Title: Quantum gravity and the Coulomb potential

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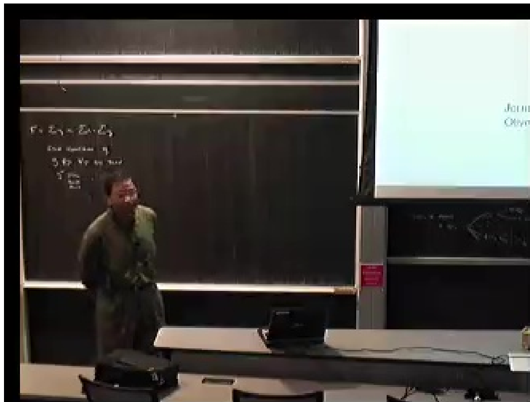
Abstract: An ingredient in recent discussions of curvature singularity avoidance in quantum gravity is the "inverse scale factor" operator and its generalizations. I describe a general lattice origin of this idea, and show how it applies to the Coulomb singularity in quantum mechanics, and more generally to lattice formulations of quantum gravity. The example also demonstrates that a lattice discretized Schrodinger or Wheeler-DeWitt equation is computationally equivalent to the so called "polymer"
quantization derived from loop quantum gravity.



Quantum gravity and the Coulomb potential

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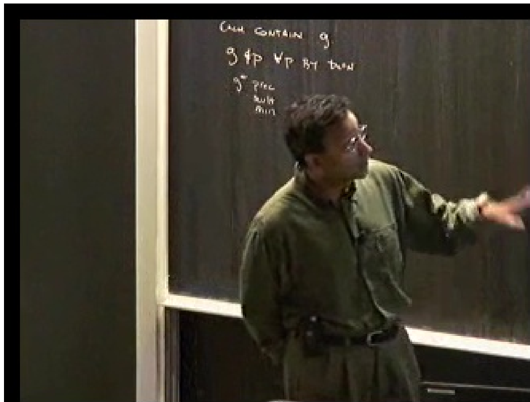
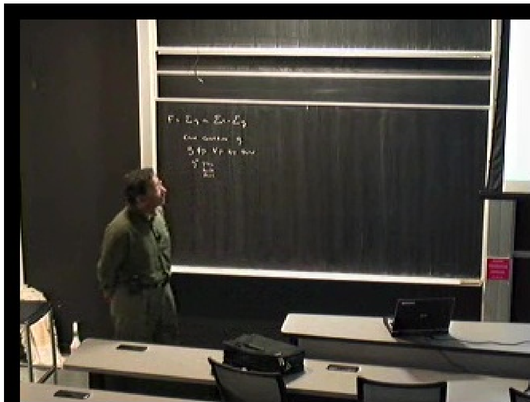
August 1, 2007
PI

with
Jorma Louko(Nottingham)
Oliver Winkler (Perimeter)



Outline

- What is "singularity avoidance" in quantum theory?
- Some developments concerning SA in quantum gravity
- Polymer quantization and the Coulomb singularity
- Schrödinger and Wheeler-Dewitt equations on lattices
- Conclusions and lessons



Singularity avoidance in quantum theory

For classically singular potentials $V(x)$, singularity avoidance consists of two components:

- Kinematical SA: In basis states of the Hilbert space

$$\langle \widehat{V(x)} \rangle < \infty$$

- Dynamical SA: Time evolution of arbitrary initial states is well defined: eq. Coulomb scattering cross sections are not divergent. Unitarity.

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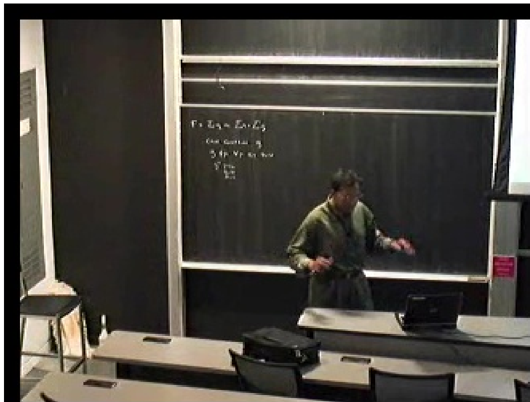


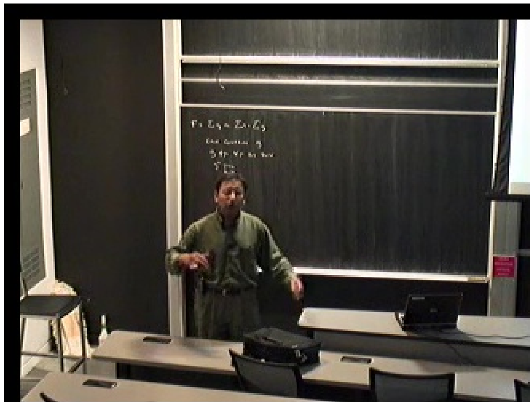
Singularity avoidance in QG

Similarly, in quantum gravity we would like

- Kinematical:

$$\langle \widehat{\text{curvature operators}} \rangle < \infty$$

- Dynamical: "scattering" from what are classically infinite curvature regions of spacetime, or "evolution through the singularity".



