Title: Mapping classical spin models to the graph state formalism

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Abstract: In this talk we discuss how large classes of classical spin models, such as the Ising and Potts models on arbitrary lattices, can be mapped to the graph state formalism. In particular, we show how the partition function of a spin model can be written as the overlap between a graph state and a complete product state. Here the graph state encodes the interaction pattern of the spin model---i.e., the lattice on which the model is defined---whereas the product state depends only on the couplings of the model, i.e., the interaction strengths. As main examples, we find that the 1D Ising model corresponds to the 1D cluster state, the 2D Ising model without external field is mapped to Kitaev's toric code state, and the 2D Ising model with external field corresponds to the 2D cluster state---but the mappings are completely general in that arbitrary graphs, and also q-state models can be treated.

These mappings allow one to make connections between concepts in
(classical) statistical mechanics and quantum information theory and to obtain a cross-fertilization between both fields. As a main application, we will prove that the classical Ising model on a 2D square lattice (with external field) is a "complete model", in the sense that the partition function of any other spin model---i.e., for q-state spins on arbitrary lattices---can be obtained as a special instance of the (q=2) 2D Ising partition function with suitably tuned (complex) couplings.

This result is obtained by invoking the above mappings from spin models to graph states, and the property that the 2D cluster states are universal resource states for one-way quantum computation.

Joint work with Wolfgang Duer and Hans Briegel, see PRL/ 98 117207 (2007)/ and quant-ph/0708.2275. For related work, see also S. Bravyi and R. Raussendorf, quant-ph/0610162.

Mapping classical spin models to the graph state formalism

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Overview

- We describe mappings which relate classical spin models (Ising, Potts) with the quantum stabilizer formalism (graph states)
- Establish connection between statistical physics and QIT
- Aim: obtain cross-fertilization between both fields
- Main application presented here is centered around measurement-based quantum computation

Quant-ph/0708.2275 and PRL 98, 117207 (2007)

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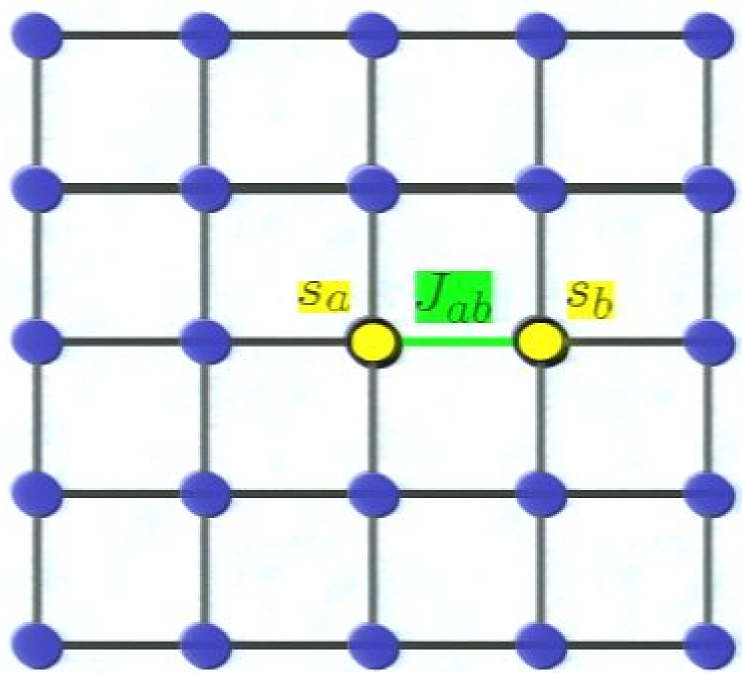
q-state Classical spin models

- Ising model on a graph G (q=2)

$$-\sum_{(a,b)\in E} J_{ab}\ s_a s_b \quad (s_a = \pm 1)$$

- Potts model (arbitrary q)

