Title: Heisenberg\'s Uncertainty Principle

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Abstract:

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This is a synopsis of the article:

- *Heisenberg's Uncertainty Principle*, Paul Busch, Teiko Heinonen, PL, arXiv:quant-ph/0609185 v2 5 Dec 2006, *Phys. Rep.*, in the press.

Important new contributions:

- R. Werner: The uncertainty relation for joint measurement of position and momentum, *Quant. Inf. Comp.*, 4:546–562, 2004.
- C. Carmeli, T. Heinonen, and A. Toigo: On the coexistence of position and momentum observables. *J. Phys. A*, 38:5253–5266, 2005.
- C. Carmeli, T. Heinonen, and A. Toigo: Intrinsic unsharpness and approximate repeatability of quantum measurements, *J. Phys. A*, 40:1303–1323, 2007.
- P. Busch and D. Pearson: Universal joint-measurement uncertainty relation for error bars, *math-ph/0612074*, 2006.

The uncertainty principle is usually described, rather vaguely, as comprising one or more of the following **no-go statements**:

(a) *It is impossible to prepare states in which position and momentum are simultaneously arbitrarily well localized.*

(b) *It is impossible to measure simultaneously position and momentum.*

(c) *It is impossible to measure position without disturbing momentum, and vice versa.*

It also has the **positive aspect**. The relation

$$\Delta q \cdot \Delta p \gtrsim \hbar,$$

describes

(A) the possibilities to prepare the system in view of the standard deviations Δq and Δp of position and momentum;

- the extent $\Delta q \Delta p$ to which they can simultaneously be localized unsharply;

(B) the accuracies Δq and Δp within which position and momentum can be measured together;

(C) the necessary amount of disturbance Δp in momentum caused by a position measurement with the accuracy Δq, and vice versa.

The uncertainty principle could be formulated as an ‘epistemic axiom’ in any of the so-called axiomatic approaches to quantum mechanics (quantum logic, convexity approach, effect algebras, algebraic approach) and typically it would rule out the classical case.

Here this ‘principle’ is considered within the Hilbert space quantum mechanics, where it is not an extra assumption.

Basic notations 1

For a spin-0-object in one dimension:

- Hilbert space $\mathcal{H} = L^2(\mathbb{R})$, unit vector $\psi \in \mathcal{H}$.

- position
  - operator $\hat{Q}$: $(\hat{Q}\psi)(x) = x\psi(x)$,
  - spectral measure Q: $\mathrm{Q}(X)\psi = \chi_X\psi$,
  - probability $\mathrm{p}^{\mathrm{Q}}_{\psi}$: $\mathrm{p}^{\mathrm{Q}}_{\psi}(X) = \langle\psi|\mathrm{Q}(X)\psi\rangle = \int_X |\psi(x)|^2\, dx$.

- momentum
  - operator $\hat{Q}$: $(\hat{P}\psi)(x) = -i\hbar\psi'(x)$,
  - spectral measure P: $\mathrm{P}(Y) = F^{-1}\mathrm{Q}(Y)F$,
  - probability $\mathrm{p}^{\mathrm{P}}_{\psi}$: $\mathrm{p}^{\mathrm{P}}_{\psi}(Y) := \int_Y |\widehat{\psi}(p)|^2\, dp$, $\widehat{\psi} = F\psi$.

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Basic notations 2

For a generic probability measure p on $\mathbb{R}$:

- the *standard deviation*

$$\Delta(\mathrm{p}) := \int_{\mathbb{R}} (x-\mu)^2\, dp, \quad \mu = \int_{\mathbb{R}} x\, dp.$$

- the *overall width*, for given $\varepsilon \in (0,1)$,

$$W_\varepsilon(\mathrm{p}) := \inf_X \{|X| \mid \mathrm{p}(X) \geq 1-\varepsilon\}, \quad X \subset \mathbb{R} \quad \text{interval.}$$

