Title: Symmetrised tomography for efficient characterisation of noisy quantum channels

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Abstract:

Symmetrised tomography for efficient characterisation of noisy quantum channels

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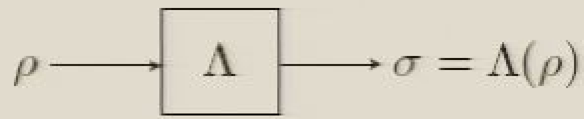
June 16, 2007

Motivation

We want to experimentally verify properties of noise in a quantum system – thresholds, as they stand, are dependent on details of the error model.



So far, the approach has been to fully characterise Λ.

$$\Lambda(\rho) = \sum_k A_k \rho A_k^\dagger$$
$$= \sum_{i,j} \chi_{ij} P_i \rho P_j, \quad P_i \in \mathcal{P}_n$$

But $i, j \in [0, d^2 - 1]$, so χ has $O(d^4)$ parameters.

No hope of characterising noise in a QC moderately larger than today's state of the art.

But without full tomography, one can

- estimate average fidelity
- estimate probability of Pauli errors with weight up to w
- observe some types non-Markovian noise behaviour
- search for some types of noiseless or unitarily recoverable subsystems

Average Gate Fidelity

A usefull figure of merit is the gate fidelity averaged uniformly over all possible pure inputs[1]

$$F(\Lambda, 1) = \int d\,|\psi\rangle\, \langle\psi|\, \Lambda(|\psi\rangle\langle\psi|)\, |\psi\rangle$$
$$= \int dU\, \langle 0|\, U^\dagger \Lambda(U|0\rangle\langle 0|U^\dagger)\, |0\rangle$$

We can greatly simplify this task by considering **2-designs**[2]

$$\int dU\, \langle 0|\, U^\dagger \Lambda(U|0\rangle\langle 0|U^\dagger)\, |0\rangle = \frac{1}{|\mathcal{C}_n|} \sum_{i=1}^{|\mathcal{C}_n|} \langle 0|\, U_i^\dagger \Lambda(U_i|0\rangle\langle 0|U_i^\dagger)\, |0\rangle$$

$U_i \in \mathcal{C}_n$, where $\mathcal{C}_n$ is the set of tensor products of single qubit Clifford operations – **Clifford twirling**.

From Chernoff bound, we can estimate $F(\Lambda, 1)$ with precision δ and **overhead independent of** n – number of experiments is $\ln 2/\delta^2$.

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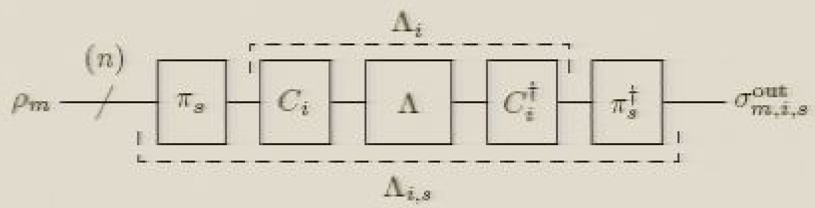
[1] Emerson, Alicki, Zyczkowski, J. Opt. B, **7** S347–S52 (2005)

[2] Renes et al., J. Math. Phys 45(6):2171–2180 (2004); A. Klappenecker and M. Rötteler, Proc. ITIS 2005, 1740–1744 (2005); Dankert, Cleve, Emerson, Livine, quant-ph/0606161

Twirled Channels

Consider now, the permutation and Clifford twirled channel $\bar{\Lambda}$



$$\bar{\Lambda}(\rho) = \frac{1}{|\mathcal{C}_n||s|}\sum_{j,s} \pi_s^\dagger C_j^\dagger \Lambda(C_j \pi_s \rho \pi_s^\dagger C_j^\dagger) C_j \pi_s$$

$$= \sum_{w=0}^{n} p_w \sum_{\text{wt}(P_i)=w} \frac{1}{3^w \binom{n}{w}} P_i \rho P_i$$

$$\equiv \sum_{w=0}^{n} p_w M_w^p(\rho)$$

where $p_w = \sum_{\text{wt}(P_i)=w} \chi_{ii}$.