$$-\sum_{(a,b)\in E} J_{ab}\ \delta(s_a, s_b) \quad (s_a = 0, \ldots, q-1)$$



- with/without magnetic fields, e.g., $-\sum_a h_a s_a$

- Partition function: $Z_G = \sum \exp(-\beta H(\{s_a\}))$

q-state Classical spin models

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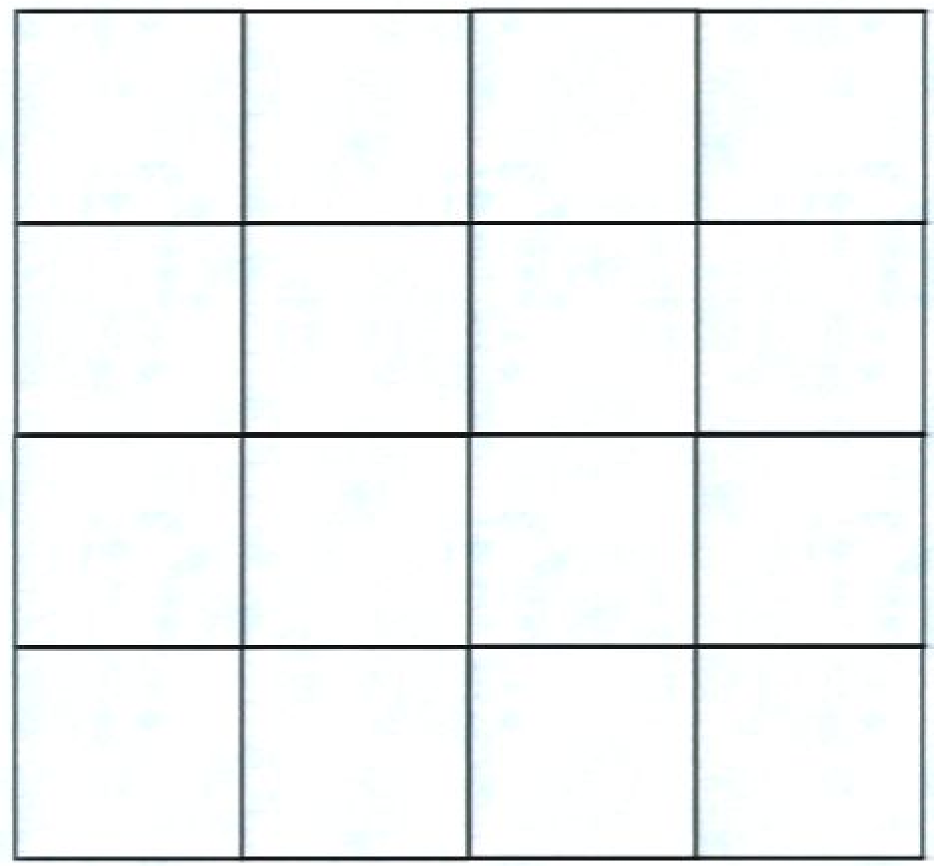
First mapping:

spin models without external fields

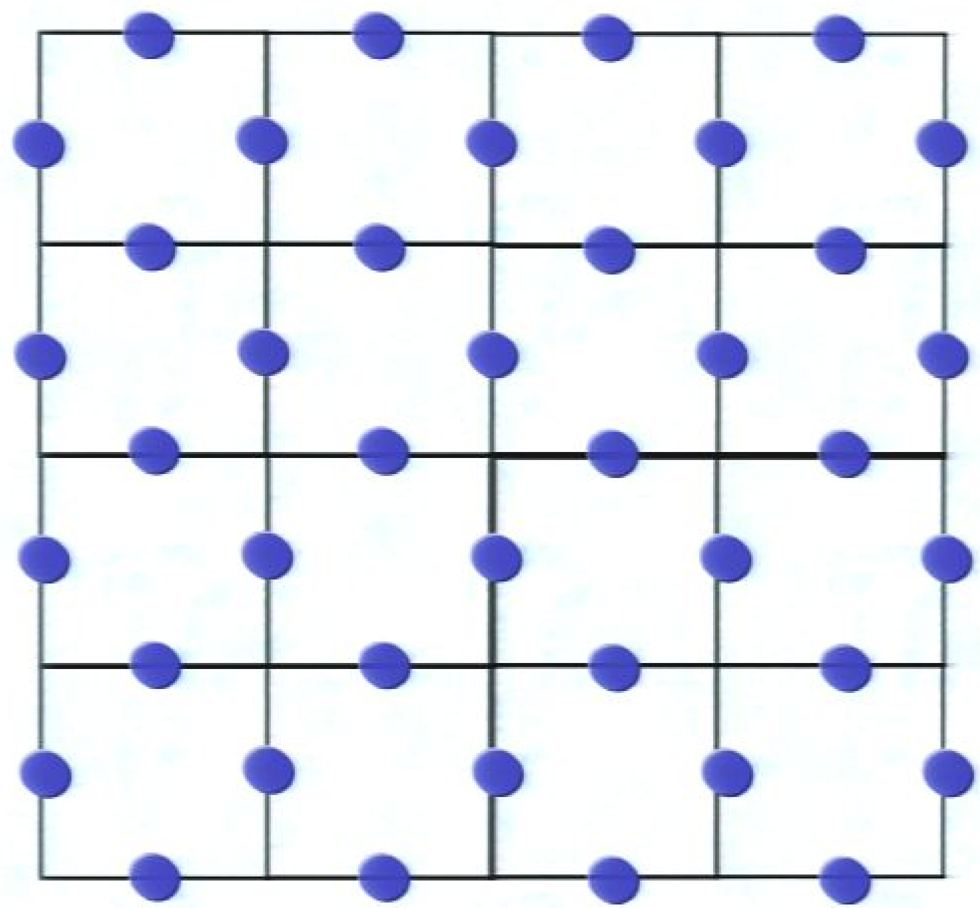
Kitaev-type states $|\psi_G\rangle$

- start with a graph G



Kitaev-type states $|\psi_G\rangle$

- ❑ start with a graph G
- ❑ qubits ● = edges of G



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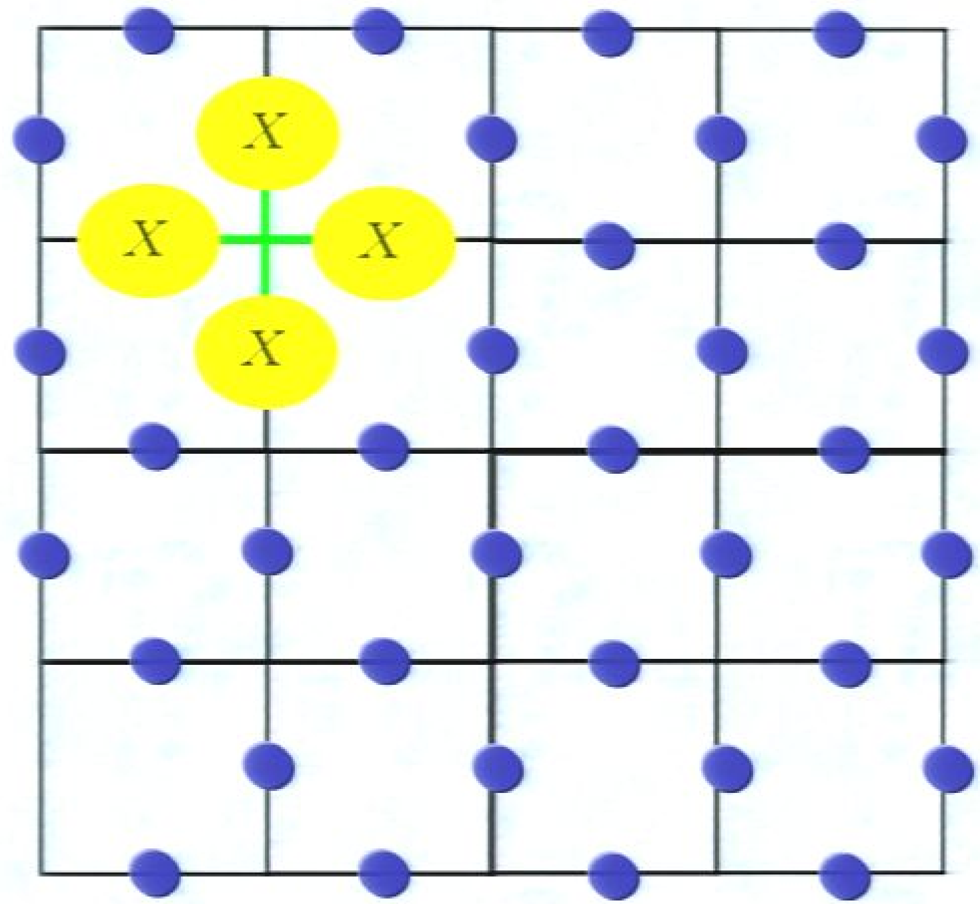
- 1 stabilizer operator per vertex:

$$V_a = \prod_{e:a\in e} X^{(e)}$$



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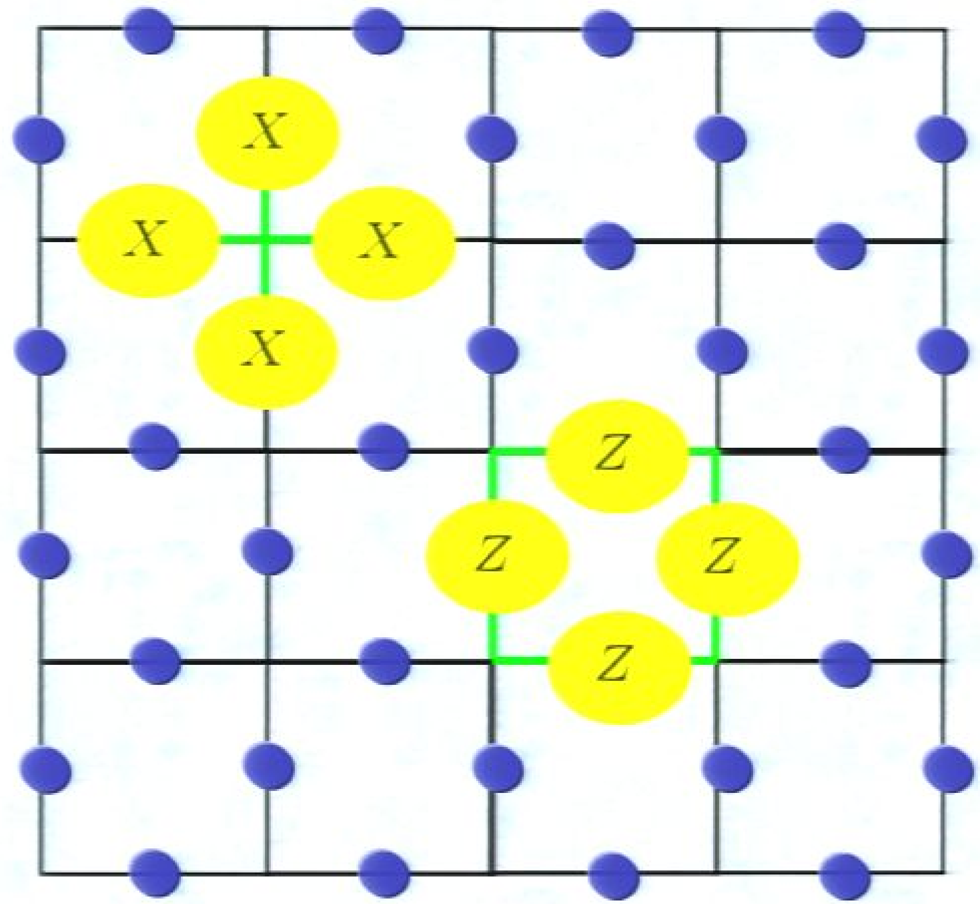
- 1 stabilizer operator per plaquette:

$$P_\square = \prod_{e\in\square} Z^{(e)}$$



Kitaev-type states $|\psi_G\rangle$

- Basis expansion: $|\psi_G\rangle \propto \sum_{t \in \{0,1\}^V} |B^T t\rangle$

- B = incidence matrix of G $\quad B^T t = \begin{bmatrix} \vdots \\ t_a - t_b \\ \vdots \end{bmatrix}$ $\quad$ (a, b) = edge of G

The mapping

- Ising model without external fields
- For every edge (a, b) of graph: $|\alpha_{ab}\rangle = e^{\beta J_{ab}}|0\rangle + e^{-\beta J_{ab}}|1\rangle$
- Then: $$Z_G \propto \langle \psi_G | \bigotimes_{ab} |\alpha_{ab}\rangle$$
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Interaction PATTERN	Interaction STRENGTH

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Examples

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- 1D Ising model with open BCs $\longrightarrow$ Product state $|\psi_G\rangle = |+\rangle^N$

$$Z_G \propto \prod_{ab} \langle + | \alpha_{ab} \rangle = \prod_{ab} \left\{ e^{\beta J_{ab}} + e^{-\beta J_{ab}} \right\}$$

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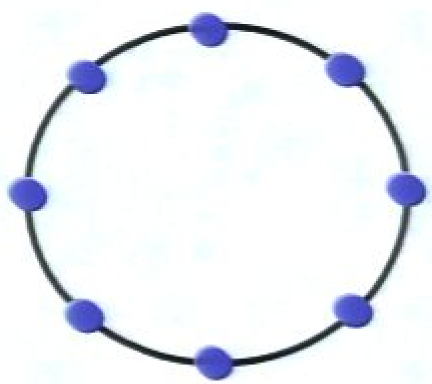
- 1D Ising model with periodic BCs $\longrightarrow$ GHZ state $|\psi_G\rangle = |+\rangle^N + |-\rangle^N$



