Title: Sequential Products of Quantum Measurements

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Abstract:

SEQUENTIAL PRODUCTS OF QUANTUM MEASUREMENTS

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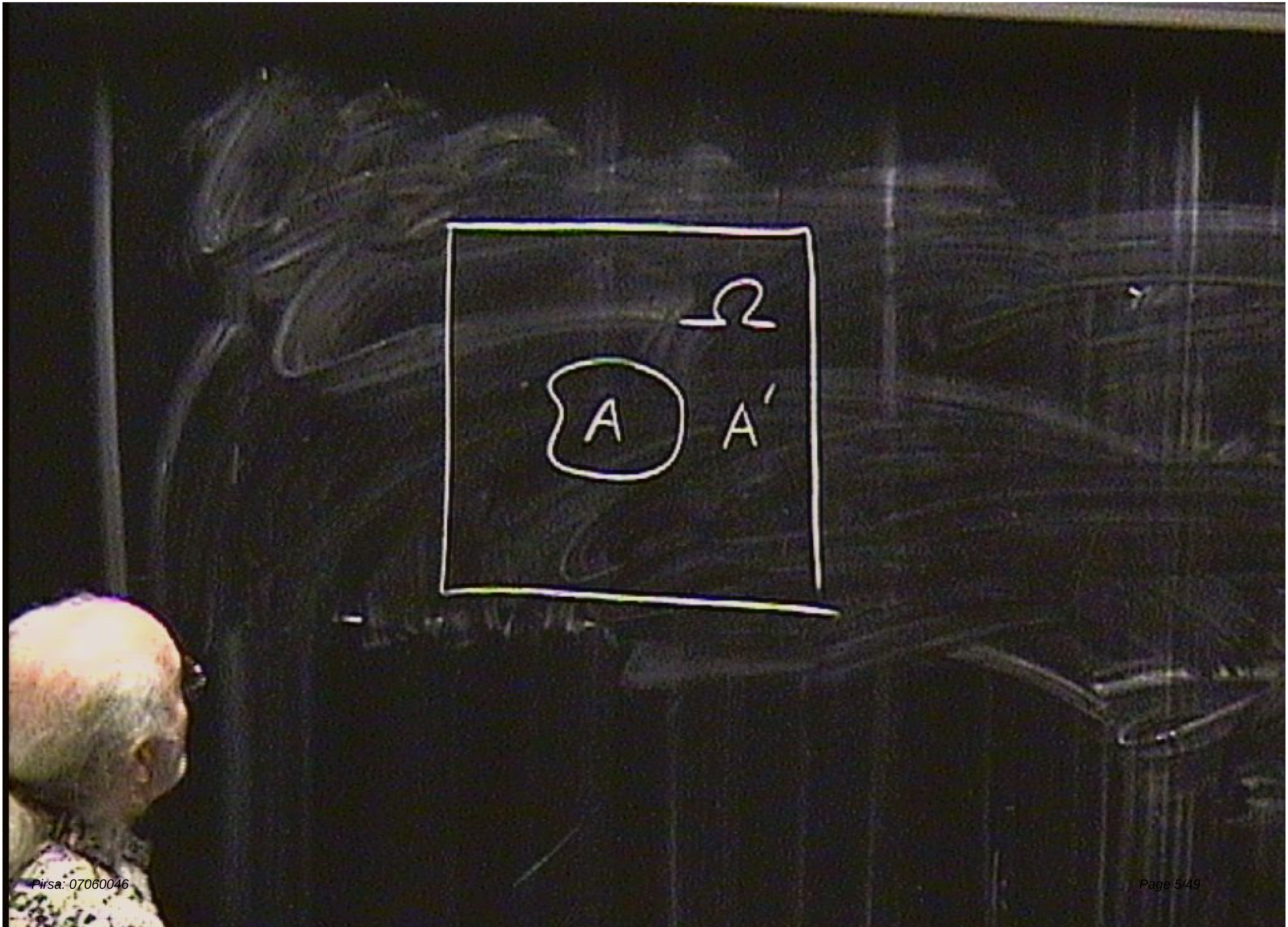
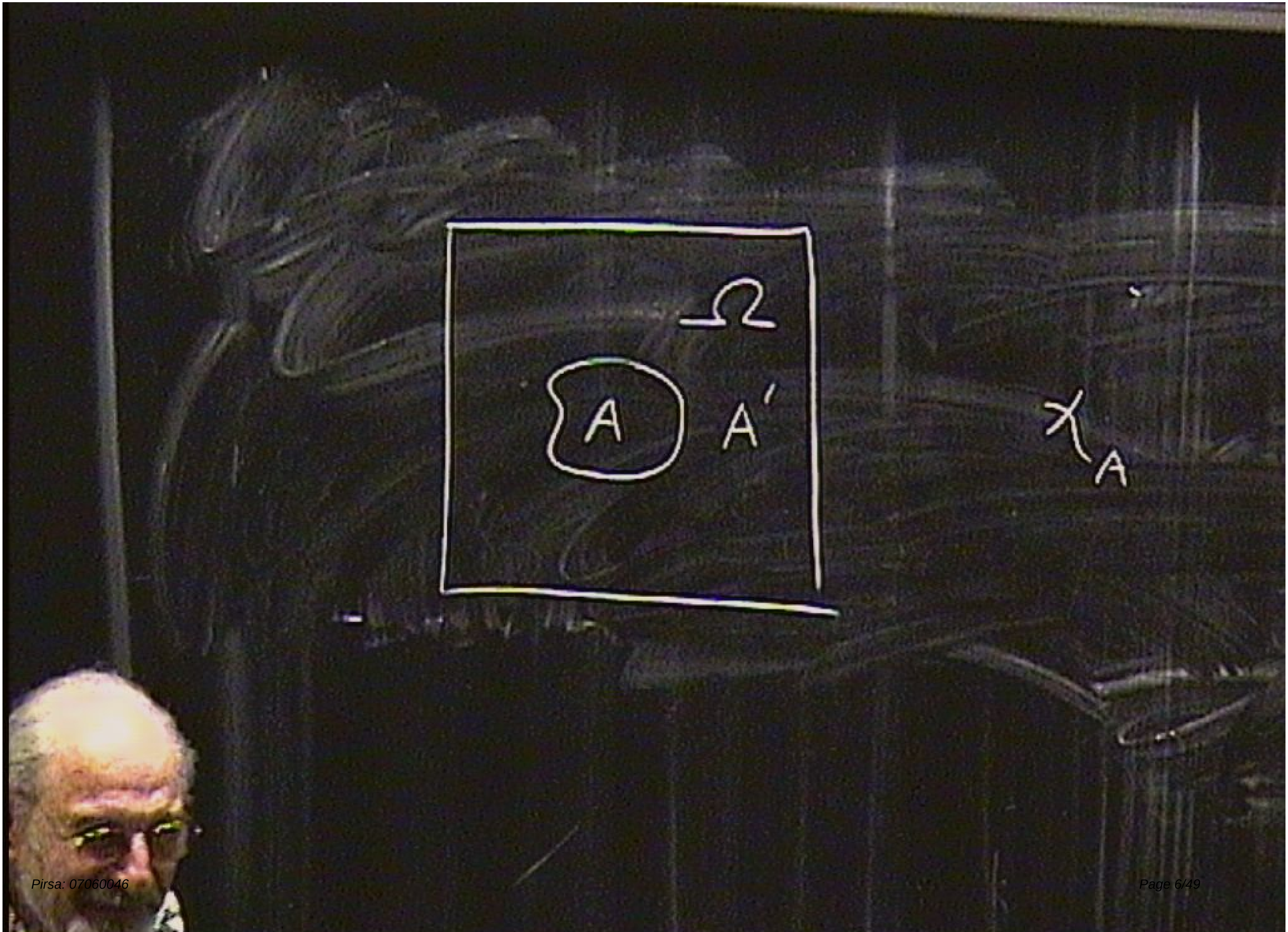
1. Notation and Definitions

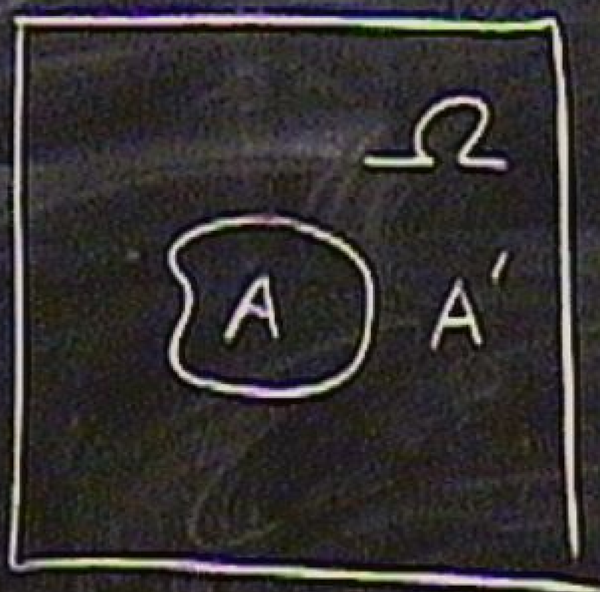
A classical two-valued measurement corresponds to a pair $\mathcal{A} = \{\chi_A, \chi_{A'}\}$ where $\chi_{A'} = 1 - \chi_A$. We interpret $\mathcal{A}$ as having the value 1 if χ_A occurs and the value 0 if χ_A does not occur. If $\mathcal{B} = \{\chi_B, \chi_{B''}\}$ is another two-valued measurement, we form their **sequential product**

$$\begin{aligned}\mathcal{A} \circ \mathcal{B} &= \{\chi_A\chi_B, \chi_A\chi_{B'}, \chi_{A'}\chi_B, \chi_{A'}\chi_{B'}\} \\ &= \{\chi_{A\cap B}, \chi_{A\cap B'}, \chi_{A'\cap B}, \chi_{A'\cap B'}\}\end{aligned}$$

We interpret $\mathcal{A} \circ \mathcal{B}$ as the measurement resulting from first performing $\mathcal{A}$ and then performing $\mathcal{B}$. Then $\mathcal{A} \circ \mathcal{B}$ has four values depending on whether χ_A and χ_B, χ_A and $\chi_{B'}$, $\chi_{A'}$ and χ_B, or $\chi_{A'}$ and $\chi_{B'}$ both occur. Since $\mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A}$ the order of performing measurements is irrelevant in the classical theory. Because of quantum interference, this is no longer true in quantum mechanics.


