Title: Cosmic Microwave Background and the Structure of hte Universe

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Abstract:

The cosmic microwave background

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2007

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CITA
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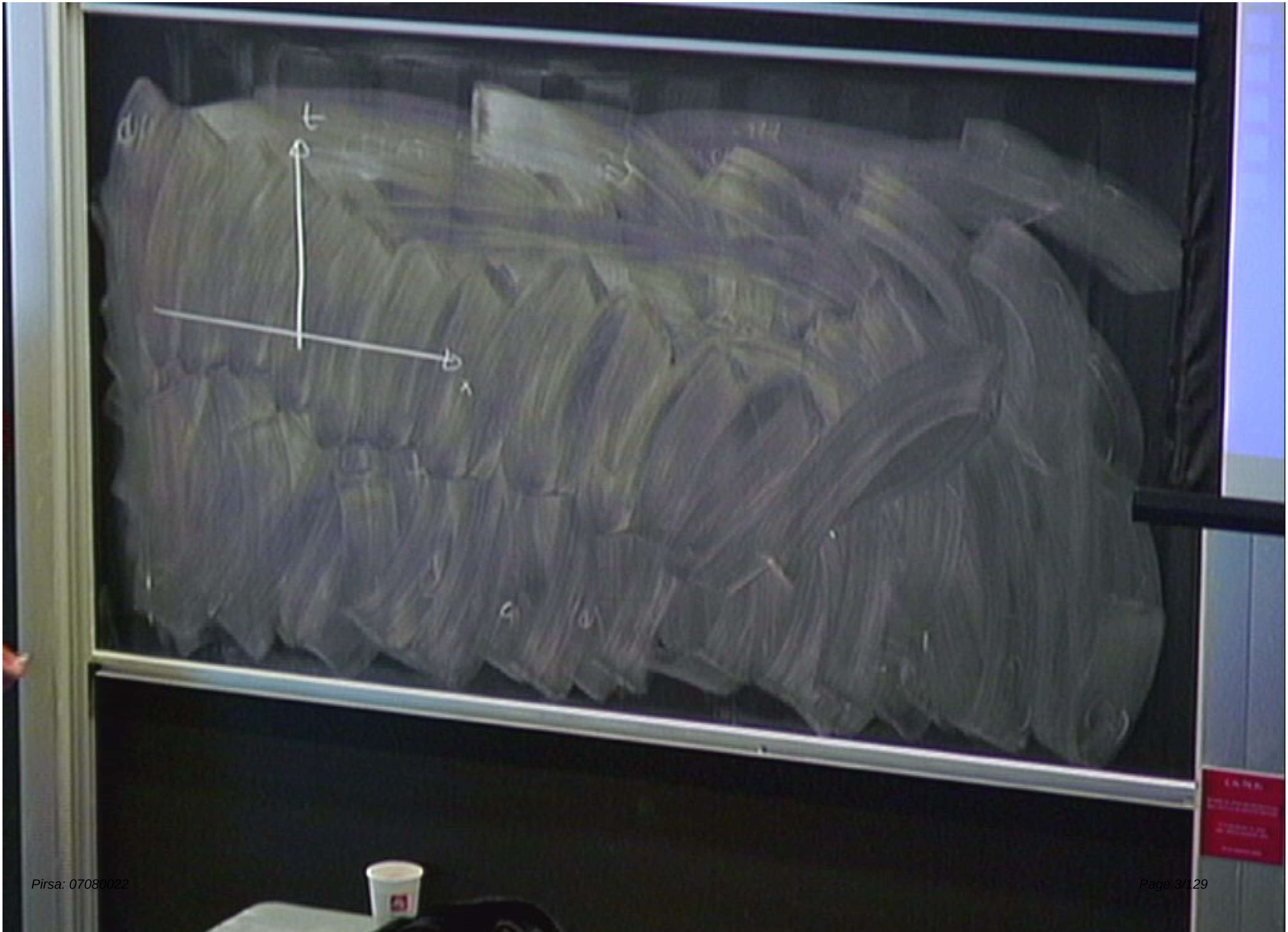
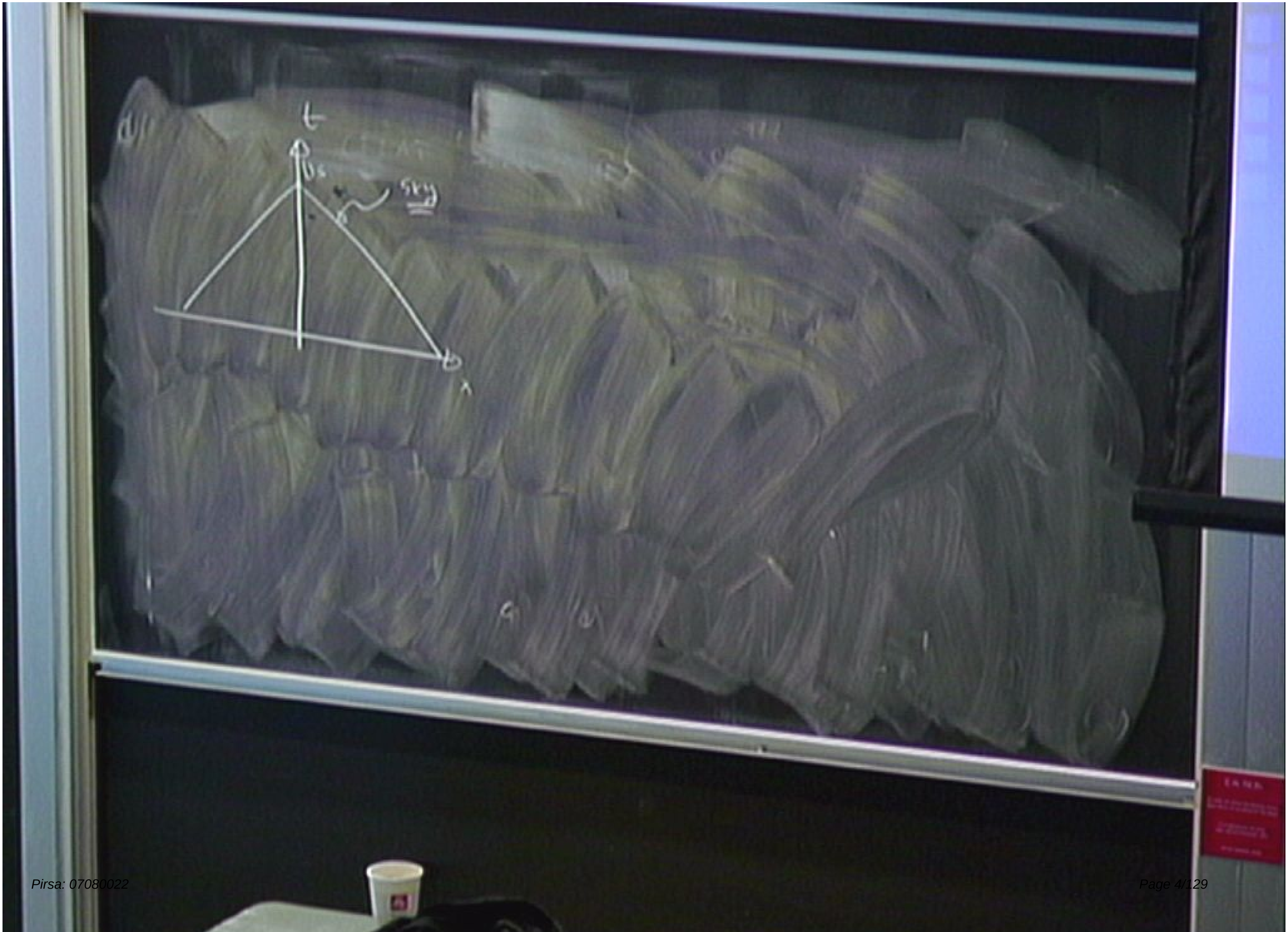
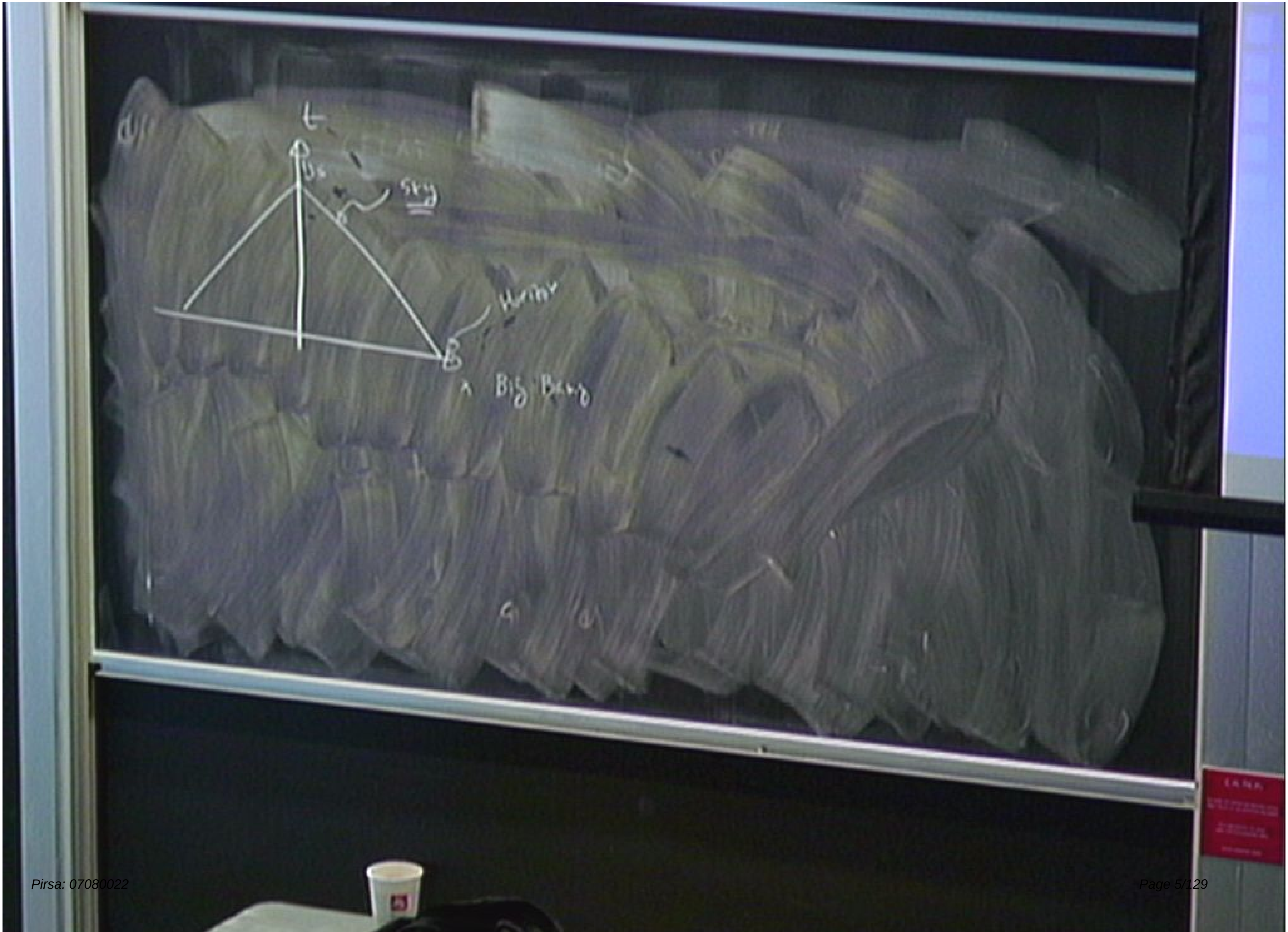
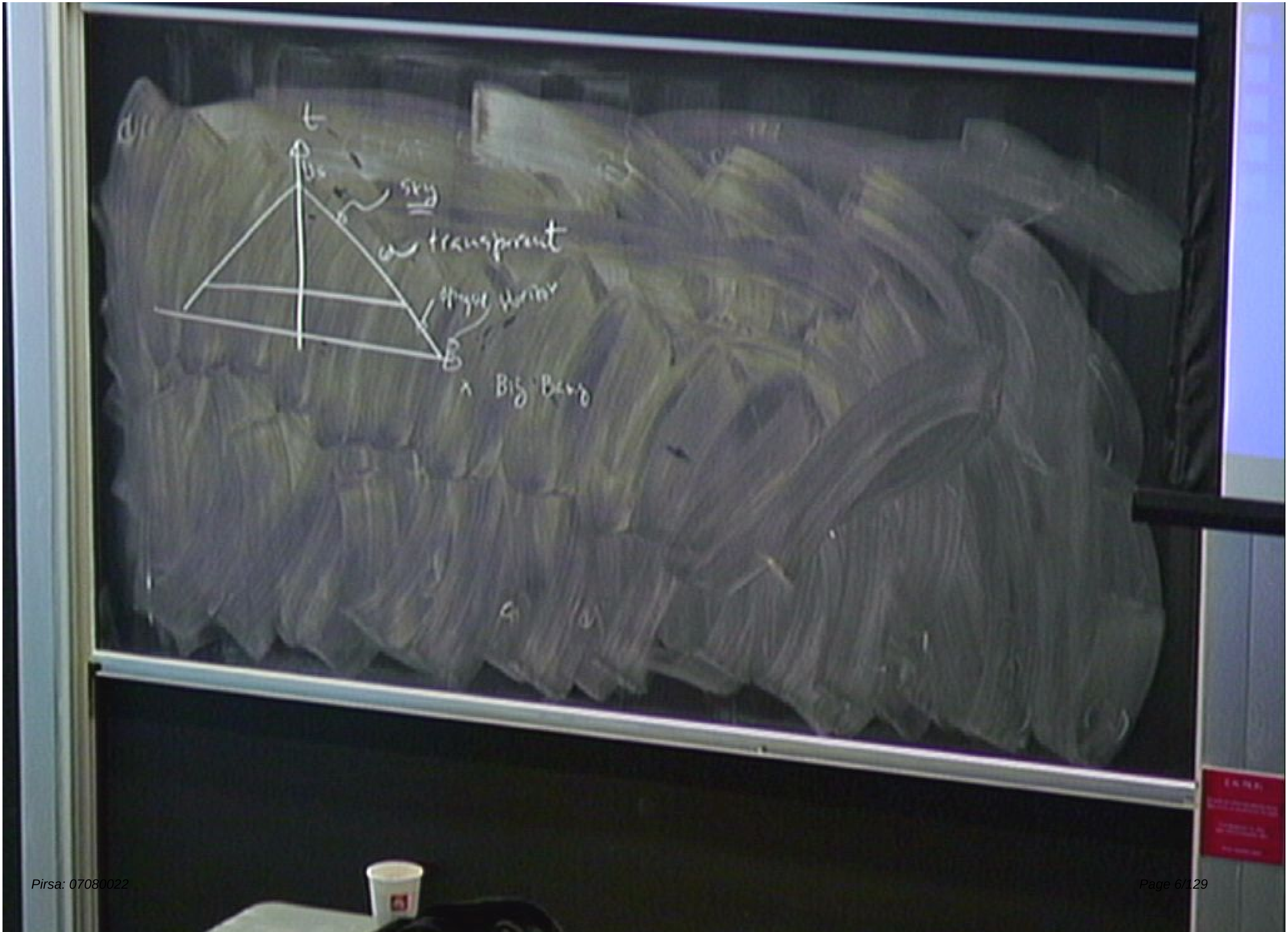
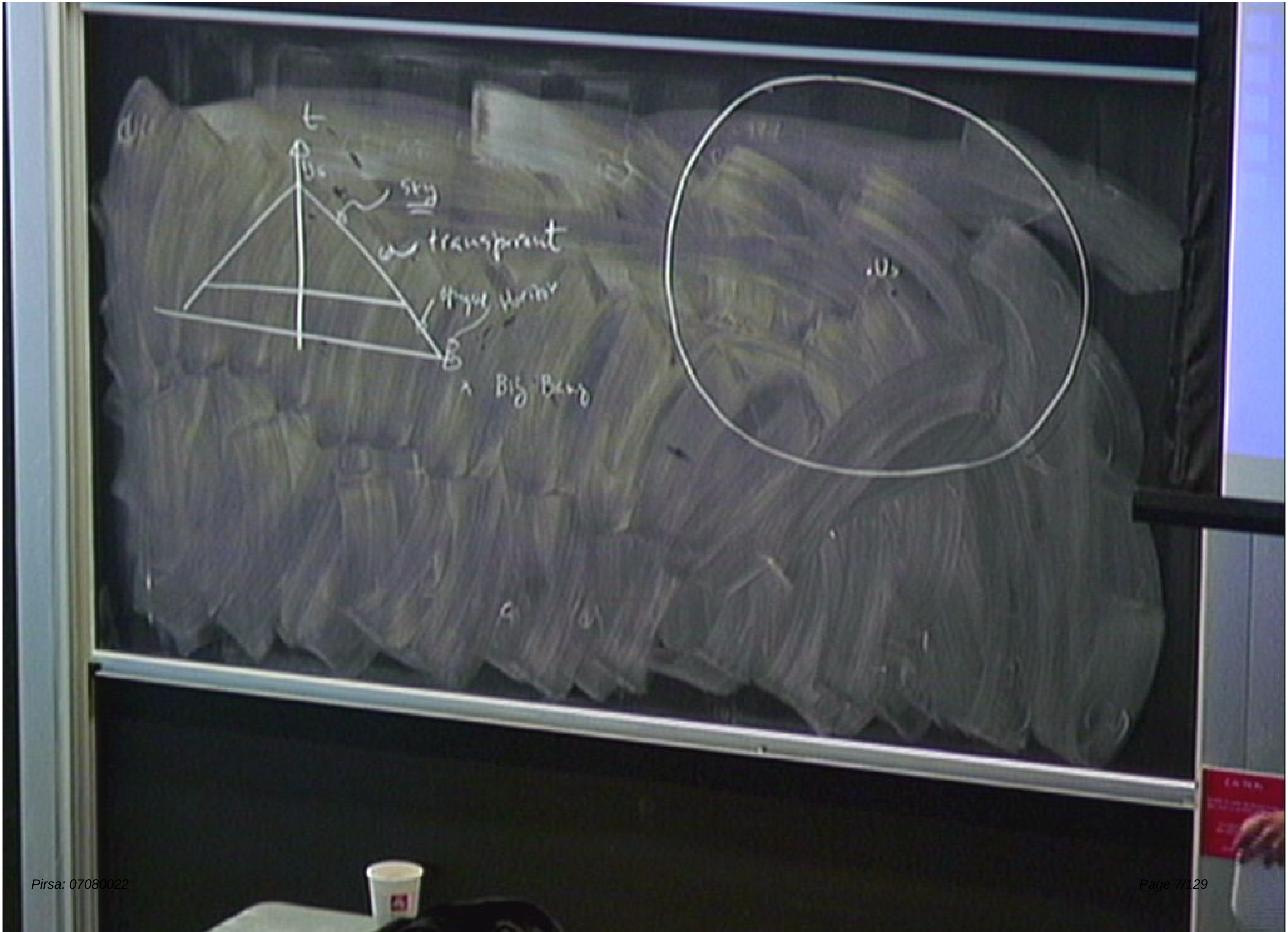
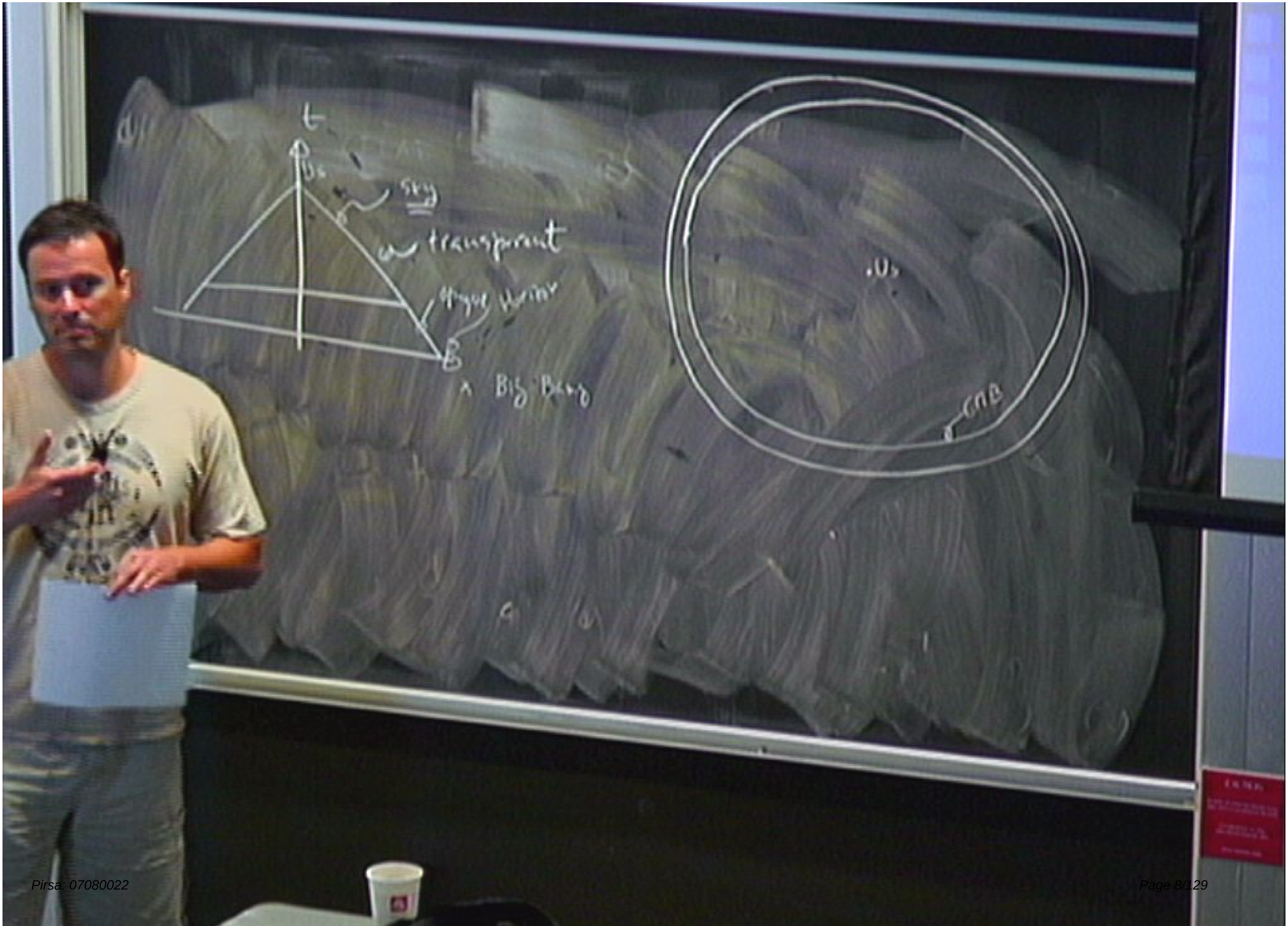