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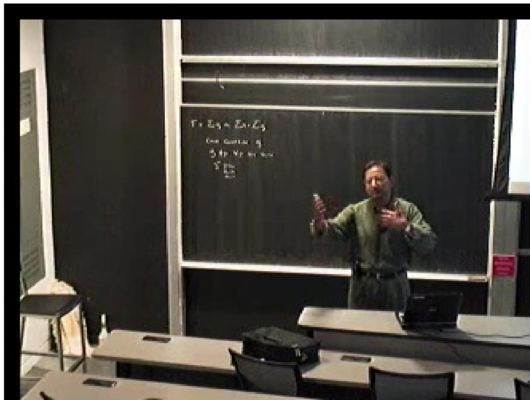

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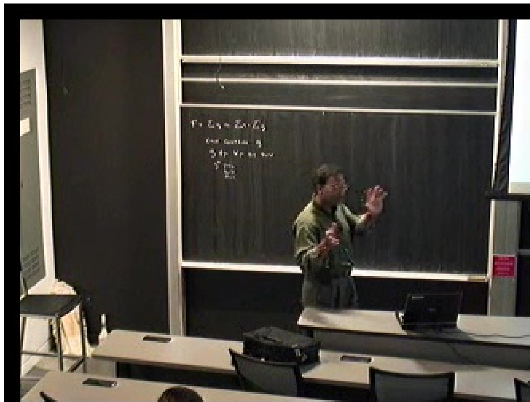


Recent developments in canonical QG I

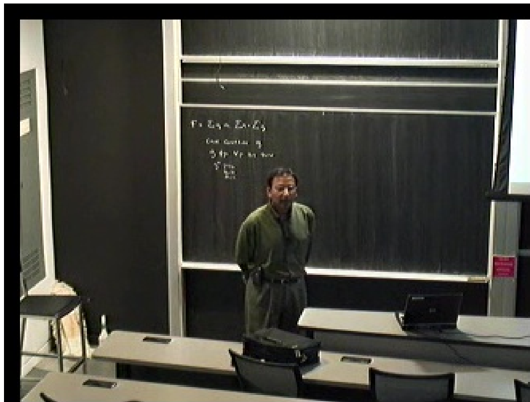
FRW cosmology with scalar field (has a long history)

$$ds^2 = -dt^2 + a(t)^2(dx^2 + y^2 + dz^2)$$

- In LQC (A^i_a, E^{ai}) (Bojowald; Ashtekar et. al,...) : bounded inverse scale factor operator

$$\left\langle \widehat{\frac{1}{a}} \right\rangle < \infty,$$

- Scattering of a wavepacket from the $a = 0$ region.
- Qualitatively similar results in metric variable (q_{ab}, π^{ab}) quantum gravity (VH, O. Winkler, ...) — results independent of loop variables, but due to fundamental discreteness.



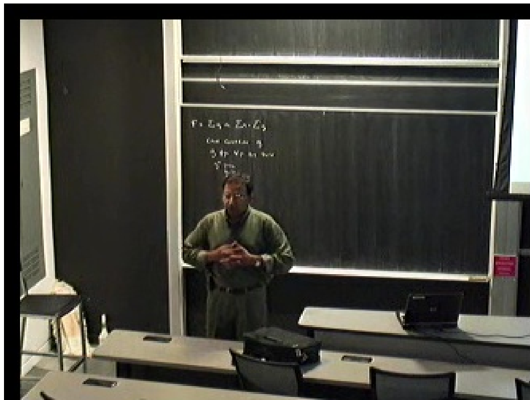
II Effective (QG corrected) dynamics

- Define a Hamiltonian constraint operator
- Represent factors of inverse powers of $\sqrt{q}$ using "Thiemann" trick
- Compute a (kinematical) state $|\psi(p,q)>$ dependent effective constraint

$$H_{eff}(p,q) := <\psi|\hat{H}|\psi>$$

- Use H_{eff} and classical Hamiltonian equations to get QG modified FRW dynamics.

An essential input in all this is the inverse scale factor operator.



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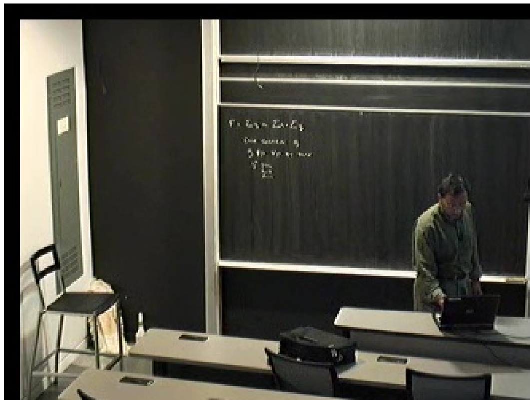