Ω
A
A'


Ω
A
A′
χ_A



$$a = \{\chi_A, \chi_{A'}\}$$

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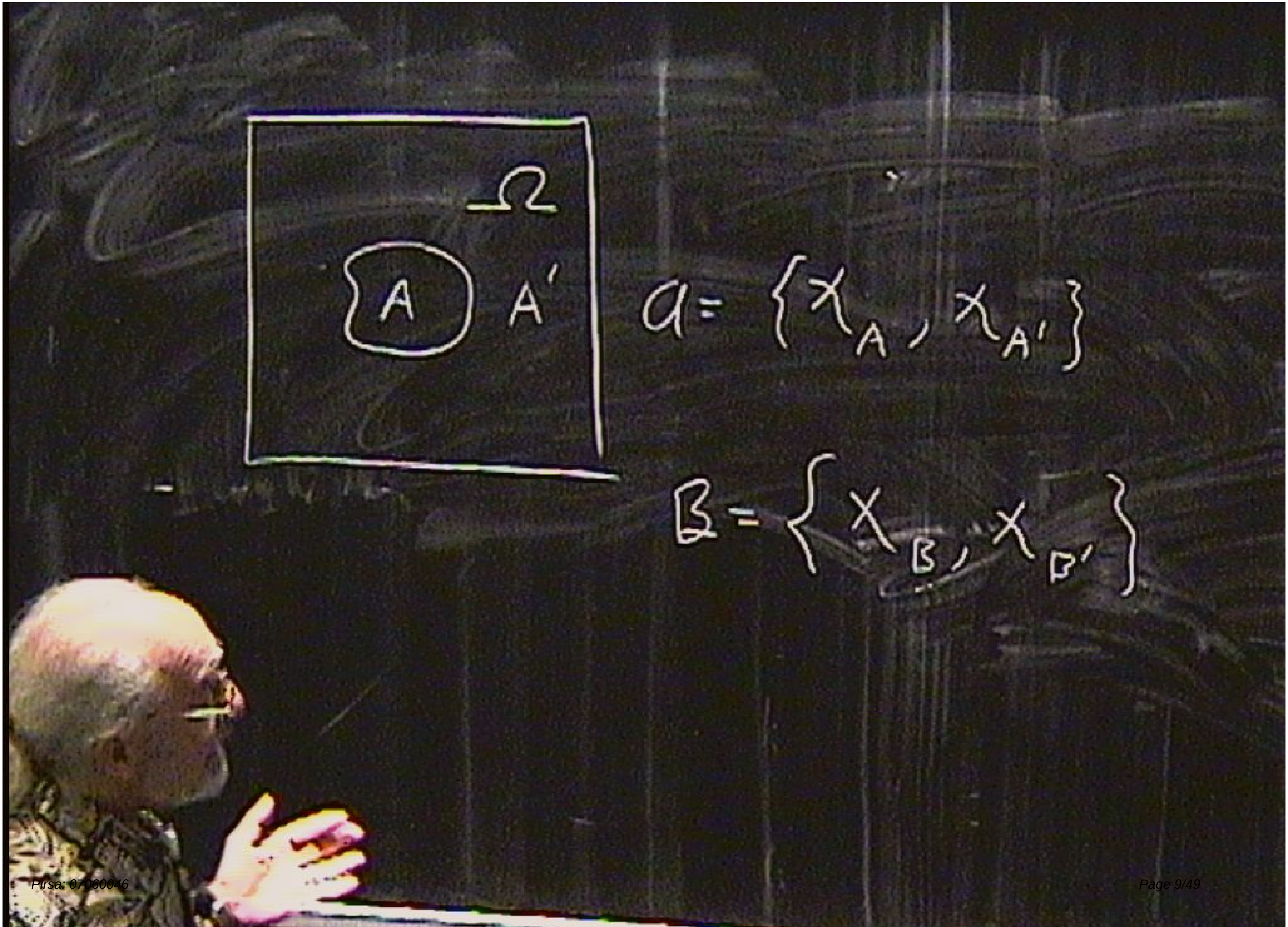
Ω
A
A'
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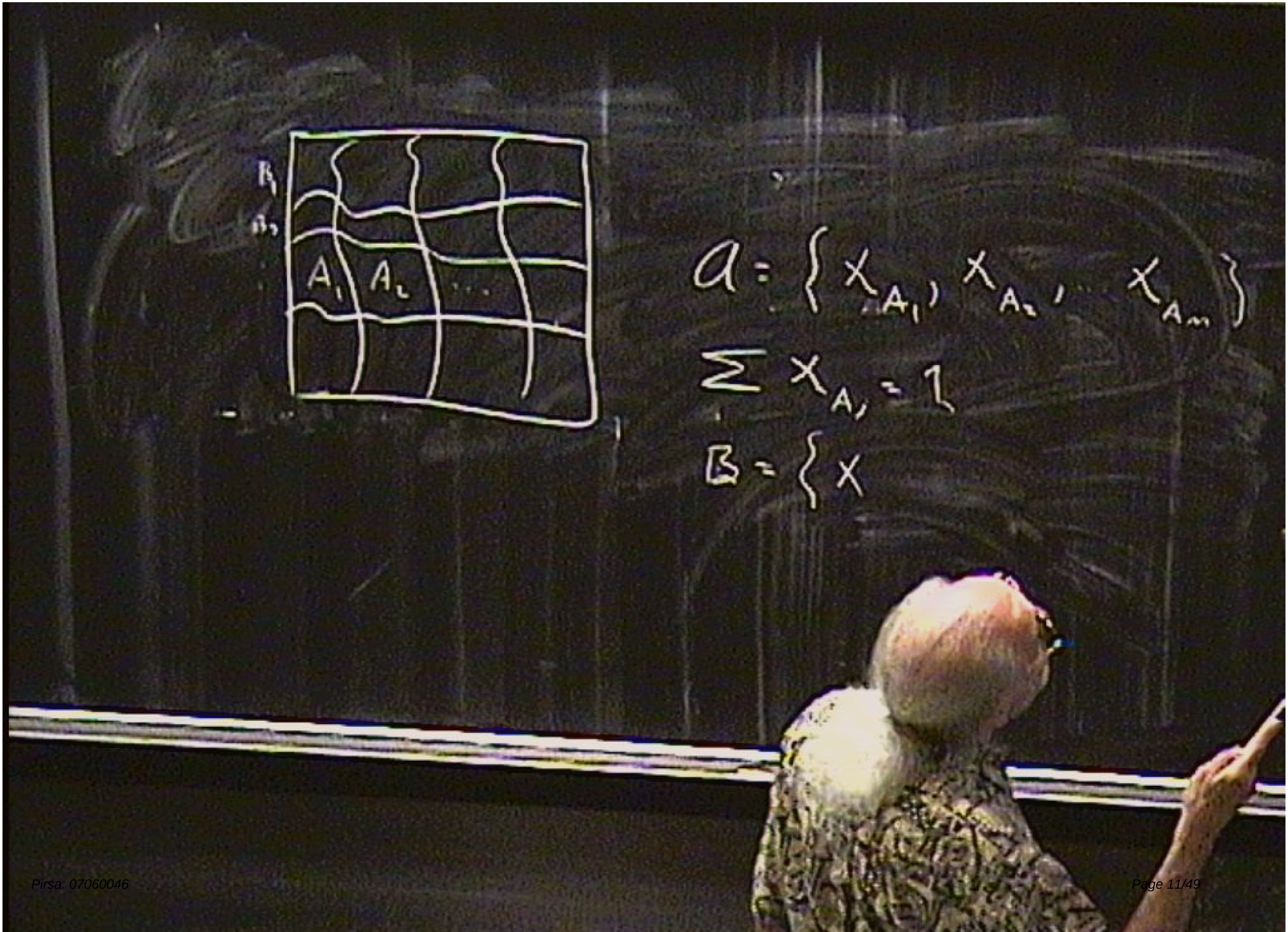
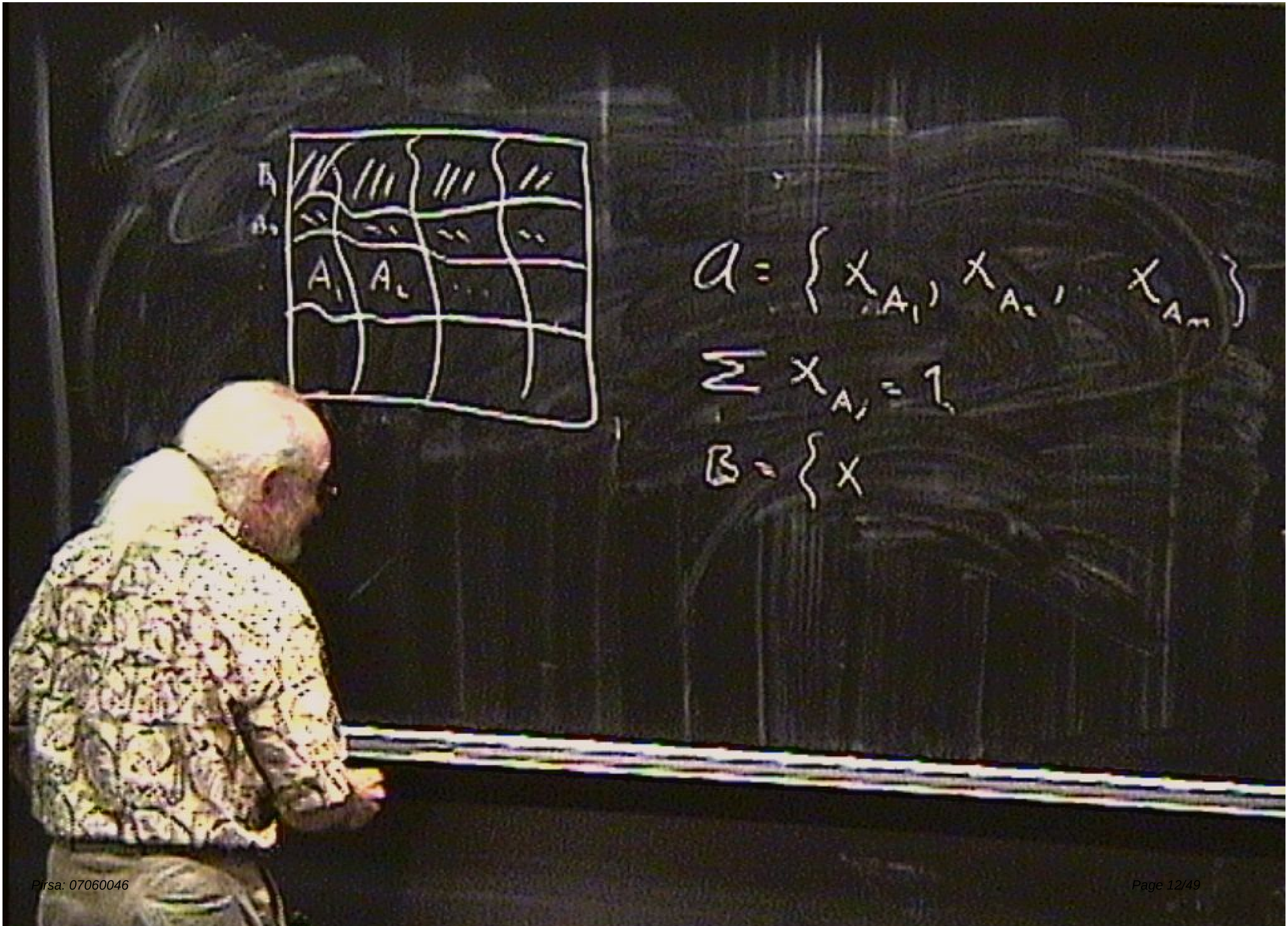

B1
B2
A1 A2 ...
a = { X_A1, X_A2, ... X_Am }
Σ X_Ai = 1
B = { X


B1
B2
A1 A2 ...
a = { X_A1 , X_A2 , X_Am }
Σ X_Ai = 1
B = { X

We can extend this discussion to discrete measurements which have a finite or countable number of values. In this case, $\mathcal{A} = \{\chi_{A_1}, \chi_{A_2}, \ldots\}$ where $\{A_1, A_2, \ldots\}$ is a partition of a sample space Ω. In this case $\mathcal{A}$ is equivalent to a discrete random variable. If $\mathcal{B} = \{\chi_{B_1}, \chi_{B_2}, \ldots\}$ is another discrete measurement we form the sequential product

$$\mathcal{A} \circ \mathcal{B} = \left\{\chi_{A_i}\chi_{B_j} : i, j = 1, 2, \ldots\right\} = \left\{\chi_{A_i \cap B_j} : i, j = 1, 2, \ldots\right\}$$

This corresponds to a finer partition of Ω

We can extend this to classical fuzzy probability theory. In this case fuzzy events are represented by functions in $[0,1]^X$. A discrete measurement is given by $\mathcal{A} = \{f_1, f_2, \ldots\}$ where $f_i \in [0,1]^X$ satisfy $\sum f_i = 1$. If $\mathcal{B} = \{g_1, g_2, \ldots\}$ is another discrete measurement their sequential product becomes

$$\mathcal{A} \circ \mathcal{B} = \left\{f_i g_j : i, j = 1, 2, \ldots\right\}$$

As before $\mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A}$

The formalism for quantum measurements is similar to that of fuzzy probability theory except now the fuzzy events are represented by positive operators on a complex Hilbert space H.

