Title: Three types of probability distributions associated to generalized quantum coherent states

Date: Jun 07, 2007 11:00 AM

URL: http://pirsa.org/07060051

Abstract:

COHERENT STATES: QUANTUM AND CLASSICAL PROBABILITIES

Perimeter Institute
June 7, 2007

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THE AIM OF THIS TALK IS TO POSE A SET OF QUESTIONS TO WHICH I SEEK YOUR HELP IN FINDING ANSWERS

- Certain mathematical constructs, namely coherent states, which seem to display a certain quantum-classical duality
- They seem to incorporate purely quantum, as well as purely classical properties
- We start with the very well known canonical coherent states
- These lie at the heart of our understanding of simultaneous measurements of incompatible observables, as well as the quantum classical transition.

CCS: $\mathfrak{H} = L^2(\mathbb{R}, dx)$, $\eta \in \mathfrak{H}$ s.t.

$$\eta(x) = \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}}$$

$\forall\ (q, p) \in \mathbb{R}^2$, define $\eta_{q,p} \in \mathfrak{H}$, $\|\eta_{q,p}\|^2 = 1 = \|\eta\|^2$

s.t. $\eta_{q,p}(x) = e^{ip(x - q/2)}\,\eta(x - q)$ CCS

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$$\Rightarrow \int_{\mathbb{R}^2} |\eta_{q,p}\rangle\langle\eta_{q,p}| \frac{dq\,dp}{2\pi} = I_{\mathfrak{H}}$$

Associated POV-measure

$\Delta \in \mathcal{B}(\mathbb{R}^2)$ $\qquad a(\Delta) = \int_\Delta |\eta_{q,p}\rangle\langle\eta_{q,p}| \frac{dq\,dp}{2\pi}$

- Allows for joint measurement of position and momentum

- $\Phi \in \mathcal{H}$ $\qquad P_\Phi(\Delta) = \langle\Phi | a(\Delta)\Phi\rangle$,

  Probability for system to be in the (fuzzy) phase space region Δ, when in state Φ

- $\eta_{q,p}$ minimal uncertainty states

$$\Delta(Q)\,\Delta(P) = \frac{1}{2}$$

- Group theoretical origin:

$$\eta_{q,p} = U(q,p)\eta = e^{i(qP - pQ)}\eta$$

$U(q,p)$ realize a multiplier rep. of the Weyl-Heisenberg group.

- Quantization: $f \mapsto \hat{f}$

$$\hat{f} = \int f(q,p)\, da(q,p)$$

– Analytic properties: Set $z = \frac{q - ip}{\sqrt{2}}$

$$\Rightarrow \quad \eta_{q,p} := |z\rangle = \mathcal{N}(|z|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$$

where,
$$\mathcal{N}(|z|^2) = e^{|z|^2}$$

$\{|n\rangle\}_{n=0}^{\infty}$ = normalized eigenvectors of the Harmonic oscillator

– $\phi \in \mathcal{H} \Rightarrow \langle \phi | z \rangle = e^{-|z|^2/2} f(\bar{z})$

$f(z)$ = entire analytic $\in L^2_{hol}(\mathbb{C}, e^{-|z|^2} \frac{dq\,dp}{2\pi})$

SOME PROBABILITY DISTRIBUTIONS ASSOCIATED TO THE CCS

(1) $P_\phi(z) := P_\phi(q, p) = |\langle \phi | z \rangle|^2$

$$\int_{\mathbb{R}^2} P_\phi(q, p) \frac{dq\,dp}{2\pi} = 1$$

Quantum probability density for localization in phase space

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Quantum probability density for localization in phase space

(2) Write $z = \sqrt{\lambda}\, e^{i\theta}$, $\lambda > 0$, $\theta \in [0, 2\pi]$

$$\Rightarrow |\langle n | z \rangle|^2 = \frac{e^{-\lambda} \lambda^n}{n!} = \mathcal{P}(n, \lambda)$$

Poisson distribution for n successes

$\mathcal{N}(|z|^2) = e^{\cdots}$

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Poisson distribution for n successes

- Parameter λ = average no. of successes.
Classical, discrete distribution

$$\sum_{n=0}^{\infty} P(n, \lambda) = 1$$

- $P(n, \lambda)$ is also the quantum probability for finding n excitons in a state

(3) Thinking of $P(n, \lambda)$ as a classical Poisson distribution, assume that λ itself is a random variable

Now,

$$\int_0^{\infty} P(n, \lambda)\, d\lambda = 1$$

$\Rightarrow$ the CCS dictate that λ be uniformly distributed on $(0, \infty)$

(4) The measure

$$d\nu(\lambda, \theta) = e^{-\lambda} d\lambda\, d\theta$$

$$\int_0^{2\pi}\int_0^{1} d\nu(\lambda, \theta) = 1$$

is a natural measure with which the CCS assigns probabilities to regions of phase space

$$\mathbb{R}^2 = T^* \mathbb{R}$$

NOTE THE DUALITY:

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NOTE THE DUAL[illegible] :

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Let us reverse the process:

Start with the Poisson distribution: purely classical

$$P(n, \lambda) = \frac{e^{-\lambda} \lambda^n}{n!}, \quad n = 0, 1, 2, 3, \ldots$$

Let's say we want to find a 'square root' of this distribution.