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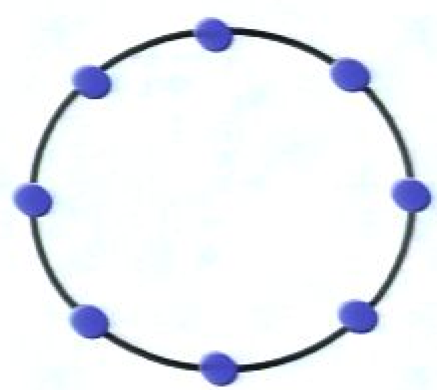


$$Z_G \propto \prod_{ab} \langle + | \alpha_{ab} \rangle + \prod_{ab} \langle - | \alpha_{ab} \rangle = \dots$$

- 2D Ising model with open BCs $\longrightarrow$ Kitaev's planar code states

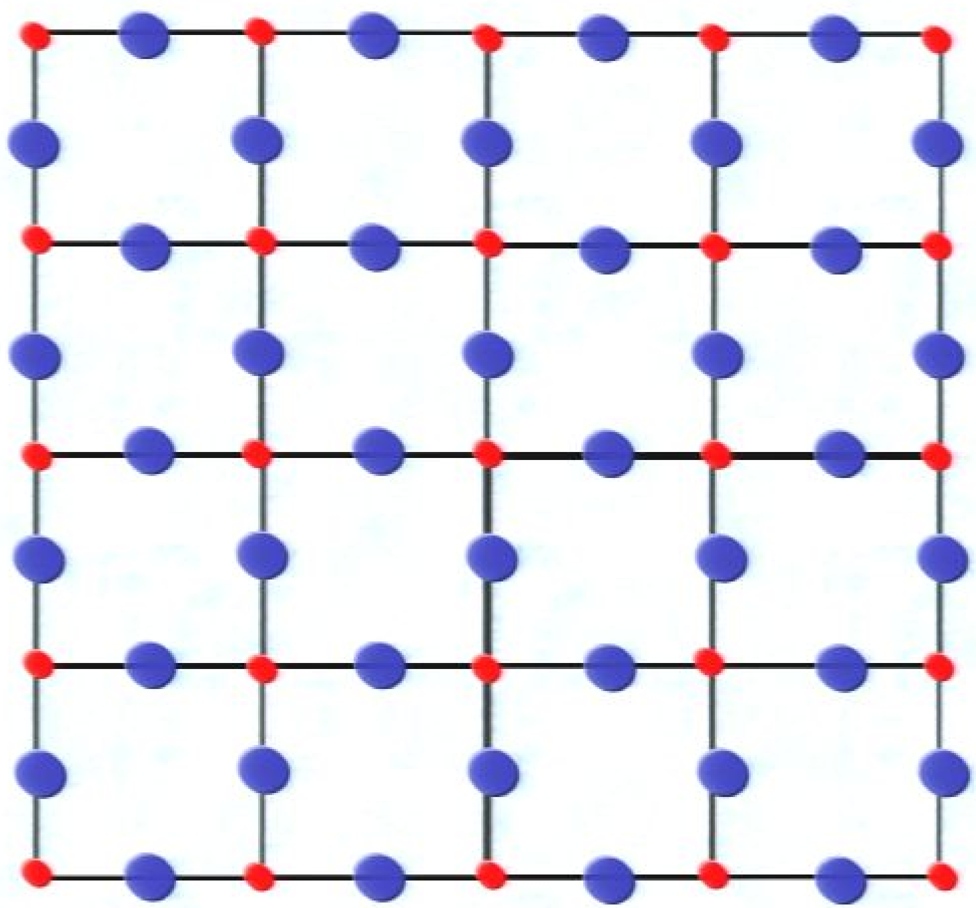
periodic toric

Second mapping:

spin models with external fields

Cluster-type states $|\varphi_G\rangle$

- Start with a graph G
- Edge-qubits
- Vertex-qubits



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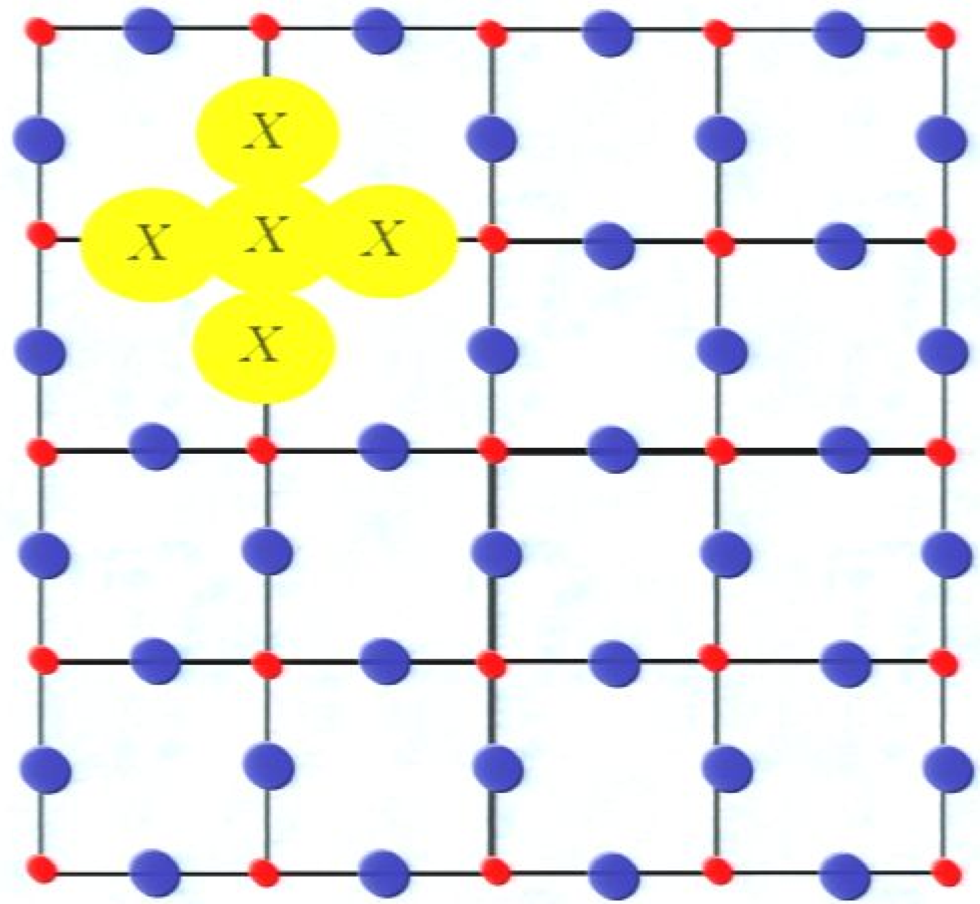

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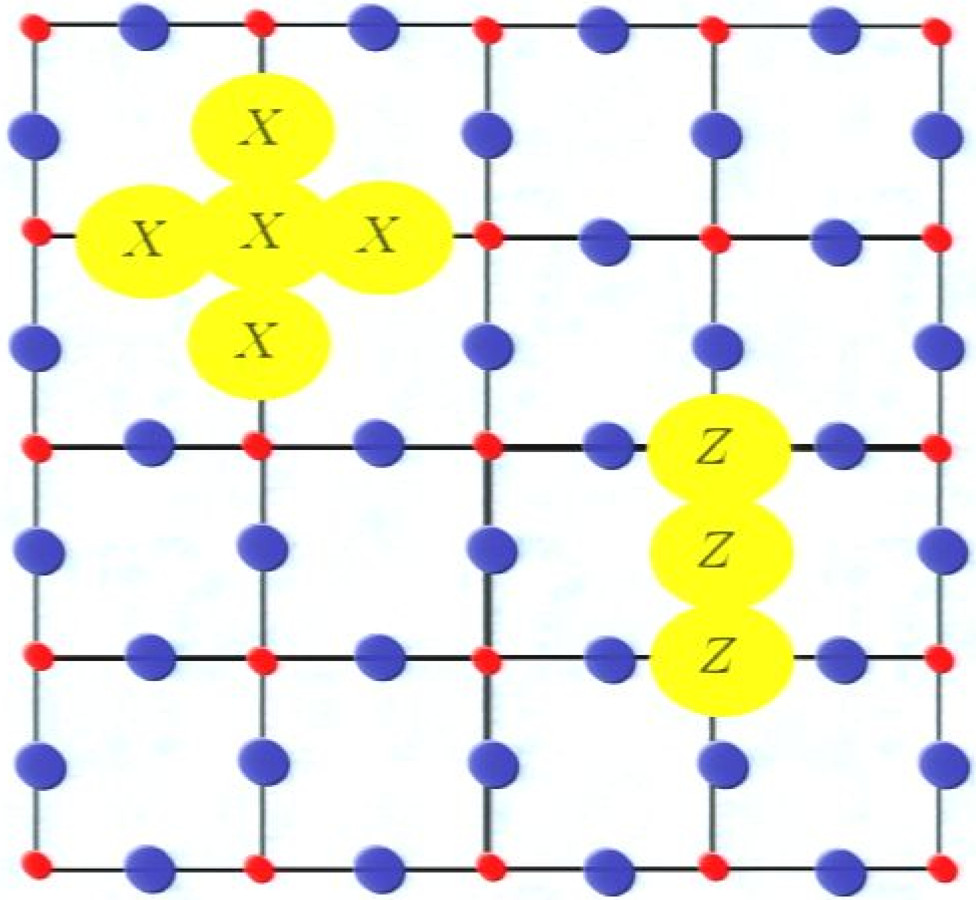
$$E_{ab} = Z^{(a)} Z^{(ab)} Z^{(b)}$$