Notation:

$$\mathcal{B}(H) = \{\text{Bounded Operators on } H\}$$
$$\mathcal{E}(H) = \{A \in \mathcal{B}(H) : 0 \le A \le I\}$$
$$\mathcal{P}(H) = \{P \in \mathcal{E}(H) : P^2 = P\}$$
$$\mathcal{D}(H) = \{\rho \in \mathcal{E}(H) : \mathrm{tr}(\rho) = 1\}$$

The elements of $\mathcal{E}(H)$ correspond to fuzzy quantum events and are called **effects.**

The elements of $\mathcal{P}(H)$ are projections corresponding to quantum events and are called **sharp effects.**

The elements of $\mathcal{D}(H)$ are density operators corresponding to probability measures and are called **states.**

If $\rho \in \mathcal{D}(H)$, $A \in \mathcal{E}(H)$ then $\mathrm{tr}(\rho A)$ is the probability that A occurs in the state ρ.

If $A, B \in \mathcal{E}(H)$, their **sequential product** is

$$A \circ B = A^{1/2} B A^{1/2} \in \mathcal{E}(H)$$

We interpret $A \circ B$ as the effect that occurs when A occurs first...

A **measurement** is a finite or infinite sequence $\{A_i\}$, where $A_i \in \mathcal{E}(H)$ satisfy $\sum A_i = I$. If $\mathcal{A} = \{A_i\}$ is a measurement, then A_i is the effect observed when $\mathcal{A}$ is performed and the result is the ith outcome. We call $A_1, A_2, \ldots$, the **elements** of $\mathcal{A}$. If the system is in the state ρ and $\mathcal{A}$ is performed, then the probability that the result is the ith outcome is $\operatorname{tr}(\rho A_i)$. Notice that $i \mapsto \operatorname{tr}(\rho A_i)$ is a probability distribution because

$$\sum \operatorname{tr}(\rho A_i) = \operatorname{tr}\left(\rho \sum A_i\right) = \operatorname{tr}(\rho) = 1$$

A measurement is also called a discrete POVM. If $A_i \in \mathcal{P}(H)$, then $\mathcal{A} = \{A_i\}$ is a **sharp measurement** or discrete PVM.
We denote the set of measurements on H by $\mathcal{M}(H)$ and the set of sharp measurements on by $\mathcal{S}(H)$.

For $\mathcal{A} = \{A_i\} \in \mathcal{M}(H)$ and $\mathcal{B} = \{B_i\} \in \mathcal{M}(H)$ we define their **sequential product** by $\mathcal{A} \circ \mathcal{B} = \left\{A_i \circ B_j\right\}$. We interpret $\mathcal{A} \circ \mathcal{B}$ to be the measurement obtained when $\mathcal{A}$ is performed first and $\mathcal{B}$ is performed second. We indeed have that $\mathcal{A} \circ \mathcal{B} \in \mathcal{M}(H)$ because

$$\sum_{i,j} A_i \circ B_j = \sum_{i,j} A_i^{1/2} B_j A_i^{1/2} = \sum_i A_i^{1/2} \sum_j B_j A_i^{1/2} = \sum A_i = I$$

2. Equivalence

For $\mathfrak{A}, \mathfrak{B} \in \mathfrak{M}(H)$, we write $\mathfrak{A} \approx \mathfrak{B}$ and say $\mathfrak{A}$ and $\mathfrak{B}$ are **equivalent** if the nonzero elements of $\mathfrak{A}$ are a permutation of the nonzero elements of $\mathfrak{B}$. We say that $\mathfrak{A} = \{A_i\}$ and $\mathfrak{B} = \left\{B_j\right\}$ are **compatible** if $A_iB_j = B_jA_i$ for all i and j. It is clear that if $\mathfrak{A}$ and $\mathfrak{B}$ are compatible, then $\mathfrak{A} \circ \mathfrak{B} \approx \mathfrak{B} \circ \mathfrak{A}$. The converse does not hold. Indeed, $\mathfrak{A} \circ \mathfrak{A} \approx \mathfrak{A} \circ \mathfrak{A}$ and yet $\mathfrak{A}$ need not be compatible with itself. The converse does hold under certain circumstances.

Theorem 2.1. If $\mathcal{P} \in \mathfrak{S}(H)$, $\mathfrak{A} \in \mathfrak{M}(H)$ and $\mathcal{P} \circ \mathfrak{A} \approx \mathfrak{A} \circ \mathcal{P}$ then $\mathcal{P}$ and $\mathfrak{A}$ are compatible.

We define the **supplement** of $A \in \mathcal{E}(H)$ by $A' = I - A$.

Theorem 2.2. Suppose $\dim H < \infty$ and $\mathfrak{A} = \{A, A'\}$, $\mathfrak{B} = \{B, B'\}$. If $\mathfrak{A} \circ \mathfrak{B} \approx \mathfrak{B} \circ \mathfrak{A}$, then $\mathfrak{A}$ and $\mathfrak{B}$ are compatible.

Theorem 2.3. For $\mathfrak{A}, \mathfrak{B} \in \mathfrak{M}(H)$ we have $\mathfrak{A} \circ \mathfrak{B} \in \mathfrak{S}(H)$ if and only if $\mathfrak{A}, \mathfrak{B} \in \mathfrak{S}(H)$ and $\mathfrak{A}$ and $\mathfrak{B}$ are compatible.

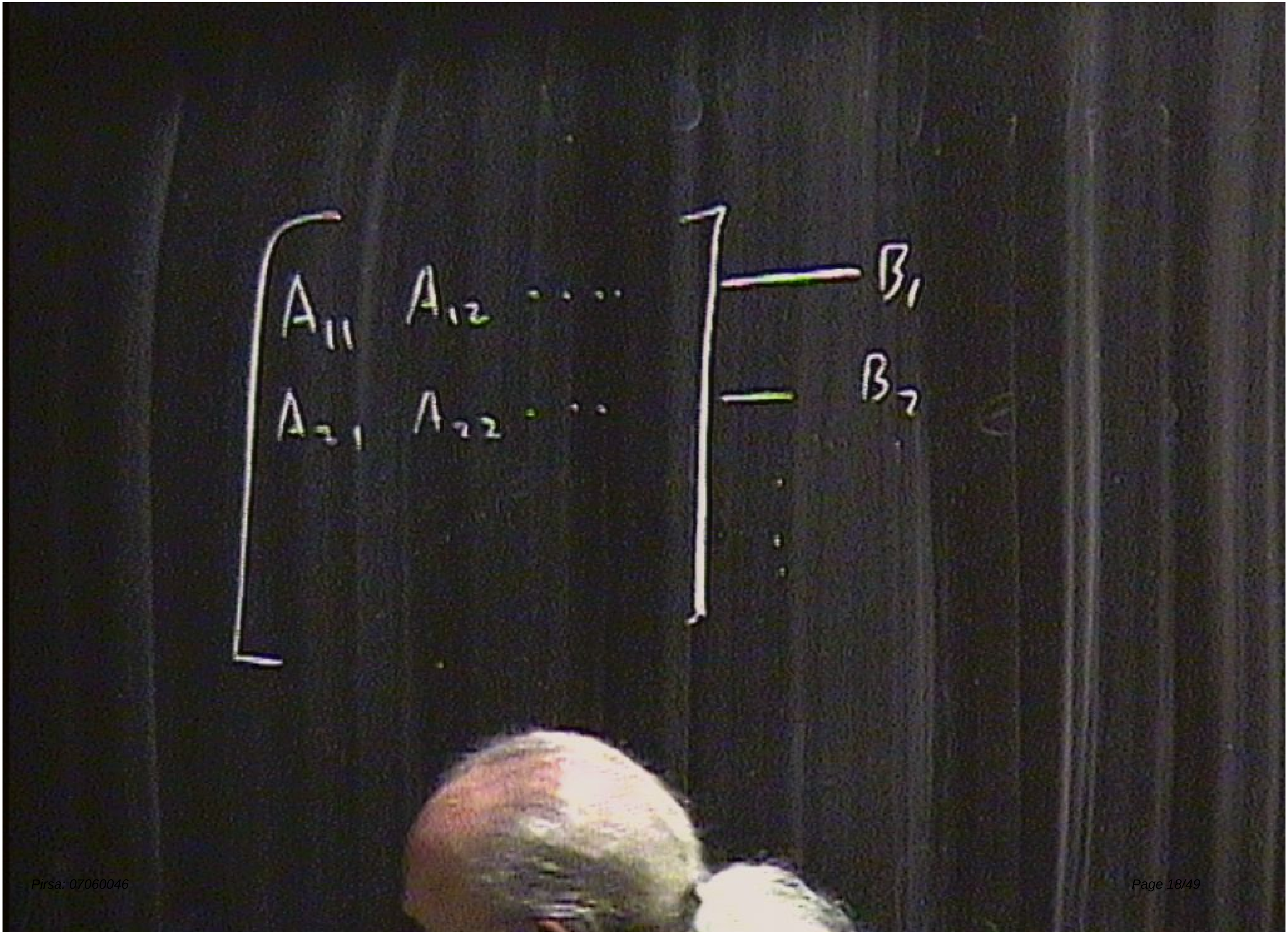
If $\mathcal{A} \in \mathcal{S}(H)$, it is clear that $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$. It is easy to prove the converse when $\mathcal{A} = \{A_1, \dots, A_n\}$ is finite, $A_i \neq 0$, $i = 1, \dots, n$. If $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$, then $\left\{A_i \circ A_j\right\} \approx \{A_i\}$ and since $A_i \neq 0$, $A_i^2 \neq 0$, $i = 1, \dots, n$. Since $\mathcal{A} \circ \mathcal{A}$ and $\mathcal{A}$ have the same number of nonzero elements, we conclude that $A_i \circ A_j = 0$, $i \neq j$. Hence,

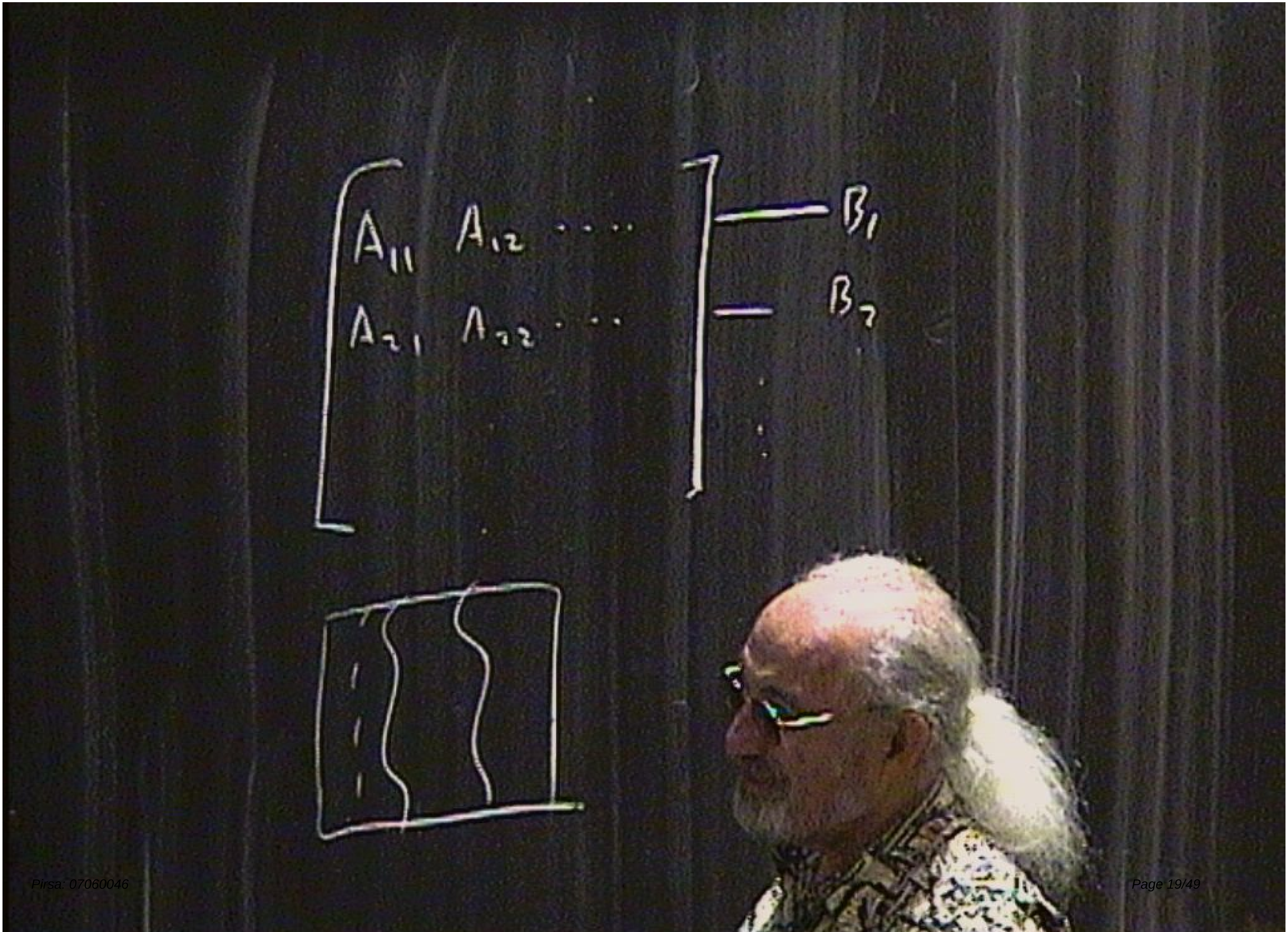
$$A_i = A_i \circ I = A_i \circ \sum_j A_j = \sum_j A_i \circ A_j = A_i^2$$