t
x

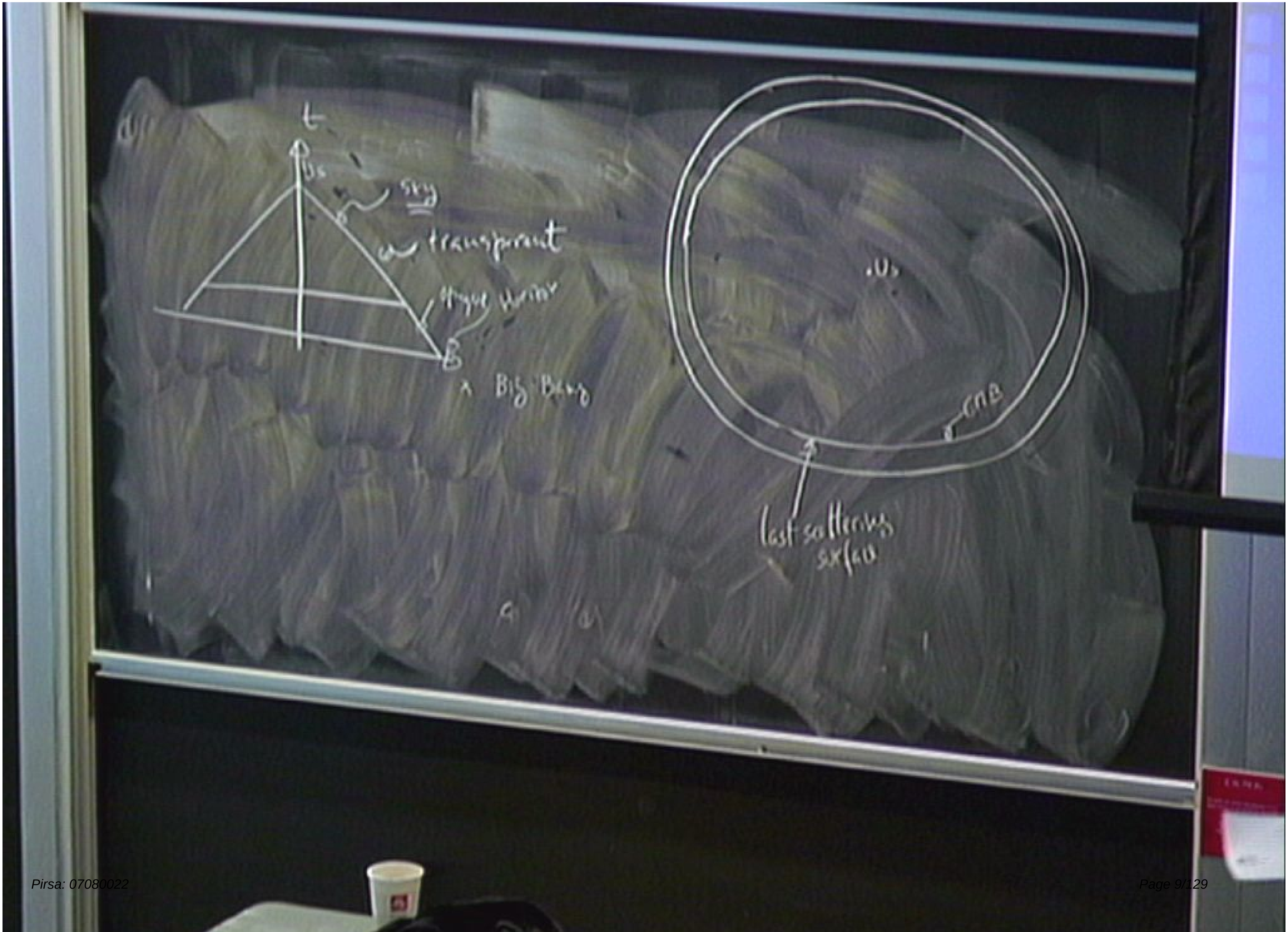

t
sky
x

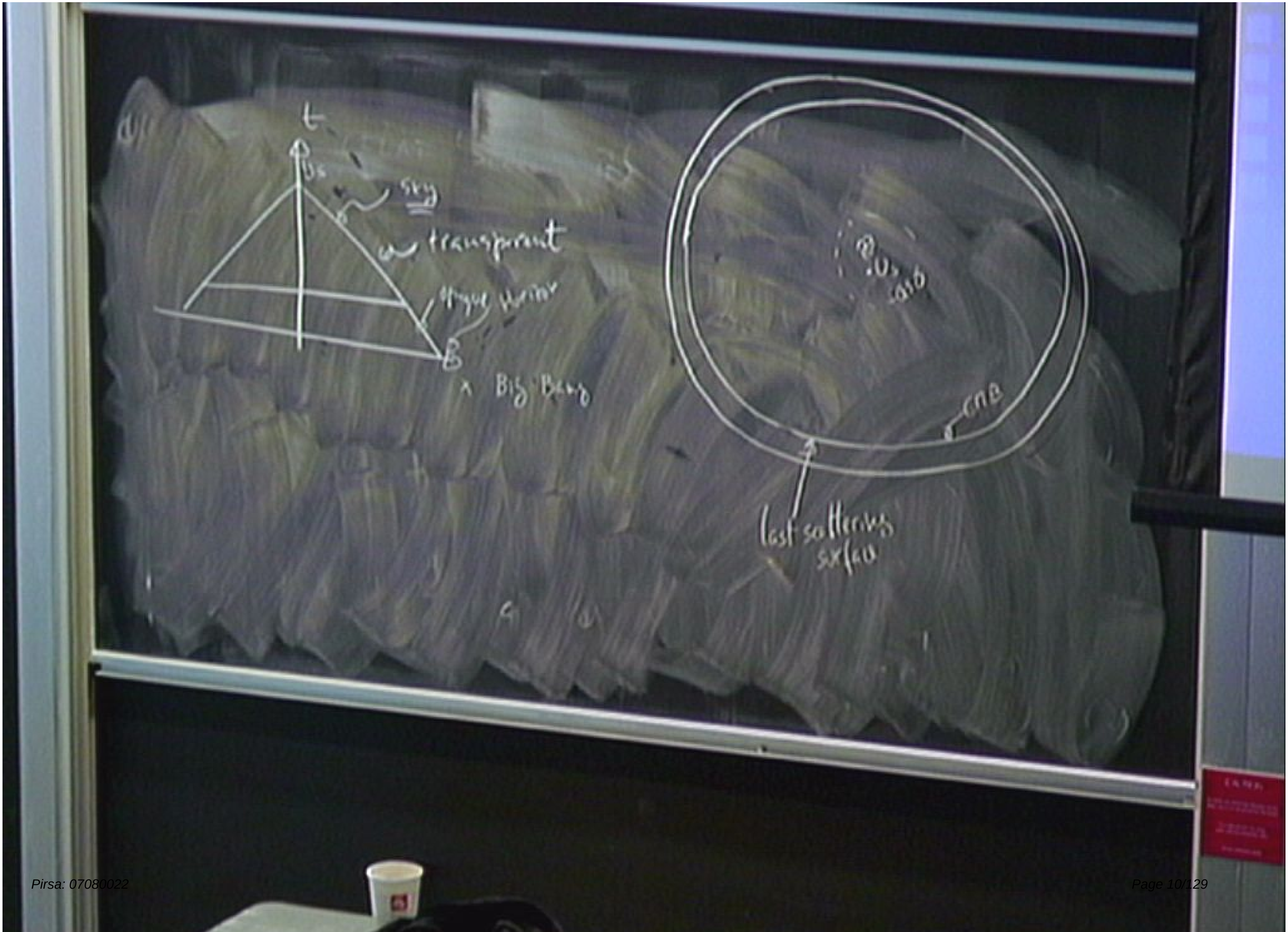

t
Sky
Big Bang

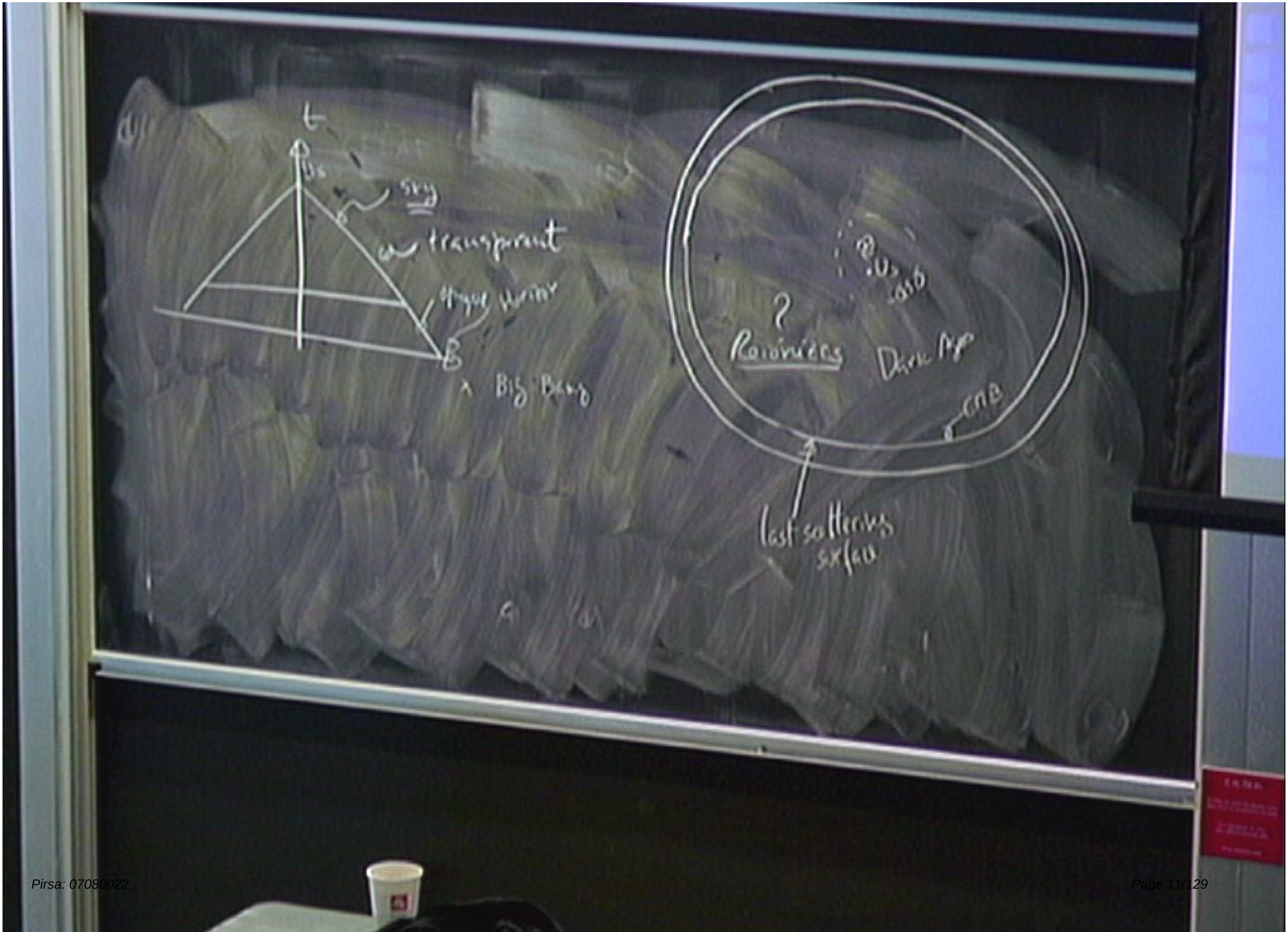

t
Us
sky
transparent
B
Big Bang


t
Sky
transparent
B
Big Bang


t
sky
transparent
B
Big Bang
Us
CMB

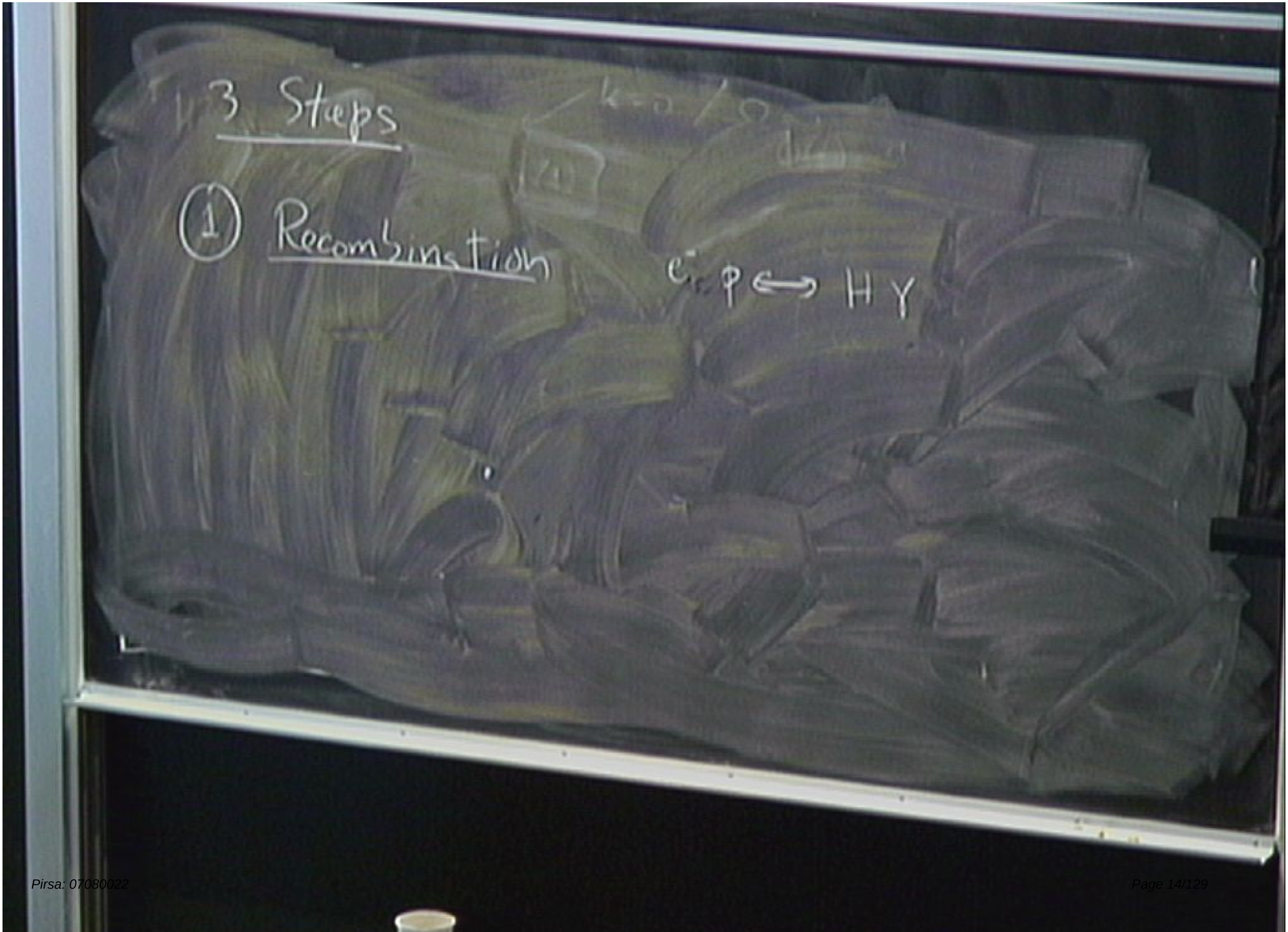

t
sky
transparent
B
Big Bang
CMB
last scattering


t
sky
transparent
B
Big Bang
CMB
last scattering


t
sky
transparent
Big Bang
?
Dark Age
CMB
last scattering


Big Bang
Dark Ages
CMB
last scattering

3 Steps

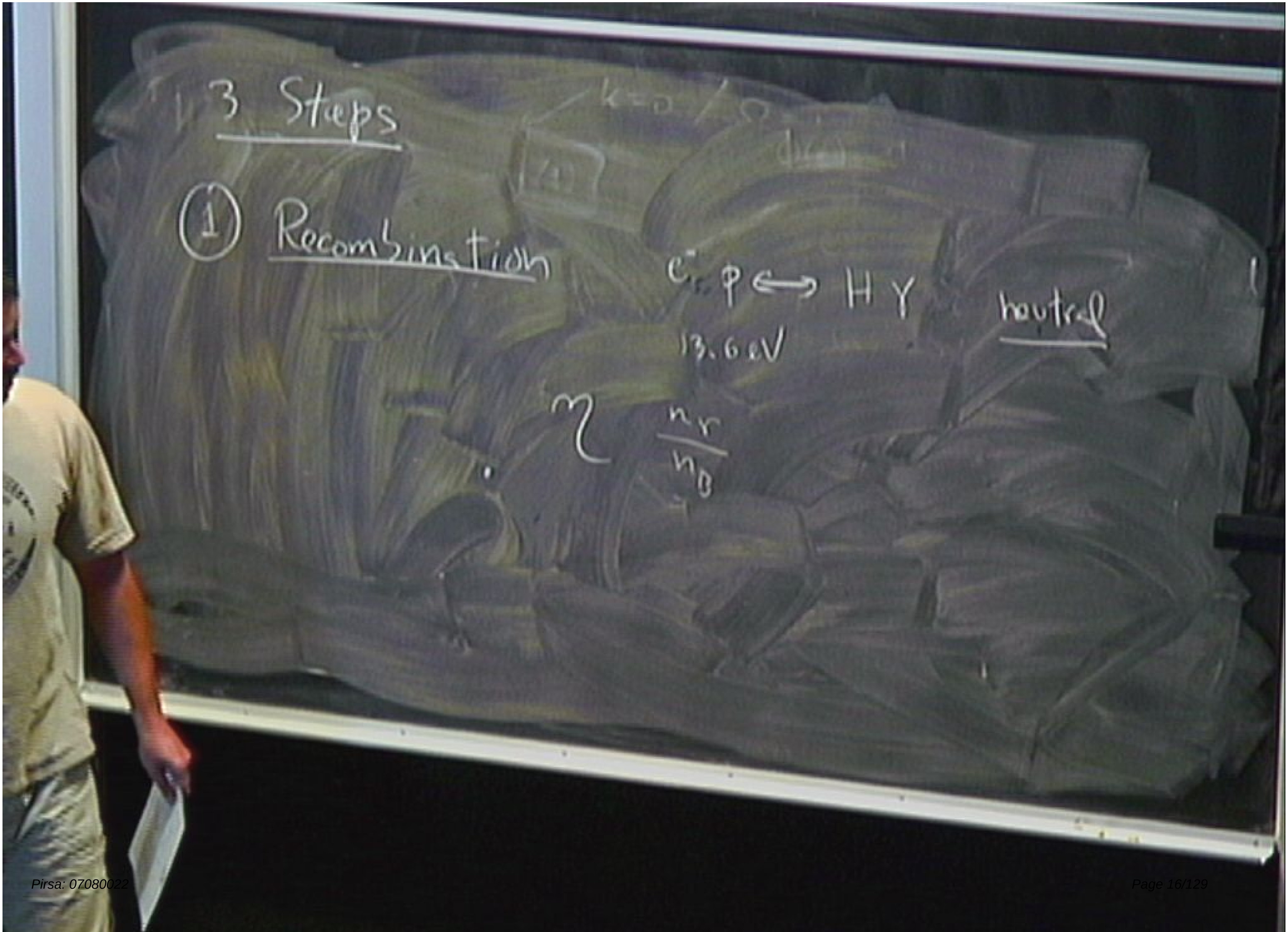

3 Steps
1 Recombination
$e^- p \leftrightarrow H\gamma$

3 Steps

1 Recombination

$e^- \; p \Leftrightarrow H \; \gamma$ neutral

13.6 eV


3 Steps
1 Recombination
e⁻ p ⟺ H γ
neutral
13.6 eV
η ~ nγ / nB

3 Steps

1 Recombination

$e^- \; p \leftrightarrow H \; \gamma$ neutral

13.6 eV

$\eta \sim \frac{n_\gamma}{n_B}$ $10^9 - 10^{10}$

3 Steps

1 Recombination

$e^- \; p \Longleftrightarrow H \; \gamma$ neutral

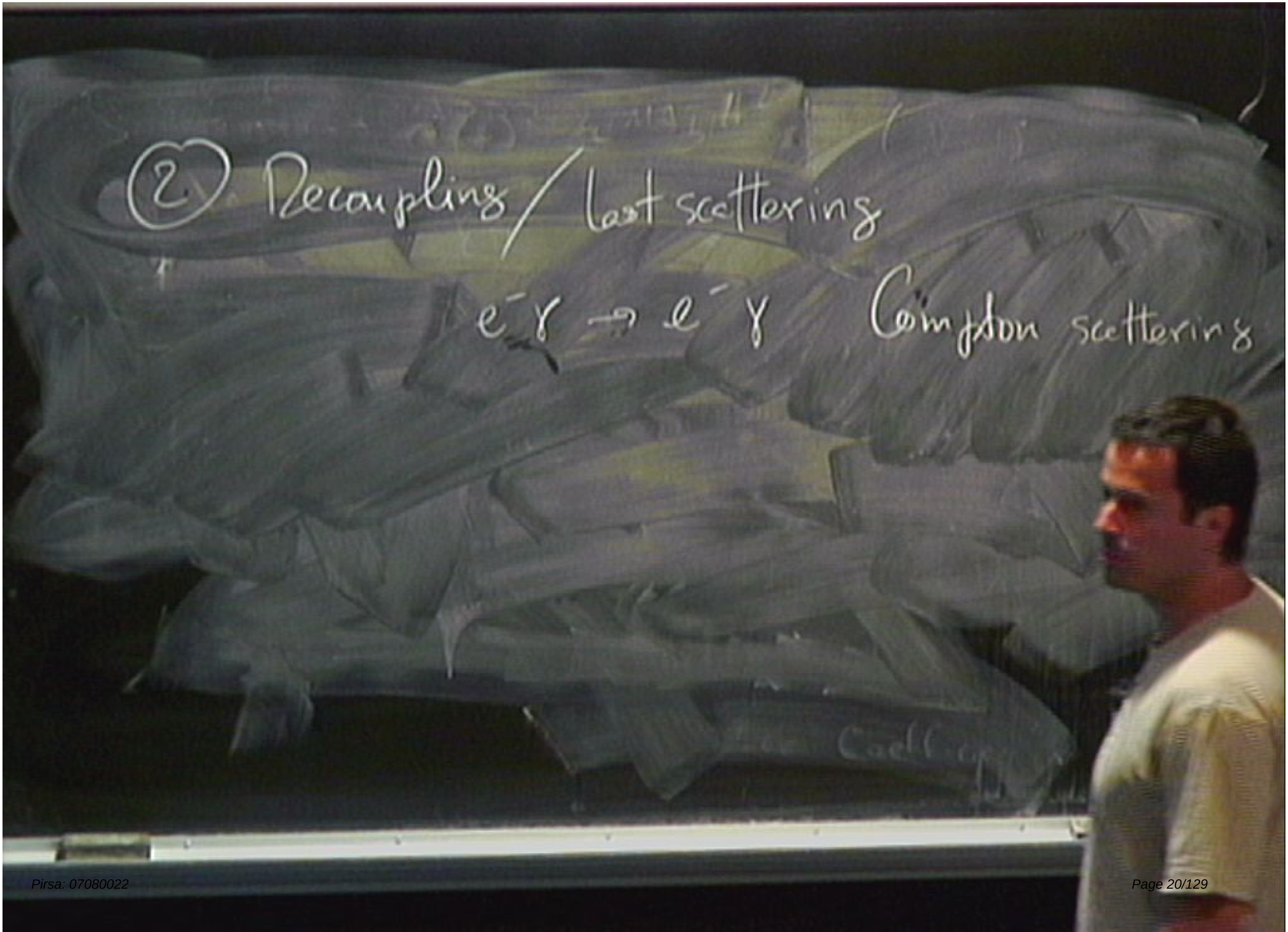
13.6 eV

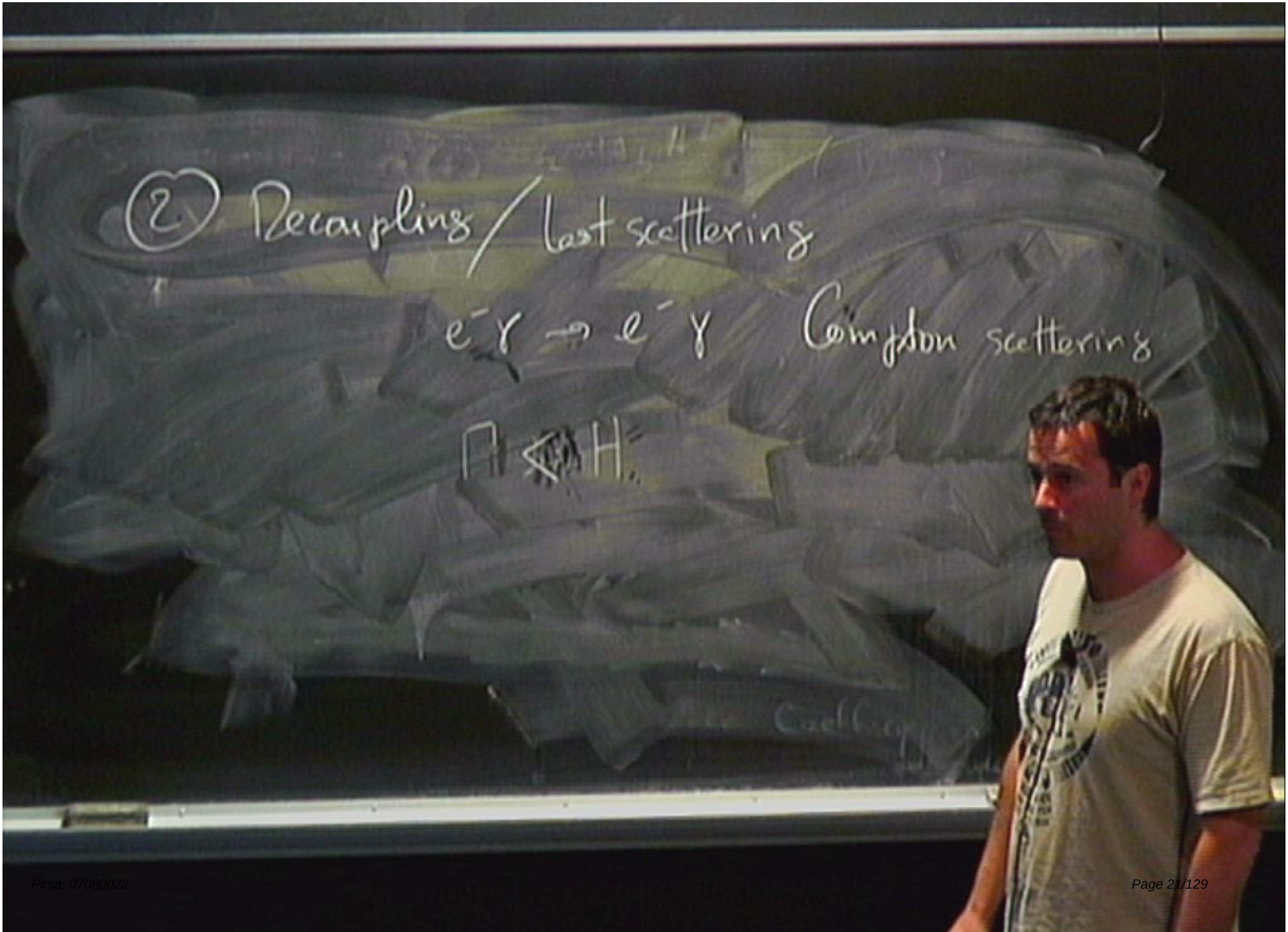
$\eta \sim \frac{n_\gamma}{n_B}$ $10^9 - 10^{10}$

→ Saha Eqn

② Decoupling / last scattering

$e^-\gamma \to e^-\gamma$

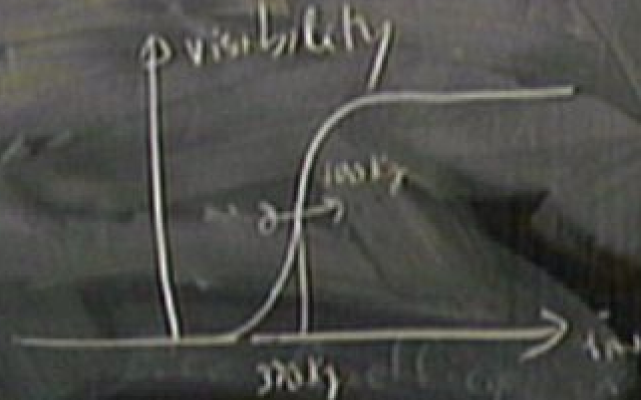

② Decoupling / last scattering
$e^- \gamma \to e^- \gamma$ Compton scattering


② Decoupling / last scattering
e⁻γ → e⁻γ Compton scattering

② Decoupling / last scattering

$e^- \gamma \to e^- \gamma$ Compton scattering

$\Gamma < H \to$ target e^- are removed by (1)



(B) Freeze-out

B Freeze-out

leave residual ionization $\longrightarrow$ $p\,e \leftrightarrow H\,\gamma$ decouples

(B) Freeze-out

leave residual ionization $\longrightarrow$ $p\,e \leftrightarrow H\,\gamma$ decouples

10^{-5}

$\downarrow$

hardly Ionized Univ.

(B) Freeze-out

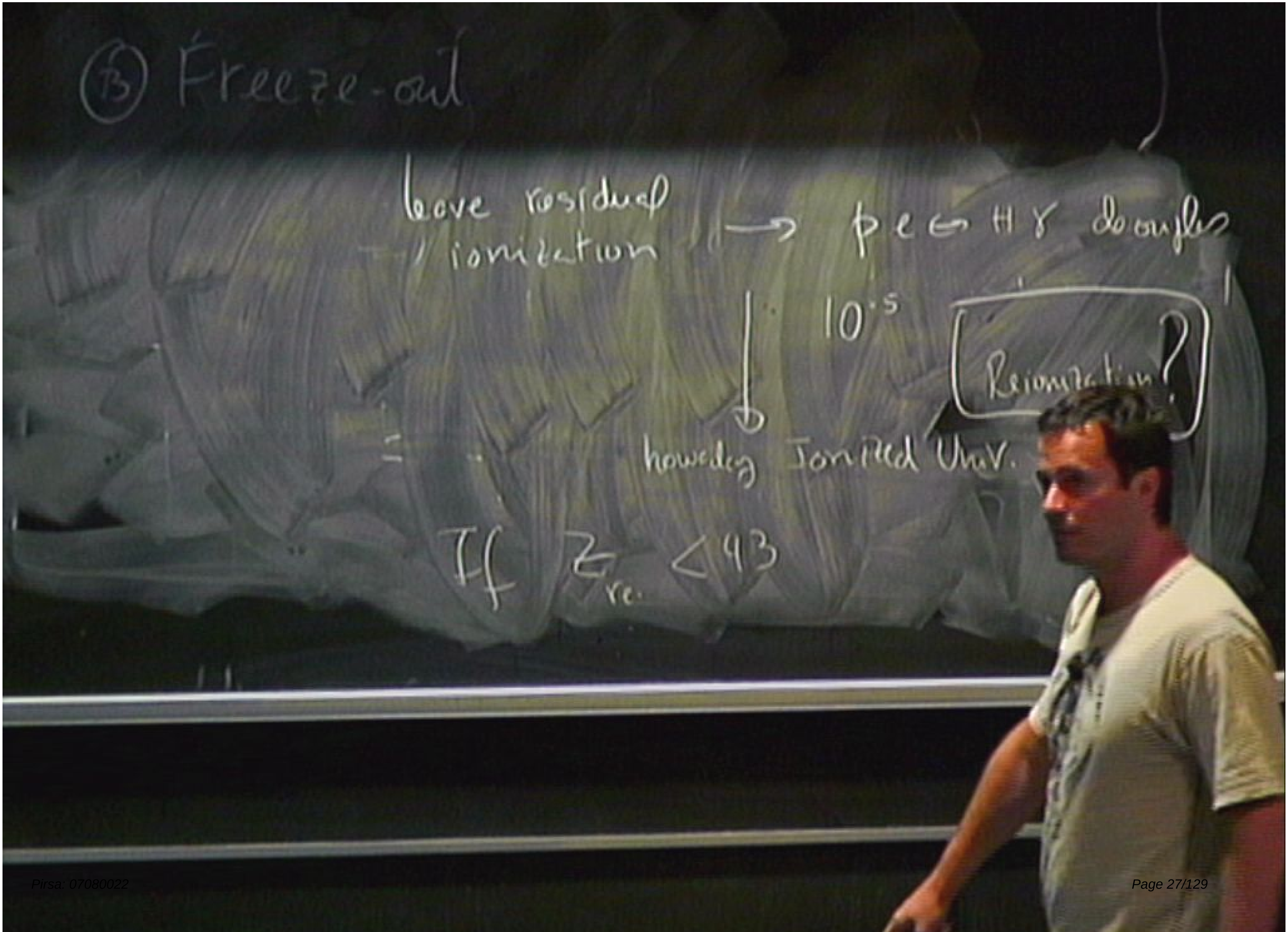
leave residual ionization $\longrightarrow$ $p\,e \leftrightarrow H\,\gamma$ decouples

10^{-5}

$\boxed{\text{Reionization?}}$

$\downarrow$

howedy Ionized Univ.



B Freeze-out
leave residual ionization
p e ↔ H γ decouples
10^5
Reionization?
Ionized Univ.
If z re. < 43

(B) Freeze-out

leave residual ionization $\longrightarrow$ $p\, e \leftrightarrow H\, \gamma$ decouples

$\downarrow$ 10^{-5}

$\boxed{\text{Reionization?}}$

$\Downarrow$

hovering Ionized Univ.

If $z_{re} < 43$ $\Longrightarrow$ No rescattering of γ

(B) Freeze-out

leave residual ionization $\longrightarrow$ $p\, e \leftrightarrow H\, \gamma$ decouples

$\downarrow$ 10^{-5}

$\boxed{\text{Reionization?}}$

however Ionized Univ.

If $z_{re} < 43$ $\Longrightarrow$ No rescattering of γ

(B) Freeze-out

leave residual ionization $\longrightarrow$ $p\,e \leftrightarrow H\,\gamma$ decouples

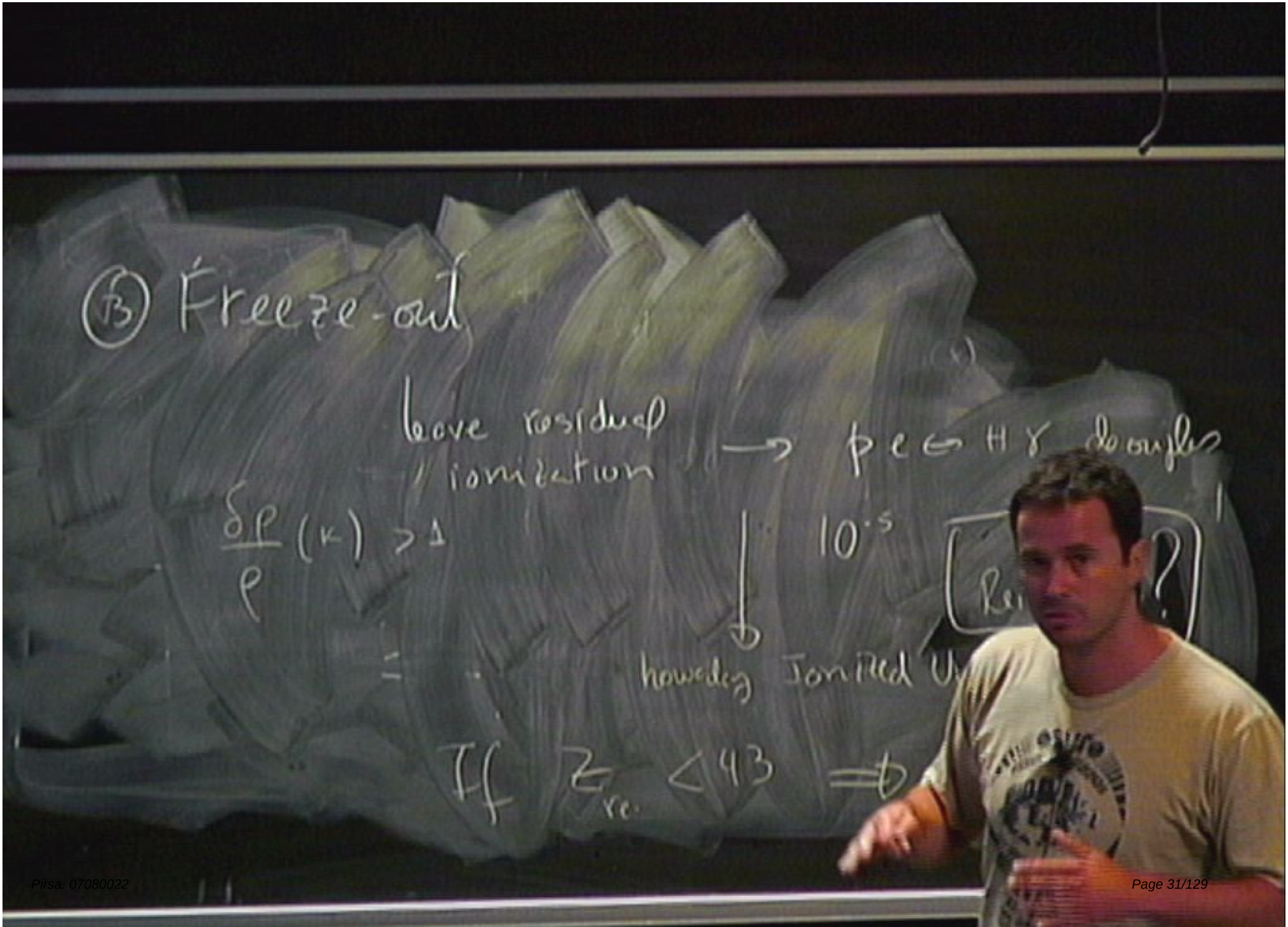
10^{-5}

$\downarrow$

howeling Ionized Univ.

$\boxed{\text{Reionization?}}$

If $z_{re.} < 43 \Longrightarrow$ No rescattering of γ


Freeze-out
leave residual
ionization
decouple
10^-5
however Ionized
If z_re < 43

(B) Freeze-out

leave residual
ionization → $p\, e \leftrightarrow H\, \gamma$ decouples

$\frac{\delta\rho}{\rho}(k) > 1$

$a \propto t^{2/3}$

$T \propto \frac{1}{a}$

10^{-5}

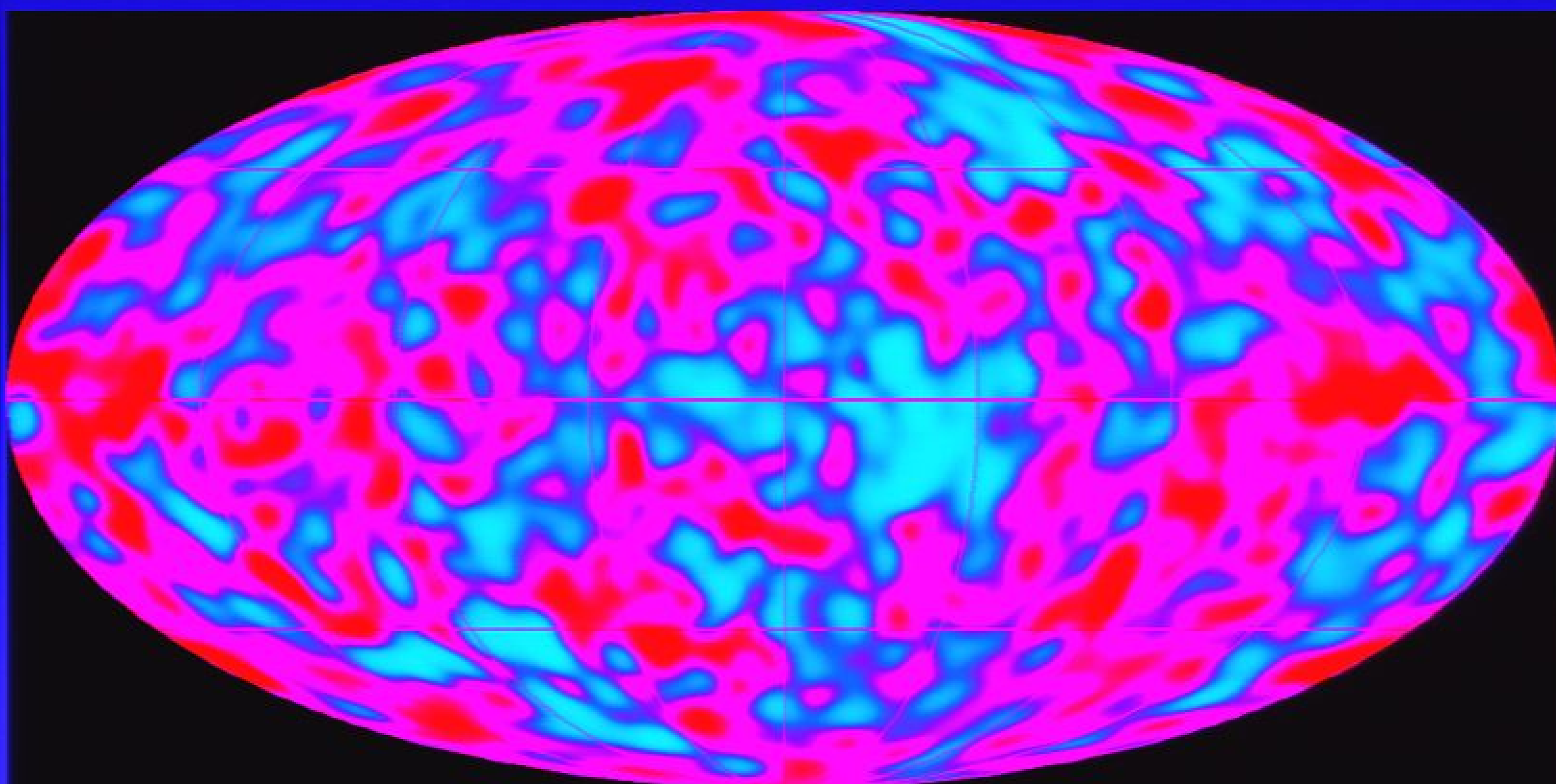
$\boxed{\text{Reionization?}}$

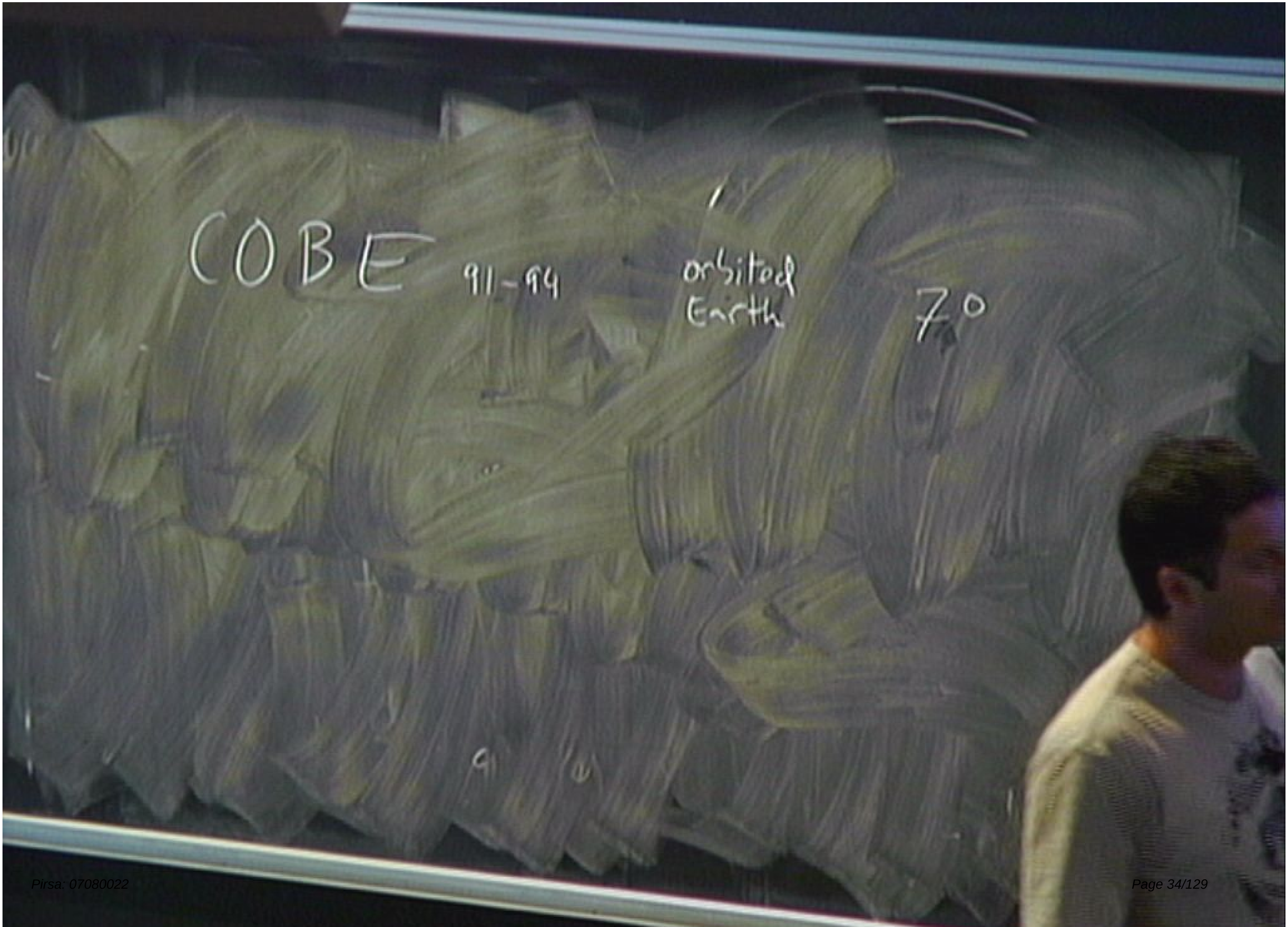
↓

howday Ionized Univ.