Note that $\bar{\Lambda}$ only has $n+1$ parameters — the probabilities p_w.

How do we estimate p_w?

The protocol

$$\bar{\Lambda}(P_k) = c_w P_k, \quad P_k \in \mathcal{P}_n \quad \text{wt}(P_k) = w$$

Thus,

$$\bar{\Lambda}(\rho) = \sum c_w M_w^c(\rho)$$

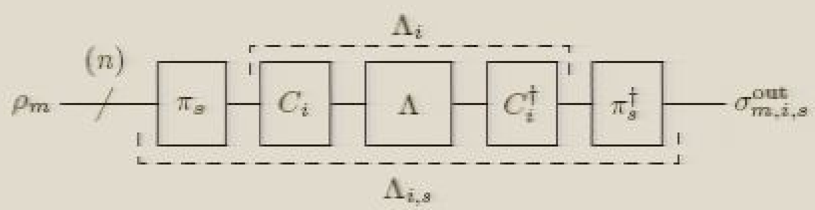
So that,

$$c_w = \sum_{w'} \Omega_{w,w'} p_{w'}, \qquad \Omega_{w,w'} \equiv \frac{\langle M_w^c, M_{w'}^p \rangle}{\langle M_w^c, M_w^c \rangle}$$

$\Omega_{w,w'}$ and $\Omega^{-1}_{w,w'}$ both known, consisting of simple combinatorial expressions.

Just measure parity of random sets of w qubits – no need for explicit permutations

$$c_w = \frac{\left\langle \pi_s Z^{\otimes w} \pi_s^\dagger \right\rangle_\sigma}{\left\langle \pi_s Z^{\otimes w} \pi_s^\dagger \right\rangle_\rho}$$



$\frac{1}{\delta^2} \ln 2(n+1)$ experiments to obtain all c_w, each with precision δ.

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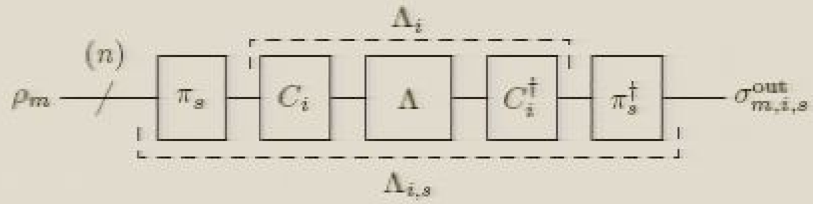

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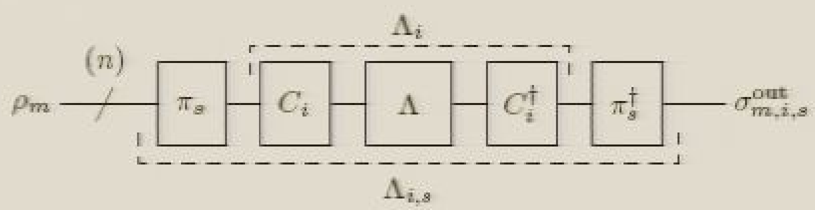