For the position and momentum distributions $\mathrm{p}_\psi^{\mathrm{Q}}$ and $\mathrm{p}_\psi^{\mathrm{P}}$ we use the notations:

$$\begin{aligned} \Delta(\mathrm{Q},\psi) &:= \Delta(\mathrm{p}_\psi^{\mathrm{Q}}), \qquad \Delta(\mathrm{P},\psi) := \Delta(\mathrm{p}_\psi^{\mathrm{P}}), \\ W_{\varepsilon_1}(\mathrm{Q},\psi) &:= W_{\varepsilon_1}(\mathrm{p}_\psi^{\mathrm{Q}}), \qquad W_{\varepsilon_2}(\mathrm{P},\psi) := W_{\varepsilon_2}(\mathrm{p}_\psi^{\mathrm{P}}). \end{aligned}$$

(a-A): version 1, standard deviations

The starting point: for any $q_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there is a vector state ψ such that $\langle \psi | \hat{Q} \psi \rangle = q_0$ and $\Delta(\mathsf{Q}, \psi) < \varepsilon$.

Theorem: For all states ψ and for any $\varepsilon > 0$,

$$\text{if} \quad \Delta(\mathsf{Q}, \psi) < \varepsilon, \quad \text{then} \quad \Delta(\mathsf{P}, \psi) > \hbar/2\varepsilon; \quad \text{and vice versa.}$$

This limitation follows directly from the uncertainty relation for standard deviations, valid for all vector states ψ:

$$\Delta(\mathsf{Q}, \psi) \cdot \Delta(\mathsf{P}, \psi) \geq \frac{\hbar}{2}.$$

Theorem: For all positive numbers $\delta q, \delta p$ for which $\delta q \cdot \delta p \geq \hbar/2$, there is a state ψ such that $\Delta(\mathsf{Q}, \psi) = \delta q$ and $\Delta(\mathsf{P}, \psi) = \delta p$.

(a-A): version 2, localizability

The starting point: for any bounded interval X (however small), there exists a vector state ψ such that $\mathsf{p}^{\mathsf{Q}}_{\psi}(X) = 1$.

Theorem: For all vector states ψ and all bounded intervals X, Y, $\mathsf{p}^{\mathsf{Q}}_{\psi}(X) = 1$ implies $0 \neq \mathsf{p}^{\mathsf{P}}_{\psi}(Y) \neq 1$, and vice versa.

Theorem: For any vector state ψ and for any bounded intervals X and Y,

$$\mathsf{p}^{\mathsf{Q}}_{\psi}(X) + \mathsf{p}^{\mathsf{P}}_{\psi}(Y) \leq 1 + \sqrt{a_0} < 2,$$

where a_0 is the largest eigenvalue of the operator $\mathsf{Q}(X)\mathsf{P}(Y)\mathsf{Q}(X)$ which is positive and trace class. There exists an optimizing vector state φ_0 such that

$$\mathsf{p}^{\mathsf{Q}}_{\varphi_0}(X) + \mathsf{p}^{\mathsf{P}}_{\varphi_0}(Y) = 1 + \sqrt{a_0}.$$

…and related trade-off relations:

Position Q is *approximately localized* in an interval X for a given state ψ whenever $\mathrm{p}_\psi^{\mathrm{Q}}(X) \geq 1 - \varepsilon$ for some (preferably small) ε, $0 < \varepsilon < 1$.

- If position and momentum are both approximately localized within X and Y, respectively, so that $\mathrm{p}_\psi^{\mathrm{Q}}(X) \geq 1 - \varepsilon_1$ and $\mathrm{p}_\psi^{\mathrm{P}}(Y) \geq 1 - \varepsilon_2$, then, for $\varepsilon_1 + \varepsilon_2 < 1$,

$$|X| \cdot |Y| \geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2,$$

and also

$$W_{\varepsilon_1}(\mathrm{Q}, \psi) \cdot W_{\varepsilon_2}(\mathrm{P}, \psi) \geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2 .$$

These results go essentially back to Landau and Pollak (1961), and Lenard (1972).

Towards the questions (b-B) and (c-C):

Observables as *positive operator measures* will be required to introduce

- viable notions of joint and sequential measurements,
- an appropriate quantification of measurement inaccuracy.

The notion of *instrument*, *operation valued measure*, is needed to describe and quantify the disturbance of an observable due to the measurement of another observable.

