Title: Mathematical structures of Quantum Mechanics and connections to operational principles

Date: Jun 05, 2007 02:30 PM

URL: http://pirsa.org/07060041

Abstract:

quantum information theory group

*quant-ph*
*0611094, 0612162,*
*0701217, 0701219*
*www.qubit.it*

# Mathematical structures for Quantum Mechanics and connections to operational principles


**Giacomo Mauro D'Ariano**

Università degli Studi di Pavia


*Operational Quantum Physics and the Quantum-Classical Contrast*

*Perimeter Institute, Waterloo, Canada*

# On experimental science

# On experimental science

**Experiments** are performed to get information on the **state** of an **object physical system**.

# On experimental science

**Experiments** are performed to get information on the **state** of an **object physical system**.

Knowledge on such state will allow us to **predict the results of forthcoming experiments** on a (similar) object in a similar situation.

# On experimental science

**Experiments** are performed to get information on the **state** of an **object physical system**.

Knowledge on such state will allow us to **predict the results of forthcoming experiments** on a (similar) object in a similar situation.

Since necessarily we work with only partial prior knowledge of both system and experimental apparatus, the rules for the experiment must be given in a **probabilistic setting**.

# What is an experiment

# What is an experiment

An experiment on a **object system** consists in making it interact with an **apparatus**.

# What is an experiment

An experiment on a **object system** consists in making it interact with an **apparatus**.

The interaction between object and apparatus produces one of a **set of possible transformations** of the object, each one occurring with some probability.

# Postulates

- **Postulate 1 (Independent systems)** *There exist **independent** systems.*
- **Postulate 2 (Symmetric faithful state)** *For every composite system made of two identical physical systems there exists a **symmetric joint state** that is both **dynamically** and **preparationally faithful**.*
- **Postulate 3 (Local observability principle)** *For composite systems **local informationally complete observables** provide **global informationally complete observables**.*
- **Postulate 4 (Info-complete discriminating observable)** *For every system there exists a minimal info-complete observable that can be achieved using a joint **discriminating observable** on system+ ancilla.*

# Postulates

- **Postulate 1 (Independent systems)** *There exist **independent** systems.*
- **Postulate 2 (Symmetric faithful state)** *For every composite system made of two identical physical systems there exists a **symmetric joint state** that is both **dynamically** and **preparationally faithful**.*
- **Postulate 3 (Local observability principle)** *For composite systems **local informationally complete observables** provide **global informationally complete observables**.*
- **Postulate 4 (Info-complete discriminating observable)** *For every system there exists a minimal info-complete observable that can be achieved using a joint **discriminating observable** on system+ ancilla.*

P1÷P4 ⟶ Hilbert space

# Postulates

- **Postulate 1 (Independent systems)** *There exist* ***independent*** *systems.*

- **Postulate 2 (Symmetric faithful state)** *For every composite system made of two identical physical systems there exists a* ***symmetric joint state*** *that is both* ***dynamically*** *and* ***preparationally faithful****.*

- **Postulate 3 (Local observability principle)** *For composite systems* ***local informationally complete observables*** *provide* ***global informationally complete observables****.*

- **Postulate 4 (Info-complete discriminating observable)** *For every system there exists a minimal info-complete observable that can be achieved using a joint* ***discriminating observable*** *on system+ ancilla.*

P1÷P4 ⟶ Hilbert space

P1,P2 ⟶ C*-algebra

# Actions and outcomes

***Experiment (or "action"):*** every experiment is described by a set $\mathbb{A} \equiv \{\mathscr{A}_j\}$ of possible transformations $\mathscr{A}_j$ having overall unit probability, with the apparatus signaling the outcome $j$ labeling which transformation actually occurred.

# States

# States

***State:*** A state $\omega$ for a physical system is a rule which provides the probability for any possible transformation within an experiment, namely:

$$\omega : \textbf{\textit{state}}, \quad \omega(\mathscr{A}) : \textit{probability that the transformation } \mathscr{A} \textit{ occurs}$$

# States

***State:*** A state $\omega$ for a physical system is a rule which provides the probability for any possible transformation within an experiment, namely:

$$\omega : \textit{\textbf{state}}, \quad \omega(\mathscr{A}) : \textit{probability that the transformation } \mathscr{A} \textit{ occurs}$$

***Normalization:***

$$\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1$$

# States

***State:*** A state $\omega$ for a physical system is a rule which provides the probability for any possible transformation within an experiment, namely:

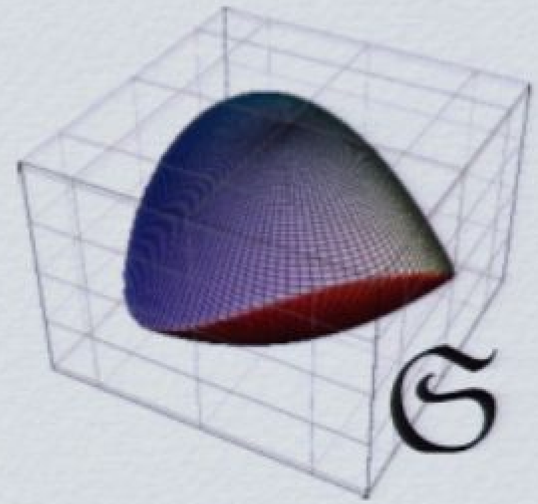
$$\omega : \textbf{\textit{state}}, \quad \omega(\mathscr{A}) : \textit{probability that the transformation } \mathscr{A} \textit{ occurs}$$

***Normalization:*** $$\sum_{\mathscr{A}_j \in \mathbb{A}} \omega(\mathscr{A}_j) = 1$$

Identity transformation: $$\omega(\mathscr{I}) = 1$$

# Convex structure of states

The possible states of a physical system make a convex set $\mathfrak{S}$




$\omega_1, \omega_2$ any two states:

$$\omega = \lambda\omega_1 + (1-\lambda)\omega_2, \quad 0 \leq \lambda \leq 1.$$

corresponding to the probability rule

$$\omega(\mathscr{A}) = \lambda\omega_1(\mathscr{A}) + (1-\lambda)\omega_2(\mathscr{A})$$

# Monoid $\mathfrak{T}$ of transformations

***Transformations make a monoid:*** the composition $\mathscr{A} \circ \mathscr{B}$ of two transformations $\mathscr{A}$ and $\mathscr{B}$ is itself a transformation. Consistency of composition of transformations requires associativity, namely

$$\mathscr{C} \circ (\mathscr{B} \circ \mathscr{A}) = (\mathscr{C} \circ \mathscr{B}) \circ \mathscr{A}$$

There exists the identical transformation $\mathscr{I}$ which leaves the physical system invariant, and which for every transformation $\mathscr{A}$ satisfies the composition rule

$$\mathscr{I} \circ \mathscr{A} = \mathscr{A} \circ \mathscr{I} = \mathscr{A}$$

# Independent systems and local transformations

***Independent systems and local experiments:*** two physical systems are "independent" if on each system it is possible to perform "local experiments" for which on every joint state one has the commutativity of the pertaining transformations

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}$$

# Independent systems and local transformations

***Independent systems and local experiments:*** two physical systems are "independent" if on each system it is possible to perform "local experiments" for which on every joint state one has the commutativity of the pertaining transformations

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}$$

$$(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) \doteq \mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} \circ \mathscr{C}^{(3)} \circ \ldots$$

# Independent systems and local transformations

***Independent systems and local experiments:*** two physical systems are "independent" if on each system it is possible to perform "local experiments" for which on every joint state one has the commutativity of the pertaining transformations

$$\mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} = \mathscr{B}^{(2)} \circ \mathscr{A}^{(1)}$$

$$(\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots) \doteq \mathscr{A}^{(1)} \circ \mathscr{B}^{(2)} \circ \mathscr{C}^{(3)} \circ \ldots$$



***Multipartite system:*** a collection of independent systems

# Local state

For a multipartite system we define the local state $\omega|_n$ of the *n*-th system the state that gives the probability of any local transformation $\mathscr{A}$ on the *n*-th system with all other systems untouched, namely

$$\omega|_n(\mathscr{A}) \doteq \Omega(\mathscr{I}, \ldots, \mathscr{I}, \underbrace{\mathscr{A}}_{nth}, \mathscr{I}, \ldots)$$

# Conditional state

When composing two transformations $\mathscr{A}$ and $\mathscr{B}$ the probability that $\mathscr{B}$ occurs conditioned that $\mathscr{A}$ occurred before is given by the **Bayes rule**

$$p(\mathscr{B}|\mathscr{A}) = \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}$$

***Conditional state:*** the conditional state $\omega_{\mathscr{A}}$ gives the probability that a transformation $\mathscr{B}$ occurs on the physical system in the state $\omega$ after the transformation $\mathscr{A}$ occurred, namely

$$\omega_{\mathscr{A}}(\mathscr{B}) \doteq \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}$$

# No-signaling from the future

[Ozawa] The definition of conditional state needs to assume that

$$\sum_{\mathscr{B}_j \in \mathbb{B}} \omega(\mathscr{B}_j \circ \mathscr{A}) = \omega(\mathscr{A}), \quad \forall \mathbb{B}, \forall \mathscr{A}.$$

This is no-signaling from the future.

# Conditional state

When composing two transformations $\mathscr{A}$ and $\mathscr{B}$ the probability that $\mathscr{B}$ occurs conditioned that $\mathscr{A}$ occurred before is given by the **Bayes rule**

$$p(\mathscr{B}|\mathscr{A}) = \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}$$

***Conditional state:*** the conditional state $\omega_{\mathscr{A}}$ gives the probability that a transformation $\mathscr{B}$ occurs on the physical system in the state $\omega$ after the transformation $\mathscr{A}$ occurred, namely

$$\omega_{\mathscr{A}}(\mathscr{B}) \doteq \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}$$

# No-signaling from the future

[Ozawa] The definition of conditional state needs to assume that

$$\sum_{\mathscr{B}_j \in \mathbb{B}} \omega(\mathscr{B}_j \circ \mathscr{A}) = \omega(\mathscr{A}), \quad \forall \mathbb{B}, \forall \mathscr{A}.$$

This is no-signaling from the future.

# Weights and Operations

***Weight:*** un-normalized state $\boxed{\omega = \frac{\tilde{\omega}}{\tilde{\omega}(\mathscr{I})}}$

$$0 \leqslant \tilde{\omega}(\mathscr{A}) \leqslant \tilde{\omega}(\mathscr{I}) < +\infty$$

# Conditional state

When composing two transformations $\mathscr{A}$ and $\mathscr{B}$ the probability that $\mathscr{B}$ occurs conditioned that $\mathscr{A}$ occurred before is given by the **Bayes rule**

$$p(\mathscr{B}|\mathscr{A}) = \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}$$

***Conditional state:*** the conditional state $\omega_{\mathscr{A}}$ gives the probability that a transformation $\mathscr{B}$ occurs on the physical system in the state $\omega$ after the transformation $\mathscr{A}$ occurred, namely

$$\omega_{\mathscr{A}}(\mathscr{B}) \doteq \frac{\omega(\mathscr{B} \circ \mathscr{A})}{\omega(\mathscr{A})}$$

# No-signaling from the future

[Ozawa] The definition of conditional state needs to assume that

$$\sum_{\mathscr{B}_j \in \mathbb{B}} \omega(\mathscr{B}_j \circ \mathscr{A}) = \omega(\mathscr{A}), \quad \forall \mathbb{B}, \forall \mathscr{A}.$$

This is no-signaling from the future.