- We introduce $z = \sqrt{\lambda}\, e^{i\theta}$, $\lambda > 0$
  $z \in \mathbb{C}$

- Let $\mathcal{H}$ be an abstract Hilbert space over $\mathbb{C}$, separable and infinite dimensional
  $\{\phi_n\}_{n=0}^{\infty}$ = orthonormal basis of $\mathcal{H}$

Define $|z\rangle \in \mathcal{H}$, $\forall z \in \mathbb{C}$ as:

$$|z\rangle = \sum_{n=0}^{\infty} \sqrt{P(n,\lambda)}\, e^{in\theta}\, \phi_n$$

$$\Rightarrow \langle z|z\rangle = \| |z\rangle \|^2 = \sum_{n=0}^{\infty} P(n,\lambda) = 1$$

$|z\rangle$ is a CCS

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$\Rightarrow$ uniform distribution of λ on $(0, \infty)$

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WHAT ARE COHERENT STATES ?

$\mathfrak{H}$ = abstract Hilbert space, $\dim \mathfrak{H}$ = finite or ∞

$(X, d\nu)$ = measure space

$\forall x \in X$, associate N vectors $\eta^i_x \in \mathfrak{H}$, $i = 1, 2, \ldots, N$
s.t.

(i) η^i_x, $i = 1, 2, \ldots, N$ for fixed $x \in X$, are linearly independent

(ii) $x \longmapsto \langle \eta^i_x | \Phi \rangle$, $\forall i$ and $\forall \Phi \in \mathfrak{H}$ is measurable

(iii) $\displaystyle\sum_{i=1}^{N} \int_X | \eta^i_x \rangle \langle \eta^i_x | \, d\nu(x) = I_{\mathfrak{H}}$, weakly

$N = 1$, usually called coherent states
$N > 1$, vector coherent states

IMMEDIATE PROPERTIES

- $W : \mathfrak{H} \longrightarrow \mathbb{C}^N \otimes L^2(X, d\nu)$

$$(W\Phi)^i(x] = \langle \eta^i_x | \Phi \rangle \quad , \quad i = 1, 2, \ldots, N$$

is a linear isometry

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- $\mathfrak{H}_K = W(\mathfrak{H})$ is a reproducing kernel Hilb. sp.

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Reproducing kernel:

$K : X \times X \longrightarrow \mathbb{C}^{N\times N}, \quad K_{ij}(x,y) = \langle \eta_x^i \mid \eta_y^j \rangle$

(i) $K_{ii}(x,x) > 0$

(ii) $K_{ij}(x,y) = \overline{K_{ji}(y,x)}$

(iii) $\sum_{k=1}^{N} \int_X K_{ik}(x,z) K_{kj}(z,y)\, d\nu(z) = K_{ij}(x,y)$

(iv) K is the integral operator: $\mathbb{P}_K : \mathbb{C}^N \otimes L^2(X,\nu) \to \mathfrak{H}_K$

$(\mathbb{P}_K \Psi)^i(x) = \sum_{j=1}^{N} \int_X K_{ij}(x,y)\, \Psi^j(y)\, d\nu(y)$

$\forall\, \Psi \in \mathbb{C}^N \otimes L^2(X, d\nu)$

POV-measure

$\Delta \subset X$, Borel set

Define the positive operator $a(\Delta)$

$a(\phi) = 0, \qquad a(X) = I_{\mathfrak{H}}$

$a(\bigcup_{i\in J} \Delta_i) = \sum_{i\in J} a(\Delta_i)$, weakly

if $\Delta_i \cap \Delta_j = \phi$ for $i \neq j$

$a(\Delta) = \sum_{i=1}^{N} \int |\eta_x^i\rangle\langle \eta_x^i|\, d\nu(x)$

THE ORIGINAL EXAMPLE: CANONICAL COHERENT STATES

$\mathfrak{H} = L^2(\mathbb{R}, dx)$, $\eta \in \mathfrak{H}$ s.t. $\eta(x) = \frac{1}{(\pi)^{1/4}} e^{-\frac{x^2}{2}}$

$X = \mathbb{R}^2$, $d\nu = \frac{dq\,dp}{2\pi}$, $(q,p) \in \mathbb{R}^2$

$\forall (q,p)$, define $\eta_{q,p}$ s.t.

$$\eta_{q,p}(x) = e^{-ipx}\,\eta(x-q)$$

$$\Rightarrow \int_{\mathbb{R}^2} |\eta_{q,p}\rangle\langle\eta_{q,p}| \frac{dq\,dp}{2\pi} = I$$

ANALYTIC PROPERTIES

Introduce complex variable,

$$z = \frac{1}{\sqrt{2}}(q - ip)$$

$$\Rightarrow |z\rangle := \eta_{q,p} = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} \eta_n ,$$

where,

$$\eta_n = \frac{(a^\dagger)^n}{\sqrt{n!}} \eta \quad , \ a^\dagger = \text{creation op. on } \mathfrak{H}$$

$$a^\dagger = \frac{Q - iP}{\sqrt{2}}$$

Q : mult. by x

P : $-i\frac{\partial}{\partial x}$

The set of all such functions, $\forall \Phi \in \mathfrak{H}$, the holomorphic functions f constitute the Hilbert space,

$$L^2\left(\mathbb{C}, e^{-|z|^2}\frac{dq\,dp}{2\pi}\right)_{hol} \subset L^2\left(\mathbb{C}, e^{-|z|^2}\frac{dq\,dp}{2\pi}\right)$$