If $z_{re.} < 43 \Rightarrow$ No rescattering of γ

The COBE-DMR map

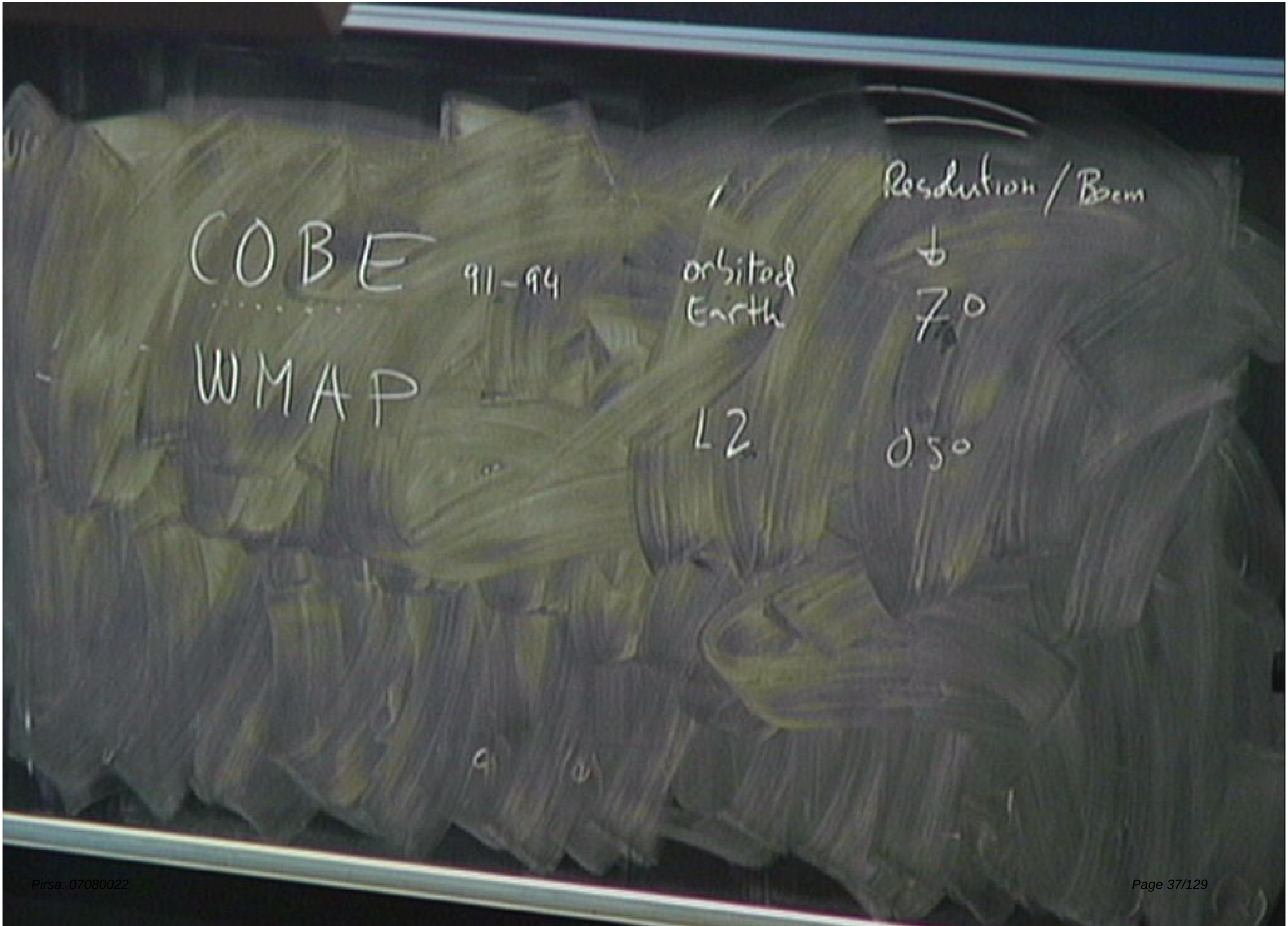


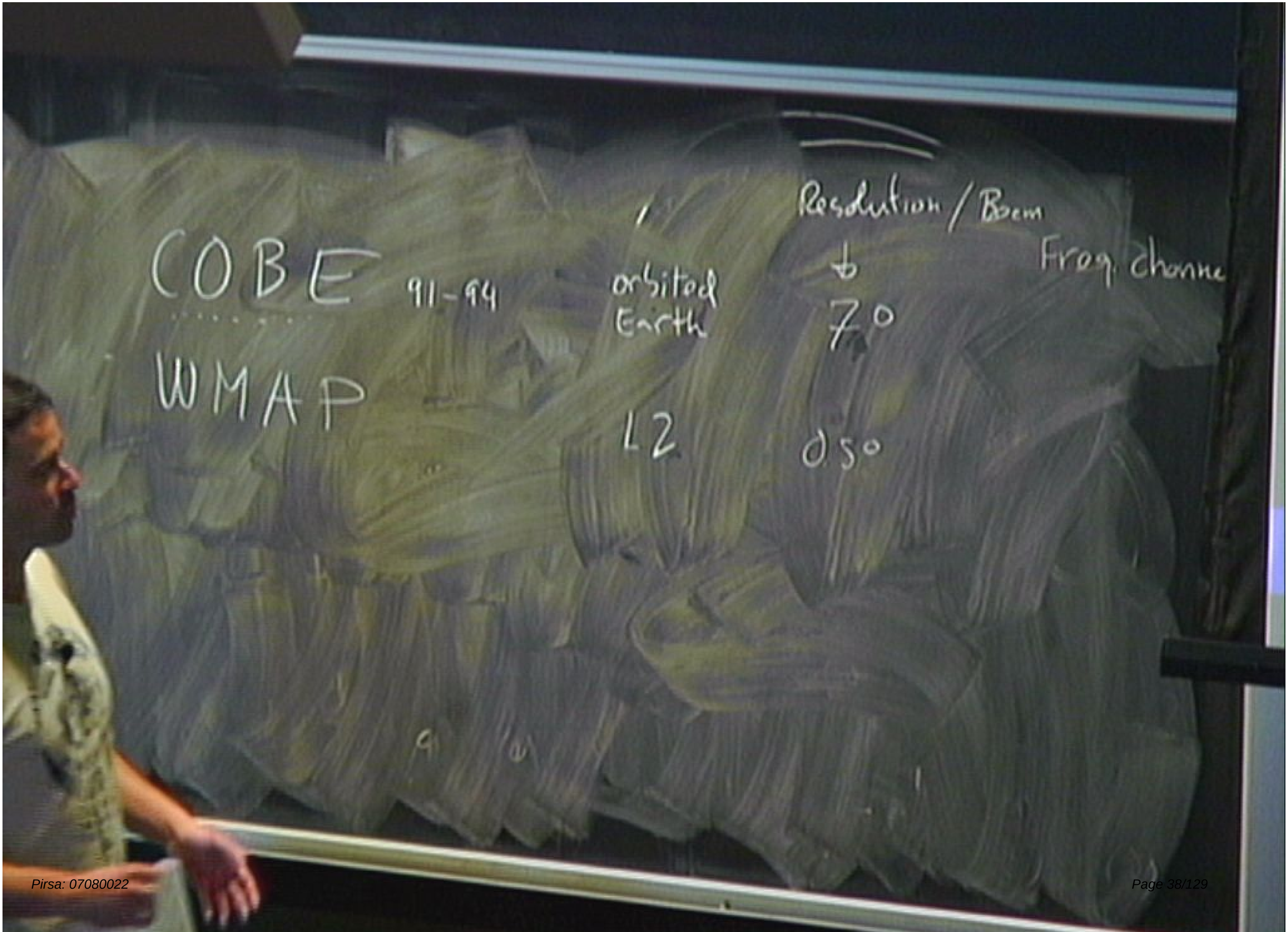

COBE
91-94
orbited
Earth
7°

COBE 91-94 orbited Earth 7°

WMAP L2

COBE	91-94	orbited Earth	7°
WMAP		L2	0.5°


Resolution / Beam
COBE
91-94
orbited Earth
7°
WMAP
L2
0.5°

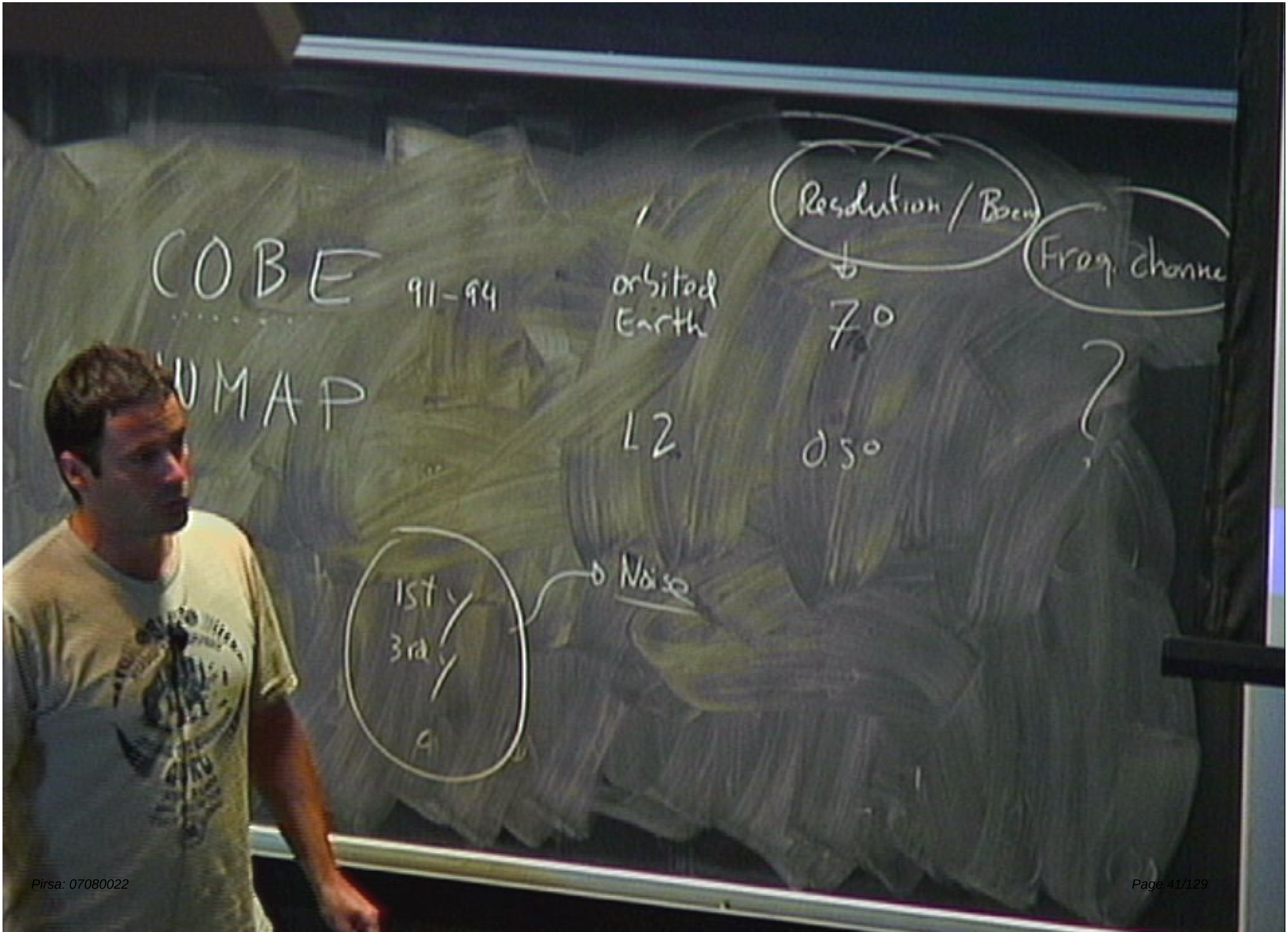

Resolution / Beam
Freq. Channe
COBE
91-94
orbited
Earth
7°
WMAP
L2
0.5°

			Resolution / Beam θ	Freq. Channe
COBE	91-94	orbited Earth	7°	?
WMAP		L2	0.5°	

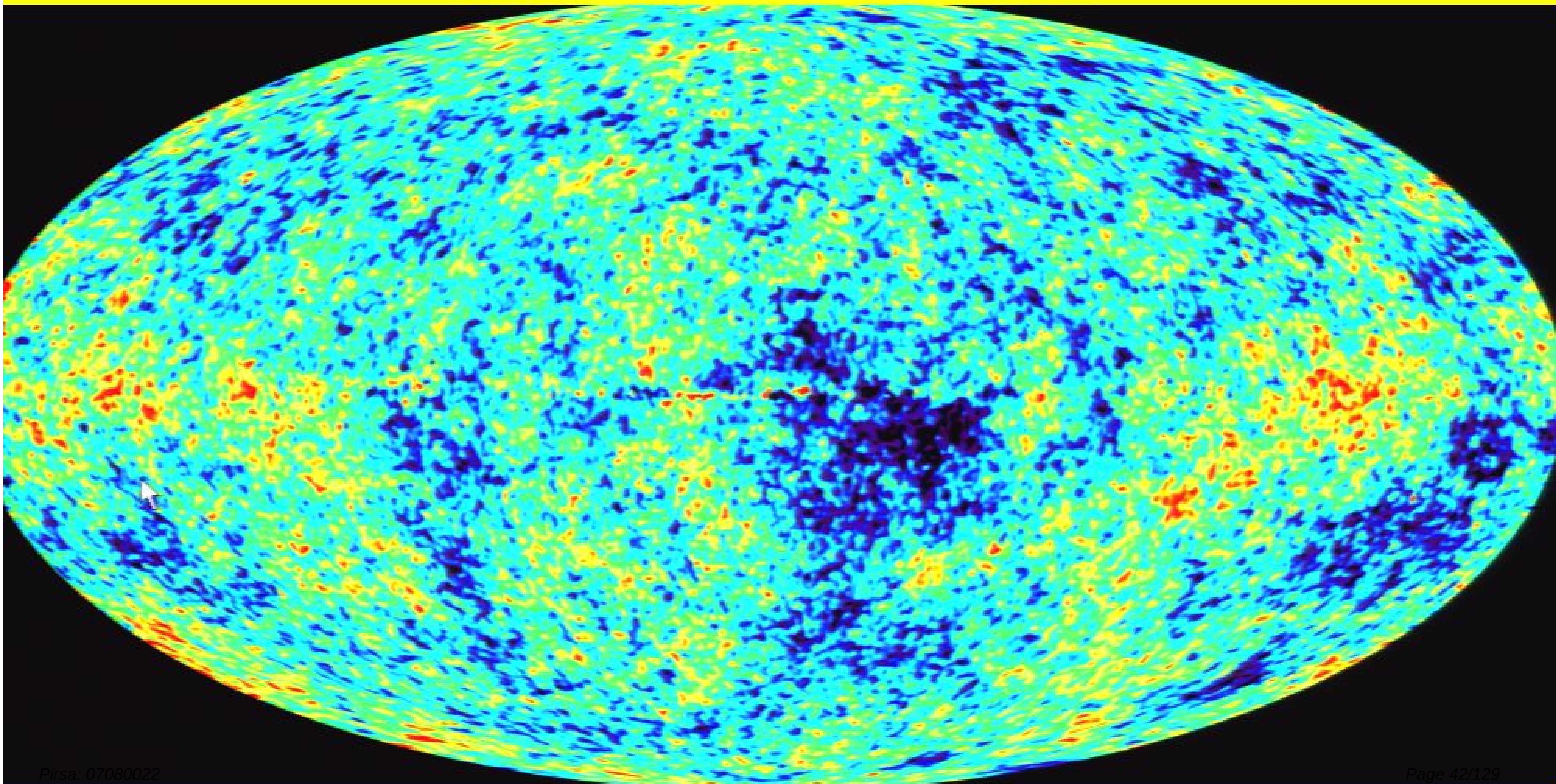
			Resolution / Beam	Freq. Channe
COBE	91-94	orbited Earth	7°	
WMAP		L2	0.5°	?

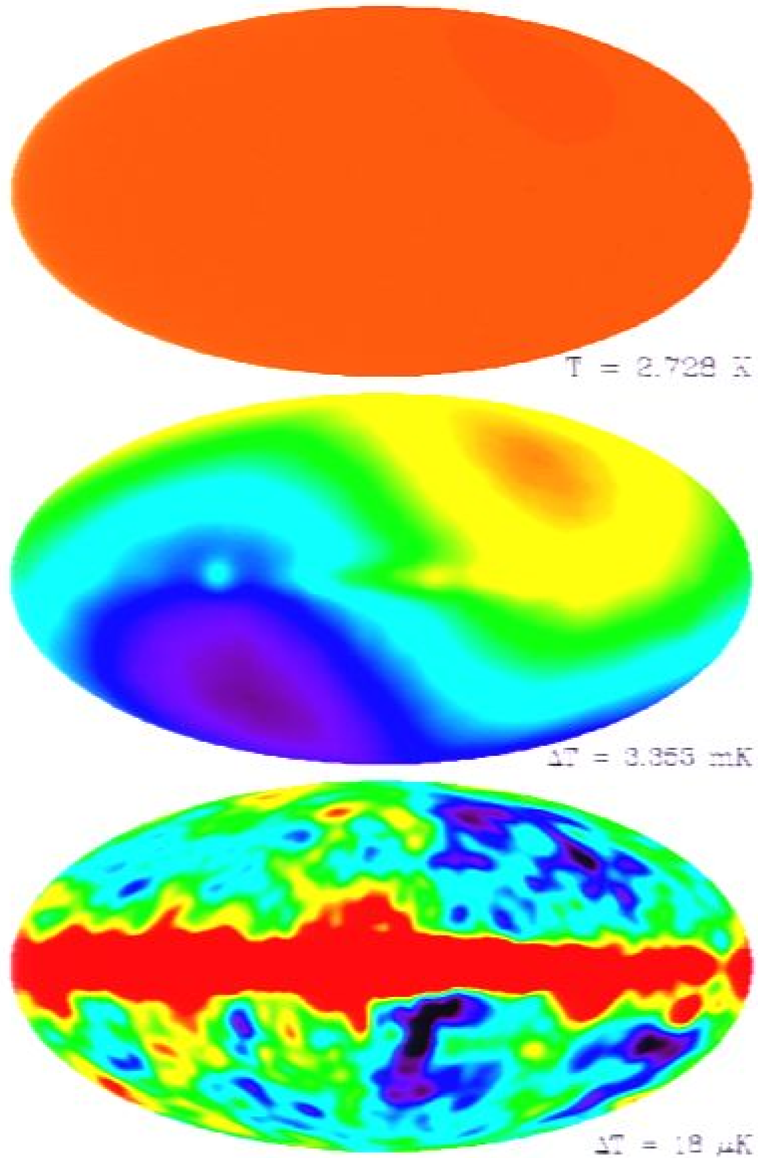
1st

3rd


COBE
91-94
orbited Earth
7°
MAP
L2
0.5°
Resolution / Beam
Freq. Channe
?
1st
3rd
Noise

The WMAP temperature map




T = 2.728 K
ΔT = 3.353 mK
ΔT = 18 μK

Resolution / Beam

Freq. channels

COBE	91-94	orbited Earth	7°
WMAP		L2	0.5°

?

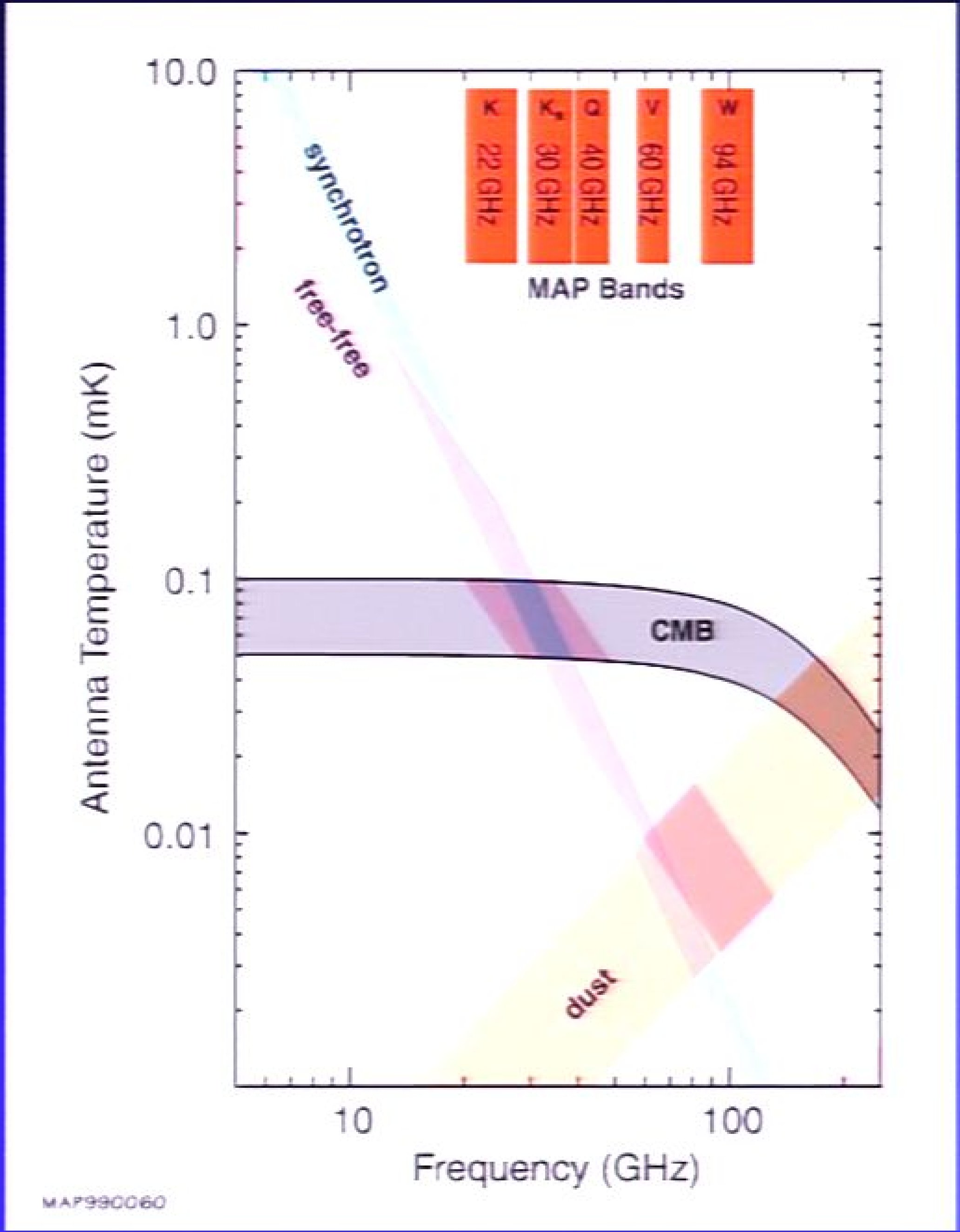
1st y

3rd y

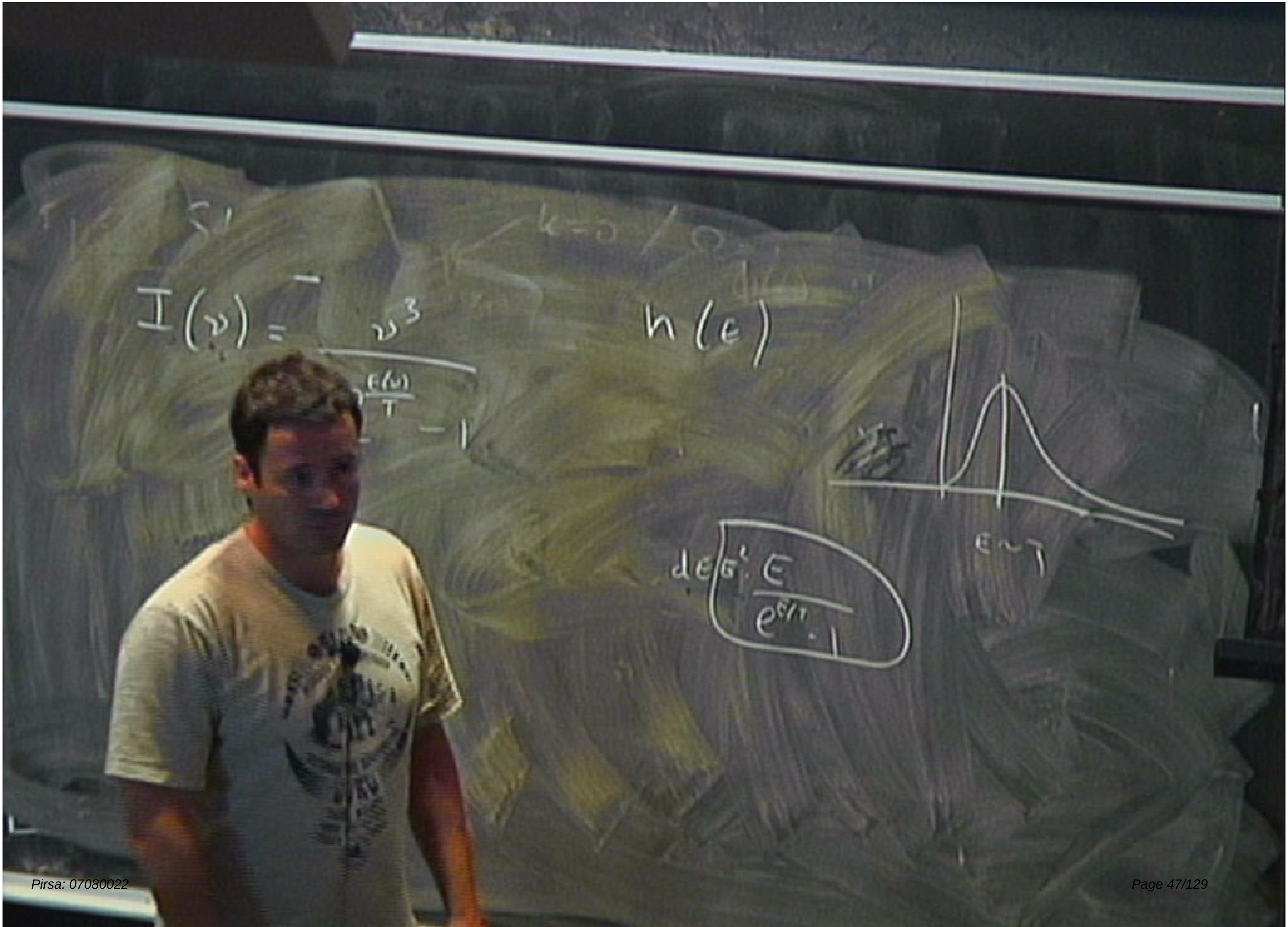
4

Noise $\sigma_N^2 = \frac{s^2}{t}$

FOREGRO


10.0
1.0
0.1
0.01
Antenna Temperature (mK)
10
100
Frequency (GHz)
K
22 GHz
Ka
30 GHz
Q
40 GHz
V
60 GHz
W
94 GHz
MAP Bands
synchrotron
free-free
CMB
dust
MAP990060

$$I(\nu) = \frac{\nu^3}{e^{\frac{E(\nu)}{T}} - 1}$$



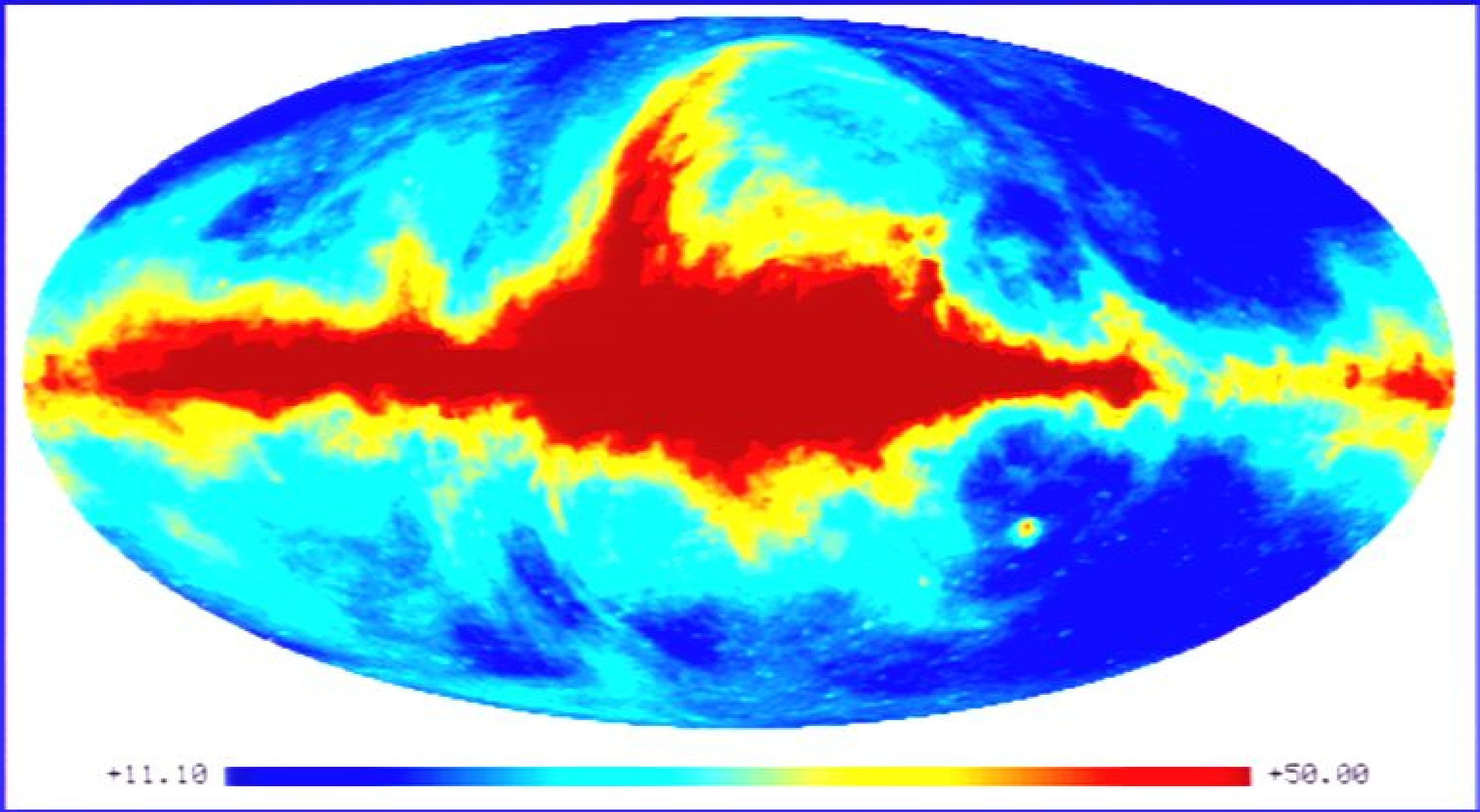
I(ν) = ν³ / (e^{E(ν)/T} − 1)
n(ε)
dε ε² ε / (e^{ε/T} − 1)
ε~T