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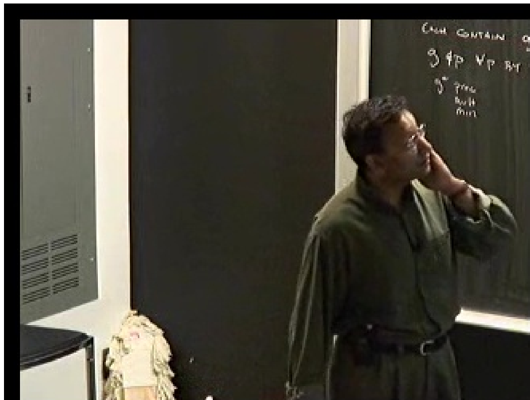
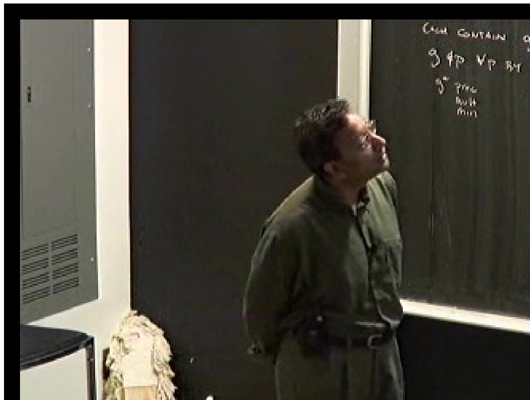


- In LQG the basic operators are a shift operator (holonomy) and a diagonal operator (triad).
- These are just like a field and its translation operators on a lattice.
- Shift operators are the central ingredient in defining inverse scale factor and curvature operators.

This is easily illustrated for a particle on a lattice.



Polymer Quantization (or "loop quantum particle")

The Hilbert space is the vector space of almost periodic functions

$$\psi(p) = \sum_{k=1}^{N} c_k e^{i\mu_k p}$$

μ_k are arbitrary real numbers (an irregular lattice). The inner product is

$$\langle \mu_k | \mu_l \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dp \; e^{ip(\mu_l - \mu_k)} = \delta_{lk}.$$

A representation of the position and translation operators:

$$\hat{x} e^{i\mu_k p} = -i \frac{\partial}{\partial p} e^{i\mu_k p} = \mu_k e^{i\mu_k p} \quad (1)$$

$$\hat{U}_\lambda e^{i\mu_k p} := \widehat{e^{i\lambda p}} e^{i\mu_k p} = e^{i(\mu_k + \lambda)p} \quad (2)$$

An operator corresponding to the momentum p does not exist.

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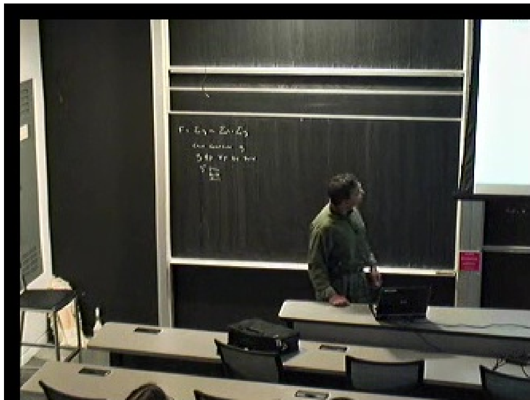
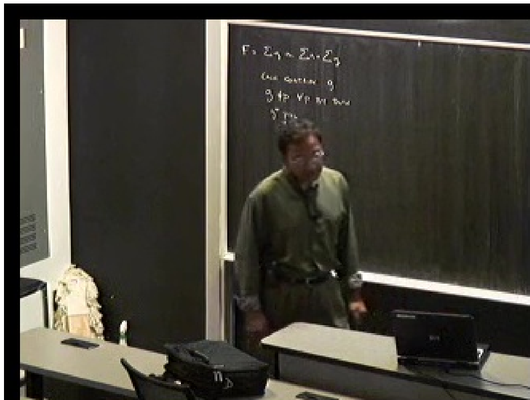


Polymer Hamiltonian: KE

The momentum and kinetic energy operators have to be defined using $\hat{U}_\lambda$. Kinetic energy operator depends on λ:

$$\hat{p^2}_\lambda := \frac{1}{\lambda^2}\left(2I - \hat{U} - \hat{U}^\dagger\right)$$

$$\hat{p^2}|\mu_k\rangle = \frac{1}{\lambda^2}(2|\mu_k\rangle - |\mu_k + \lambda\rangle - |\mu_k - \lambda\rangle)$$



Polymer Hamiltonian: Potential

The Coulomb potential $1/x$ is quantized by representing the r.h.s. of

$$\frac{1}{|x|} = \left(2\frac{1}{i\lambda}U^*\{U, \sqrt{|x|}\}\right)^2$$

as an operator using the definitions of $\hat{U}$ and $\hat{x}$ (Thiemann trick). The resulting operator is bounded.

$$\widehat{\frac{1}{x}}|\mu_k\rangle = \frac{1}{\lambda^2}\left(\sqrt{|\mu_k+\lambda|} - \sqrt{|\mu_k-\lambda|}\right)^2|\mu_k\rangle$$

Similar relations have been used to define curvature operators in quantum gravity. (VH, O. Winkler; A. Dasgupta)

