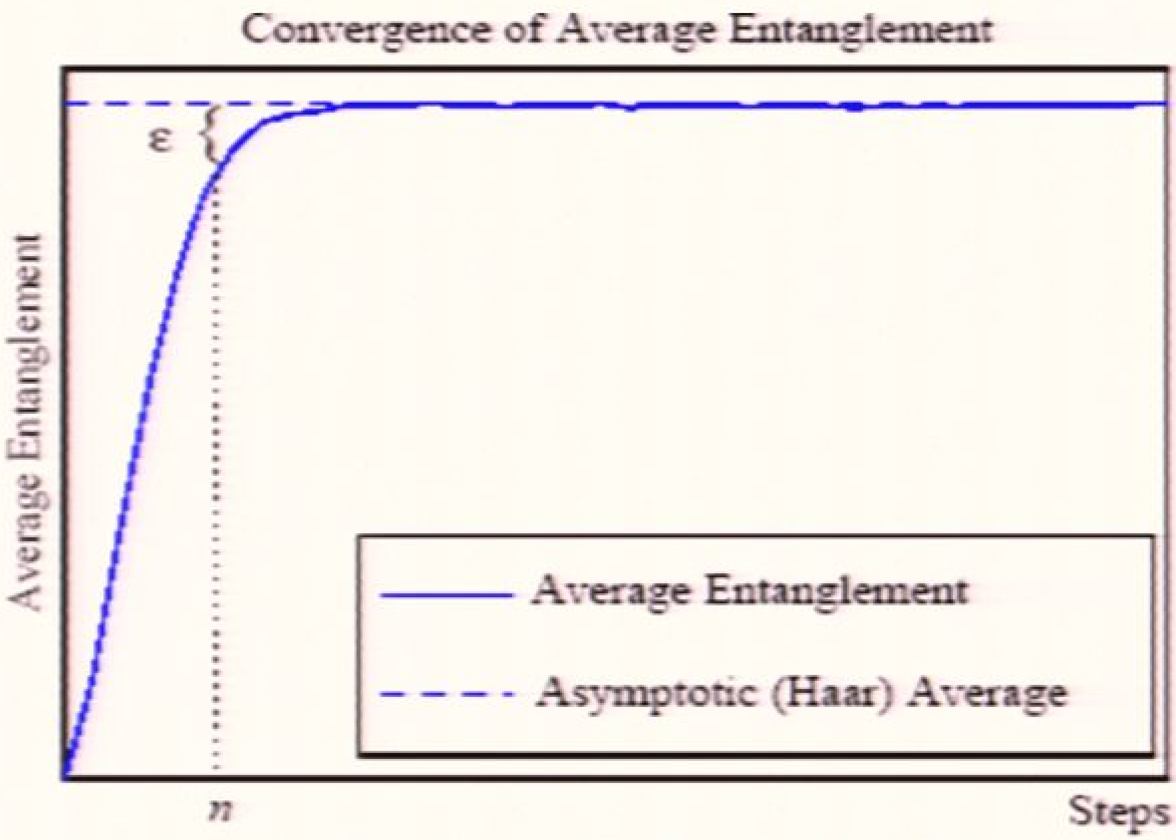

Simplest example of entanglement
N_A << N_B
N_A < N_B
N_A = N_B
→∞


Entanglement
A
B
|ψ_AB>
Entanglement
S(ρ_A) = - Tr ρ_A log ρ_A
ρ_A = tr_B ρ_AB
N=2, N_A=1, N_B=100
~0.44


Entanglement
A
B
|ψAB>
Entanglement
S(ρA) = - Tr ρA log ρA
ρA = trB ρAB
N=2, NA=1, NB=100
~0.44

Result 1

- **Theorem:** The average entanglement of the unitarily invariant measure is reached to a fixed arbitrary accuracy ε within *O(N^3)* steps.



- In other words the circuit is expected to make the input state maximally entangled in a physical number of steps.

Result 1, proper statement

Theorem:

Let some arbitrary $\varepsilon < 1$ be given.

Then for a number n of gates in the random circuit satisfying

$$n \geq 9N(N-1)[(4 \ln 2)N + \ln \varepsilon^{-1}]/4$$

we have $\langle E(\psi_n) \rangle \geq (\min(N_A, N_B) - 2^{-|N_B - N_A|} + \varepsilon)/\ln 2$

and, for $|\phi\rangle$ maximally entangled

$$\langle \max_{|\phi\rangle} |\langle \psi_n | \phi \rangle| \rangle \geq 1 - \sqrt{\frac{2^{-(N_B - N_A)} + \varepsilon}{2 \ln 2}}$$

