Title: Operationalism, hidden variable models, and contextuality (Part 1A)

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Abstract:

Operationalism, Hidden Variable Models, and Contextuality

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August 27, 2007
QFSS, PI

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Outline

Problems with the standard postulates of quantum theory and two strategies to deal with them

Realism and the traditional notion of a noncontextual hidden variable model of quantum theory

Operational quantum mechanics and operationalism in general

An operational definition of noncontextuality
-- comparison with the traditional notion
-- new no-go theorems
-- what contextuality is useful for

Do quantum states describe reality or our knowledge of reality?

"Orthodox" postulates of quantum theory

The rays of Hilbert space ψ correspond one-to-one with the physical states of the system.

Measurements are associated with Hermitian operators $A=\sum_{\kappa} a_k P_{\kappa}$. Outcomes are indeterministic; a_k occurs with probability $\langle\psi|P_k|\psi\rangle$.

The physical state of an isolated system evolves unitarily, i.e. deterministically and continuously

If a measurement associated with $A=\sum_k a_k P_k$ yields outcome a_k, the physical state of the system changes discontinuously as: $|\psi\rangle \rightarrow P_k|\psi\rangle$.

Inconsistencies of the orthodox interpretation

By the collapse postulate (applied to the system)	By unitary evolution postulate (applied to isolated system that includes the apparatus)
Indeterministic and discontinuous evolution	Deterministic and continuous evolution
Determinate properties	Indeterminate properties

Operationalism vs. Realism

More generally, the term “measurement” is not defined in terms of the more primitive “physical states of systems”. Isn’t a measurement just another kind of physical interaction?

Two strategies:

(1) Realist strategy: Eliminate measurement as a primitive concept
Elements of the formalism represent reality or our knowledge of reality

(2) Operational strategy: Eliminate “the physical state of a system” as a primitive concept
Elements of the formalism represent lists of instructions of what to do in the lab

“It would seem that the theory is exclusively concerned about "results of measurement", and has nothing to say about anything else. What exactly qualifies some physical systems to play the role of "measurer"? ”

- John Bell

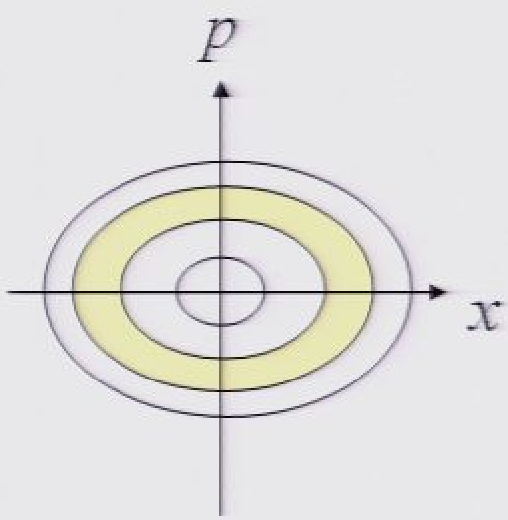
“In a strict sense, quantum theory is a set of rules allowing the computation of probabilities for the outcomes of tests which follow specified preparations.”

- Asher Peres

A realist strategy:
Hidden variable models

The idea of a deterministic hidden variable model of quantum mechanics

In a classical theory, properties are associated with regions of the state space



$$E(x,p) = \frac{p^2}{2m} + \frac{kx^2}{2}$$

$$E_1 \leq E \leq E_2$$

Consider
$$\begin{aligned} \alpha(x,p) &= a_1 \text{ if } x < x_1, \\ &= a_2 \text{ if } x_1 \leq x \leq x_2, \\ &= a_3 \text{ if } x > x_2. \end{aligned}$$

Consider
$$
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&= a_2 \text{ if } x_1 \le x \le x_2, \\
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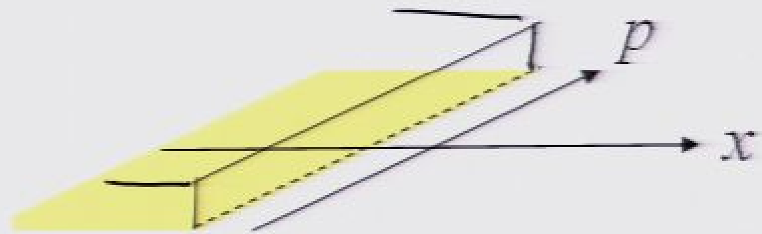
\end{aligned}
$$

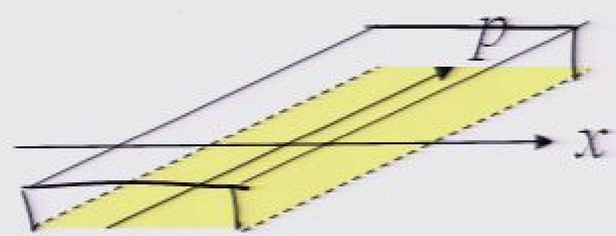
Equivalently, $\alpha(x,p) = \sum_k a_k \chi_k(x,p)$

where

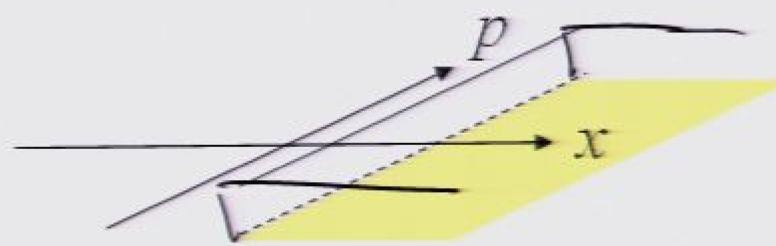
$$
\begin{aligned}
\chi_1(x,p) &= 1 \text{ if } x < x_1 \\
&= 0 \text{ otherwise,}
\end{aligned}
$$



$$
\begin{aligned}
\chi_2(x,p) &= 1 \text{ if } x_1 \le x \le x_2 \\
&= 0 \text{ otherwise,}
\end{aligned}
$$



$$
\begin{aligned}
\chi_3(x,p) &= 1 \text{ if } x > x_2 \\
&= 0 \text{ otherwise,}
\end{aligned}
$$



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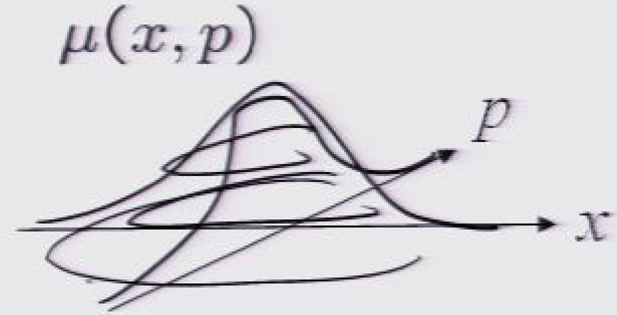

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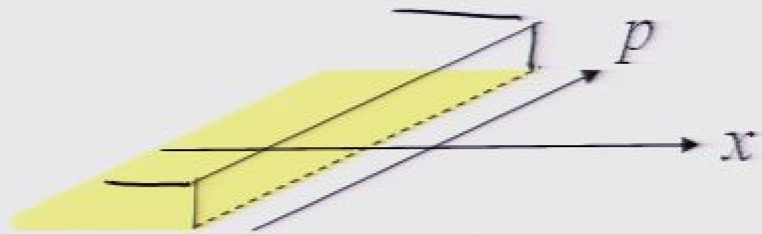

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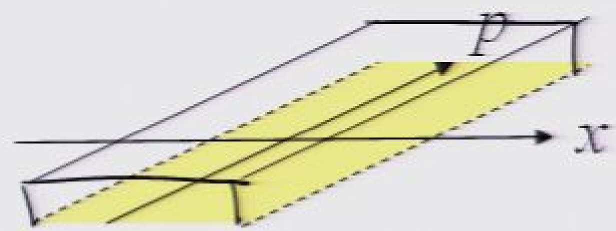

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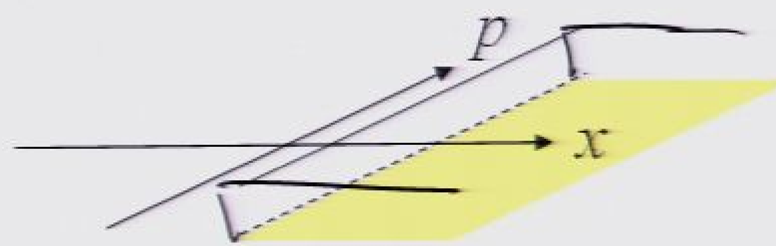

Can still have probabilistic outcomes if x,p is unknown

$$\text{Prob}(k) = \int dx\, dp\, \mu(x,p)\, \chi_k(x,p)$$

In quantum theory, we have

$$A = \sum_k a_k P_k$$

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The idea of a deterministic hidden variable theory is that

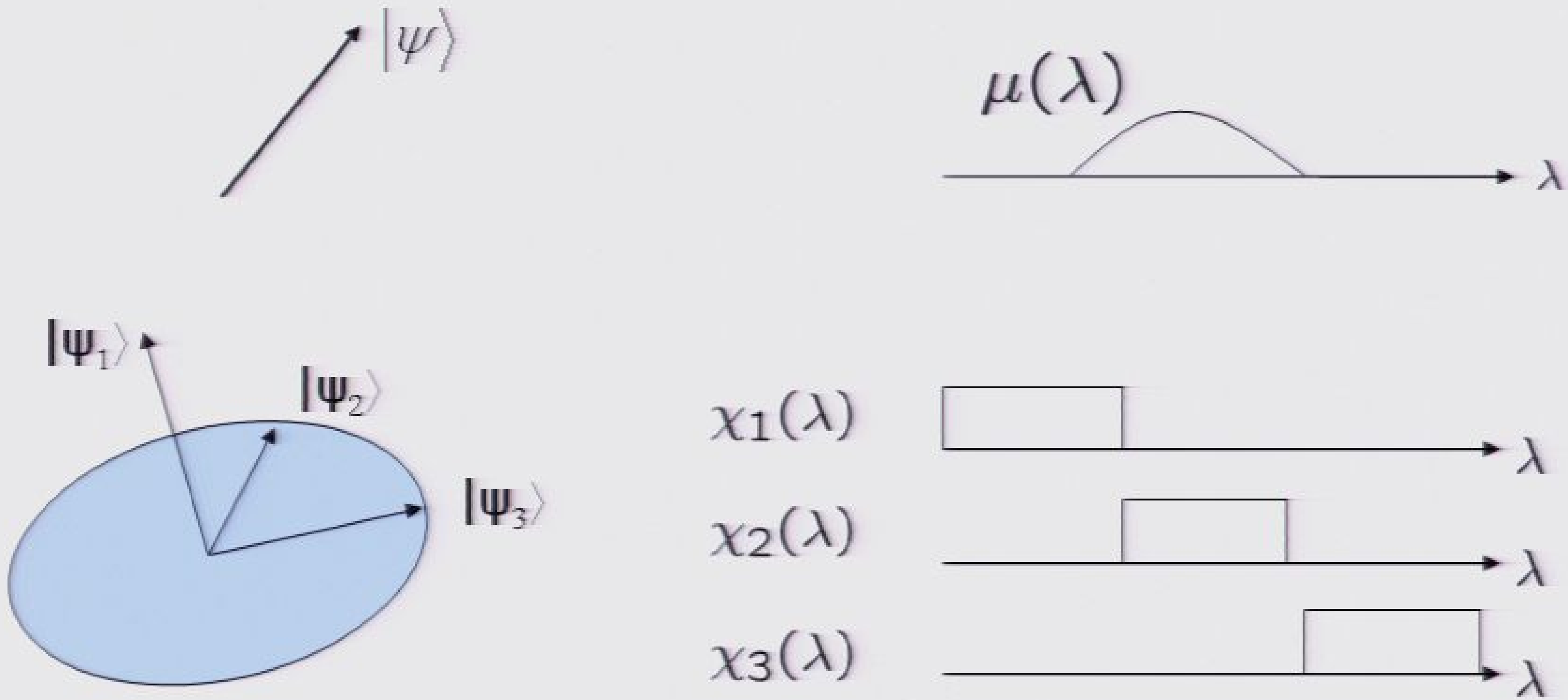
$$|\psi\rangle \leftrightarrow \mu(\lambda)$$

$$A \leftrightarrow \alpha(\lambda)$$

$$\{P_k\} \leftrightarrow \{\chi_k(\lambda)\}$$

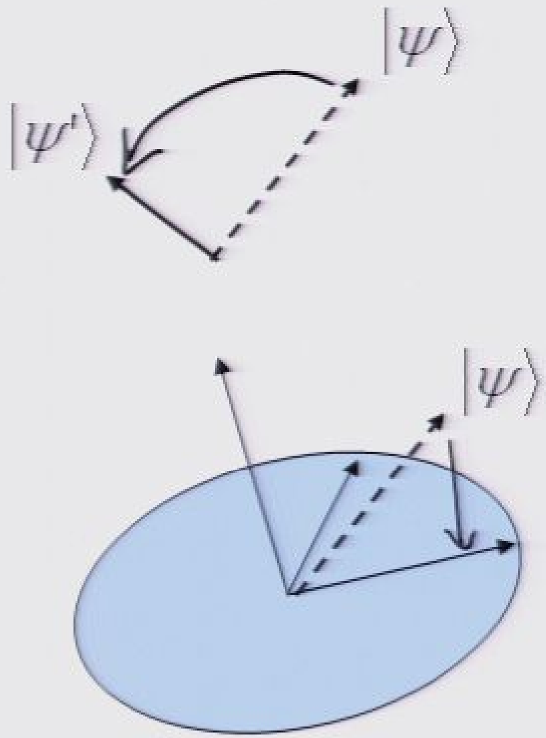
Such that

$$\langle\psi|P_k|\psi\rangle = \int d\lambda \mu(\lambda) \chi_k(\lambda)$$

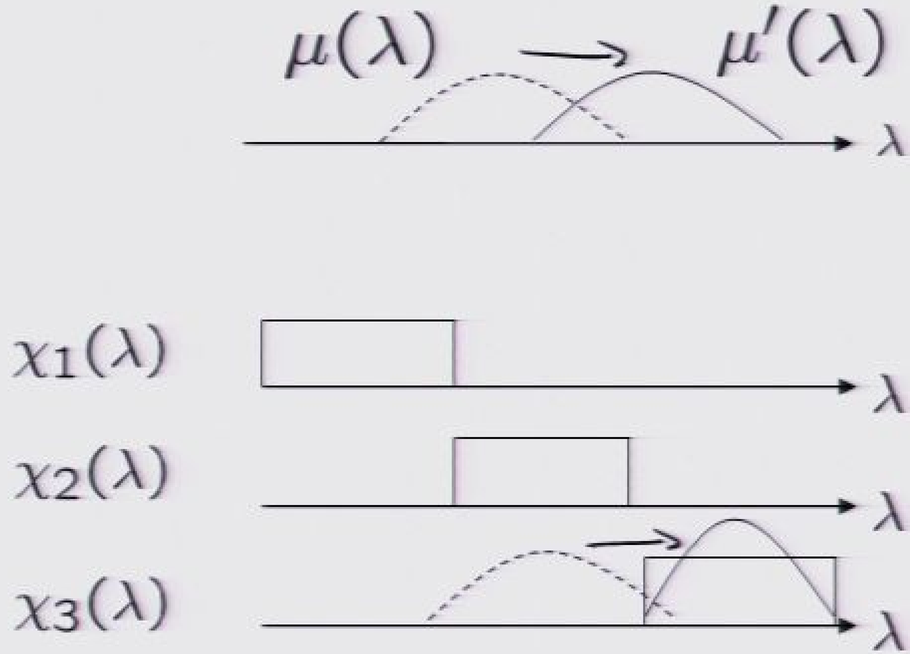


$$|\langle\psi|\chi_k\rangle|^2 = \int d\lambda \mu(\lambda)\chi_k(\lambda)$$

Aside:

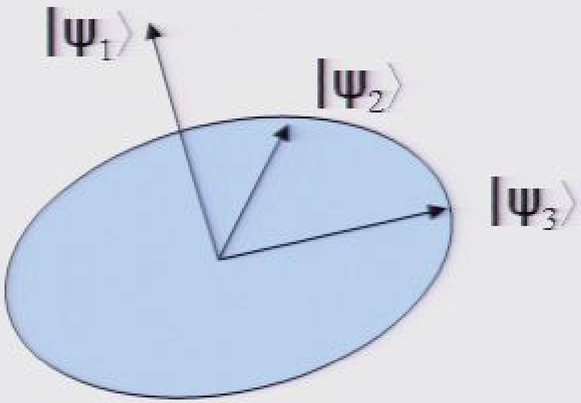


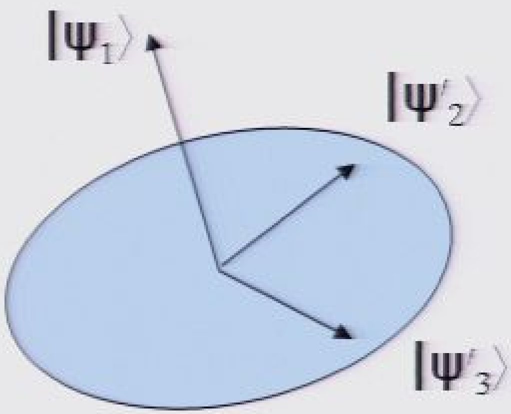
$|\psi\rangle$
$|\psi'\rangle$
$|\psi\rangle$



$\mu(\lambda)$
$\mu'(\lambda)$
λ
$\chi_1(\lambda)$
λ
$\chi_2(\lambda)$
λ
$\chi_3(\lambda)$
λ