Observable as a POM

- *Observables* are identified with POMs $E : X \mapsto E(X)$ on the (Borel) subsets of $\mathbb{R}$ or $\mathbb{R}^2$, that is, with the *families of the probability measures* $X \mapsto \langle\psi|E(X)\psi\rangle =: \mathrm{p}_\psi^E(X)$, ψ unit vector.

- E is *sharp* if it is a spectral measure.
- For any E on $\mathbb{R}$, the moment operators are $E[k] := \int x^k E(dx)$.
- The standard deviation $\Delta(E,\psi)$ of p_ψ^E gets the form

$$\Delta(E,\psi)^2 = \langle\psi|E[2]\psi\rangle - \langle\psi|E[1]\psi\rangle^2.$$

- E need not be commutative; that is, it is not always the case that $E(X)E(Y) = E(Y)E(X)$ for all sets X, Y.

Intrinsic noise and resolution width of E

- The *noise operator* of E is the operator

$$N_i(E) := E[2] - E[1]^2 \geq 0.$$

If $E[1]$ is selfadjoint, then $N_i(E) = 0$ exactly when E is sharp.

- The *intrinsic noise* of E in a vector state ψ:

$$\mathcal{N}_i(E;\psi) := \langle\psi|N_i(E)\psi\rangle;$$

the *overall intrinsic noise* of E:

$$\mathcal{N}_i(E) := \sup_{\psi\in\mathcal{H}_1} \mathcal{N}_i(E;\psi).$$

- The *resolution width* of E at confidence level $1-\varepsilon$:

$$\gamma_\varepsilon(E) := \inf\{d > 0 \,|\, \forall x \in \mathbb{R}\ \exists\, \psi \in \mathcal{H}_1 \ \curvearrowright\ \mathrm{p}^E_\psi(J_{x;d}) \geq 1-\varepsilon\}.$$

Here $J_{x;d} := [x - \frac{d}{2}, x + \frac{d}{2}]$.

Instruments as OVMs

A QTM-measurement $\langle \mathcal{K}, \Phi, Z, U \rangle$ determines an instrument $\mathcal{I}$ which defines the measured observable E:

$$\langle U(\psi \otimes \Phi) | I \otimes Z(X) U(\psi \otimes \Phi) \rangle =: \mathrm{tr}\,[\mathcal{I}(X)(P[\psi])] =: \langle \psi | E(X) \psi \rangle.$$

Using the dual operations (Heisenberg picture) the measured observable is given by

$$E(X) = \mathcal{I}(X)^*(I).$$

The (nonselective) state change

$$P[\psi] \mapsto \mathcal{I}(\mathbb{R})(P[\psi])$$

caused by a measurement, with the instrument $\mathcal{I}$, results in the change

$$\mathsf{p}^F_\psi \mapsto \mathsf{p}^F_{\mathcal{I}(\mathbb{R})(P[\psi])}$$

of the probabilities of any observable F, which in the H-representation, yields

$$F(X) \mapsto \mathcal{I}(\mathbb{R})^*(F(X)) =: F'(X), \quad X \in \mathcal{B}(\mathbb{R}),$$

showing the "distortion" of an observable

$$F \mapsto F'$$

under a measurement by $\mathcal{I}$.

Joint measurements

- Observables E_1 and E_2 on $\mathbb{R}$ are *jointly measurable* if there is an observable M on $\mathbb{R}^2$ such that

$$E_1(X) = M(X \times \mathbb{R}), \quad E_2(Y) = M(\mathbb{R} \times Y)$$

for all (Borel) sets X, Y.
E_1 and E_2 are the marginal observables M_1 and M_2 of M.

- If either E_1 or E_2 is sharp, then they are jointly measurable exactly when they commute mutually. The unique joint observable M is then given by $M(X \times Y) = E_1(X)E_2(Y)$.

- Position Q and momentum P are sharp and they do not commute. Therefore, they cannot be measured jointly. **This is the no-go statement (b).**

- An observable M on $\mathbb{R}^2$ is an *approximate joint observable* for Q and P if its marginal observables M_1 and M_2 are *approximations* (in some suitably defined sense) of Q and P, respectively.