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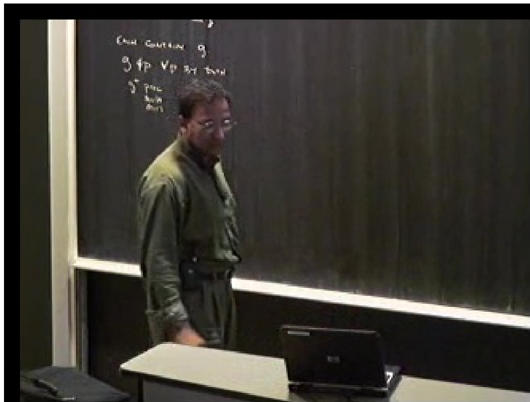


Eigenvalue of 1/x operator

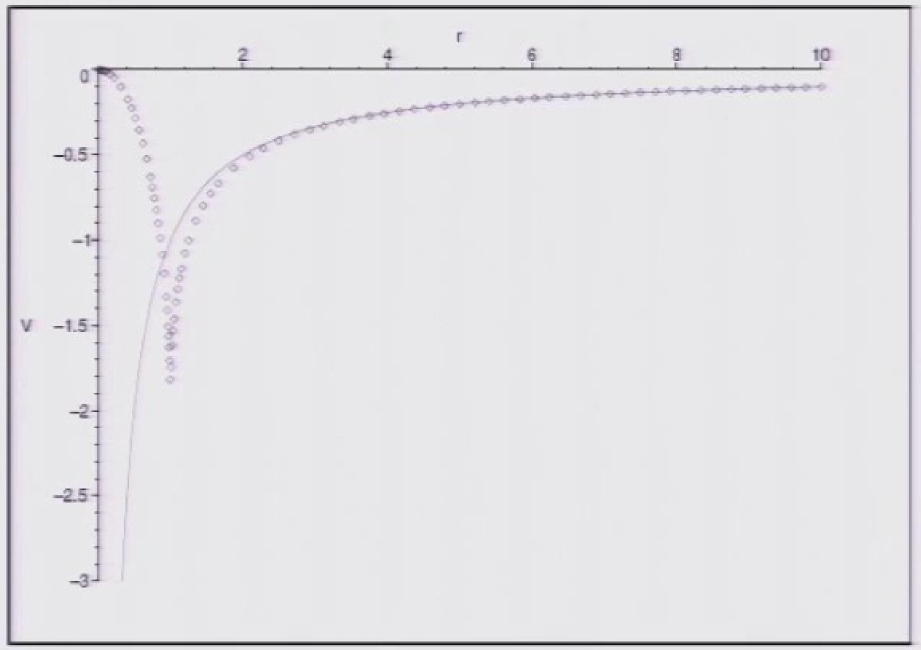
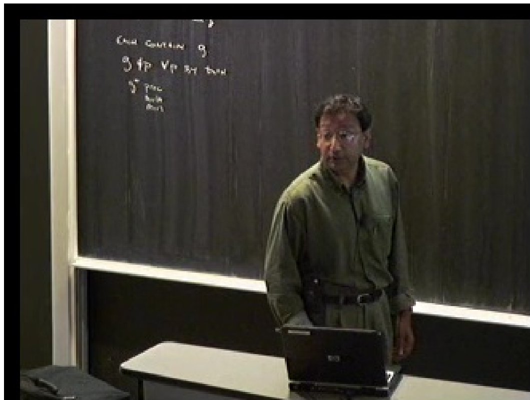


Figure: The Coulomb potential and the eigenvalue of $1/x$ operator.



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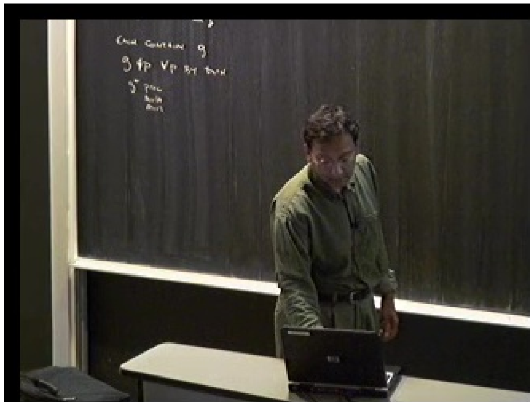

