Title: An algorithmic approach to heterotic compactification

Date: Sep 04, 2007 11:00 AM

URL: http://pirsa.org/07090004

Abstract: In this talk, I will describe recent work in string phenomenology from the perspective of computational algebraic geometry. I will begin by reviewing some of the long-standing issues in heterotic model building and the goal of producing realistic particle physics from string theory. This goal can be approached by creating a large class of heterotic models which can be algorithmically scanned for physical suitability. I will outline a well-defined set of heterotic compactifications over complete intersection Calabi-Yau manifolds using the monad construction of vector bundles.
Further, I will describe how a combination of analytic methods and computer algebra can provide efficient techniques for proving stability and calculating particle spectra.

An Algorithmic Approach to Heterotic Compactification:

Monads, CICYs, and particle spectra

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Perimeter Institute - September 4th, 2007

Outline

- Introduction
  - Heterotic Phenomenology
  - Why we're interested (and the problems)
- The monad construction
  - The Calabi-Yau Spaces
  - Building vector bundles
  - Particle spectra and bundle stability
- Finding physically relevant bundles - An algorithmic approach
- Future directions - Symmetry breaking, yukawa couplings, fermion masses.

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Motivations

Challenge of String Phenomenology

- Is string theory a model of the real world? Can it not only 'inspire' new physics but produce testable models as well?
- the theory must correspond with what we already know
- "Occam's Razor" - A philosophy of parsimony, a model should involve the minimal number of assumptions
- How to produce the SUSY standard model (e.g. MSSM) from string theory -the holy grail of string phenomenology?
  - Is there a string model that gives exactly the real world? We need gauge unification, fermion masses, yukawa couplings, etc.

Many Approaches

- Major stringy approaches to realistic particle physics include
  - D-brane models, type II
  - M-theory on G_2 manifolds
  - Heterotic model building

- Most approaches to phenomenology in string theory have limitations. It's easy to come close to the real world, but very hard to get the details exactly right.
  - special points in moduli-space $\Rightarrow$ enhanced gauge groups and spectra

    $\Rightarrow$ want generic points
  - Large numbers of exotic particles
    1. particles with the wrong quantum numbers
    2. vector-like pairs

    $\Rightarrow$ Heterotic string theory produces the correct quantum numbers
  - Moduli stabilization?

Heterotic Model building

- Despite being one of the oldest approaches to particle phenomenology in string theory, heterotic model building is still one of the more promising avenues

  Features include:

  - Gauge unification is automatic (GUTS)
  - Standard Model families originate from an underyling spinor rep of a GUT group (E_6, $SO(10)$, $SU(5)$)

- However, heterotic model building is hard.

  - Inherent mathematical difficulty - Defining bundles and manifolds. Algebraic geometry
  - Computational difficulty - bundle cohomology and particle spectra

A heterotic model

We begin with the $E_8 \times E_8$ Heterotic string in 10-dimensions

- One E_8 gives rise to the "Visible" sector, the other to the "Hidden" sector
- Compactify on a Calabi-Yau 3-fold, X - leads to $\mathcal{N} = 1$ SUSY in $4D$
- Also have a vector bundle V on X (with structure group $G \subset E_8$)

  V breaks E_8 to Low Energy GUT group
- The weakly coupled theory has been studied since the 80's
- First attempt at string phenomenology - Candelas-Horowitz-Strominger-Witten 1986
- Soon after discovery of $E_8 \times E_8$ - found CYs that gave E_6 GUTS
- The strongly coupled theory is dual to M-Theory on a manifold with boundary - Hořava-Witten Theory