$$I(\nu) = \frac{\nu^3}{e^{\frac{E(\nu)}{T}} - 1}$$

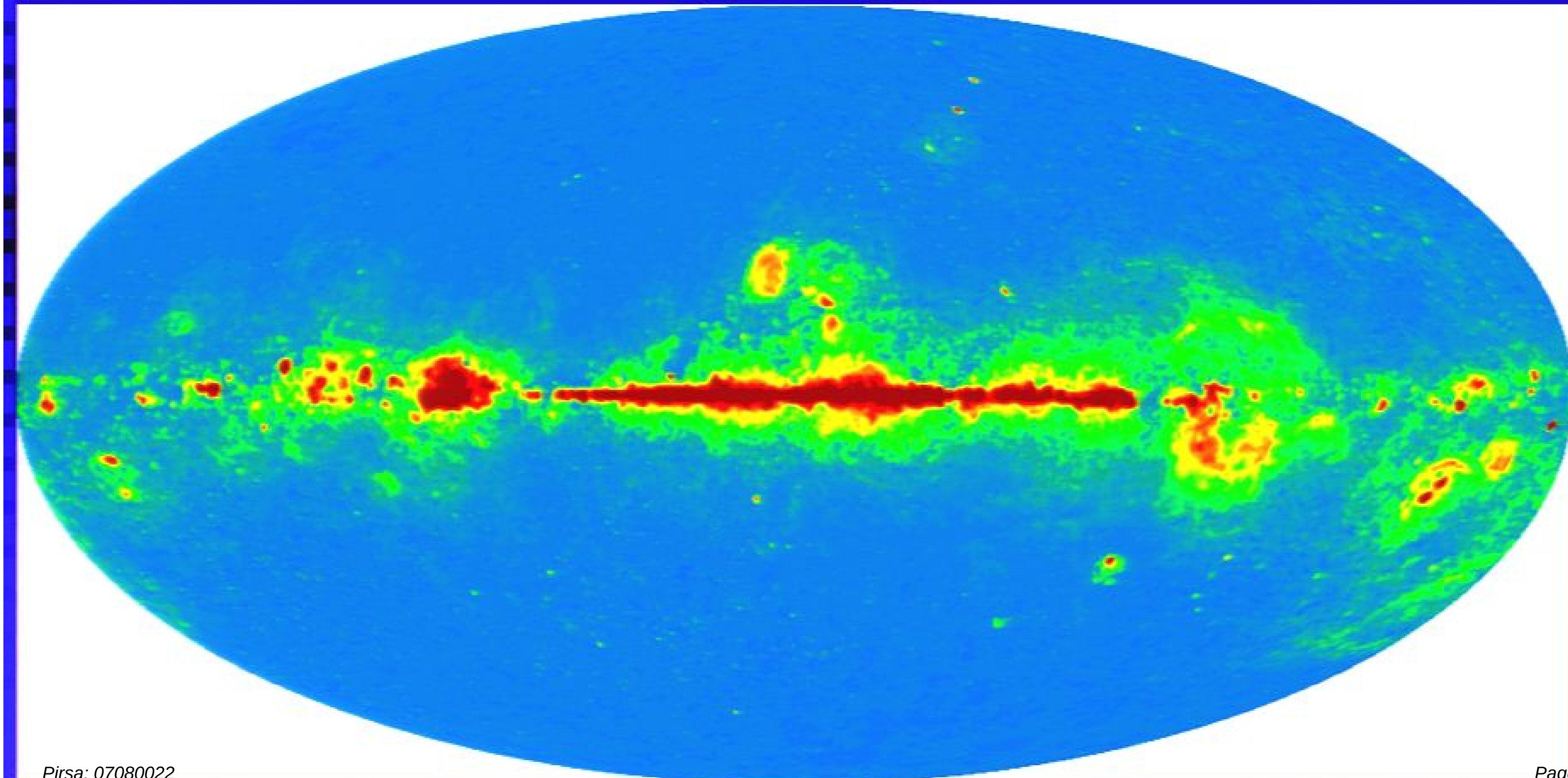
$$n(\epsilon)$$

$$d\epsilon\,\epsilon^2\,\frac{\epsilon}{e^{\epsilon/T} - 1}$$

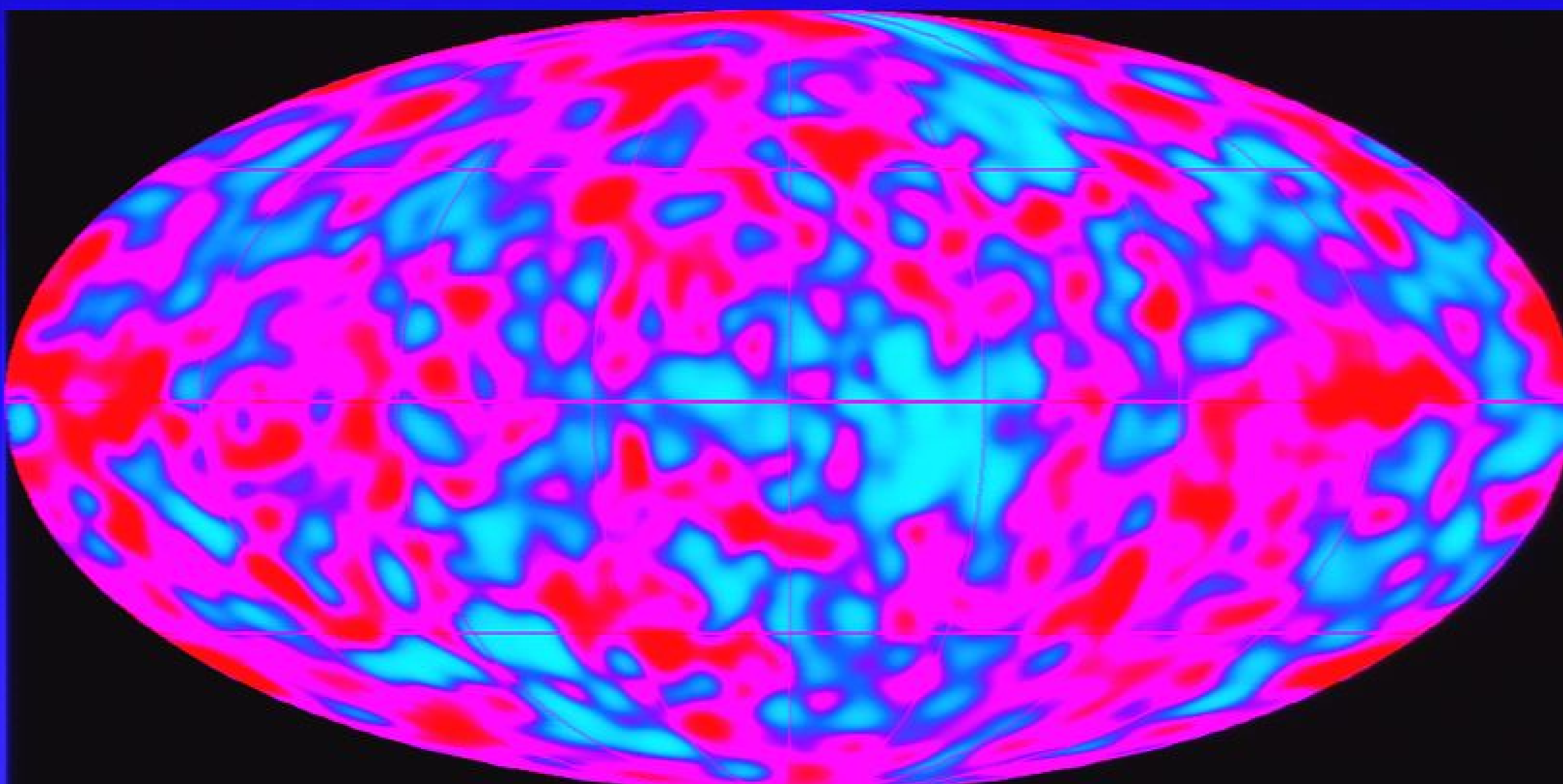
Synchrotron foreground



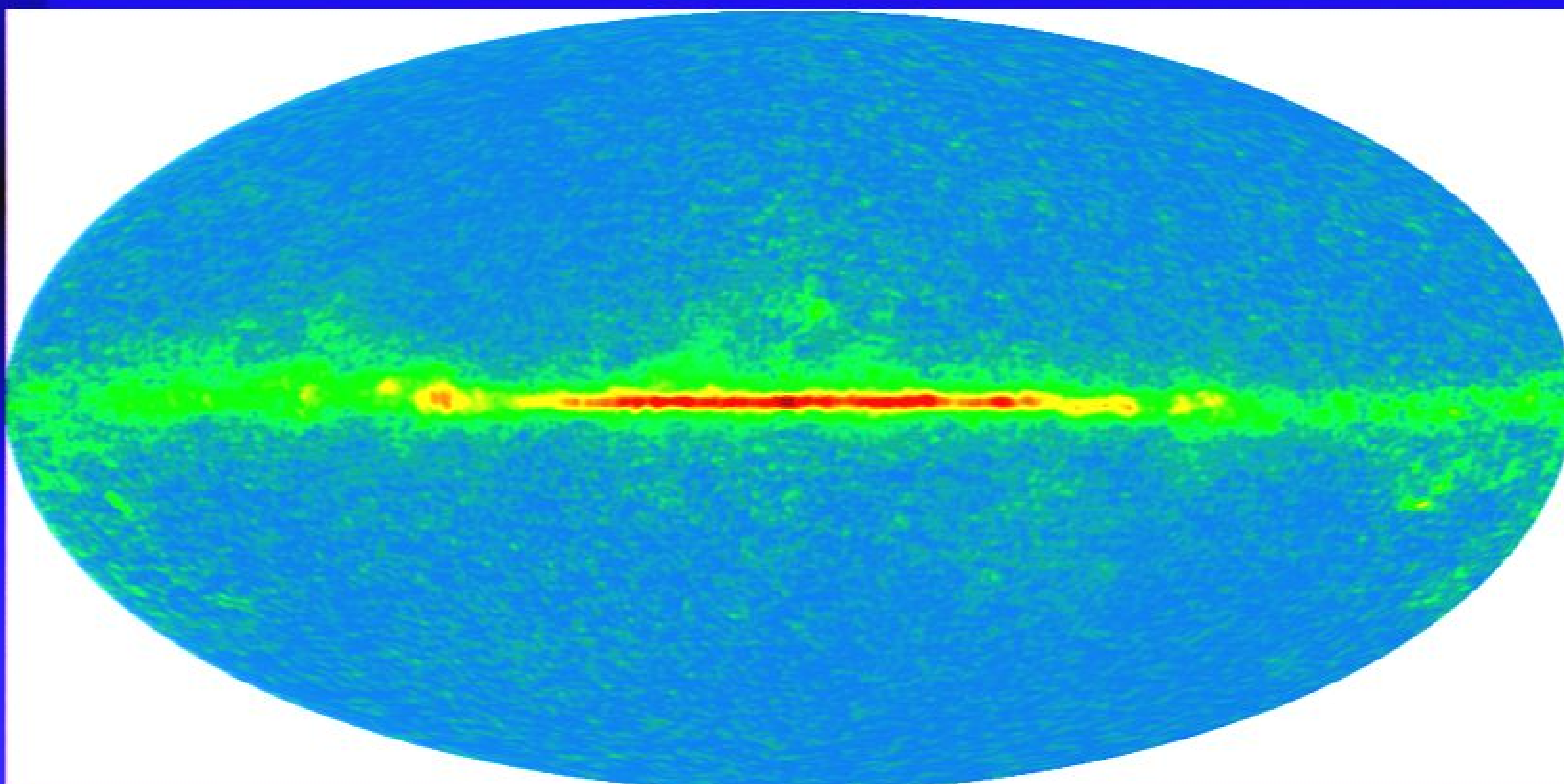
Free-free foreground



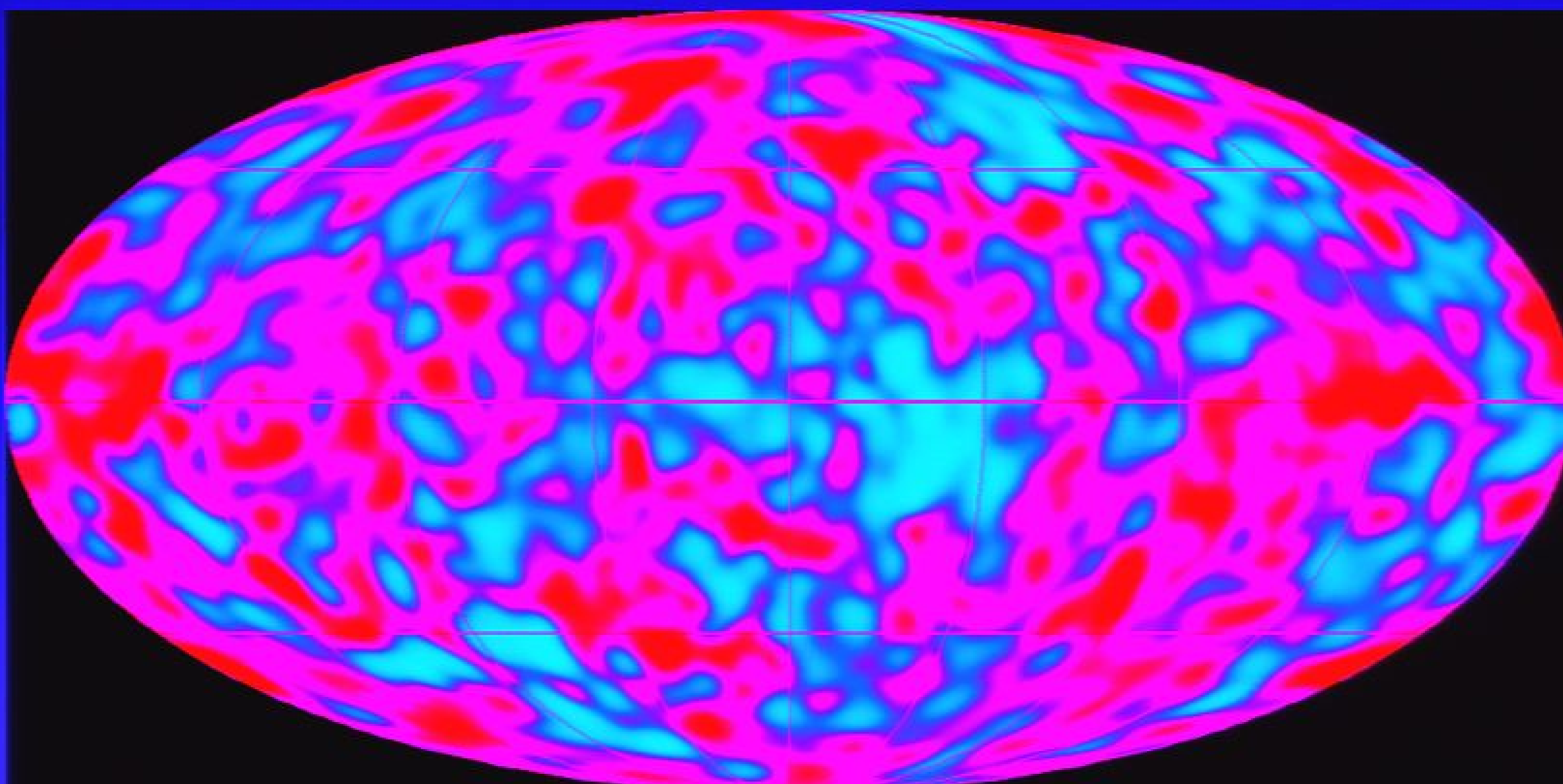
The COBE-DMR map

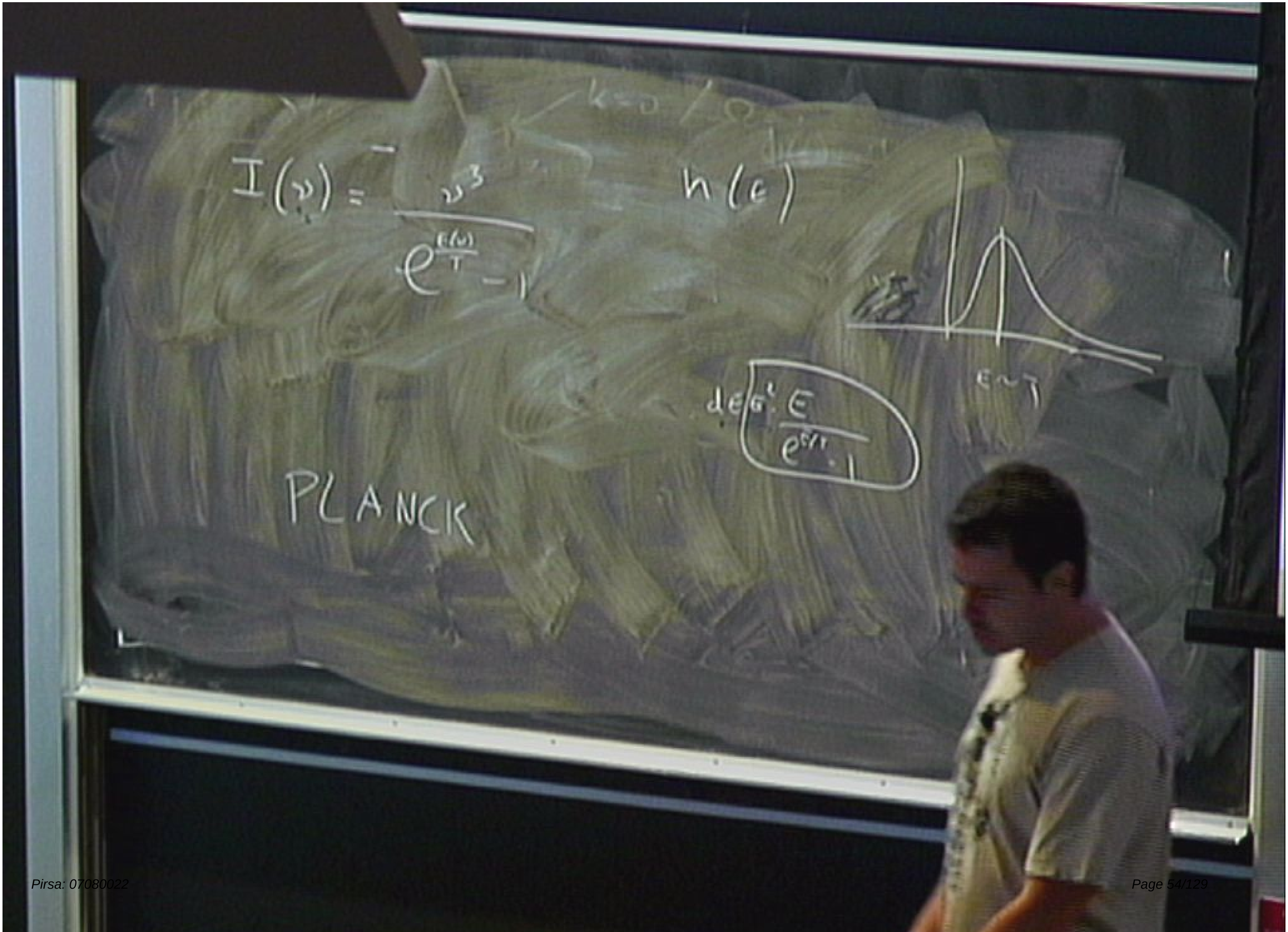


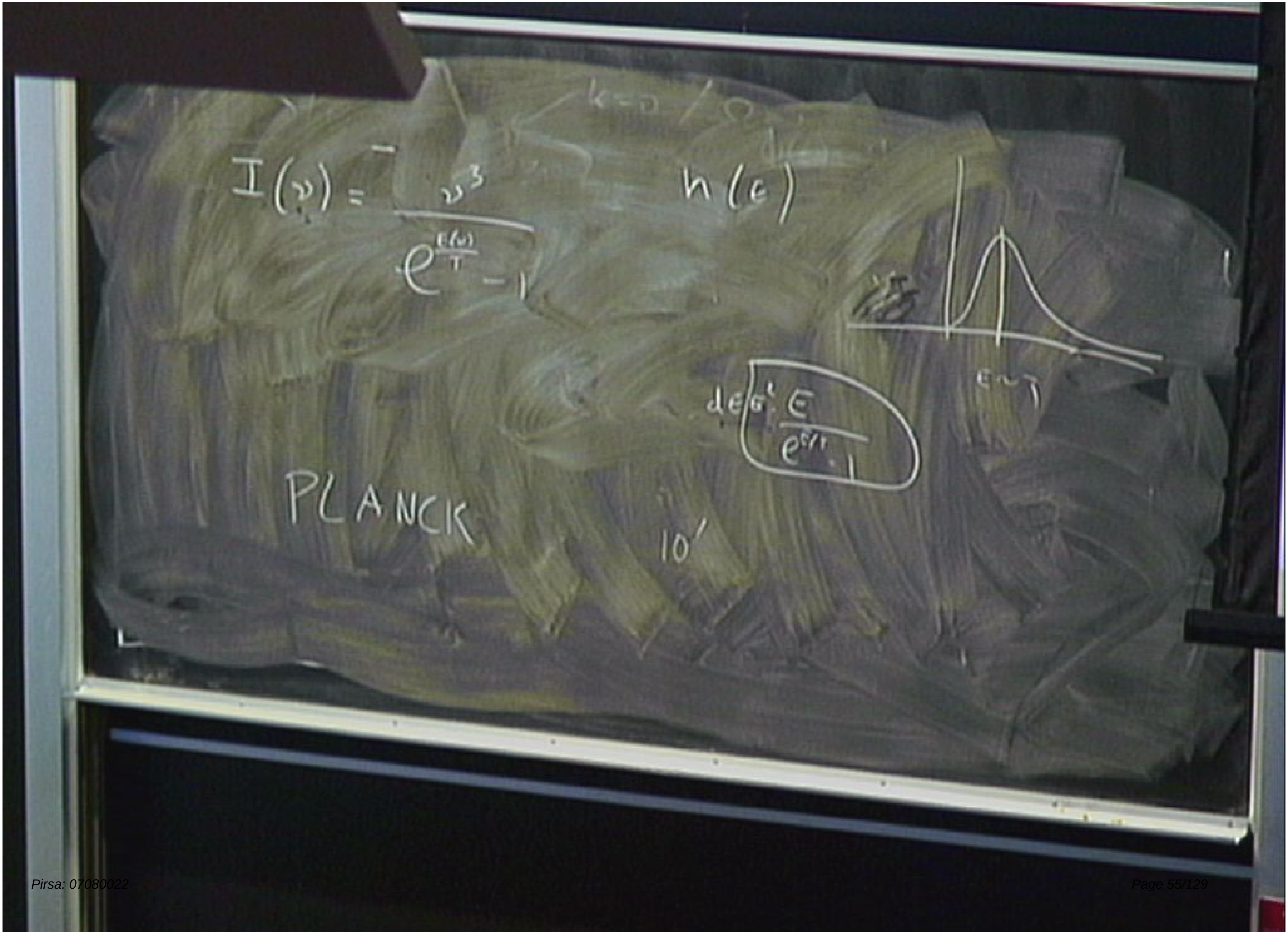
Synchrotron foreground



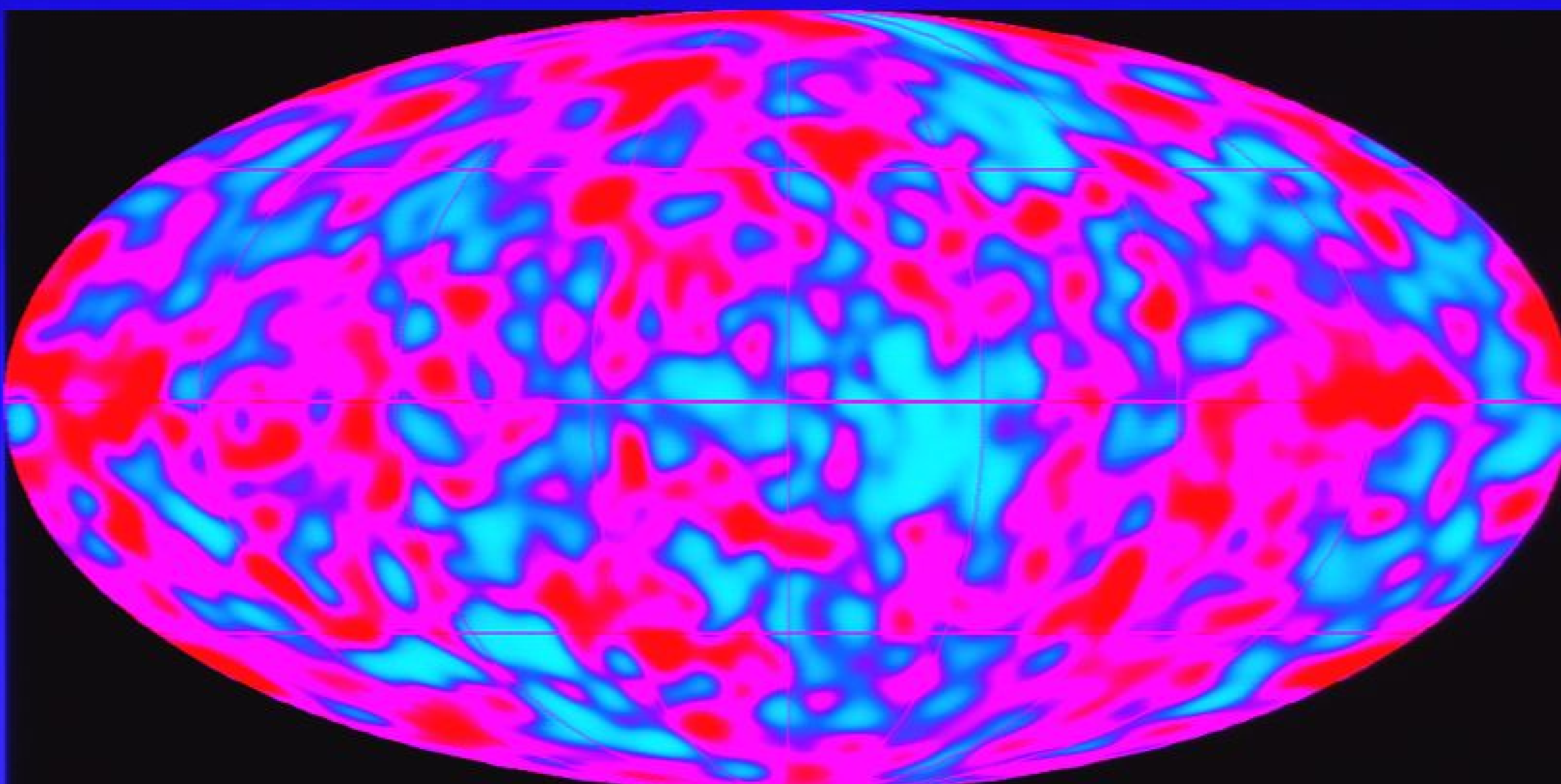
The COBE-DMR map




I(ν) = ν³ / (e^{E(ν)/T} − 1)
n(ε)
ε~T
PLANCK


I(ν) = ν³ / (e^{E(ν)/T} − 1)
n(ε)
ε ~ T
PLANCK
10′

The COBE-DMR map



$$I(\nu) = \frac{\nu^3}{e^{\frac{E(\nu)}{T}} - 1}$$

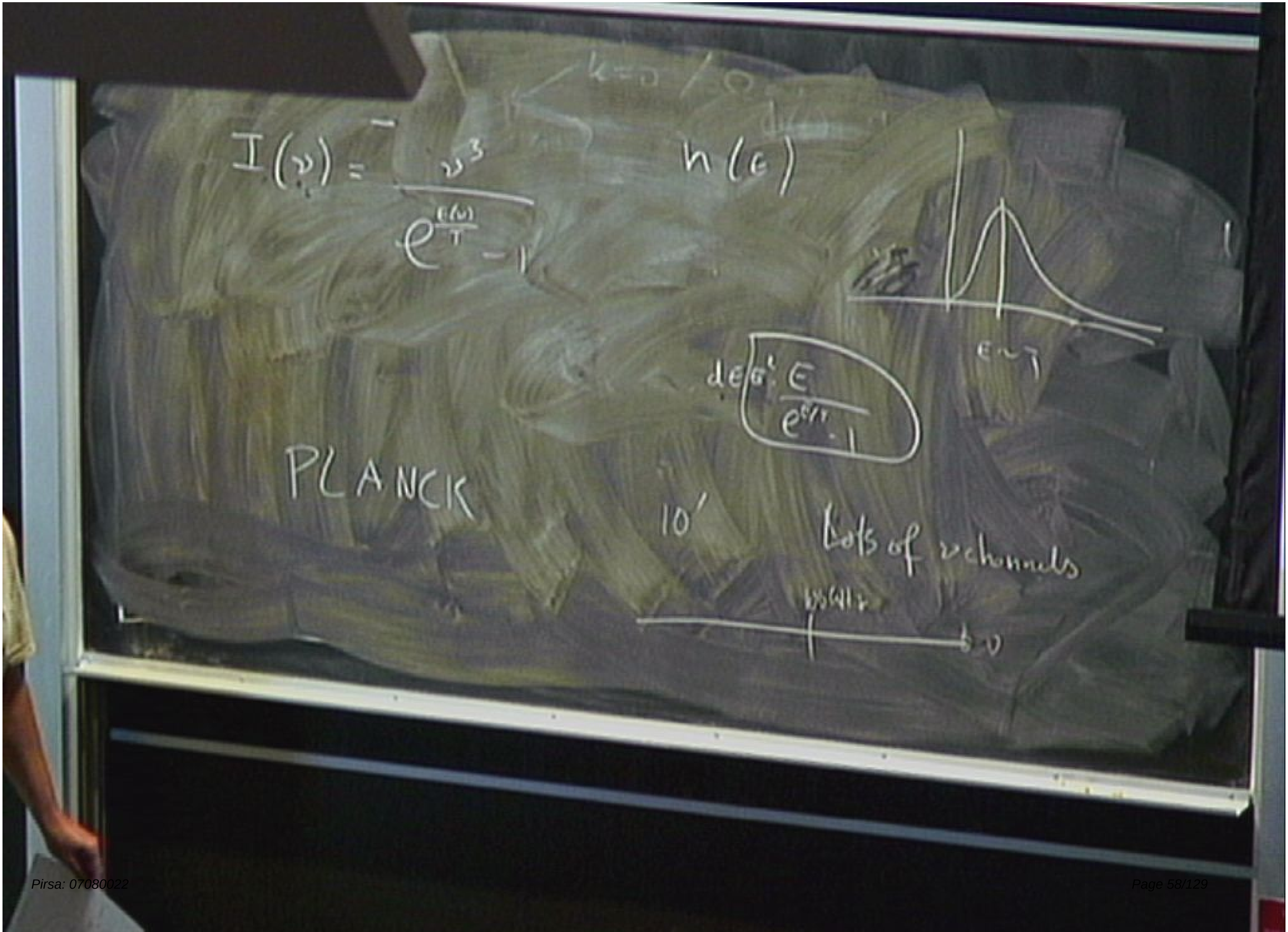
$n(\epsilon)$

$$d\epsilon\, \epsilon^2 \boxed{\frac{\epsilon}{e^{\epsilon/T} - 1}}$$

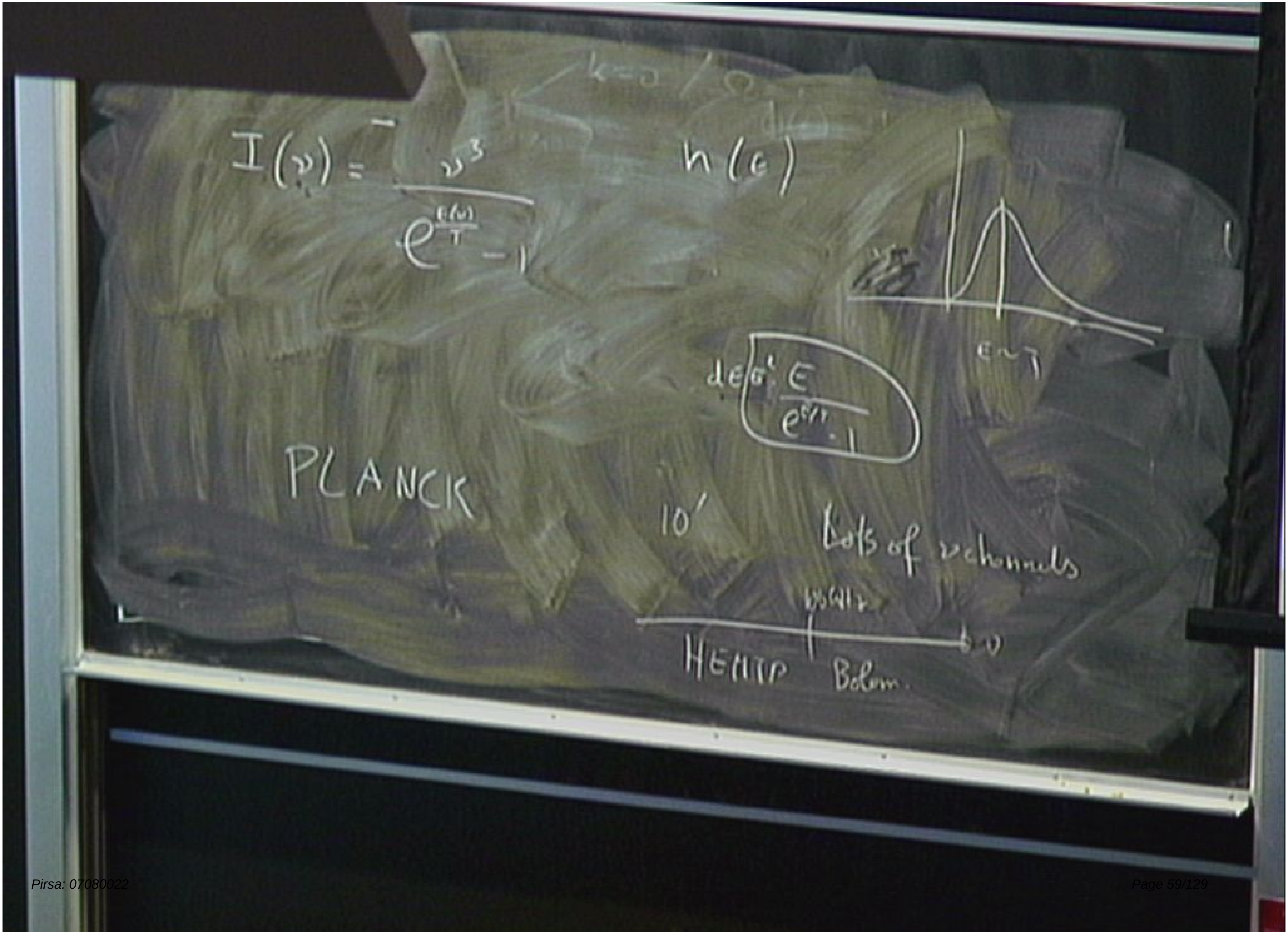
PLANCK

10′

Lots of ν channels



$I(\nu) = \frac{\nu^3}{e^{E(\nu)/T} - 1}$
$n(\epsilon)$
$\epsilon \sim T$
$\frac{\epsilon}{e^{\epsilon/T} - 1}$
PLANCK
10′
Lots of ν channels


$I(\nu) = \frac{\nu^3}{e^{\frac{E(\nu)}{T}} - 1}$
$n(\epsilon)$
$d\epsilon\, \epsilon^2\, \frac{\epsilon}{e^{\epsilon/T} - 1}$
$\epsilon \sim T$
PLANCK
10′
Lots of ν channels
HEMT
Bolom.