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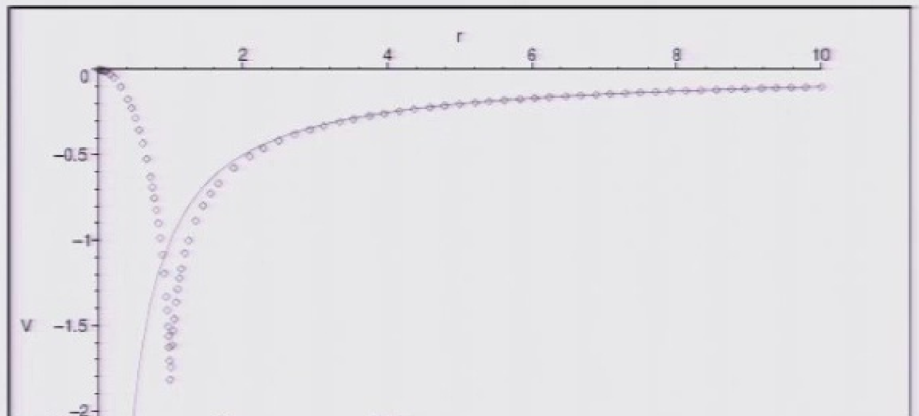
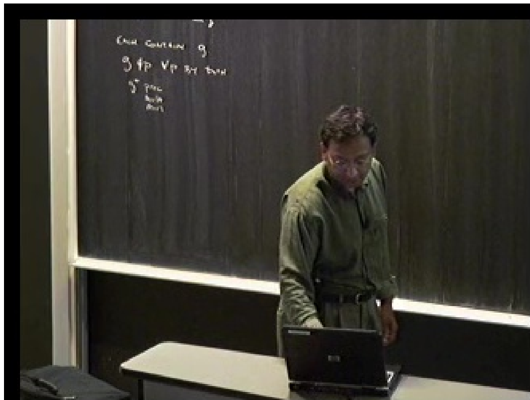


and the antisymmetry condition $c_k = -c_{-k}$.
The polymer eigenvalues are to be compared with the Schrödinger spectrum

$$e_n = -\frac{1}{4n^2}$$



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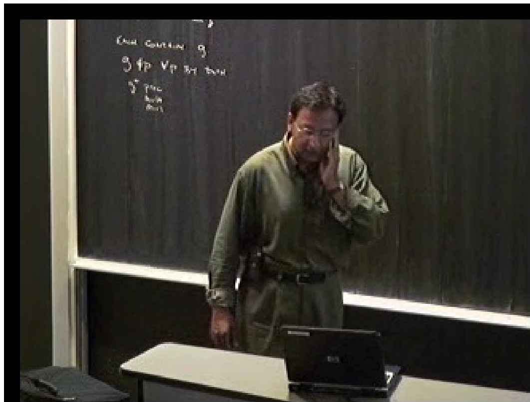

Polymer eigenvalue problem

We look for energy eigenstates in the form

$$\sum_k c_k |k\lambda, \rangle$$

where the coefficients c_n are subject to the normalizability condition

$$\sum_k |c_k|^2 < \infty$$

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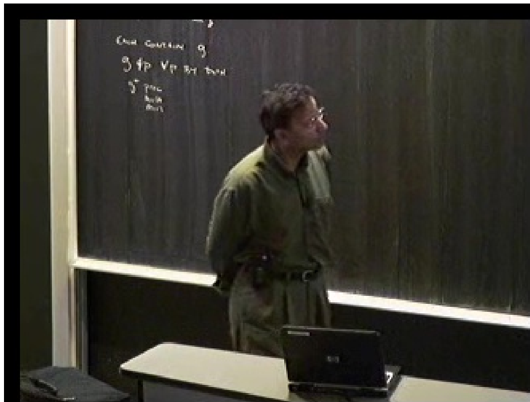


Numerical method

- The eigenvalue problem leads to the recursion relation

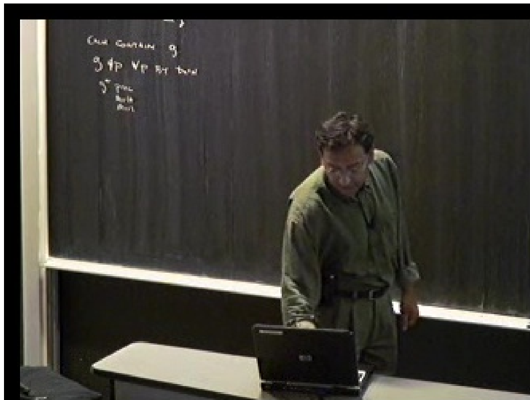
$$c_n(2 - \lambda f_n - \lambda^2 E) = c_{n+1} + c_{n-1}.$$

where $f_n = (\sqrt{|n-1|} + \sqrt{|n+1|})^2$.

- For $E < 0$ and $n \to \infty$ this has solution

$$c_n = \left[1 - \frac{1}{2}\lambda^2 E + \sqrt{(1 - \frac{1}{2}\lambda^2 E)^2 - 1}\right]^{\pm n}$$

- Use shooting method: Pick an E value for some large n_0. Compute c_{n_0}. Use recursion relation to find c_0. Repeat to get the function $c_0(E)$.