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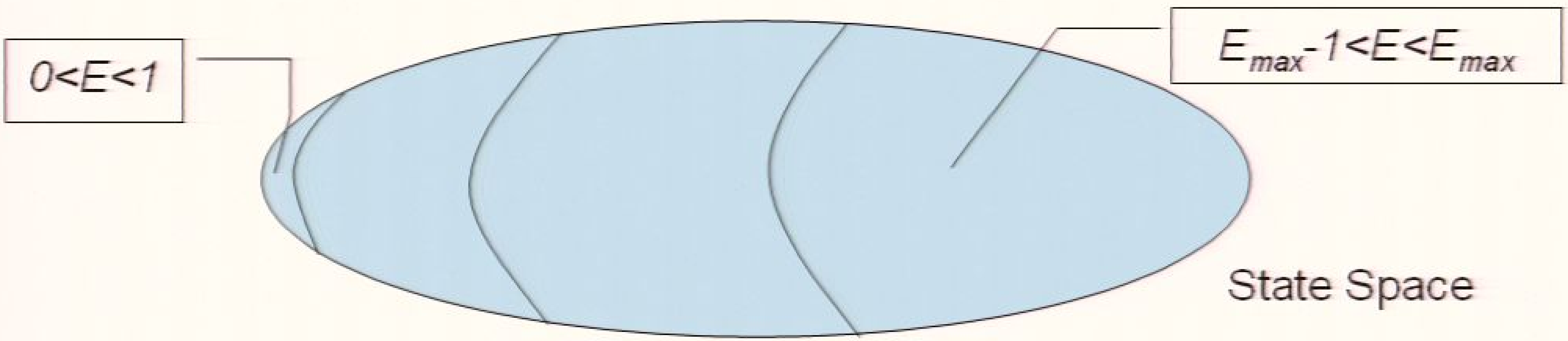
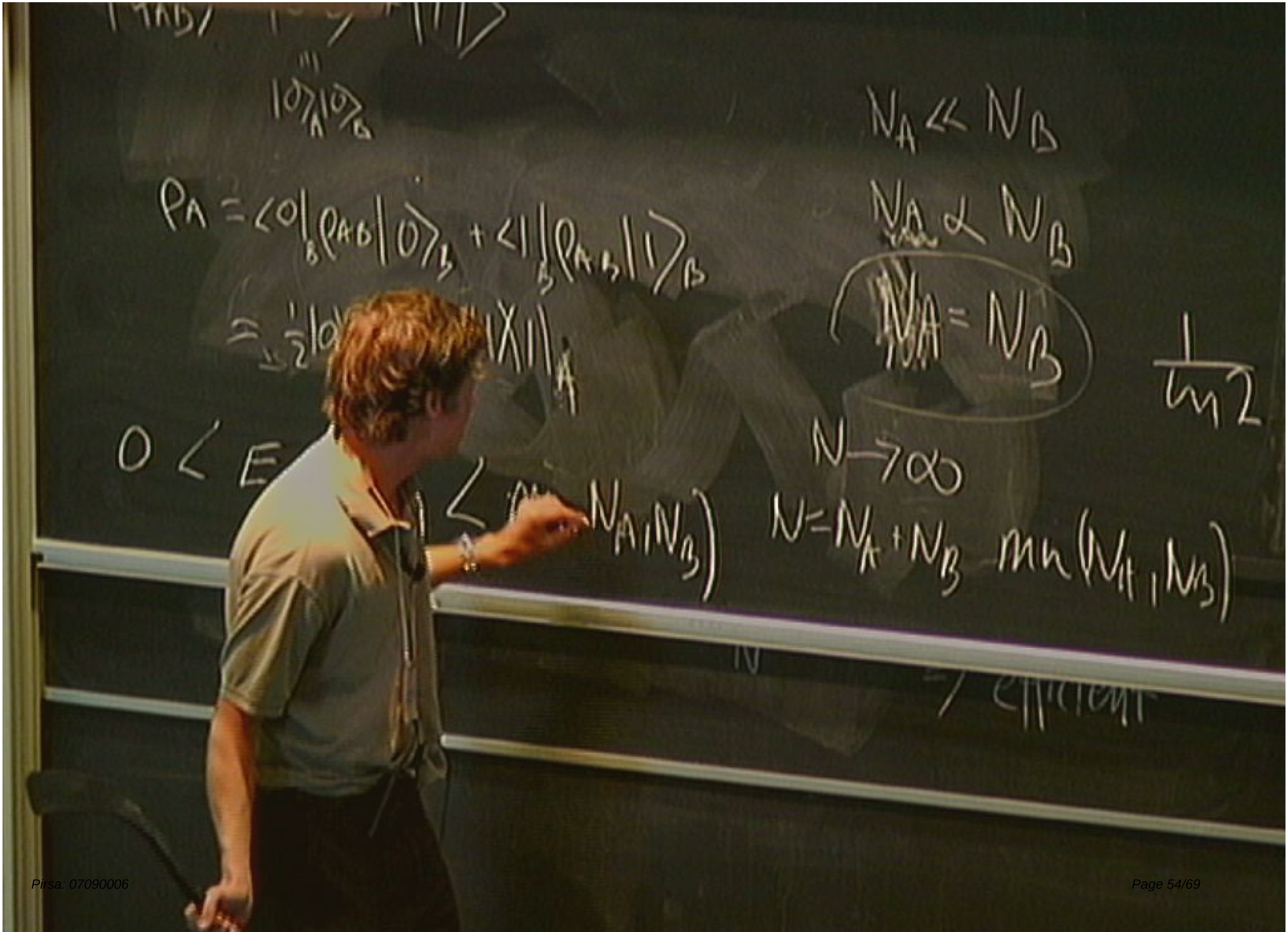
Result 1, proof outline

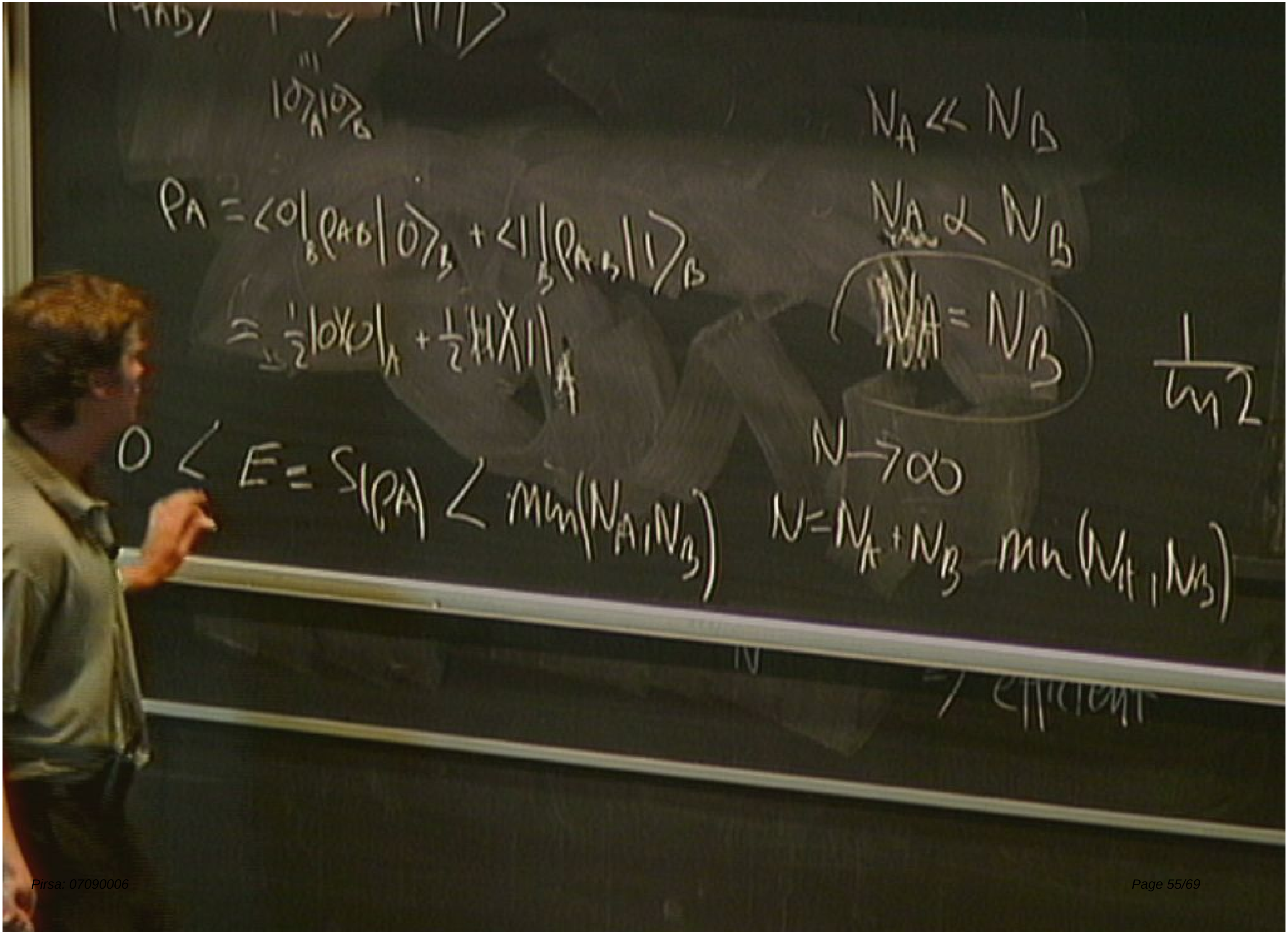
- The random circuit does a random walk on a massive state space.