A given projector may appear in many different measurements





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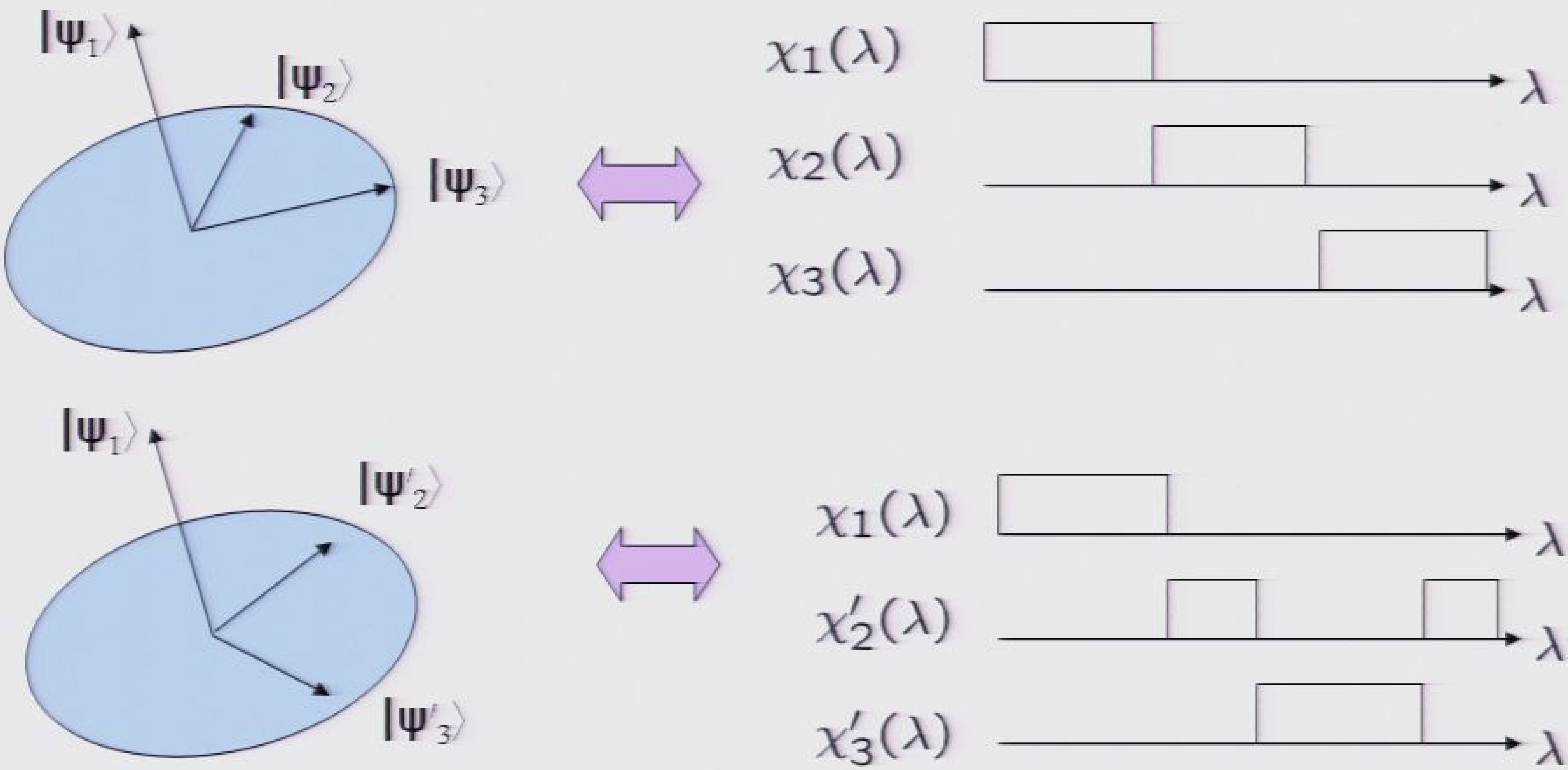
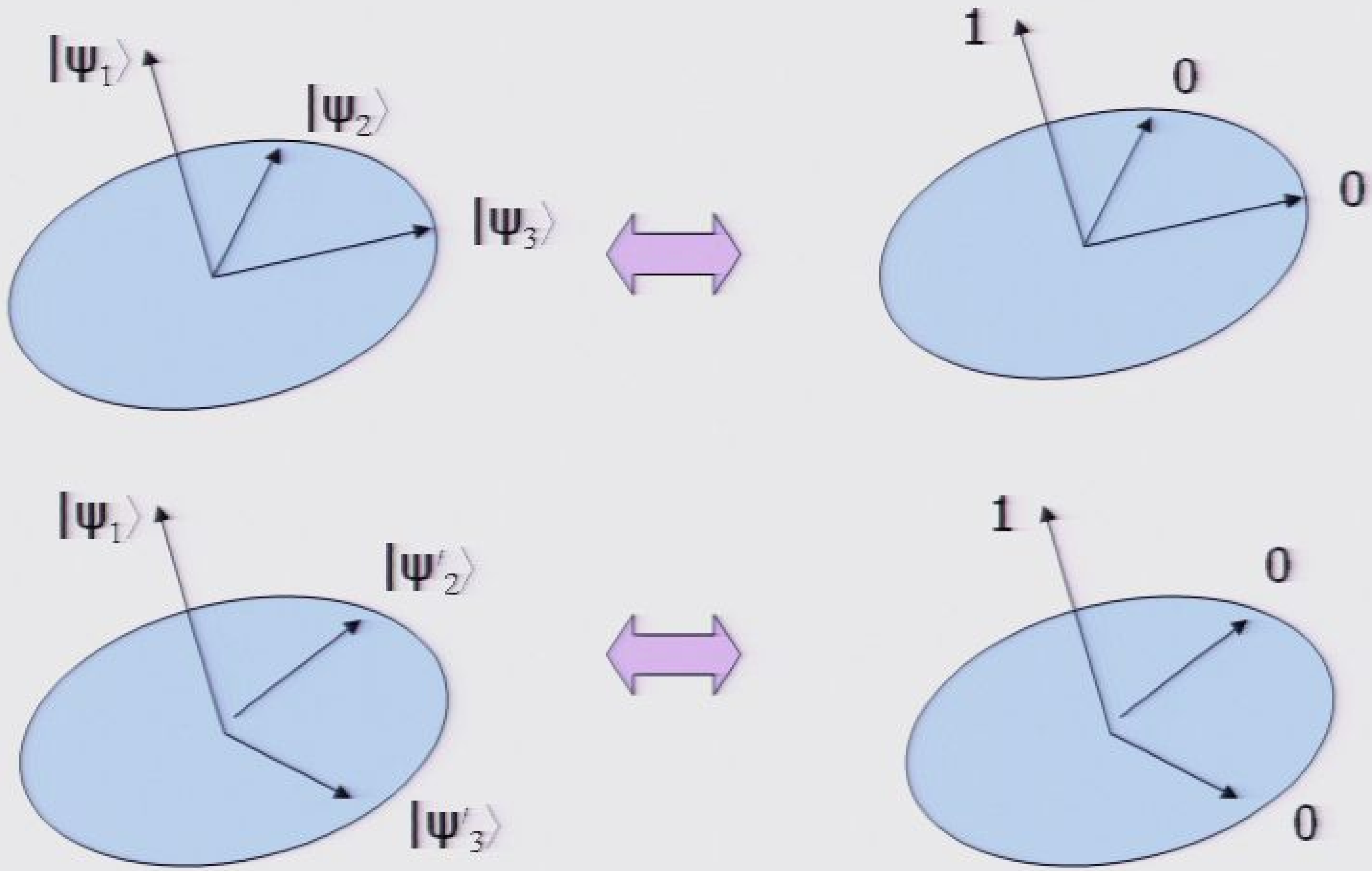


The traditional notion of noncontextuality:
Every P is associated with the same $\chi(\lambda)$ regardless of how it is measured

Alternatively, for a given λ

$$\chi_\psi(\lambda) = 0 \text{ or } 1$$

$$\sum_k \chi_{\psi_k}(\lambda) = 1$$



It was shown by Bell (1966) and Kochen and Specker (1967) that a noncontextual hidden variable model of quantum theory for Hilbert spaces of dimensionality 3 or greater is impossible. That is, quantum theory is contextual

This is the Bell-Kochen-Specker theorem

Example (Cabello's algebraic 18 ray proof in 4d):

Each of the 18 rays appears twice in the following list

0,0,0,1	0,0,0,1	1,-1,1,-1	1,-1,1,-1	0,0,1,0	1,-1,-1,1	1,1,-1,1	1,1,-1,1	1,1,1,-1
0,0,1,0	0,1,0,0	1,-1,-1,1	1,1,1,1	0,1,0,0	1,1,1,1	1,1,1,-1	-1,1,1,1	-1,1,1,1
1,1,0,0	1,0,1,0	1,1,0,0	1,0,-1,0	1,0,0,1	1,0,0,-1	1,-1,0,0	1,0,1,0	1,0,0,1
1,-1,0,0	1,0,-1,0	0,0,1,1	0,1,0,-1	1,0,0,-1	0,1,-1,0	0,0,1,1	0,1,0,-1	0,1,-1,0

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0,0,0,1	0,0,0,1	1,-1,1,-1	1,-1,1,-1	0,0,1,0	1,-1,-1,1	1,1,-1,1	1,1,-1,1	1,1,1,-1
0,0,1,0	0,1,0,0	1,-1,-1,1	1,1,1,1	0,1,0,0	1,1,1,1	1,1,1,-1	-1,1,1,1	-1,1,1,1
1,1,0,0	1,0,1,0	1,1,0,0	1,0,-1,0	1,0,0,1	1,0,0,-1	1,-1,0,0	1,0,1,0	1,0,0,1
1,-1,0,0	1,0,-1,0	0,0,1,1	0,1,0,-1	1,0,0,-1	0,1,-1,0	0,0,1,1	0,1,0,-1	0,1,-1,0

In each of the 9 columns, one ray is assigned 1, the other three 0
Therefore, 9 rays must be assigned 1

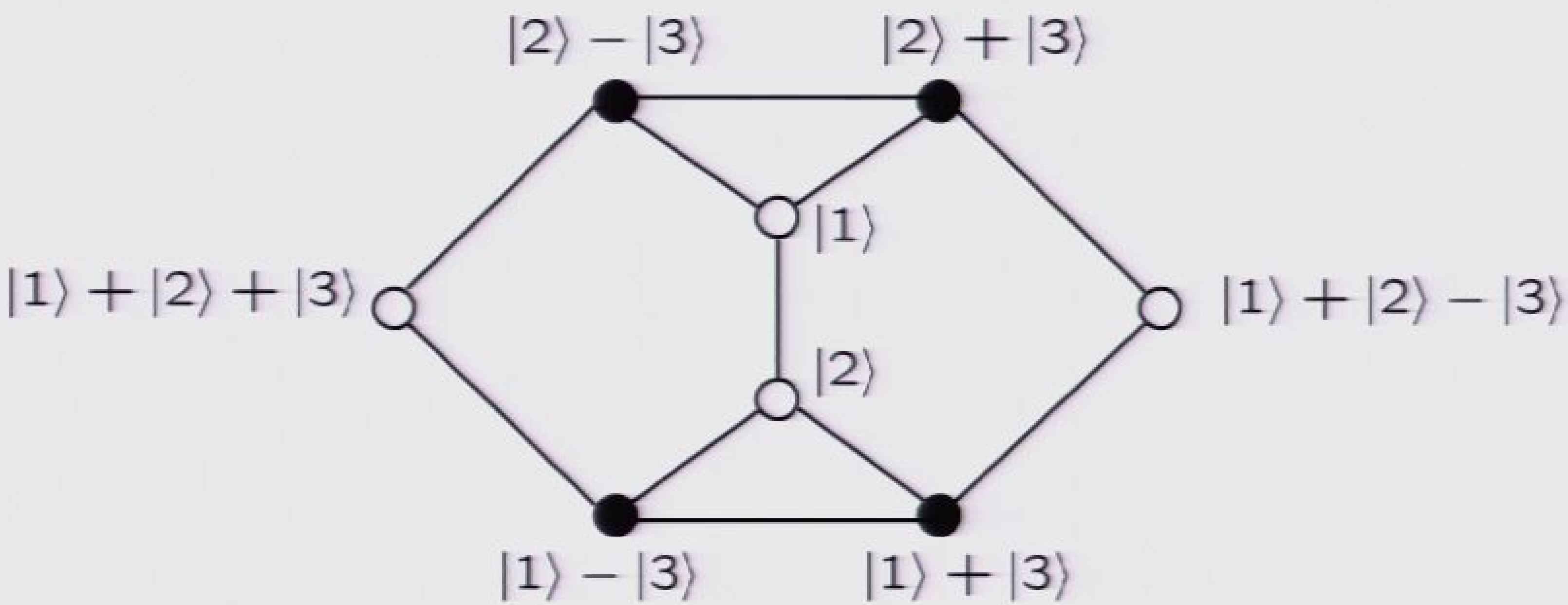
But each ray appears twice and so there must be an even number of rays assigned 1

CONTRADICTION!

Example (Clifton's statistical 8 ray proof in 3d):

$$|\psi\rangle \; \circ \longrightarrow \chi_{|\psi\rangle}(\lambda) = 1$$

$$|\psi\rangle \; \bullet \longrightarrow \chi_{|\psi\rangle}(\lambda) = 0$$



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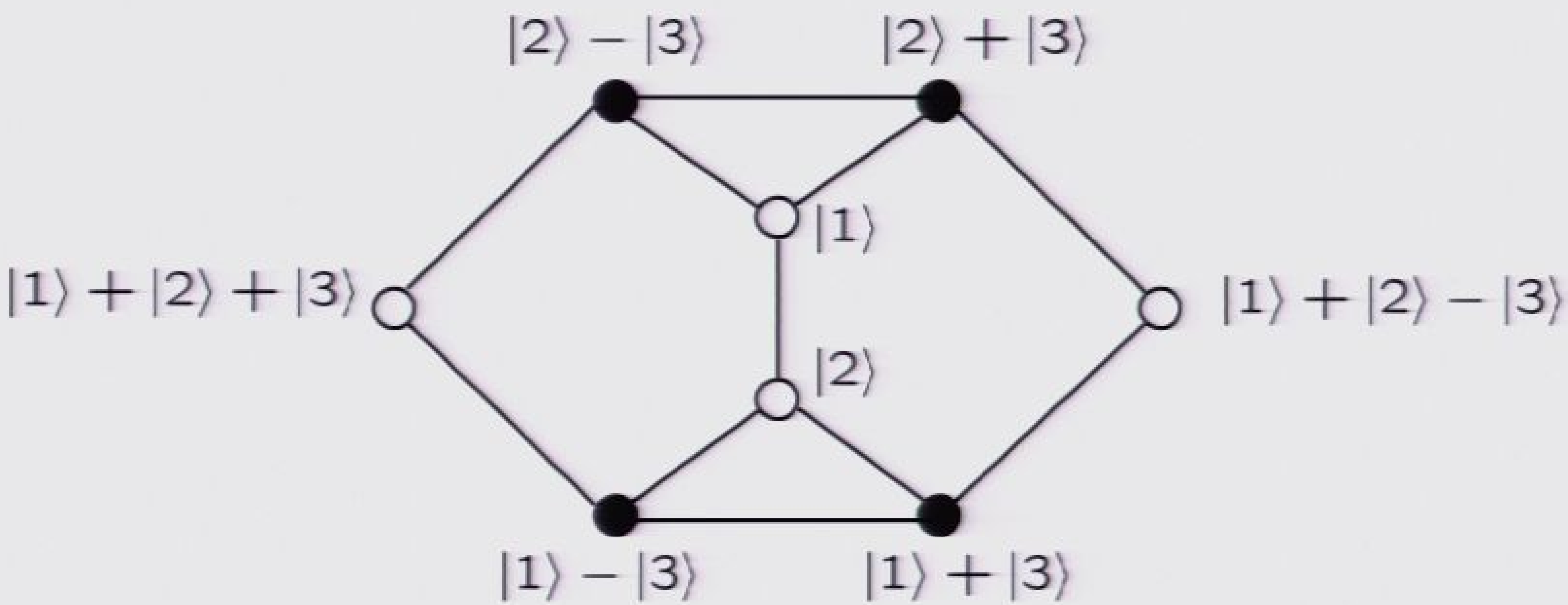


CONTRADICTION!

Problems with the traditional definition of noncontextuality:

- applies only to deterministic hidden variable models
- applies only to quantum theory

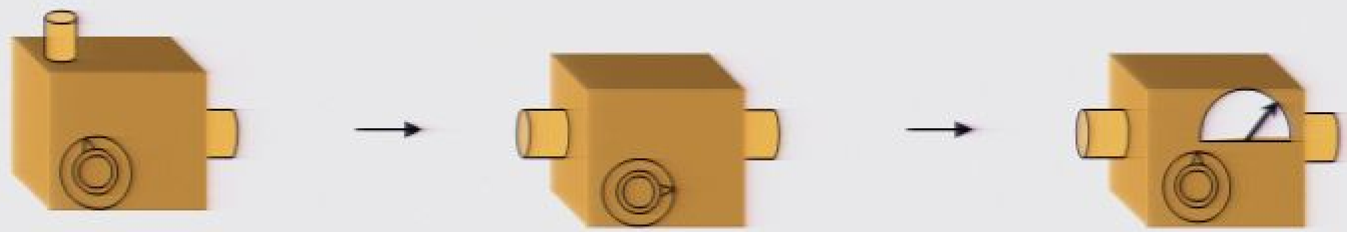
Can we define it in such a way that we can judge

- whether any given theory is contextual or not
- whether any given data requires contextuality for its explanation

Yes, by being operational.