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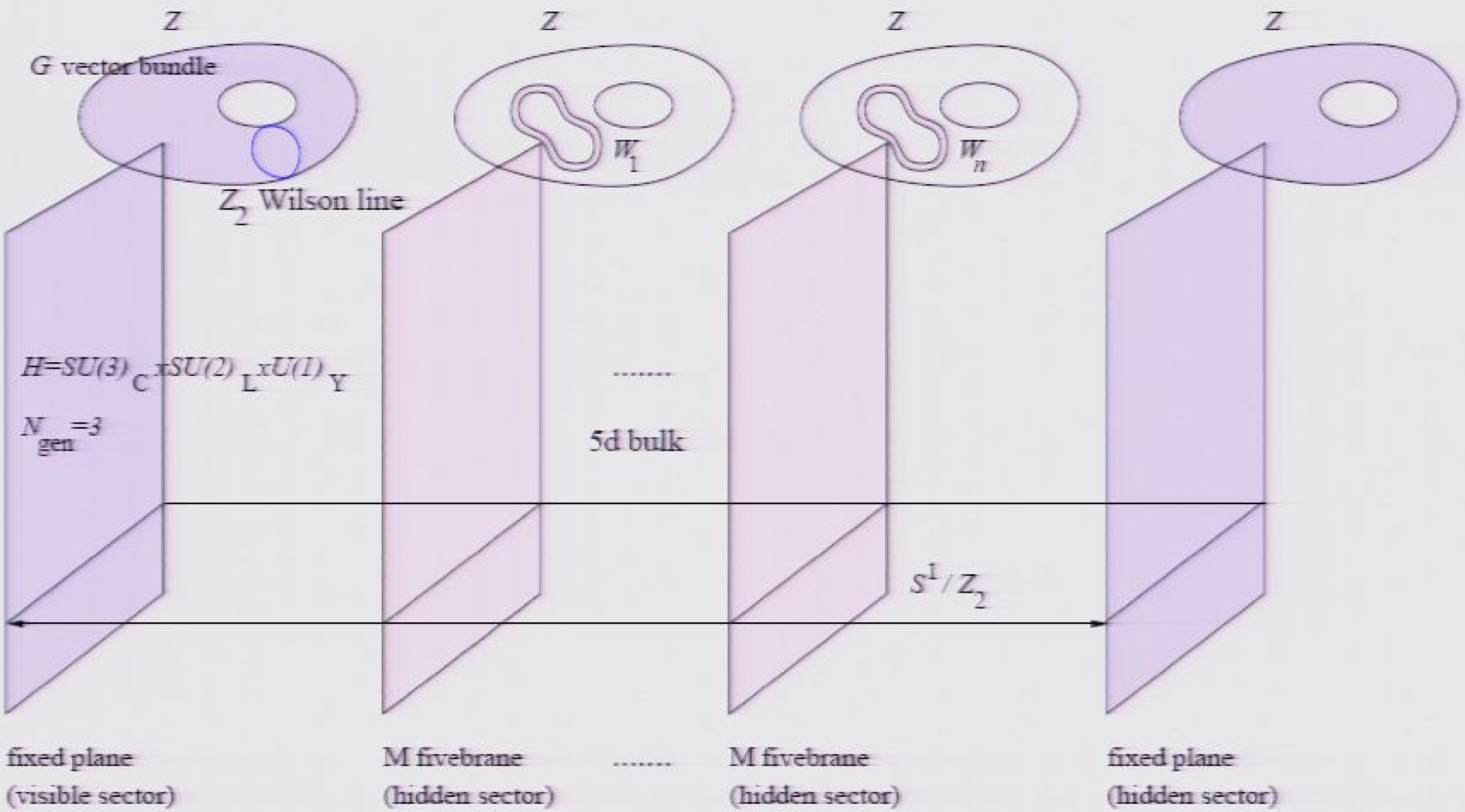

Donagi-Ovrut-Pantev-Waldram [9912208]

The idea...

- Finding the correct string vacuua to model realistic particle physics is a difficult task. How to choose? How to design the right model?
- A new approach:
  Formulate an algorithmic and systematic search for the correct vacuum using computational algebraic geometry
  1. Produce a computer database of thousands of CY spaces and their topological data
  2. Construct well-defined natural vector bundles over them
  3. Scan through literally hundreds of billions of potential candidates in the vast landscape of string vacua for those that are physically relevant
- How many are close to nature? Study these models...

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The "Standard" Embedding

- The first attempt at a heterotic model was the so-called "standard" embedding (CHSW, Greene, Ross, et all (1980's))
- Take $V = TX$ and $G = SU(3) = hol(X)$
- Spin connection of X becomes the connection of the vector bundle and must satisfy the Hermitian Yang-Mills equations

  $F_{ab} = F_{\bar{a}\bar{b}} = g^{a\bar{b}} F_{\bar{b}a} = 0$

  where F is the gauge field strength of the holomorphic bundle V
- Generalization of Ricci-flatness
- Obtain a $N = 1$, E_6 SUSY GUT in $4D$
- Particle spectra directly determined by choice of Calabi-Yau

The resultant model is not phenomenologically favored

- Non-minimal - Extra $U(1)'s$
- 27 of E_6, $27 \to (16 \oplus 10 \oplus 1)$, $10 \oplus 1$ exotic.
- Number of generations $N_{Gen} = 3c_3(TX)$ (hard to find good CYs)
- Need 3 net families: $|h^{1,1} - h^{2,1}| = 3$
- Extensive search for such CYs in the 80's found very few;

So, we need something more...