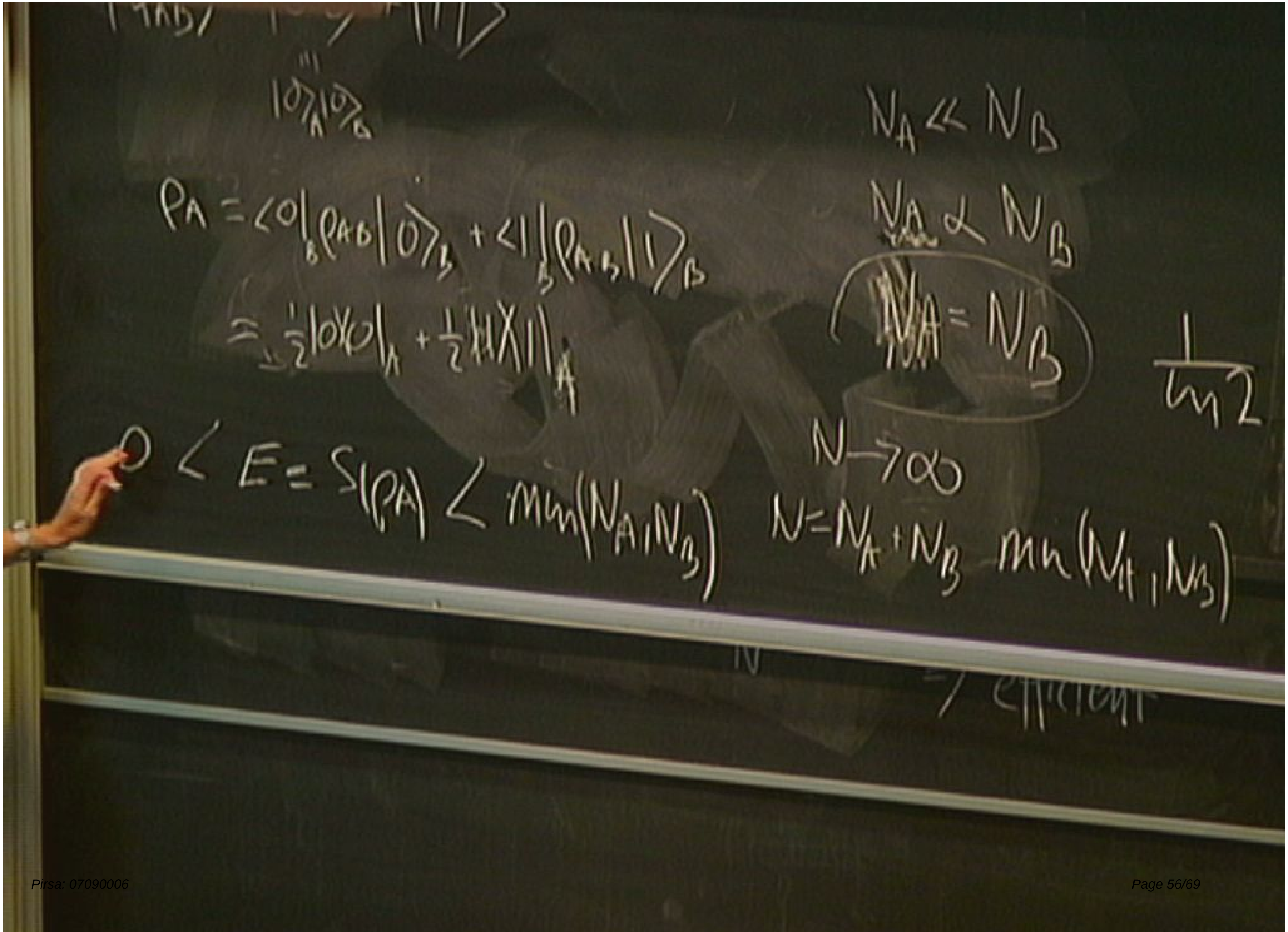


- One could consider mapping the random walk onto an associated, faster converging, random walk on the entanglement state space.
- It is a bit more complicated though. In fact we map it onto a random walk relating to the purity.
- We then use known Markov Chain methods to bound the rate of convergence of this smaller walk.



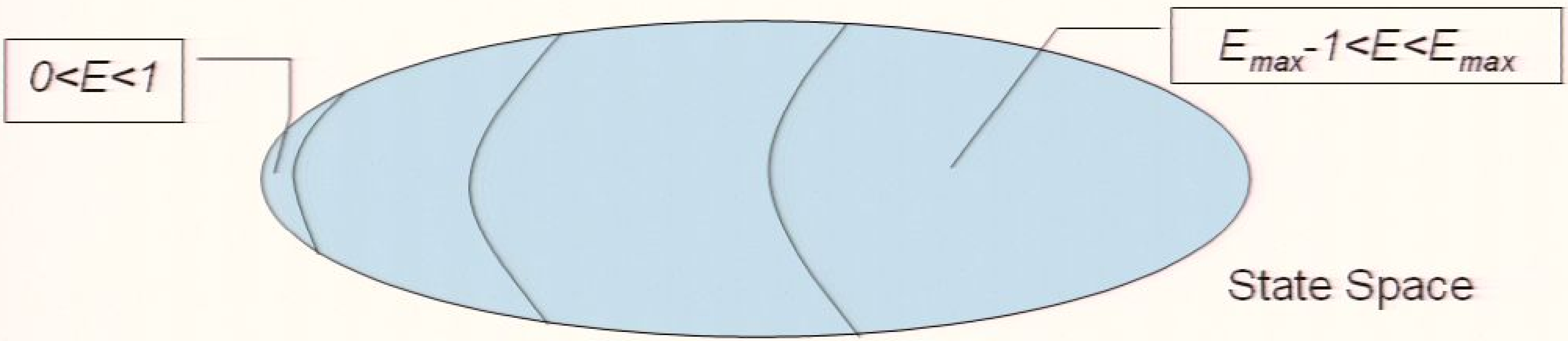
NA << NB
NA < NB
NA = NB
N → ∞
N = NA + NB
min(NA, NB)
1/ln 2
efficient