A purely operational formulation of quantum theory

Operational Quantum Mechanics



Preparation	Transformation	Measurement
P	T	M
Vector	Unitary map	Projector-valued measure (PVM)
$\|\psi\rangle$	U	$\{P_k\}$

$$\boxed{Pr(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \langle\psi|U^\dagger P_k U|\psi\rangle}$$

More general preparations

Probability p, prepare $|\psi\rangle$
Probability q, prepare $|\chi\rangle$
Measure $\{P_k\}$

$$\begin{aligned}\text{Prob}(k) &= p\langle\psi|P_k|\psi\rangle + q\langle\chi|P_k|\chi\rangle \\ &= p\,\text{Tr}(|\psi\rangle\langle\psi|P_k) + q\,\text{Tr}(|\chi\rangle\langle\chi|P_k) \\ &= \text{Tr}(\rho P_k)\end{aligned}$$

$$\rho = p|\psi\rangle\langle\psi| + q|\chi\rangle\langle\chi|$$

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$$\rho = p|\psi\rangle\langle\psi| + q|\chi\rangle\langle\chi|$$

A density operator

$$\rho \in \mathcal{L}(\mathbb{C}_d)$$

$$\langle\psi|\rho|\psi\rangle \geq 0, \forall\psi$$

$$\text{Tr}(\rho) = 1$$

$\rho = |\psi\rangle\langle\psi| \quad \leftrightarrow$ Pure preparation

$\rho \neq |\psi\rangle\langle\psi| \quad \leftrightarrow$ Mixed preparation

More general measurements

Prepare ρ

Probability p, measure the PVM $\{P_k\}$

Probability q, measure the PVM $\{Q_k\}$

$$\begin{aligned}\mathrm{Prob}(k) &= p\,\mathrm{Tr}(\rho P_k) + q\,\mathrm{Tr}(\rho Q_k)\\ &= \mathrm{Tr}(\rho E_k)\end{aligned}$$

$$E_k = pP_k + qQ_k$$

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A Positive operator valued measure (POVM)

$$E_k \in \mathcal{L}(\mathbb{C}_d)$$

$$\langle\psi|E_k|\psi\rangle \geq 0, \forall\psi$$

$$\sum_{k=1}^{d} E_k = I$$

$\{E_k\} = \{P_k\}$ $\leftrightarrow$ Sharp measurement
$\{E_k\} \neq \{P_k\}$ $\leftrightarrow$ Unsharp measurement

More general transformations

Prepare ρ
Probability p, transform with U
Probability q, transform with V
measure $\{\mathrm{E_k}\}$

$$\begin{aligned}\mathrm{Prob}(k) &= p\,\mathrm{Tr}(U\rho U^\dagger E_k) + q\,\mathrm{Tr}(V\rho V^\dagger E_k)\\ &= \mathrm{Tr}(\mathcal{T}(\rho)E_k)\end{aligned}$$

$$\mathcal{T}(\cdot) = pU(\cdot)U^\dagger + qV(\cdot)V^\dagger$$

More general transformations

Prepare ρ
Probability p, transform with U
Probability q, transform with V
measure $\{\mathrm{E_k}\}$

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$$\mathcal{T}(\cdot) = pU(\cdot)U^\dagger + qV(\cdot)V^\dagger$$

A completely positive map (CP map)

$$\mathcal{T} : \mathcal{L}(\mathbb{C}_d) \rightarrow \mathcal{L}(\mathbb{C}_d)$$

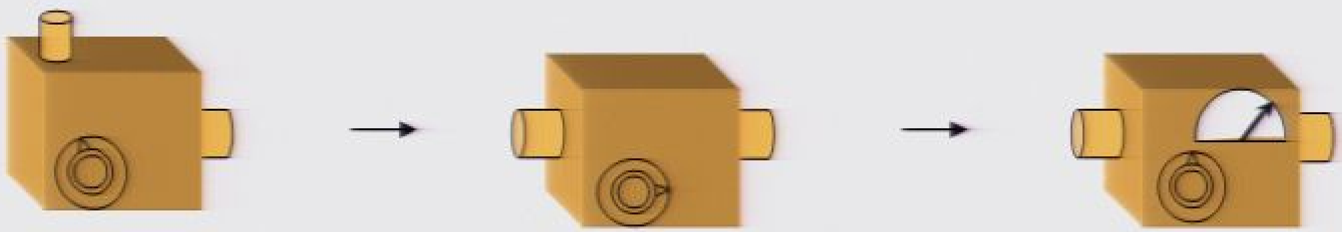
$$\mathcal{T}(\rho) = \sum_\mu A_\mu \rho A_\mu^\dagger$$

$$\sum_\mu A_\mu^\dagger A_\mu = I$$

$\mathcal{T}(\rho) = U\rho U^\dagger \quad \leftrightarrow$ Reversible transformation

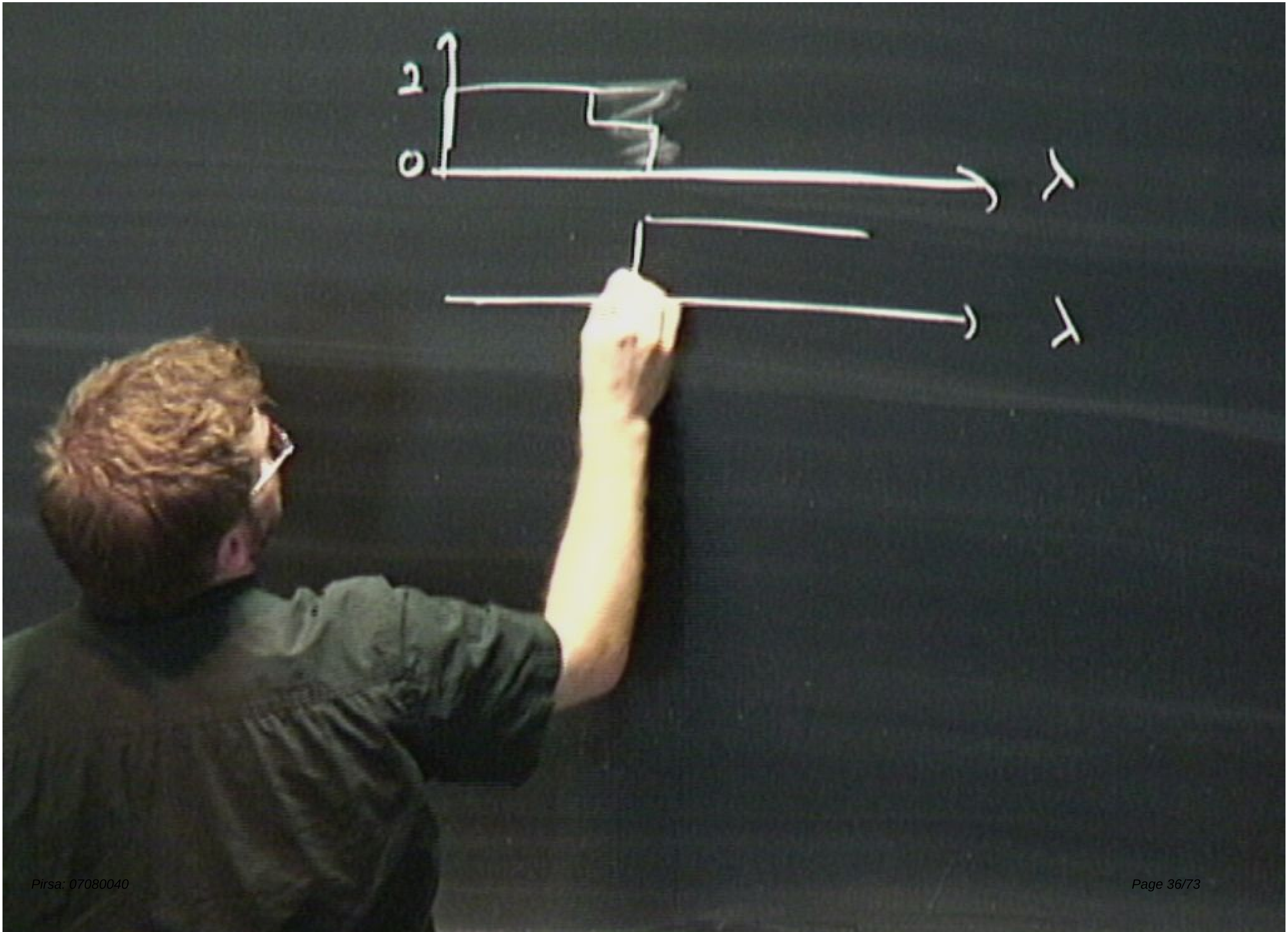
$\mathcal{T}(\rho) \neq U\rho U^\dagger \quad \leftrightarrow$ Irreversible transformation

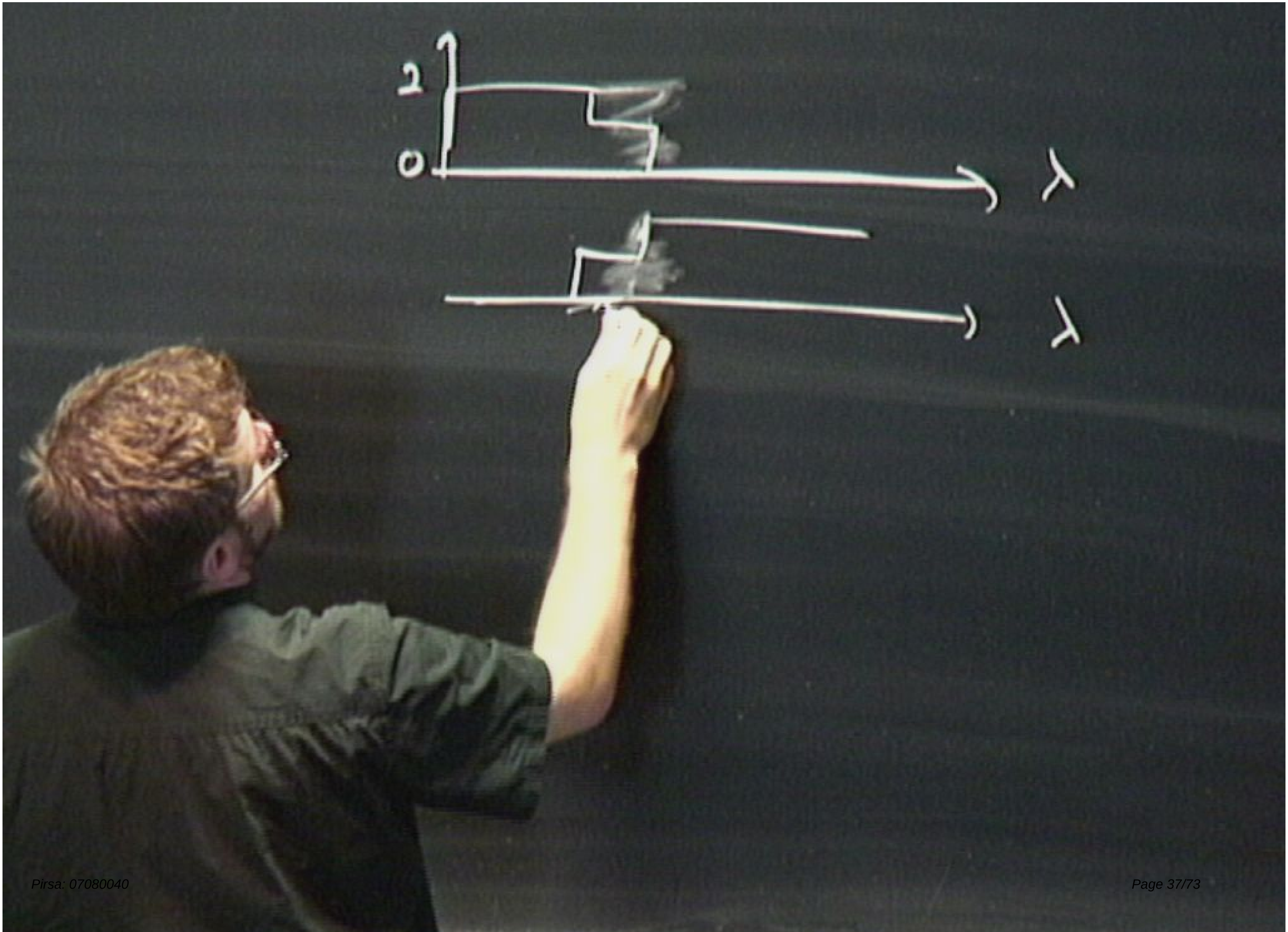
Operational Quantum Mechanics

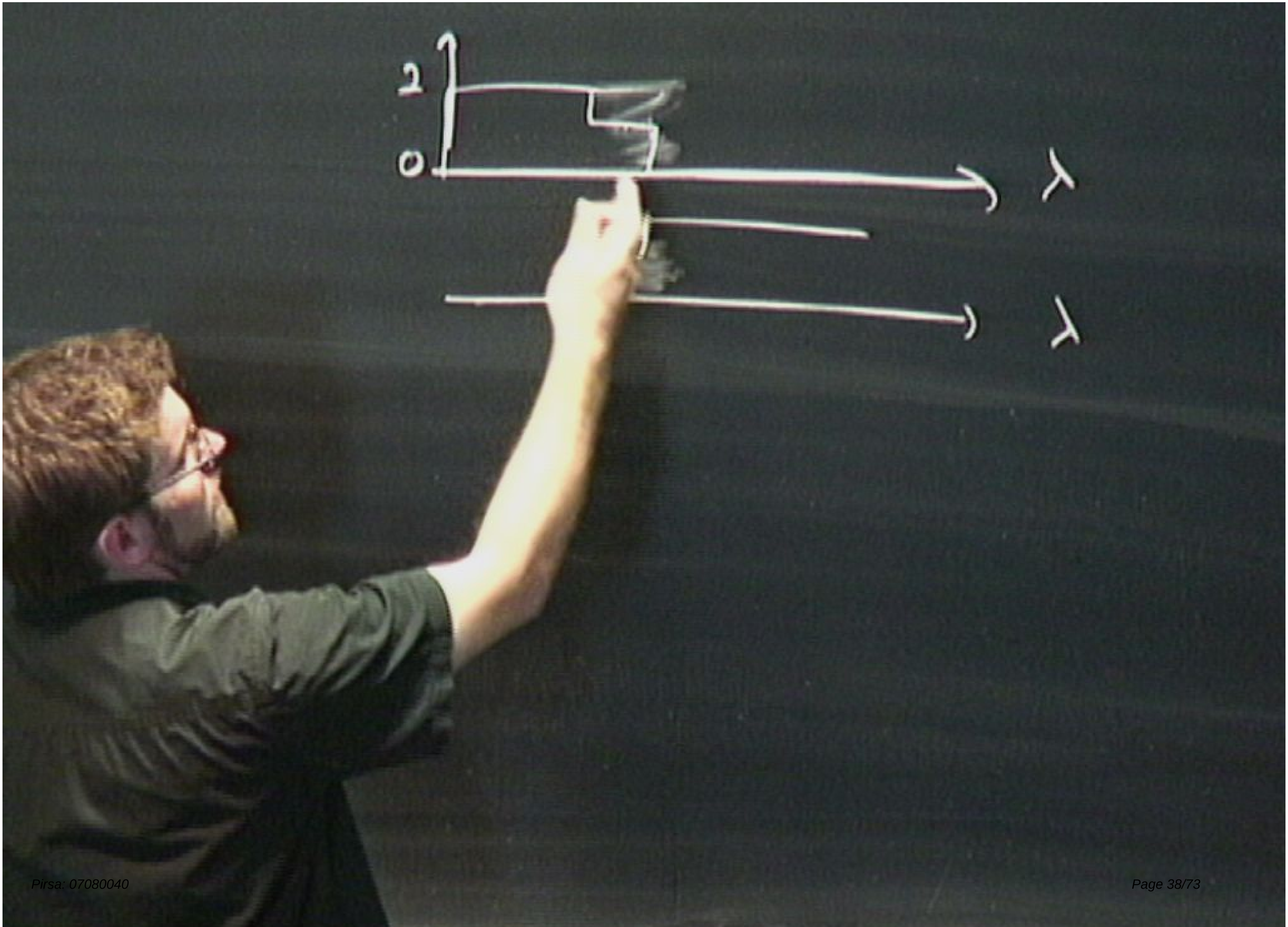


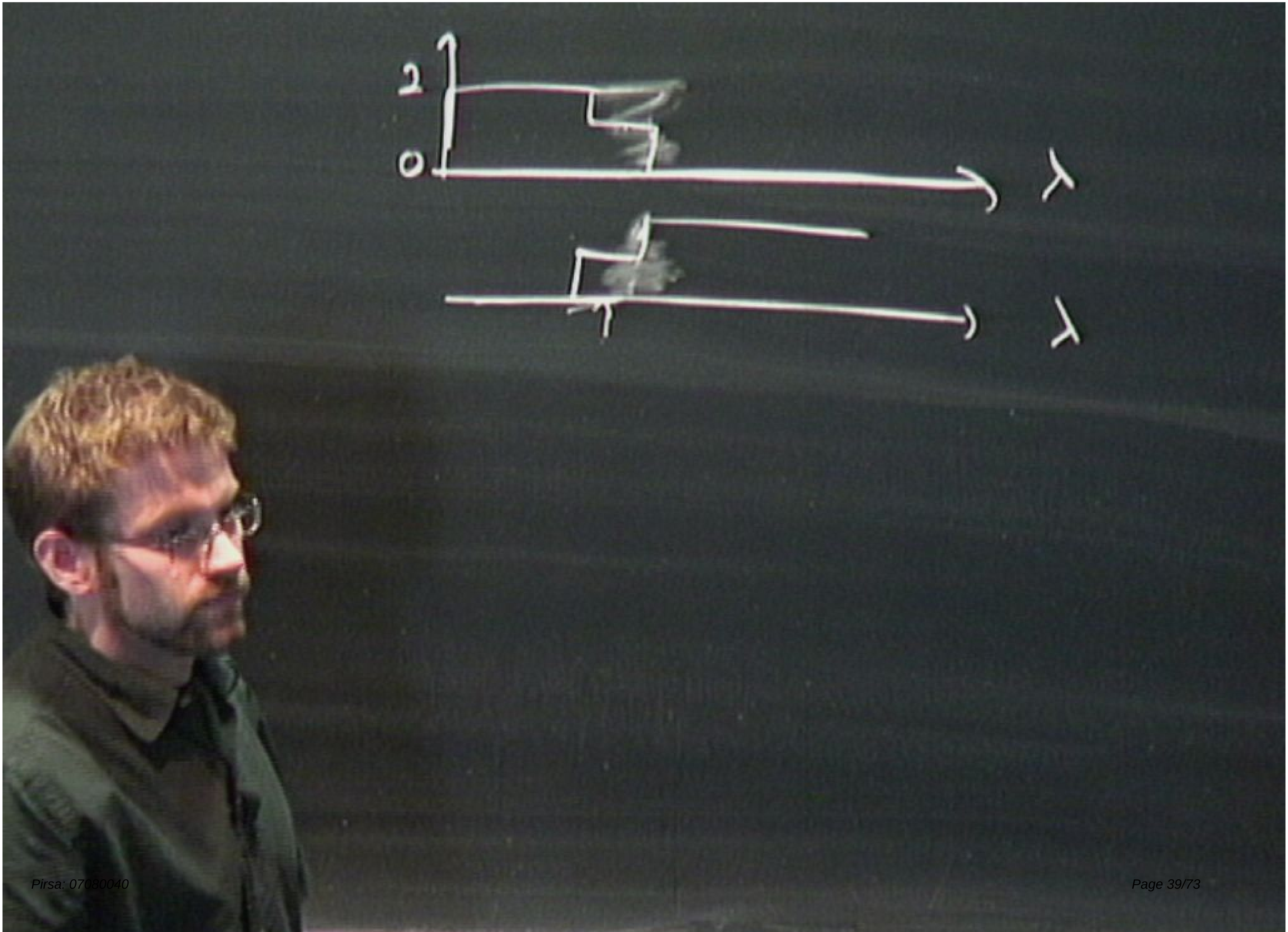
Preparation	Transformation	Measurement
P	T	M
Density operator	Trace-preserving completely positive linear map (CP map)	Positive operator-valued measure (POVM)
ρ	$\mathcal{T}$	$\{E_k\}$

$$\boxed{Pr(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \mathrm{Tr}[E_k \mathcal{T}(\rho)]}$$


2
0
λ
λ


2
0
λ
λ


2
0
λ
λ


2
0
λ
λ

More general preparations

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Measure $\{P_k\}$

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$$\rho = p|\psi\rangle\langle\psi| + q|\chi\rangle\langle\chi|$$

A density operator

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Operational Quantum Mechanics

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$$\boxed{Pr(k|\mathsf{P},\mathsf{T},\mathsf{M}) = \mathrm{Tr}[E_k \mathcal{T}(\rho)]}$$

Operational postulates of quantum theory

Every preparation P is associated with a density operator ρ

Every measurement M is associated with a positive operator-valued measure $\{E_k\}$. The probability of M yielding outcome k given a preparation P is $p_k = \mathrm{Tr}(E_k\rho)$.