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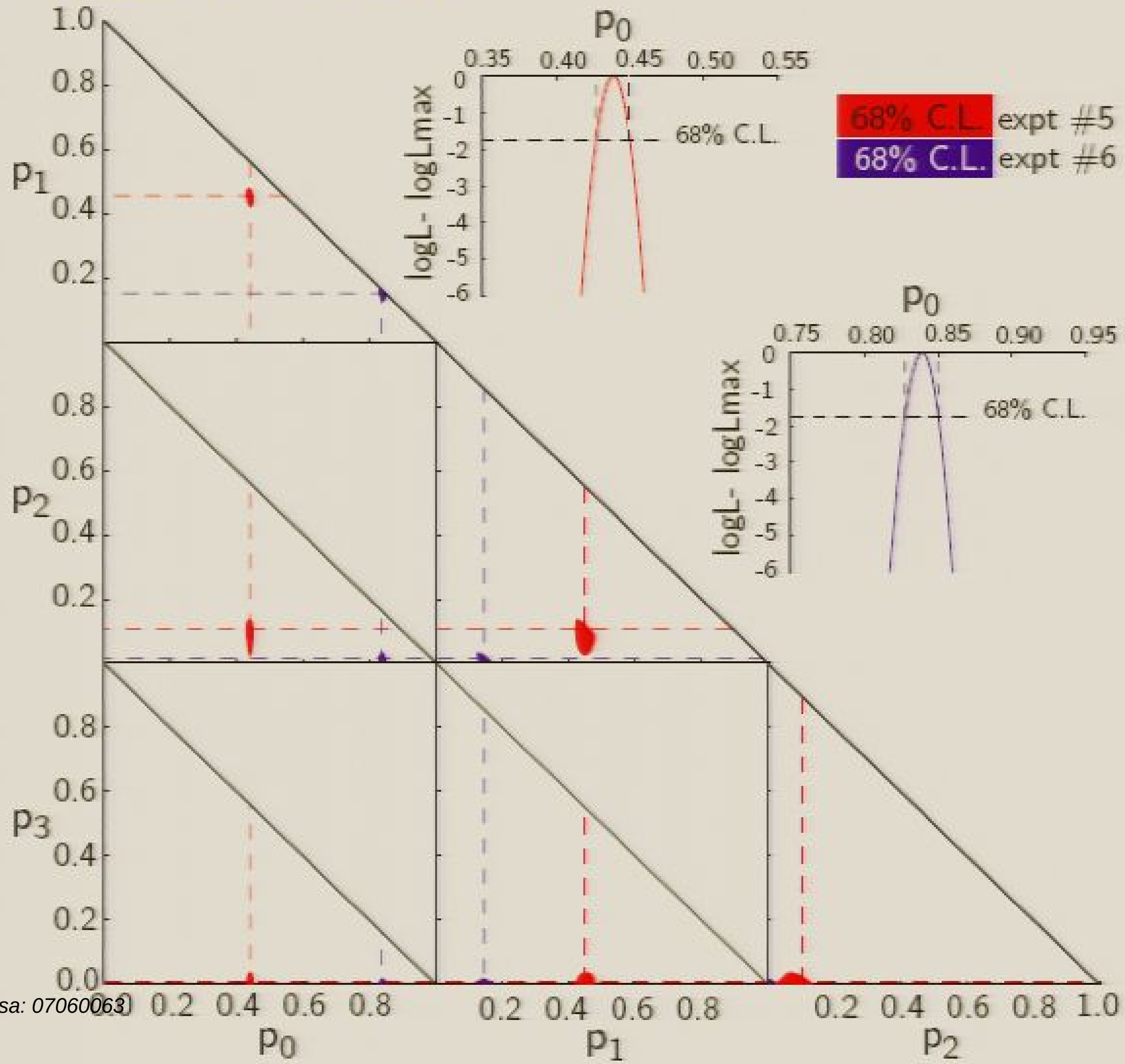
Experimental results in NMR

3 qubits (solid state)



Twirled channel parameters for two time-suspension sequences which differ only in the pulse spacing (10 μs vs. 5 μs).

Performance improvement evident in significantly larger p_0.

2×432 experiments needed to characterize each sequence.

Scalable even with current NMR technology – does not need pseudo-pure preparation.

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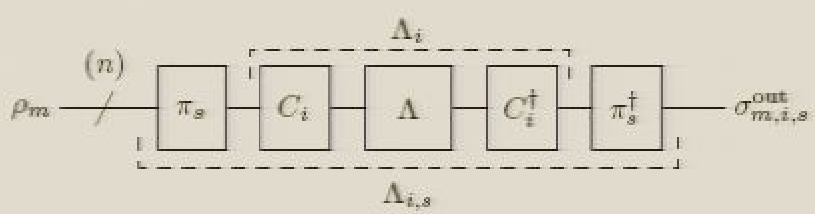