$$I(\nu) = \frac{\nu^3}{e^{\frac{E(\omega)}{T}} - 1}$$

$n(\epsilon)$

$$d\epsilon\, \epsilon^2 \boxed{\frac{\epsilon}{e^{\epsilon/T} - 1}}$$

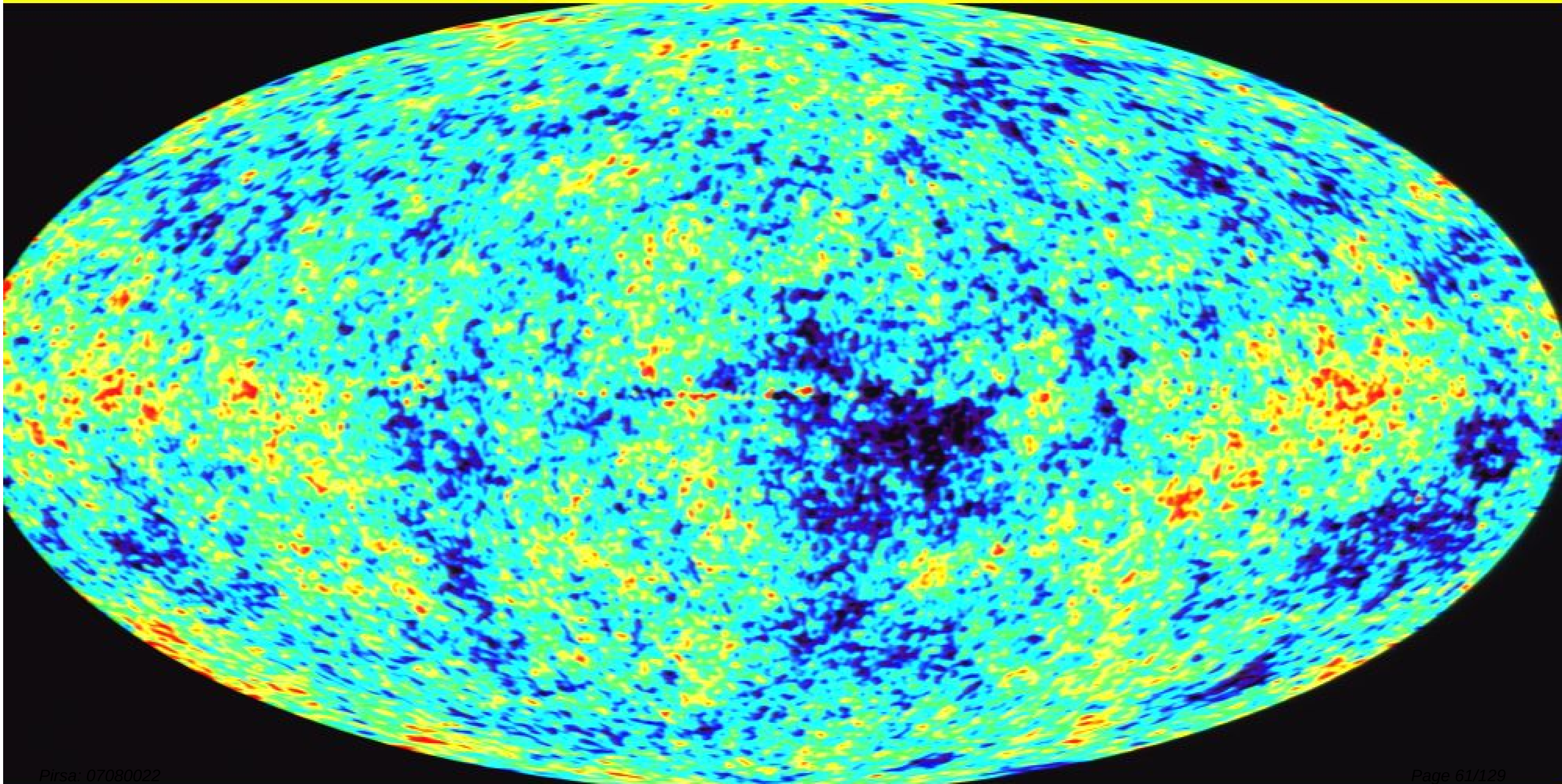
PLANCK

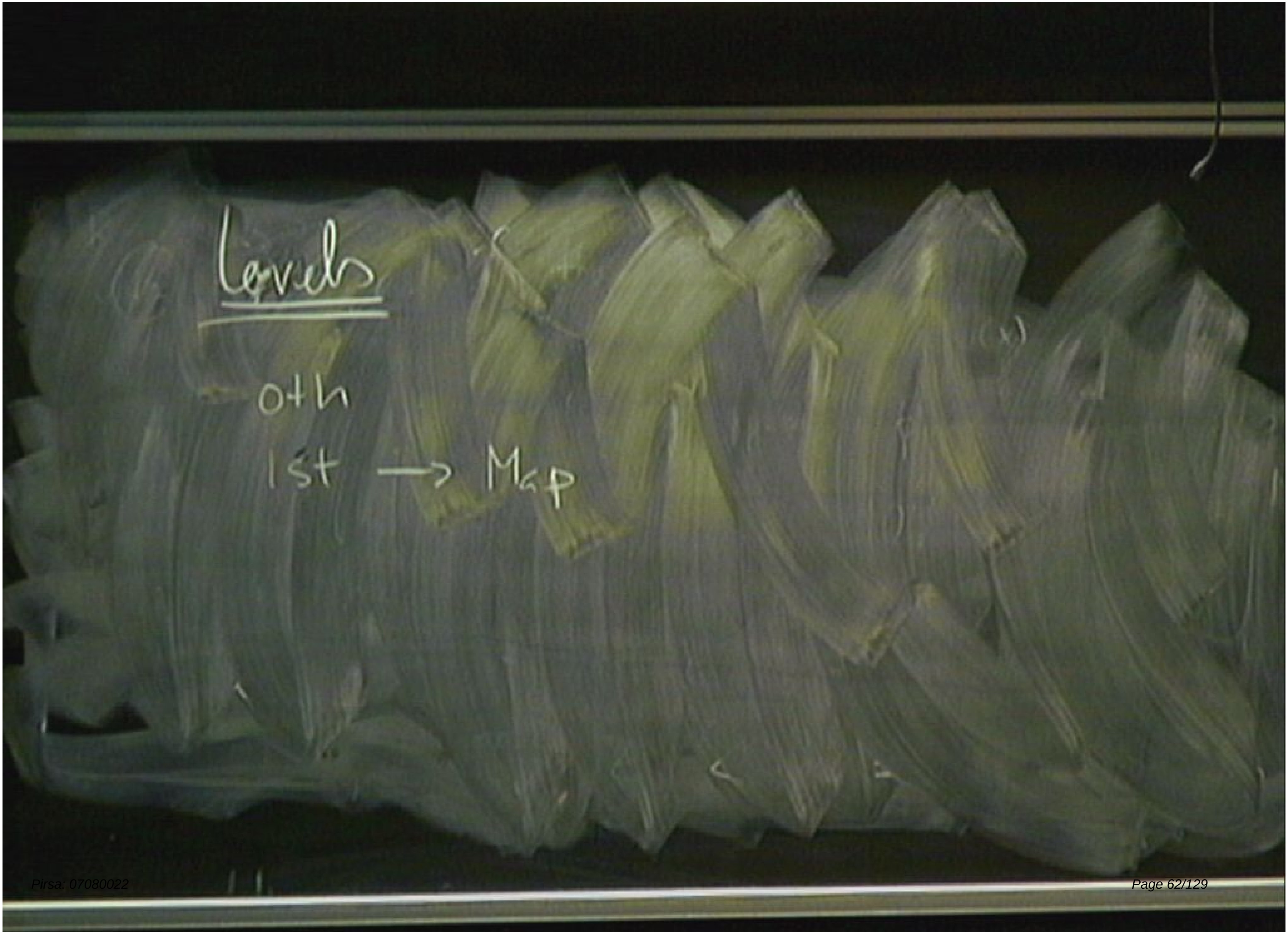
10'

Lots of ν channels

HEMTP Bolom.

The WMAP temperature map




Levels
0th
1st → Map

Levels

0th

1st $\longrightarrow$ Map

2nd $\longrightarrow$ Power spectrum C_ℓ

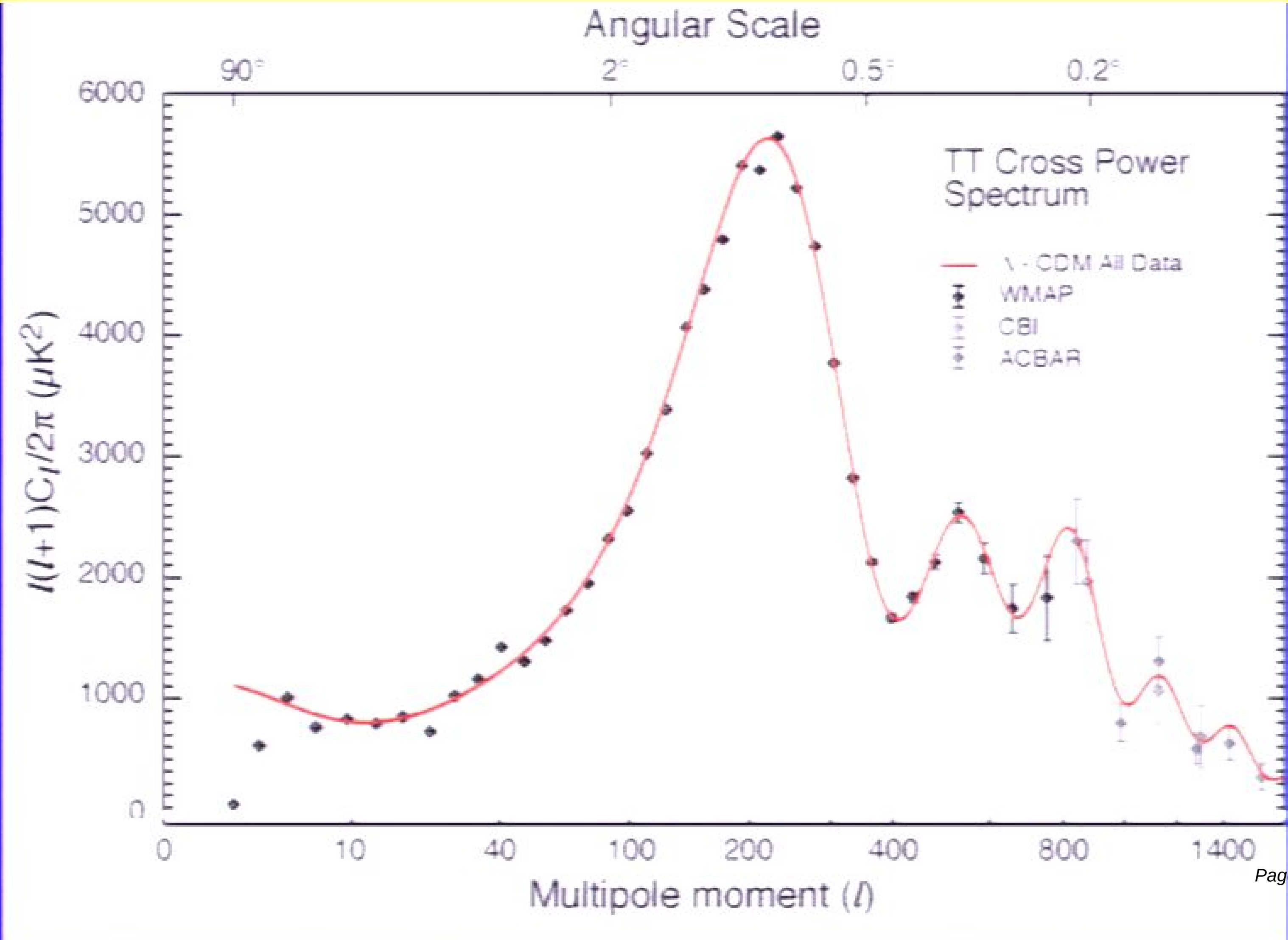
Levels

0th

1st $\longrightarrow$ Map

2nd $\longrightarrow$ Power spectrum C_ℓ $\Big\}$ gaussian (isotropic)

The power spectrum



The cosmological parameters

WMAP Cosmological Parameters Model: lcdm Data: wmap	
$10^2\Omega_b h^2$	2.229 ± 0.073
$\Delta^2_{\mathcal{R}}(k = 0.002/\mathrm{Mpc})$	$(23.5 \pm 1.3) \times 10^{-10}$
h	0.732 [illegible]
H_0	73.2 [illegible] km/s/Mpc
$\log(10^{10} A_s)$	3.156 ± 0.056
$n_s(0.002)$	0.958 ± 0.016
$\Omega_b h^2$	0.02229 ± 0.00073
$\Omega_c h^2$	0.1054 [illegible]
Ω_Λ	0.759 ± 0.034
Ω_m	0.241 ± 0.034
$\Omega_m h^2$	0.1277 [illegible]
σ_8	0.761 [illegible]
τ	0.089 ± 0.030
θ_A	0.5952 ± 0.0021 [illegible]
z_r	11.0 [illegible]

Levels

0th

1st $\longrightarrow$ Map

gaussian (isotropic)

2nd $\longrightarrow$ Power spectrum C_ℓ

C_ℓ (model params)

3rd $\longrightarrow$ Cosmo parameters

Levels

0th → Time-ordered series

1st → Map

2nd → Power spectrum C_ℓ gaussian (isotropic)

3rd → Cosmo parameters C_ℓ (model param)

Levels

0th → Time-ordered series ??

1st → Map

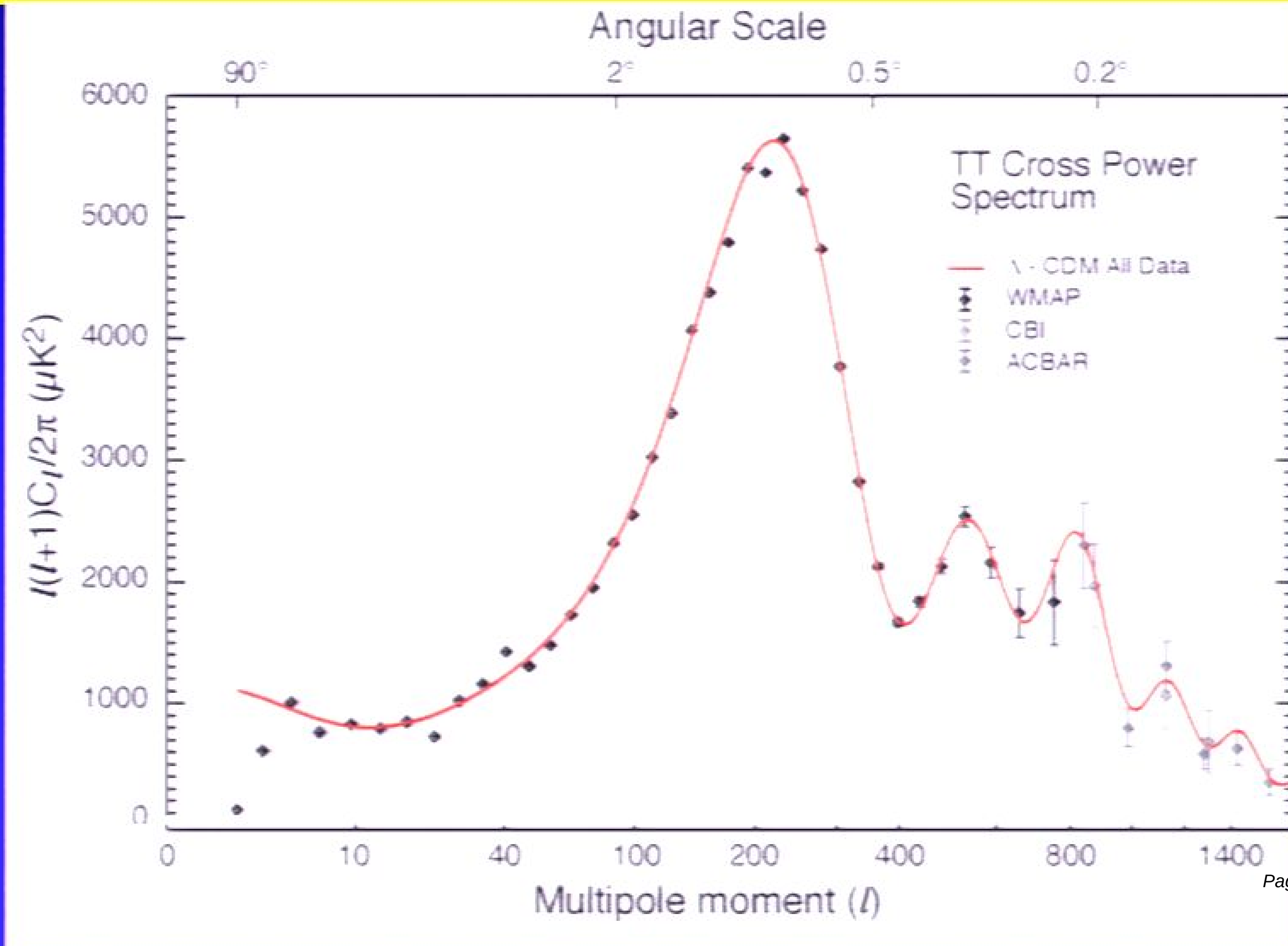
gaussian (isotropic)

2nd → Power spectrum C_ℓ

C_ℓ (model param)

3rd → Cosmo parameters

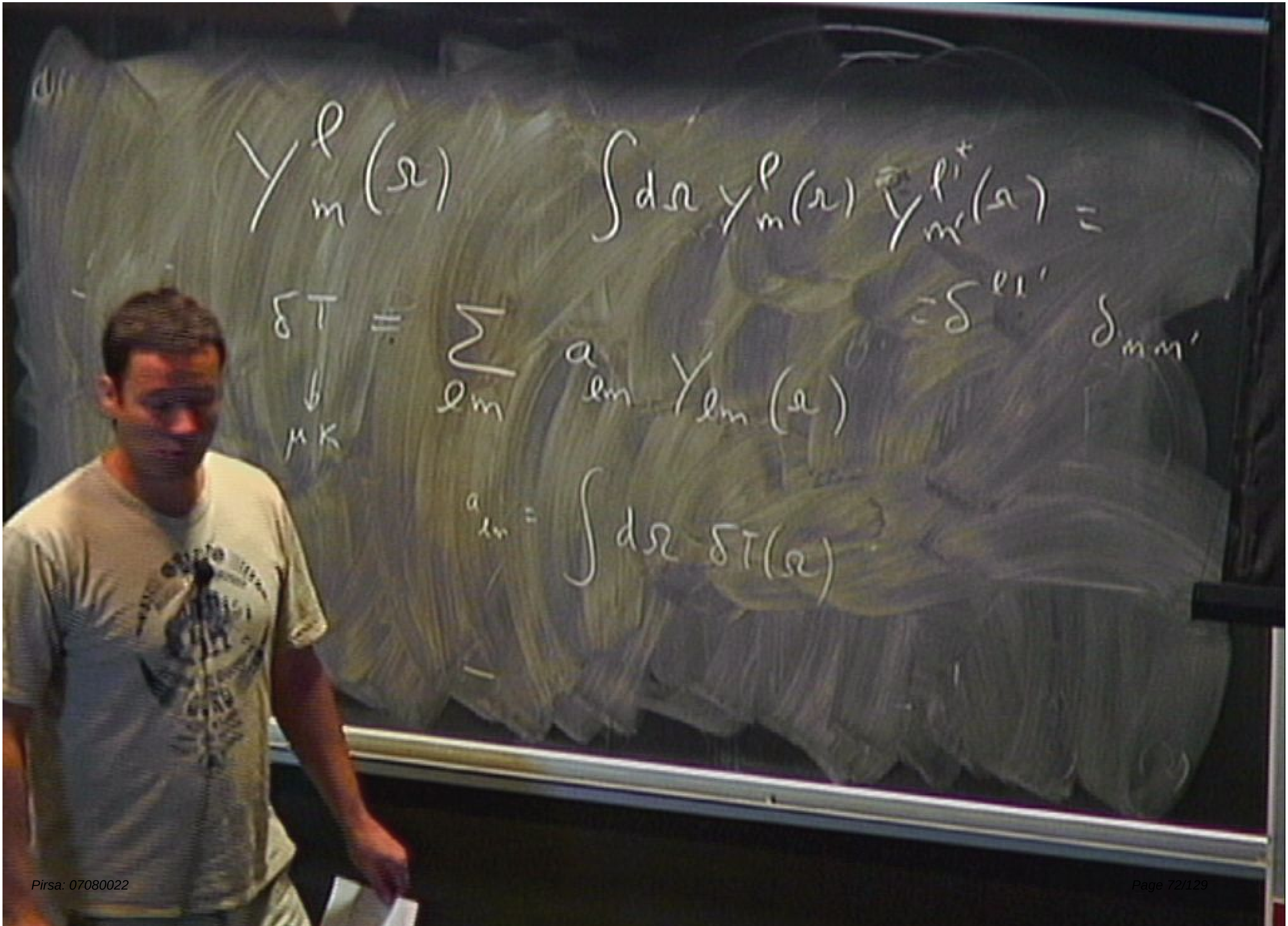
The power spectrum



Harmonic analysis on the sphere

$$\frac{\delta T}{T}(\theta,\varphi) = \sum_{l} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\varphi)$$

Spherical Harmonics



$Y^{\ell}_{m}(\Omega)$
$\int d\Omega\, Y^{\ell}_{m}(\Omega)\, Y^{\ell'*}_{m'}(\Omega) = \delta^{\ell\ell'}\, \delta_{mm'}$
$\delta T = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\Omega)$
μK
$a_{\ell m} = \int d\Omega\, \delta T(\Omega)$

$$Y^{\ell}_{m}(\Omega) \qquad \int d\Omega\, Y^{\ell}_{m}(\Omega)\, Y^{\ell' *}_{m'}(\Omega) = \delta^{\ell\ell'}\,\delta_{mm'}$$

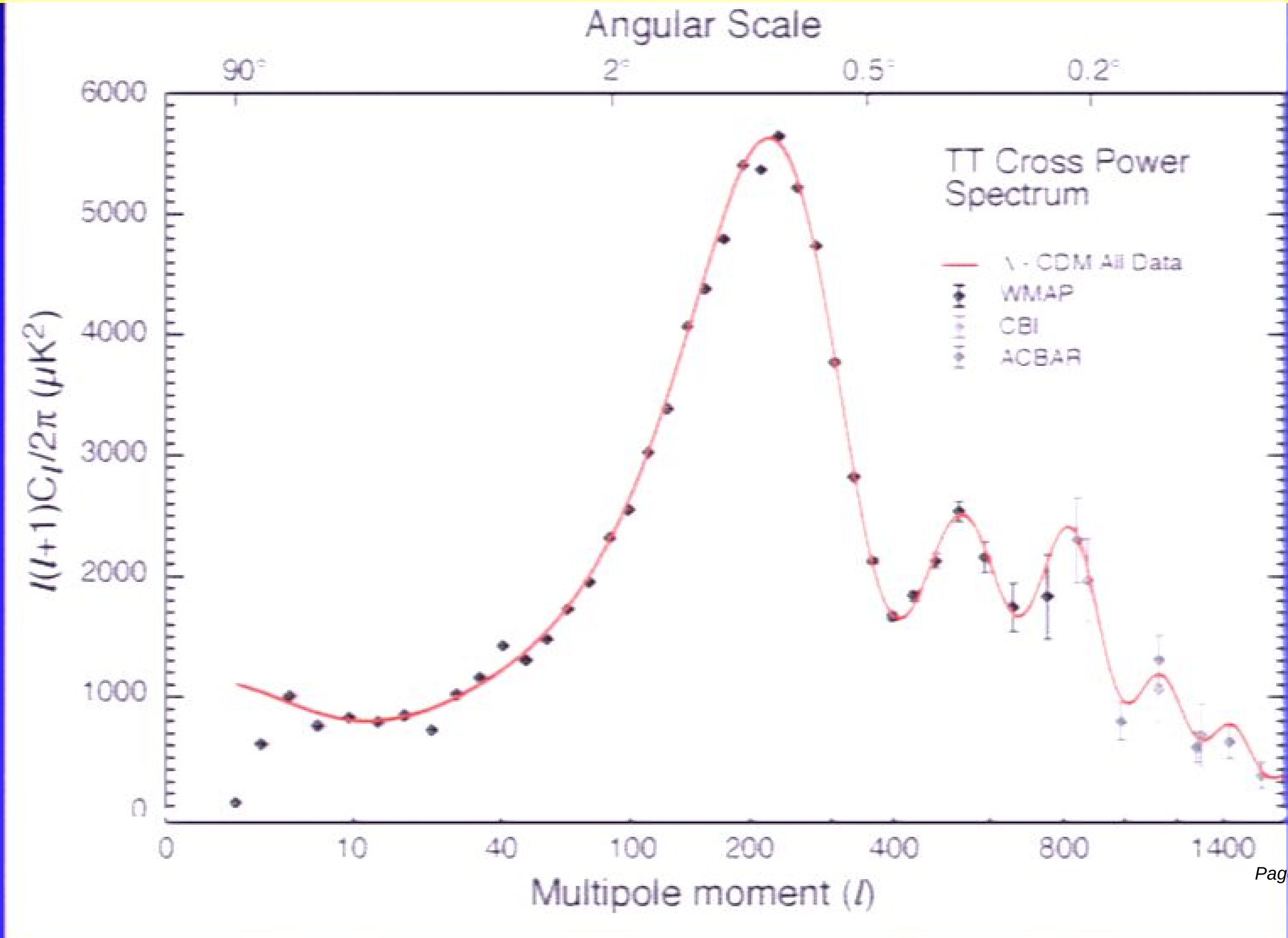
$$\delta T = \sum_{\ell m} a_{\ell m}\, Y_{\ell m}(\Omega)$$