Every transformation is associated with a trace-preserving completely-positive linear map $\rho \rightarrow \rho' = T(\rho)$,

Every measurement outcome k is associated with a trace-nonincreasing completely-positive linear map $\rho \rightarrow \rho' = T_k(\rho)$.

The formalism of density operators, POVMs, and CP maps is critical in quantum information theory and arguably quantum foundations as well.

Operational alternatives to quantum theory

Axiomatization

L. Hardy, "Quantum theory from five reasonable axioms", quant-ph/0101012

Foil theories

J. Barrett, "Information Processing in Generalized Probabilistic Theories", quant-ph/0508211

RS, "In defense of the epistemic view of quantum states: a toy theory" quant-ph/0401052

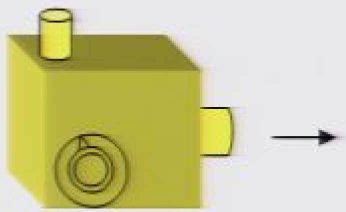
For more references, see:
http://qubit.damtp.cam.ac.uk/users/rob/foilswebpage.htm

Operationalism as a tool for the realist:

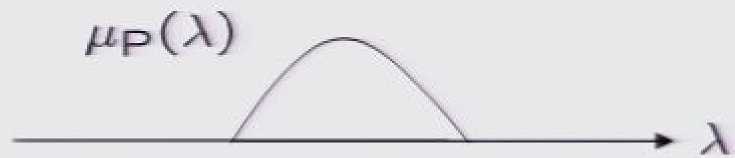
Devising a theory-independent definition of contextuality

A hidden variable model of an operational theory assumes primitives of systems and properties

Preparation
P

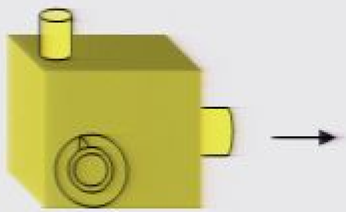


$$\int \mu_{\mathsf{P}}(\lambda)\, d\lambda = 1$$



A hidden variable model of an operational theory assumes primitives of systems and properties

Preparation
P



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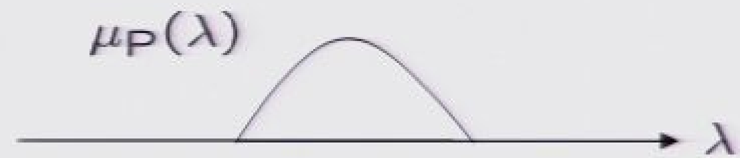
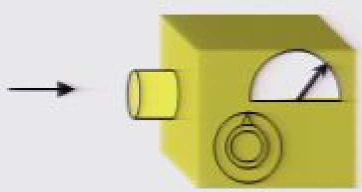
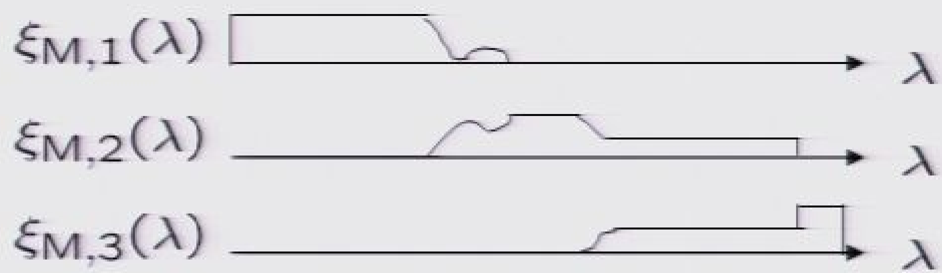


Measurement
M



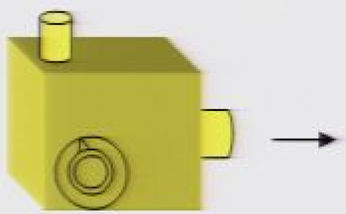
$0 \leq \xi_{\mathrm{M},k} \leq 1$

$\sum_k \xi_{\mathrm{M},k}(\lambda) = 1$ for all λ

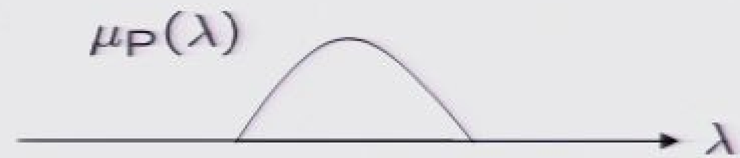


A hidden variable model of an operational theory assumes primitives of systems and properties

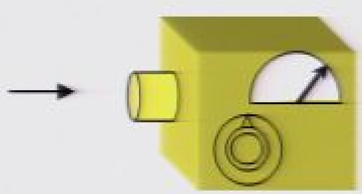
Preparation
P



$\int \mu_{\mathsf{P}}(\lambda) d\lambda = 1$



Measurement
M



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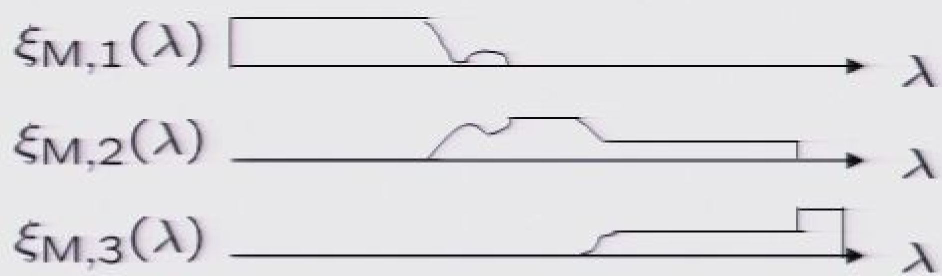


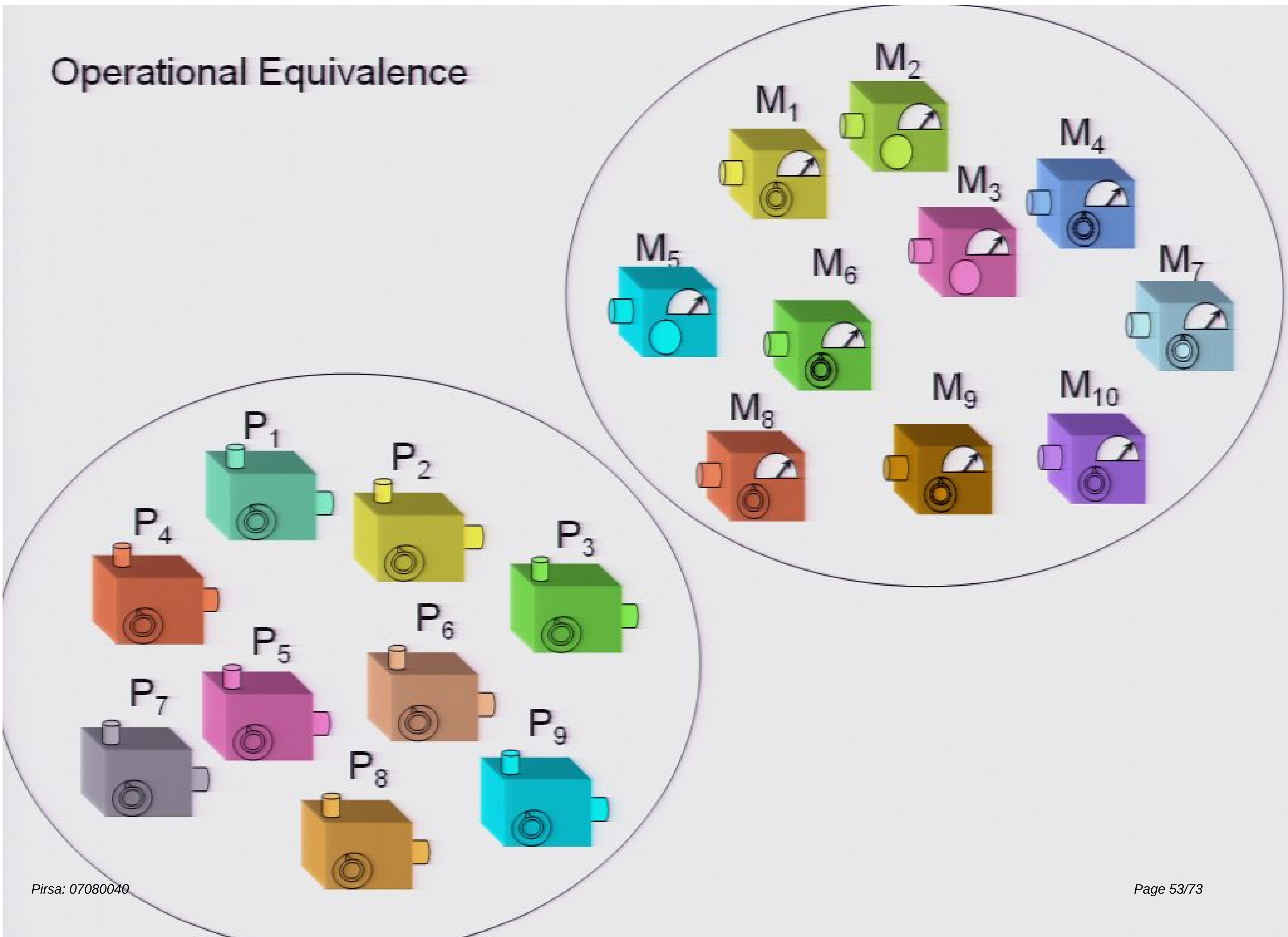
$$\boxed{p(k|\mathsf{P},\mathsf{M}) = \int d\lambda \, \xi_{\mathsf{M},k}(\lambda) \, \mu_{\mathsf{P}}(\lambda)}$$

Proposed new definition of noncontextuality:

A HV model of an operational theory is noncontextual if

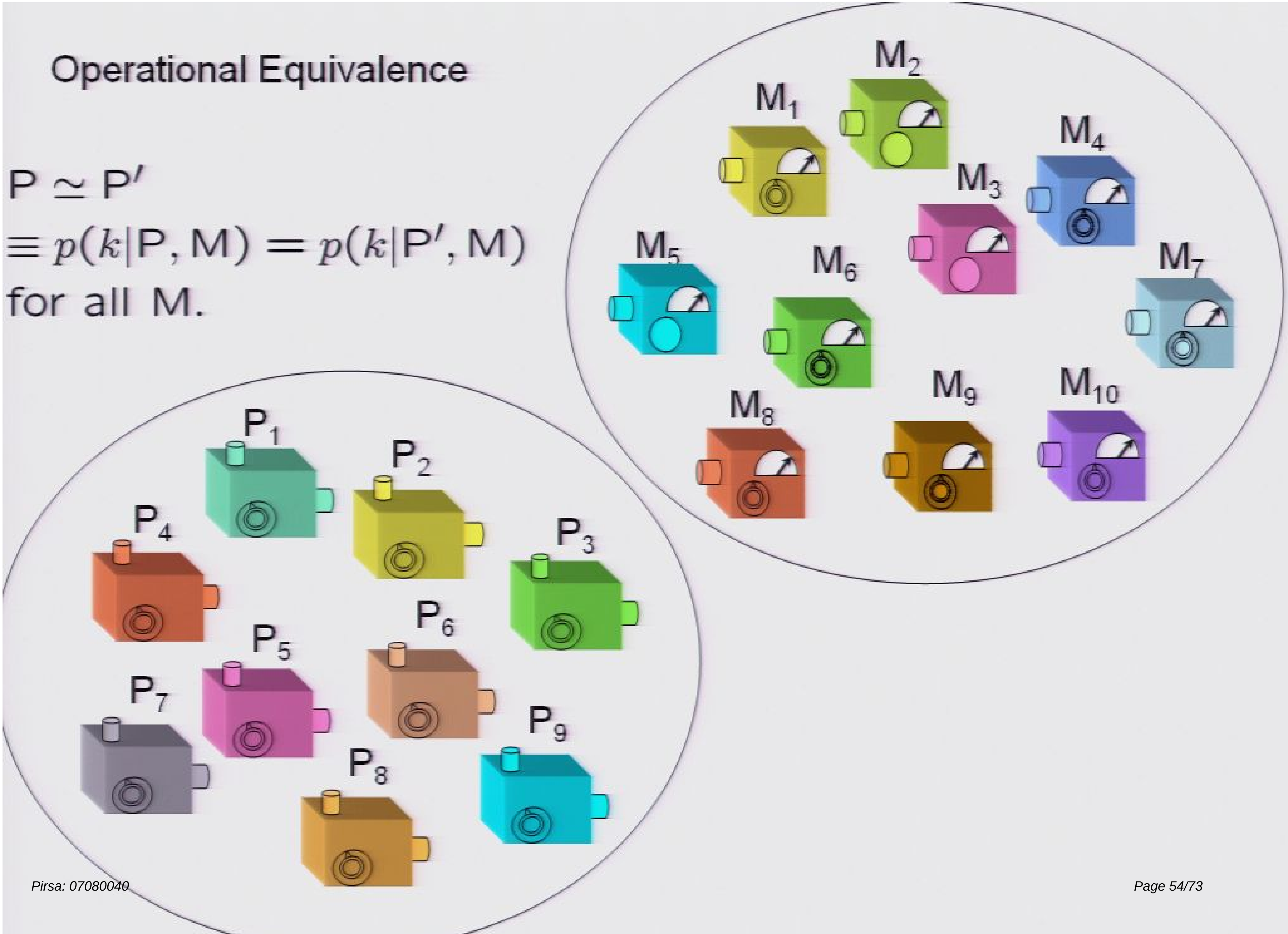
Operational equivalence of two experimental procedures $\longrightarrow$ Equivalent representations in the HV model

Operational Equivalence



Operational Equivalence

$\mathsf{P} \simeq \mathsf{P}'$
$\equiv p(k|\mathsf{P},\mathsf{M}) = p(k|\mathsf{P}',\mathsf{M})$
for all M.



Operational Equivalence

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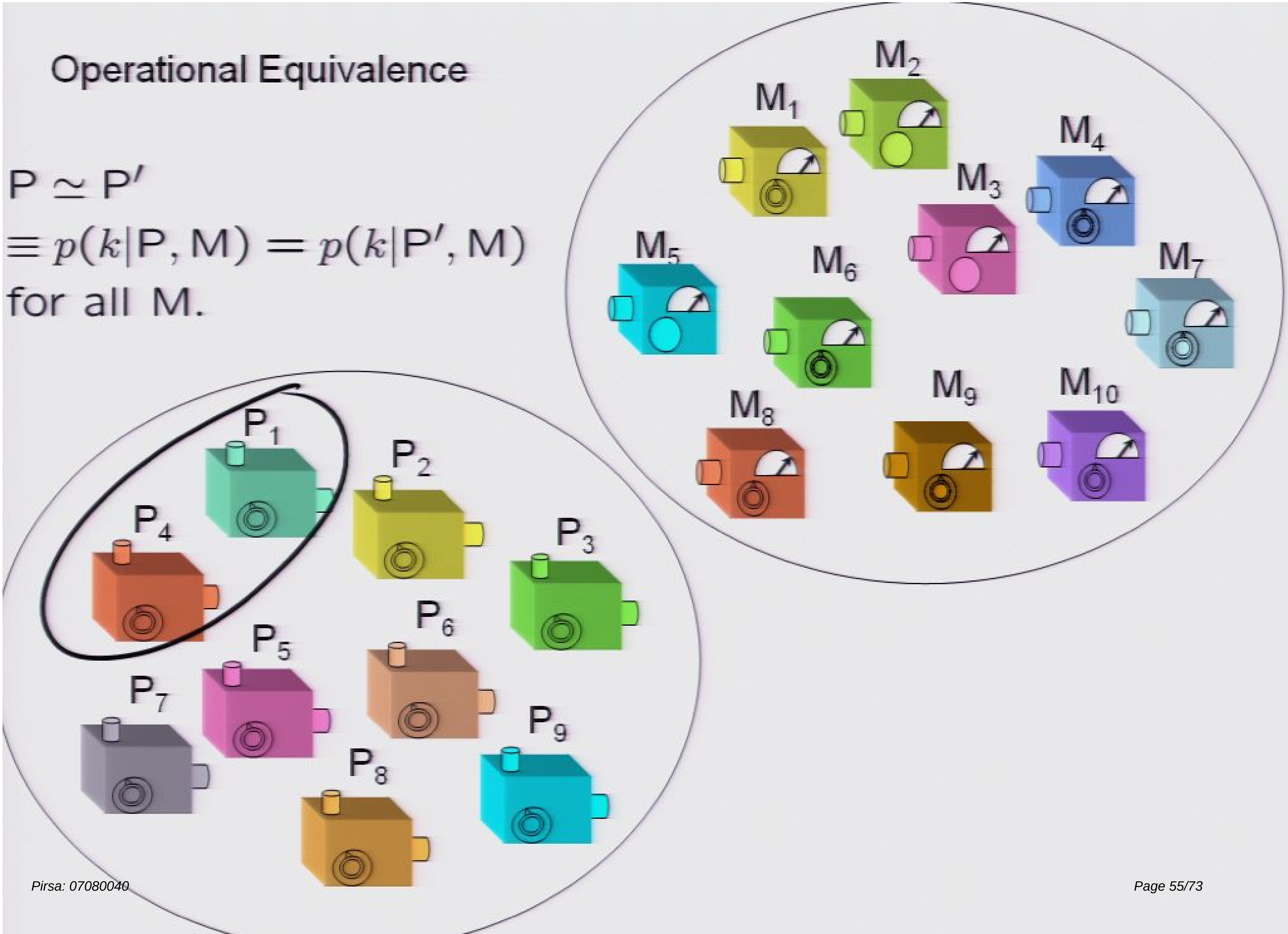