A POSSIBLE GENERALIZATION : NON-LINEAR COHERENT STATES

Let $\Phi_0, \Phi_1, \Phi_2, \ldots \Phi_n, \ldots$ be any orthonormal basis of $\mathfrak{H}$

Let $x_0 = 0, x_1, x_2, \ldots, x_n, \ldots > 0$ be a positive sequence such that,

$$x_0! := 1, \quad x_n! = x_1 x_2 x_3 \ldots x_n, \qquad n = 1, 2, 3, \cdots$$

is a moment sequence, i.e., $\exists$ measure $d\lambda$ on $\mathbb{R}$ s.t.

$$\int_0^{L^{1/2}} r^{2n}\, d\lambda(r) = \frac{x_n!}{2\pi} \qquad \forall n,$$

where $L = \lim_{n\to\infty} x_n$. We assume $L \neq 0$

Then, $\mathcal{N}(|z|^2) = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{x_n!} < \infty$

$\forall z \in \mathcal{D} = \{ z \in \mathbb{C} \mid |z| < L \}$

Define non-linear coherent states:

$$|z\rangle = \mathcal{N}(|z|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{x_n!}} \phi_n$$

They satisfy,

$$\int_{\mathcal{D}} \mathcal{N}(|z|)|z\rangle\langle z| \, d\lambda(r) \, d\theta = I_{\mathcal{H}}$$

The functions $z \longmapsto \mathcal{N}(|z|^2)^{1/2} \langle \phi | z\rangle$, $\forall \phi \in \mathcal{H}$, are holomorphic.

Associated orthogonal polynomials:

$$a|z\rangle = z|z\rangle \Rightarrow a\phi_n = \sqrt{x_n}\, \phi_{n-1}$$

$$a^{\dagger}\phi_n = \sqrt{x_{n+1}}\, \phi_{n+1}$$

Define $\quad Q = \dfrac{a + a^{\dagger}}{\sqrt{2}}$

$$\Rightarrow Q\phi_n = \sqrt{\frac{x_n}{2}}\, \phi_{n-1} + \sqrt{\frac{x_{n+1}}{2}}\, \phi_{n+1}$$

If $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{x_n}} = \infty$, $\quad Q$ is ess. self-adjoint

Diagonalizing, Q becomes multiplication op. on some

$L^2(\mathbb{R}, d\mu)$: $\quad (Q\phi)(\lambda) = \lambda \phi(\lambda)$

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$$a^{+}\phi_n = \sqrt{x_{n+1}}\, \phi_{n+1}$$

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$L^2(\mathbb{R}, d\mu)$: $\quad (Q\phi)(\lambda) = \lambda\phi(\lambda)$

$$\phi \in L^2(\mathbb{R}, d\mu)$$

$$\int_{\mathbb{R}} \overline{\Phi_n(x)}\, \Phi_m(x)\, d\mu(x) = \delta_{nm}$$

EXAMPLES OF NON-LINEAR COHERENT STATES

- $|z\rangle = (1-|z|^2)^{\kappa} \sum_{n=0}^{\infty} \left[\frac{(2\kappa)_n}{n!}\right]^{1/2} z^n \Phi_n$

  $\kappa = 1, 3/2, 5/2, \ldots \qquad \mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$

  $(a)_m = \dfrac{\Gamma(a+m)}{m!}$

$\Rightarrow \dfrac{2\kappa-1}{\pi}\displaystyle\int_{\mathcal{D}} |z\rangle\langle z| \frac{r\,dr\,d\theta}{(1-r^2)^2} = I \qquad z = re^{i\theta}$

Gilmore - Perelomov type : SU(1,1)

- $|z\rangle = \dfrac{|z|^{2\kappa-1}}{[I_{2\kappa-1}(2|z|)]^{1/2}} \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!\,(2\kappa+n-1)!}}\, \Phi_n$

Barut Girardello type : SU(1,1)

$I_\nu(x)$ = modified Bessel fn. of order ν

EXTENSION TO MATRIX DOMAINS: VECTOR CS

We want to replace z by $\mathfrak{Z}$, an $N \times N$ matrix in the above to get vector coherent states of the type:

$$|\mathfrak{Z}; i\rangle = \mathcal{N}(\|\mathfrak{Z}\|^2)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\mathfrak{Z}^n}{\sqrt{x_n!}} \chi^i \otimes \phi_n ,$$

$$i = 1, 2, 3, \ldots, N$$

$\mathfrak{Z} \in \mathcal{D}$, some $N \times N$ complex matrix domain

$\{\chi^i\}_{i=1}^{N}$ = orthonormal basis in $\mathbb{C}^N$

$|\mathfrak{Z}; i\rangle \in \mathbb{C}^N \otimes \mathcal{H}$

We also want

$$\sum_{i=1}^{N} \int_{\mathcal{D}} |\mathfrak{Z}; i\rangle\langle \mathfrak{Z}; i| \, \mathcal{N}(\|\mathfrak{Z}\|^2) \, d\mu(\mathfrak{Z}, \bar{\mathfrak{Z}})$$

$$= I \text{ on } \mathbb{C}^N \otimes \mathcal{H}$$

Note: $\mathcal{N}(\|\mathfrak{Z}\|^2) = \sum_{n=0}^{\infty} \mathrm{Tr}[\mathfrak{Z}^{*n} \mathfrak{Z}^n] / x_n!$