General Embedding

A more general choice of vector bundle can be made

- Take $G = SU(n)$, $n = 3, 4, 5$ low energy gauge group
- $4D$ structure group, $H =$Commutant(G, E_8)

$E_8 \to G \times H$	Residual Group Structure
SU(3) × E$_6$	$248 \to (1,78) \oplus (3,27) \oplus (\overline{3},\overline{27}) \oplus (8,1)$
SU(4) × SO(10)	$248 \to (1,45) \oplus (4,16) \oplus (\overline{4},\overline{16}) \oplus (6,10) \oplus (15,1)$
SU(5) × SU(5)	$248 \to (1,24) \oplus (5,\overline{10}) \oplus (\overline{5},10) \oplus (10,5) \oplus (\overline{10},\overline{5}) \oplus (24,1)$

- We expect "Two-step" Symmetry breaking

1. E_8 breaks to GUT group (E_6,$SO(10)$, or $SU(5)$)
2. Wilson lines break GUT symmetry

Wilson line $\Rightarrow SU(3)_c \times SU(2)_l \times U(1)_Y \times U(1)_{B-L}$ symmetry

The Elements of the construction

- X: A Calabi-Yau 3–fold, X
- V: A holomorphic vector bundle, satisfying the Hermitian YM equations
- G: The structure group of V ($G \subset E_8$)
- H: The low energy $4D$, $\mathcal{N}=1$ GUT symmetry (H is the commutant of G in E_8)
- V + Wilson line leads to symmetry containing MSSM
- Heterotic vacuua contain $M5$-branes which can wrap a holomorphic effective 2-cycle, W ($\in H_2(X,\mathbb{Z})$), of X. Leads to anomaly cancellation condition

$$c_2(X) - c_2(V) = W_{M5}$$

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Stability

- Hermitian YM equations are a set of wickedly complicated PDE's

  $F_{ab} = F_{\bar{a}\bar{b}} = g^{a\bar{b}} F_{\bar{b}a} = 0$

- We are saved by the Donaldson-Uhlenbeck-Yau Theorem:

  On each stable, holomorphic vector bundle V, there exists a Hermitian YM connection satisfying HYM.

- The slope, $\mu(V)$, of a vector bundle is

$$\mu(V) \equiv \frac{1}{\mathrm{rk}(V)} \int_X c_1(V) \wedge J^{d-1}$$

  where J is a Kahler form on X

- V is Stable if for every sub-sheaf, $\mathcal{F}$, of V, $\mu(\mathcal{F}) < \mu(V)$
- Semi-stability ($\mu(\mathcal{F}) \leq \mu(V)$) is sufficient for SUSY, but full stability is computationally useful

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So...

We want a stable holomorphic vector bundle V on X

- For stable bundles on a CY 3-fold, X, stability implies that $H^0(X, V) = H^0(X, V^*) = 0$
- In general: V with $SU(n)$

INDEX THEOREM and particle generations:

- $\text{index}(\nabla_X) = \sum_{i=0}^{3}(-1)^i h^i(X, V) = \int_X (V)(X) = \frac{1}{2}\int_X c_3(V)$
- Serre Duality: $h^i(X, V) = h^{3-i}(X, V^* \otimes K_X)$
- $\Rightarrow$ 3-families: $3 = -h^1(X, V) + h^1(X, V^*)$
- Unfortunately, "conservation of misery" $\Rightarrow$ stability still very hard to show!

Spectra and Cohomology

In heterotic models, $4D$ particle spectra is determined by bundle cohomology

- LE particles $\sim$ massless modes of V-twisted Dirac Operator: $\not{\nabla}_X \Psi = 0$
- massless modes of $\not{\nabla}_X \overset{1:1}{\longleftrightarrow}$ V-valued cohomology groups

Decomposition	Cohomologies
$SU(3) \times E_6$	$n_{27} = h^1(V)$, $n_{\overline{27}} = h^1(V^*) = h^2(V)$, $n_1 = h^1(V \otimes V^*)$
$SU(4) \times SO(10)$	$n_{16} = h^1(V)$, $n_{\overline{16}} = h^2(V)$, $n_{10} = h^1(\wedge^2 V)$, $n_1 = h^1(V \otimes V^*)$
$SU(5) \times SU(5)$	$n_{10} = h^1(V^*)$, $n_{\overline{10}} = h^1(V)$, $n_5 = h^1(\wedge^2 V)$, $n_{\bar{5}} = h^1(\wedge^2 V^*)$ $n_1 = h^1(V \otimes V^*)$

- We need to compute various bundle cohomologies in order to proceed!