Operational Equivalence

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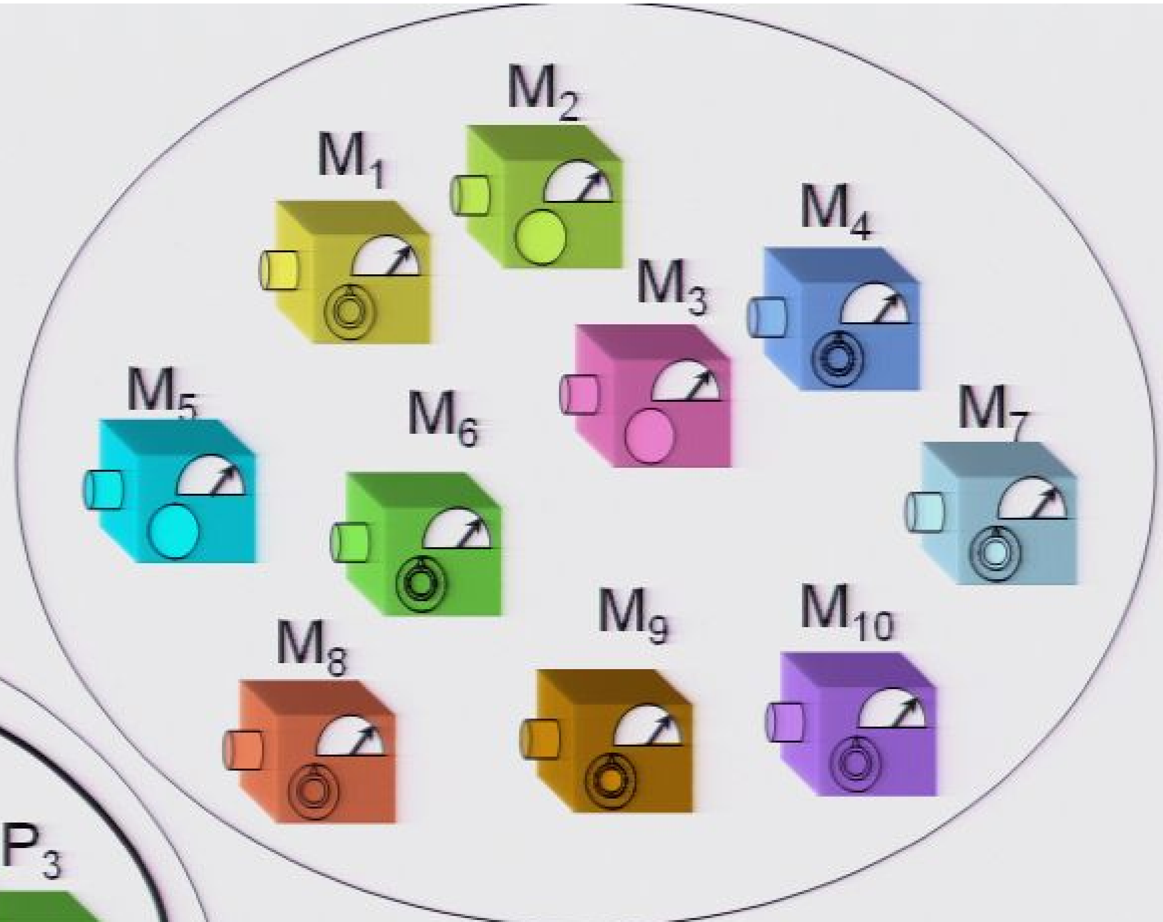
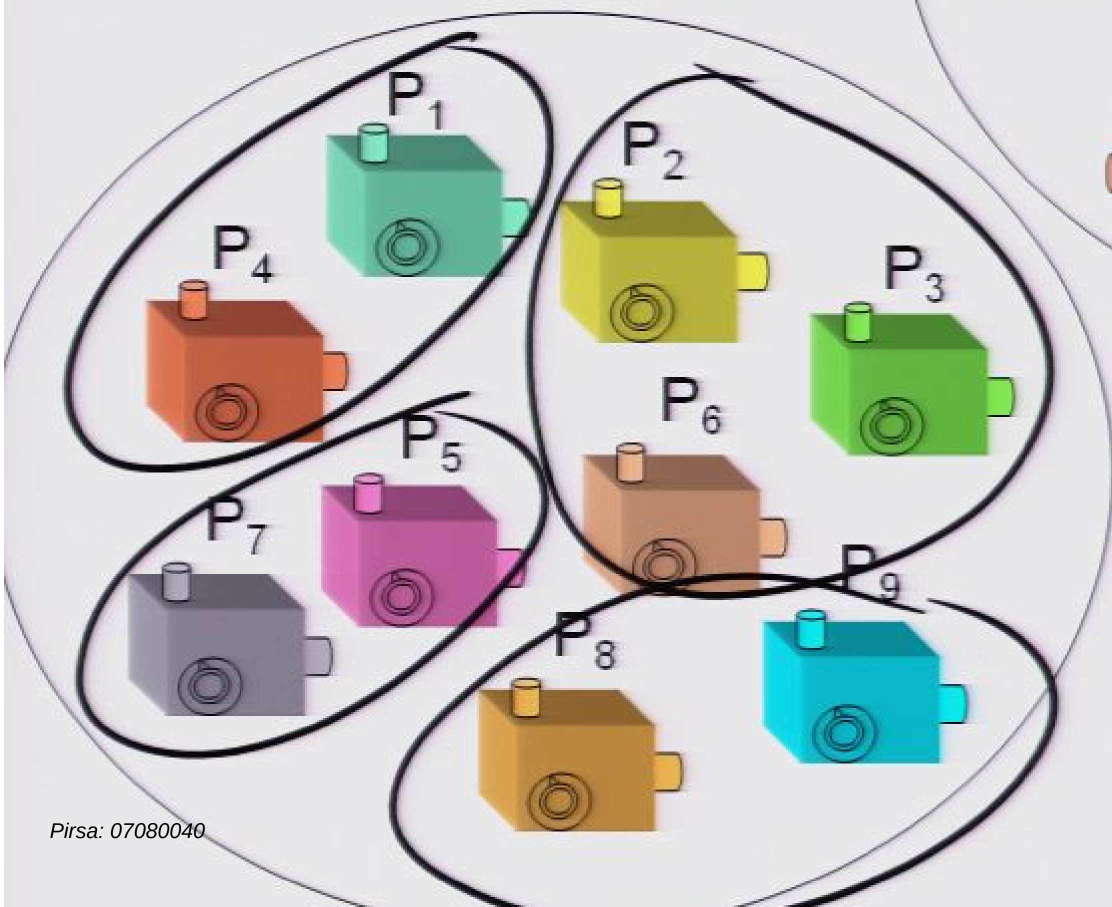
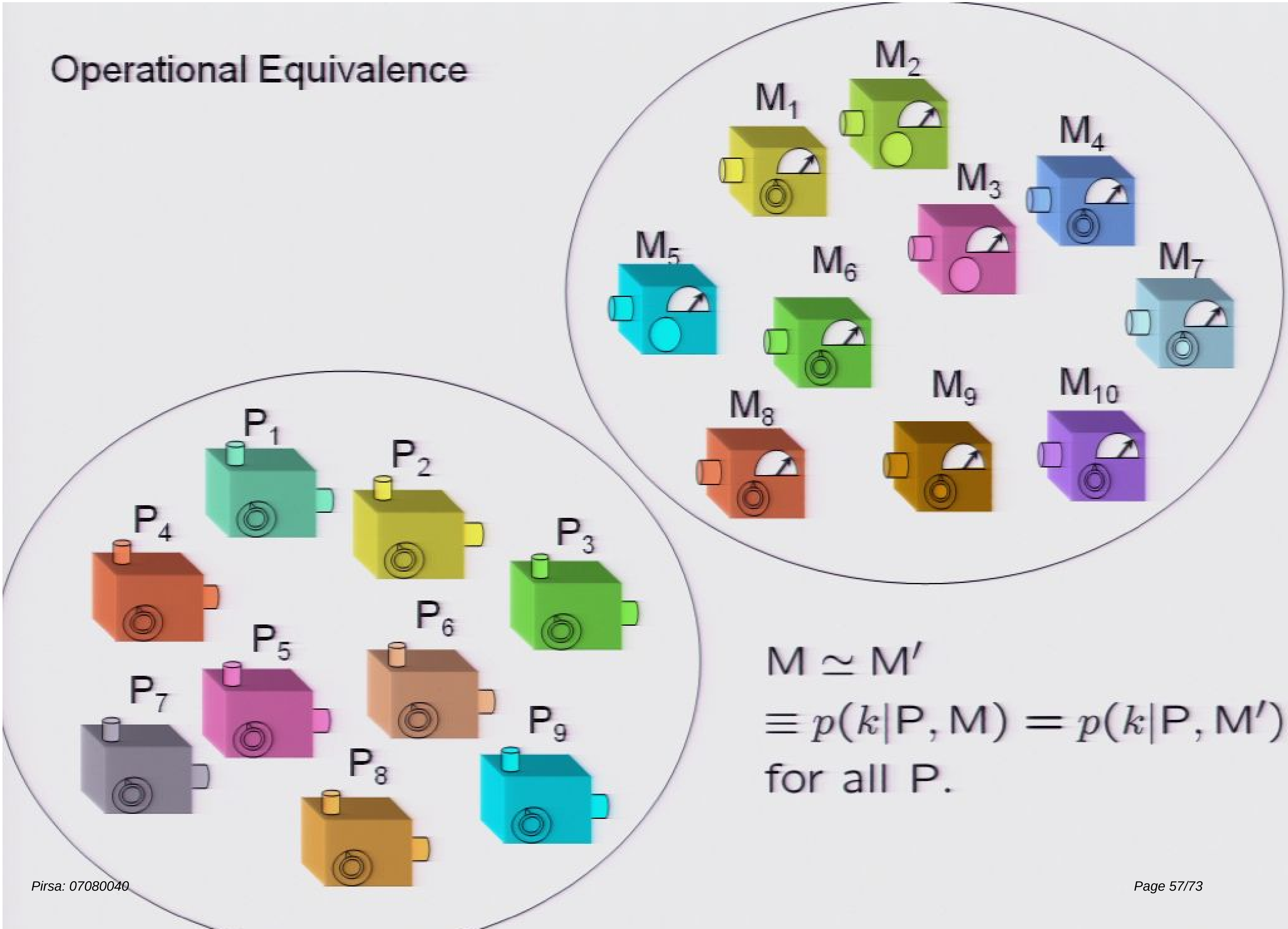




Operational Equivalence

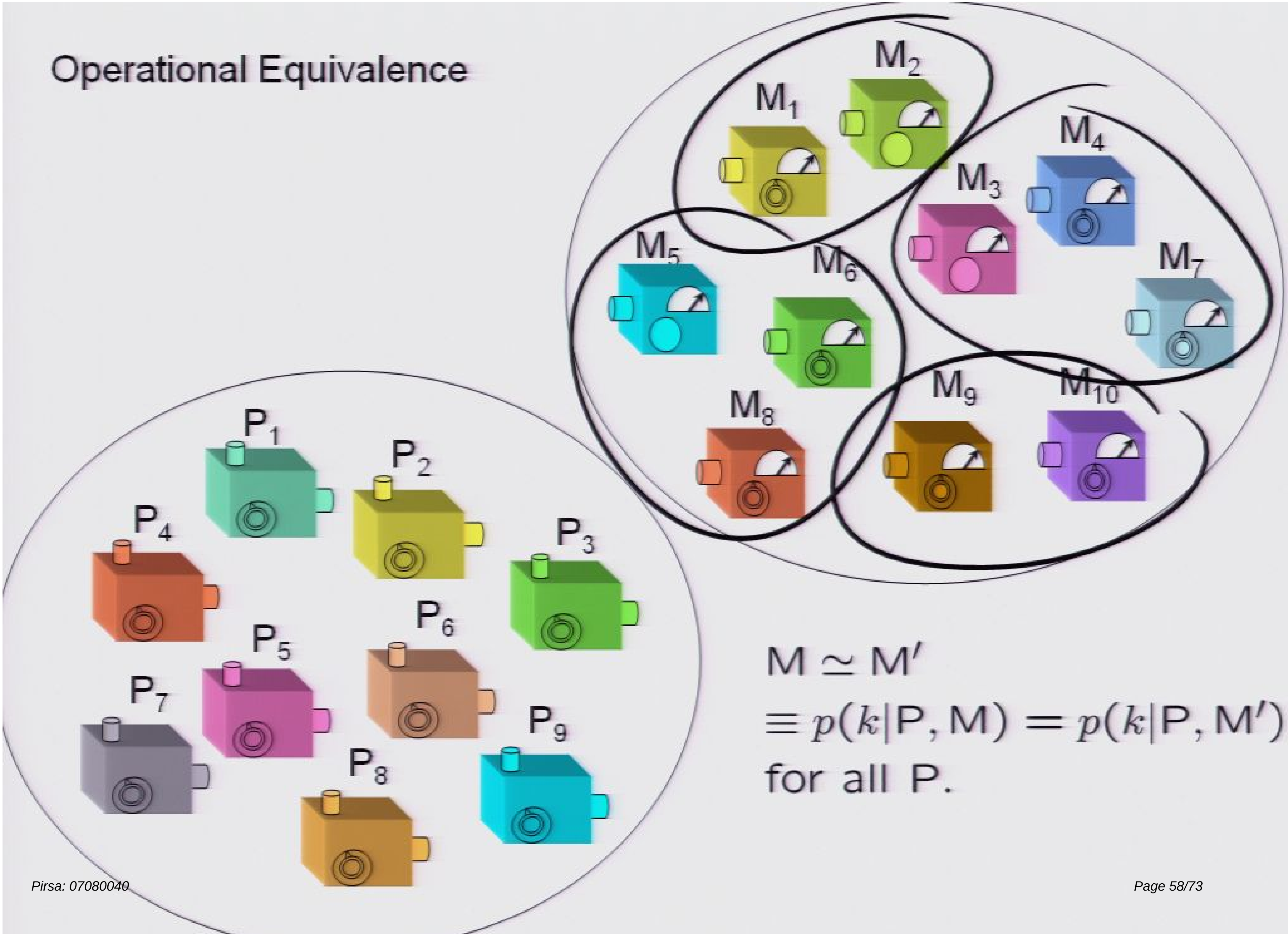


$\mathsf{M} \simeq \mathsf{M}'$

$\equiv p(k|\mathsf{P},\mathsf{M}) = p(k|\mathsf{P},\mathsf{M}')$

for all P.

Operational Equivalence



$\mathsf{M} \simeq \mathsf{M}'$

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Operational Equivalence

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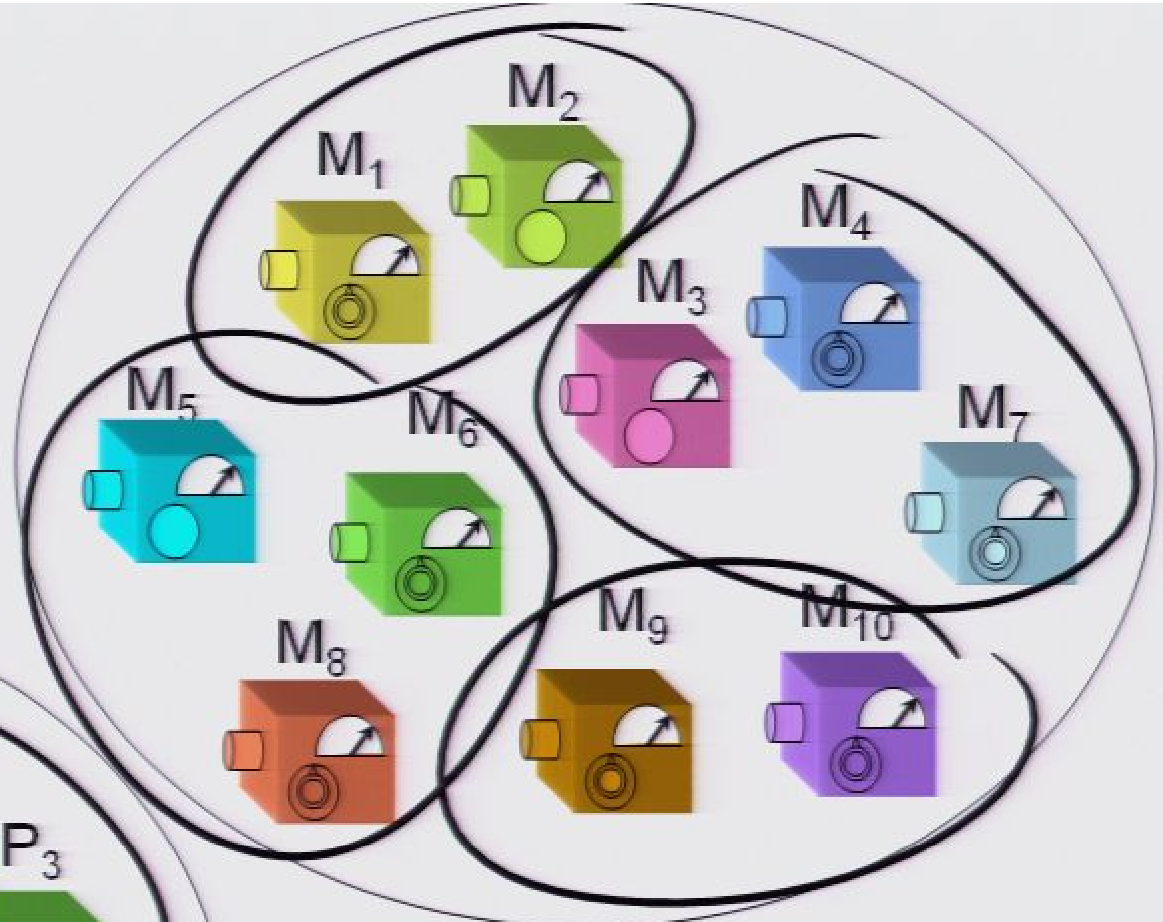