# Weights and Operations

***Weight:*** un-normalized state

$$\omega = \frac{\tilde{\omega}}{\tilde{\omega}(\mathscr{I})}$$

$$0 \leqslant \tilde{\omega}(\mathscr{A}) \leqslant \tilde{\omega}(\mathscr{I}) < +\infty$$

# Weights and Operations

***Weight:*** un-normalized state $$\omega = \frac{\tilde{\omega}}{\tilde{\omega}(\mathscr{I})}$$

$$0 \leqslant \tilde{\omega}(\mathscr{A}) \leqslant \tilde{\omega}(\mathscr{I}) < +\infty$$

***convex cone of weights***: $\mathfrak{W}$

# Weights and Operations

***Weight:*** un-normalized state $\omega = \dfrac{\tilde{\omega}}{\tilde{\omega}(\mathscr{I})}$

$$0 \leqslant \tilde{\omega}(\mathscr{A}) \leqslant \tilde{\omega}(\mathscr{I}) < +\infty$$

***convex cone of weights***: $\mathfrak{W}$

***Operation:*** $\mathrm{Op}_{\mathscr{A}}\,\omega \doteq \tilde{\omega}_{\mathscr{A}} = \omega(\cdot \circ \mathscr{A})$

# Weights and Operations

***Weight:*** un-normalized state $\omega = \frac{\tilde{\omega}}{\tilde{\omega}(\mathscr{I})}$

$$0 \leqslant \tilde{\omega}(\mathscr{A}) \leqslant \tilde{\omega}(\mathscr{I}) < +\infty$$

***convex cone of weights***: $\mathfrak{W}$

***Operation:*** $\mathrm{Op}_{\mathscr{A}}\, \omega \doteq \tilde{\omega}_{\mathscr{A}} = \omega(\cdot \circ \mathscr{A})$ $\tilde{\omega}_{\mathscr{A}}(\mathscr{B}) = \omega(\mathscr{B} \circ \mathscr{A})$

***Action of a transformation over a state ("Schrödinger picture"):***

$$\mathscr{A}\,\omega := \mathrm{Op}_{\mathscr{A}}\, \omega$$



$$(\mathscr{A}\,\omega)(\mathscr{B}) := \omega(\mathscr{B} \circ \mathscr{A})$$

# Evolution as conditioning

*Axioms*

*Theorems*

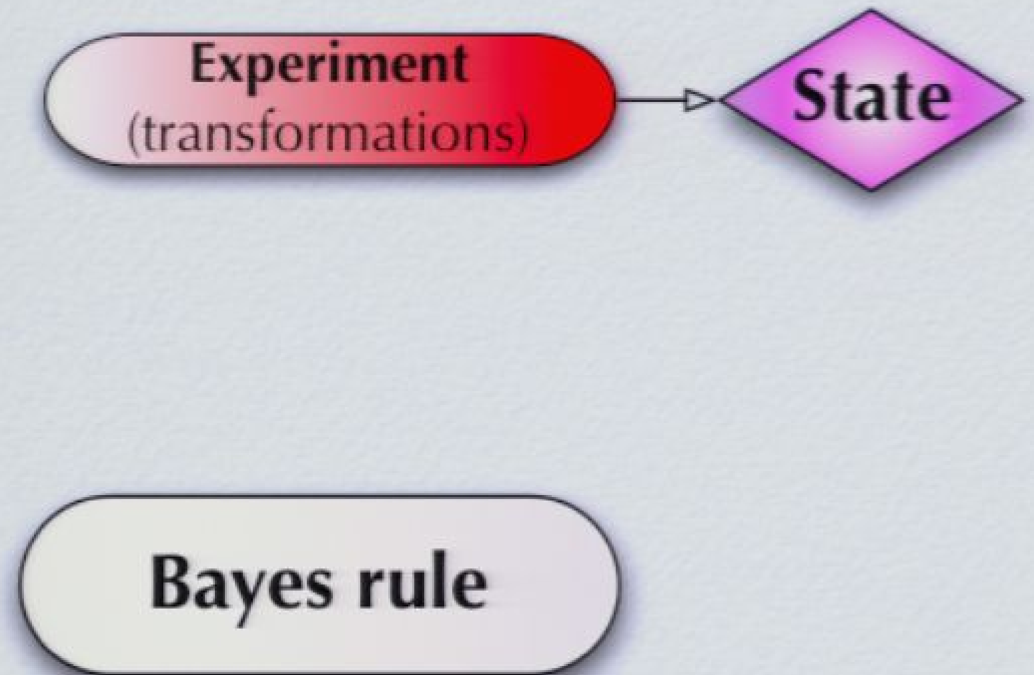
# Evolution as conditioning


*Axioms*

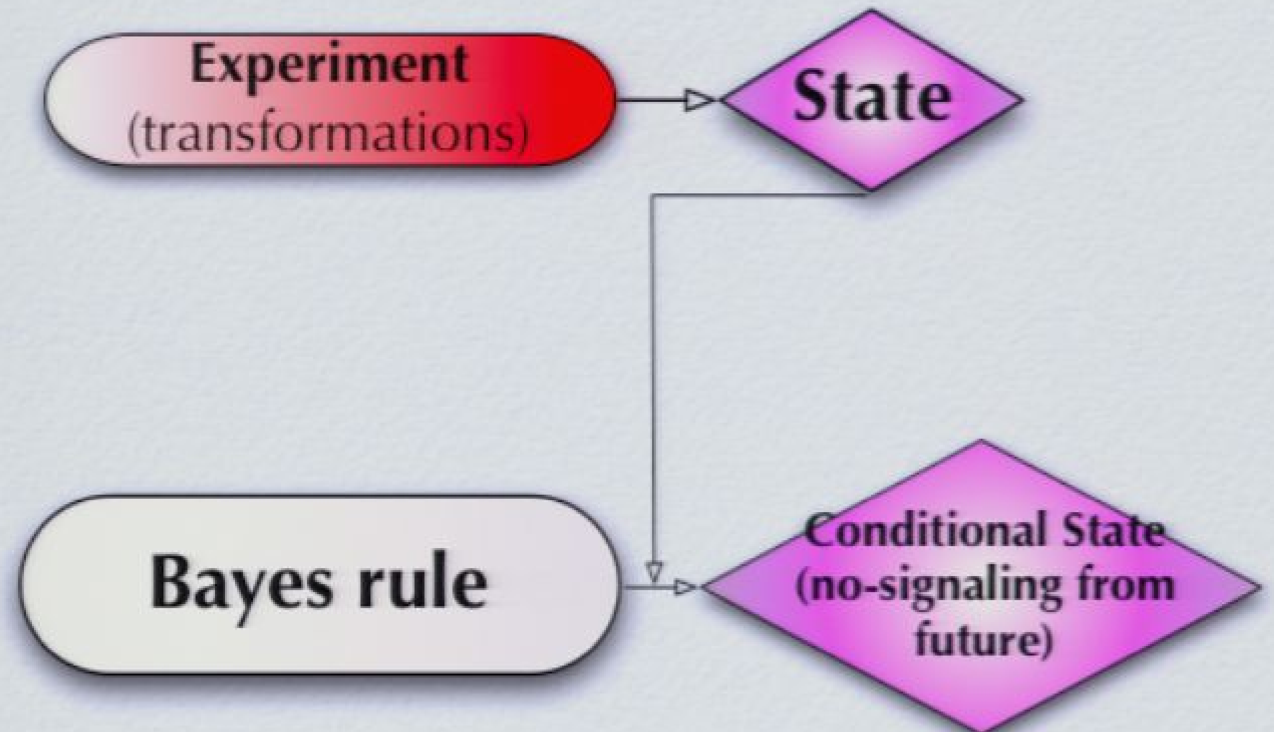
*Theorems*

# Evolution as conditioning

*Axioms*
*Theorems*

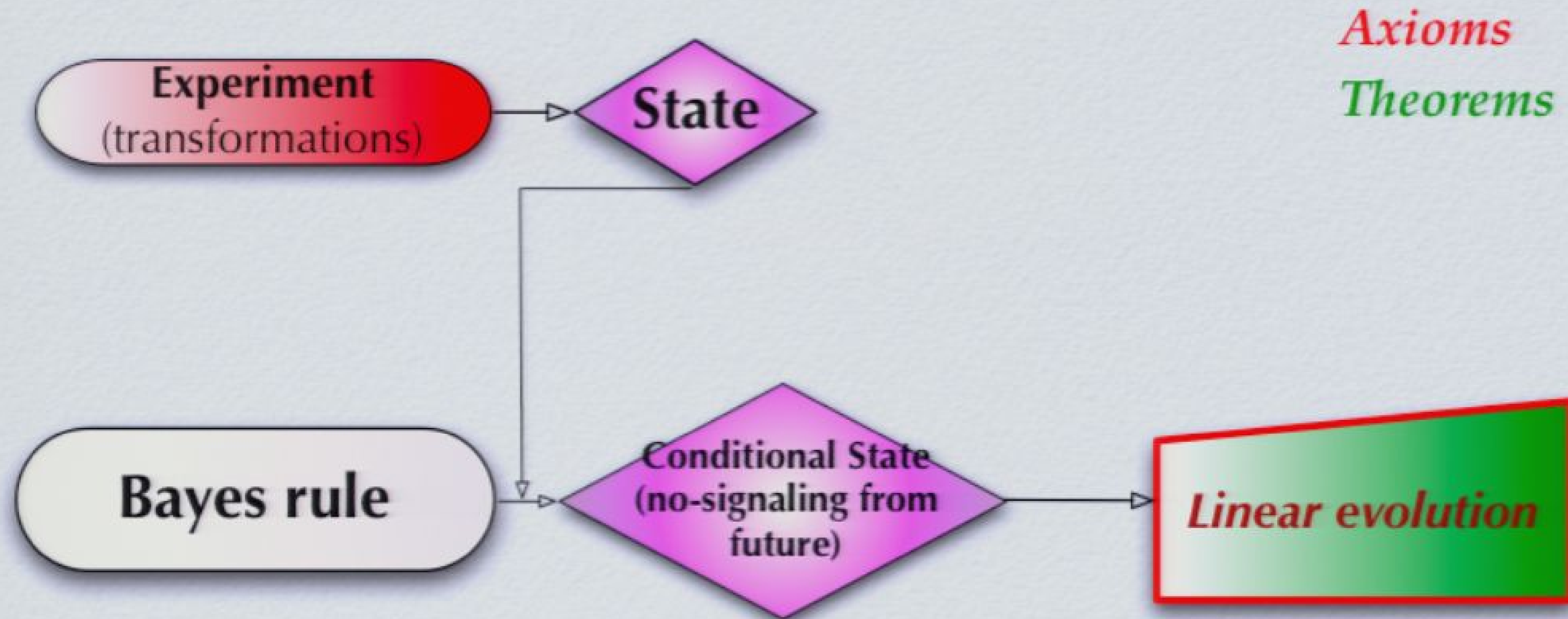


# Evolution as conditioning




*Axioms*
*Theorems*

# Evolution as conditioning



# Dynamical and informational equivalence

From the definition of conditional state we have:

# Dynamical and informational equivalence

From the definition of conditional state we have:

- there are different transformations which always produce the **same state change, but** generally occur with **different probabilities**

# Dynamical and informational equivalence

From the definition of conditional state we have:

- there are different transformations which always produce the **same state change, but** generally occur with **different probabilities**
- there are different transformations which always occur with the **same probability, but** generally affect a **different state change**

# Dynamical and informational equivalence

***Dynamical equivalence of transformations***: two transformations $\mathscr{A}$ and $\mathscr{B}$ are dynamically equivalent if

$$\omega_{\mathscr{A}} = \omega_{\mathscr{B}} \qquad \forall \omega \in \mathfrak{S}$$

# Dynamical and informational equivalence

***Dynamical equivalence of transformations***: two transformations $\mathscr{A}$ and $\mathscr{B}$ are dynamically equivalent if

$$\omega_{\mathscr{A}} = \omega_{\mathscr{B}} \qquad \forall \omega \in \mathfrak{S}$$

***Informational equivalence of transformations:*** two transformations $\mathscr{A}$ and $\mathscr{B}$ are informationally equivalent if

$$\omega(\mathscr{A}) = \omega(\mathscr{B}) \qquad \forall \omega \in \mathfrak{S}$$

# Addition of transformations

Two transformations $\mathscr{A}$ and $\mathscr{B}$ are ***informationally compatible*** (or coexistent) if for every state $\omega$ one has

$$\omega(\mathscr{A}) + \omega(\mathscr{B}) \leq 1$$

# Addition of transformations

Two transformations $\mathscr{A}$ and $\mathscr{B}$ are ***informationally compatible*** (or coexistent) if for every state $\omega$ one has

$$\omega(\mathscr{A}) + \omega(\mathscr{B}) \leq 1$$

For any two coexistent transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ we define the transformation $\boxed{\mathscr{A}_1 + \mathscr{A}_2}$ as the transformation corresponding to the event $e = \{1, 2\}$ namely the apparatus signals that either $\mathscr{A}_1$ or $\mathscr{A}_2$ occurred, but doesn't specify which one:

$$\forall \omega \in \mathfrak{S} \qquad \omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2) \qquad \textit{(info-class)}$$

$$\forall \omega \in \mathfrak{S} \qquad \omega_{\mathscr{A}_1 + \mathscr{A}_2} = \frac{\omega(\mathscr{A}_1)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)} \omega_{\mathscr{A}_1} + \frac{\omega(\mathscr{A}_2)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)} \omega_{\mathscr{A}_2}$$

***(dyn-class)***

# Addition of transformations

Two transformations $\mathscr{A}$ and $\mathscr{B}$ are ***informationally compatible*** (or coexistent) if for every state $\omega$ one has

$$\omega(\mathscr{A}) + \omega(\mathscr{B}) \leq 1$$

For any two coexistent transformations $\mathscr{A}_1$ and $\mathscr{A}_2$ we define the transformation $\boxed{\mathscr{A}_1 + \mathscr{A}_2}$ as the transformation corresponding to the event $e = \{1, 2\}$ namely the apparatus signals that either $\mathscr{A}_1$ or $\mathscr{A}_2$ occurred, but doesn't specify which one:

$$\forall \omega \in \mathfrak{S} \qquad \omega(\mathscr{A}_1 + \mathscr{A}_2) = \omega(\mathscr{A}_1) + \omega(\mathscr{A}_2)$$

***(info-class)***

$$\forall \omega \in \mathfrak{S} \qquad \omega_{\mathscr{A}_1 + \mathscr{A}_2} = \frac{\omega(\mathscr{A}_1)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)} \omega_{\mathscr{A}_1} + \frac{\omega(\mathscr{A}_2)}{\omega(\mathscr{A}_1 + \mathscr{A}_2)} \omega_{\mathscr{A}_2}$$

***(dyn-class)***

$$(\mathscr{A}_1 + \mathscr{A}_2)\omega = \mathscr{A}_1\omega + \mathscr{A}_2\omega$$

# Rescaling of transformations

***Multiplication by a scalar:*** for each transformation $\mathscr{A}$ the transformation $\lambda\mathscr{A}$ for $0 \leq \lambda \leq 1$ is defined as the transformation which is dynamically equivalent to $\mathscr{A}$ but occurs with probability $\omega(\lambda\mathscr{A}) = \lambda\omega(\mathscr{A})$

# Rescaling of transformations

***Multiplication by a scalar:*** for each transformation $\mathscr{A}$ the transformation $\lambda\mathscr{A}$ for $0 \leq \lambda \leq 1$ is defined as the transformation which is dynamically equivalent to $\mathscr{A}$ but occurs with probability $\omega(\lambda\mathscr{A}) = \lambda\omega(\mathscr{A})$

***Convex structure for transformations*** $\mathfrak{T}$
***and for actions***

# Effect

We call ***effect*** an informational equivalence class $\underline{\mathscr{A}}$ of transformations $\mathscr{A}$

# Effect

We call ***effect*** an informational equivalence class $\underline{\mathscr{A}}$ of transformations $\mathscr{A}$

***"Heisenberg picture":***
(from the notion of conditional state)

$$\mathrm{Op}_{\mathscr{A}}\, \underline{\mathscr{B}} = \underline{\mathscr{B}} \circ \mathscr{A} = \underline{\mathscr{B} \circ \mathscr{A}}$$

# Effect

We call ***effect*** an informational equivalence class $\underline{\mathscr{A}}$ of transformations $\mathscr{A}$

***"Heisenberg picture":***
(from the notion of conditional state)

$$\mathrm{Op}_{\mathscr{A}}\, \underline{\mathscr{B}} = \underline{\mathscr{B}} \circ \mathscr{A} = \underline{\mathscr{B} \circ \mathscr{A}}$$

**duality**

effects as positive linear $l$ functionals over states:

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

# Effect

We call ***effect*** an informational equivalence class $\underline{\mathscr{A}}$ of transformations $\mathscr{A}$

***"Heisenberg picture":***
(from the notion of conditional state)

$$\mathrm{Op}_{\mathscr{A}}\, \underline{\mathscr{B}} = \underline{\mathscr{B}} \circ \mathscr{A} = \underline{\mathscr{B} \circ \mathscr{A}}$$

**duality**

effects as positive linear $l$ functionals over states:

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

***Convex structure for effects*** $\mathfrak{P}$

# No-signaling

The occurrence of the transformation $\mathscr{B}$ on system 1 generally affects the local state on system 2, i. e.

$$\Omega_{\mathscr{B},\mathscr{I}}|_2 \neq \Omega_2$$

# No-signaling

The occurrence of the transformation $\mathscr{B}$ on system 1 generally affects the local state on system 2, i. e.

$$\Omega_{\mathscr{B},\mathscr{I}}|_2 \neq \Omega_2$$

However a local action $\mathbb{A} \equiv \{\mathscr{A}_j\}$ on system 2 does not affect the local state on system 1, more precisely:

# No-signaling

The occurrence of the transformation $\mathscr{B}$ on system 1 generally affects the local state on system 2, i. e.

$$\Omega_{\mathscr{B},\mathscr{I}}|_2 \neq \Omega_2$$

However a local action $\mathbb{A} \equiv \{\mathscr{A}_j\}$ on system 2 does not affect the local state on system 1, more precisely:

***acausality of local actions***: any local action on a system is equivalent to the identity transformation on another independent system.

$$\mathbb{A} \equiv \mathscr{S}(\mathbb{A}) := \sum_{\mathscr{A}_j \in \mathbb{A}} \mathscr{A}_j$$

$$\forall \Omega \in \mathfrak{S}^{\times 2}, \forall \mathbb{A}, \qquad \Omega_{\mathbb{A},\mathscr{I}}|_2 = \Omega|_2$$

# No-signaling

**Theorem 1 (No-signaling)** *Any local action on a system does not affect another independent system. More precisely, any local action on a system is equivalent to the identity transformation when viewed from another independent system. In equations one has*

$$\forall \Omega \in \mathfrak{S}^{\times 2}, \forall \mathbb{A}, \qquad \Omega_{\mathbb{A},\mathscr{I}}|_2 = \Omega|_2. \tag{1}$$

**Proof.** Since the two systems are dynamically independent, for every two local transformations one has $\mathscr{A}^{(1)} \circ \mathscr{A}^{(2)} = \mathscr{A}^{(2)} \circ \mathscr{A}^{(1)}$, which implies that $\Omega(\mathscr{A}^{(1)} \circ \mathscr{A}^{(2)}) = \Omega(\mathscr{A}^{(2)} \circ \mathscr{A}^{(1)}) \equiv \Omega(\underline{\mathscr{A}}^{(1)}, \underline{\mathscr{A}}^{(2)})$. By definition, for $\mathscr{B} \in \mathfrak{T}$ one has $\Omega|_2(\mathscr{B}) = \Omega(\mathscr{I}, \mathscr{B})$, and using the addition rule for transformations and reminding the identification $\mathbb{A} \equiv \sum_j \mathscr{A}_j$, one has

$$\Omega(\mathbb{A}, \mathscr{B}) = \Omega(\underline{\mathbb{A}}, \underline{\mathscr{B}}) = \Omega(\underline{\mathscr{I}}, \underline{\mathscr{B}}) =: \Omega|_2(\mathscr{B}). \tag{2}$$