The Plan...

- We need a large class of CY manifolds and a systematic way to construct bundles over them

  $\Rightarrow$ 7890 CICYs + Monad construction

- We require an explicit construction compatible with "Two-step" symmetry breaking, Wilson lines, etc.

  $\Rightarrow$ CICYs are the simplest and most explict form of CY construction. Relatively easy to find discrete symmetries, Wilson lines, etc.

- Computerizability. Millions of models cannot be analyzed by hand, need an algorithmic approach that makes good use of existing technology in computational algebraic geometry.

  $\Rightarrow$ Teach computers how to analyze monads! (C code, Mathematica, Singular, Macaulay)

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- Need to be able to compute bundle cohomology (Koszul and Leray sequences)
- Need to be able to check bundle stability (Hoppe's Criterion and generalization)
- Scan millions of bundles for physical suitability!
- How many bundles are there? What distributions? What properties?...

Complete Intersection CYs

- Begin with an ambient space composed as a product of projective space $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_m}$.
- Next, add K defining polynomials $\{p_{j=1,\ldots,K}\}$.
- The projective coordinates of the $\mathbb{P}^{n_i}$ factor are $[x_0^i : x_1^i : \ldots : x_{n_i}^i] \Rightarrow$ each polynomial constraint specified by its (homogeneous) degree in the variables x^i.
- A convenient way to encode this information is by a configuration matrix (columns $\leftrightarrow$ constraints)

$$\left[\begin{array}{c|cccc} \mathbb{P}^{n_1} & q_1^1 & q_1^2 & \ldots & q_1^K \\ \mathbb{P}^{n_2} & q_2^1 & q_2^2 & \ldots & q_2^K \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbb{P}^{n_m} & q_m^1 & q_m^2 & \ldots & q_m^K \end{array}\right]_{m\times K} \quad (1)$$

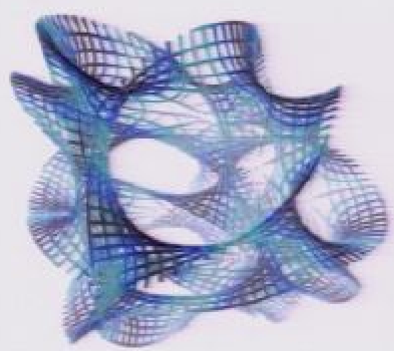
- Favorable CICYs: Those for which $h^{1,1} =$ of embedding $\mathbb{P}^n$'s (4515 manifolds)

  $\Rightarrow$ The Kahler forms J on the CY descend from those on the ambient space. Computationally useful!

- Classic Example: The Quintic 3-manifold (A quintic hypersurface in $\mathbb{P}^4$). Written

$$[4|5]_{-200}$$

The Quintic is "Cyclic", (i.e. $Pic(X) = \mathbb{Z}$) $h^{1,1} = 1$, $h^{2,1} = 101$



Line Bundles

- Line Bundles on $\mathbb{P}^n$
  - Written as $\mathcal{O}(k)$ (the kth power of the hyperplane bundle $\mathcal{O}(1)$)
  - By definition, $\mathcal{O}(k)$ is the line bundle who's first chern class is $c_1(\mathcal{O}(k)) = k$
- Similarly, on a product space $\mathbb{P}^n_1 \times \mathbb{P}^n_2 \times \ldots \mathbb{P}^n_m$: $\mathcal{O}(k_1, k_2, ..., k_m)$
- On a "favorable" CY we have $\mathcal{O}_X(k_1, k_2, ..., k_m)$
- Kodaira vanishing theorem

  For a positive line bundle, P, $H^q(X, P \times K_X) = 0 \forall q > 0$
- Note: If $k_i > 0$ $\forall i$ then $\mathcal{O}_X(k_1, k_2, ..., k_m)$ is a positive bundle.

What is a Monad?

- For this work, we consider monads defined by a short exact sequence of vector bundles (sheaves)

$$0 \to V \xrightarrow{f} B \xrightarrow{g} C \to 0$$

where short exact implies that $ker(g) = im(f)$.