$$\downarrow$$

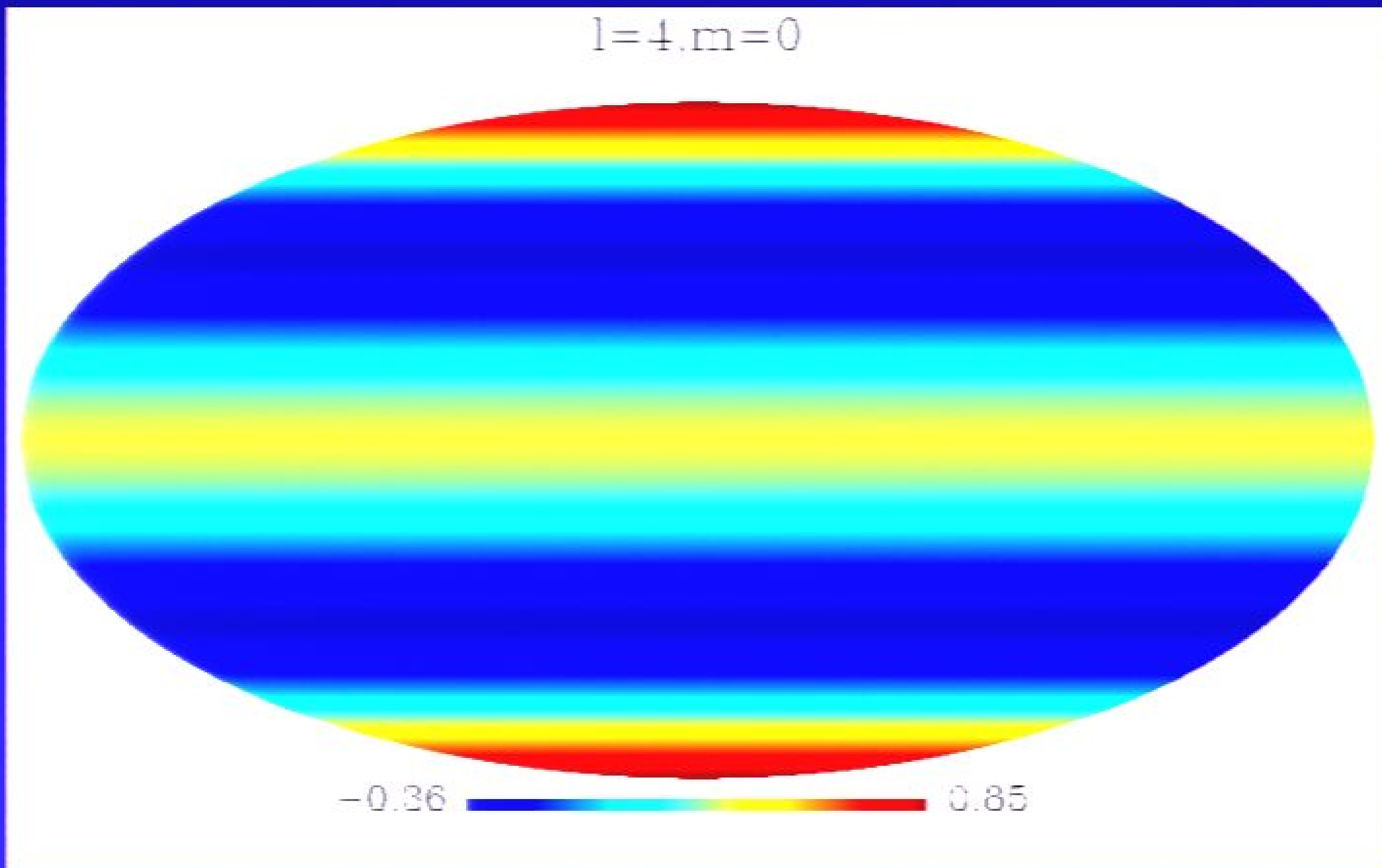
$$\mu K$$

$$a_{\ell m} = \int d\Omega\, \delta T(\Omega)\, Y^{*\ell}_{m}(\Omega)$$

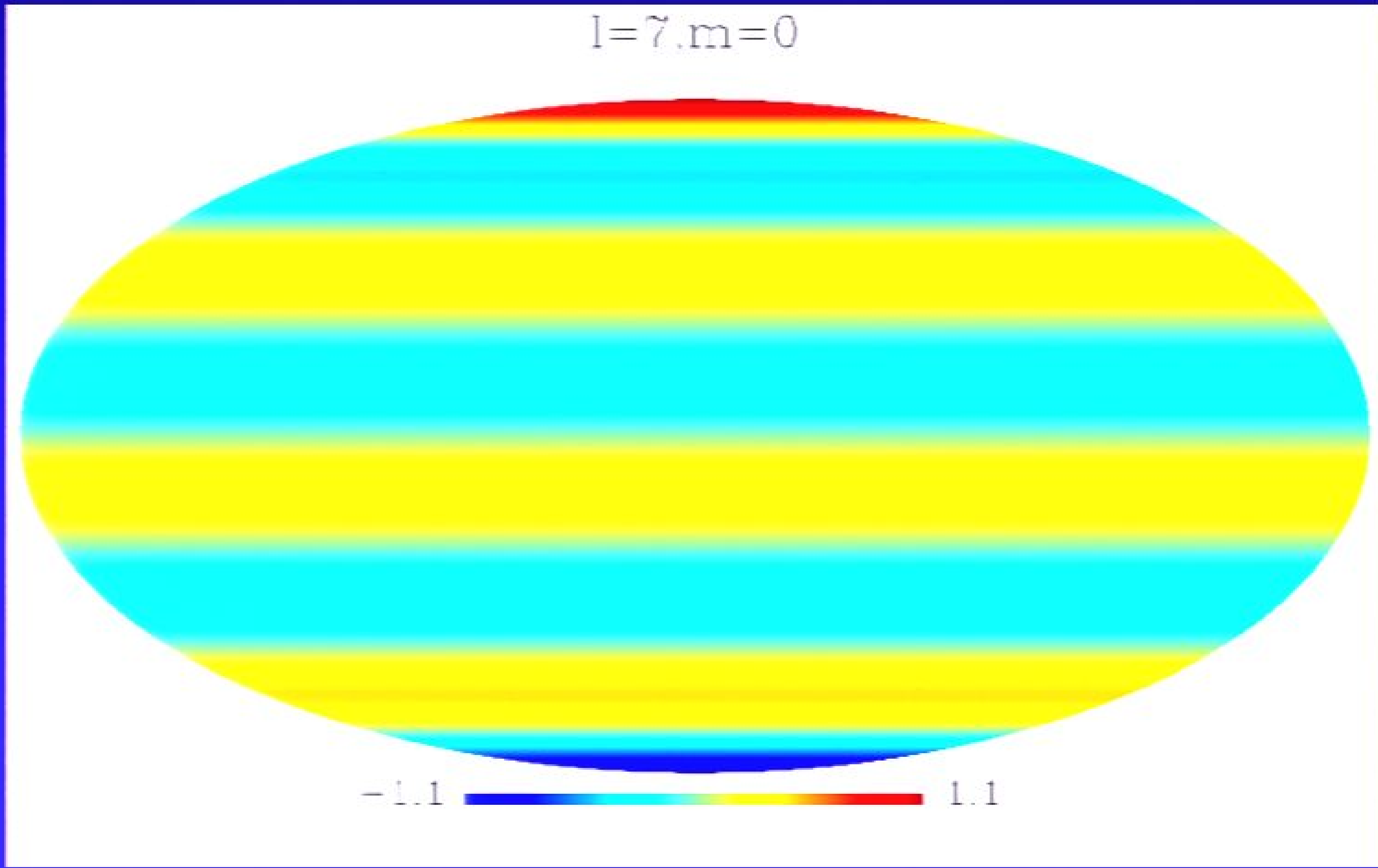
The power spectrum



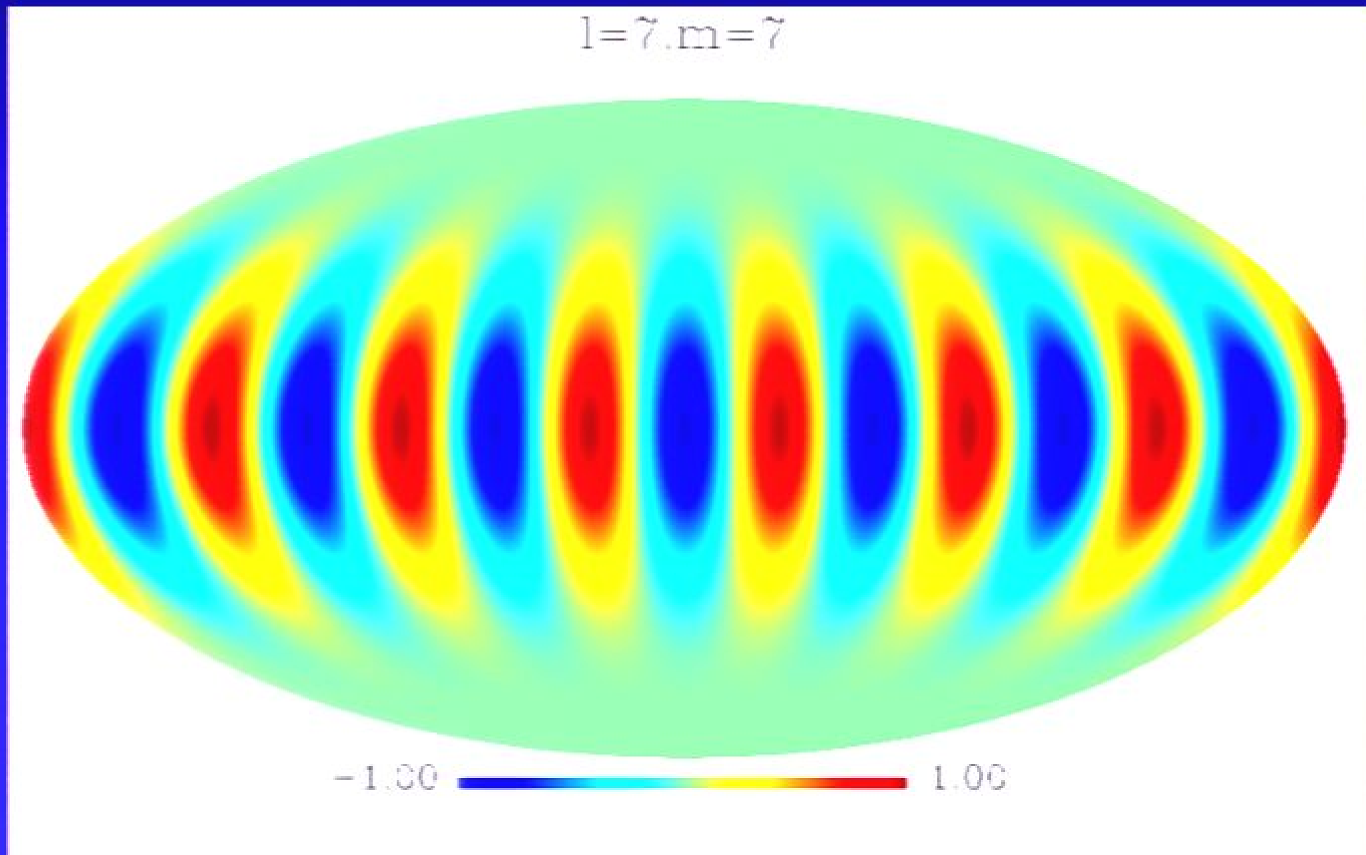
Cylindrically symmetric modes



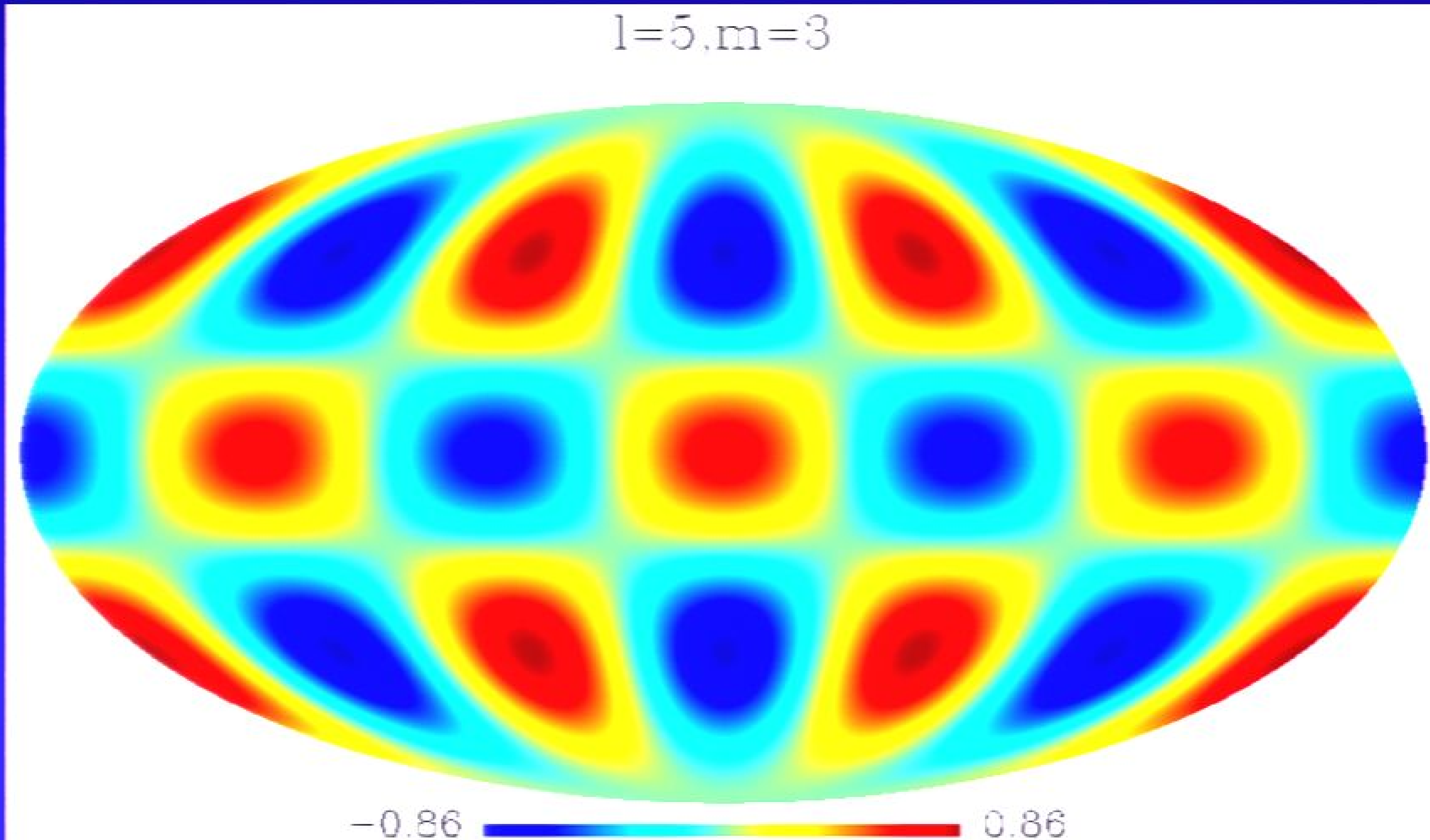
Cylindrically symmetric modes



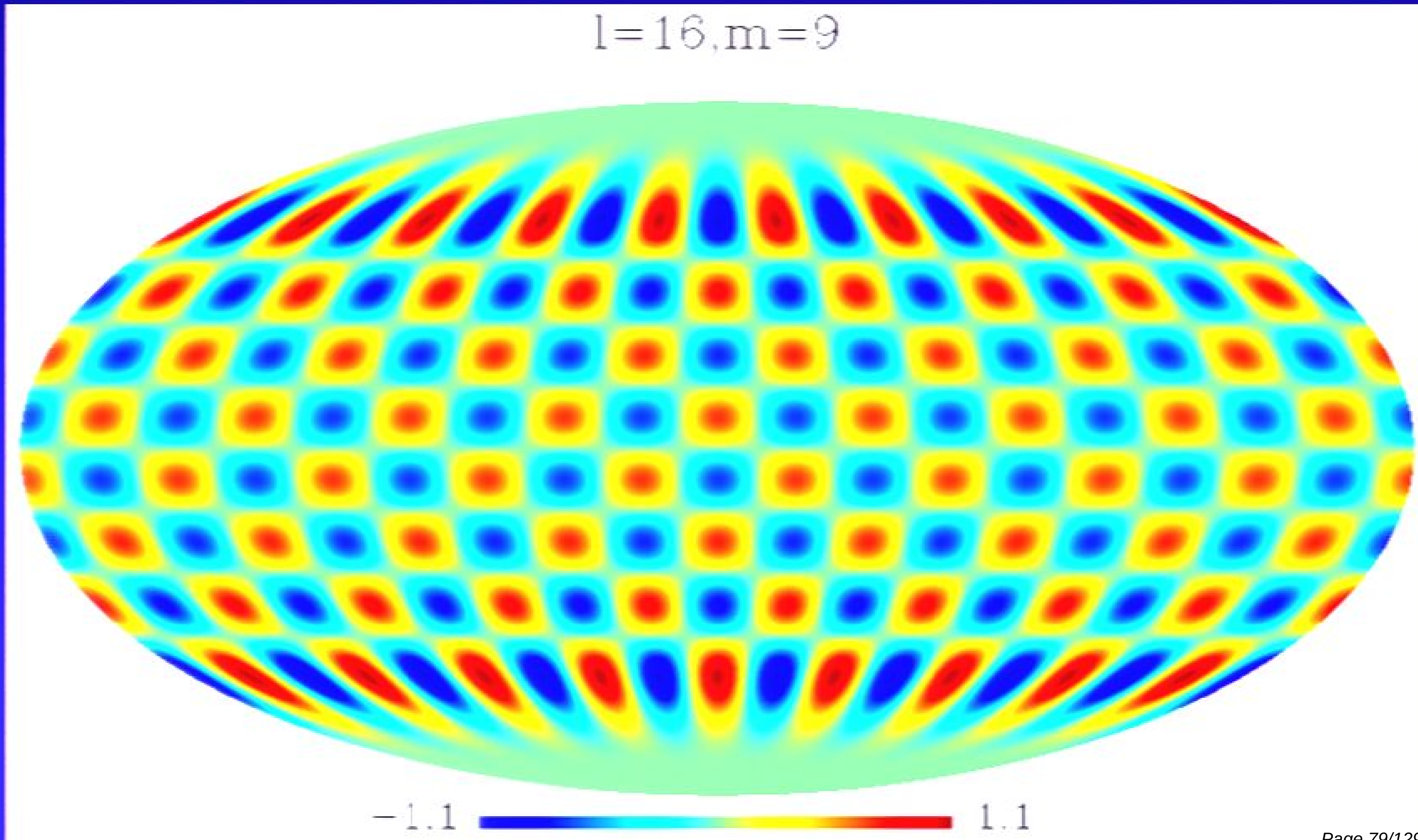
"Planar" modes



“In between” asymmetric modes



“In between” asymmetric modes



$$Y^{\ell}_{m}(\Omega)$$

$$\int d\Omega \, Y^{\ell}_{m}(\Omega) \, Y^{\ell'*}_{m'}(\Omega) = \delta^{\ell\ell'} \delta_{mm'}$$

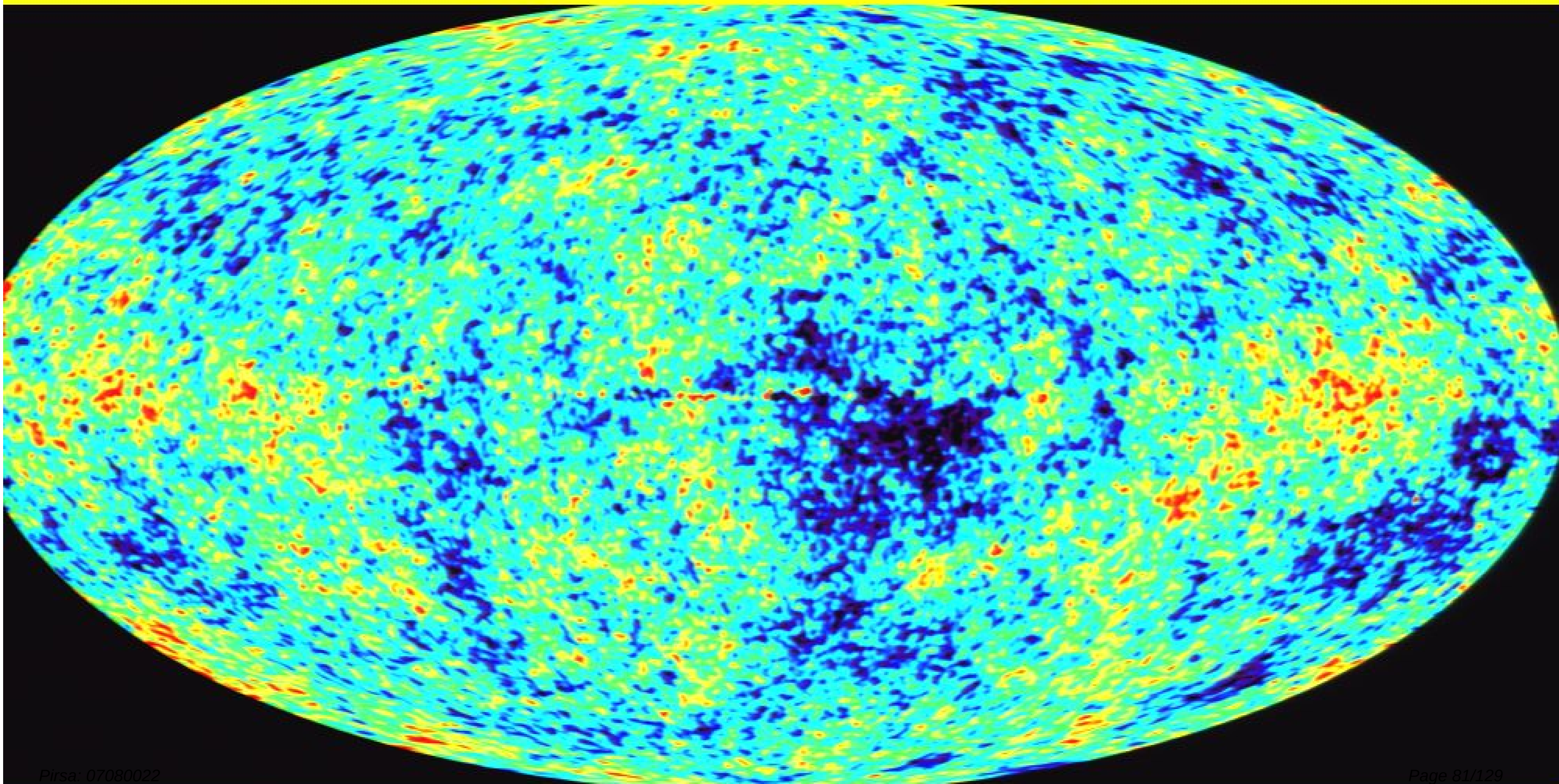
$$\delta T = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\Omega)$$

$$\delta T \downarrow \mu K$$

$$\ell \sim \frac{60^\circ - 120^\circ}{\theta}$$

$$a_{\ell m} = \int d\Omega \, \delta T(\Omega) \, Y^{*\ell}_{m}(\Omega)$$

The WMAP temperature map

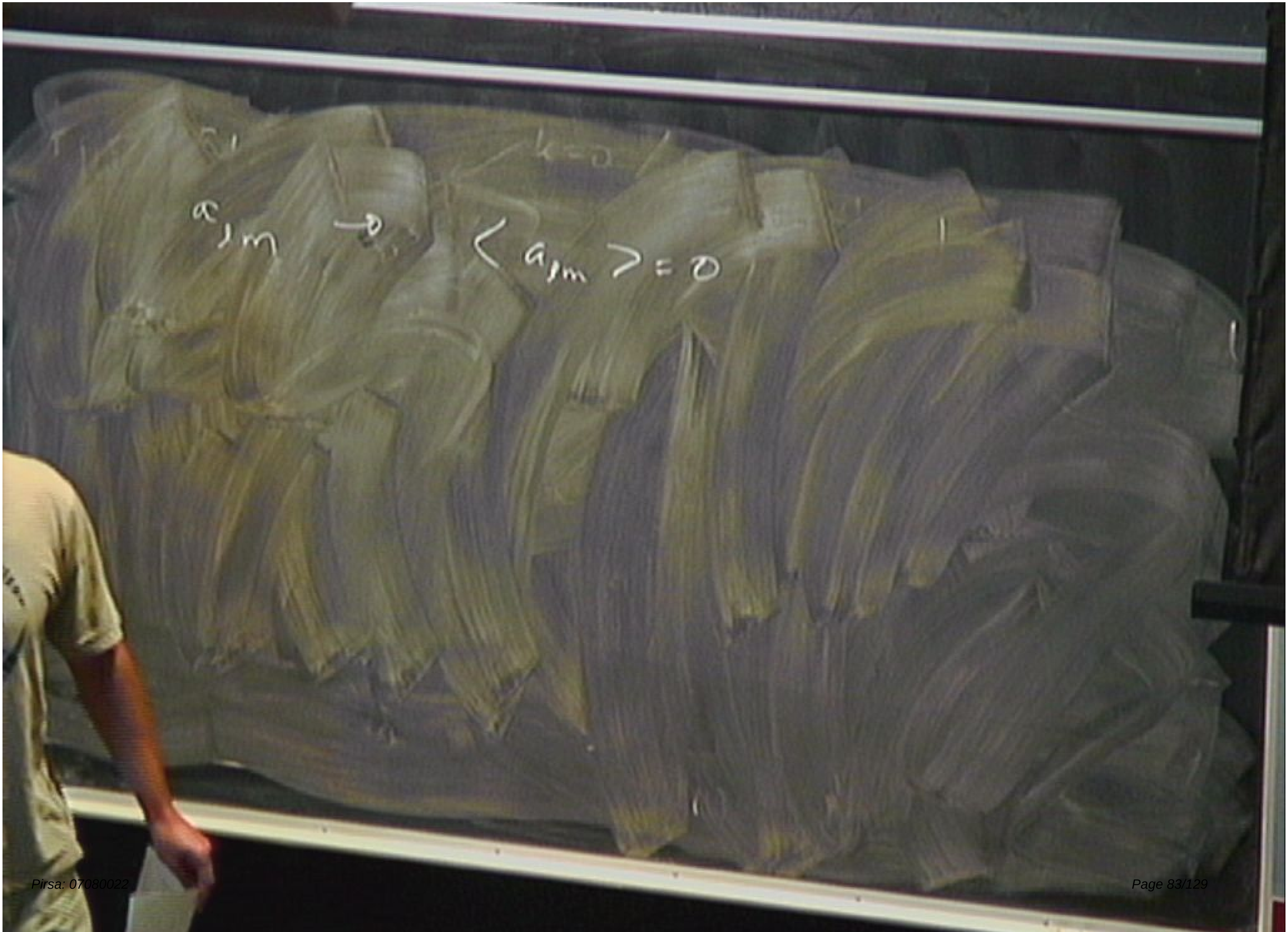


$$Y^{\ell}_{m}(\Omega) \qquad \int d\Omega \, Y^{\ell}_{m}(\Omega) \, Y^{\ell' *}_{m'}(\Omega) = \delta^{\ell \ell'} \, \delta_{m m'}$$

$$\underset{\mu K}{\delta T} = \sum_{\ell m} a_{\ell m} \, Y_{\ell m}(\Omega)$$

$$\ell \sim \frac{60^\circ - 120^\circ}{\theta} \qquad a_{\ell m} = \int d\Omega \, \delta T(\Omega) \, Y^{* \ell}_{m}(\Omega)$$

Random variables


$a_{\ell m} \to 0$
$\langle a_{\ell m} \rangle = 0$

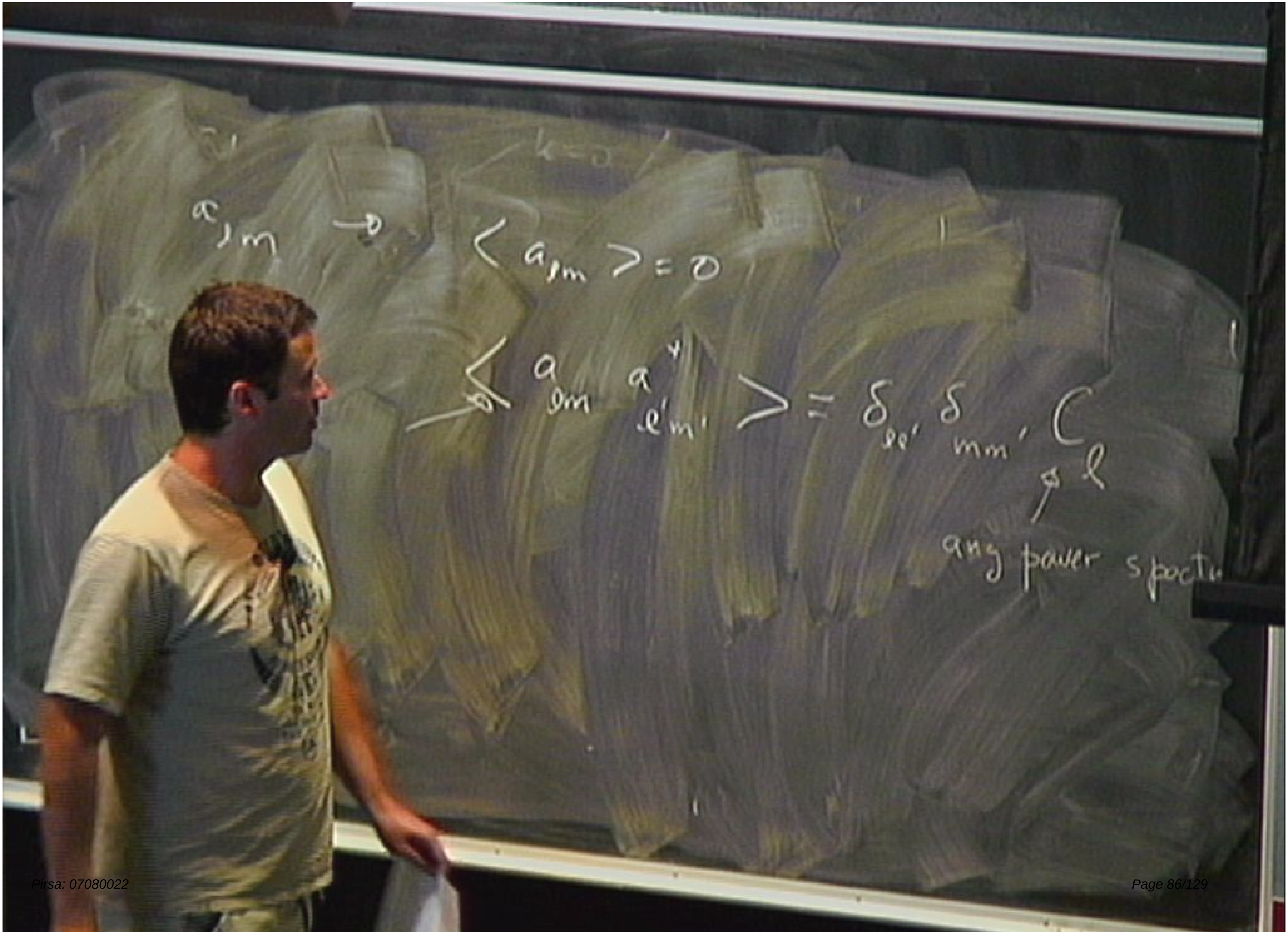
$a_{lm} \to 0$

$\langle a_{lm} \rangle = 0$

$\langle a_{lm} \, a^{\dagger}_{l'm'} \rangle$

$a_{\ell m} \to 0, \quad \langle a_{\ell m} \rangle = 0$

$$\langle a_{\ell m} \, a^{*}_{\ell' m'} \rangle = \delta_{\ell \ell'} \, \delta_{m m'}$$

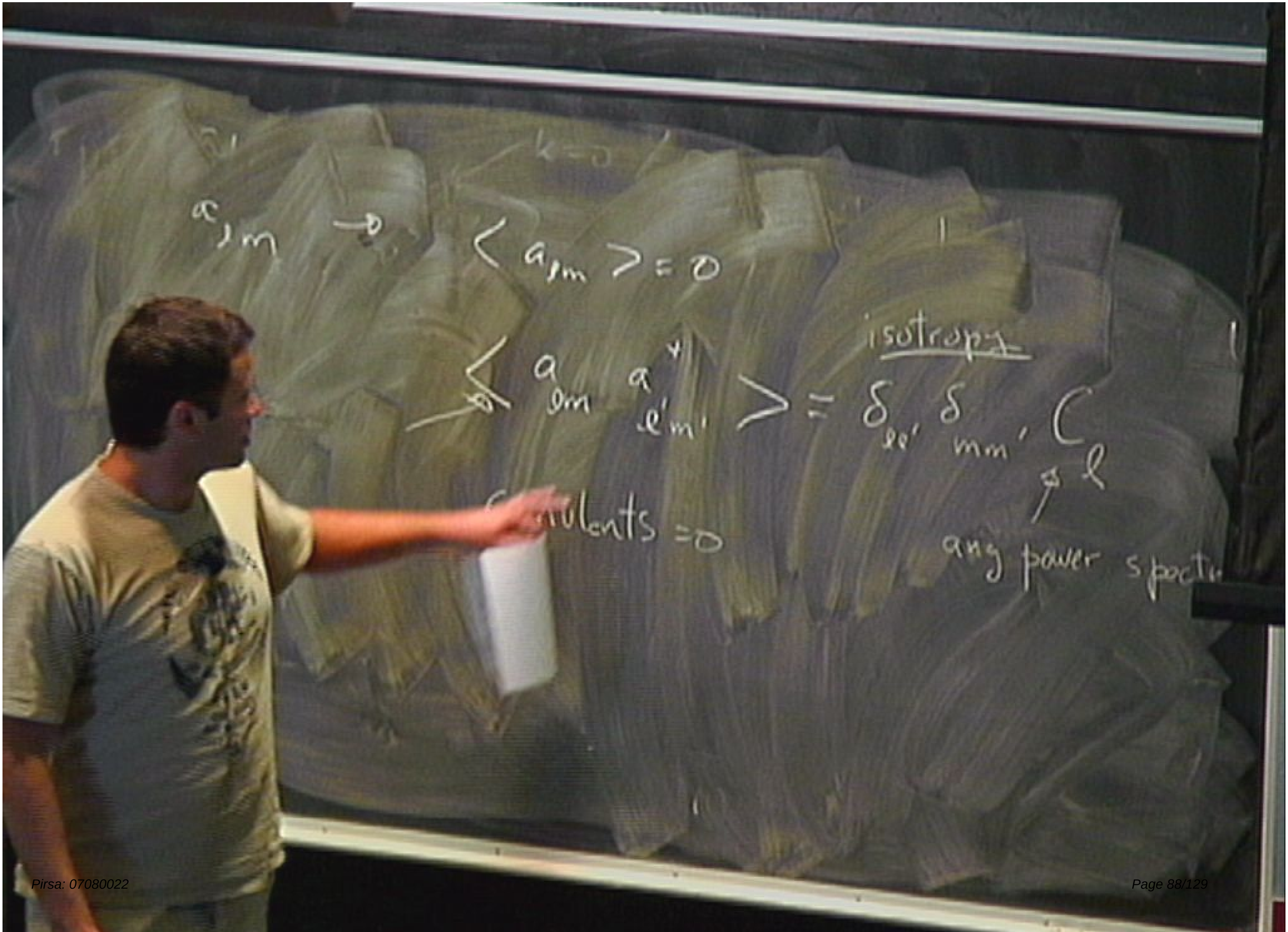

$a_{\ell m} \to 0$
$\langle a_{\ell m} \rangle = 0$
$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$
ang power spectru

$a_{\ell m} \to 0$ $\langle a_{\ell m} \rangle = 0$

isotropy

$$\langle a_{\ell m} \, a^*_{\ell' m'} \rangle = \delta_{\ell \ell'} \, \delta_{m m'} \, C_\ell$$

ang power spectr


$a_{\ell m} \rightarrow 0$
$\langle a_{\ell m} \rangle = 0$
isotropy
$\langle a_{\ell m} a^*_{\ell' m'} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell$
ulants =0
ang power spectr

$a_{\ell m} \to 0$

$\langle a_{\ell m} \rangle = 0$

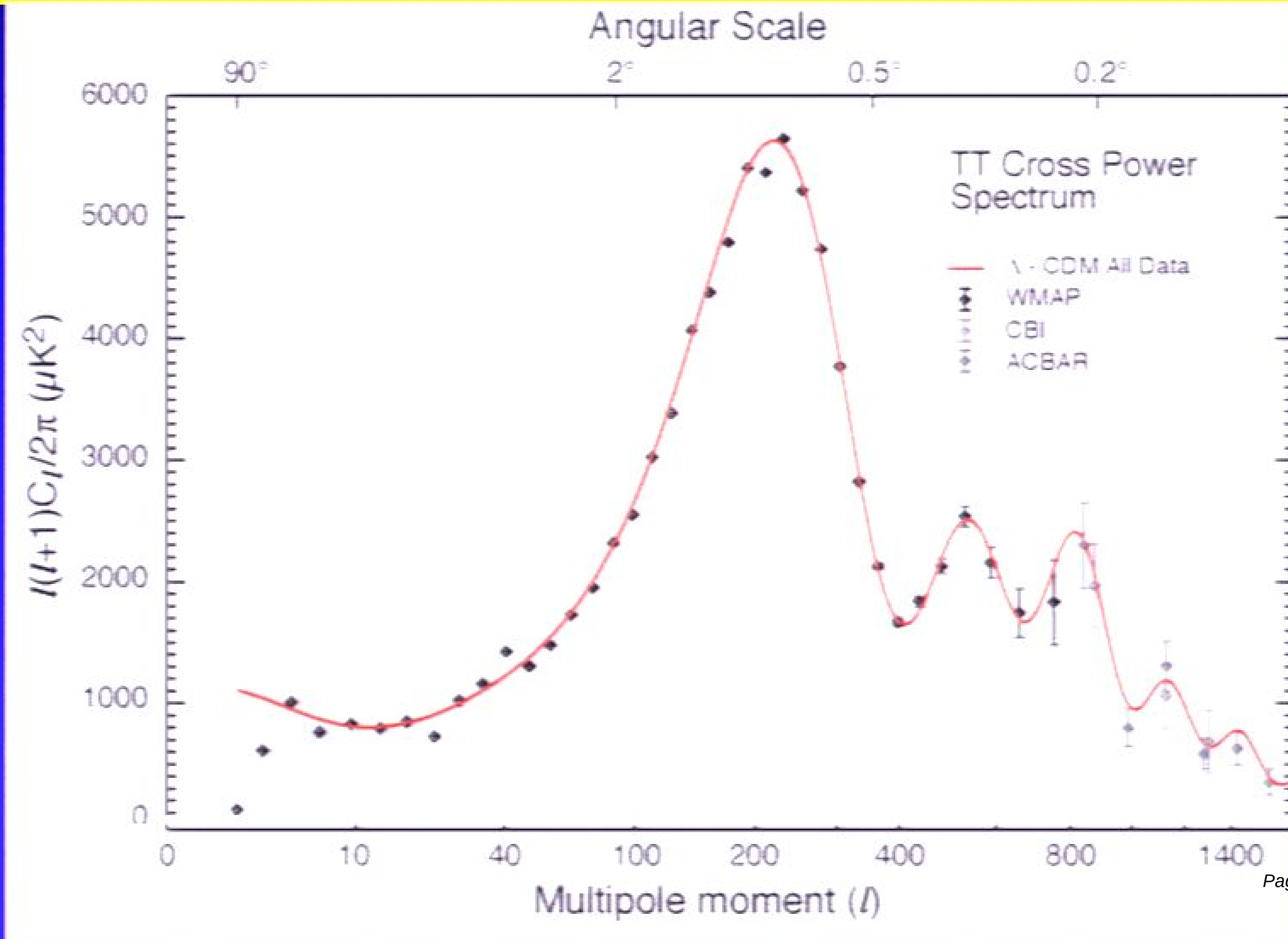
isotropy

$$\langle a_{\ell m}\, a^*_{\ell' m'} \rangle = \delta_{\ell\ell'}\, \delta_{mm'}\, C_\ell$$

cumulants = 0

ang power spectr

The power spectrum



$a_{lm} \to$ $\langle a_{lm} \rangle = 0$

<u>isotropy</u>

$$\langle a_{lm} \, a^*_{l'm'} \rangle = \delta_{ll'} \, \delta_{mm'} \, C_l$$

ang power spectru

cumulants $= 0$

<u>Cosmic Variance</u>

$\hat{C}_l =$

$a_{\ell m} \to \langle a_{\ell m} \rangle = 0$

isotropy

$$\langle a_{\ell m} \, a^*_{\ell' m'} \rangle = \delta_{\ell\ell'} \, \delta_{mm'} \, C_\ell$$

ang power spectrum

cumulants $= 0$

Cosmic variance

$$\hat{C}_\ell = \frac{1}{2\ell+1} \sum_m \left| a^{\text{obs}}_{\ell m} \right|^2$$

$a_{\ell m} \to 0$, $\langle a_{\ell m} \rangle = 0$

isotropy

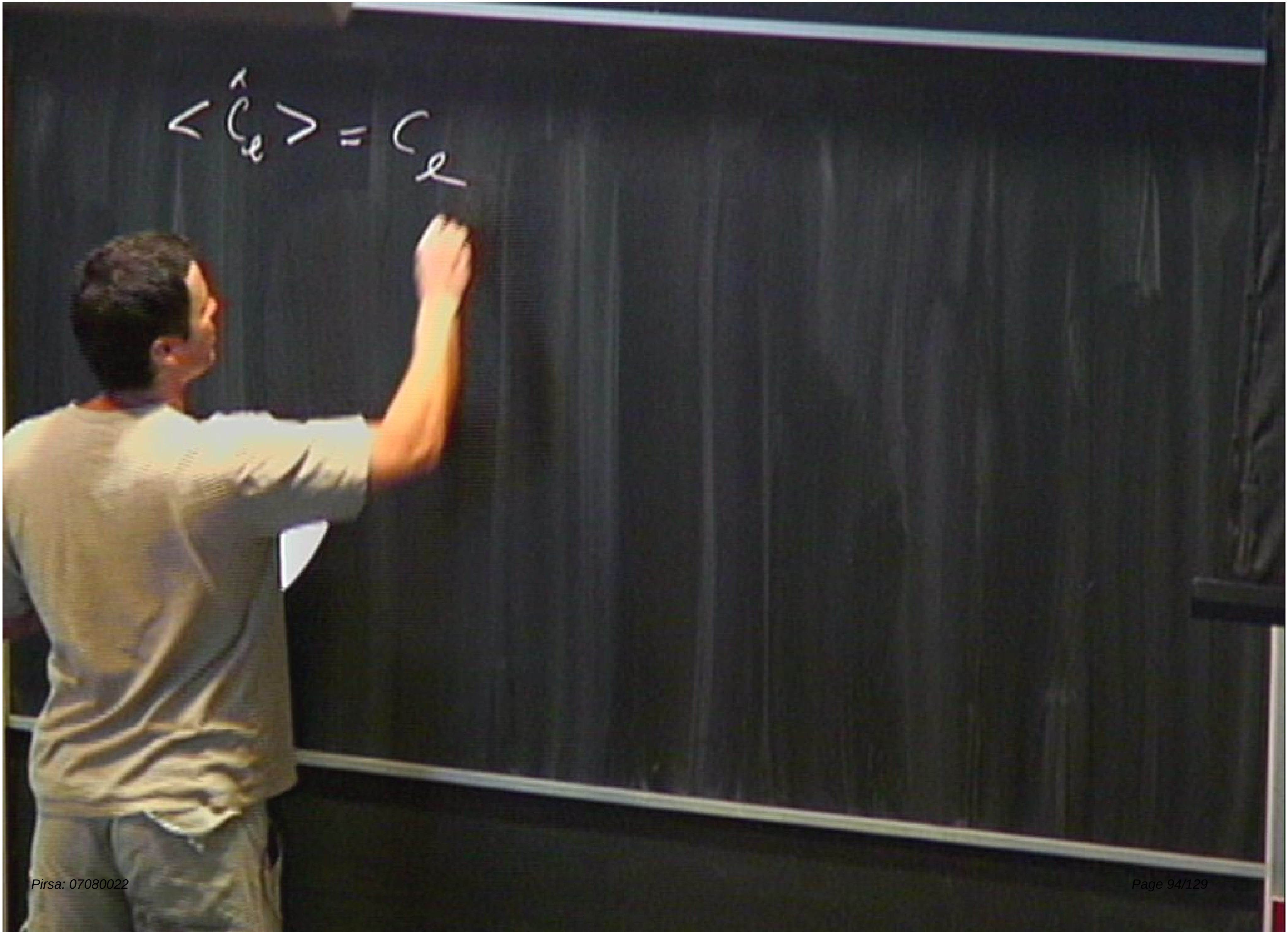
$$\langle a_{\ell m}\, a^{*}_{\ell' m'} \rangle = \delta_{\ell \ell'}\, \delta_{m m'}\, C_\ell$$

ang power spectrum

cumulants $= 0$

Cosmic variance

$$\hat{C}_\ell = \frac{1}{2\ell+1} \sum_m \left| a^{\text{obs}}_{\ell m} \right|^2$$