ρA = ⟨0|B ρAB |0⟩B + ⟨1|B ρAB |1⟩B
= ½ |0⟩⟨0|A + ½ |1⟩⟨1|A
0 < E = S(ρA) < min(NA, NB)
NA << NB
NA < NB
NA = NB
N → ∞
N = NA + NB
min(NA, NB)
1/ln 2
efficient

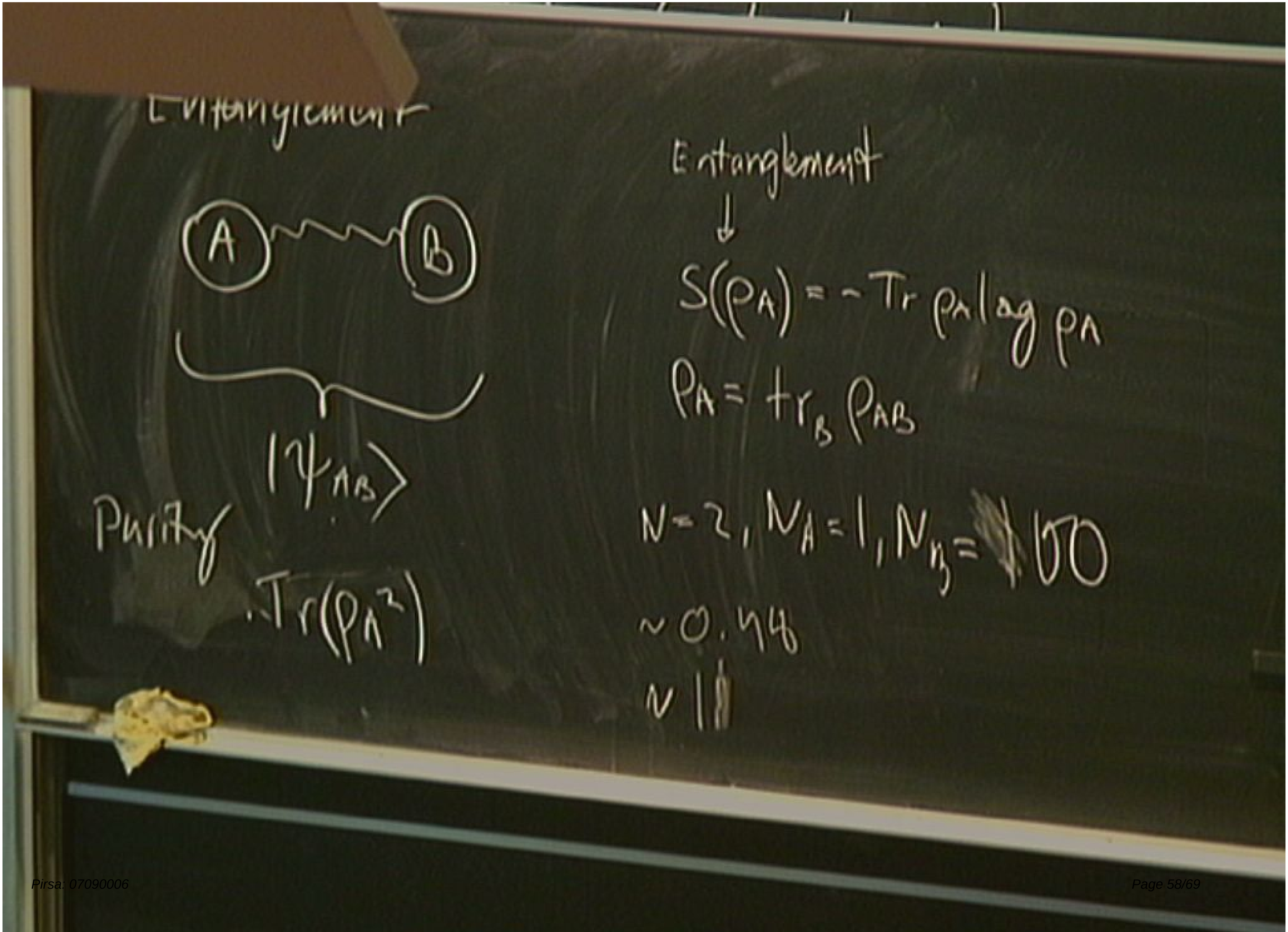

|0⟩_A|0⟩_B
N_A ≪ N_B
ρ_A = ⟨0|_B ρ_AB |0⟩_B + ⟨1|_B ρ_AB |1⟩_B
= ½|0⟩⟨0|_A + ½|1⟩⟨1|_A
N_A < N_B
N_A = N_B
1/ln 2
0 < E = S(ρ_A) < min(N_A, N_B)
N → ∞
N = N_A + N_B
min(N_A, N_B)

Result 1, proof outline

- The random circuit does a random walk on a massive state space.



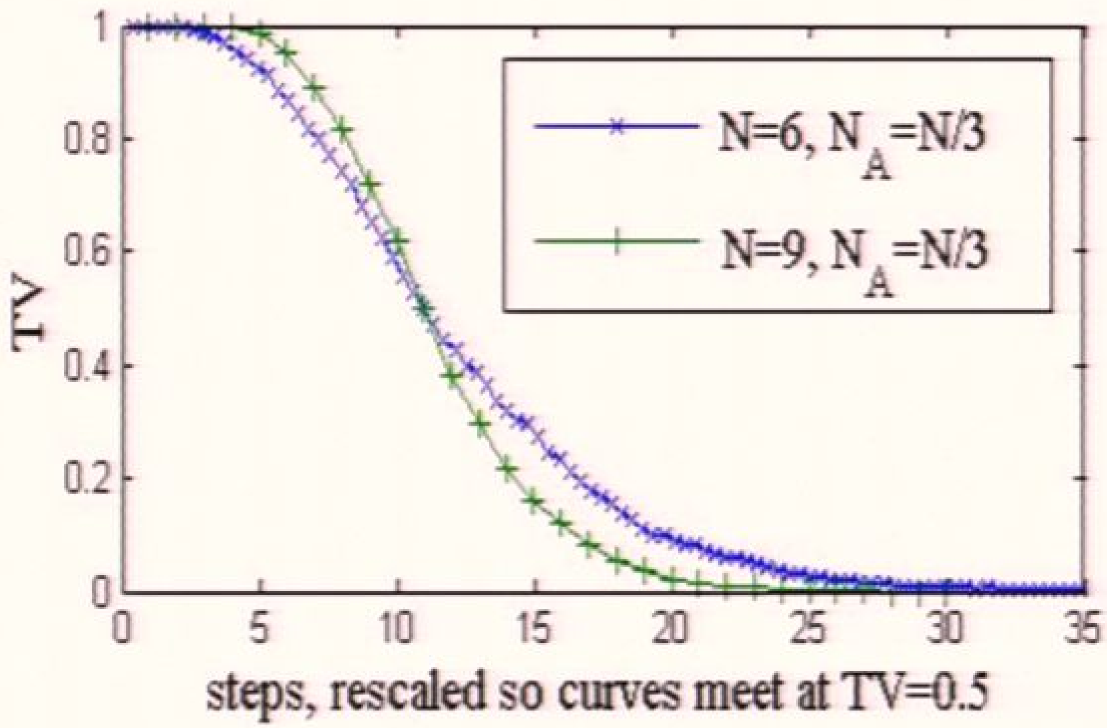
- One could consider mapping the random walk onto an associated, faster converging, random walk on the entanglement state space.
- It is a bit more complicated though. In fact we map it onto a random walk relating to the purity.
- We then use known Markov Chain methods to bound the rate of convergence of this smaller walk.