so $A_i \in \mathcal{P}(H)$, $i = 1, \dots, n$. Hence, $\mathcal{A} \in \mathcal{S}(H)$. This result holds in general.

Theorem 3.1. For $\mathcal{A} \in \mathcal{M}(H)$ we have that $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$ if and only if $\mathcal{A} \in \mathcal{S}(H)$.

For $\mathcal{A}, \mathcal{B} \in \mathcal{M}(H)$ we call $\mathcal{A}$ a **refinement** of $\mathcal{B}$ and write $\mathcal{A} \leq \mathcal{B}$ if we can adjoin 0s to $\mathcal{A}$ if necessary and organize the elements of $\mathcal{A}$ so that $\mathcal{A} \approx \left\{A_{ij}\right\}$ and $B_i = \sum_j A_{ij}$ for all i.
For example, $\mathcal{A} \circ \mathcal{B} \leq \mathcal{A}$. Indeed, $\mathcal{A} \circ \mathcal{B} = \left\{A_i \circ B_j\right\}$ and $A_i = \sum_j A_i \circ B_j$ for all i. There are examples which show that the converse does not hold. That is $\mathcal{C} \leq \mathcal{A}$ does not imply that $\mathcal{C} \approx \mathcal{A} \circ \mathcal{B}$ for some $\mathcal{B} \in \mathcal{M}(H)$.


A11 A12 ....
A21 A22 ...
B1
B2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If $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$, then $\{A_i \circ A_j\} \approx \{A_i\}$ and since $A_i \neq 0$, $A_i^2 \neq 0$, $i = 1, \ldots, n$. Since $\mathcal{A} \circ \mathcal{A}$ and $\mathcal{A}$ have the same number of nonzero elements, we conclude that $A_i \circ A_j = 0$, $i \neq j$. Hence,

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so $A_i \in \mathcal{P}(H)$, $i = 1, \ldots, n$. Hence, $\mathcal{A} \in \mathcal{S}(H)$. This result holds in general.

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The next result shows that $\le$ gives a partial order on $\mathcal{M}(H)$. Strictly speaking, we are considering equivalence classes because we use $\approx$ instead of equality.

Theorem 3.2. $(\mathcal{M}(H), \le)$ is a poset with largest element $\mathcal{I}$ and $\mathcal{A} \le \mathcal{B}$ implies $\mathcal{C} \circ \mathcal{A} \le \mathcal{C} \circ \mathcal{B}$.

Theorem 3.3. (a) If $\mathcal{A} \in \mathcal{S}(H)$, $\mathcal{B} \in \mathcal{M}(H)$ and $\mathcal{B} \le \mathcal{A}$, then $\mathcal{B} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$. (b) If $\mathcal{A} \in \mathcal{S}(H)$, $\mathcal{B} \in \mathcal{M}(H)$ and $\mathcal{A} \le \mathcal{B}$, then $\mathcal{B} \in \mathcal{S}(H)$ and $\mathcal{A} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$. (c) If $\mathcal{A}, \mathcal{B} \in \mathcal{M}(H)$ and $\mathcal{A} \circ \mathcal{B} \approx \mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are compatible, $\mathcal{B} \in \mathcal{S}(H)$ and $\mathcal{A} \le \mathcal{B}$.

Theorem 3.4. For $\mathcal{A} \in \mathcal{M}(H)$ and $\mathcal{B} \in \mathcal{S}(H)$, $\mathcal{A} \wedge \mathcal{B}$ exist if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible. In this case $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \circ \mathcal{B}$.

The characterization of pairs $\mathcal{A}, \mathcal{B}$ for which $\mathcal{A} \wedge \mathcal{B}$ (or $\mathcal{A} \vee \mathcal{B}$) exist is an open problem. It is easy to show that if $\mathcal{A}, \mathcal{B} \in \mathcal{M}(H)$, $\mathcal{B}$ has two elements and $\mathcal{A} \not\le \mathcal{B}$, then $\mathcal{A} \vee \mathcal{B} = \mathcal{I}$.

4. Coexistence

Two effects $A, B \in \mathcal{E}(H)$ **coexist** if there exist effects $C_1, C_2, C_3 \in \mathcal{E}(H)$ such that $C_1 + C_2 + C_3 \leq I$ and $A = C_1 + C_2$, $B = C_2 + C_3$. It is well known that if $A, B \in \mathcal{E}(H)$ are compatible then A, B coexist. Moreover, if $A \in \mathcal{E}(H)$, $P \in \mathcal{P}(H)$ and A, P coexist, then A, P are compatible. We say that $A \in \mathcal{E}(H)$ is **associated** with $\mathcal{A} \in \mathcal{M}(H)$ if $\mathcal{A} \leq \{A, A'\}$.

Lemma 4.1. Two effects $A, B \in \mathcal{E}(H)$ coexist if and only if A, B are associated with a common measurement $\mathcal{A} \in \mathcal{M}$.

We say that $\mathcal{A}, \mathcal{B} \in \mathcal{M}(H)$ **coexist** if they have a common refinement. Notice that if $\mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$, then $\mathcal{A} \circ \mathcal{B} \leq \mathcal{A}, \mathcal{B}$ so $\mathcal{A}, \mathcal{B}$ coexist. By Theorem 2.2, the converse does not hold. The next lemma shows that this definition generalizes the definition of coexistence of effects.

Lemma 4.2. $A, B \in \mathcal{E}(H)$ coexist if and only if $\{A, A'\}$, $\{B, B'\}$ coexist.

If $\{A_i\}$, $\{B_j\}$ coexist, then A_i, B_j coexist for all i, j. The next example shows that the converse does not hold. Since $\mathcal{A} \leq \mathcal{B}$

then A, B coexist. Moreover, if $A \in \mathcal{E}(H)$, $P \in \mathcal{P}(H)$ and A, P coexist, then A, P are compatible. We say that $A \in \mathcal{E}(H)$ is **associated** with $\mathfrak{A} \in \mathcal{M}(H)$ if $\mathfrak{A} \leq \{A, A'\}$.

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If $\{A_i\}$, $\{B_j\}$ coexist, then A_i, B_j coexist for all i, j. The next example shows that the converse does not hold. Since $\mathfrak{A} \leq \mathfrak{B}$ implies $\mathfrak{C} \circ \mathfrak{A} \leq \mathfrak{C} \circ \mathfrak{B}$, it follows that if $\mathfrak{A}, \mathfrak{B}$ coexist, then $\mathfrak{C} \circ \mathfrak{A}$, $\mathfrak{C} \circ \mathfrak{B}$ coexist. In contrast, if $\mathfrak{A}, \mathfrak{B}$ are compatible, then $\mathfrak{C} \circ \mathfrak{A}$, $\mathfrak{C} \circ \mathfrak{B}$ need not be compatible.

Example. Let $P, Q \in \mathcal{P}(H)$ with $PQ \neq QP$ and let $A = \frac{1}{2}P$, $B = \frac{1}{2}Q$. Then $A, B \in \mathcal{E}(H)$ and since $A + B \leq I$ we have that A, B coexist. Since $B + B = Q \in \mathcal{P}(H)$ and A is not compatible with $B + B$, A and $B + B$ do not coexist. We conclude that an effect A can coexist with two effects B_1, B_2 where $B_1 + B_2 \leq I$ and yet A and $B_1 + B_2$ do not coexist. Letting $A = \frac{1}{2}P$, $B = \frac{1}{2}Q$ as before, define the measurements $\mathcal{A} = \{A, A'\}$,

$$\mathcal{B} = \left\{B, B, \tfrac{1}{2}I - B, \tfrac{1}{2}I - B\right\}$$

The elements of $\mathcal{A}$ and $\mathcal{B}$ mutually coexist, but since A does not coexist with $B + B$ we conclude that $\mathcal{A}$ and $\mathcal{B}$ do not coexist.

Theorem 3.4. If $\mathcal{A} \in \mathcal{M}(H)$, $\mathcal{B} \in \mathcal{S}(H)$ coexist, then $\mathcal{A}$ and $\mathcal{B}$ are compatible.

5. Quantum Markov Chains

A **transition effect matrix** (TEM) is a (possibly infinite) square matrix of effects $E = [E_{ij}]$ whose row sums are I, i.e. $\sum_j E_{ij} = I$ for all i. Thus, each row is a POVM. If $E = [E_{ij}]$, $F = [F_{ij}]$ are TEM's we define $E \circ F$ by

$$(E \circ F)_{ij} = \sum_k E_{ik} \circ E_{kj}$$

The following calculation shows that $E \circ F$ is again a TEM.

$$\sum_j (E \circ F)_{ij} = \sum_j \sum_k E_{ik} \circ E_{kj} = \sum_k E_{ik} \circ \sum_j F_{kj} = \sum_k E_{ik} = I$$

A **vector state** is a column vector $A = (A_1, A_2, \ldots)$ where A_i is a positive trace class operator, $i = 1, 2, \ldots$, and $\sum \operatorname{tr}(A_i) = 1$. We use the notation $\operatorname{tr}(A) = (\operatorname{tr}(A_1), \operatorname{tr}(A_2), \ldots)$ and denote the set of vector states by $\mathcal{V}(H)$. If E is a TEM and $A \in \mathcal{V}(H)$, we define $E * A$ to be the column vector $E * A = E^T \circ A$; that is,

$$(E * A)_i = \sum_j (E^T)_{ij} \circ A_j = \sum_j E_{ji} \circ A_j$$

$$\mathrm{tr}\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} \mathrm{tr}(A_1) \\ \mathrm{tr}(A_2) \\ \vdots \\ \vdots \end{bmatrix}$$

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$$(E * A)_i = \sum_j (E^T)_{ij} \circ A_j = \sum_j E_{ji} \circ A_j$$

The multiplication $E \circ F$ of TEMs is a generalization of the sequential product of measurements. For example, let $\mathcal{A} = \{A, A'\}$, $\mathcal{B} = \{B, B'\}$ be 2-valued measurements. Form the TEMs

$$E = \begin{bmatrix} A & A' & 0 & 0 \\ A & A' & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \qquad F = \begin{bmatrix} B & B' & 0 & 0 \\ 0 & 0 & B & B' \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