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– $\mathcal{D} = \mathbb{C}^{N\times N}$, all $N\times N$ complex matrices

$$x_n! = \frac{1}{(n+1)(n+2)}\left[\prod_{j=1}^{n+1}(N+j) - \prod_{j=1}^{n+1}(N-j)\right]$$

Not a moment sequence. But one has a resolution of the identity with

$$d\mu(z, z^*) = e^{-\operatorname{Tr}[z z^*]}\,\pi^{-N^2}\prod_{i,j=1}^{N} dz$$

dz = Lebesgue measure on $\mathbb{C}^{N\times N}$

Interesting matrix identity:

$$\int_{\mathbb{C}^{N\times N}} z^{*k} z^m \, d\mu = \delta_{km}\, x_m!\, \mathbb{I}_N$$

Reproducing kernel:

$$K(X, Y) = \sum_{n=0}^{\infty} \frac{X^n Y^{*n}}{x_n!}$$

- $\mathcal{D} = \{ \mathfrak{z} \in \mathbb{C}^{N \times N} \mid \mathfrak{z}\mathfrak{z}^* = \mathfrak{z}^*\mathfrak{z} \}$, normal

$$\Rightarrow \quad \mathfrak{z} = U^* D U , \qquad U \in U(N)$$

$$D = \mathrm{diag}(z_1, z_2, \dots, z_N)$$

In this case, with

$$d\mu = e^{-\mathrm{Tr}[\mathfrak{z}^*\mathfrak{z}]} \, dU \, dz_1 \, dz_2 \cdots dz_N$$

dU = invariant measure of $U(N)$
dz_i = Lebesgue measure of $\mathbb{C}$, $\forall i$

we get $\quad x_n! = n!$ and

$$\int_{\mathcal{D}} \mathfrak{z}^{*j} \mathfrak{z}^k \, d\mu = \delta_{jk} \, k! \, I_N$$

$\Rightarrow$

$$|\mathfrak{z}; i\rangle = e^{-\mathrm{Tr}[\mathfrak{z}^*\mathfrak{z}]/2} \sum_{n=0}^{\infty} \frac{\mathfrak{z}^n}{\sqrt{n!}} \chi^i \otimes \phi_n$$

$$= e^{-\frac{1}{2} U^* T U} \, \mathbb{D}(\mathfrak{z}) \, \chi^i \otimes \phi_0$$

where, $\quad \mathbb{D}(\mathfrak{z}) = e^{\mathfrak{z} \otimes a^\dagger - \mathfrak{z}^* \otimes a}$

$T = \mathrm{diag}(a_1, a_2, \dots, a_N)$, $\; a_i = \mathrm{Tr}[\mathfrak{z}^*\mathfrak{z}] - |z_i|^2$

Matrix generalization of canonical CS

N = number of trials, q = probability of success
$p = 1 - q$ = prob. of failure

$$P(n, q) = {}^N C_n \, q^n p^{N-n} = \frac{N!}{(N-n)!\, n!} q^n p^{N-n}$$

$$n = 0, 1, 2, \ldots, N$$

$$\sum_{n=0}^{N} P(n, q) = (q + p)^N = 1 \qquad 0 \le q \le 1$$

- Can we find associated coherent states?
- If q is a random variable, what could be its distribution?

Set $\lambda = \frac{q}{p}$

$$P(n, q) := P(n, \lambda) = \frac{\Gamma(N+1)}{\Gamma(N-n+1)\Gamma(n+1)} \frac{\lambda^n}{(1+\lambda)^N}$$

To build CS let $z = \sqrt{\lambda}\, e^{i\theta} \in \mathbb{C}$

Let $\mathfrak{H} = (N+1)$-dim Hilb. sp.

$\{\phi_n\}_{n=0}^{N}$ = o.n.b. of $\mathfrak{H}$

Define $|z\rangle \in \mathfrak{H}$ s.t.

$$|z\rangle = \sum_{n=0}^{N} \sqrt{P(n, \lambda)}\, e^{in\theta} \phi_n = \mathcal{N}(|z|^2) \sum_{n=0}^{N} \frac{\sqrt{\Gamma(N+1)}\; z^n}{\sqrt{\Gamma(N-n+1)\Gamma(n+1)}} \phi_n$$

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$$\mathcal{N}(|z|^2) = (1+|z|^2)^N = \sum_{n=0}^{N} \frac{\Gamma(N+1)\,|z|^{2n}}{\Gamma(N-n+1)\,\Gamma(n+1)}$$

Of course: $\langle z|z\rangle = \||z\rangle\|^2 = 1 = \sum_{n=0}^{N} P(n,\lambda)$

These CS satisfy:

$$\frac{N+1}{2\pi}\int_0^\infty\int_0^{2\pi} |z\rangle\langle z|\,\frac{d\lambda\, d\theta}{(1+\lambda)^2} = I_{\mathcal{H}}$$