Entanglement
A
B
$|\psi_{AB}\rangle$
Purity
$Tr(\rho_A^2)$
Entanglement
$S(\rho_A) = -Tr\, \rho_A \log \rho_A$
$\rho_A = tr_B\, \rho_{AB}$
$N=2,\ N_A=1,\ N_B=100$
~ 0.48
~ 1

Result 2

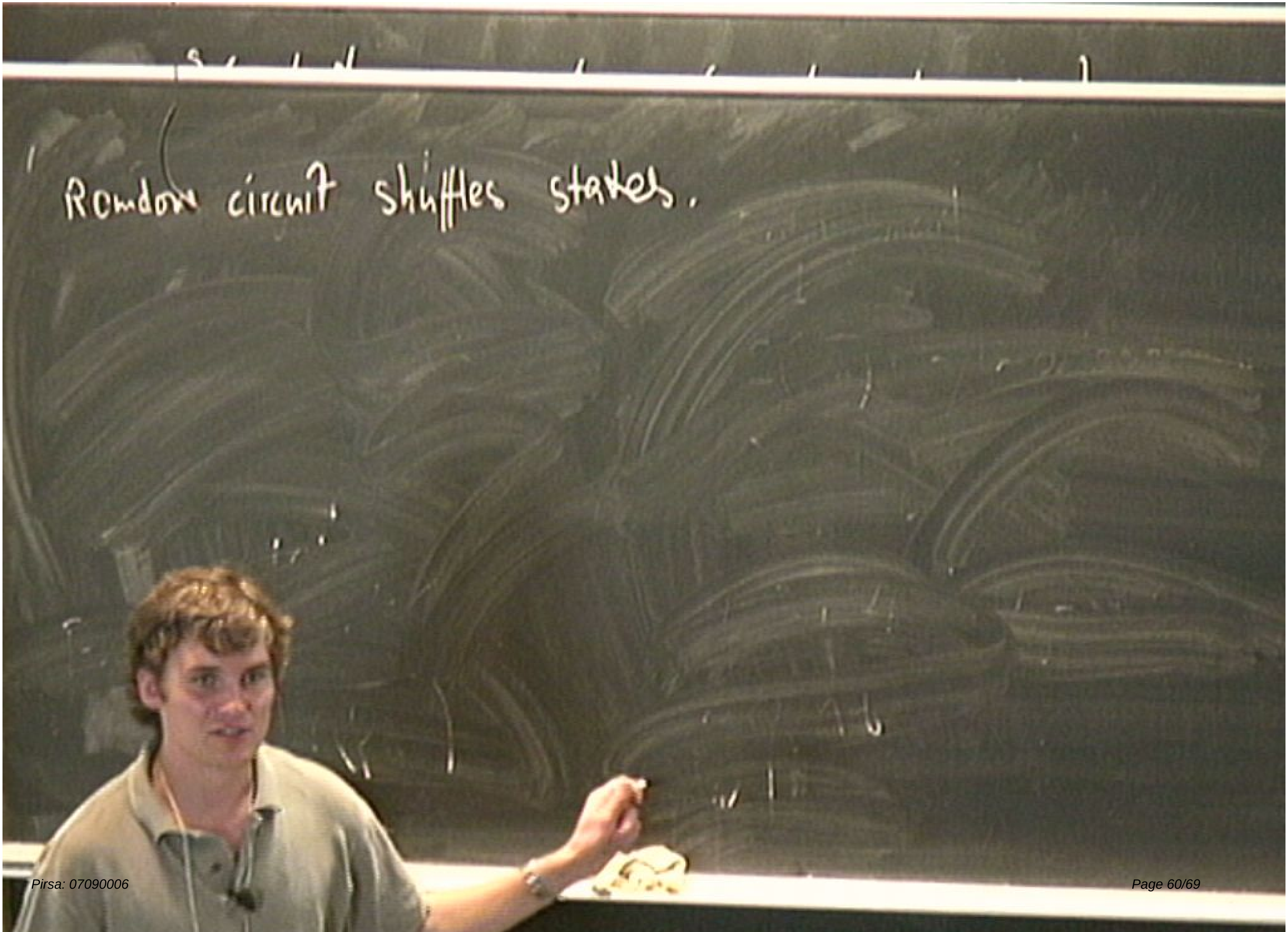
- **Numerical observation:** can associate a specific time with achievement of generic entanglement.

- This figure shows the total variation, TV, distance to the asymptotic entanglement probability distribution. It tends to a step function with increasing N.