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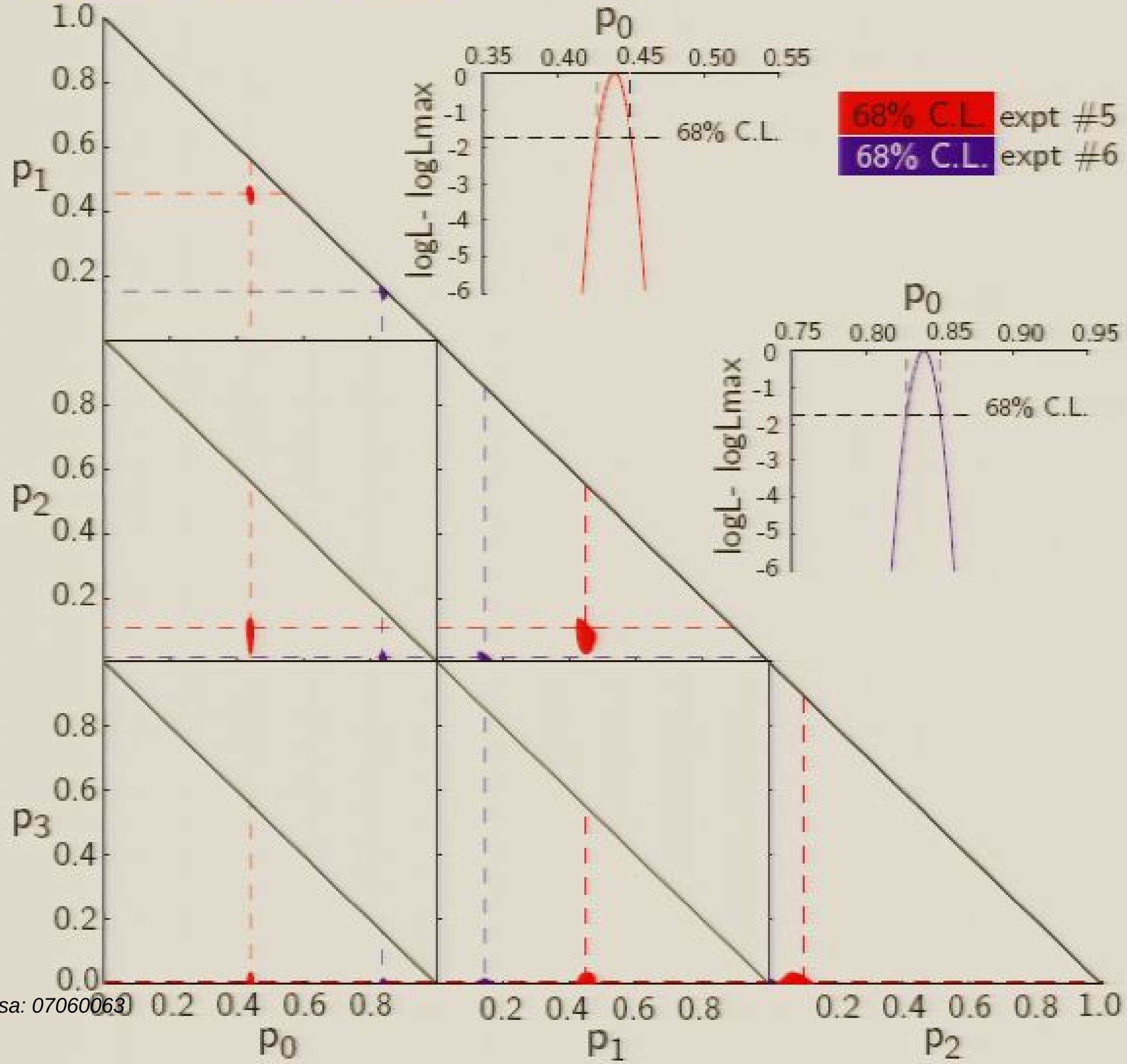