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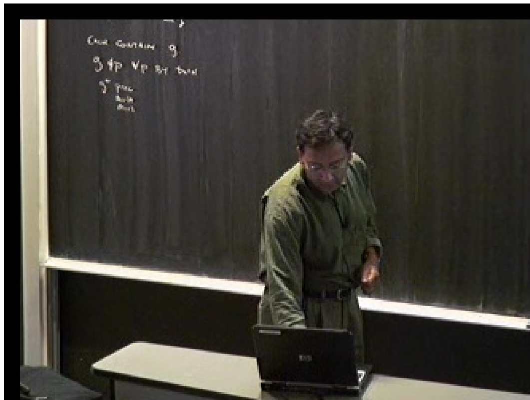


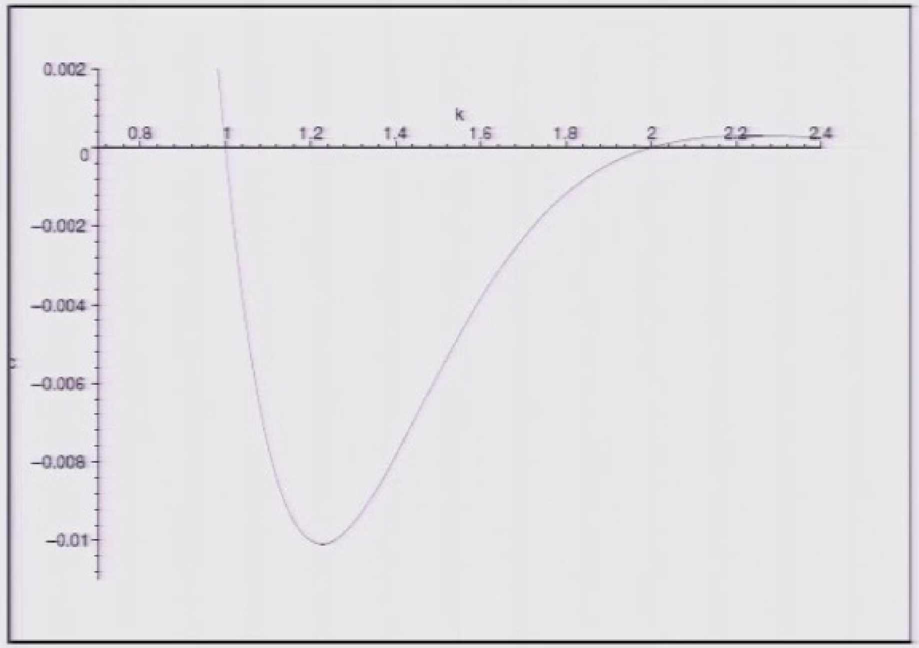
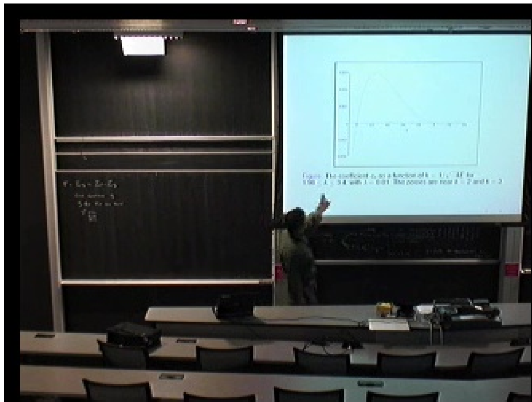
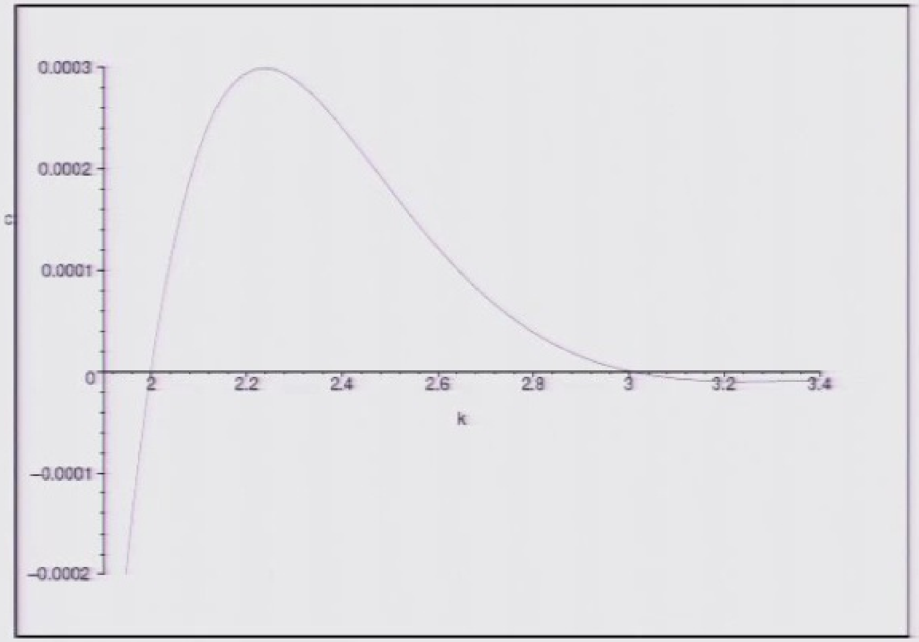


Figure: The coefficient c_0 as a function of $k = 1/\sqrt{-4E}$ for $0.98 \leq k \leq 2.4$, with $\lambda = 0.01$. The zeroes are near $k = 1$ and $k = 2$.