- The vector bundle V is defined as

$$V = ker(g) \text{ with } rk(V) = rk(B) - rk(C)$$

- Where B and C are taken to be direct sums of line bundles

$$B = \bigoplus_{i=1}^{r_B} \mathcal{O}(b_r^i)\ , \qquad C = \bigoplus_{j=1}^{r_C} \mathcal{O}(c_r^j)$$

- The map g can be written as a matrix of polynomials. (e.g. on $\mathbb{P}^n$ the ij-th entry is a homogeneous polynomial of degree $c_i - b_j$)
- The monad construction is a powerful and general way of definining vector bundles. For example, every bundle on $\mathbb{P}^n$ can be written as a monad

Physical Constraints

- $SU(n)$ bundles - (Structure group $SU(n)$, $c_1(V) = 0$)
- Anomaly cancellation condition
- $Ind(V) = 3k$ for $k \in \mathbb{Z}$

  $k > 1 \Rightarrow$ need Wilson lines and discrete symmetries
- Stable bundles
- Monads must define bundles (i.e. defining exact sequences should produce bundles rather than just sheaves)

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$$V = ker(g) \text{ with } rk(V) = rk(B) - rk(C)$$

- Where B and C are taken to be direct sums of line bundles

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- The map g can be written as a matrix of polynomials. (e.g. on $\mathbb{P}^n$ the ij-th entry is a homogeneous polynomial of degree $c_i - b_j$)

- The monad construction is a powerful and general way of definining vector bundles. For example, every bundle on $\mathbb{P}^n$ can be written as a monad

Physical Constraints

- $SU(n)$ bundles - (Structure group $SU(n)$, $c_1(V) = 0$)
- Anomaly cancellation condition
- $Ind(V) = 3k$ for $k \in \mathbb{Z}$

  $k > 1 \Rightarrow$ need Wilson lines and discrete symmetries
- Stable bundles
- Monads must define bundles (i.e. defining exact sequences should produce bundles rather than just sheaves)

Classification

The physical and Mathematical constraints can be written as constraints on the integers defining the line bundles of the monad

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- Is this a finite class? What are the properties of the bundles defined by these constraints?

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Constraints

- $b_r^i \leq c_r^j$ for all $i, j, s \Leftrightarrow ker(g)$ defines a bundle.
- The map g can be taken to be *generic* so long as exactness of the sequence is maintained.
- $c_1(V) = 0 \Leftrightarrow$
  $\sum_{i=1}^{r_B} b_i^r - \sum_{j=1}^{r_C} c_j^r = 0$
- Anomaly cancellation $\Leftrightarrow$
  $c_2(TX) - c_2(V) = c_2(TX) - \frac{1}{2}(\sum_{i=1}^{r_B} b_s^i b_t^i - \sum_{i=1}^{r_C} c_s^j c_i^t) J^s J^t \geq 0$
- 3 Generations $\Leftrightarrow$
  $c_3(V) = \frac{1}{3}(\sum_{i=1}^{r_B} {b_r}^i {b_s}^i {b_t}^i - \sum_{j=1}^{r_C} {c_r}^j {c_s}^j {c_t}^j) J^r J^s J^t$
  is divisible by 3.
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What is a Monad?

- For this work, we consider monads defined by a short exact sequence of vector bundles (sheaves)

$$0 \longrightarrow V \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

where short exact implies that $ker(g) = im(f)$.

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Hoppe's Criterion

Over a projective manifold X with Picard group $Pic(X) \simeq \mathbb{Z}$, let V be a vector bundle. If $H^0(X, [\bigwedge^p V]_{norm}) = 0$ for all $p = 1, 2, \ldots, (V) - 1$, then V is stable.

- Those manifolds where $Pic(X) \simeq \mathbb{Z}$ are called *cyclic*
- Where $[V]_{norm} = V(i) := V \otimes O_X(i)$ for a unique i such that $c_1(V(i)) \in [-rk(V) + 1, \ldots, -1, 0]$
- *normalize* V so that the slope $\mu(V)$ is between -1 and 0.
- Hoppe's criterion applies directly to the 5 cyclic CYs
- Generalization to arbitrary CICYs?...
- We will begin with this criterion and the cyclic manifolds...

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Warming Up...The cyclic CICYs

The mathematical technology of producing bundles and computing their spectra is difficult, so we begin with the most straightforward possible cases...

- There are 5 cyclic ($Pic(X) = \mathbb{Z}$) CICYs

  $[4|5]$, $[5|2\ 4]$, $[5|3\ 3]$, $[6|3\ 2\ 2]$, $[7|2\ 2\ 2\ 2]$
- These are the simplest known CYs.
- We can find a complete classification of *all* physical monad bundles on these spaces.
  - Demanding $SU(n)$ and anomaly-free bundles is sufficient to bound the problem
  - A finite class - We find only 37 bundles over these 5 spaces

Cyclic CICYs

- For the cyclic CYs, we find only positive monads to be physical (e.g. those for which $b_i, c_j > 0$)
  - $b_i, c_j \leq 0 \Leftrightarrow$ unstable
- Can compute the full spectra of these bundles using exact and spectral sequences (and results can be checked using Macaulay, Singular)
- No anti-generations (limits exotics)
- The Higgs content is dependent on the choice of map (where we are in moduli space)
- Using Hoppe, we find that all positive monads on CICYs are stable!