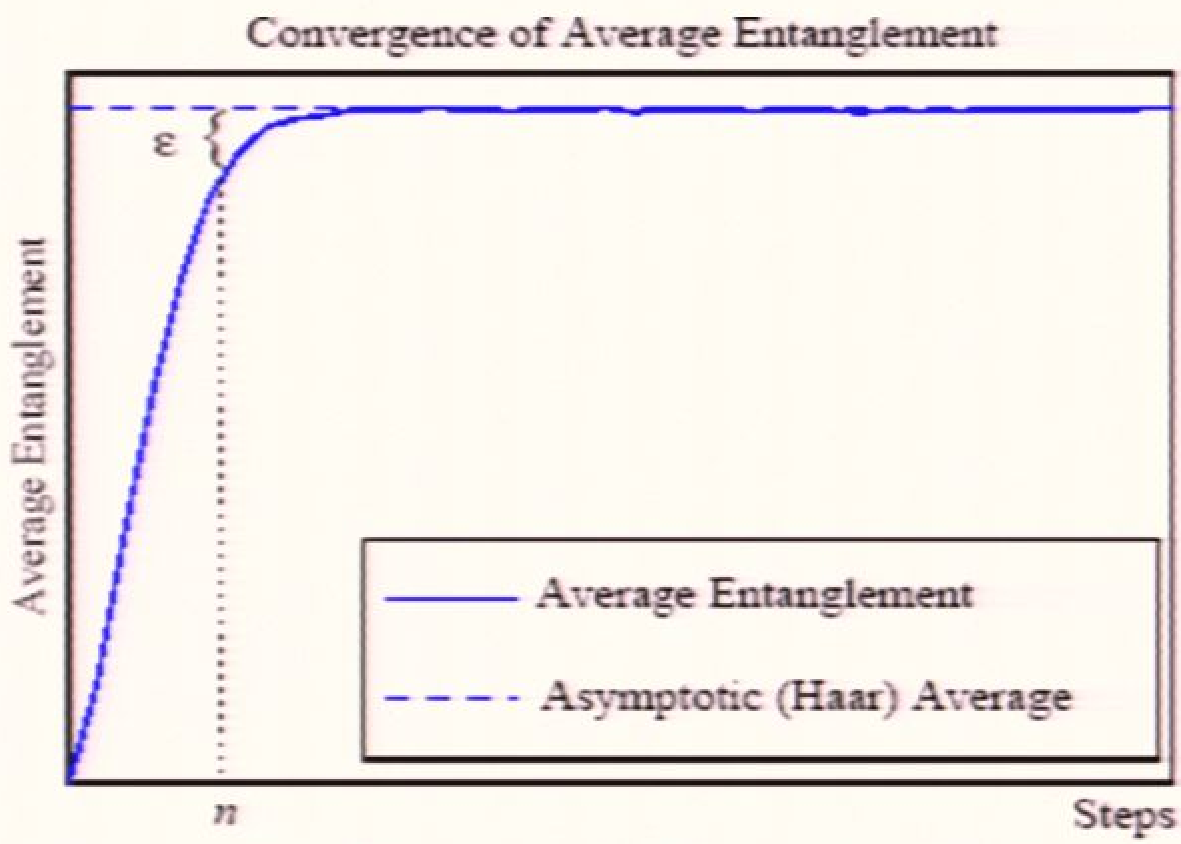
- We term this a *variation cut-off* after a known effect in Markov Chains

[Diaconis, Cut-off effect in Markov chains]


Random circuit shuffles states.

Result 1

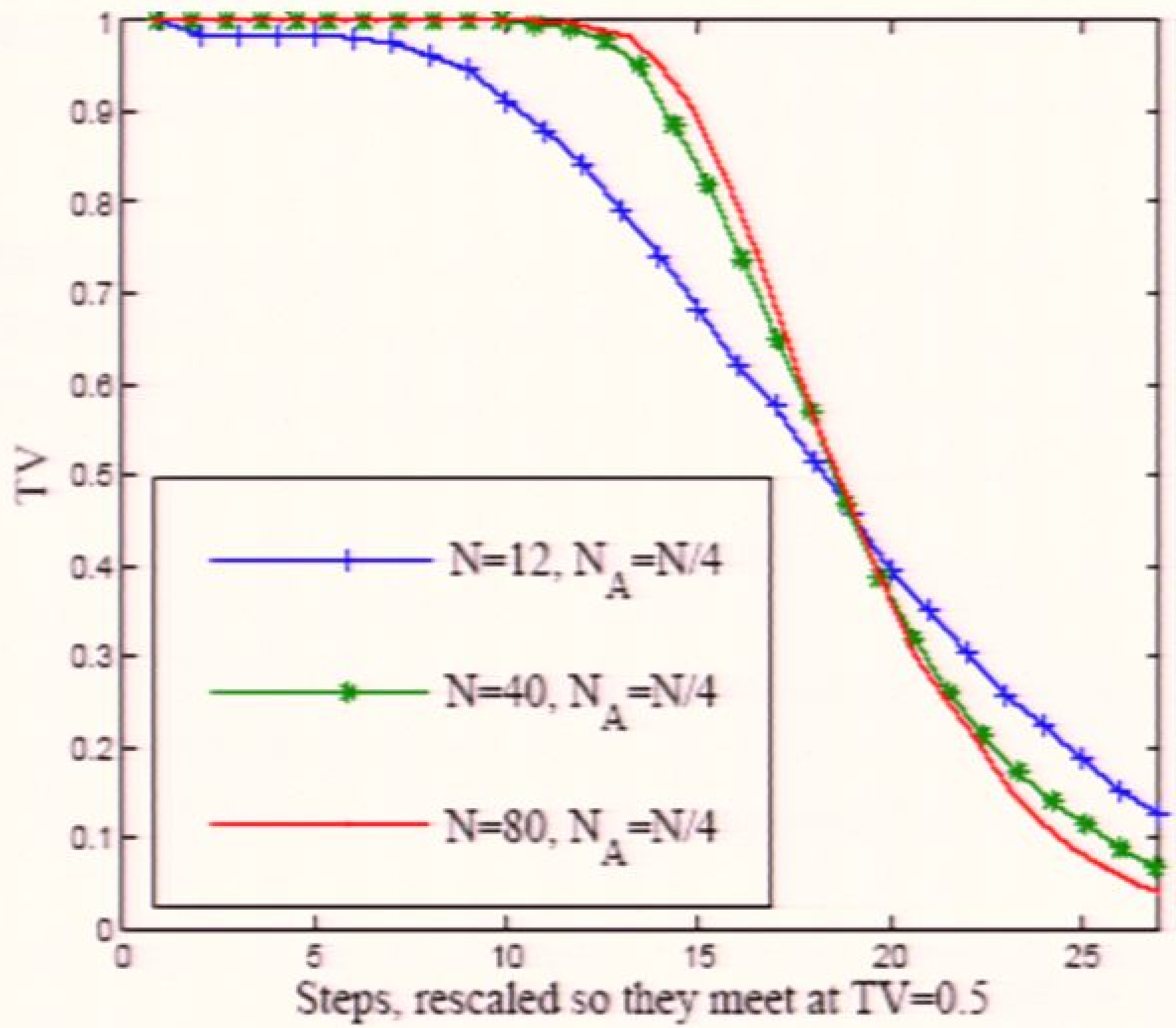
- **Theorem:** The average entanglement of the unitarily invariant measure is reached to a fixed arbitrary accuracy ε within $O(N^3)$ steps.



- In other words the circuit is expected to make the input state maximally entangled in a physical number of steps.