- An appropriate quantification of the differences between M_1 and Q and between M_2 and P may serve as a measure of the *(in)accuracy* of the joint approximate measurement represented by M.

Sequential measurements

To analyze measurements of two observables E_1 and E_2 performed in immediate succession, one needs to take into account the influence of the first measurement on the object system.

A sequential measurement "first E_1, with $\mathcal{I}_1$, and then E_2" defines a unique (sequential) joint observable M on $\mathbb{R}^2$ via

$$\begin{aligned} M(X\times Y) &= \mathcal{I}_1^*(X)(E_2(Y)). \\ M_1(X) &= \mathcal{I}_1^*(X)(E_2(\mathbb{R})) = \mathcal{I}_1^*(X)(I) = E_1(X), \\ M_2(Y) &= \mathcal{I}_1^*(\mathbb{R})(E_2(Y)) =: E_2'(Y). \end{aligned}$$

Thus the first marginal is the first-measured observable E_1 and the second marginal is a distorted version E_2' of E_2.

If E_1 is sharp, the distorted effects $\mathcal{I}_1^*(\mathbb{R})(E_2(Y))$ commute with $E_1(X)$ for all X, Y, whatever the second observable E_2 is.

Sequential position-momentum measurements

A Q-measurement, with an instrument $\mathcal{I}_{\mathsf{Q}}$, distorts the momentum P to the extent that all

$$\mathsf{P}'(Y) := \mathcal{I}_{\mathsf{Q}}^{*}(\mathbb{R})(\mathsf{P}(Y))$$

are functions of the position operator $\hat{Q}$.

In this sense, a measurement of sharp position completely destroys any information about the momentum distribution in the input state. **This result formalizes the no-go statement (c).**

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In this sense, a measurement of sharp position completely destroys any information about the momentum distribution in the input state. **This result formalizes the no-go statement (c).**

One may be able to control and limit the disturbance due to a Q-measurement, by measuring an observable Q′ which is an approximation to Q. One can then hope to achieve that the distorted momentum P′ is an approximation to P.

This amounts to defining a sequential joint observable M with marginals $M_1 = \mathsf{Q}'$ and $M_2 = \mathsf{P}'$. Any appropriate quantification of the difference between M_1 and Q is a measure of the inaccuracy of the first (approximate) position measurement; similarly any appropriate quantification of the difference between M_2 and P is a measure of the disturbance of the momentum due to the position measurement.

The problem (c-C) reduces to the problem (b-B).

Error / (in)accuracy in a measurement

Every measurement, whether classical or quantum, is subject to *noise*, which results in a deviation of the actually measured observable E_1 from that intended to be measured, E.

We will refer to this deviation and any measure of it as *error* or *inaccuracy*.

In general there can be systematic errors, or *bias*, leading to a shift of the mean values, and random errors, resulting in a broadening of the distributions.

Any measure of measurement noise should be *operationally significant* in the sense that it should be determined by the probability distributions p_ψ^E and $p_\psi^{E_1}$.

Disturbance of a measurement

Any measurement of any observable disturbs any other observable F:

$$F \mapsto F', \quad F'(X) = \mathcal{I}(\mathbb{R})(F(X)).$$

An operationally significant measure of *disturbance* is to be determined by the probability distributions p_ψ^F and $\mathrm{p}_\psi^{F'}$.

Formally, the notions of error, (in)accuracy, and disturbance are the same.

In the following we will discuss three different approaches to quantifying such measures.

Standard measures of error and disturbance

E - the observable actually measured (QTM), $\langle \mathcal{K}, \Phi, Z, U \rangle$.

A - the sharp observable intended to be measured.

- The *standard error* (from QTM)

$$\begin{aligned}\epsilon(E, A; \psi)^2 &= \langle U(\psi \otimes \Phi) | (Z - A)^2 U(\psi \otimes \Phi) \rangle \\ &= \langle \psi | (E[1] - A)^2 \psi \rangle + \langle \psi | (E[2] - E[1]^2) \psi \rangle \\ &= \langle \psi | N_r(E, A)^2 \psi \rangle + \langle \psi | N_i(E) \psi \rangle,\end{aligned}$$

$N_r(E, A) := E[1] - A$ - the *relative noise* operator.