On the other hand, we have

$$\Omega_{\mathbb{A},\mathscr{I}}|_2(\mathscr{B}) = \Omega((\mathscr{I}, \mathscr{B}) \circ (\mathbb{A}, \mathscr{I})) = \Omega(\mathbb{A}, \mathscr{B}), \tag{3}$$



namely the statement. ■

# No-signaling from dynamical independence

*Axioms*

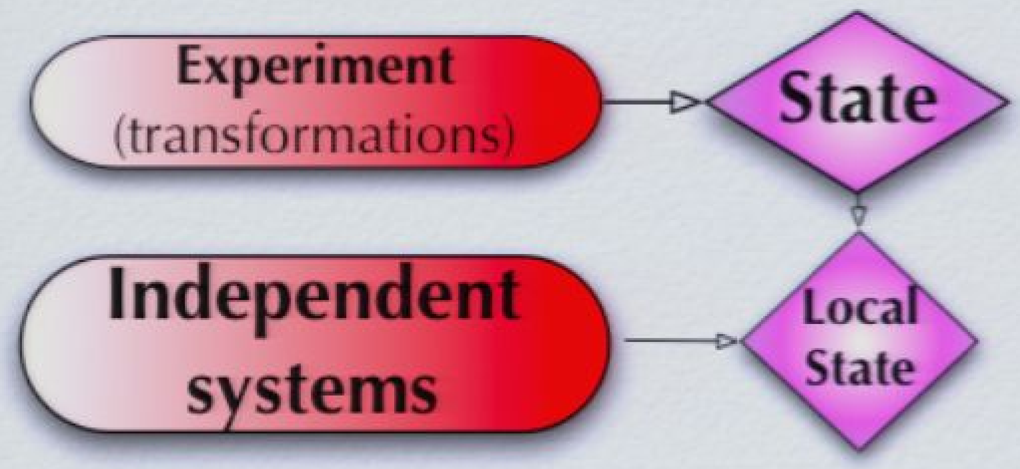
*Theorems*

# No-signaling from dynamical independence


*Axioms*

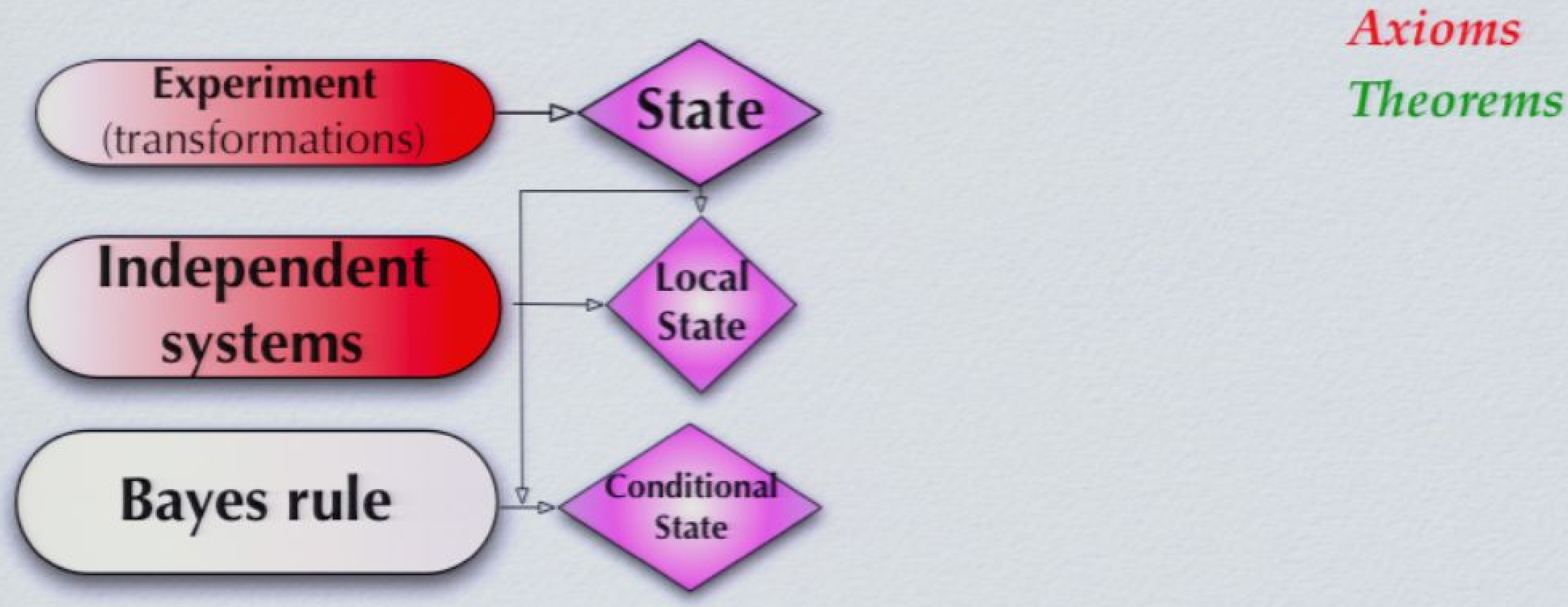
*Theorems*

# No-signaling from dynamical independence

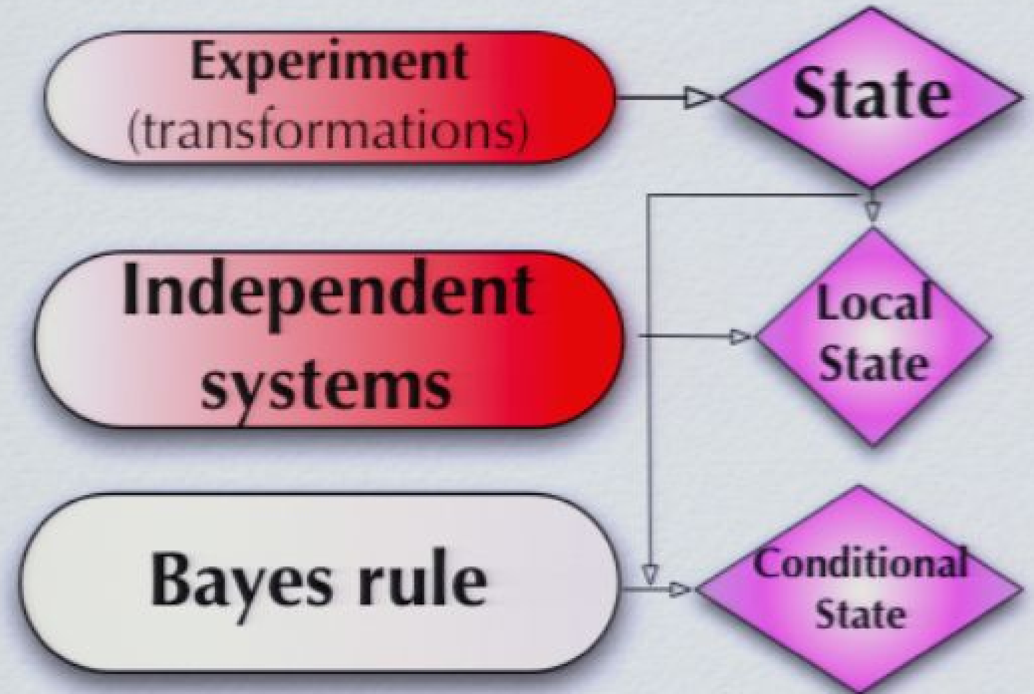



*Axioms*
*Theorems*

# No-signaling from dynamical independence

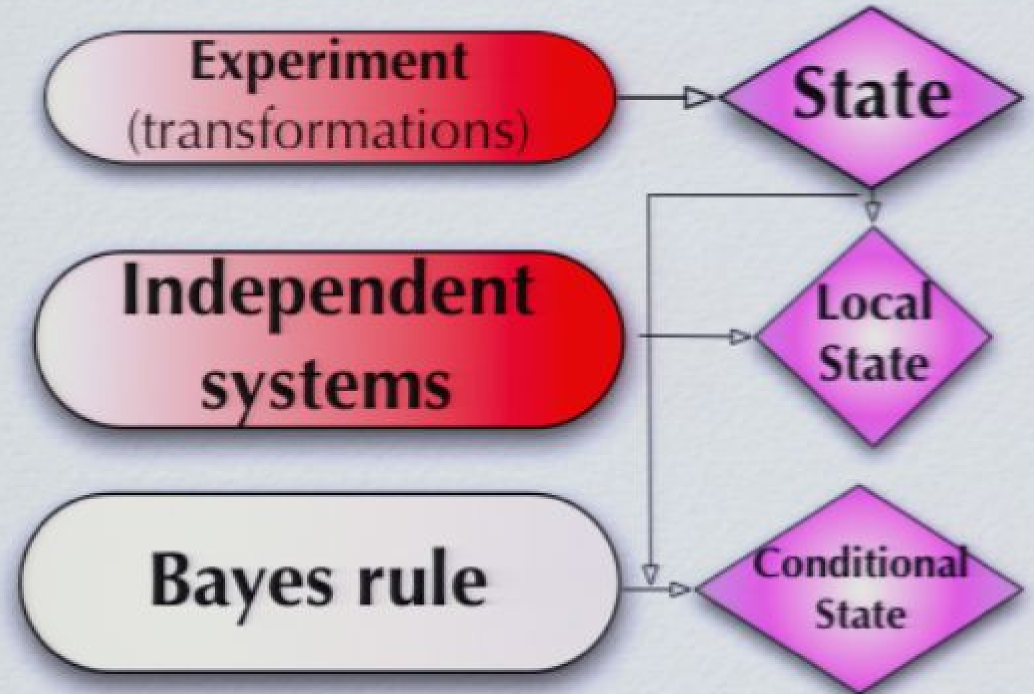


# No-signaling from dynamical independence

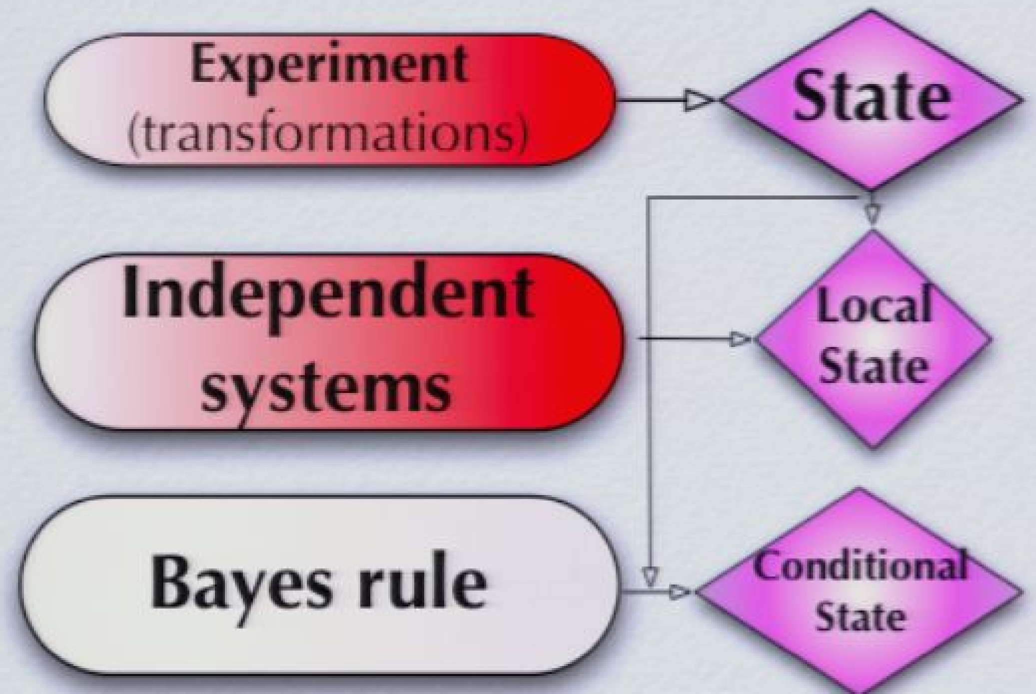


# No-signaling from dynamical independence

*Axioms*
*Theorems*



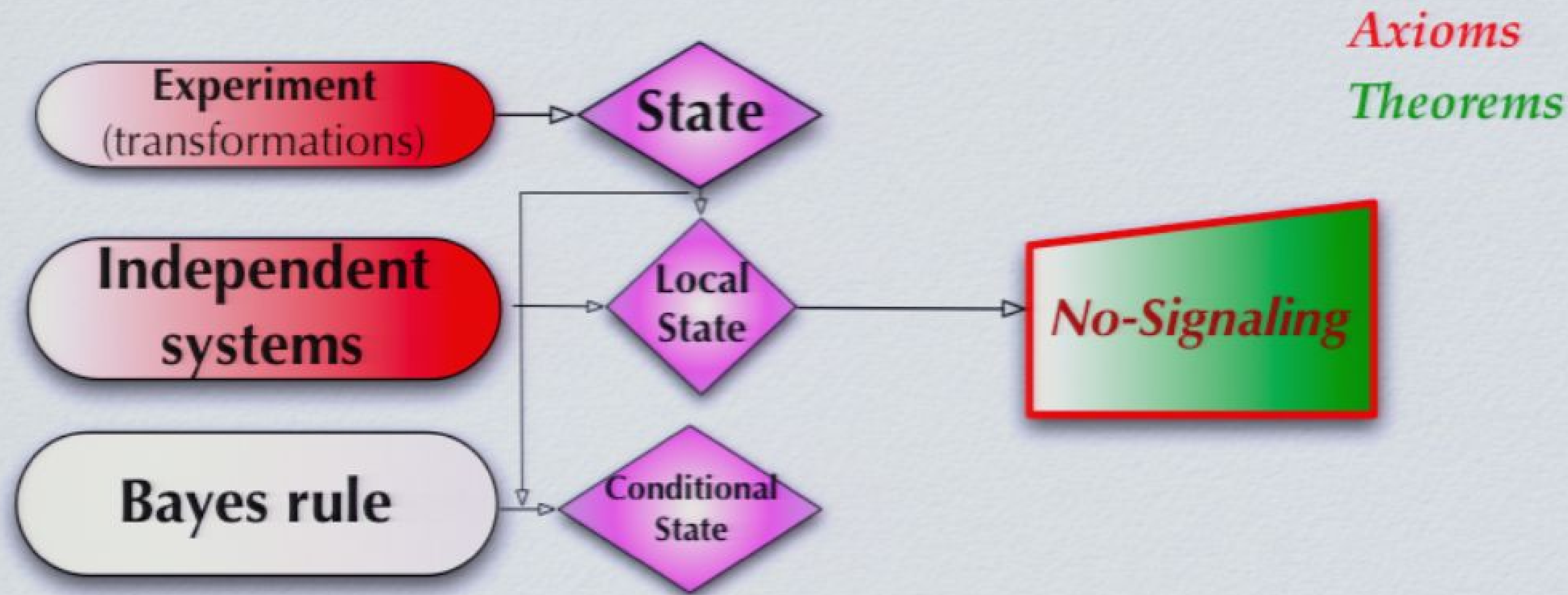
# No-signaling from dynamical independence




*Axioms*

*Theorems*

# No-signaling from dynamical independence



# Generalized weights, transformations, and effects

# Generalized weights, transformations, and effects

***Generalize by taking differences:***

convex sets/cones → (affine) linear spaces

# Generalized weights, transformations, and effects

***Generalize by taking differences:***

| convex sets/cones → (affine) linear spaces |
|---|

weights $\mathfrak{W}$ → gen. weights $\mathfrak{W}_{\mathbb{R}}$

trans

# Generalized weights, transformations, and effects

***Generalize by taking differences:***

convex sets/cones → (affine) linear spaces

weights $\mathfrak{W}$ → gen. weights $\mathfrak{W}_{\mathbb{R}}$

transformations $\mathfrak{T}$ → gen. transformations $\mathfrak{T}_{\mathbb{R}}$
**(real algebra)**

# Generalized weights, transformations, and effects

***Generalize by taking differences:***

convex sets/cones → (affine) linear spaces

weights $\mathfrak{W}$ → gen. weights $\mathfrak{W}_{\mathbb{R}}$

transformations $\mathfrak{T}$ → gen. transformations $\mathfrak{T}_{\mathbb{R}}$
**(real algebra)**

# Generalized weights, transformations, and effects

***Generalize by taking differences:***

| convex sets/cones → (affine) linear spaces |
|---|

weights $\mathfrak{W}$ → gen. weights $\mathfrak{W}_{\mathbb{R}}$

transformations $\mathfrak{T}$ → gen. transformations $\mathfrak{T}_{\mathbb{R}}$

**(real algebra)**

# Generalized weights, transformations, and effects

***Generalize by taking differences:***

| convex sets/cones → (affine) linear spaces |
|---|

weights $\mathfrak{W}$ → gen. weights $\mathfrak{W}_{\mathbb{R}}$

transformations $\mathfrak{T}$ → gen. transformations $\mathfrak{T}_{\mathbb{R}}$

**(real algebra)**

effects $\mathfrak{P}$ → gen. effects $\mathfrak{P}_{\mathbb{R}}$

# Real Banach spaces

***norms:***

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$ :

$$\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

# Real Banach spaces

**norms:**

gen. effects $\mathfrak{P}_{\mathbb{R}}$ : $\quad \|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$

gen. weights $\mathfrak{W}_{\mathbb{R}}$: $\quad \|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$ :

$$\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

gen. weights $\mathfrak{W}_{\mathbb{R}}$:

$$\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$$

gen. transformations $\mathfrak{T}_{\mathbb{R}}$ :

$$\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \|\underline{\mathscr{B}} \circ \mathscr{A}\|$$

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$ : $\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$

gen. weights $\mathfrak{W}_{\mathbb{R}}$: $\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$

gen. transformations $\mathfrak{T}_{\mathbb{R}}$ : $\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \|\underline{\mathscr{B}} \circ \mathscr{A}\|$

$\mathfrak{W}_{\mathbb{R}}$ $\mathfrak{P}_{\mathbb{R}}$ ***dual B***

$l_{\underline{\mathscr{A}}}($

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$:

$$\|\underline{\mathscr{A}}\| := \sup_{\omega\in\mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

gen. weights $\mathfrak{W}_{\mathbb{R}}$:

$$\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}}\ni\|\underline{\mathscr{A}}\|\leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$$