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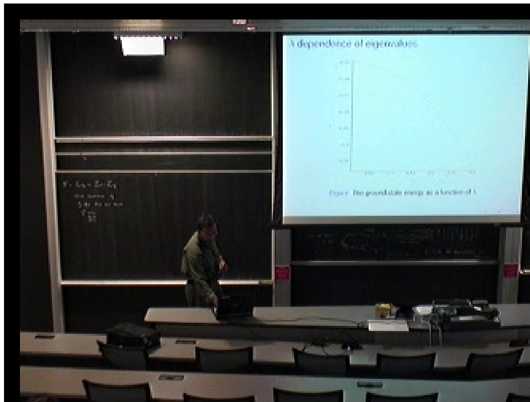


λ dependence of eigenvalues

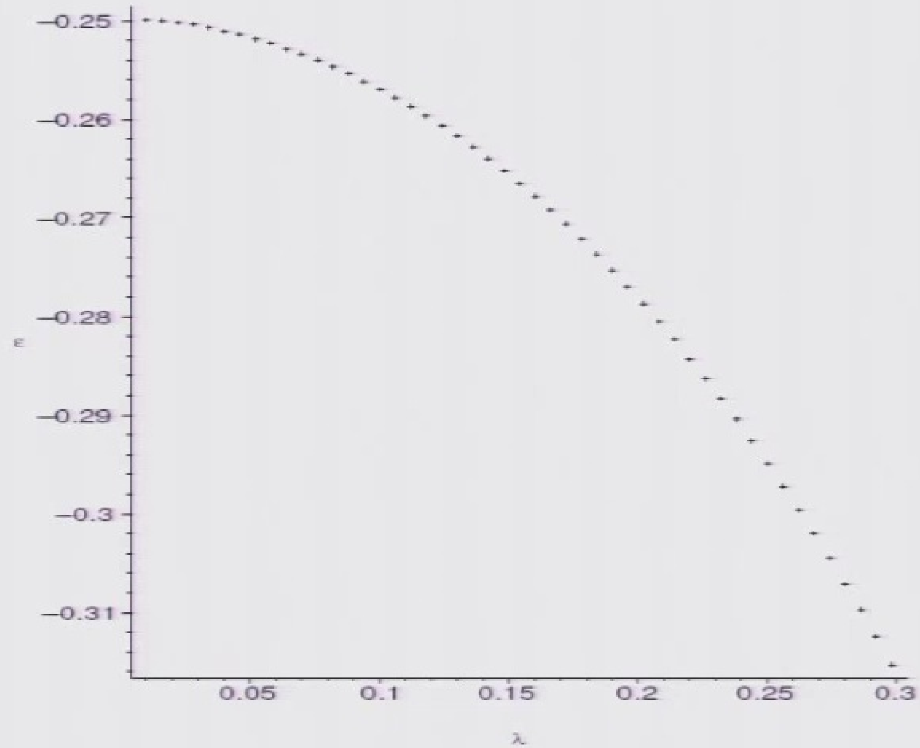
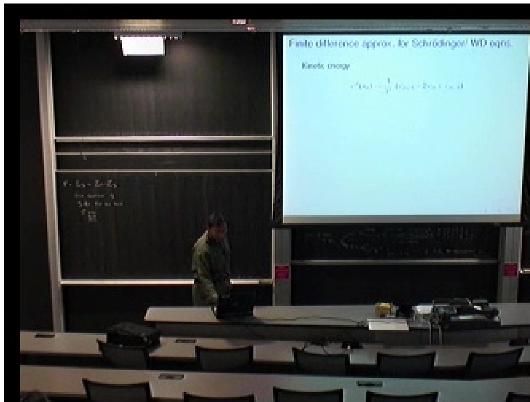


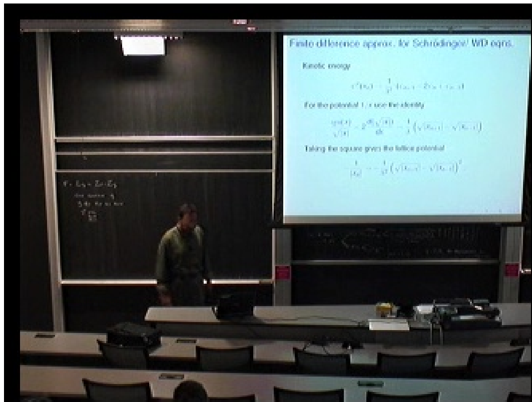
Figure: The ground state energy as a function of λ



Finite difference approx. for Schrödinger/ WD eqns.