Monads on CICYS: An example

- For example consider the monad

  $$0 \rightarrow V \longrightarrow \mathcal{O}(1)^7 \xrightarrow{g} \mathcal{O}(3) \oplus \mathcal{O}(2)^2 \rightarrow 0$$

- this is a rank 4 bundle on $[4|5]$
- $SU(4)$ bundle, stable, anomaly-free
- $c_3 = -30 \Rightarrow \mathbb{Z}_5 \times \mathbb{Z}_5$ symmetry for Wilson lines
- Cohomology calculation gives us $n_{16} = 30$, $n_1 = 112$
- Number of Higgs depends on choice of map

Rank	$\{b_i\}$	$\{c_i\}$	$c_2(V)/J^2$	$ind(V)$
3	(1, 1, 1, 1)	(4)	6	-90
3	(1, 1, 1, 1, 1)	(3, 2)	4	-45
3	(2, 1, 1, 1, 1)	(3, 3)	5	-63
3	(1, 1, 1, 1, 1, 1)	(2, 2, 2)	3	-27
3	(2, 2, 2, 1, 1, 1)	(3, 3, 3)	6	-81
4	(1, 1, 1, 1, 1, 1)	(3, 3)	6	-72
4	(1, 1, 1, 1, 1, 1, 1)	(3, 2, 2)	5	-54
4	(1, 1, 1, 1, 1, 1, 1, 1)	(2, 2, 2, 2)	4	-36
5	(1, 1, 1, 1, 1, 1, 1, 1, 1)	(3, 2, 2, 2)	6	-63
5	(1, 1, 1, 1, 1, 1, 1, 1, 1, 1)	(2, 2, 2, 2, 2)	5	-45

Table: *Positive monad bundles on* [5|3 3].

Rank	$\{b_i\}$	$\{c_i\}$	$c_2(V)/J^2$	$ind(V)$
3	(2, 2, 1, 1, 1)	(4, 3)	7	-60
3	(2, 2, 2, 1, 1)	(5, 3)	10	-105
3	(3, 2, 1, 1, 1)	(4, 4)	8	-75
3	(1, 1, 1, 1, 1, 1)	(2, 2, 2)	3	-15
3	(2, 2, 2, 1, 1, 1)	(3, 3, 3)	6	-45
3	(3, 3, 3, 1, 1, 1)	(4, 4, 4)	9	-90
3	(2, 2, 2, 2, 2, 2, 2, 2)	(4, 3, 3, 3, 3)	10	-90
3	(2, 2, 2, 2, 2, 2, 2, 2, 2)	(3, 3, 3, 3, 3, 3)	9	-75
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4	(1, 1, 1, 1, 1, 1, 1)	(3, 2, 2)	5	-30
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5	(1, 1, 1, 1, 1, 1, 1, 1)	(3, 3, 2)	7	-45

Finding a Higgs doublet

- Higgs at special points - The "Jumping phenomena".
- Example: Specifically, let is consider the following $SU(4)$ bundle on $[4|5]$:

  $$0 \to V \to \mathcal{O}_X^{\oplus 2}(2) \oplus \mathcal{O}_X^{\oplus 4}(1) \xrightarrow{g} \mathcal{O}_X^{\oplus 2}(4) \to 0$$

  with ($x_{0,\dots,4}$ are the homogeneous coordinates on $\mathbb{P}^4$)
- $$g = \begin{pmatrix} 4x_3^2 & 9x_0^2 + x_2^2 & 8x_2^3 & 2x_3^3 & 4x_1^3 & 9x_1^3 \\ x_0^2 + 10x_2^2 & x_1^2 & 9x_2^3 & 7x_3^3 & 9x_1^3 + x_2^3 & x_1^3 + 7x_4^3 \end{pmatrix}.$$
  - We can calculate that

    $n_{16} = h^1(X, V) = 90$, $n_{10} = h^1(X, \wedge^2 V) = 13$

    $n_1 = 277$

Positive Monads on favorable CICYs

- For the monads with strictly positive entries ($b^i, c^j > 0$) the $SU(n)$ and anomaly conditions are sufficient to bound the problem. The class is finite and all physical bundles can be classified.
- Over the CICYs we find thousands of bundles and their spectra
- No anti-generations!
- Unfortunately, we are limited by Wilson Lines and discrete symmetries of the Calabi-Yau