Let us introduce the complex fcns.

$$\Psi_n(\bar{z}) = \left[\frac{\Gamma(N+1)}{\Gamma(N-n+1)\Gamma(n+1)}\right]^{\frac{1}{2}} \bar{z}^n,$$

$$n = 0, 1, 2, \cdots, N$$

$$\Rightarrow \int_0^\infty\int_0^{2\pi} \overline{\Psi_n(\bar{z})}\,\Psi_m(\bar{z})\, d\nu(\lambda)\, d\theta = \delta_{nm}$$

$$d\nu(\lambda) = \frac{N+1}{2\pi}\frac{d\lambda}{(1+\lambda)^{N+2}}$$

$$\Rightarrow \Psi_n \in L^2_{\text{anti-hol}}(\mathbb{C}, d\nu\, d\theta)$$

Thus, we may replace ϕ_n in the CS by Ψ_n

$$|z\rangle = (1+|z|^2)^{-N/2}\sum_{k=-j}^{j}\left[\frac{\Gamma(2j+1)}{\Gamma(j-k+1)\Gamma(j+k+1)}\right]^{1/2} \times z^{k+j}\,\Psi_k$$

$N = 2j$, $k = \ldots j$

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These are the CS of the rotation group

$$|z\rangle = e^{zJ_+} e^{\eta J_3} e^{-\bar{z} J_-} \Psi_{-j}$$

$$\eta = \log(1+|z|^2), \quad (n+1)dq \text{ over } (0,1]$$

- Negative binomial distribution : SU(1,1) coherent states
- Hypergeometric distributions
- Beta, Gamma distributions
- Other discrete distributions

- CS over continuous parameter spaces
- Always leads to a distribution over parameter

MULTIPLE POISSON PROCESSES

Suppose we have N Poisson processes with parameters $\lambda_1, \lambda_2, \ldots, \lambda_N$

$$P(n, \lambda_j) = e^{-\lambda_j} \lambda_j^n / n!$$

We look for n ...

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$$P(n, \lambda_j) = e^{-\lambda_j} \lambda_j^n / n!$$

We look for n successes, coming from any one of these N processes:

$$P(n, \lambda_1, \lambda_2, \ldots, \lambda_N) = \frac{1}{N}\sum_{j=1}^{N} P(n, \lambda_j)$$

$$\sum_{n=0}^{\infty} P(n, \lambda_1, \lambda_2, \dots, \lambda_N) = 1$$

Can we build CS for this system?

Set $\quad z_j = \sqrt{\lambda_j}\, e^{i\theta_j} \in \mathbb{C}, \quad j = 1, 2, \dots, N$

$$\mathfrak{z} = V \,\mathrm{diag}(z_1, z_2, \dots, z_N)\, V^*, \quad V \in U(N)$$

$$\Rightarrow \quad \mathfrak{z}\mathfrak{z}^* = \mathfrak{z}^*\mathfrak{z}$$

$$\mathbb{N} = \mathrm{diag}(e^{-\lambda_1}, e^{-\lambda_2}, \dots, e^{-\lambda_N})$$

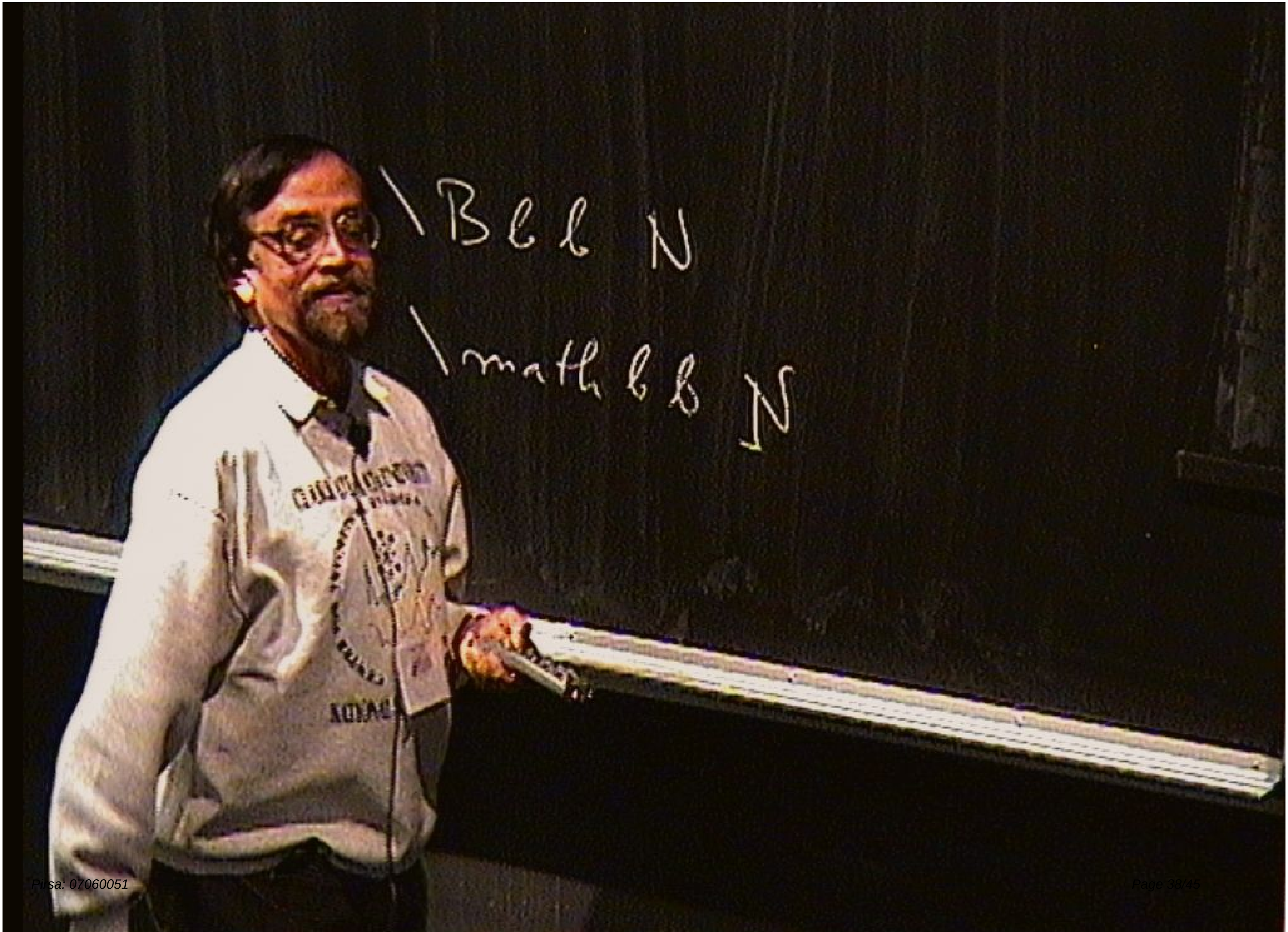
Let $\{\phi_n\}_{n=0}^{\infty}$ be o.n.b. in $\mathfrak{H}$