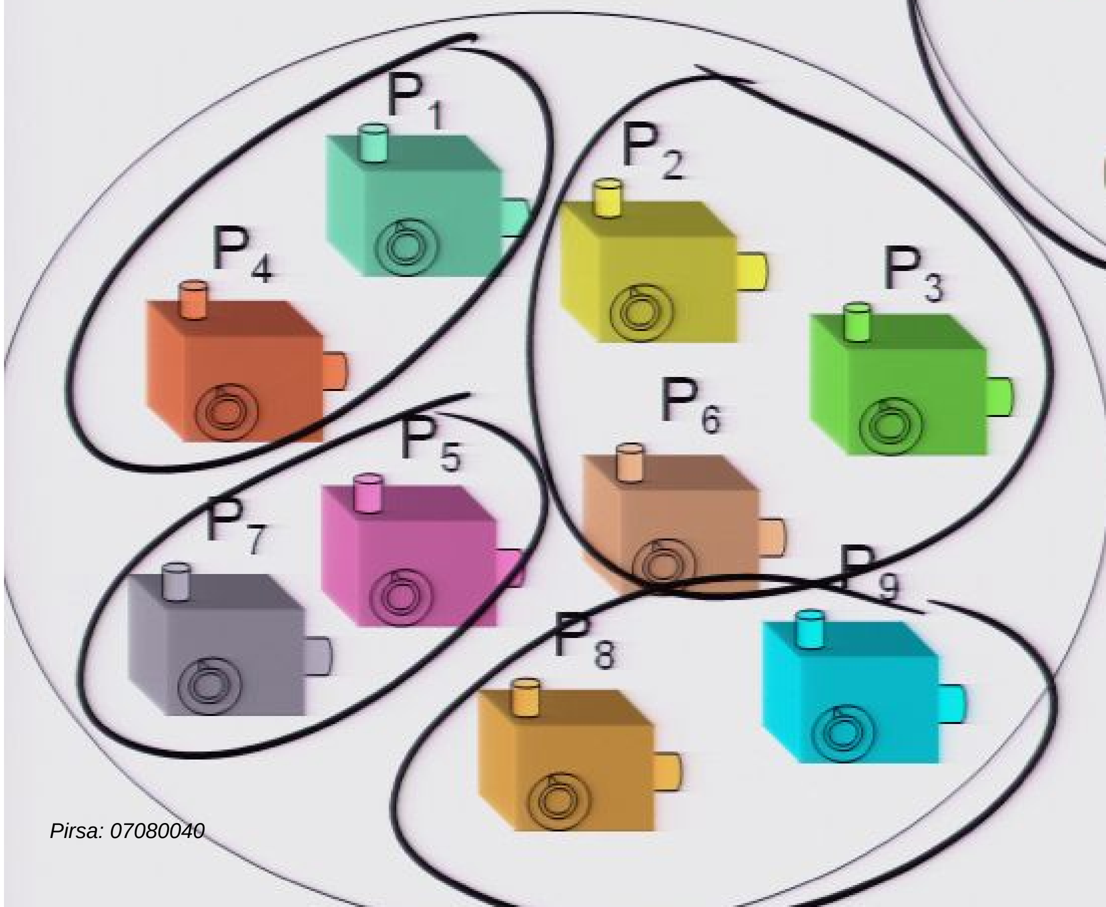

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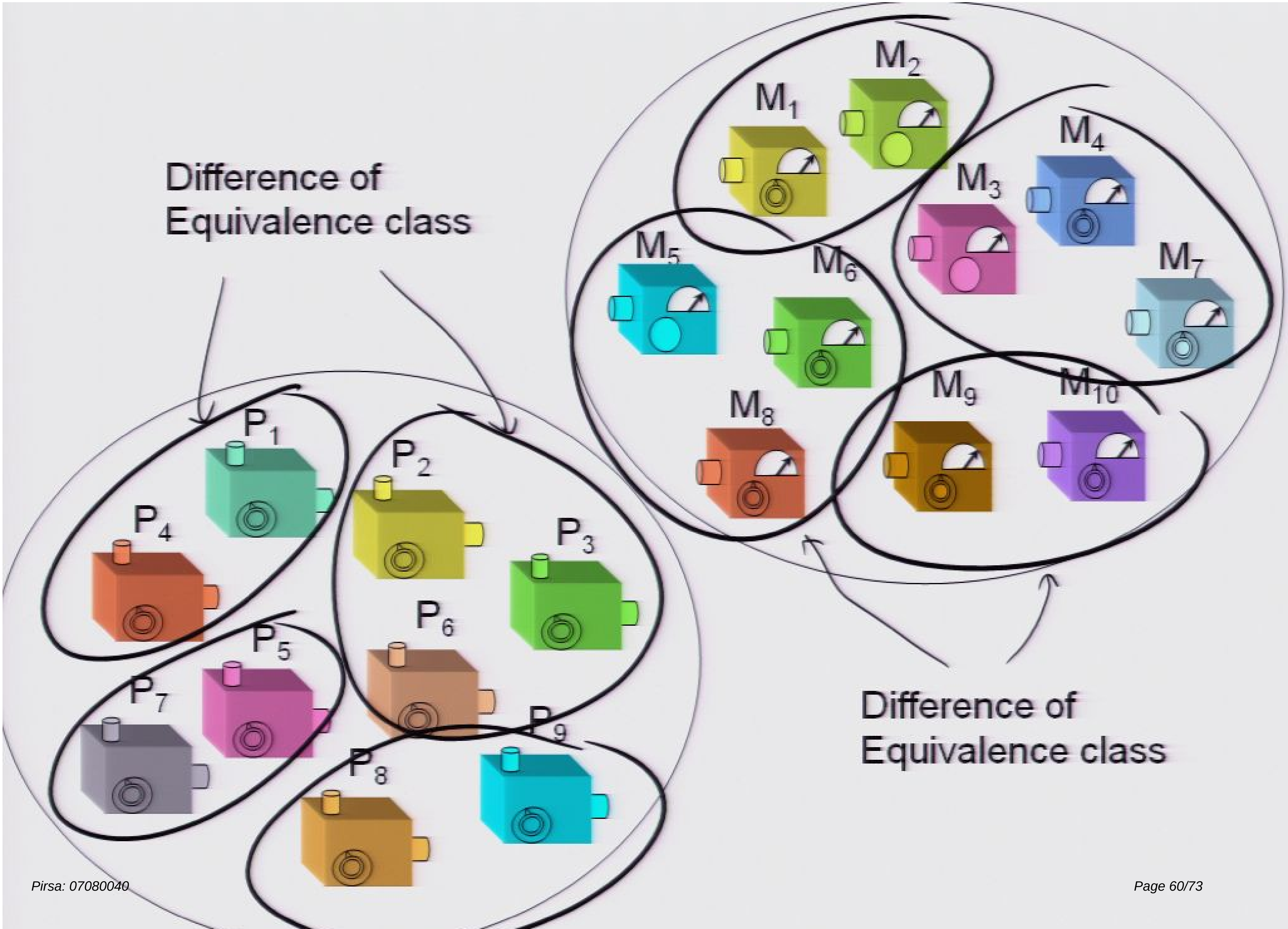
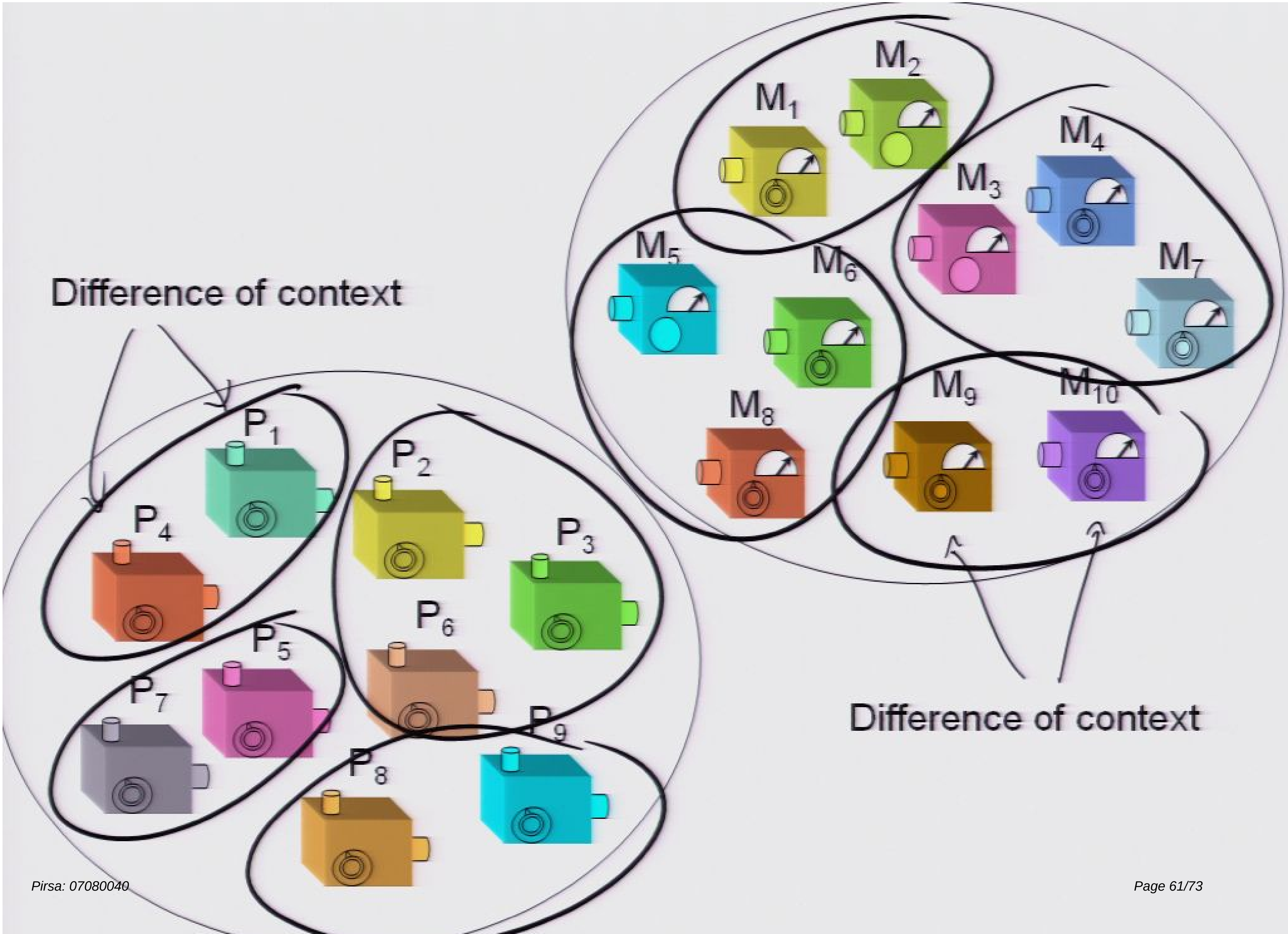


Difference of
Equivalence class
P1
P2
P3
P4
P5
P6
P7
P8
P9
M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
Difference of
Equivalence class

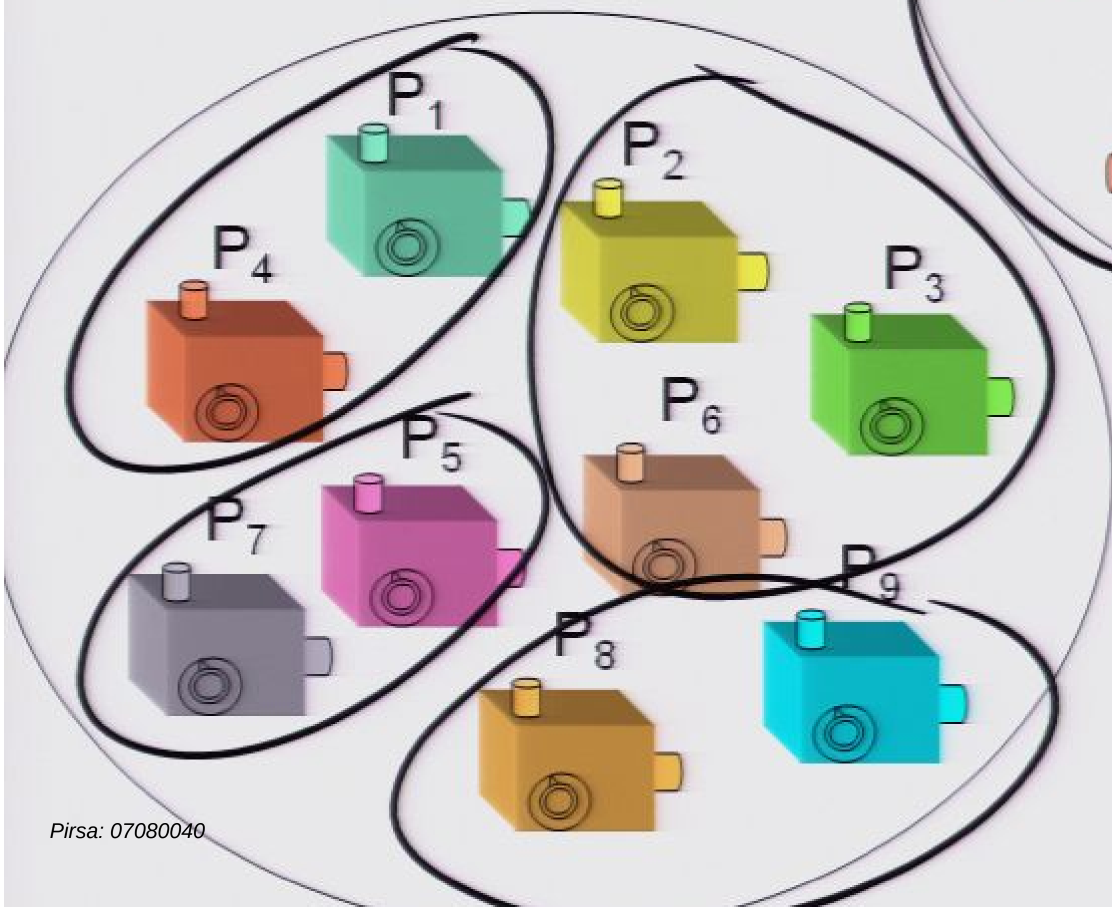


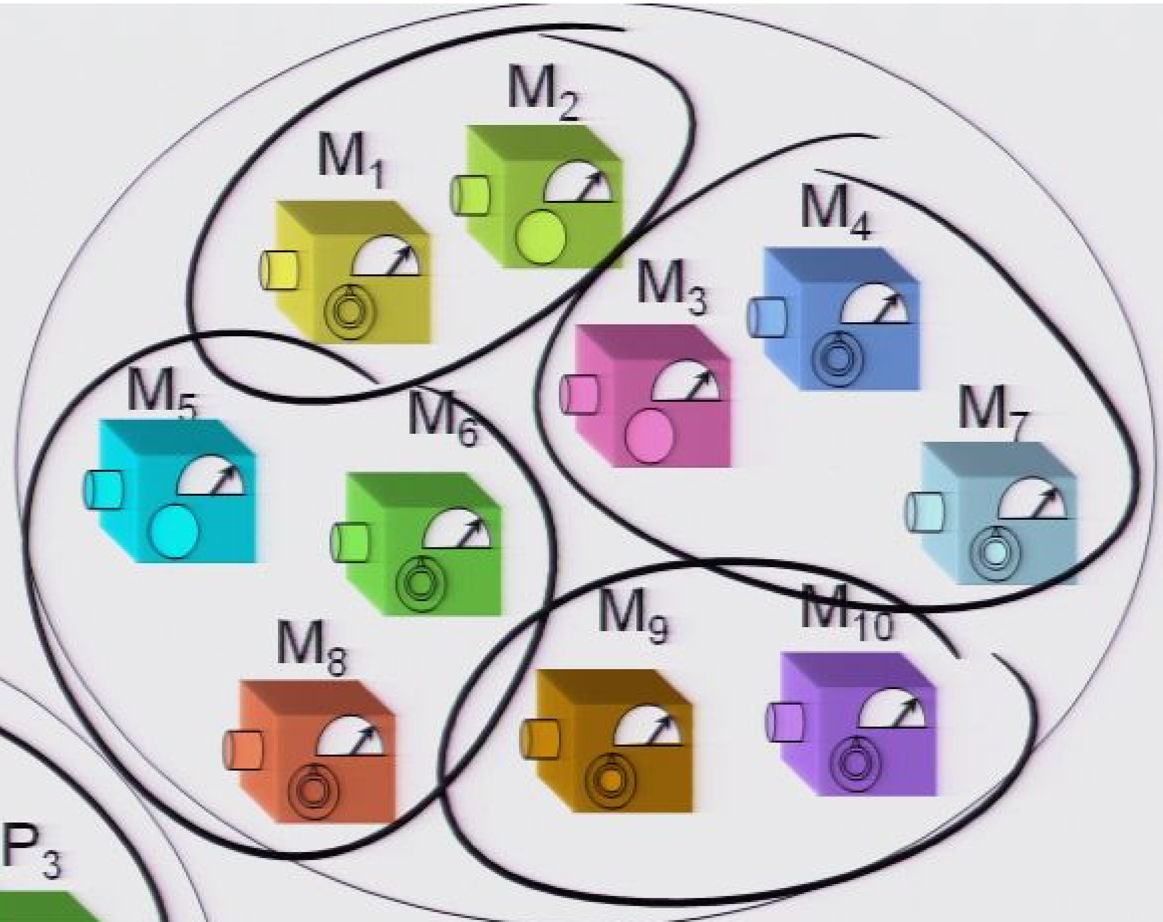
Difference of context
M_1 M_2 M_3 M_4 M_5 M_6 M_7 M_8 M_9 M_{10}
P_1 P_2 P_3 P_4 P_5 P_6 P_7 P_8 P_9
Difference of context