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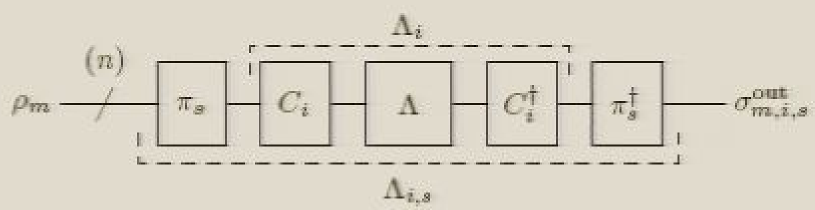

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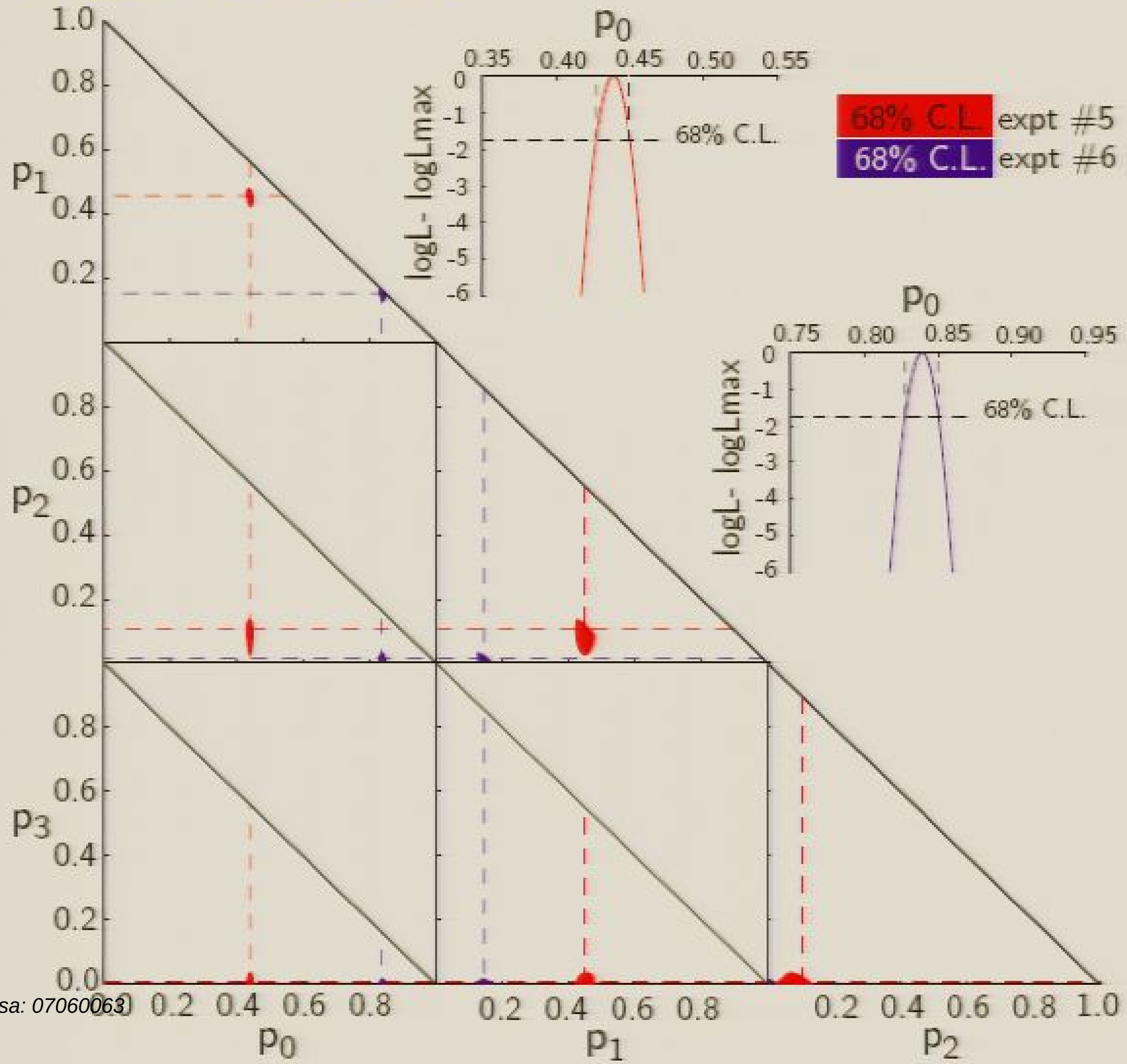