Define $|\mathfrak{z}\rangle \in \mathbb{C}^{N\times N} \otimes \mathfrak{H}$:

$$|\mathfrak{z}\rangle = \frac{1}{N} \sum_{n=0}^{\infty} V^* \,\mathrm{diag}\left(\sqrt{P(n,\lambda_1)}\, e^{i\theta_1}, \sqrt{P(n,\lambda_2)}\, e^{i\theta_2}, \dots, \sqrt{P(n,\lambda_N)}\, e^{i\theta_N}\right) V \otimes \phi_n$$

$$= \frac{\mathbb{N}^{-\frac{1}{2}}}{(N)^{1/2}} \sum_{n=0}^{\infty} \frac{\mathfrak{z}^n}{\sqrt{n!}} \otimes \phi_n$$

$$\langle \mathfrak{z} | \mathfrak{z} \rangle = 1, \quad \mathrm{Tr}\left[\langle \phi_n | \mathfrak{z}\rangle \langle \mathfrak{z} | \phi_n \rangle\right] = \frac{1}{N} \sum_{j=1}^{N} P(n, \lambda_j)$$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\mathbb N

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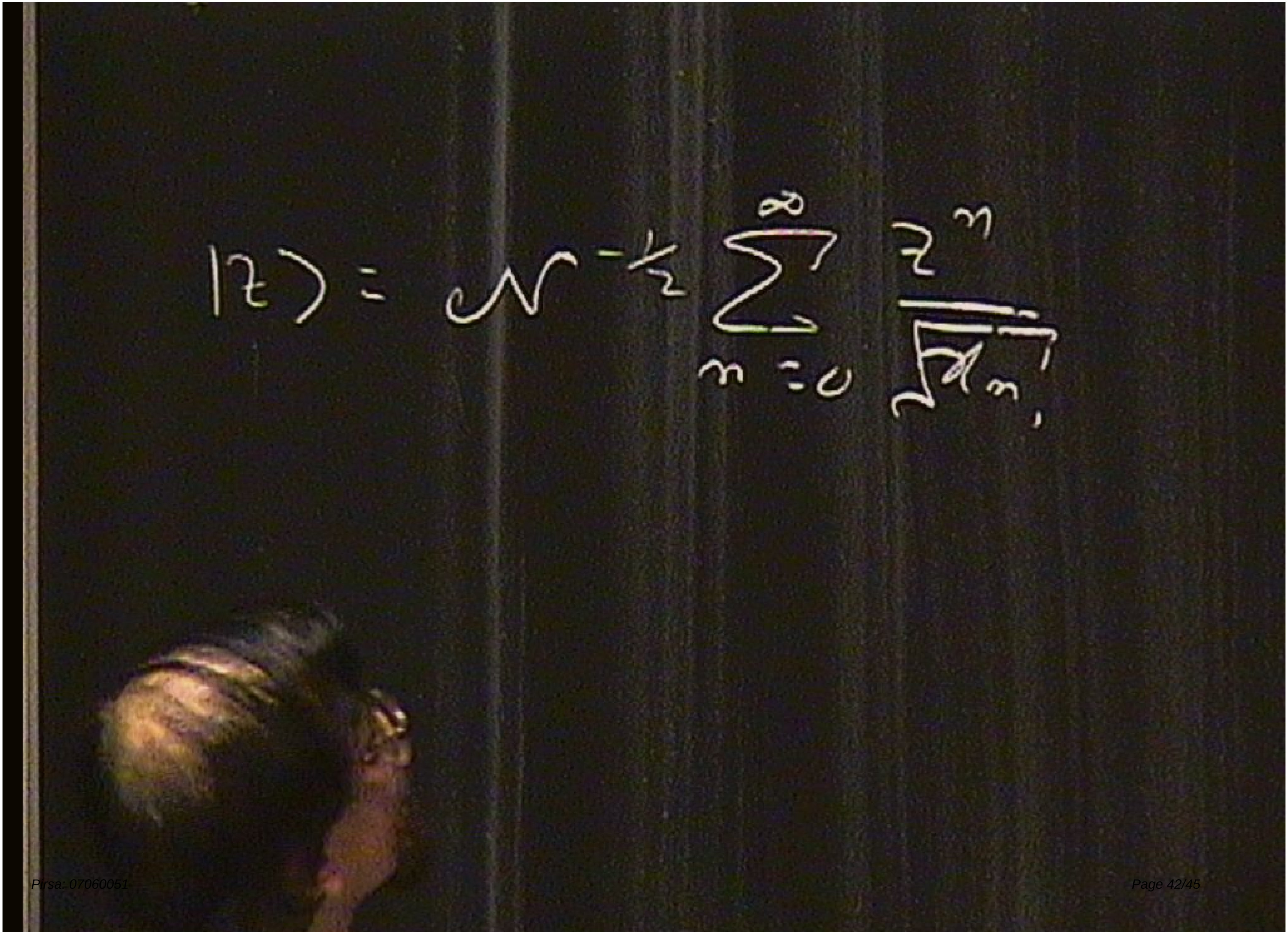
$$\int_{\mathbb{C}^N} |\mathfrak{z}\rangle\langle\mathfrak{z}|\, \mathbb{N}\, e^{-\operatorname{Tr}[\mathfrak{z}^*\mathfrak{z}]} \prod_{j=1}^{N} \frac{d\lambda_j\, d\theta_j}{(2\pi)^N} = I$$