Cluster-type states

$$|\varphi_G\rangle \propto \sum_{t\in\{0,1\}^V} \underbrace{|B^T t\rangle}_{E} \otimes \underbrace{|t\rangle}_{V}$$

Edge-qubits Vertex-qubits

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$$\Big\downarrow\ \boxed{H^E \otimes I^V}$$

$|G^{\mathrm{dec}}\rangle =$ Graph state associated with <u>decorated graph</u>

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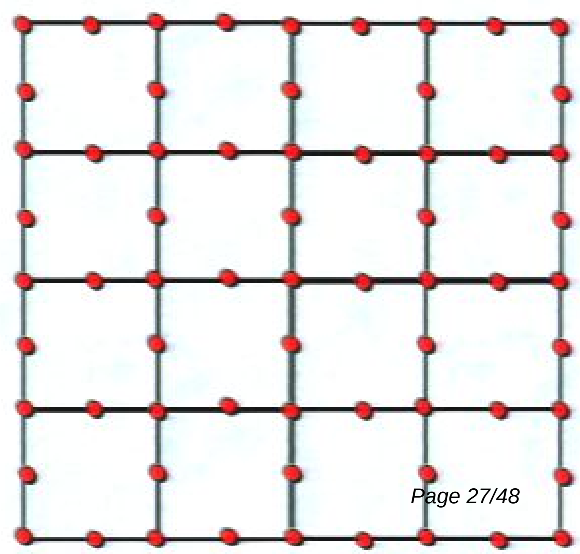

The mapping

- Ising model with external fields
- $|\alpha_{ab}\rangle = e^{\beta J_{ab}}|0\rangle + e^{-\beta J_{ab}}|1\rangle$ Also: $|\alpha_a\rangle = e^{\beta h_a}|0\rangle + e^{-\beta h_a}|1\rangle$
- Then: $$Z_G \propto \langle \varphi_G | \bigotimes_{ab} |\alpha_{ab}\rangle \bigotimes_{a} |\alpha_a\rangle$$
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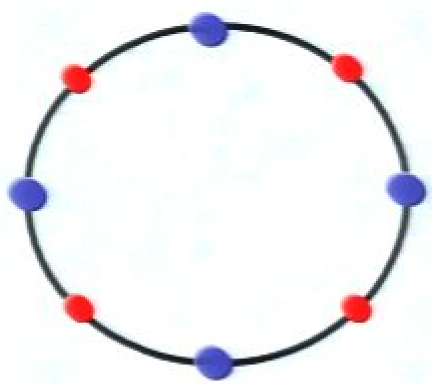
- 1D Ising model with open BCs ⟶ 1D cluster state

- 1D Ising model with periodic BCs ⟶ 1D ring cluster state



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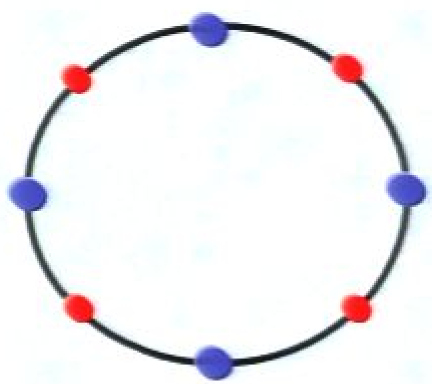

- 2D Ising model with open BCs ⟶ Decorated 2D cluster states with OBC/PBC

periodic

Applications – What can we do with this?