Result 2 continued

- For larger *N* we used tricks to do the simulation efficiently
- We used stabilizer states and tools for efficient evaluation of stabilizer state entanglement. [Gottesmann, Caltech PhD] [Audenaert,Plenio, NJP 2005]
- The final entanglement distribution is known, and result 1 applies here too. [Dahlsten, Plenio, QIC 2006][in preparation by authors]
- We find a continued trend towards a step function.



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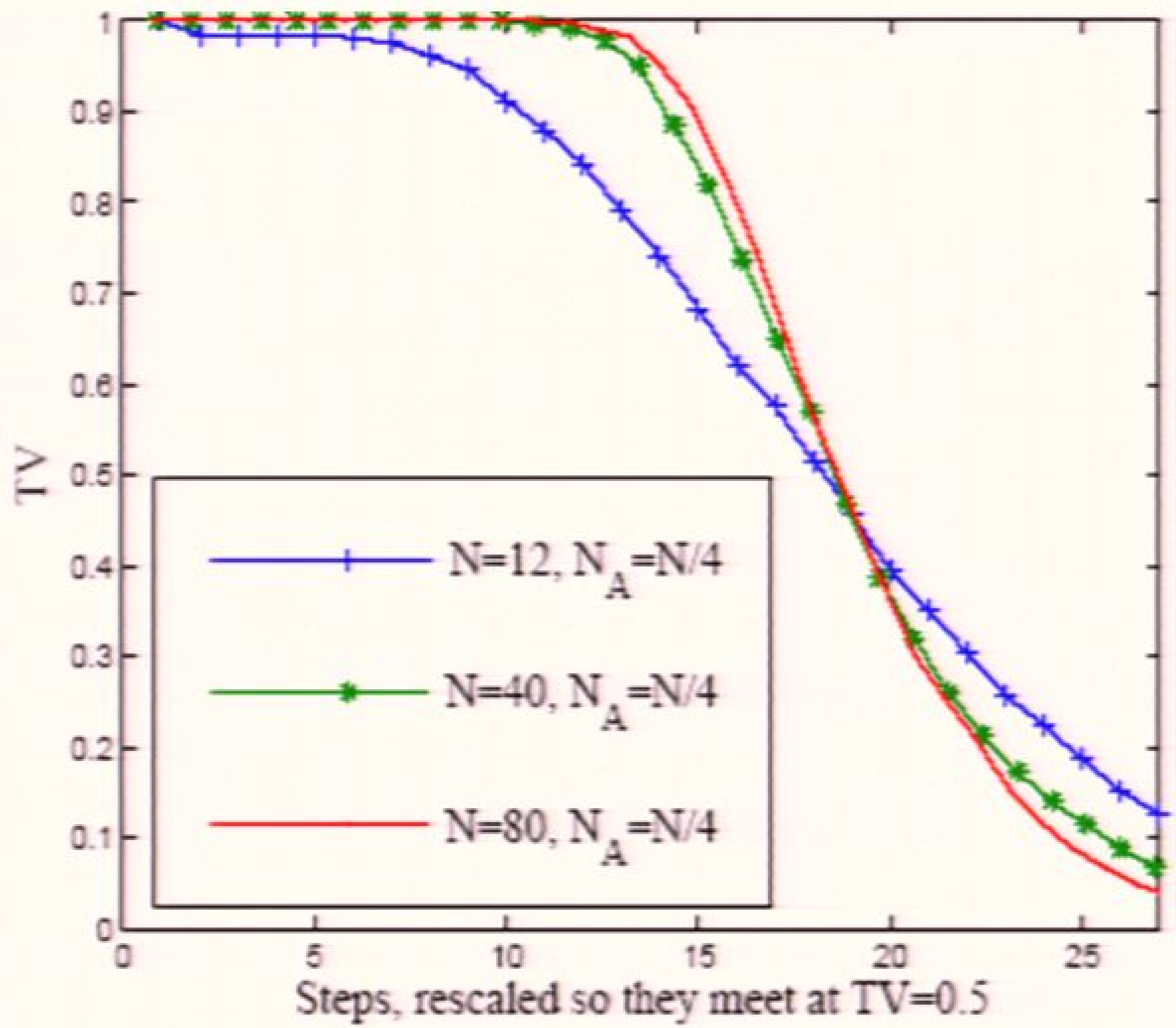


Waterloo, 7 Sep 2007

Conclusion

- **Result 1:** Proof that generic entanglement is physical as it can be generated using poly(N) two-qubit gates.
- **Implication:** arguments and protocols assuming generic entanglement gain relevance.

  [Abeyesinghe,Hayden,Smith, Winter, quant-ph/0407061][Harrow, Hayden, Leung, PRL 2004]
- **Result 2:** Numerical observation that generic entanglement is achieved at a particular instant.
- Question: how does this volume-scaling picture relate to area-scaling?

Acknowledgements.

- Discussions with J. Oppenheim as well as T.Rudolph, G.Smith and J.Smolin
- Funding by The Leverhulme Trust, EPSRC QIP-IRC, EU Integrated Project QAP, the Royal Society, the NSA and the ARDA

END

Result 1, proper statement

Theorem:

Let some arbitrary $\varepsilon < 1$ be given.