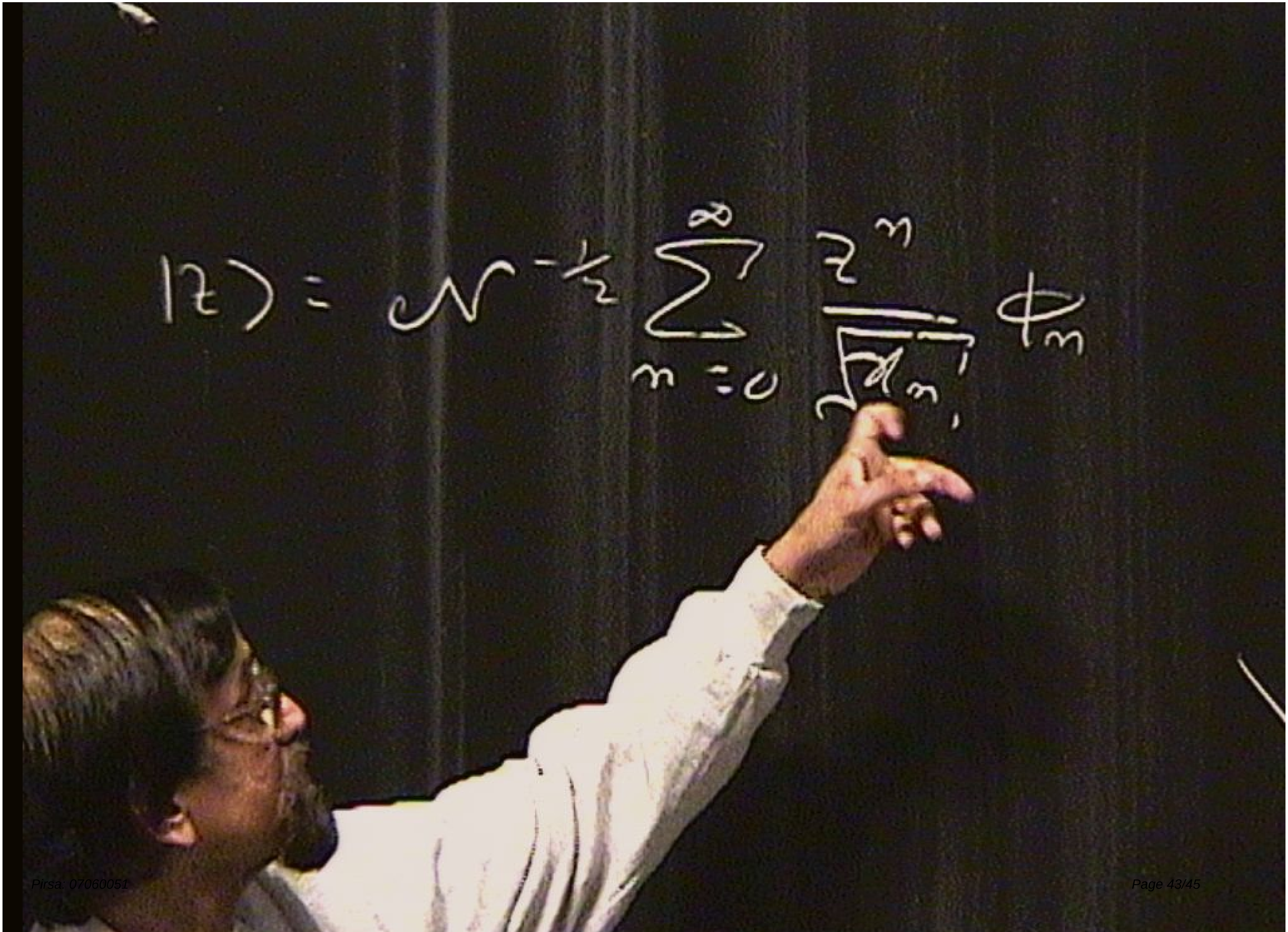
– Possible to do such an analysis for any discrete compound process!