We then have

$$E \circ F = \begin{bmatrix} A \circ B & A \circ B' & A' \circ B & A' \circ B' \\ A \circ B & A \circ B' & A' \circ B & A' \circ B' \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

Thus, $E \circ F$ contains two copies of the sequential product $\mathcal{A} \circ \mathcal{B}$ together with two identity measurements. It is straightforward to generalize this to measurements with more than two values. Another possibility is to let

$$E = \begin{bmatrix} A & A' & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \qquad F = \begin{bmatrix} B & B' & 0 & 0 \\ 0 & 0 & B & B' \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

the TEMs

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Then

$$E \circ F = \begin{bmatrix} A \circ B & A \circ B' & A' \circ B & A' \circ B' \\ 0 & 0 & B & B' \\ 0 & 0 & I & 0 \end{bmatrix}$$

In general, the product of TEM's $\circ$ is nonassociative and when we write $E_n \circ \cdots \circ E_2 \circ E_1$ we mean

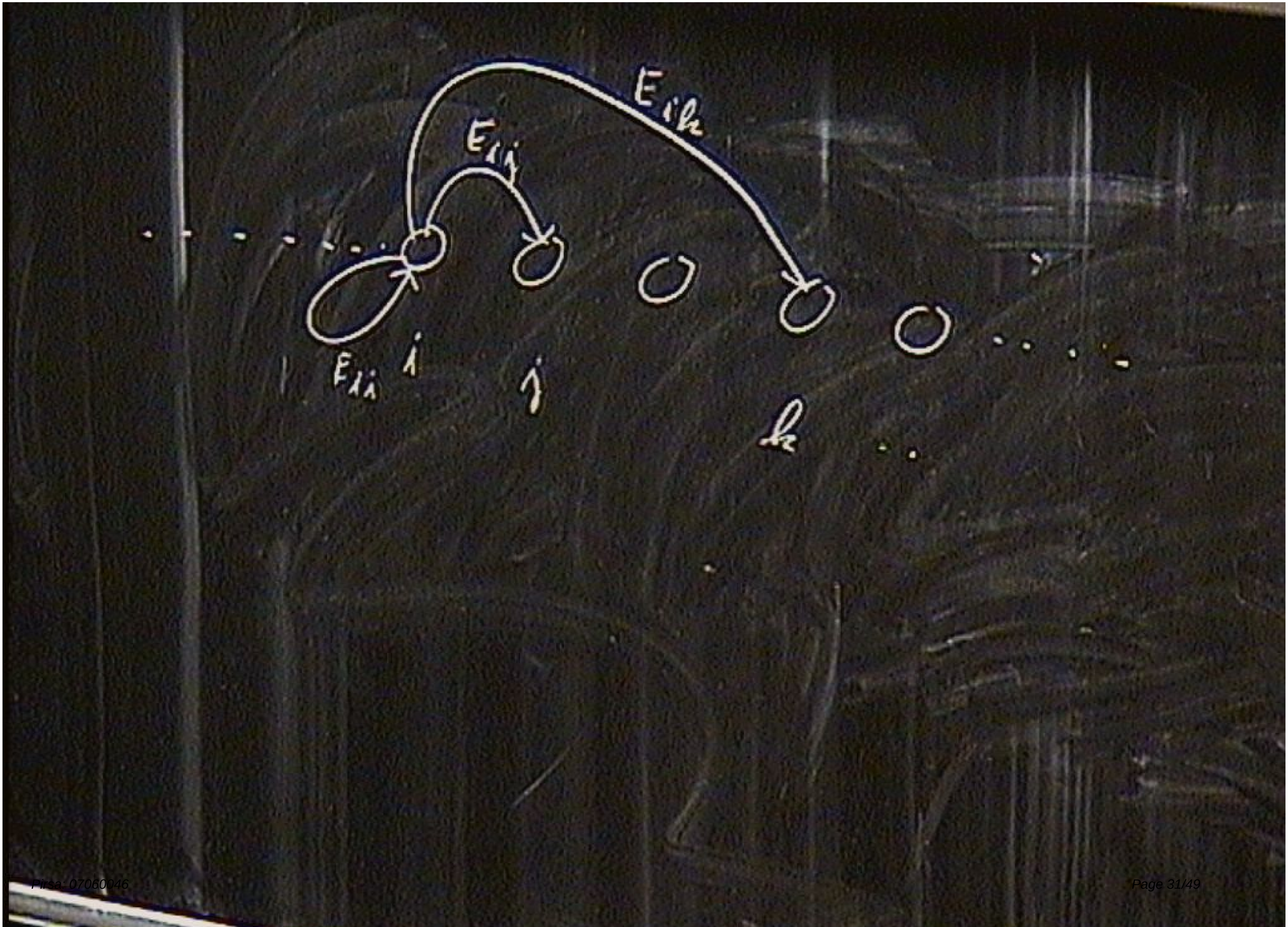
$$E_n \circ \cdots \circ \left\{E_4 \circ [E_3 \circ (E_2 \circ E_1)]\right\}$$

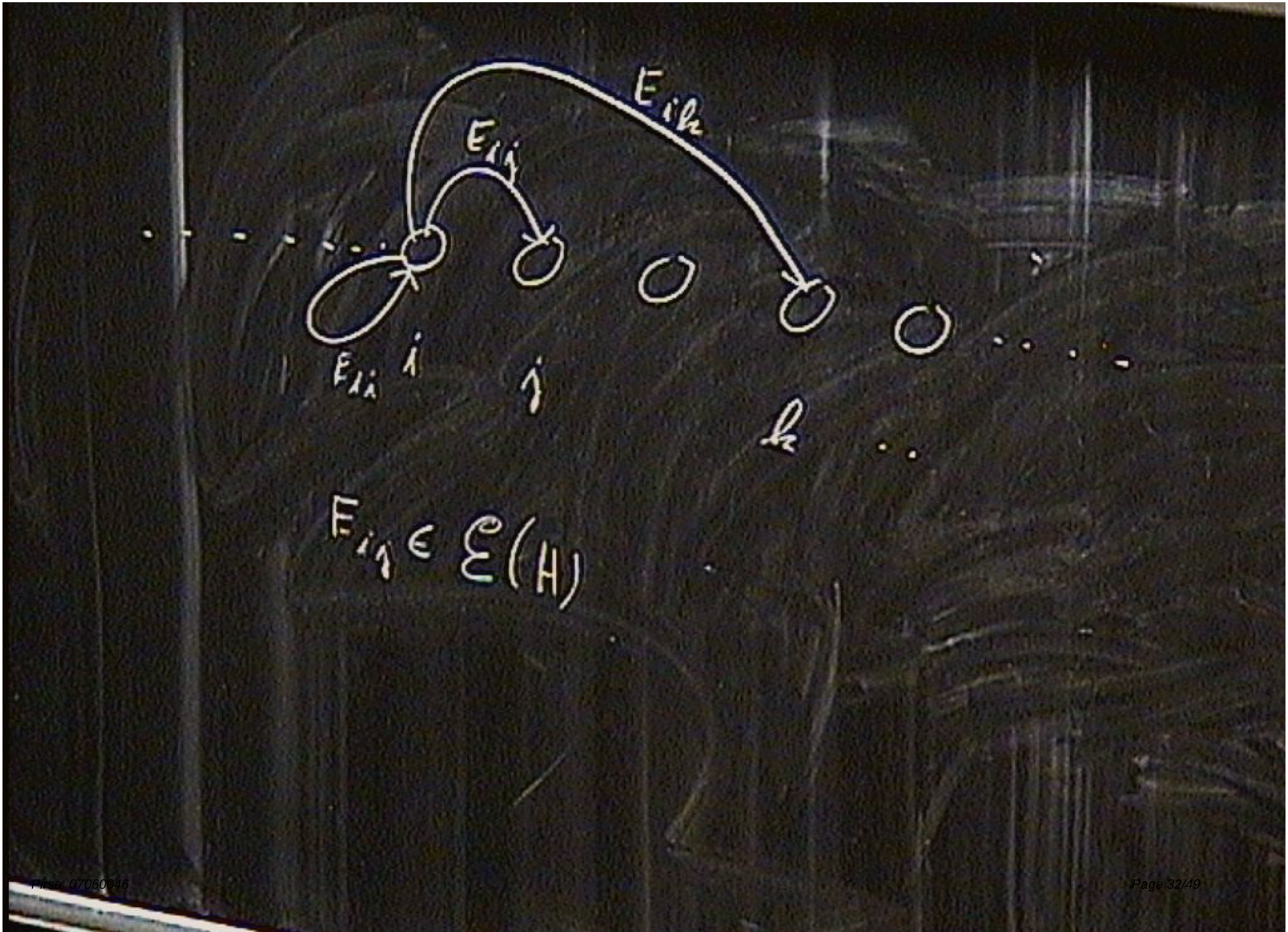
We associate a TEM $E = [E_{ij}]$ with a quantum Markov chain and interpret E_{ij} as the effect that a quantum system evolves (or performs a transition) from site i to site j in one time step. The fact that $\sum_j E_{ij} = I$ shows that if the system is presently at site i, then it will evolve to some site j in one time step with certainty. The probability distribution of the system is given by a vector state $A = (A_1, A_2, \ldots)$ where $\mathrm{tr}(A_i)$ is the probability the system is at site i. The n-**step** TEM is given by

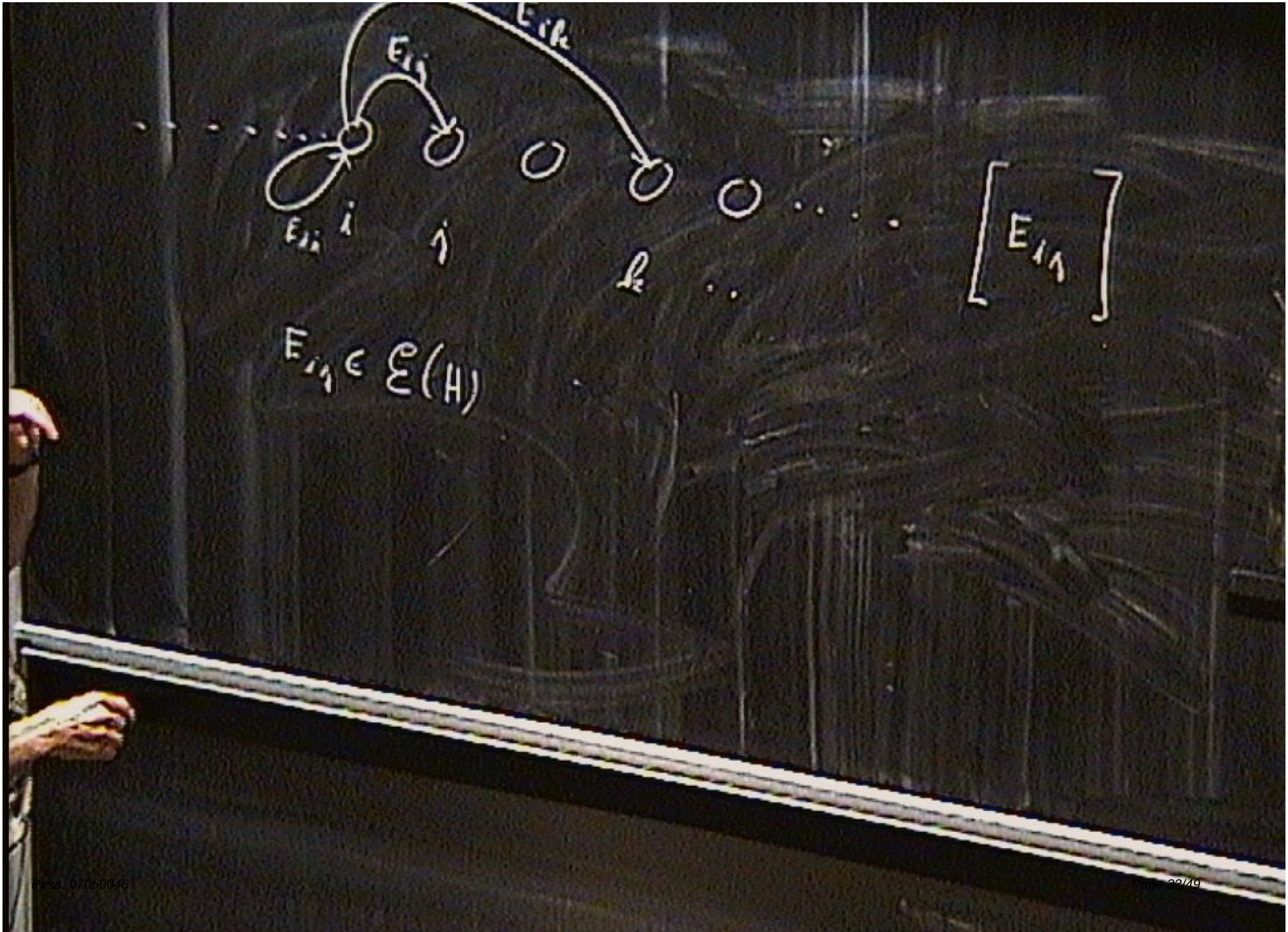
$$E^{(n)} = E \circ E \circ \cdots \circ E \quad (n \text{ factors})$$