- The *global standard error* of an observable E relative to A:

$$\epsilon(E, A) := \sup_{\psi \in \mathcal{H}_1} \epsilon(E, A; \psi).$$

- If $\epsilon(E, A) < \infty$, then E is a *standard approximation* to A.

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- If $\epsilon(E, A) < \infty$, then E is a *standard approximation* to A.

Geometric measure of approximation and disturbance

- $\Lambda := \{h : \mathbb{R} \to \mathbb{R} \,|\, \text{bounded measurable}, \, |h(x) - h(y)| \leq |x - y|\}$,
- $E[h] := \int_{\mathbb{R}} h \, dE$ (weakly) - bounded selfadjoint operator.

- The (Werner) distance between the observables E_1 and E_2

$$d(E_1, E_2) := \sup_{\psi \in \mathcal{H}_1} \sup_{h \in \Lambda} |\langle \psi | (E_1[h] - E_2[h]) \, \psi \rangle| \, .$$

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$$F \mapsto F', \quad F'(X) = \mathcal{I}(\mathbb{R})(F(X)).$$

$$= \langle\psi|(E[1] - A)^2\psi\rangle + \langle\psi|(E[2] - E[1]^2)\psi\rangle$$
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Error bars

Error bars give the minimal average interval lengths that one has to allow to contain all output values with a given confidence level.

- For each $\varepsilon \in (0,1)$, an observable E_1 is an ε-*approximation* to a sharp observable E if for all $\delta > 0$ there is a positive number $w < \infty$ such that for all $x \in \mathbb{R}$, $\psi \in \mathcal{H}_1$, the condition $\mathrm{p}_\psi^E(J_{x;\delta}) = 1$ implies that $\mathrm{p}_\psi^{E_1}(J_{x,w}) \geq 1 - \varepsilon$.
- The infimum of all such w will be called the *inaccuracy* of E_1 with respect to E and will be denoted $\mathcal{W}_{\varepsilon,\delta}(E_1, E)$. Thus,

$$\mathcal{W}_{\varepsilon,\delta}(E_1, E) := \inf\{w \mid \mathrm{p}_\psi^E(J_{x;\delta}) = 1 \Rightarrow \mathrm{p}_\psi^{E_1}(J_{x,w}) \geq 1 - \varepsilon\}. \quad (1)$$

- The inaccuracy describes the range within which the input values can be inferred from the output distributions, with confidence level $1 - \varepsilon$, given initial localizations within δ.

- We note that the inaccuracy is an increasing function of δ, so that we can define the *error bar width* of E_1 relative to E:

$$\mathcal{W}_\varepsilon(E_1, E) := \inf_\delta \mathcal{W}_{\varepsilon,\delta}(E_1, E) = \lim_{\delta\to 0} \mathcal{W}_{\varepsilon,\delta}(E_1, E). \qquad (2)$$

  If $\mathcal{W}_\varepsilon(E_1, E) < \infty$ for all $\varepsilon \in (0, \frac{1}{2})$, we will say that E_1 approximates E in the sense of *finite error bars*.
- We note that the finiteness of either $\epsilon(E_1, E)$ or $d(E_1, E)$ implies the finiteness of $\mathcal{W}_\varepsilon(E_1, E)$. Therefore, among the three measures of inaccuracy, the condition of finite error bars gives the most general criterion for selecting approximations of Q and P.

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From (b) to (B)

Position Q and momentum P have no joint observable, they cannot be measured together.
However, one may ask for an approximate joint measurement, that is, for an observable M on $\mathbb{R}^2$ such that the marginals M_1 and M_2 are appropriate approximations of Q and P.

Commuting functions of Q and P

Theorem (e.g. Ylinen, 1987)

Let g and h be essentially bounded Borel functions such that neither $g(\hat{Q})$ nor $h(\hat{P})$ is a constant operator. The functions Q^g of position and P^h of momentum commute if and only if g and h are both periodic with minimal positive periods a, b satisfying $\frac{2\pi}{ab} \in \mathbb{N}$.