MYSELF, WHEN YOUNG, DID EAGERLY FREQUENT
DOCTOR AND SAINT AND HEARD GREAT ARGUMENT
ABOUT IT AND ABOUT, BUT EACH TIME CAME OUT
BY THE SAME DOOR AS IN I WENT!

—OMAR KHAYYAM



$|z\rangle = \mathcal{N}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}$


$|z\rangle = \mathcal{N}^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\epsilon_n!}} \phi_n$

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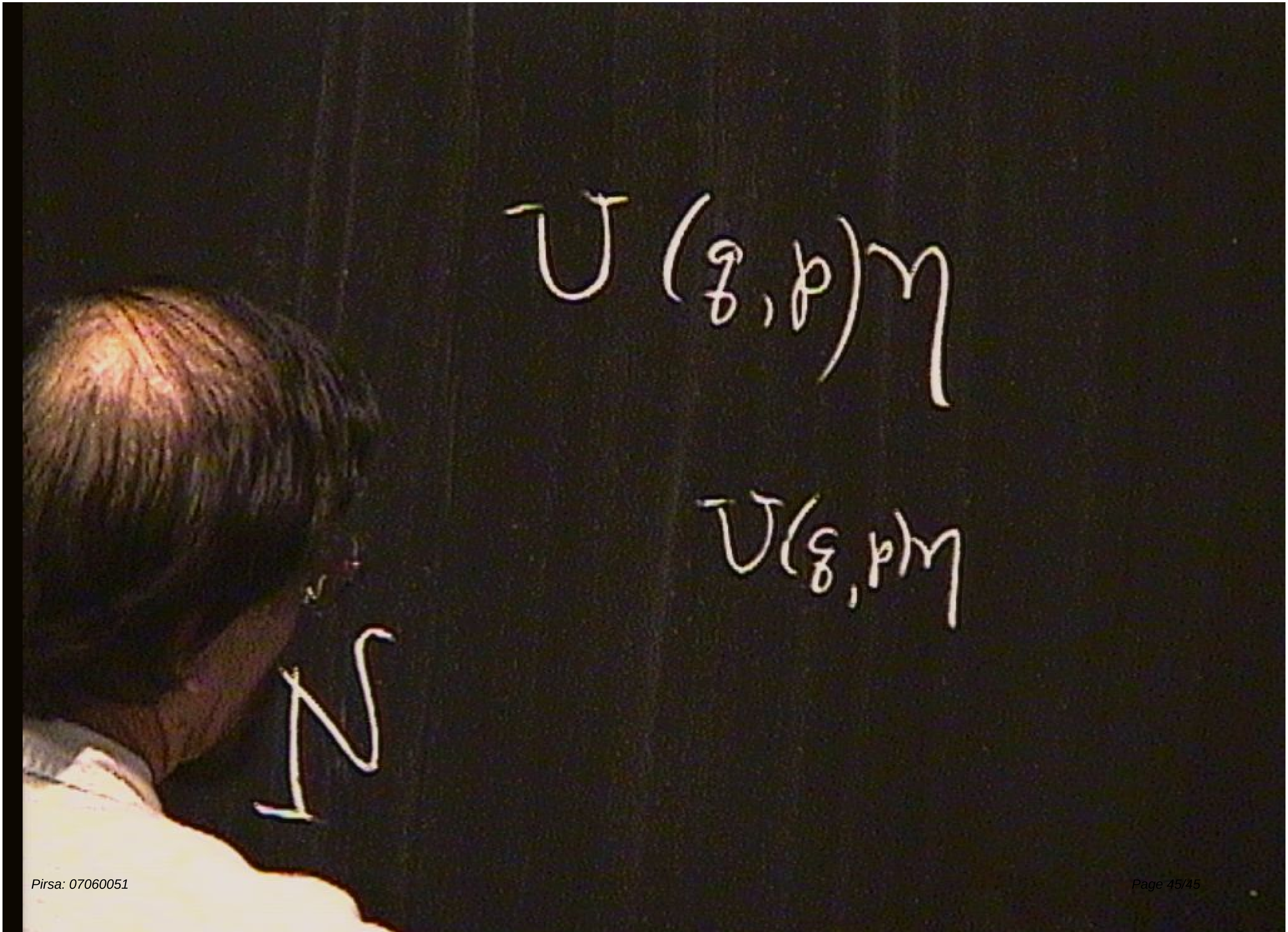
$$P(n,\lambda) = \frac{\lambda^n}{\mathcal{N}(\lambda)\, x_n!}$$


U(q,p)η
U(q,p)η
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