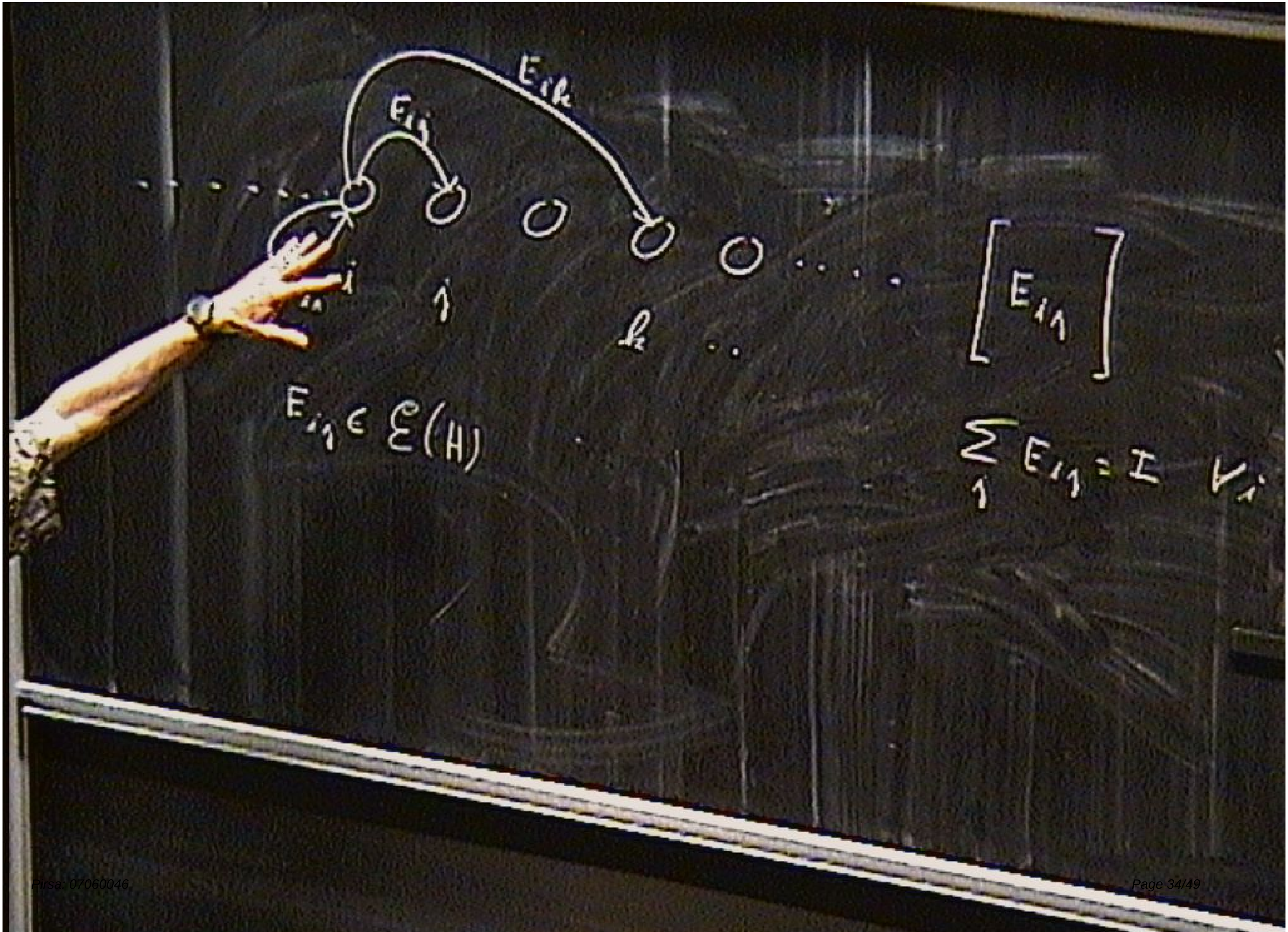
We interpret $E_{ij}^{(n)}$ as the effect that the system evolves from site i to site j in n time steps.

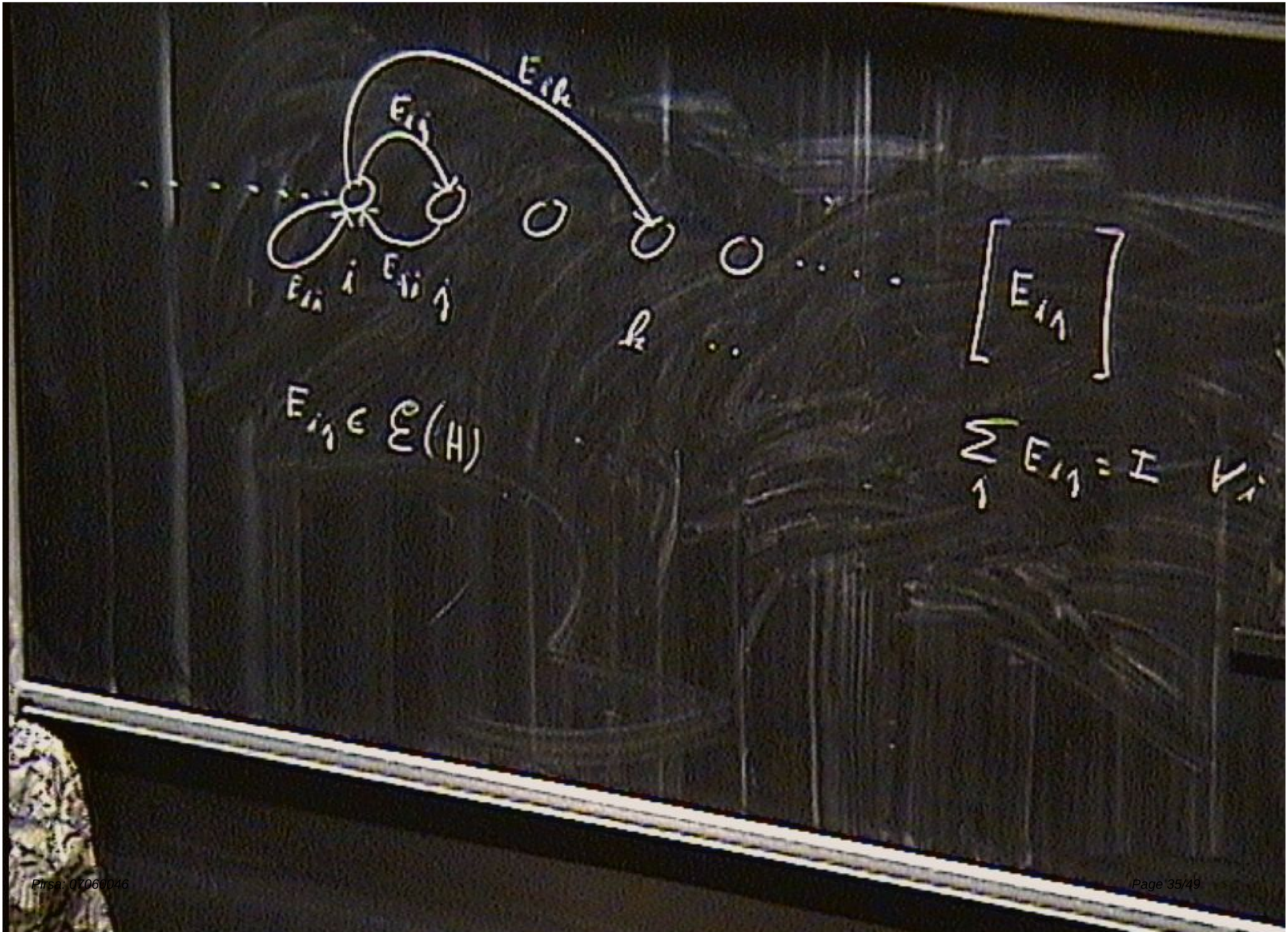
This formalism includes classical Markov chains as a special case. Just let $E = [p_{ij} I]$ where $p_{ij} \geq 0$, $\sum_j p_{ij} = 1$ for all i. In this case E is essentially a stochastic or transition matrix.