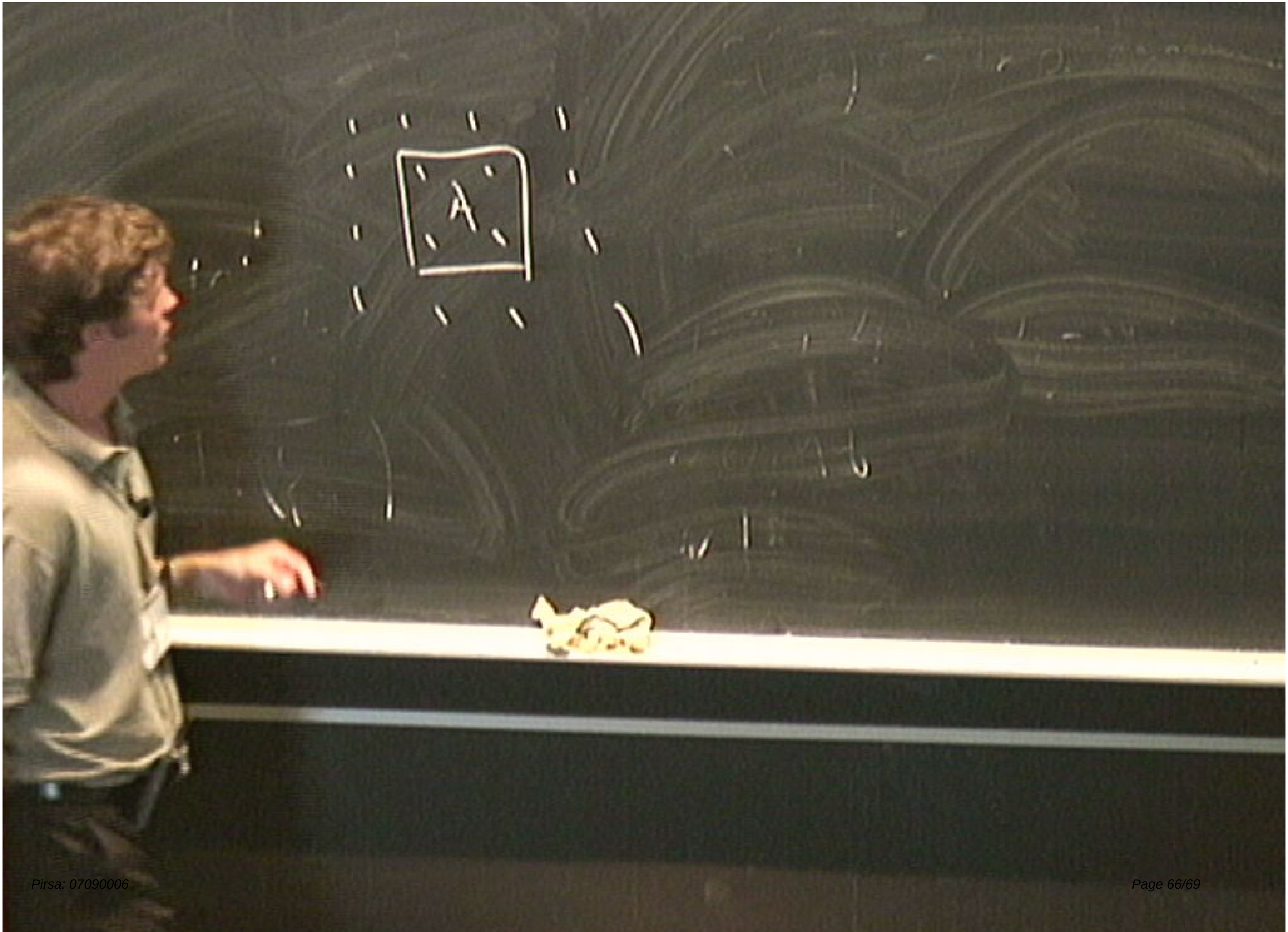
Then for a number n of gates in the random circuit satisfying

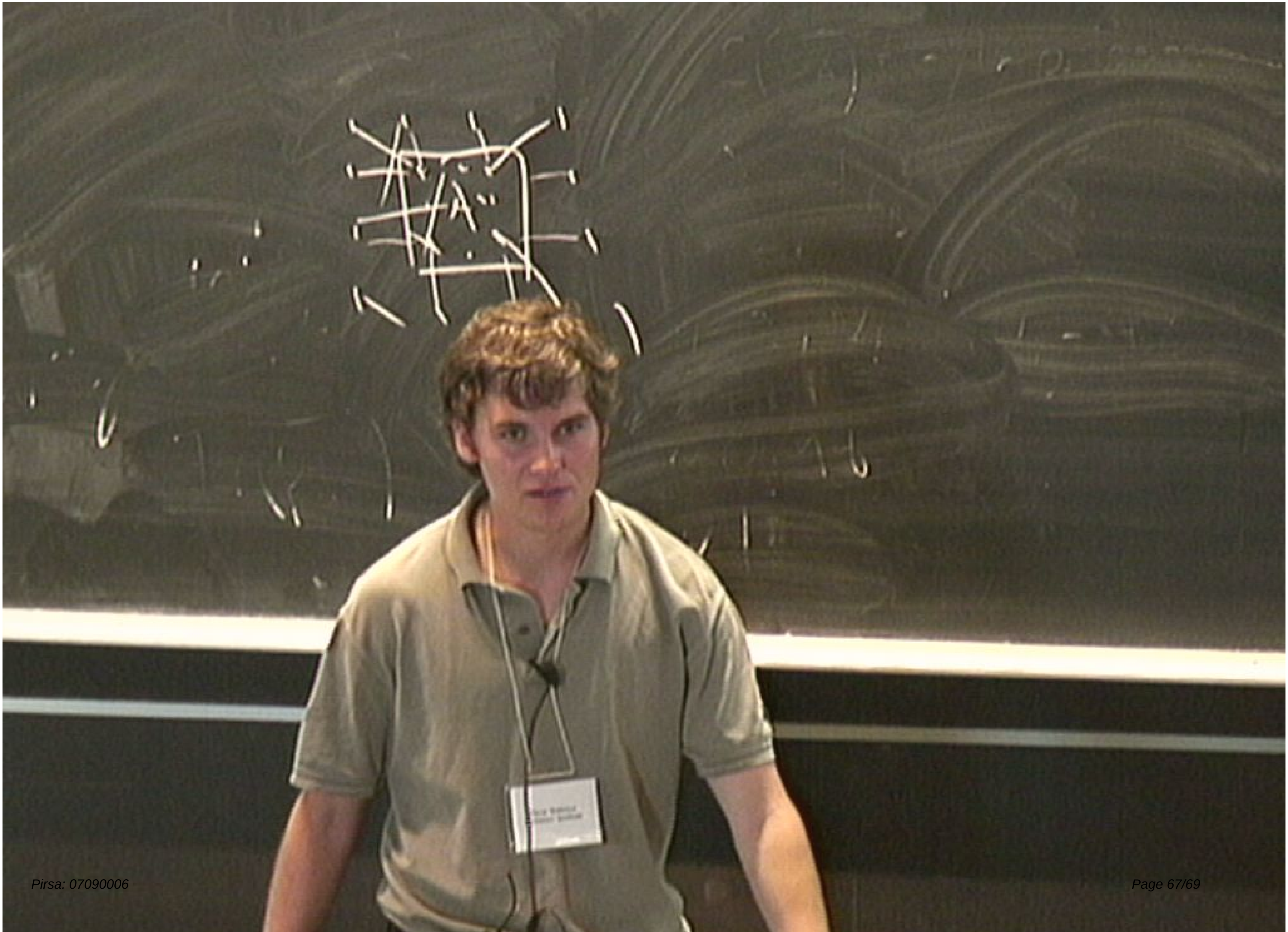
$$n \geq 9N(N-1)[(4 \ln 2)N + \ln \varepsilon^{-1}]/4$$

we have $\langle E(\psi_n) \rangle \geq (\min(N_A, N_B) - 2^{-|N_B - N_A|} + \varepsilon)/\ln 2$

and, for $|\phi\rangle$ maximally entangled

$$\left\langle \max_{|\phi\rangle} \left| \langle \psi_n | \phi \rangle \right| \right\rangle \geq 1 - \sqrt{\frac{2^{-(N_B - N_A)} + \varepsilon}{2 \ln 2}}$$


A



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