- If Q^g and P^h are commuting observables, then they have the joint observable M, with $M(X \times Y) = \mathsf{Q}^g(X)\mathsf{P}^h(Y)$, meaning that Q^g and P^h can be measured jointly.
- $\epsilon(Q^f, Q) = d(\mathsf{Q}^f, \mathsf{Q}) = \mathcal{W}_{\varepsilon,\delta}(Q^f, Q) = \infty$.

Covariant approximations of Q and P

Let μ, ν be probability measures on $\mathbb{R}$, and define $\mathsf{Q}_\mu, \mathsf{P}_\nu$ via

$$\mathsf{Q}_\mu(X) = \int_{\mathbb{R}} \mu(X-q)\,\mathsf{Q}(dq), \qquad \mathsf{P}_\nu(Y) = \int_{\mathbb{R}} \nu(Y-p)\,\mathsf{P}(dp).$$

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Corollary (Busch *et al*, 2006)

If M is an observable on $\mathbb{R}^2$, which approximates both Q *and* P *in the sense of finite standard error, or geometric distance, or error bar width, then the universally valid Heisenberg uncertainty relations hold*

$$\epsilon(M_1, \mathrm{Q}) \cdot \epsilon(M_2, \mathrm{P}) \geq \frac{\hbar}{2},$$

$$d(M_1, \mathrm{Q}) \cdot d(M_2, \mathrm{P}) \geq C\hbar,$$

$$\mathcal{W}_{\varepsilon_1}(M_1, \mathrm{Q}) \cdot \mathcal{W}_{\varepsilon_2}(M_2, \mathrm{P}) \geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2 .$$

Distances of G_1^T, G_2^T from Q, P:

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Theorem (Carmeli et al, 2005)

An approximate position Q_μ and an approximate momentum P_ν are jointly measurable if and only if they have a covariant joint observable G^T. This is the case exactly when there is a positive operator T of trace equal to 1 such that $\mu = \mu_T, \nu = \nu_T$.

Any translation covariant boost invariant pom on $\mathbb{R}$ is of the form Q_μ.
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The degrees of approximations

Standard deviations:

$$\Delta(G_1^T,\psi)^2 = \Delta(\mathsf{Q},\psi)^2 + \Delta(\mu_T)^2, \quad \Delta(G_2^T,\psi)^2 = \Delta(\mathsf{P},\psi)^2 + \Delta(\nu_T)^2.$$

with the state-preparation uncertainty relation with respect to G^T,

$$\Delta(G_1^T,\psi)\cdot\Delta(G_2^T,\psi) \geq \hbar.$$

Intrinsic noise operators (for appropriate T):

$$N_i(G_1^T) = \Delta(\mu_T)^2\, I, \quad N_i(G_2^T) = \Delta(\nu_T)^2\, I.$$

with the uncertainty relation for intrinsic noise, valid for any G^T:

$$\mathcal{N}_i(G_1^T)\cdot\mathcal{N}_i(G_2^T) = \Delta(\mu_T)^2\cdot\Delta(\nu_T)^2 \geq \frac{\hbar^2}{4}.$$

Resolution widths:

$$\gamma_{\varepsilon_1}(G_1^T) = W_{\varepsilon_1}(\mu_T), \quad \gamma_{\varepsilon_2}(G_2^T) = W_{\varepsilon_2}(\nu_T),$$

and thus:

$$\gamma_{\varepsilon_1}(G_1^T) \cdot \gamma_{\varepsilon_2}(G_2^T) = W_{\varepsilon_1}(\mu_T) \cdot W_{\varepsilon_2}(\nu_T) \geq 2\pi\hbar \cdot (1 - \varepsilon_1 - \varepsilon_2)^2.$$

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General approximate joint measurements

Theorem (Werner, 2004)

Let M be an observable on $\mathbb{R}^2$. If $d(M_1, \mathrm{Q}) < \infty$ and $d(M_2, \mathrm{P}) < \infty$, then there is a covariant phase space observable G^T associated with M with the property:

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The same kind of argument can be carried out in the case of the global standard error and the error bar width.

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