$\langle \hat{C}_\ell \rangle = C_\ell$

$$\langle \hat{C}_\ell \rangle = C_\ell \quad \text{unbiased}$$

$$\sigma^2(C_\ell) = \frac{1}{(2\ell+1)^2} \sum_m \sigma^2\left(|a^\ell_m|^2\right)$$

$\hookrightarrow$ $2\ell+1$ real modes

$$\langle \hat{C}_\ell \rangle = C_\ell \quad \text{unbiased}$$

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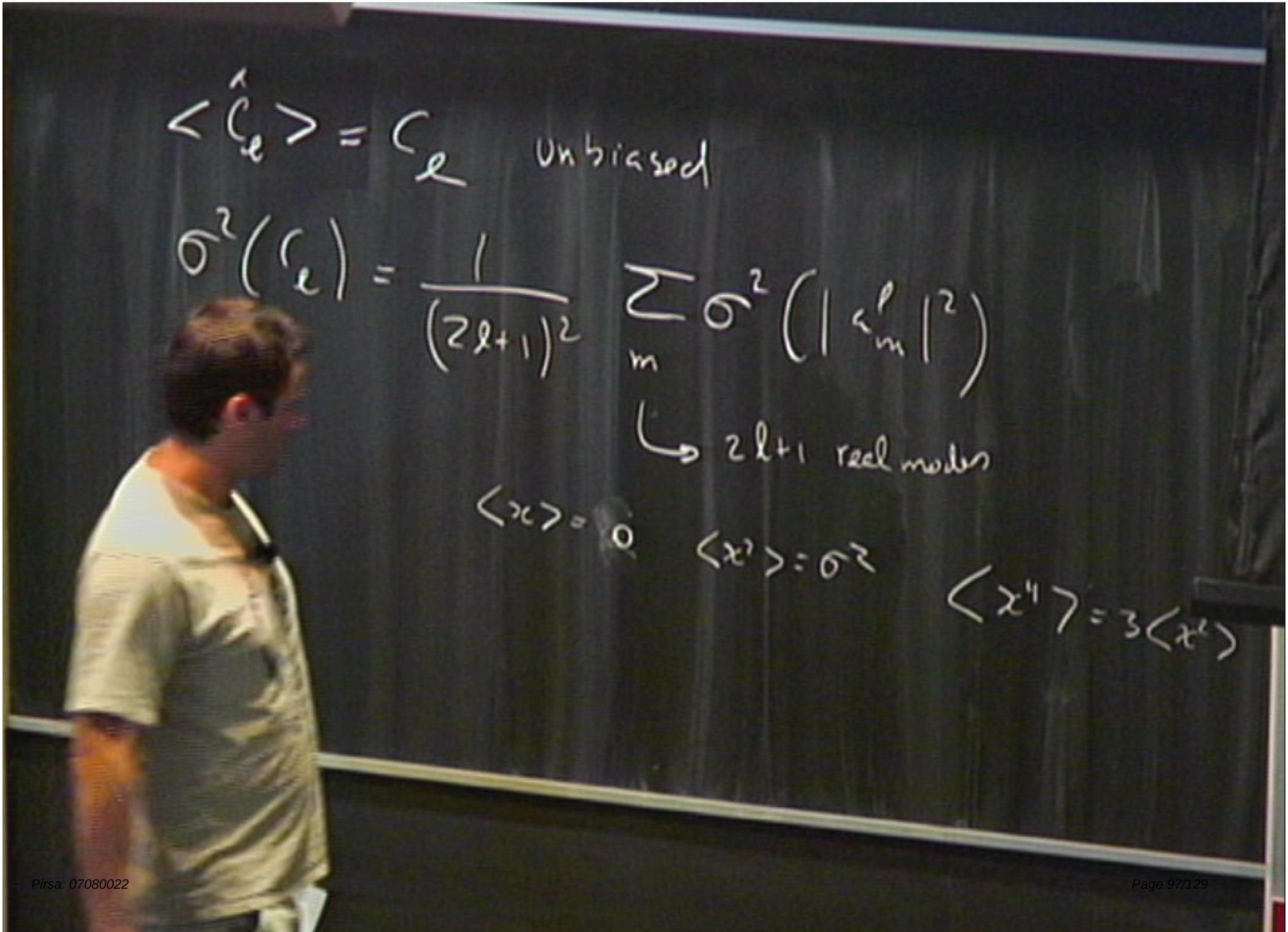
$\hookrightarrow$ $2\ell+1$ real modes


$\langle \hat{C}_\ell \rangle = C_\ell$ unbiased
$\sigma^2(C_\ell) = \frac{1}{(2\ell+1)^2} \sum_m \sigma^2(|a^\ell_m|^2)$
$\hookrightarrow$ $2\ell+1$ real modes
$\langle x \rangle = 0$
$\langle x^2 \rangle = \sigma^2$
$\langle x^4 \rangle = 3\langle x^2 \rangle$

$$\langle \hat{C}_\ell \rangle = C_\ell \quad \text{unbiased}$$

$$\sigma^2(C_\ell) = \frac{1}{(2\ell+1)^2} \sum_m \sigma^2\left(|a^\ell_m|^2\right) = \frac{2\ell+1}{(2\ell+1)^2} \cdot 2C_\ell$$

$\hookrightarrow$ $2\ell+1$ real modes

$\langle x \rangle = 0$ $\quad \langle x^2 \rangle = \sigma^2$ $\quad \langle x^4 \rangle = 3\langle x^2 \rangle$

$\sigma^2(x^2) = 2\sigma^4$

$$\langle \hat{C}_\ell \rangle = C_\ell \quad \text{unbiased}$$

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$\hookrightarrow$ $2\ell+1$ real modes

$$\sigma^2(\hat{C}_\ell) = \frac{2}{2\ell+1} C_\ell^2$$

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$\hookrightarrow$ $2\ell+1$ real modes

$$\sigma^2(\hat{C}_\ell) = \frac{2}{2\ell+1} C_\ell^2$$

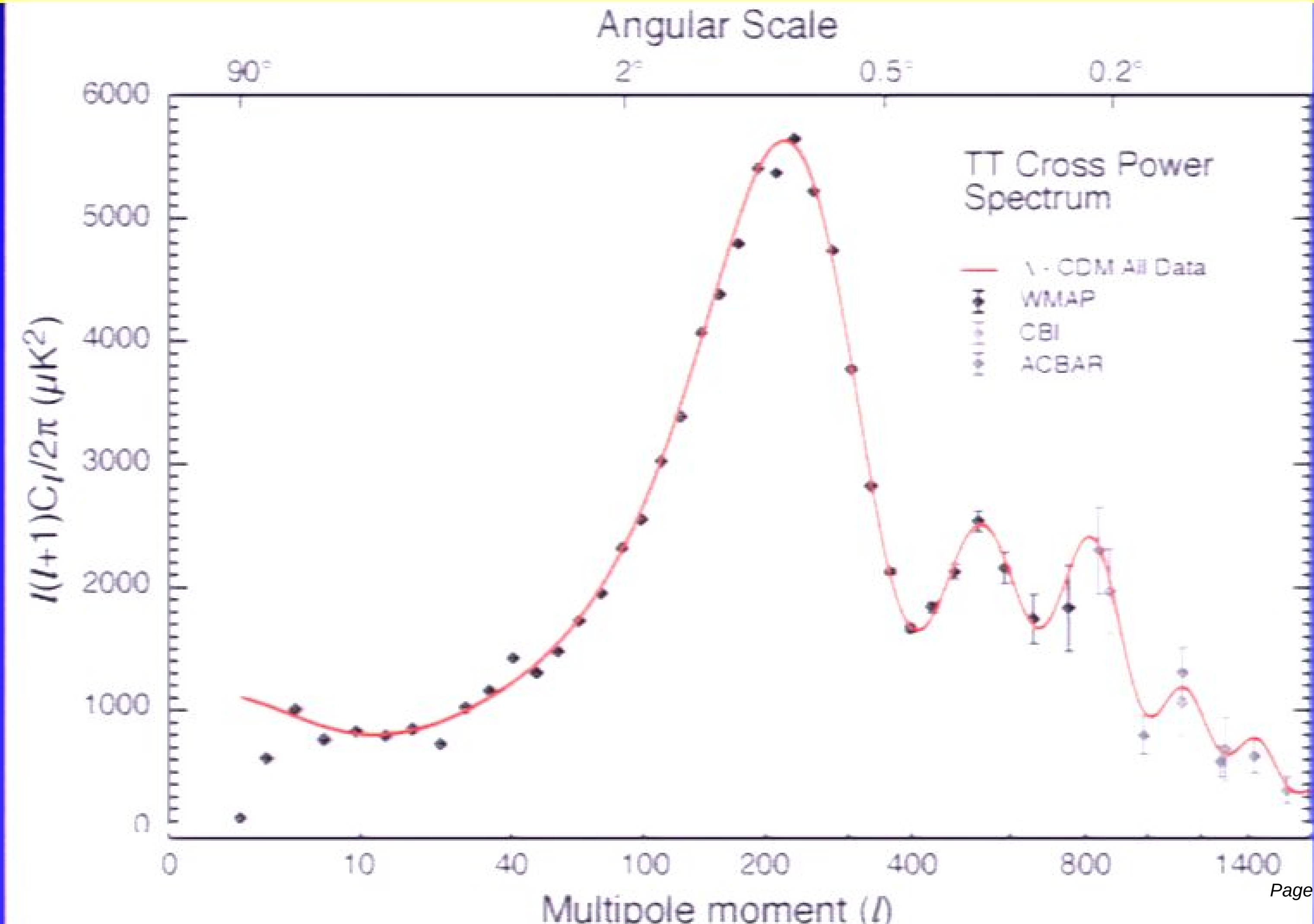
$$\langle x \rangle = 0 \qquad \langle x^2 \rangle = \sigma^2$$

$$\langle x^4 \rangle = 3 \langle x^2 \rangle$$

$$\frac{\sigma(\hat{C}_\ell)}{C_\ell} \sim \frac{1}{\sqrt{\ell}}$$

$$\sigma^2(x^2) = 2\sigma^4$$

The power spectrum



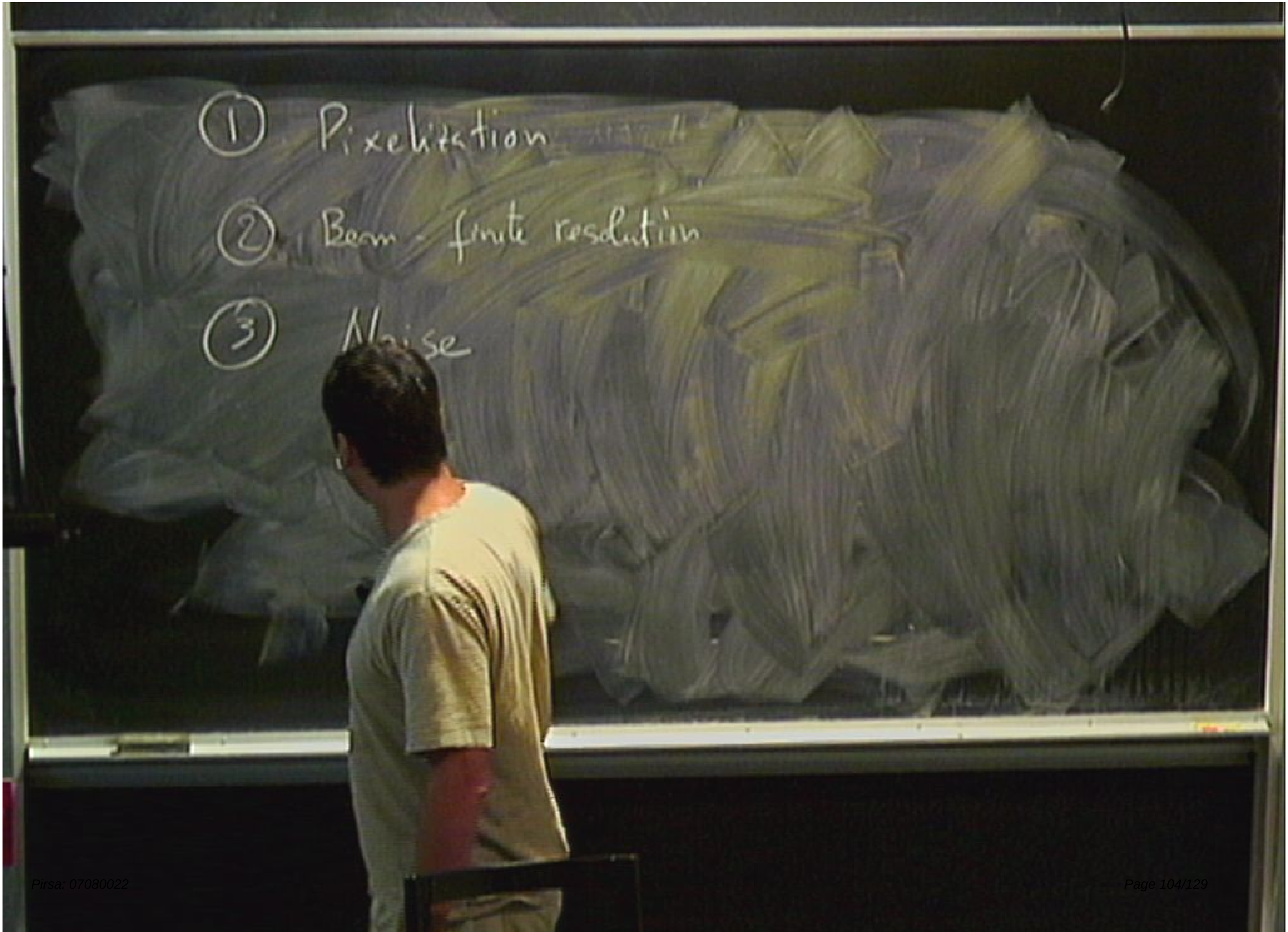
1

2 Beam - finite resolution

1.

2. Beam - finite resolution

3. Noise


① Pixelization
② Beam - finite resolution
③ Noise

1. Pixelization
2. Beam - finite resolution
3. Noise

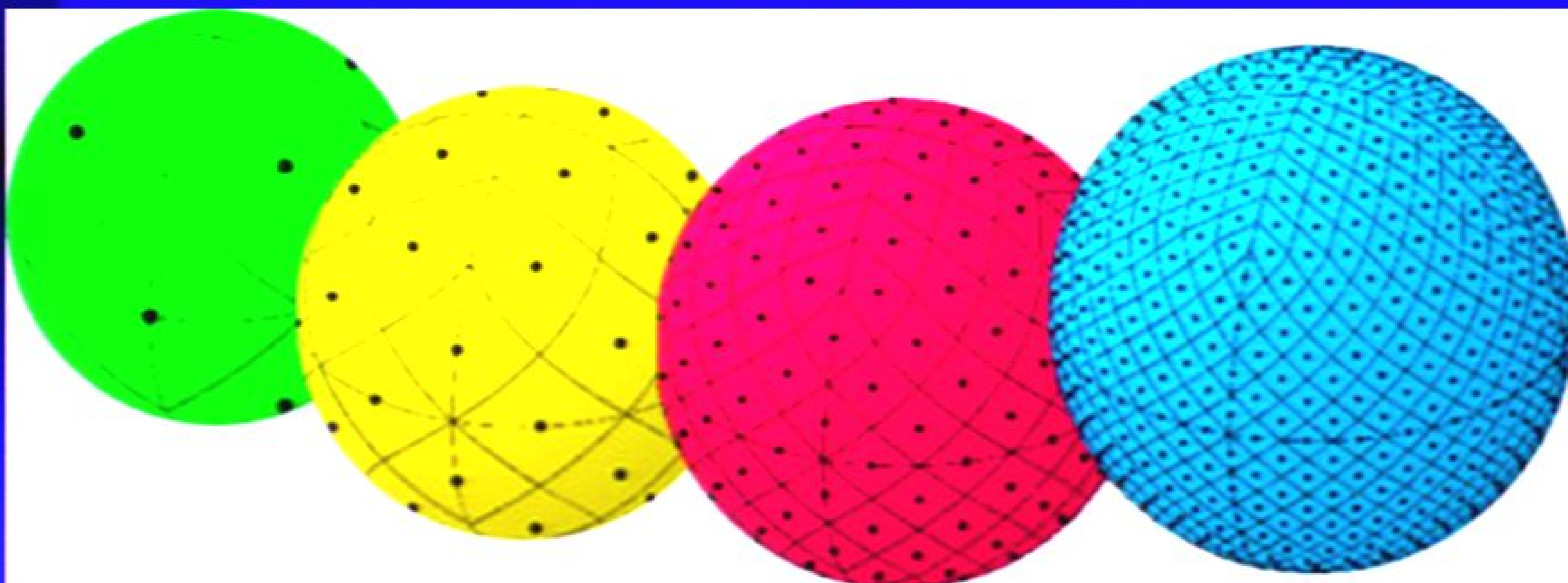
$$a_{\ell m} = \sum_{pix} \frac{4\pi}{N_{pix}} \, \delta T_{pix} \, Y^*_{\ell m}(\Omega)$$

1. Pixelization
2. Beam - finite resolution
3. Noise

QUADCUBE

$$a_{\ell m} = \sum_{pix} \frac{4\pi}{N_{pix}} \delta T_{pix} Y^*_{\ell m}(\Omega)$$

Healpix – pixelizing the sphere



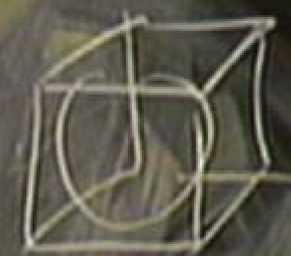
1. Pixelization
2. Beam - finite resolution
3. Noise

QUADCUBE

$$a_{\ell m} = \sum_{pix} \frac{4\pi}{N_{pix}} \, \delta T_{pix} \, Y^*_{\ell m}(\Omega)$$

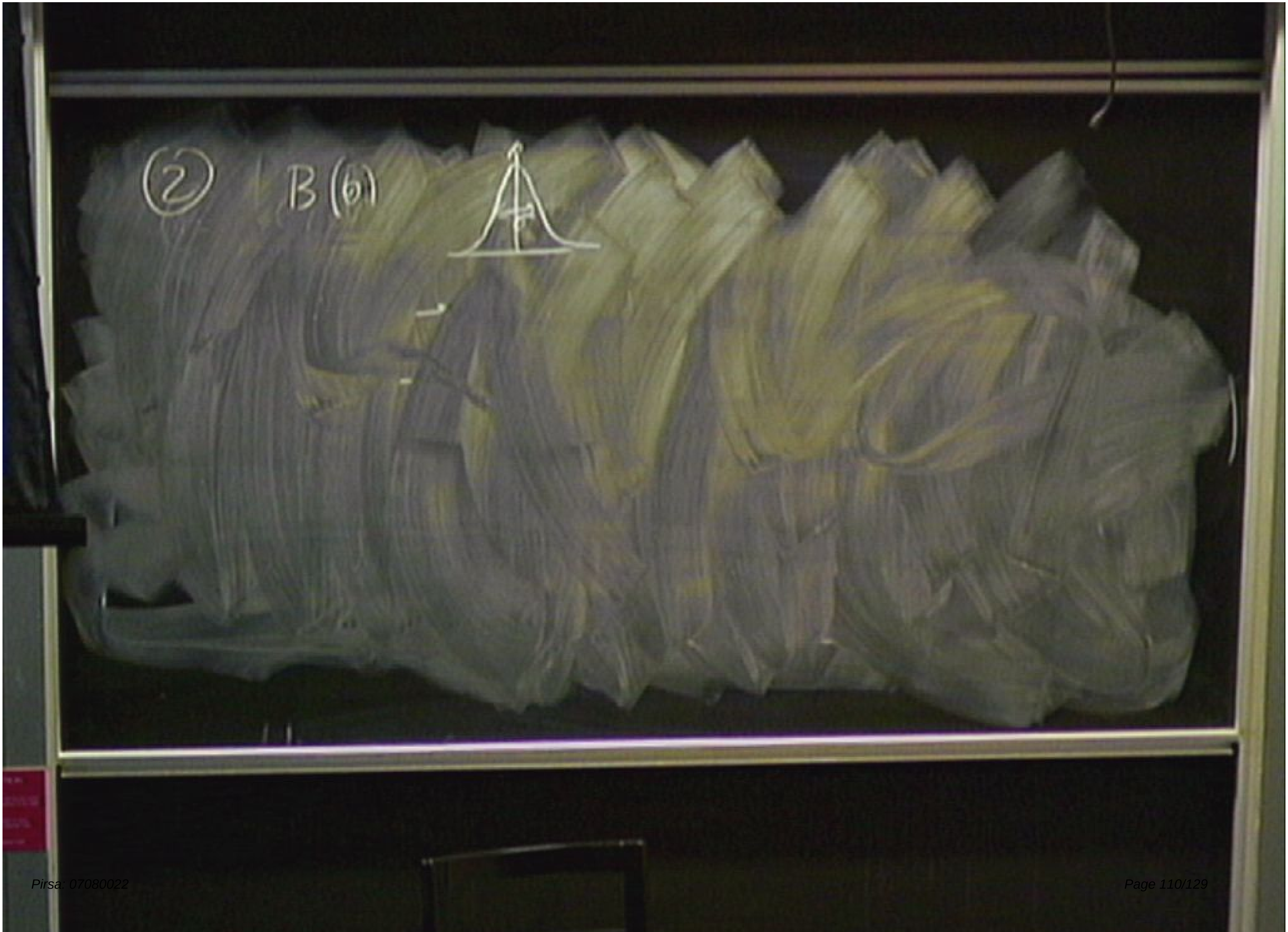
$$\delta T_{pix} = \delta T^{sky} * B$$

1. Pixelization ~~δ~~
2. Beam - finite resolution
3. Noise

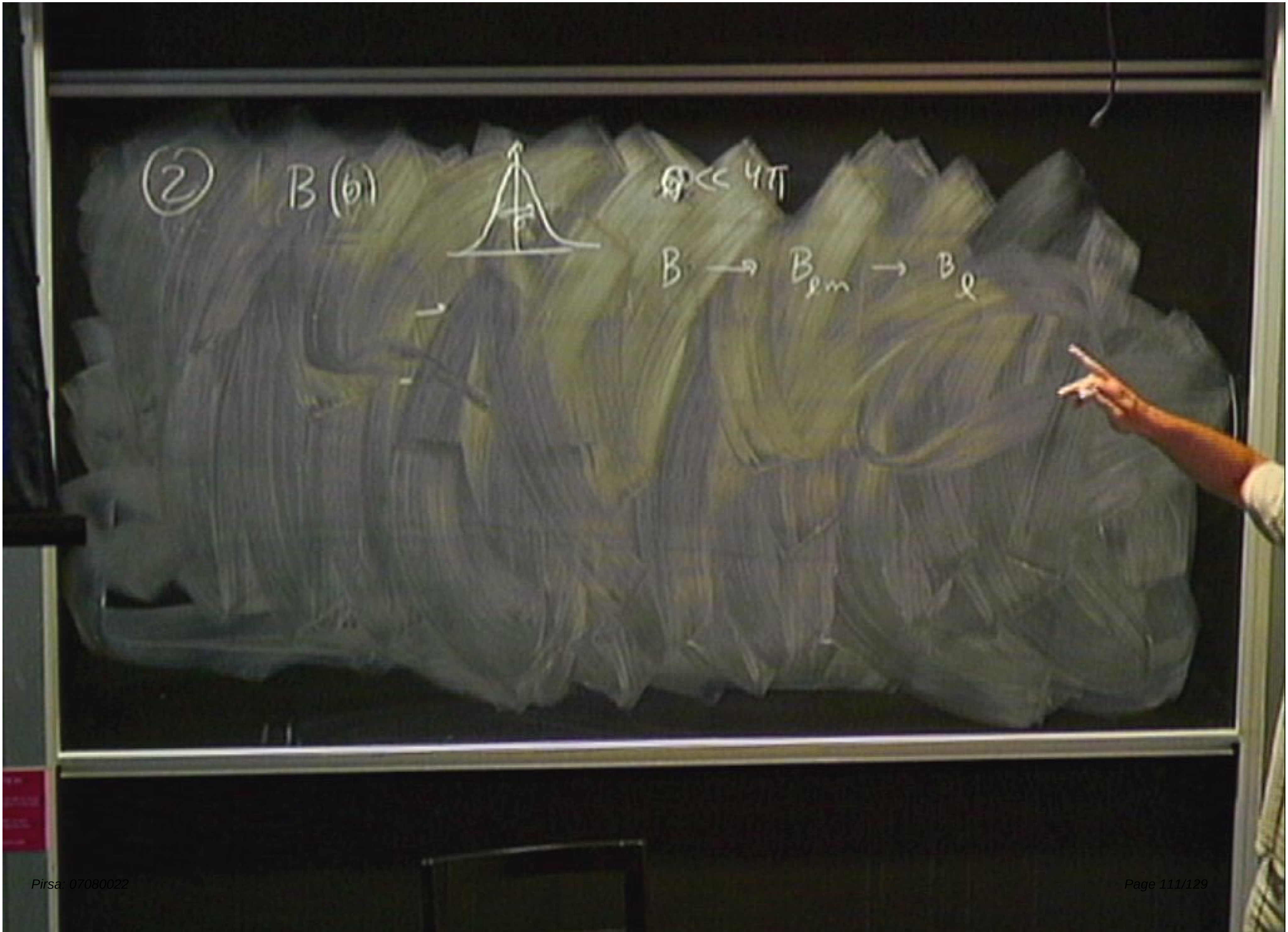


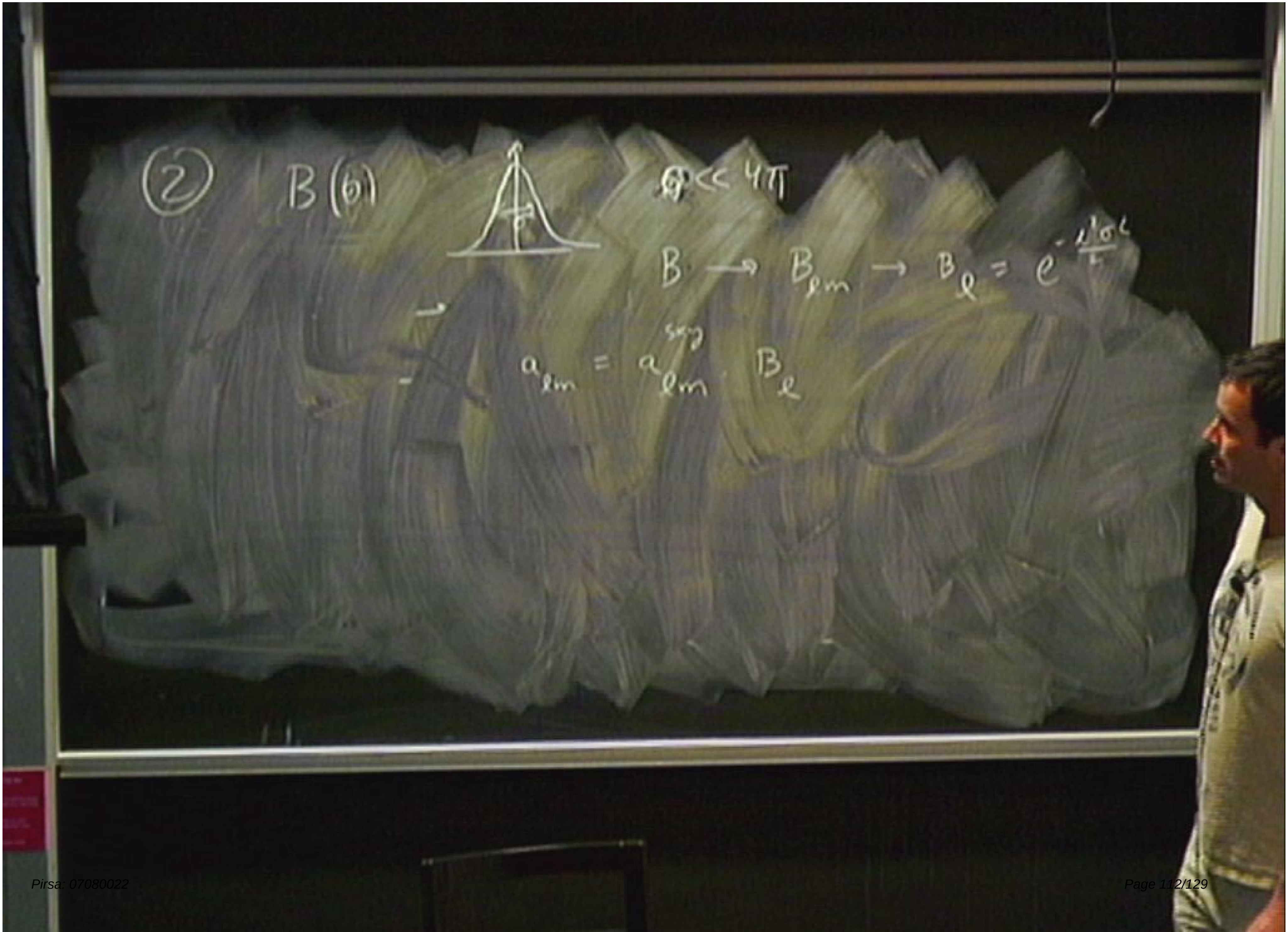
$$a_{\ell m} = \sum_{pix} \frac{4\pi}{N_{pix}} \delta T_{pix} Y^{*}_{\ell m}(\Omega)$$

$$\delta T_{pix} = \delta T^{sky} * B + N$$



2
B(6)


2
B(b)
B → Bℓm → Bℓ


2
B(θ)
θ << 4π
B → B_lm → B_l = e^{-l²σ²/2}
a_lm = a^sky_lm B_l

(2) $B(\theta)$

$\theta \ll 4\pi$

$$B \rightarrow B_{\ell m} \rightarrow B_\ell = e^{-\frac{\ell^2\sigma^2}{2}}$$

$$a_{\ell m} = a^{sky}_{\ell m} \cdot B_\ell \longrightarrow \langle \hat{C}_\ell \rangle = C_\ell\, e^{-\ell^2\sigma^2}$$

(2) $B(\theta)$

$\theta \ll 4\pi$

$$B \longrightarrow B_{\ell m} \longrightarrow B_\ell = e^{-\frac{\ell^2\sigma^2}{2}}$$

$$a_{\ell m} = a^{sky}_{\ell m} \cdot B_\ell \longrightarrow \langle \hat{C}_\ell \rangle = C_\ell e^{-\ell^2\sigma^2}$$

Deconvolving the beam $\quad \hat{C}_\ell \longrightarrow \hat{C}_\ell / B^2$

(2) $B(\theta)$

$\theta \ll 4\pi$

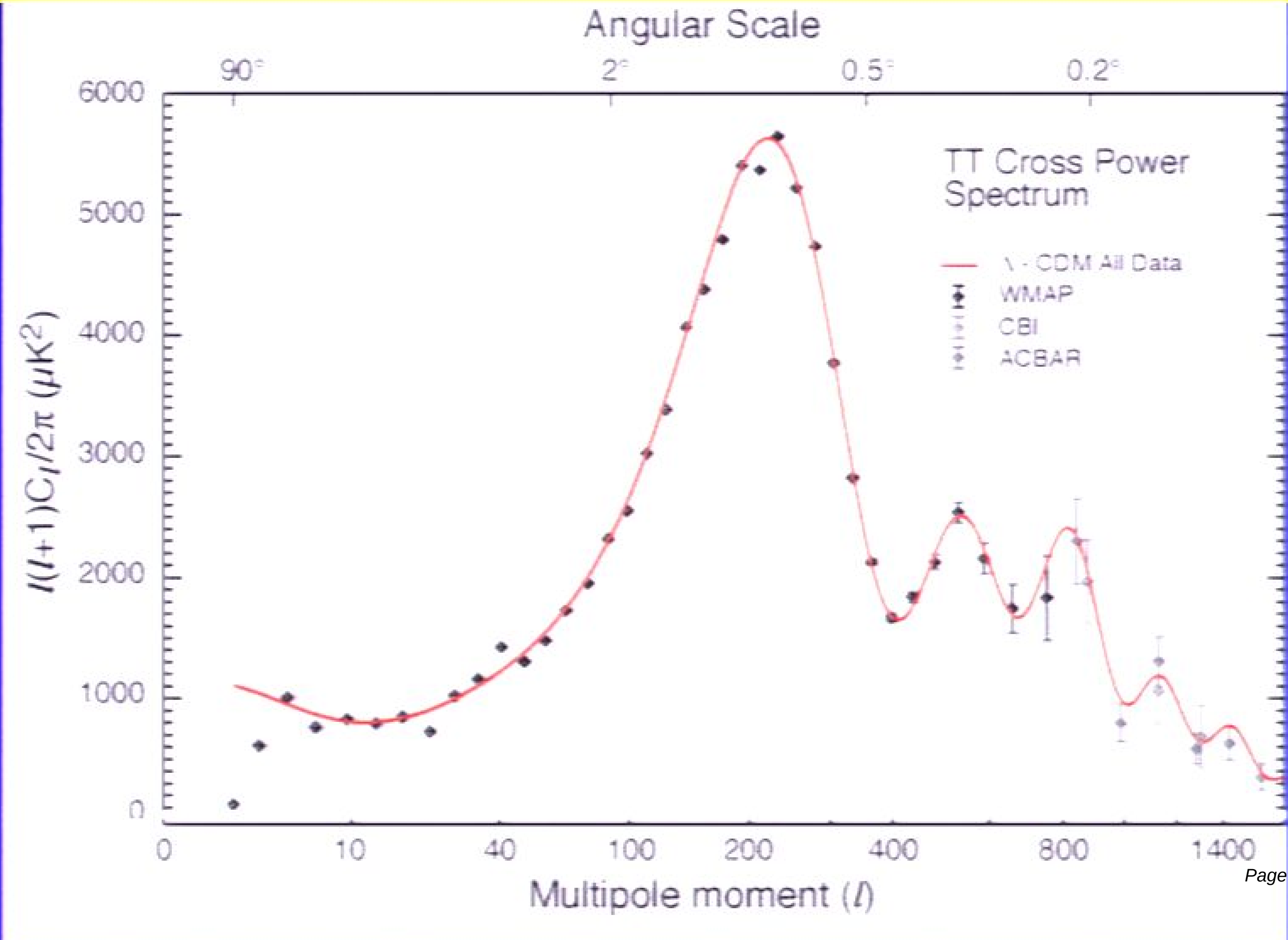
$$B \longrightarrow B_{\ell m} \longrightarrow B_\ell = e^{-\frac{\ell^2\sigma^2}{2}}$$