  Number of generations $c_3 = 3n$, $n \in \mathbb{Z}$

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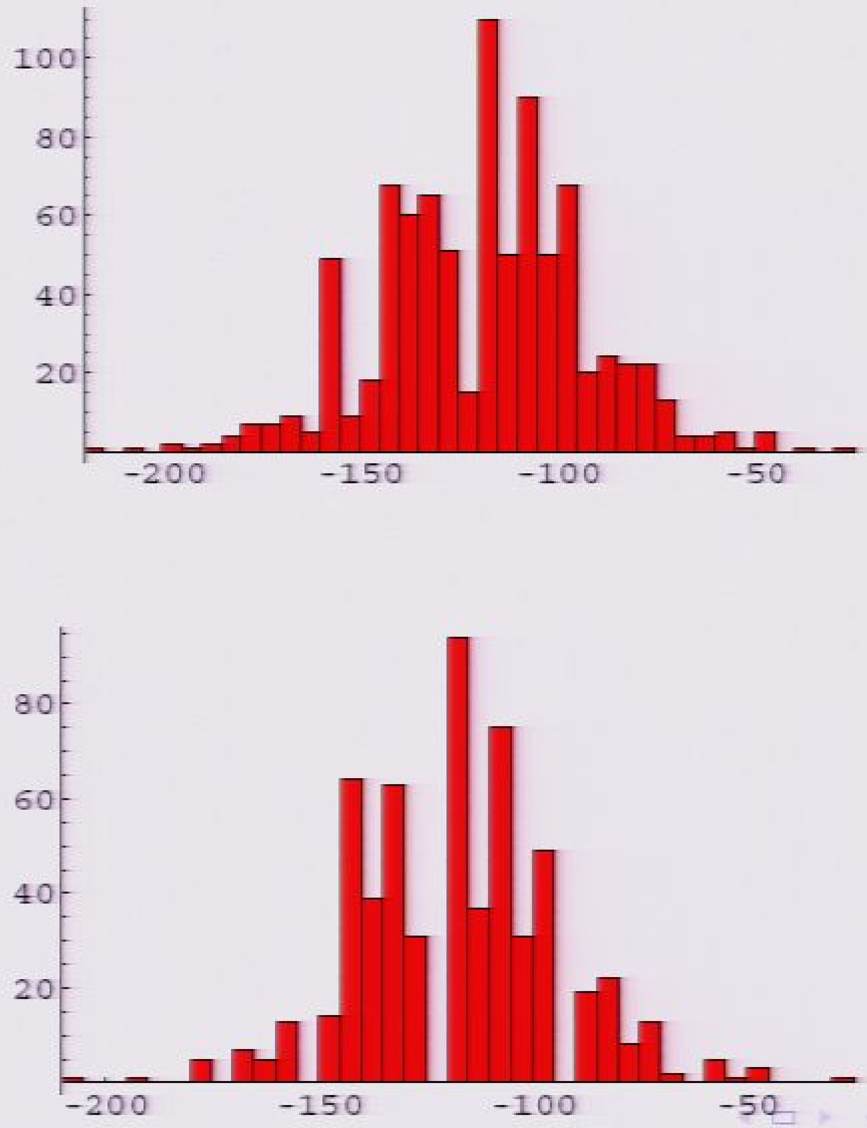
- In order to produce *exactly* 3 generations, need to divide the CY by a discrete symmetry of size n
- If c_3 too large, no symmetries of the right order exist on the CY
- The 37 monads on the cyclic CYs have the smallest $c_3 \Leftrightarrow$ most plausible models for Wilson line symmetry breaking.

Positive monads and 3-generations



Extending the search...