Kinetic energy:

$$\psi''(x_n) \rightarrow \frac{1}{\lambda^2}\left(\psi_{n+1} - 2\psi_n + \psi_{n-1}\right)$$



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$$\frac{\operatorname{sgn}(x)}{\sqrt{|x|}} = 2\frac{d(\sqrt{|x|})}{dx} \to \frac{1}{\lambda}\left(\sqrt{|x_{n+1}|} - \sqrt{|x_{n-1}|}\right)$$

Taking the square gives the lattice potential

$$-\frac{1}{|x_n|} \to -\frac{1}{\lambda^2}\left(\sqrt{|x_{n+1}|} - \sqrt{|x_{n-1}|}\right)^2.$$

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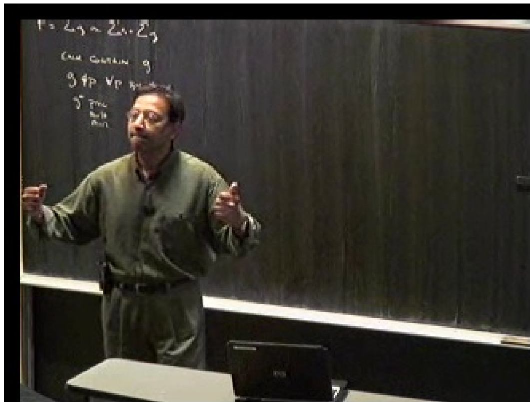
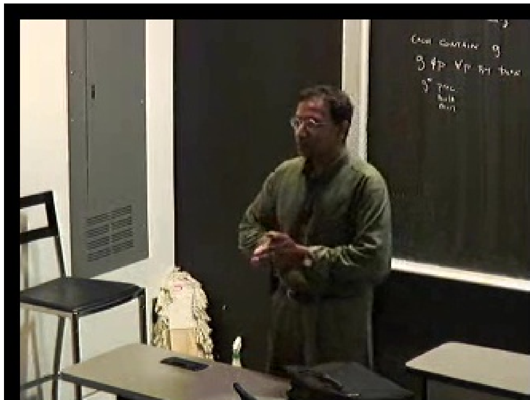


ADM constraint

- phase space $(a, \phi; p_a, p_\phi)$

$$H = -\frac{3}{8}\frac{p_a^2}{|a|^3} + |a|^3\Lambda + 8\pi G\frac{p_\phi^2}{2|a|^3} + |a|^3 V(\phi)$$

- On a regular (or irregular lattice) the essential contribution to quantum corrections near the big bang come from writing $1/a$ factors using shift operators.
- The "Wheeler-DeWitt limit" for large a is immediate (which is the equation discretized) to get the difference equation.



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Identical to Polymer quantization

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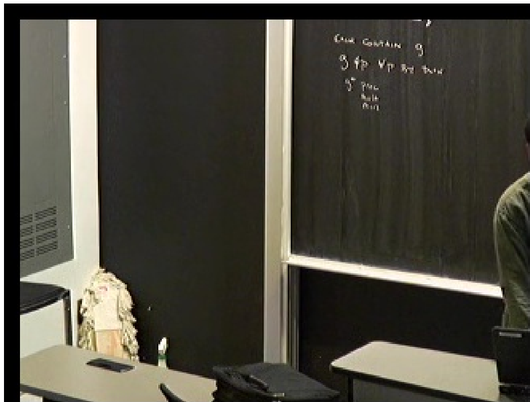

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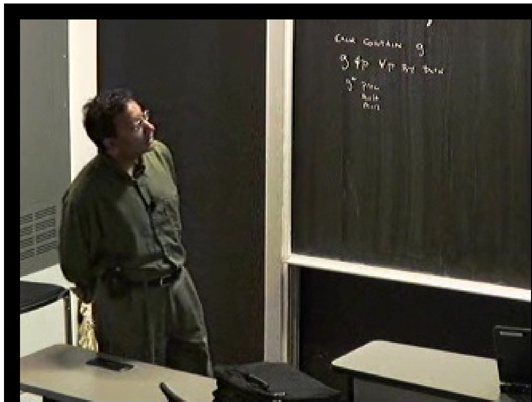

Conclusions and lessons

- Spectra for the antisymmetric case converge to standard qm as $\lambda \to 0$. But the spectra for the symmetric case depend strongly on λ even though the singularity has been kinematical resolved.
- Mechanism of singularity resolution is ultimately the same as in standard quantum mechanics: a boundary condition on the wave function and not a new type of quantization.
- Every finite difference approximation is a polymer quantization. The converse also holds but may be hard to compute with for irregular lattices.

But there is a conceptual difference:

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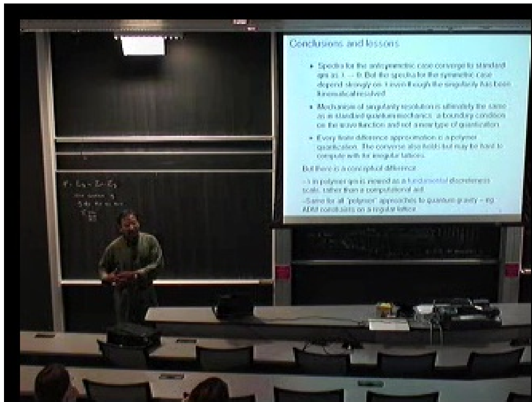

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But there is a conceptual difference:

–λ in polymer qm is viewed as a fundamental discreteness scale, rather than a computational aid.

–Same for all "polymer" approaches to quantum gravity – eg. ADM constraints on a regular lattice.