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Scaling

The variance σ_w in the estimate of p_w is given by

$$\sigma_w^2 = \langle (p_w - \langle p_w \rangle)^2 \rangle = \sum_{i=0}^{n} \sum_{j=0}^{n} \Omega_{w,i}^{-1} \Omega_{w,j}^{-1} \mathrm{Cov}(c_i, c_j)$$

We can place upper bounds on the variance

$$\sigma_w^2 \leq \sigma^2 \sum_{i=0}^{n} \sum_{j=0}^{n} |\Omega_{w,i}^{-1} \Omega_{w,j}^{-1}| \leq \sigma^2 3^{2w} \binom{n}{w}^2$$

In other words

$$\sigma_w \leq \sigma 3^w \binom{n}{w} \leq 3^w e^w n^w$$

With polynomial overhead we can only access p_w for $w \leq k$.

Correlations in distribution of error locations

We can also think of $\bar{\Lambda}$ as the result of throwing away information about where errors occurred, and details about the type of errors that occurred.

If error locations are independent, $\bar{\Lambda} = \mathcal{D}^{\otimes n}$.

Thus, the eigenvalues are

$$c_w = c_1^w$$

Statistically significant deviation from a exponential distribution of eigenvalues implies correlations in the distribution of error locations.

Can we translate other conditions for threshold theorems in terms of efficiently accessible p_w/c_w?

Observation of non-Markovian noise

All Pauli channels share same eigenvectors – channel composition is easy.

The Markovian condition

$$\bar{\Lambda}_{t_a+t_b} = \bar{\Lambda}_{t_a} \circ \bar{\Lambda}_{t_b}$$

translates to

$$c_w^{(t_a+t_b)} = c_w^{(t_a)} c_w^{(t_b)}$$

Statistically significant deviations indicate non-Markovian behaviour in both $\bar{\Lambda}$ and Λ.

Smaller symmetry groups may allow for more non-Markovian behaviour to be visible.

Finding Noiseless Encodings

Symmetry group can be arbitrary.

Choose, e.g. group generated by $\mathcal{P}_n$ and all qubit permutations.

In that case the parameterisation of the resulting channel is[3]

$$\bar{\Lambda}(\rho) = \sum_{\mathbf{w}=(x,y,z)} p_{\mathbf{w}} \sum_{P_{\mathbf{w}}} \frac{1}{\binom{n}{x+y+z}\binom{x+y+z}{x}\binom{y+z}{z}} P_{\mathbf{w}} \rho P_{\mathbf{w}}$$

so we have $\frac{1}{6}n^3 + n^2 + \frac{11}{6}n + 1$ parameters – still manageble.

$\bar{\Lambda}$ is still a Pauli channel, but now it is possible the $c_{\mathbf{w}} = \pm 1$ for non-trivial $\mathbf{w}$. The isomorphism $\Omega_{\mathbf{w},\mathbf{w}'}$ is also similarly simple.

Because of the well known commutation relations between different Pauli operators, we can look for observable in the fixed-point set above with the right commutation relations.

Theorem[4]: Any noiseless encoding we find for $\mathcal{P}_n$ twirles $\bar{\Lambda}$ is also a noiseless encoding for Λ.

[3] Dankert, Cleve, Emerson and Livine, quant-ph/0606161.

[4] Nielsen and Poulin, quant-ph/0506069

Summary

- Symmetrisation of channels via twirling provides general, efficient way to obtain valuable partial information about noisy quantum channels
- We can detect deviations from independent error location model.
- We can detect some deviations from Markovian behaviour.
- We can find noiseless subsystems, and test how noiseless they are.

Questions:

- Using just a constant number of p_w, can we confirm proximity to models used in threshold theorem?
- Are there more general conditions in terms of the c_w?
- What other symmetry groups yield interesting information?

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