Applications

- The mapping connects fundamental concept from statistical physics (partition function) with fundamental concept from quantum mechanics (amplitude)

- We present some applications of how this connection can be used in <u>two directions</u>

  - Use vast knowledge of stat phys to gain insight in properties of the states $|\psi_G\rangle$

  - Use insights from QIT to understand properties of classical models

- Main application (...so far):

Statistical physics $\longleftrightarrow$ Measurement-based QC

From stat phys to MQC

- Mappings connect computational power of resource state with solvability of corresponding spin model
- Solvable model (1D, 2D without field) gives rise to simulatable resource state: e.g. planar code state, see Bravyi and Raussendorf, quant-ph/0610162
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Ising model		Resource state
Without field: solvable With field: solvable	1D	product state: simulatable (trivial) 1D cluster state: simulatable
Without field: solvable With field: NP-hard	2D	Planar code state: simulatable 2D cluster state: UNIVERSAL

From MQC to stat phys

- 2D Ising model with field is mapped to 2D cluster state

$$Z_{2D} \propto \langle \varphi_{2D} | \bigotimes_{ab} |\alpha_{ab}\rangle \bigotimes_{a} |\alpha_a\rangle$$

- 2D cluster state are universal resources for MQC. Every n-qubit state $|\psi\rangle$ can be written as:

$$\Sigma|\psi\rangle \propto (I \otimes \langle\beta|)\,|\varphi_{2D}\rangle$$

  - Σ = local Pauli correction ("byproduct operator")
  - $|\varphi_{2D}\rangle$ = 2D cluster state on M qubits
  - $|\beta\rangle$ = product state on M – n measured qubits (basis + outcome)

- If $|\psi\rangle$ is stabilizer state: M = poly(n) AND $|\beta\rangle$ and Σ can be determined efficiently

From MQC to stat phys

- Now take $|\psi\rangle \equiv |\varphi_G\rangle$ and consider at the same time

$$\Sigma|\varphi_G\rangle \propto (I \otimes \langle\beta|)\,|\varphi_{2D}\rangle \qquad\qquad Z_G \propto \langle\varphi_G| \bigotimes_{ab} |\alpha_{ab}\rangle \bigotimes_{a} |\alpha_a\rangle$$

- Combination yields:
  - Z_G can be written as the overlap between $|\varphi_{2D}\rangle$ and product state
  - The latter is instance of 2D partition function

- Thus: any Ising partition function can be seen as a special instance of a 2D Ising partition function with polynomially more spins and suitable (complex) couplings

$$\boxed{Z_G(\{J_{ab}, h_a\}) \propto Z_{\text{2D}}(\{J'_{ij}, h'_i\})}$$

The 2D Ising model is "complete"

From MQC to stat phys

- 2D couplings are typically complex "non-physical"
- Inhomogeneous magnetic fields are needed
- Method is constructive

- Generalization to q-state models possible:

partition function of 2D Ising model contains all q-state models!

Polynomial overhead for Potts-type models

Possibly exponential overhead needed for other models

- Idea of proof: similar to above + embed q-dimensional quantum systems into blocks of qubits

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From MQC to stat phys

- Main message: result from QIT is used to obtain result in stat phys via our mappings

  - Universality of cluster states = well known
  - proof of universality uses typical QIT concepts: universal gate sets, measurements/teleportation, stabilizer formalism, etc
  - These powerful techniques are not available in classical stat phys
  - Our mappings allow to use these techniques for stat phys and conclude that 2D partition function is complete (this is even almost trivial corollary)

- This is not the only application

  - Simulation techniques to evaluate partition functions
  - Duality relations

Conclusion

- We have introduced mapping from classical spin models to quantum systems
- Partition function as overlap between product state and stabilizer state
- Mappings are general: arbitrary graphs, q-state models, inhomogeneous couplings and magnetic fields
- Connect well-known spin models with well-known quantum states (GHZ, cluster, Kitaev, ...)
- Allows cross-fertilization between fields:
  - From stat phys to MQC: solvable models give simulatable states
  - From MQC to stat phys: universality of 2D cluster state implies completeness of 2D Ising model

Thank you very much!