Preparation Noncontextuality

if P$\simeq$P$'$ then

$$\mu_{P}(\lambda) = \mu_{P'}(\lambda)$$

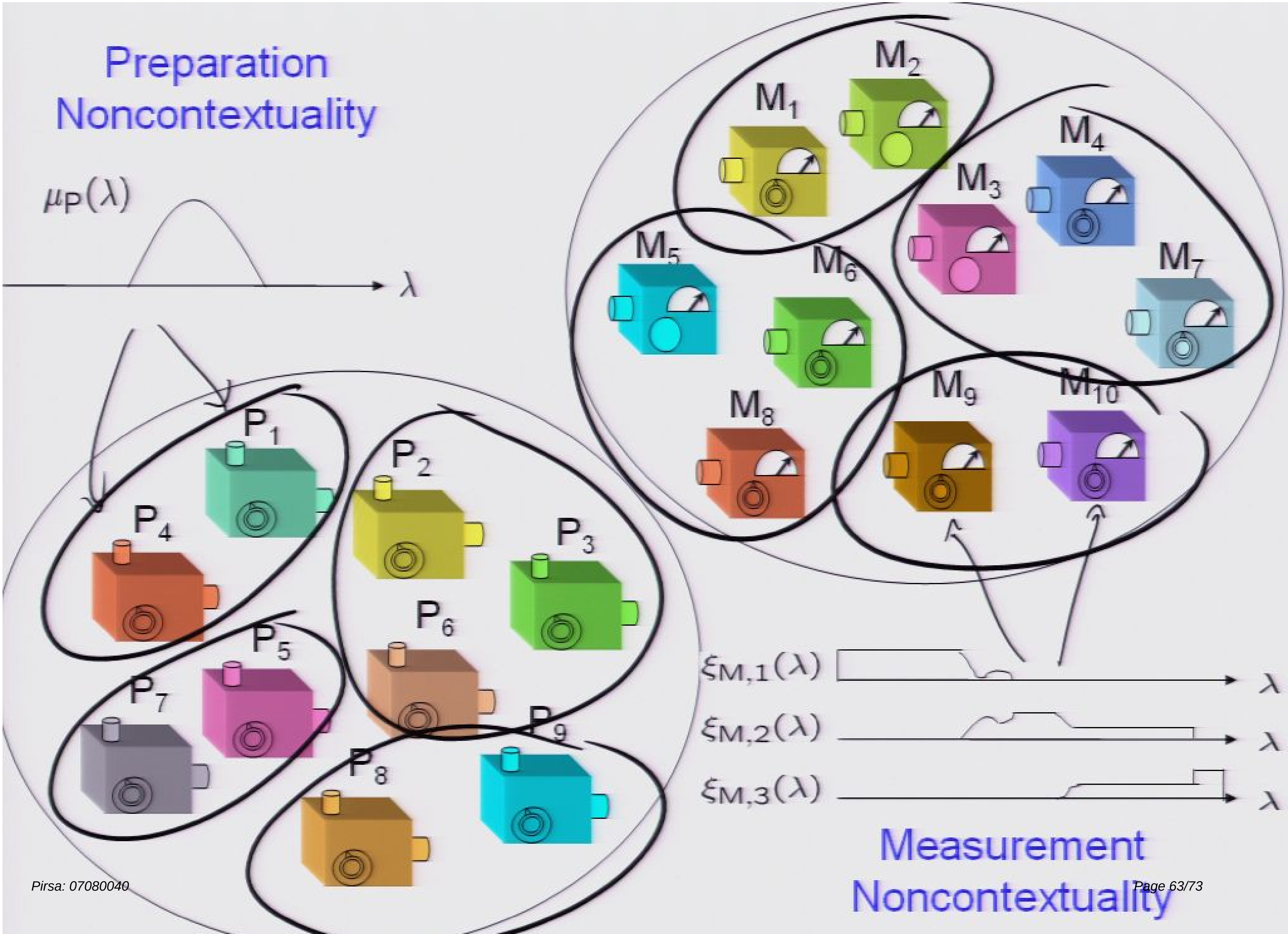




Measurement Noncontextuality

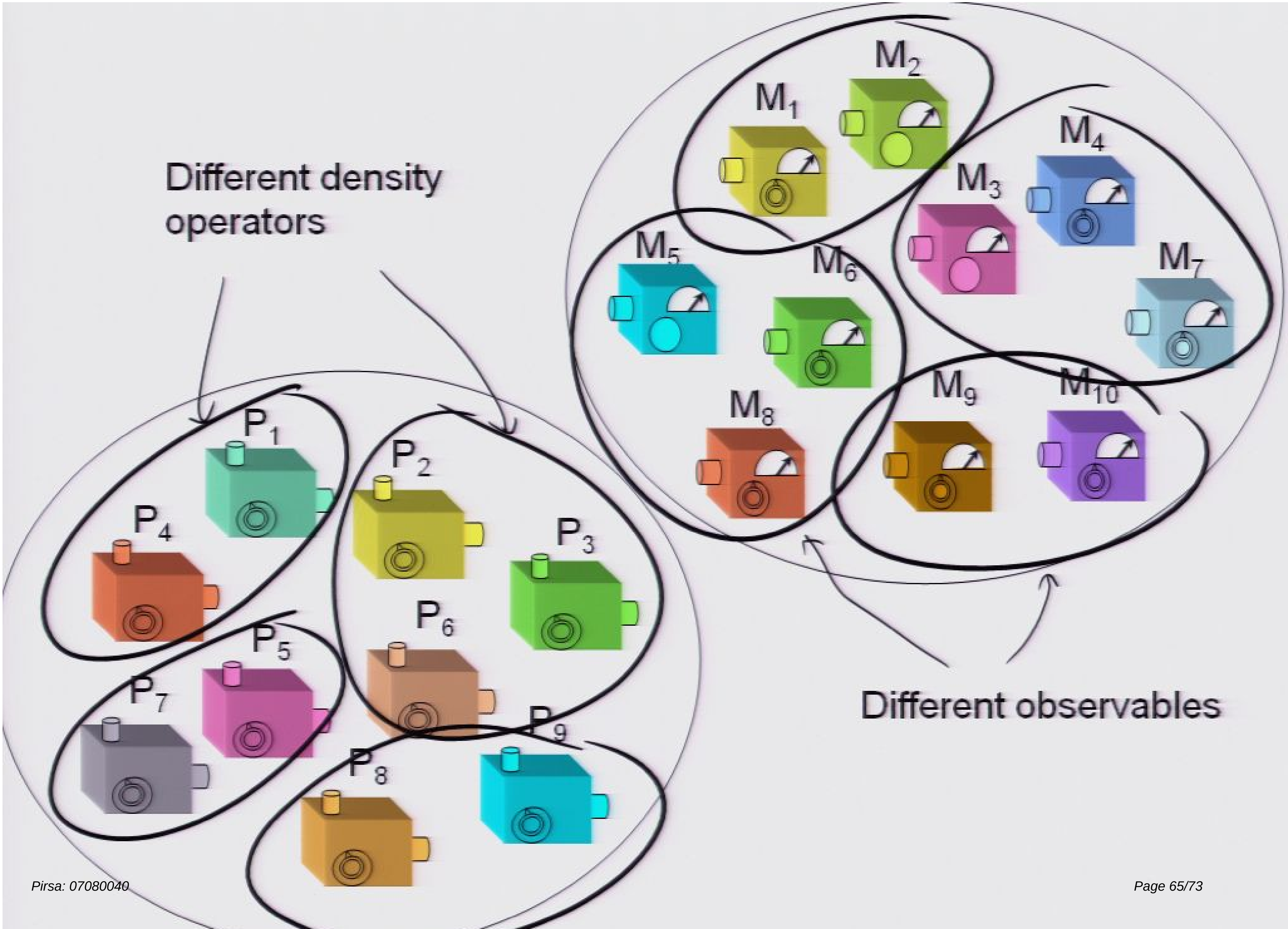
if M$\simeq$M$'$ then

$$\xi_{M,k}(\lambda) = \xi_{M',k}(\lambda)$$



Preparation
Noncontextuality
$\mu_P(\lambda)$
λ
P_1
P_2
P_3
P_4
P_5
P_6
P_7
P_8
P_9
M_1
M_2
M_3
M_4
M_5
M_6
M_7
M_8
M_9
M_{10}
$\xi_{M,1}(\lambda)$
$\xi_{M,2}(\lambda)$
$\xi_{M,3}(\lambda)$
Measurement
Noncontextuality

Contextuality in quantum theory revisited



Different density operators
P1
P2
P3
P4
P5
P6
P7
P8
P9
M1
M2
M3
M4
M5
M6
M7
M8
M9
M10
Different observables

Ex: different convex decompositions of ρ

Many $\{p_j, |\psi_j\rangle\}$ such that

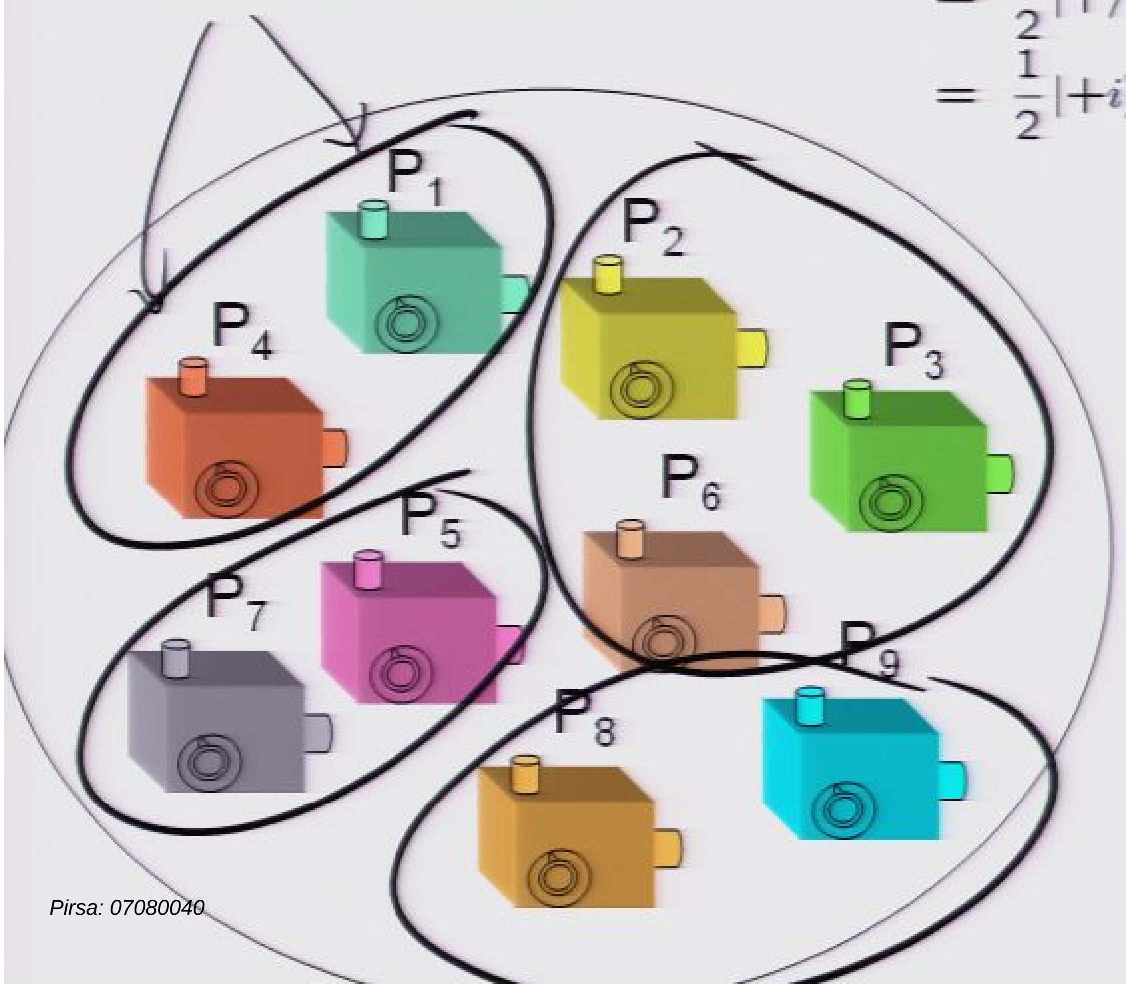
$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

$$\begin{aligned} \frac{1}{2}I &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \\ &= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \\ &= \frac{1}{2}|+i\rangle\langle +i| + \frac{1}{2}|-i\rangle\langle -i| \end{aligned}$$

$|\pm\rangle = |0\rangle \pm |1\rangle$

$|\pm i\rangle = |0\rangle \pm i|1\rangle$

Difference of context



Ex: different convex decompositions of ρ

Many $\{p_j, |\psi_j\rangle\}$ such that

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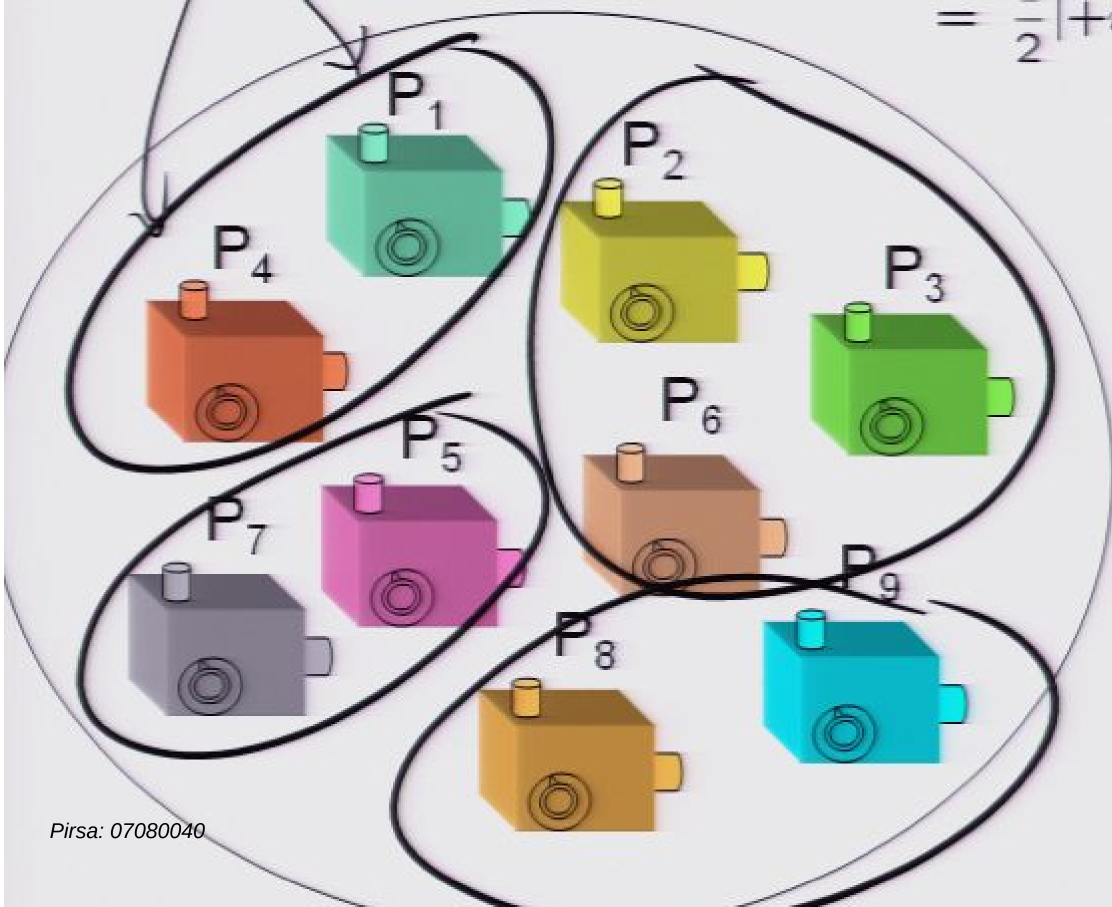
Difference of context



Ex: Different purifications of ρ

Many $|\Psi\rangle_{AB}$ such that

$\rho = \mathrm{Tr}_B(|\Psi\rangle_{AB}\langle\Psi|)$

Ex: different convex decompositions of ρ

Many $\{p_j, |\psi_j\rangle\}$ such that

$\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|$

$$\begin{aligned} \frac{1}{2}I &= \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| \\ &= \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -| \\ &= \frac{1}{2}|+i\rangle\langle +i| + \frac{1}{2}|-i\rangle\langle -i| \end{aligned}$$

$$|\pm\rangle = |0\rangle \pm |1\rangle$$
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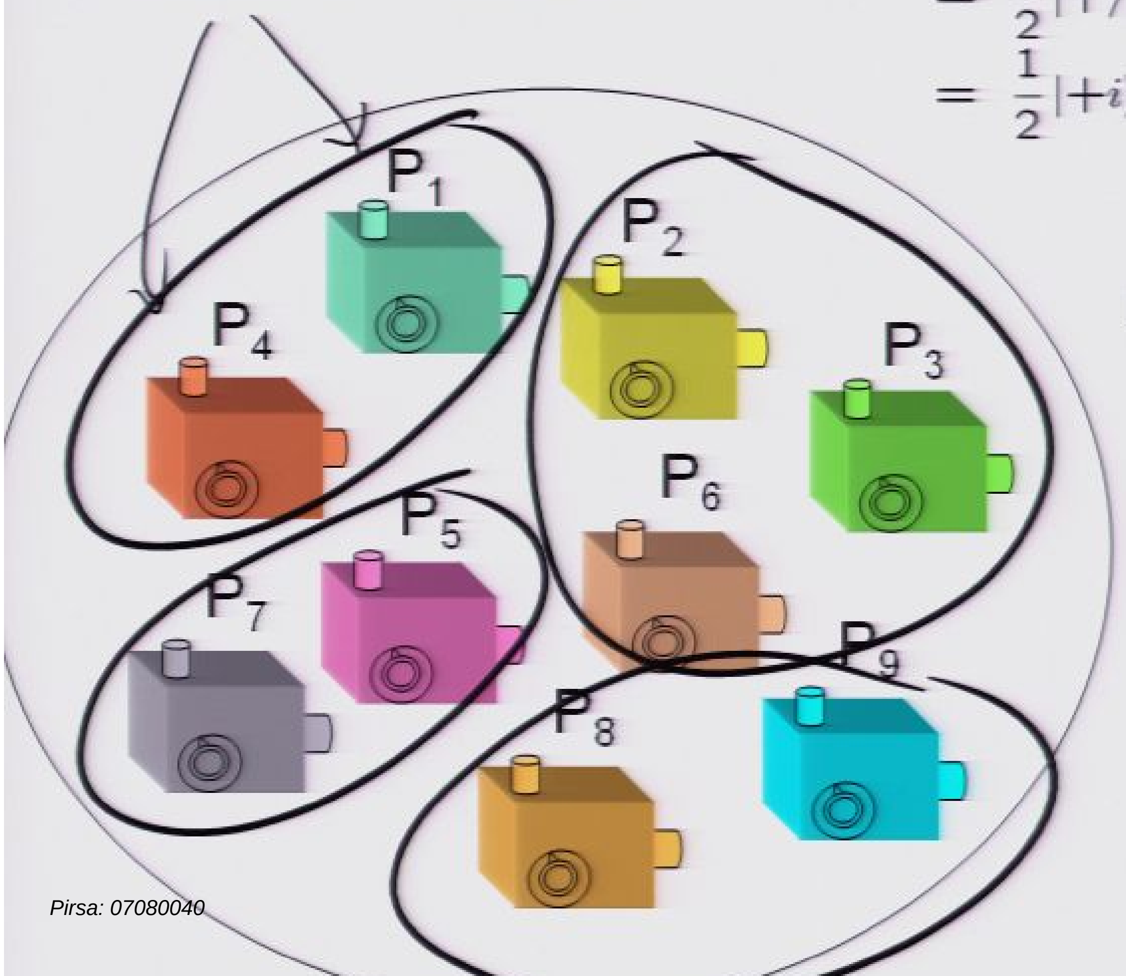
Difference of context