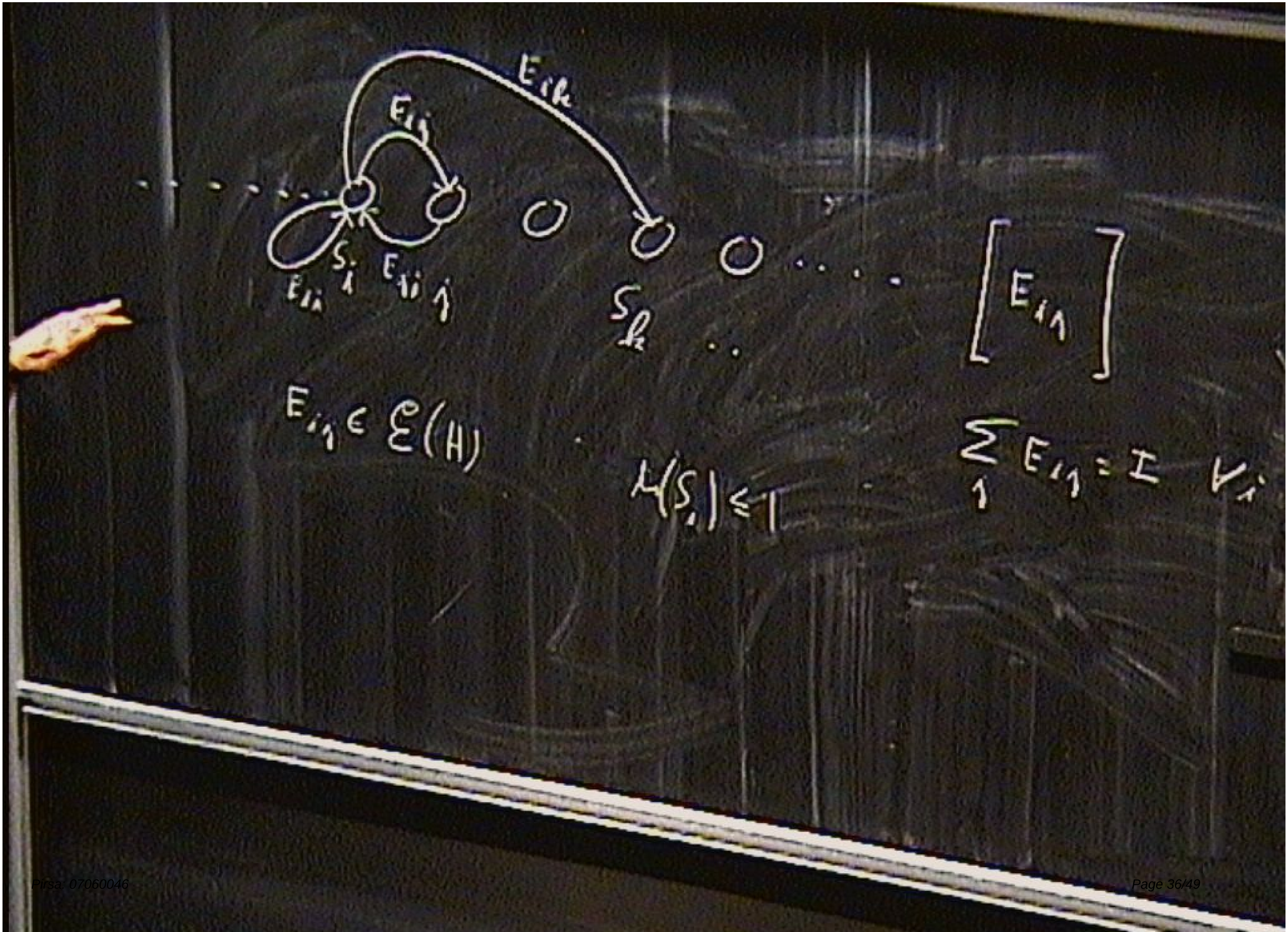

E ik
E ij
E ii
i
j
k

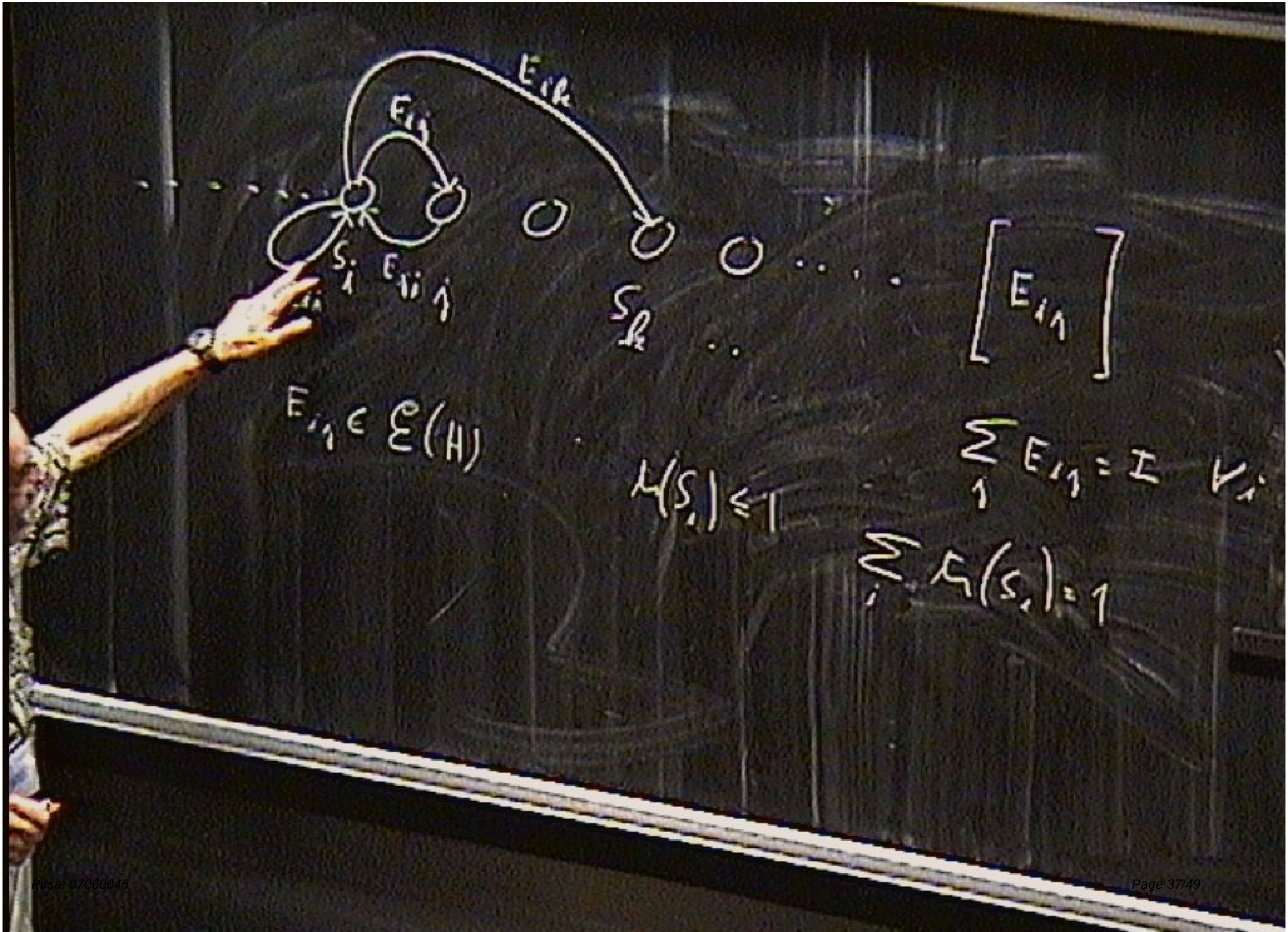

E ik
E ij
E ii
i
j
k
E ij ∈ ℰ(H)


E_ik
E_ij
E_ii
i
j
k
[E_11]
E_ij ∈ ℰ(H)


E_i1
E_ik
i
1
k
E_i1 ∈ ℰ(H)
[E_i1]
Σ_1 E_i1 = I ∀i


E_{ik}
E_{ij}
E_{ii}
i
j
k
$E_{ij} \in \mathcal{E}(H)$
$[E_{ij}]$
$\sum_j E_{ij} = I \quad \forall i$


E ik
E ij
S i
E ii
E ii j
S k
E i1
E i1 ∈ ℰ(H)
μ(S i) ≤ 1
Σ E i1 = I ∀i


E_ik
E_ij
S_i
E_ii
j
S_k
E_i1
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We interpret $E_{ij}^{(n)}$ as the effect that the system evolves from site i to site j in n time steps.

This formalism includes classical Markov chains as a special case. Just let $E = [p_{ij} I]$ where $p_{ij} \geq 0$, $\sum_j p_{ij} = 1$ for all i. In this case E is essentially a stochastic or transition matrix.

In this formalism, there are two types of possible dynamics, the state dynamics and the operator dynamics. For a state dynamics, the vector state A evolves and the TEM E is considered fixed. For an operator dynamics, the TEM evolves and the vector state is considered fixed. This is analogous to the Schrödinger and Heisenberg pictures for quantum dynamics and in this framework the two types of dynamics are statistically equivalent.

Suppose the initial state vector is A and the evolution is described by the TEM E. In the state dynamics the system will be in the state

$$E_{(n)}(A) = E * \cdots * [E * (E * A)] \quad (n\ E \text{ factors})$$

after n time steps. In the operator dynamics, the system will be in the state $E^{(n)} * A$ after n time steps. Because of nonassociativity these are not identical. For example, $E * (E * A) \neq E^{(2)} * A$ in general. However, they are statistically equivalent in the sense we now define.

A TEM is **equitable** if $\operatorname{tr}\left(E^{(n)} * A\right) = \operatorname{tr}\left(E_{(n)}(A)\right)$ for every $n \in \mathbb{N}$ and $A \in \mathcal{V}(H)$.

Lemma 5.2. If E and F are TEM's of the same size on H, then $\operatorname{tr}[(E \circ F) * A] = \operatorname{tr}[F * (E * A)]$ for all $A \in \mathcal{V}(H)$.

Theorem 5.3. Any TEM E is equitable.

Proof. We show by induction on n that

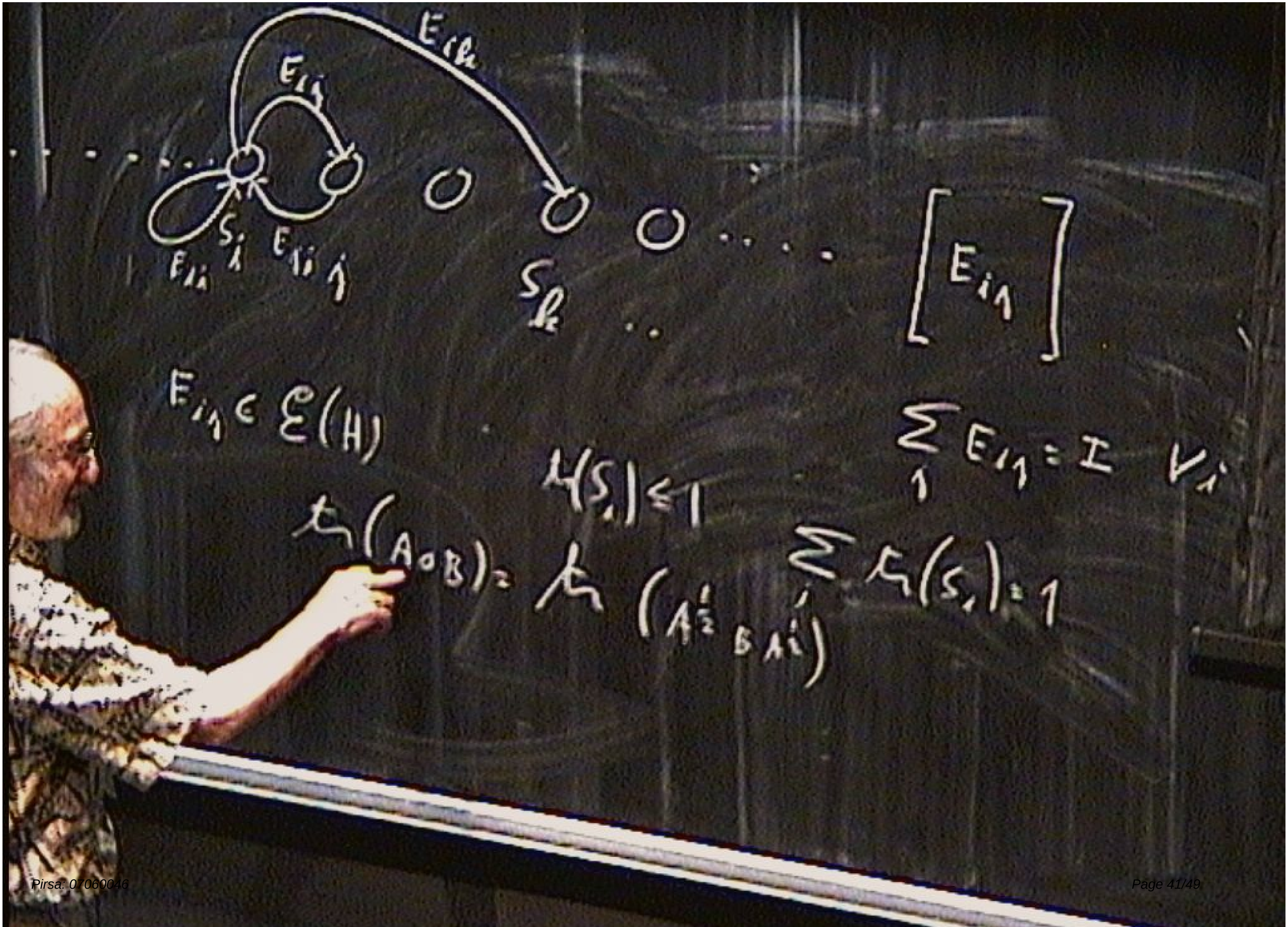
$$\operatorname{tr}\left(E^{(n)} * A\right) = \operatorname{tr}\left(E_{(n)}(A)\right)$$

for every $A \in \mathcal{V}(H)$. The result clearly holds for $n = 1$. Suppose the result holds for $n \in \mathbb{N}$. Applying Lemma 5.2 and the induction hypothesis gives

$$\begin{aligned} \operatorname{tr}\left(E^{(n+1)} * A\right) &= \operatorname{tr}\left[\left(E \circ E^{(n)}\right) * A\right] = \operatorname{tr}\left[E^{(n)} * (E * A)\right] \\ &= \operatorname{tr}\left[E_{(n)}(E * A)\right] = \operatorname{tr}\left[E_{(n+1)}(A)\right] \end{aligned}$$

The result follows by induction. □

We now give an example which shows that the two types of dynamics need not be identical.