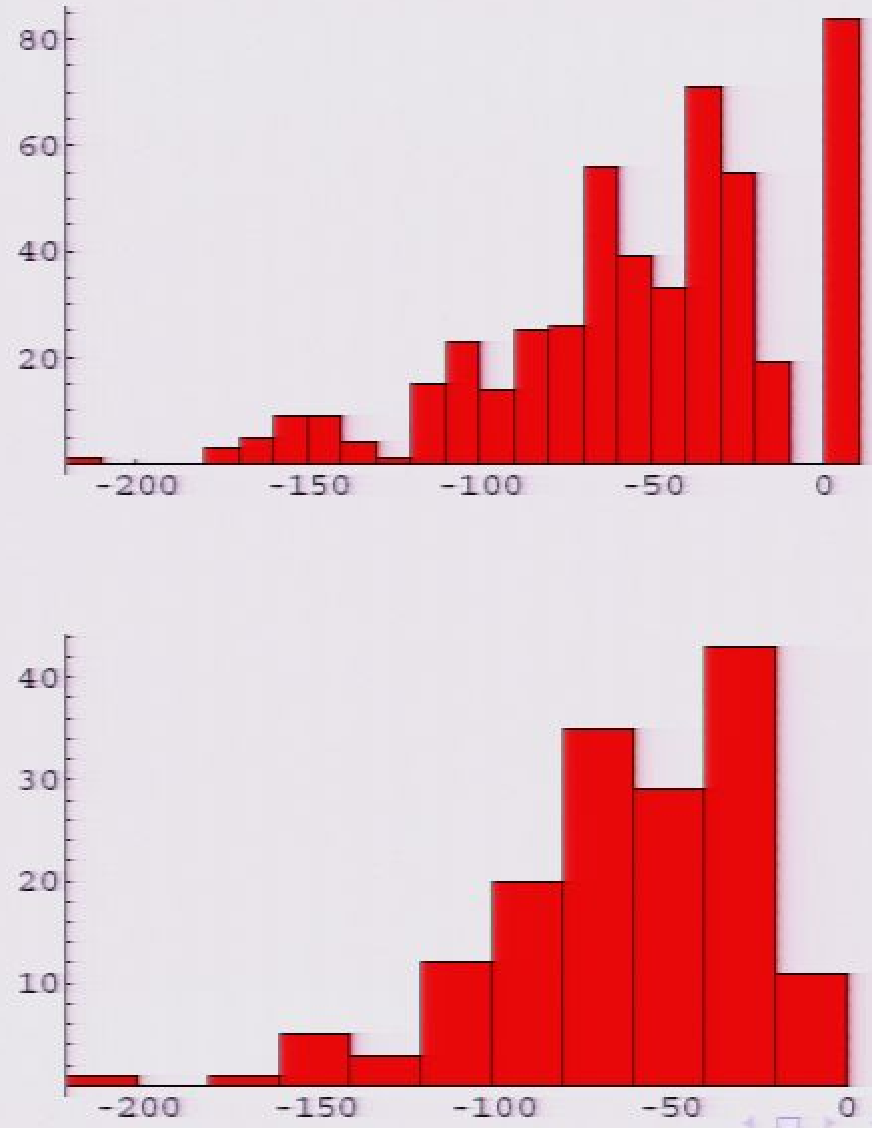
Since the positive bundles on CICYs are highly restricted, in order to produce a large class for an algorithmic scan, we must extend our search...

Zero-Entry Monads

For those with entries greater than or equal to zero ($b^i, c^j \geq 0$) the construction is much bigger (and more interesting!)

- Clearly not constrained as before, can produce unbounded sets of bundles
  Example: $B = \mathcal{O}(1,0)^3 \oplus \mathcal{O}(t-3,0)$ and $C = \mathcal{O}(t,0)$ is an anomaly-free bundle for each integer $t > 1$ on $\left[\begin{array}{c|cc} \mathbb{P}^1 & 0 & 2 \\ \mathbb{P}^4 & 4 & 1 \end{array} \right]$
- Much more physically suitable
  - smaller $c_3(V)$
  - Generically have a Higgs
- Much harder to work with. Cohomology more difficult to compute in general
- Can do checks of stability (scan for $H^{0,3}(X,V) = 0$)
- Generalization of Hoppe (in progress...)

Zero-entry distributions



An algorithmic approach...

- We have successfully constructed a LARGE class of vector bundles
- Previous attempts have been made to create classes of stable bundles (e.g. Friedman-Morgan-Witten). However, this is the first effort to create an extensive class of stable, physically relevant bundles suitable for algorithmic scans.
- So, far we have scanned for "higher-level" physical constraints - i.e. particle spectra
- We can verify low-energy SUSY (i.e. bundle stability) for a sub-class
- We are now ready to impose more detailed physical constraints - Wilson lines, discrete symmetries, etc.

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Future work

We are only at the beginning of this effort...

- Extend techniques to the 473, 800, 776 toric CY manifolds (in progress)
- Develop computational analysis of monad bundles (Mathematica, Macaulay2 - CICYs)
- Add Wilson lines, explore realistic 4D models (in progress)
- Compute yukawa couplings? Fermion masses?