Ex: Different purifications of ρ

Many $|\Psi\rangle_{AB}$ such that

$\rho = \mathrm{Tr}_B(|\Psi\rangle_{AB}\langle\Psi|)$

Preparation Noncontextuality

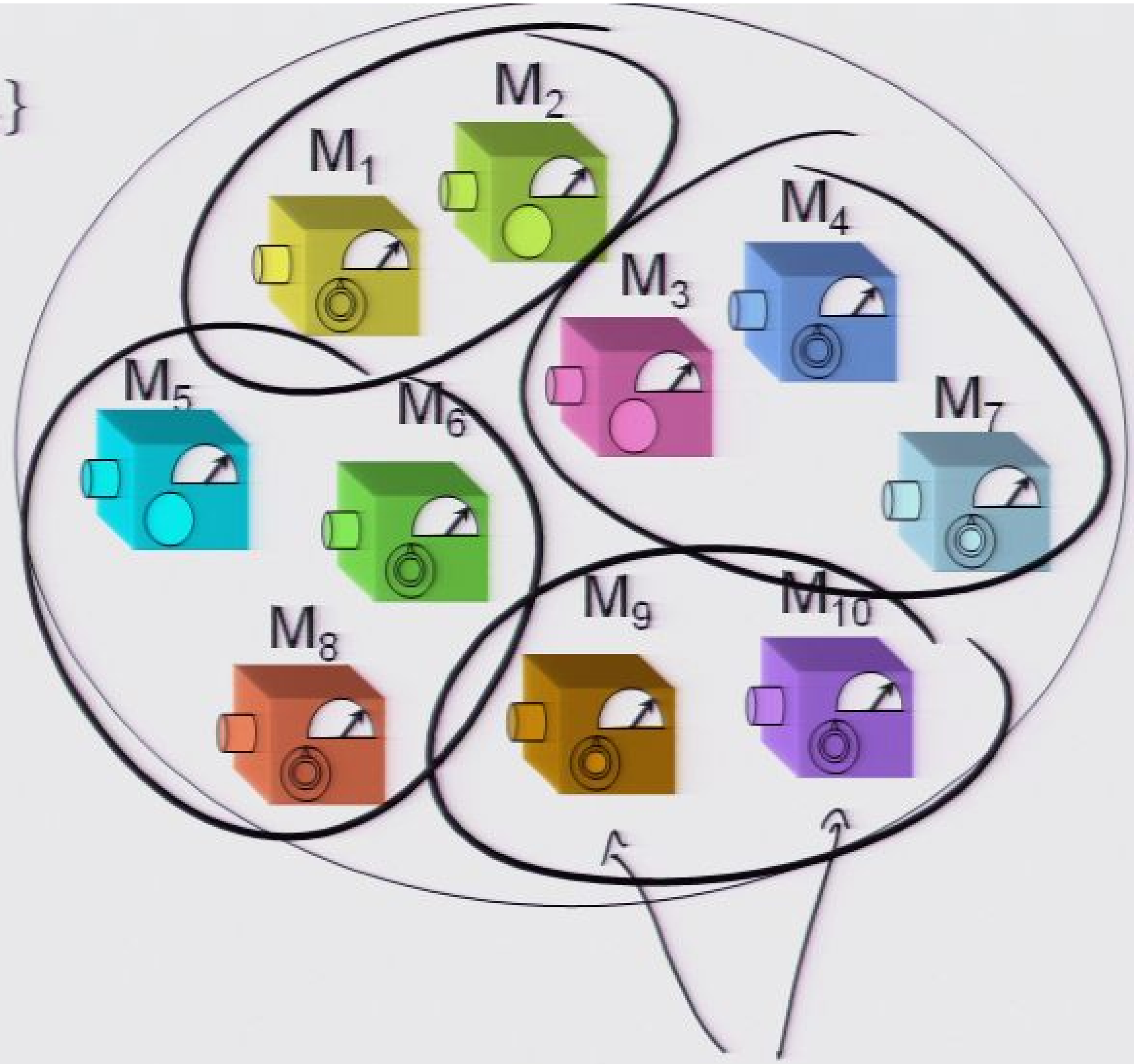
$$\mu_{\mathrm{P}}(\lambda) = \mu_{\rho}(\lambda)$$

Different fine-grainings of $\{E_k\}$

Many $\{E_{k,s}\}$ such that

$E_k = \sum_s E_{k,s}$

$$I - |\psi_1\rangle\langle\psi_1|$$
$$= |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|$$
$$= |\psi'_2\rangle\langle\psi'_2| + |\psi'_3\rangle\langle\psi'_3|$$



Difference of context

Different fine-grainings of $\{E_k\}$

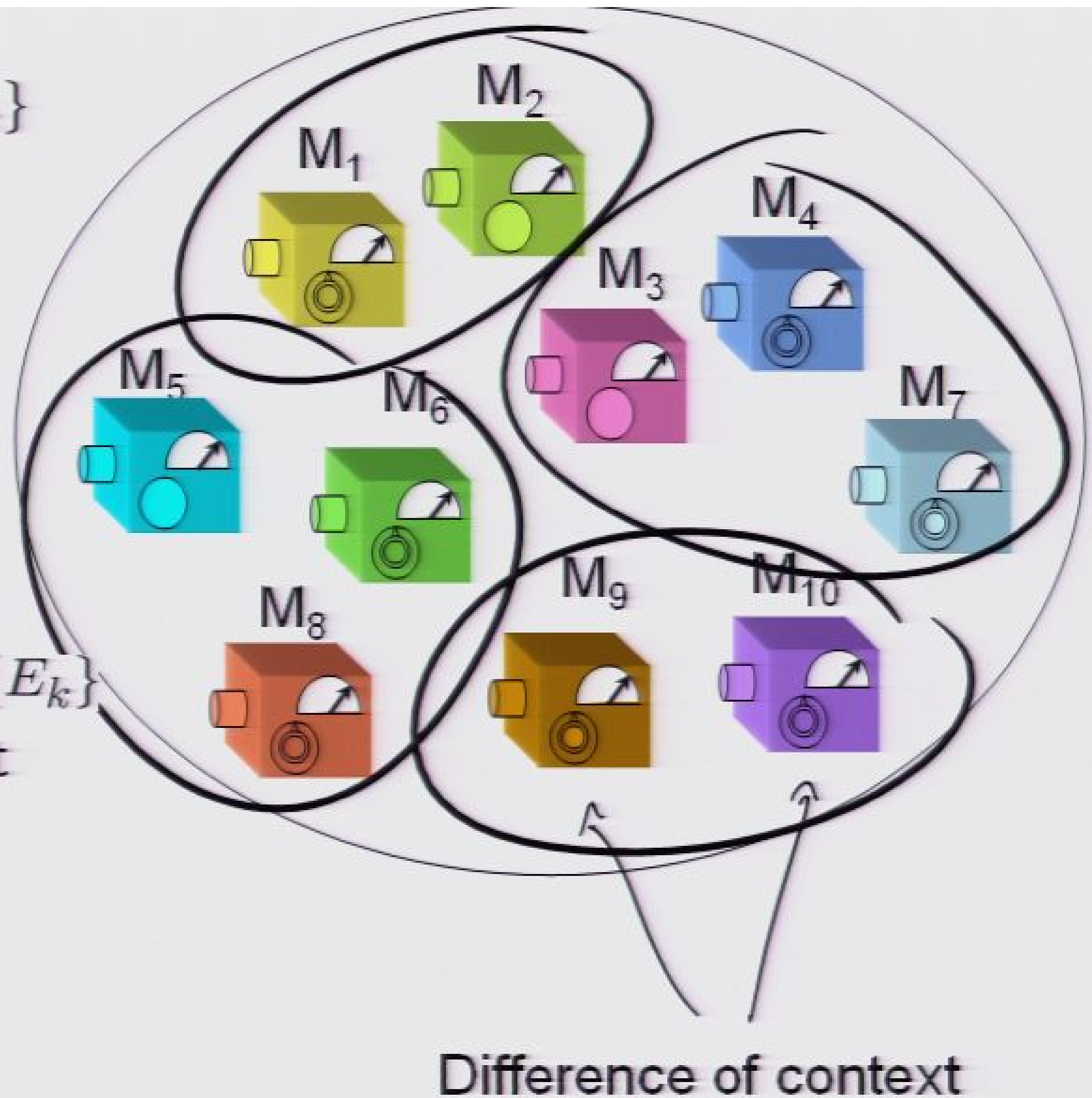
Many $\{E_{k,s}\}$ such that

$E_k = \sum_s E_{k,s}$

$I - |\psi_1\rangle\langle\psi_1|$

$= |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|$

$= |\psi'_2\rangle\langle\psi'_2| + |\psi'_3\rangle\langle\psi'_3|$



Ex: different convex decoms of $\{E_k\}$

Many $\{p_j, \{E_k^j\}\}$ such that

$E_k = \sum_j p_j E_k^j$

Different fine-grainings of $\{E_k\}$

Many $\{E_{k,s}\}$ such that

$E_k = \sum_s E_{k,s}$

$I - |\psi_1\rangle\langle\psi_1|$

$= |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|$

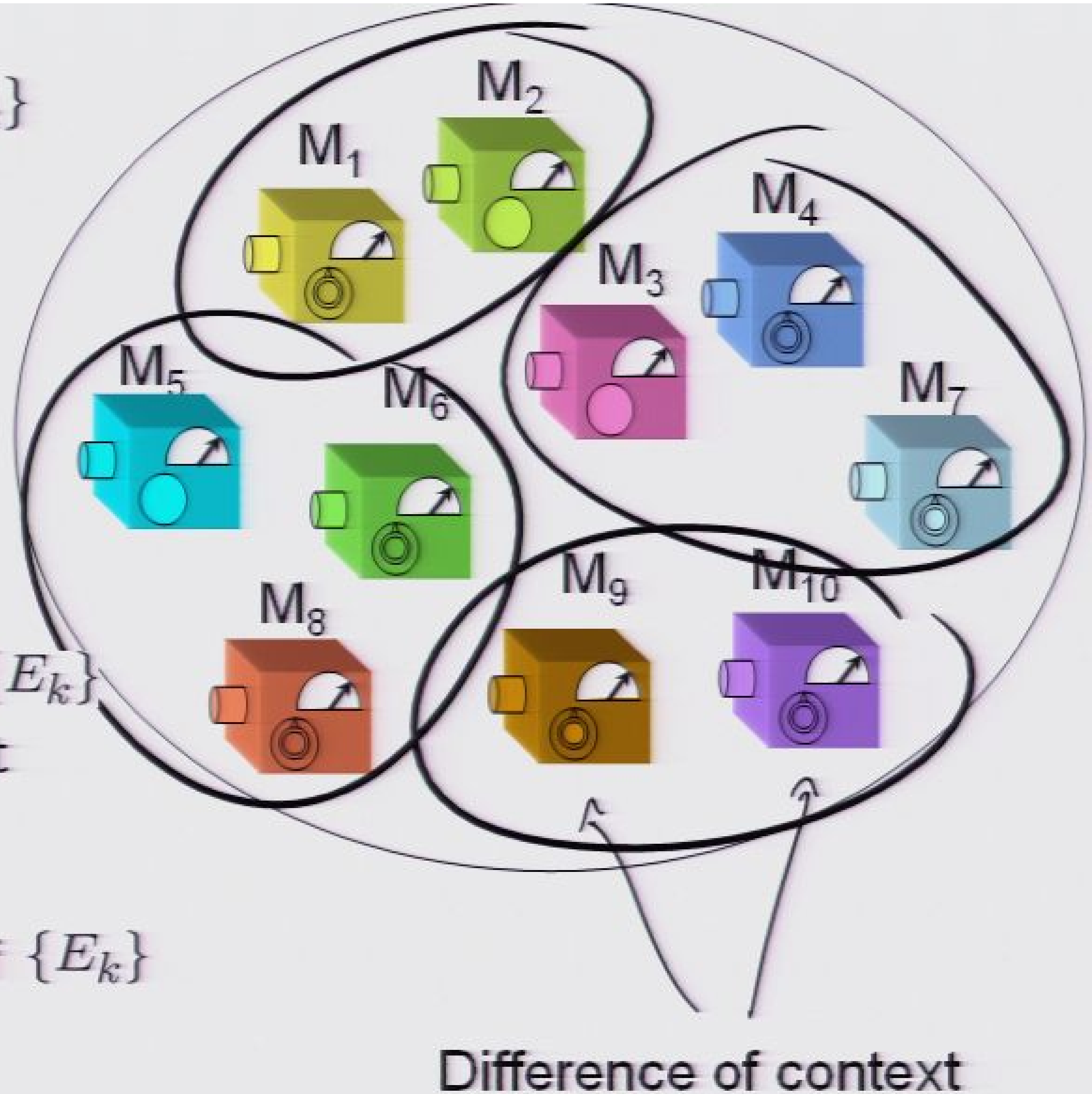
$= |\psi'_2\rangle\langle\psi'_2| + |\psi'_3\rangle\langle\psi'_3|$

Ex: different convex decoms of $\{E_k\}$

Many $\{p_j, \{E_k^j\}\}$ such that

$E_k = \sum_j p_j E_k^j$

Different Neumark extensions of $\{E_k\}$



Different fine-grainings of $\{E_k\}$

Many $\{E_{k,s}\}$ such that

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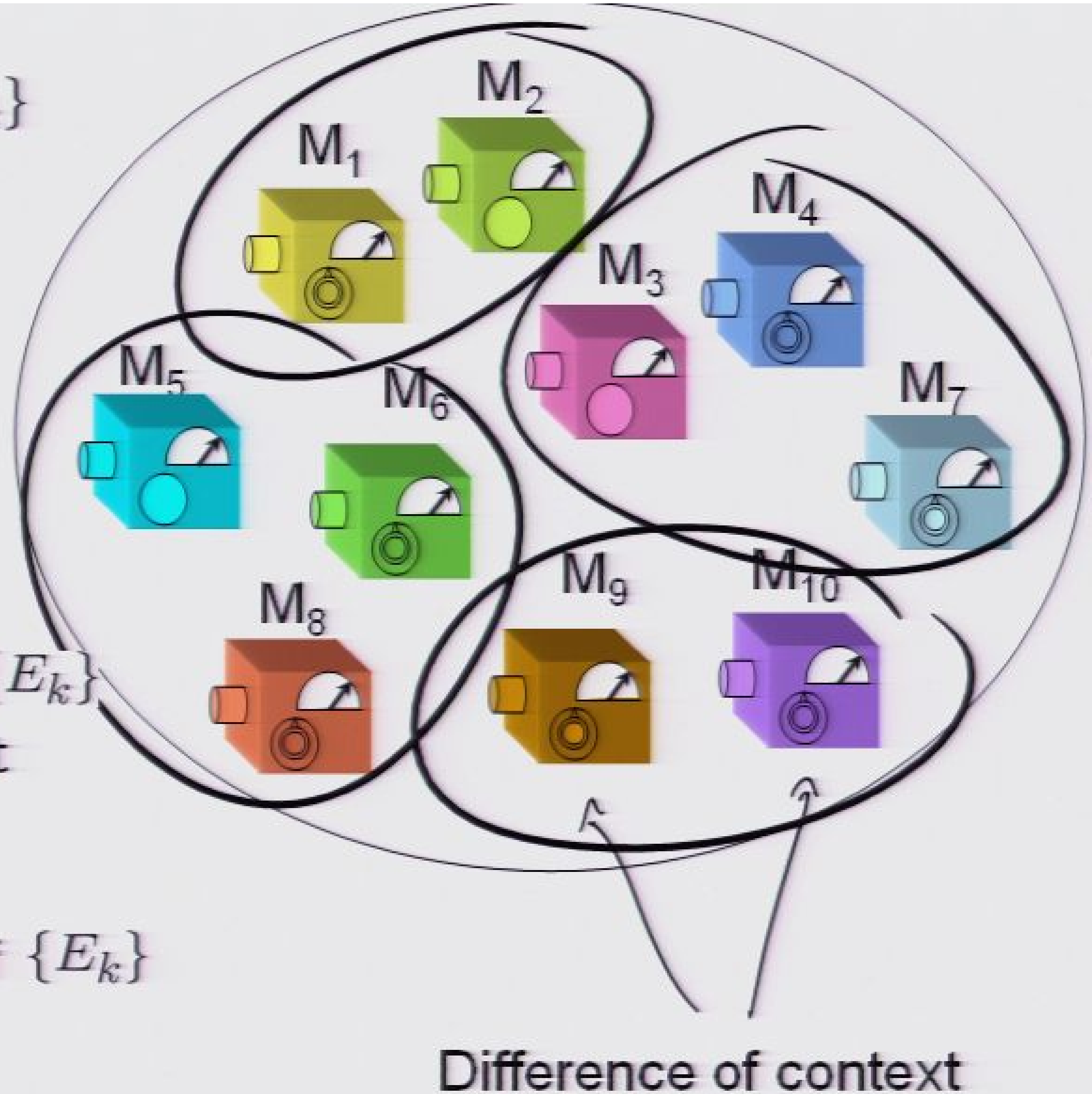
$$I - |\psi_1\rangle\langle\psi_1|$$
$$= |\psi_2\rangle\langle\psi_2| + |\psi_3\rangle\langle\psi_3|$$
$$= |\psi_2'\rangle\langle\psi_2'| + |\psi_3'\rangle\langle\psi_3'|$$



Ex: different convex decoms of $\{E_k\}$

Many $\{p_j, \{E_k^j\}\}$ such that

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Different Neumark extensions of $\{E_k\}$

Measurement
Noncontextuality

$$\xi_{\mathrm{M},j}(\lambda) = \xi_{\{E_k\},j}(\lambda)$$

Proof of preparation contextuality

(a preparation noncontextual hidden variable model is impossible)