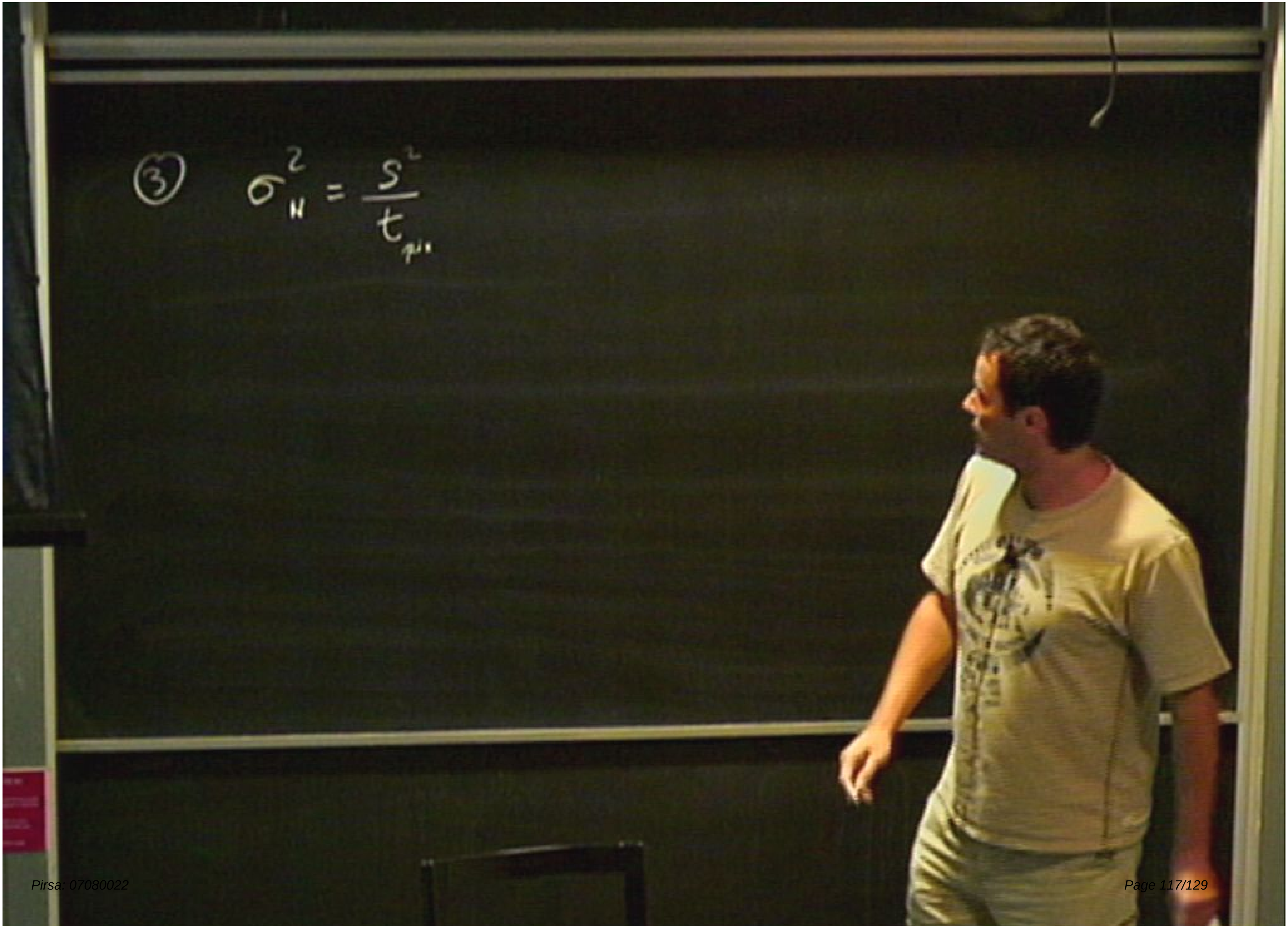
$$a_{\ell m} = a^{sky}_{\ell m} \cdot B_\ell \longrightarrow \langle \hat{C}_\ell \rangle = C_\ell \, e^{-\ell^2\sigma^2}$$

Deconvolving the beam $\quad \hat{C}_\ell \longrightarrow \hat{C}_\ell / B^2 = \hat{C}_\ell^{uns}$

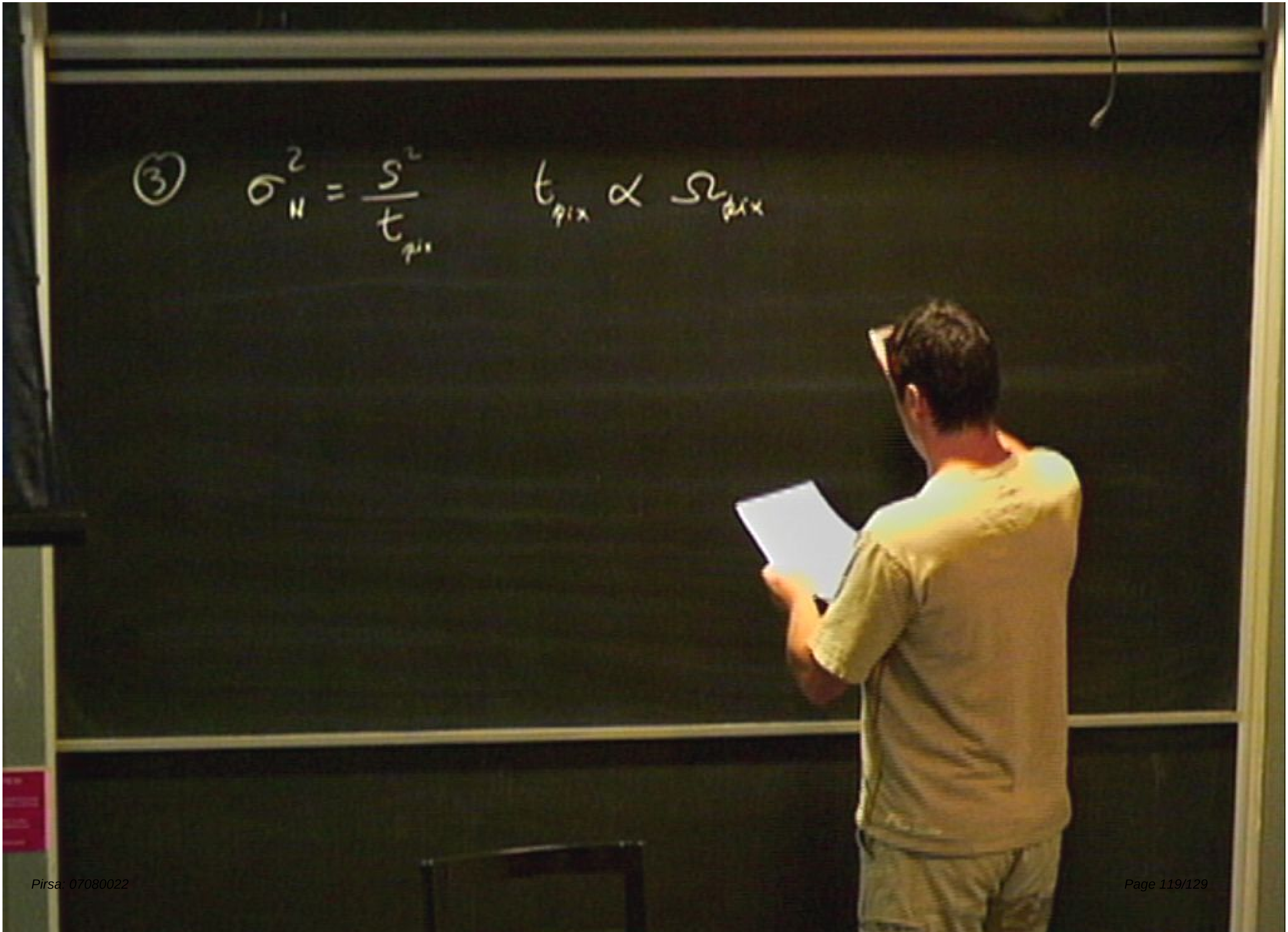
$$\langle \hat{C}_\ell^{uns} \rangle = C_\ell$$

The power spectrum



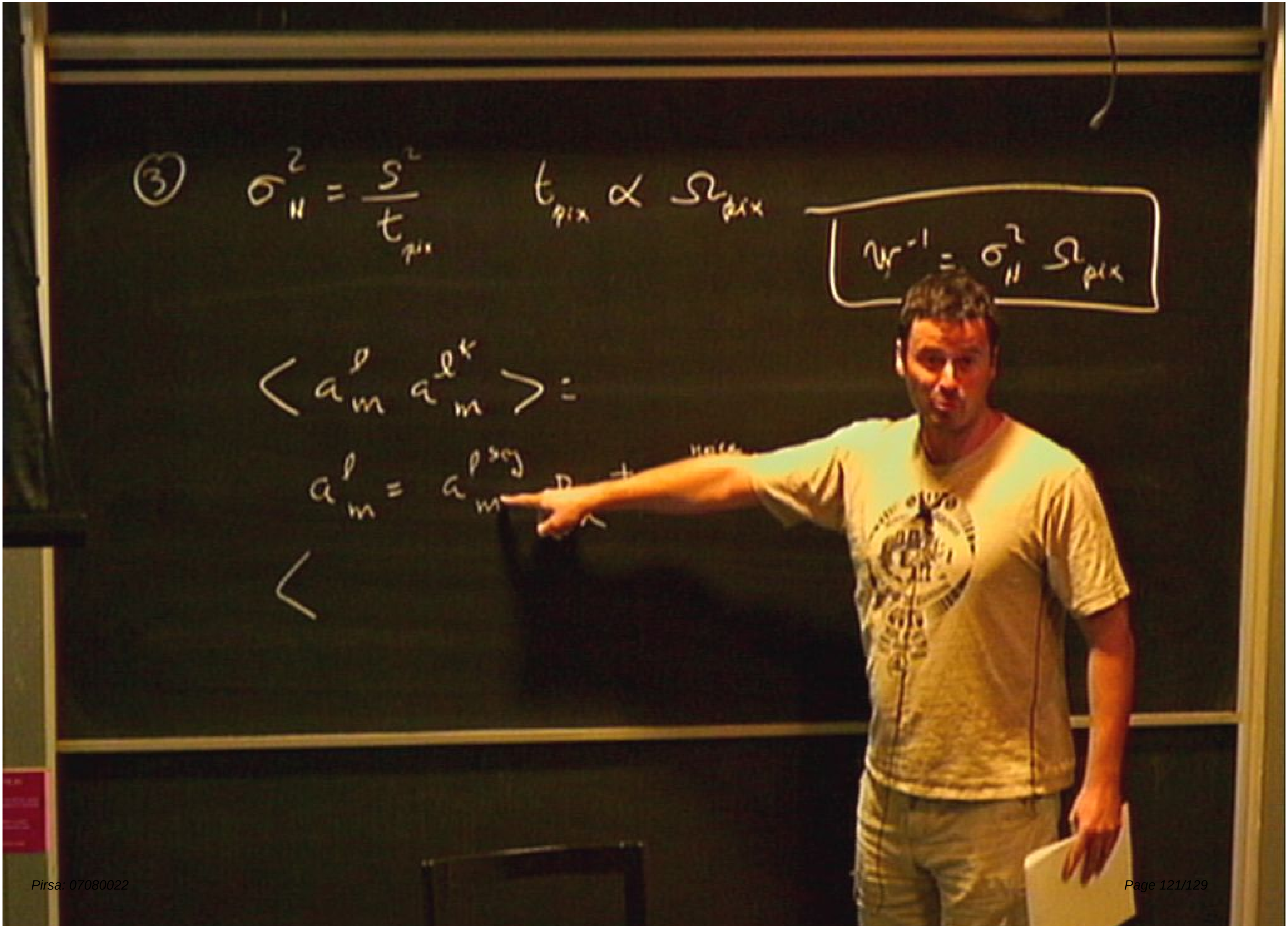

3
$\sigma_N^2 = \frac{s^2}{t_{pix}}$

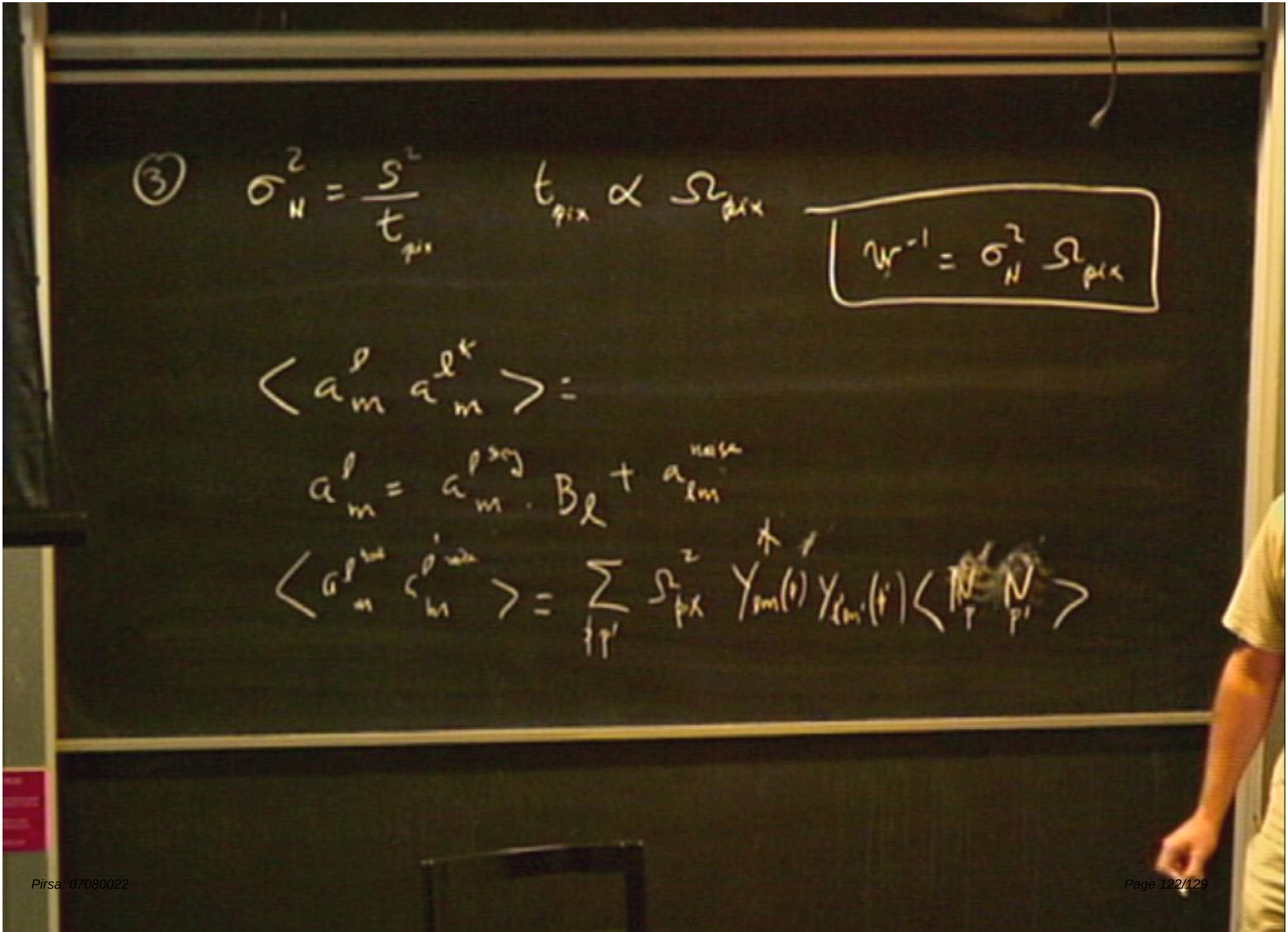
3 $\sigma_N^2 = \frac{s^2}{t_{pix}}$ t_{pix}


③ $\sigma_N^2 = \frac{s^2}{t_{pix}}$
$t_{pix} \propto \Omega_{pix}$

(3) $\sigma_N^2 = \frac{s^2}{t_{pix}}$ $\quad t_{pix} \propto \Omega_{pix}$

$$\boxed{w^{-1} = \sigma_N^2 \, \Omega_{pix}}$$


③ $\sigma_N^2 = \frac{s^2}{t_{pix}}$
$t_{pix} \propto \Omega_{pix}$
$\boxed{w^{-1} = \sigma_N^2 \Omega_{pix}}$
$\langle a^\ell_m a^{\ell *}_m \rangle =$
$a^\ell_m = a^{\ell\, sky}_m$
$\langle$


③ $\sigma_N^2 = \frac{s^2}{t_{pix}}$ $t_{pix} \propto \Omega_{pix}$
$\boxed{w^{-1} = \sigma_N^2 \, \Omega_{pix}}$
$\langle a_{\ell m} \, a_{\ell m}^* \rangle =$
$a_{\ell m} = a_{\ell m}^{sky} \cdot B_\ell + a_{\ell m}^{noise}$
$\langle a_{\ell m}^{noise} \, a_{\ell m'}^{noise} \rangle = \sum_{p p'} \Omega_{pix}^2 \, Y_{\ell m}^*(p) \, Y_{\ell m'}(p') \langle N_p \, N_{p'} \rangle$

(3) $\sigma_N^2 = \dfrac{s^2}{t_{pix}}$ $\qquad t_{pix} \propto \Omega_{pix}$

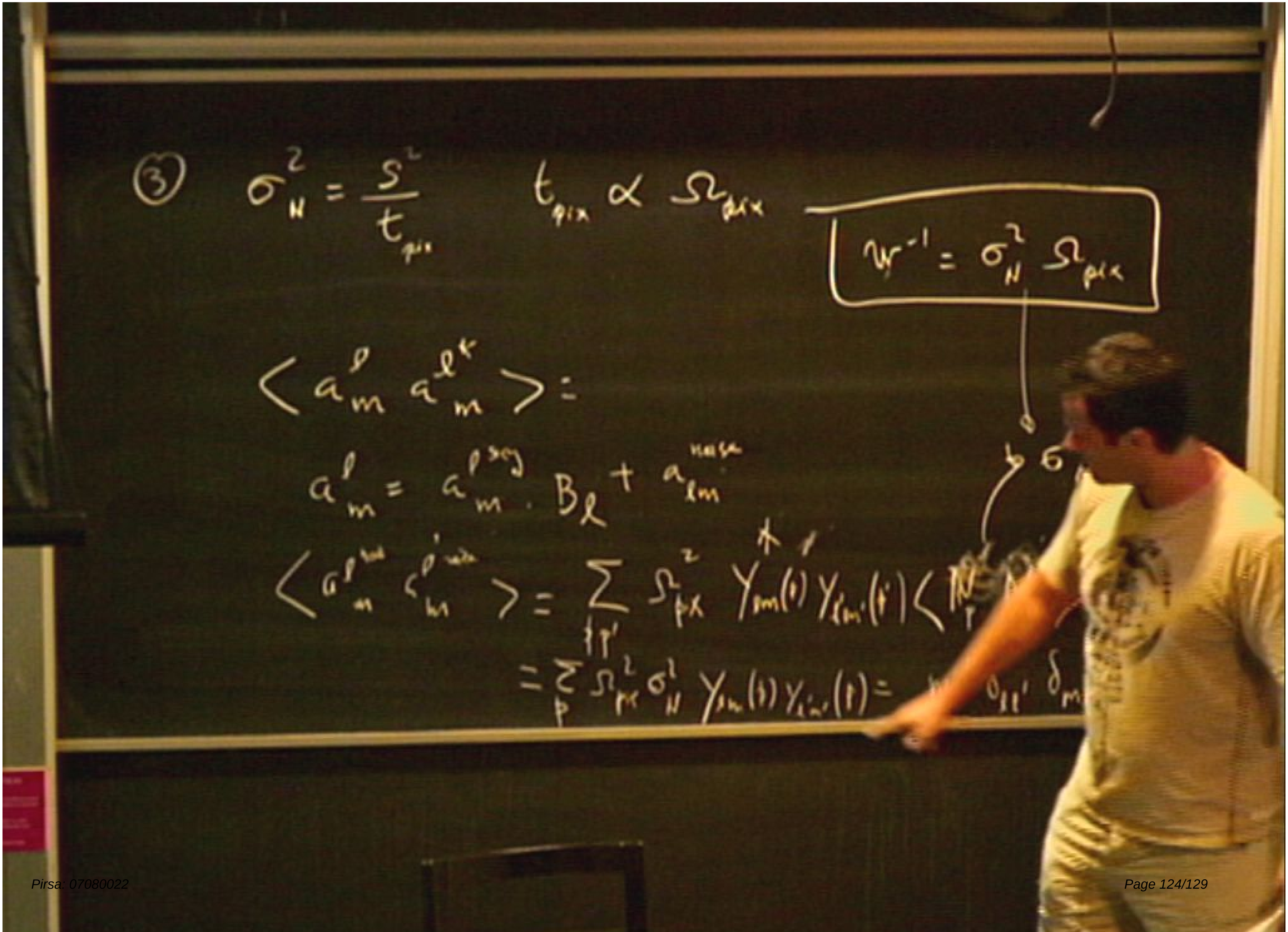
$$\boxed{w^{-1} = \sigma_N^2 \, \Omega_{pix}}$$

$$\langle a_{\ell m} \, a^{*}_{\ell m} \rangle =$$

$$a_{\ell m} = a^{sky}_{\ell m} \cdot B_\ell + a^{noise}_{\ell m}$$

$$\langle a^{noise}_{\ell m} \, a^{noise\,*}_{\ell m'} \rangle = \sum_{p p'} \Omega_{pix}^2 \, Y^*_{\ell m}(p) \, Y_{\ell m'}(p') \, \langle N_p N_{p'} \rangle$$

$$\sigma_N^2 \, \delta_{pp'}$$


③ $\sigma_N^2 = \frac{s^2}{t_{pix}}$
$t_{pix} \propto \Omega_{pix}$
$\boxed{w^{-1} = \sigma_N^2 \Omega_{pix}}$
$\langle a^{\ell}_{m} a^{\ell *}_{m} \rangle =$
$a^{\ell}_{m} = a^{\ell\, sky}_{m} \cdot B_\ell + a^{noise}_{\ell m}$
$\langle a^{\ell\, noise}_{m} a^{\ell'\, noise *}_{m'} \rangle = \sum_{p p'} \Omega_{pix}^2 Y_{\ell m}(p) Y_{\ell' m'}(p') \langle N_p$
$= \sum_p \Omega_{pix}^2 \sigma_N^2 Y_{\ell m}(p) Y_{\ell' m'}(p) =$

③ $\sigma_N^2 = \dfrac{s^2}{t_{pix}} \qquad t_{pix} \propto \Omega_{pix}$

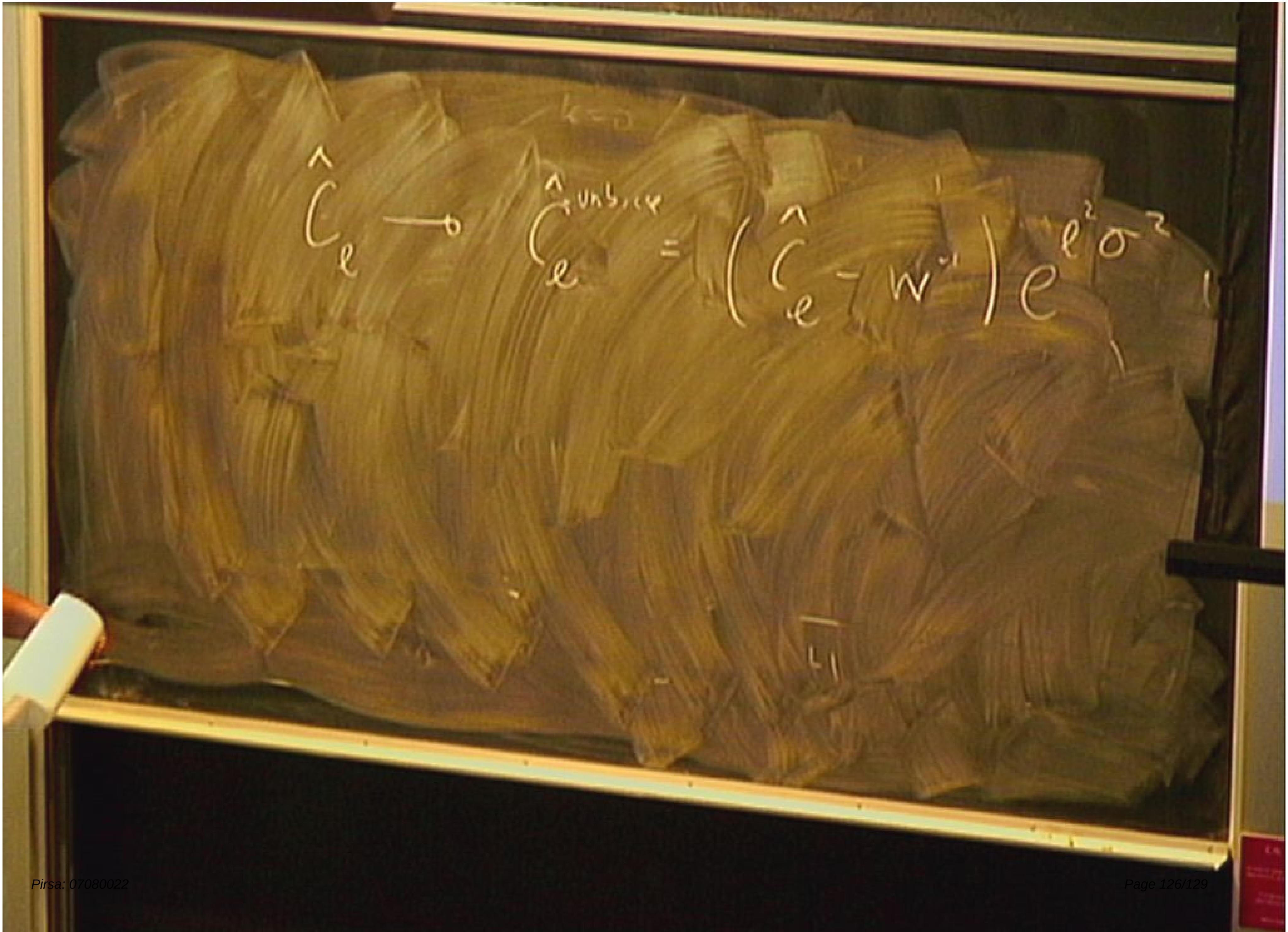
$$\boxed{w^{-1} = \sigma_N^2 \, \Omega_{pix}}$$

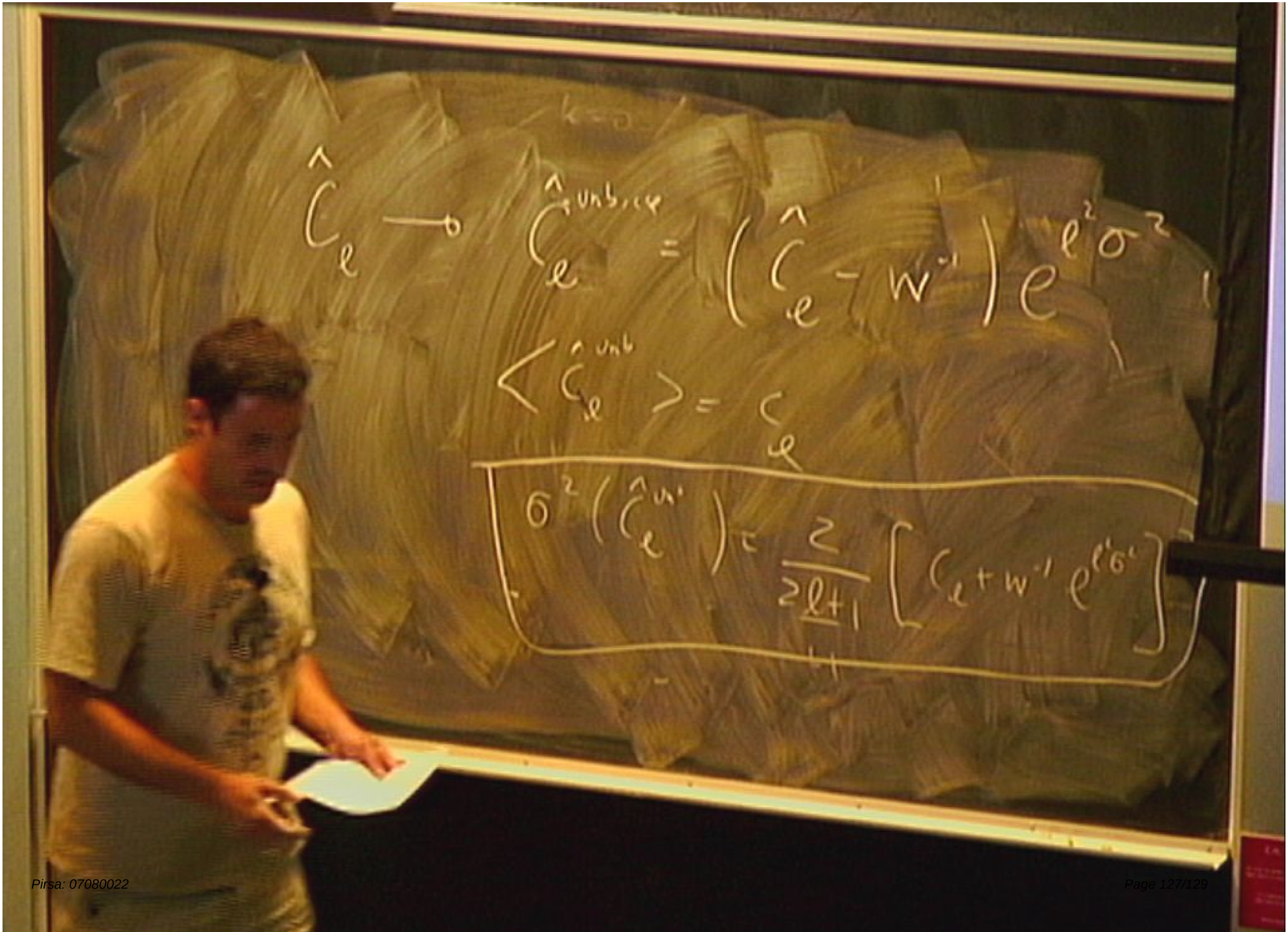
$$\boxed{\langle a^{\ell}_{m} \, a^{\ell *}_{m} \rangle = \delta_{\ell\ell'} \delta_{mm'} \left(C_\ell B_\ell^{\overset{\text{beam}}{2}} + \overset{\text{noise}}{w^{-1}} \right)}$$

$$a^{\ell}_{m} = a^{\ell\,\text{sky}}_{m} \cdot B_\ell + a^{\text{noise}}_{\ell m}$$

$$\langle a^{\ell\,\text{noise}}_{m} \, a^{\ell'\,\text{noise}\,*}_{m'} \rangle = \sum_{pp'} \Omega_{px}^2 \, Y^*_{\ell m}(p) \, Y_{\ell' m'}(p') \, \langle N_p N_{p'} \rangle \qquad \langle N_p N_{p'} \rangle \to \sigma_N^2 \, \delta_{pp'}$$

$$= \sum_p \Omega_{px}^2 \sigma_N^2 \, Y^*_{\ell m}(p) \, Y_{\ell' m'}(p) = w^{-1} \delta_{\ell\ell'} \delta_{mm'}$$


$\hat{C}_\ell \longrightarrow \hat{C}_\ell^{\,\text{unbias}} = \left(\hat{C}_\ell - w^{-1}\right) e^{\ell^2 \sigma^2}$


$\hat{C}_\ell \rightarrow \hat{C}_\ell^{unbias} = (\hat{C}_\ell - w^{-1}) e^{\ell^2\sigma^2}$
$\langle \hat{C}_\ell^{unb} \rangle = C_\ell$
$\sigma^2(\hat{C}_\ell^{unb}) = \frac{2}{2\ell+1}\left[C_\ell + w^{-1} e^{\ell^2\sigma^2}\right]$

$$\hat{C}_\ell \longrightarrow \hat{C}_\ell^{\,\text{unbiased}} = \left(\hat{C}_\ell - w^{-1}\right) e^{\ell^2 \sigma^2}$$

$$\left\langle \hat{C}_\ell^{\,\text{unb}} \right\rangle = C_\ell$$

$$\boxed{\sigma^2\left(\hat{C}_\ell^{\,\text{unb}}\right) = \frac{2}{2\ell+1}\left[C_\ell + w^{-1} e^{\ell^2\sigma^2}\right]^2}$$

The power spectrum