gen. transformations $\mathfrak{T}_{\mathbb{R}}$:

$$\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}}\ni\|\underline{\mathscr{B}}\|\leqslant 1} \|\underline{\mathscr{B}}\circ\mathscr{A}\|$$

$\mathfrak{W}_{\mathbb{R}}$ $\mathfrak{P}_{\mathbb{R}}$ ***dual Banach pair*** under the pairing

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

$\mathfrak{T}_{\mathbb{R}}$ ***Banach algebra***

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$ : $$\|\underline{\mathscr{A}}\| := \sup_{\omega\in\mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

gen. weights $\mathfrak{W}_{\mathbb{R}}$: $$\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}}\ni\|\underline{\mathscr{A}}\|\leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$$

gen. transformations $\mathfrak{T}_{\mathbb{R}}$ : $$\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}}\ni\|\underline{\mathscr{B}}\|\leqslant 1} \|\underline{\mathscr{B}}\circ\mathscr{A}\|$$

$\mathfrak{W}_{\mathbb{R}}$ $\mathfrak{P}_{\mathbb{R}}$ ***dual Banach pair*** under the pairing

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$



$\mathfrak{T}_{\mathbb{R}}$ ***Banach algebra***

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$:

$$\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

gen. weights $\mathfrak{W}_{\mathbb{R}}$:

$$\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$$

gen. transformations $\mathfrak{T}_{\mathbb{R}}$:

$$\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \|\underline{\mathscr{B}} \circ \mathscr{A}\|$$

$\mathfrak{W}_{\mathbb{R}}$ $\mathfrak{P}_{\mathbb{R}}$ ***dual Banach pair*** under the pairing

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

$\mathfrak{T}_{\mathbb{R}}$ ***Banach algebra***

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$:
$$\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

gen. weights $\mathfrak{W}_{\mathbb{R}}$:
$$\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$$

gen. transformations $\mathfrak{T}_{\mathbb{R}}$:
$$\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \|\underline{\mathscr{B}} \circ \mathscr{A}\|$$

$\mathfrak{W}_{\mathbb{R}}$ $\mathfrak{P}_{\mathbb{R}}$ ***dual Banach pair*** under the pairing

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

$\mathfrak{T}_{\mathbb{R}}$ ***Banach algebra***

# Banach-space structures

*Axioms*

*Theorems*

# Real Banach spaces

**norms:**

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$ : $$\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

# Real Banach spaces

**norms:**

gen. effects $\mathfrak{P}_{\mathbb{R}}$ : $$\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

gen. weights $\mathfrak{W}_{\mathbb{R}}$: $$\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$$

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$ :

$$\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

gen. weights $\mathfrak{W}_{\mathbb{R}}$:

$$\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$$

ge

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$ : $\quad \|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$

gen. weights $\mathfrak{W}_{\mathbb{R}}$: $\quad \|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$

gen. transformations $\mathfrak{T}_{\mathbb{R}}$ : $\quad \|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \|\underline{\mathscr{B}} \circ \mathscr{A}\|$

$\mathfrak{W}_{\mathbb{R}}$ $\mathfrak{P}_{\mathbb{R}}$ ***dual Banach pair*** under the pairing

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

# Real Banach spaces

***norms:***

gen. effects $\mathfrak{P}_{\mathbb{R}}$ :

$$\|\underline{\mathscr{A}}\| := \sup_{\omega \in \mathfrak{S}} |\omega(\underline{\mathscr{A}})|$$

gen. weights $\mathfrak{W}_{\mathbb{R}}$:

$$\|\tilde{\omega}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{A}}\| \leqslant 1} |\tilde{\omega}(\underline{\mathscr{A}})|$$

gen. transformations $\mathfrak{T}_{\mathbb{R}}$ :

$$\|\mathscr{A}\| := \sup_{\mathfrak{P}_{\mathbb{R}} \ni \|\underline{\mathscr{B}}\| \leqslant 1} \|\underline{\mathscr{B}} \circ \mathscr{A}\|$$