E ik
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Σ E i1 = I ∀i
μ(S i) ≤ 1
tr(A∘B) = tr(A^½ B A^½)
Σ tr(S i) = 1

$S_i \quad E_{i1} \qquad S_k \quad \ldots \qquad [E_{i1}]$

$E_{i1} \in \mathcal{E}(H)$

$\sum_1 E_{i1} = I$

$\mu(S_i) \leq 1$

$tr(A \circ B) = tr(A^{\frac{1}{2}} B A^{\frac{1}{2}}) = tr(AB)$

$\sum_i tr(S_i) = 1$

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We now give an example which shows that the two types of dynamics need not be identical.

Example. Let $\dim H = 2$, $A, B \in \mathcal{E}(H)$, $S = \left(0, \frac{1}{2}I\right)$.

$$E = \begin{bmatrix} A & A' \\ B & B' \end{bmatrix}, \qquad E^T = \begin{bmatrix} A & B \\ A' & B' \end{bmatrix}$$

$$E * S = \begin{bmatrix} A & B \\ A' & B' \end{bmatrix} \circ \begin{bmatrix} 0 \\ \frac{1}{2}I \end{bmatrix} = \begin{bmatrix} \frac{1}{2}B \\ \frac{1}{2}B' \end{bmatrix}$$

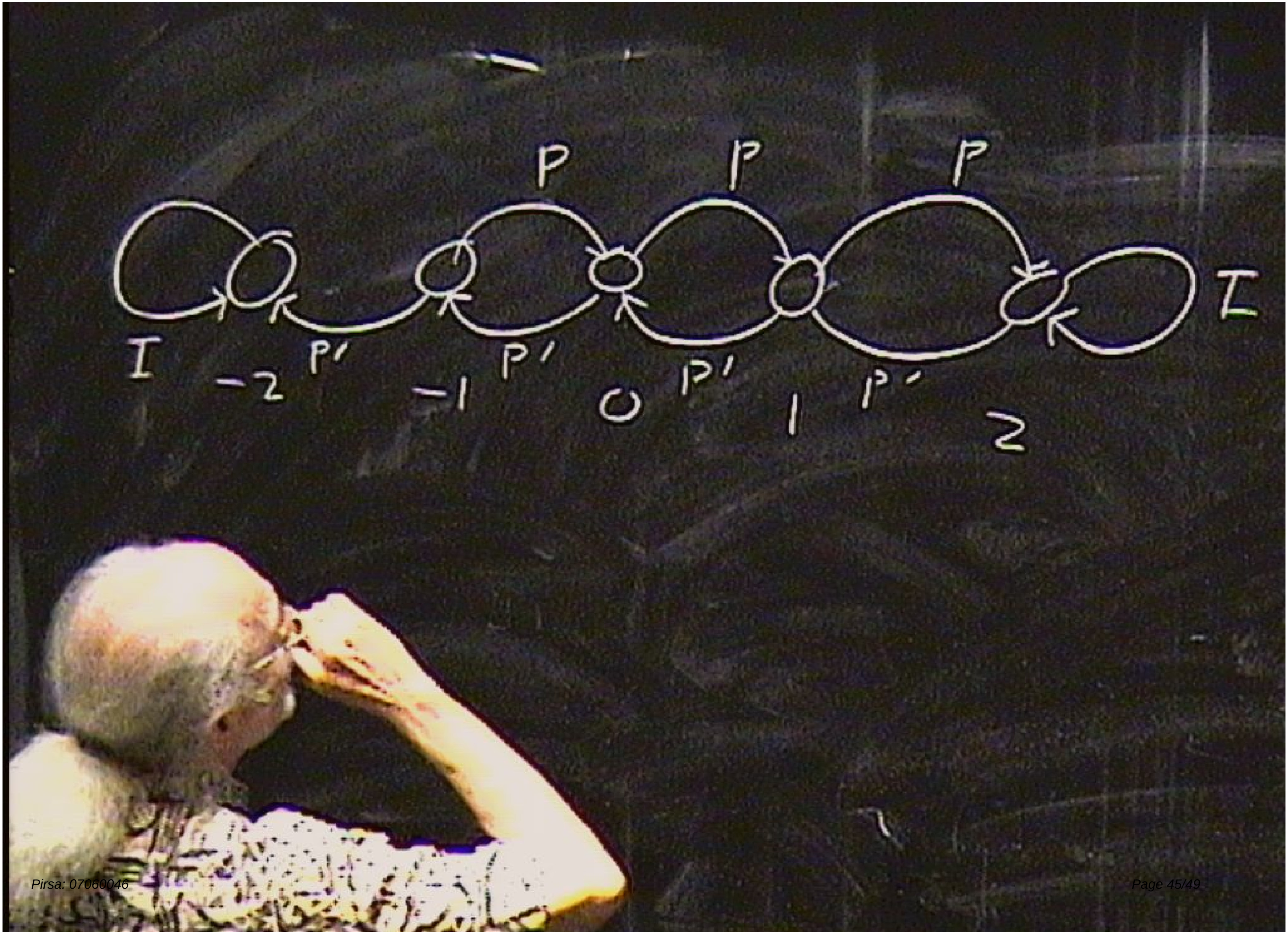
$$E * (E * S) = \frac{1}{2}\begin{bmatrix} A \circ B + B \circ B' \\ A' \circ B + B' \circ B' \end{bmatrix}$$

$$E^{(2)} * S = \frac{1}{2}\begin{bmatrix} B \circ A + B' \circ B \\ B \circ A' + B' \circ B' \end{bmatrix}$$

We see that $E^{(2)} * S = E * (E * S)$ if and only if $B \circ A = A \circ B$; tha
is, $BA = AB$.

Example. This example is a simple quantum random walk with absorbing boundaries. In this example, $P \in \mathcal{P}(H)$ and the sites ar
labeled by $-2, -1, 0, 1, 2$.

$$E = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ P' & 0 & P & 0 & 0 \\ 0 & P' & 0 & P & 0 \\ 0 & 0 & P' & 0 & P \\ \end{bmatrix}, \qquad E^{(2)} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ P' & 0 & 0 & P & 0 \\ P' & 0 & 0 & 0 & P \\ \end{bmatrix}$$


P
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1
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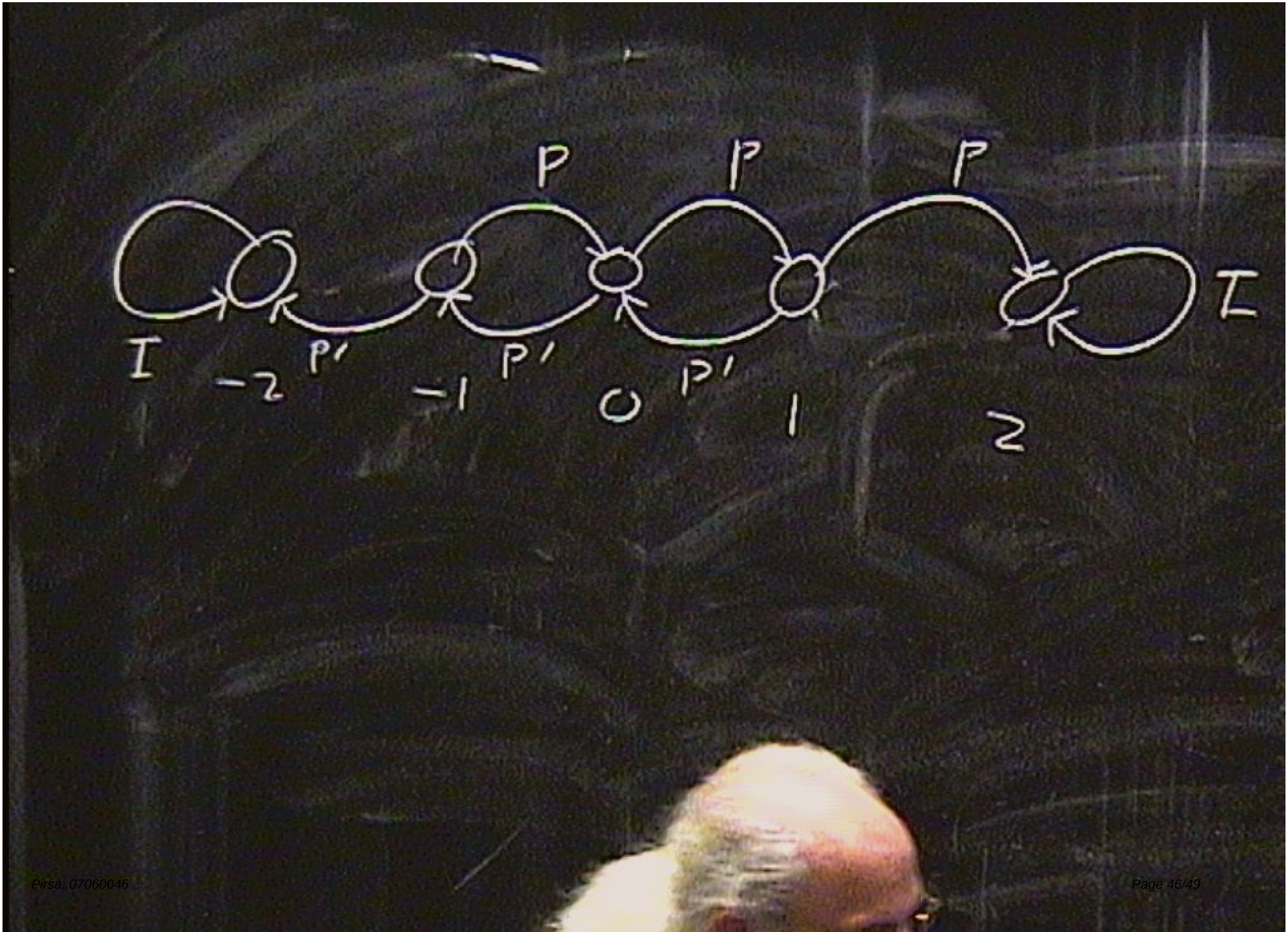
2
I


P
P
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-2
P'
-1
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0
P'
1
2
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$$E^{(3)} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ P' & 0 & 0 & 0 & P \\ P' & 0 & 0 & 0 & P \\ P' & 0 & 0 & 0 & P \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \qquad E^{(3)} = E^{(4)} = E^{(5)} = \cdots$$

Suppose the initial vector state is $A = (0, 0, P_x, 0, 0)$. Then

$$\begin{aligned} \operatorname{tr}(E * A) &= \left(0, \langle P'x, x\rangle, 0, \langle Px, x\rangle, 0\right) \\ \operatorname{tr}\left(E^{(2)} * A\right) &= \left(\langle P'x, x\rangle\, 0, 0, 0, \langle Px, x\rangle\right) \\ \operatorname{tr}\left(E^{(2)} * A\right) &= \operatorname{tr}\left(E^{(3)} * A\right) = \operatorname{tr}\left(E^{(4)} * A\right) = \cdots \end{aligned}$$

The dynamics shows that if the system is initially at the site 0, it moves directly to the right or to the left and is absorbed at the boundary sites ± 2 in two time steps. There is no classical random walk that would produce this type of dynamics.

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