$\mathfrak{W}_{\mathbb{R}}$ $\mathfrak{P}_{\mathbb{R}}$ ***dual Banach pair*** under the pairing

$$l_{\underline{\mathscr{A}}}(\omega) \doteq \omega(\mathscr{A})$$

$\mathfrak{T}_{\mathbb{R}}$ ***Banach algebra***

# Banach-space structures

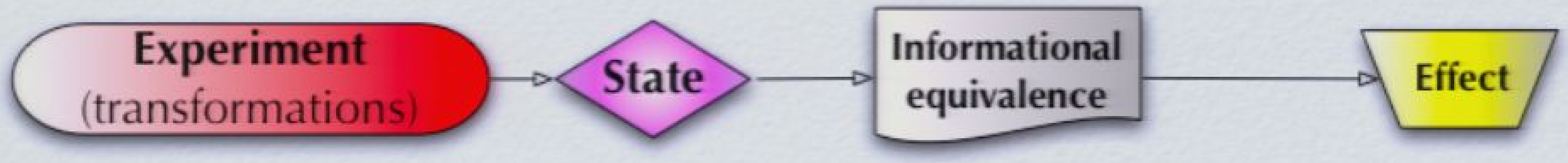
*Axioms*

*Theorems*

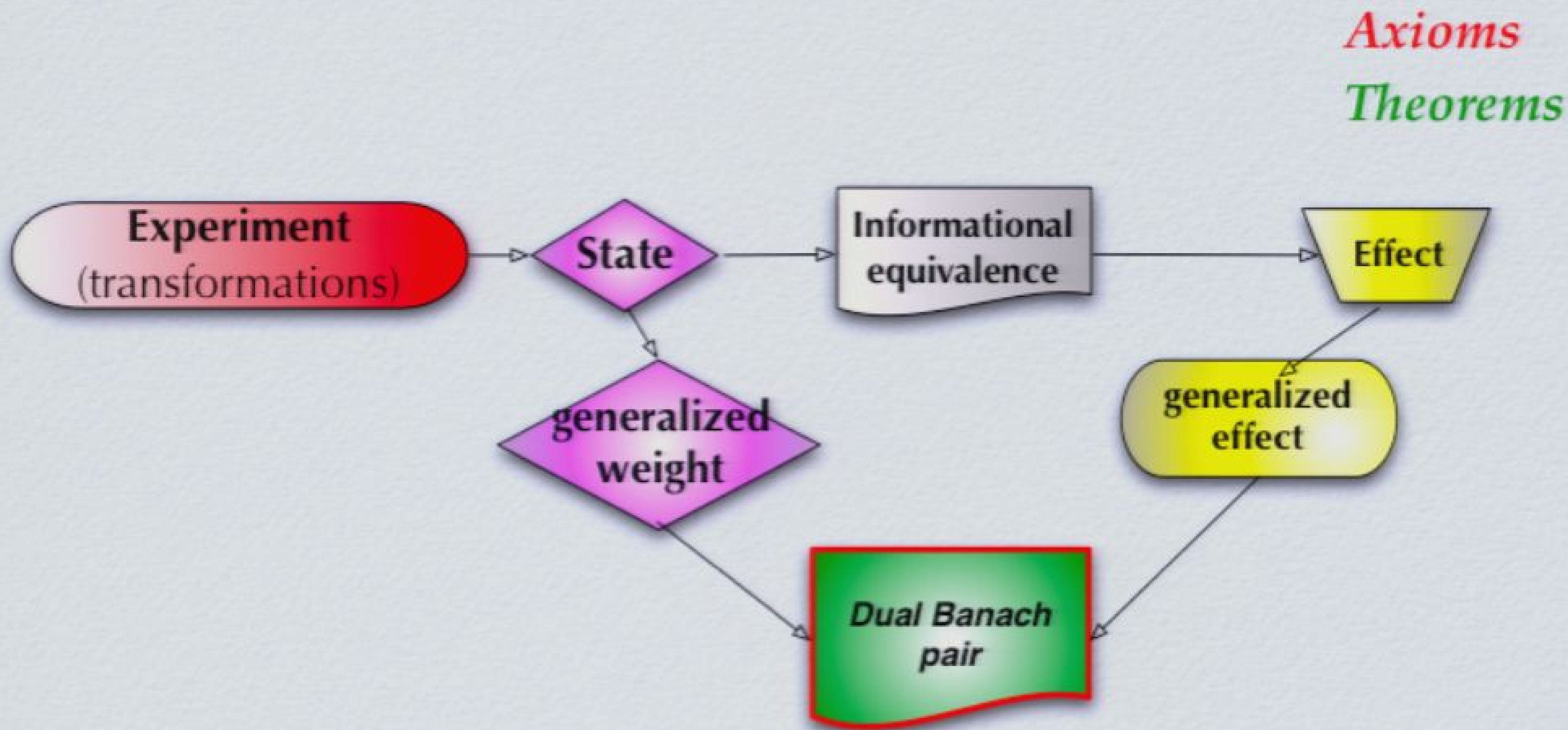
# Banach-space structures

*Axioms*

*Theorems*



# Banach-space structures



# Observable

***Observable:*** a complete set of effects $\mathbb{L} = \{l_i\}$

$$\sum_j l_j = \underline{\mathscr{I}}$$

# Informationally complete observable

***Informationally complete observable:*** an observable $\mathbb{L} = \{l_i\}$ is informationally complete if any effect $l$ can be written as linear combination of elements of $\mathbb{L}$, namely there exist coefficients $c_i(l)$ such that

$$l = \sum_{i=1}^{|\mathbb{L}|} c_i(l) l_i$$

# Informationally complete observable

***Informationally complete observable:*** an observable $\mathbb{L} = \{l_i\}$ is informationally complete if any effect $l$ can be written as linear combination of elements of $\mathbb{L}$, namely there exist coefficients $c_i(l)$ such that

$$l = \sum_{i=1}^{|\mathbb{L}|} c_i(l) l_i$$

***affine dimension:*** $\dim(\mathfrak{S}) = |\mathbb{L}| - 1$, for $\mathbb{L}$ minimal informationally complete on $\mathfrak{S}$

# Bloch representation

$$l_{\underline{\mathscr{A}}} = \sum_j m_j(\underline{\mathscr{A}}) n_j \qquad l_{\underline{\mathscr{A}}}(\omega) = \boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})$$

# Bloch representation

$$l_{\underline{\mathscr{A}}} = \sum_j m_j(\underline{\mathscr{A}}) n_j \qquad l_{\underline{\mathscr{A}}}(\omega) = \boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})$$

**Conditioning: fractional affine transformation**

$$\boldsymbol{n}(\omega) \longrightarrow \boldsymbol{n}(\omega_{\mathscr{A}})$$

$$\boldsymbol{n}(\omega_{\mathscr{A}}) = \frac{\boldsymbol{M}(\mathscr{A})\boldsymbol{n}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})}$$

$$M_{ij}(\mathscr{A}) = \begin{pmatrix} q(\underline{\mathscr{A}}) & \boldsymbol{m}(\underline{\mathscr{A}}) \\ \boldsymbol{k}(\mathscr{A}) & \boldsymbol{M}(\mathscr{A}) \end{pmatrix}$$

# Informationally complete observable

***Theorem:*** *there always exists a minimal informationally complete observable.*

**Proof.** By definition $\mathfrak{P}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}}(\mathfrak{P})$, whence there must exists a spanning set for $\mathfrak{P}_{\mathbb{R}}$ that is contained in $\mathfrak{P}$. The maximal number of elements of this set that are linearly independent will constitute a *basis*, which we suppose has finite cardinality $\dim(\mathfrak{P}_{\mathbb{R}})$. It remains to be shown that it is possible to have a basis with sum of elements equal to $\underline{\mathscr{I}}$, and that such basis is obtained operationally starting from the available observables from which we constructed $\mathfrak{P}$.

If all observables are *uninformative* (i. e. with all effects proportional to $\underline{\mathscr{I}}$) , then $\mathfrak{P}_{\mathbb{R}} = \mathsf{Span}(\underline{\mathscr{I}})$, $\underline{\mathscr{I}}$ is a minimal infocomplete observable, and the statement of the theorem is proved. Otherwise, there exists at least an observable $\mathbb{E} = \{l_i\}$ with $n \geqslant 2$ linearly independent effects. If this is the only observable, again the theorem is proved. Otherwise, take a new binary observable $\mathbb{E}_2 = \{x, y\}$ from the set of available ones (you can take different binary observables out of a given observable with more than two outcomes by summing up effects to yes-no observables). If $x \in \mathsf{Span}(\mathbb{E})$ discard it. If $x \notin \mathsf{Span}(\mathbb{E})$, then necessarily also $y \notin \mathsf{Span}(\mathbb{E})$ [since if there exists coefficients $\lambda_i$ such that $y = \sum_i \lambda_i l_i$, then $x = \sum_i (1 - \lambda_i) l_i$]. Now, consider the observable

$$\mathbb{E}' = \left\{\tfrac{1}{2}y, \tfrac{1}{2}(l_1 + x), \tfrac{1}{2}l_2, \ldots, l_n\right\} \tag{1}$$

(which operationally corresponds to the random choice between the observables $\mathbb{E}$ and $\mathbb{E}_2$ with probability $\frac{1}{2}$, and with the events corresponding to $x$ and $l_1$ made indistinguishable). This new observable has now $|\mathbb{E}'| = n + 1$ linearly independent effects (since $y$ is linearly independent on the $l_i$ and one has $y = \sum_{i=1}^{n} l_i - x = \sum_{i=2}^{n} l_i + l_1 - x$). By iterating the above procedure we reach $|\mathbb{E}'| = \dim(\mathfrak{P}_{\mathbb{R}})$, and we have so realized an apparatus that measures a minimal informationally complete observable. ■

# Bloch representation

$$l_{\underline{\mathscr{A}}} = \sum_j m_j(\underline{\mathscr{A}}) n_j \qquad l_{\underline{\mathscr{A}}}(\omega) = \boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})$$

Conditioning: fractional affine transformation

$$n(\omega) \longrightarrow n(\omega_{\mathscr{A}})$$

# Bloch representation

$$l_{\underline{\mathscr{A}}} = \sum_j m_j(\underline{\mathscr{A}}) n_j \qquad l_{\underline{\mathscr{A}}}(\omega) = \boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})$$

**Conditioning: fractional affine transformation**

$$\boldsymbol{n}(\omega) \longrightarrow \boldsymbol{n}(\omega_{\mathscr{A}})$$

$$\boldsymbol{n}(\omega_{\mathscr{A}}) = \frac{\boldsymbol{M}(\mathscr{A})\boldsymbol{n}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})}$$

$$M_{ij}(\mathscr{A}) = \begin{pmatrix} q(\underline{\mathscr{A}}) & \boldsymbol{m}(\underline{\mathscr{A}}) \\ \boldsymbol{k}(\mathscr{A}) & \boldsymbol{M}(\mathscr{A}) \end{pmatrix}$$

# Informationally complete observable

***Informationally complete observable:*** an observable $\mathbb{L} = \{l_i\}$ is informationally complete if any effect $l$ can be written as linear combination of elements of $\mathbb{L}$, namely there exist coefficients $c_i(l)$ such that

$$l = \sum_{i=1}^{|\mathbb{L}|} c_i(l) l_i$$

# Informationally complete observable

***Informationally complete observable:*** an observable $\mathbb{L} = \{l_i\}$ is informationally complete if any effect $l$ can be written as linear combination of elements of $\mathbb{L}$, namely there exist coefficients $c_i(l)$ such that

$$l = \sum_{i=1}^{|\mathbb{L}|} c_i(l) l_i$$

***affine dimension:*** $\dim(\mathfrak{S}) = |\mathbb{L}| - 1$, for $\mathbb{L}$ minimal informationally complete on $\mathfrak{S}$

# Bloch representation

$$l_{\underline{\mathscr{A}}} = \sum_j m_j(\underline{\mathscr{A}}) n_j \qquad l_{\underline{\mathscr{A}}}(\omega) = \boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})$$

**Conditioning: fractional affine transformation**

$$\boldsymbol{n}(\omega) \longrightarrow \boldsymbol{n}(\omega_{\mathscr{A}})$$

$$\boldsymbol{n}(\omega_{\mathscr{A}}) = \frac{\boldsymbol{M}(\mathscr{A})\boldsymbol{n}(\omega) + \boldsymbol{k}(\mathscr{A})}{\boldsymbol{m}(\underline{\mathscr{A}}) \cdot \boldsymbol{n}(\omega) + q(\underline{\mathscr{A}})}$$

$$M_{ij}(\mathscr{A}) = \begin{pmatrix} q(\underline{\mathscr{A}}) & \boldsymbol{m}(\underline{\mathscr{A}}) \\ \boldsymbol{k}(\mathscr{A}) & \boldsymbol{M}(\mathscr{A}) \end{pmatrix}$$

# Informationally complete observable

**Theorem:** *there always exists a minimal informationally complete observable.*

**Proof.** By definition $\mathfrak{P}_{\mathbb{R}} = \mathsf{Span}_{\mathbb{R}}(\mathfrak{P})$, whence there must exists a spanning set for $\mathfrak{P}_{\mathbb{R}}$ that is contained in $\mathfrak{P}$. The maximal number of elements of this set that are linearly independent will constitute a *basis*, which we suppose has finite cardinality $\dim(\mathfrak{P}_{\mathbb{R}})$. It remains to be shown that it is possible to have a basis with sum of elements equal to $\underline{\mathscr{I}}$, and that such basis is obtained operationally starting from the available observables from which we constructed $\mathfrak{P}$.

If all observables are *uninformative* (i. e. with all effects proportional to $\underline{\mathscr{I}}$) , then $\mathfrak{P}_{\mathbb{R}} = \mathsf{Span}(\underline{\mathscr{I}})$, $\underline{\mathscr{I}}$ is a minimal infocomplete observable, and the statement of the theorem is proved. Otherwise, there exists at least an observable $\mathbb{E} = \{l_i\}$ with $n \geqslant 2$ linearly independent effects. If this is the only observable, again the theorem is proved. Otherwise, take a new binary observable $\mathbb{E}_2 = \{x, y\}$ from the set of available ones (you can take different binary observables out of a given observable with more than two outcomes by summing up effects to yes-no observables). If $x \in \mathsf{Span}(\mathbb{E})$ discard it. If $x \notin \mathsf{Span}(\mathbb{E})$, then necessarily also $y \notin \mathsf{Span}(\mathbb{E})$ [since if there exists coefficients $\lambda_i$ such that $y = \sum_i \lambda_i l_i$, then $x = \sum_i (1-\lambda_i) l_i$]. Now, consider the observable

$$\mathbb{E}' = \left\{ \tfrac{1}{2}y, \tfrac{1}{2}(l_1 + x), \tfrac{1}{2}l_2, \ldots, l_n \right\} \tag{1}$$

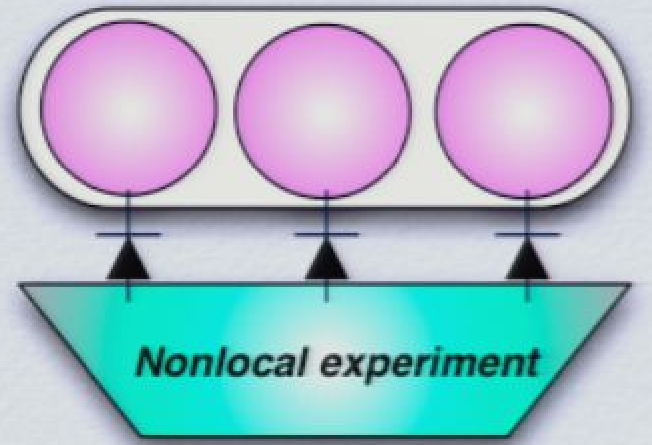
(which operationally corresponds to the random choice between the observables $\mathbb{E}$ and $\mathbb{E}_2$ with probability $\frac{1}{2}$, and with the events corresponding to $x$ and $l_1$ made indistinguishable). This new observable has now $|\mathbb{E}'| = n+1$ linearly independent effects (since $y$ is linearly independent on the $l_i$ and one has $y = \sum_{i=1}^{n} l_i - x = \sum_{i=2}^{n} l_i + l_1 - x$). By iterating the above procedure we reach $|\mathbb{E}'| = \dim(\mathfrak{P}_{\mathbb{R}})$, and we have so realized an apparatus that measures a minimal informationally complete observable. ■

# Postulate 3: Local observability principle

*For composite systems local info-complete observables provide global info-complete observables.*

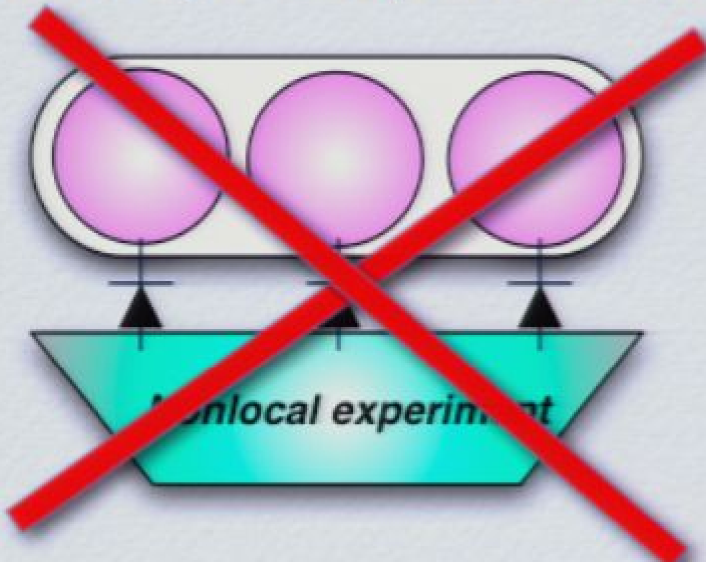
# Postulate 3: Local observability principle

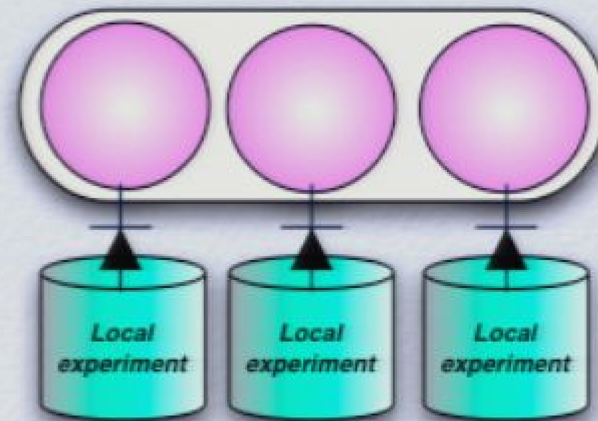
*For composite systems local info-complete observables provide global info-complete observables.*



# Postulate 3: Local observability principle

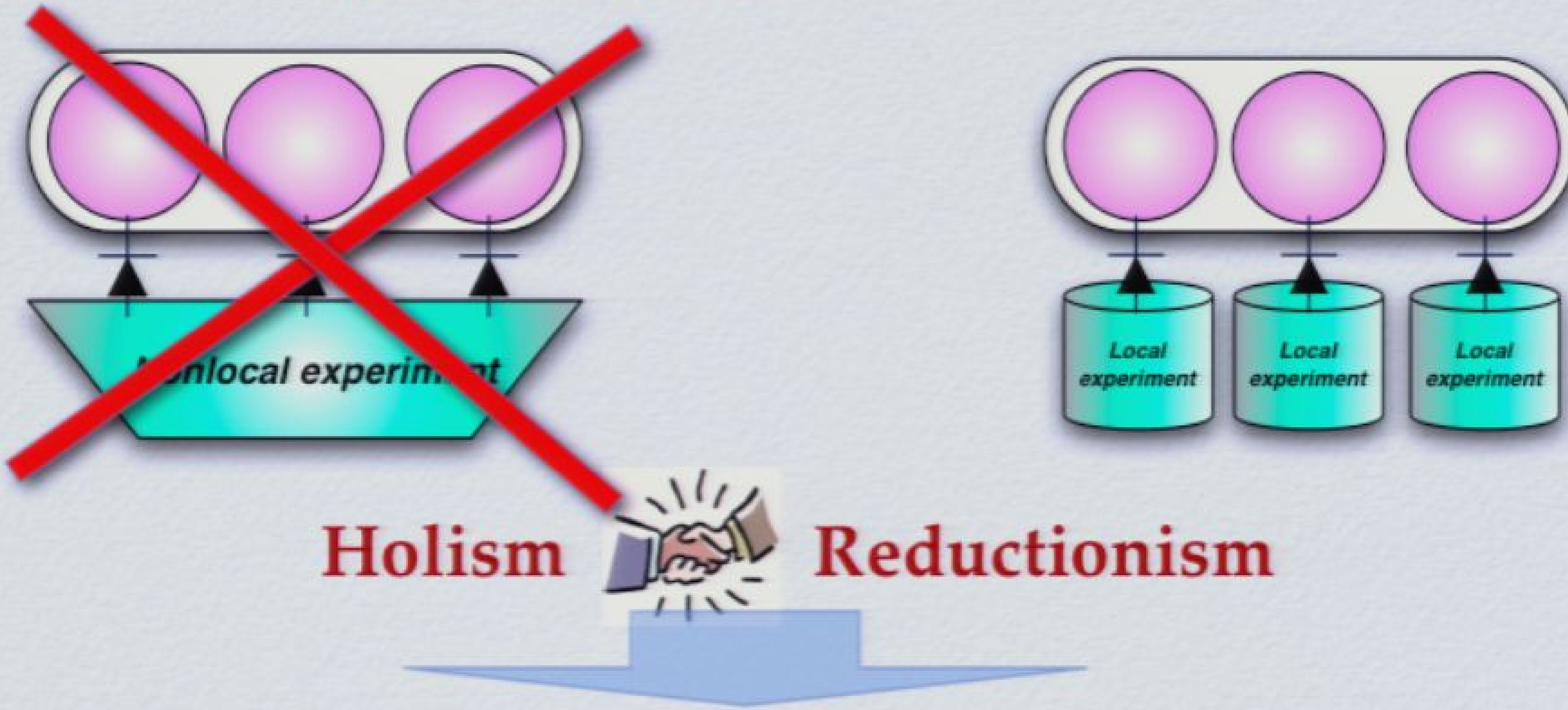
*For composite systems local info-complete observables provide global info-complete observables.*





# Postulate 3: Local observability principle

*For composite systems local info-complete observables provide global info-complete observables.*




identity for the affine dimension of composite systems



$$\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$$

# Postulate 3: Local observability principle

## identity for the affine dimension of composite systems

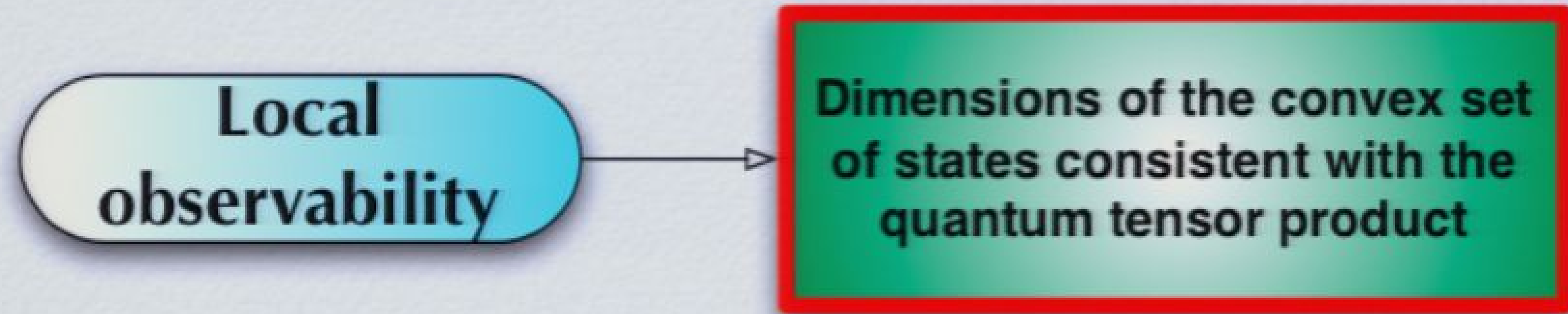
$$\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$$

**Proof.** We first prove that the left side is a lower bound for the right side. Indeed, the number of outcomes of a minimal informationally complete observable is $\dim(\mathfrak{S}) + 1$, since it equals the dimension of the affine space embedding the convex set of states $\mathfrak{S}$ plus an additional dimension for normalization. Now, consider a global informationally complete measurement made of two local minimal informationally complete observables measured jointly. It has number of outcomes $[\dim(\mathfrak{S}_1) + 1][\dim(\mathfrak{S}_2) + 1]$. However, we are not guaranteed that the joint observable is itself minimal, whence the bound.

The opposite inequality can be easily proved by considering that a global informationally incomplete measurement made of minimal local informationally complete measurements should belong to the linear span of a minimal global informationally complete measurement. ■

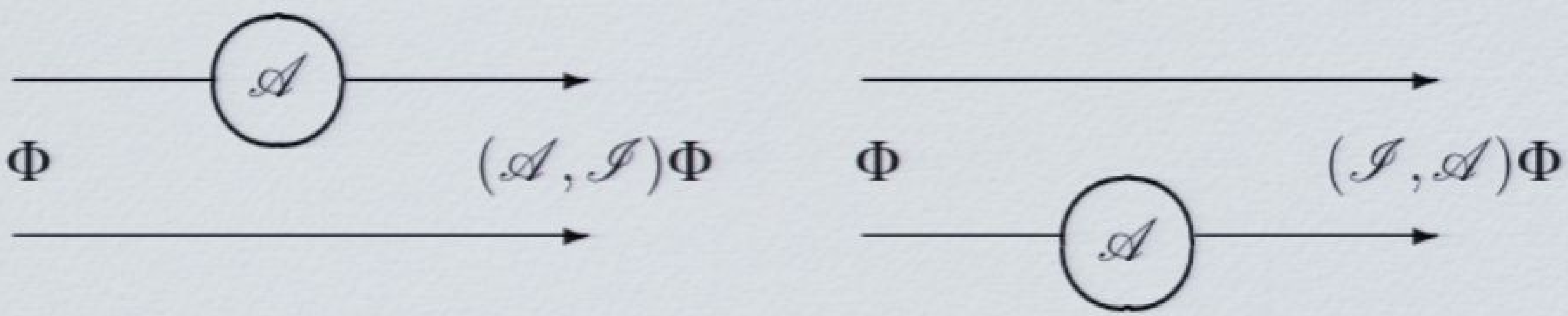
# Postulate 3: Local observability principle

*Postulates*
*Axioms*
*Theorems*



# Faithful states

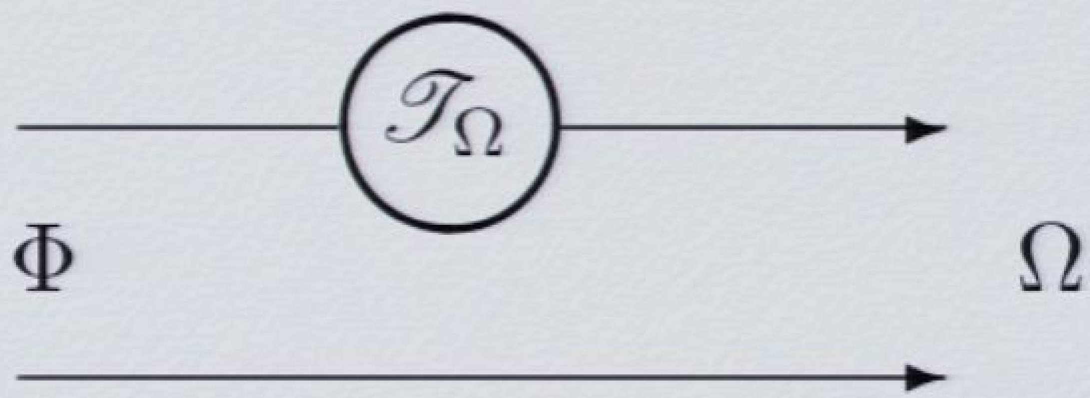
***Dynamically faithful state:*** we say that a state $\Phi$ of a bipartite system is dynamically faithful if when acting on it with a local transformation $\mathscr{A}$ on one system the output conditioned weight $(\mathscr{A},\mathscr{I})\Phi$ is in 1-to-1 correspondence with the transformation $\mathscr{A}$




$$(\mathscr{A},\mathscr{I})\Phi = 0 \Longrightarrow \mathscr{A} = 0, \quad \forall \mathscr{A} \in \mathfrak{T}_{\mathbb{R}}$$

# Faithful states

***Preparationally faithful state:*** we say that a state $\Phi$ of a bipartite system is preparationally faithful if every joint state $\Omega$ can be achieved by a suitable local transformation $\mathscr{T}_\Omega$ on one system occurring with nonzero probability



# Faithful states

***Symmetric bipartite state:*** we call a joint state $\Phi$ of a bipartite system symmetric if

$$\Phi(\mathscr{A}, \mathscr{B}) = \Phi(\mathscr{B}, \mathscr{A})$$

# Perfectly discriminating observable

***Perfectly discriminable observable/states:***
an observable $\mathbb{L} = \{l_i\}$ such that there exist states $\{\omega_j\}$ satisfying

$$l_i(\omega_j) = \delta_{ij}$$

# Perfectly discriminating observable

***Perfectly discriminable observable/states:***
an observable $\mathbb{L} = \{l_i\}$ such that there exist states $\{\omega_j\}$ satisfying

$$l_i(\omega_j) = \delta_{ij}$$

***Informational dimension*** $\dim_{\#}(\mathfrak{S})$: maximal number of perfectly discriminable states

# Postulate 4: Informationally complete discriminating observable

*For every system there exists a minimal info-complete observable that can be achieved by means of a* ***joint discriminating observable on system+ancilla***†



† "*ancilla*":= independent system

# Postulate 4: Informationally complete discriminating observable

*For every system there exists a minimal info-complete observable that can be achieved by means of a* ***joint discriminating observable on system+ancilla***†

$$\dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S}^{\times 2}) - 1$$



† *"ancilla"*:= independent system

# Dimensionality identities

| $\Longrightarrow$ | | |
|---|---|---|
| state-effect duality | $\dim(\mathfrak{P}_{\mathbb{R}}) = \dim(\mathfrak{S}) + 1$ | (D1) |
| P3 (loc. observability) | $\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$ | (D2) |
| P4 (infoc. as joint discr.) | $\dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S}^{\times 2}) - 1$ | (D4) |
| (D2)+(D4) | $\dim(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S}^{\times 2})^2 - 1$ | (D24) |
| (D24) | $\dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S})^2 - 1$ | (D24b) |
| (D4+D24b) | $\dim_{\#}(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S})^2$ | ($\otimes$) |
| P2 (faith.) | $\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1$ | ($\mathfrak{T}$) |
| (D1)+(D24b) | $\dim(\mathfrak{P}_{\mathbb{R}}) = \dim_{\#}(\mathfrak{S})^2$ | ($\mathfrak{P}$) |

# Dimensionality identities

| $\Longrightarrow$ | | |
|---|---|---|
| state-effect duality | $\dim(\mathfrak{P}_{\mathbb{R}}) = \dim(\mathfrak{S}) + 1$ | (D1) |
| P3 (loc. observability) | $\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$ | (D2) |
| P4 (infoc. as joint discr.) | $\dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S}^{\times 2}) - 1$ | (D4) |
| (D2)+(D4) | $\dim(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S}^{\times 2})^2 - 1$ | (D24) |
| (D24) | $\dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S})^2 - 1$ | (D24b) |
| (D4+D24b) | $\dim_{\#}(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S})^2$ | $(\otimes)$ |
| P2 (faith.) | $\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1$ | $(\mathfrak{T})$ |
| (D1)+(D24b) | $\dim(\mathfrak{P}_{\mathbb{R}}) = \dim_{\#}(\mathfrak{S})^2$ | $(\mathfrak{P})$ |

# Dimensionality identities

| | | |
|---|---|---|
| $\Longrightarrow$ | | |
| state-effect duality | $\dim(\mathfrak{P}_{\mathbb{R}}) = \dim(\mathfrak{S}) + 1$ | (D1) |
| P3 (loc. observability) | $\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$ | (D2) |
| P4 (infoc. as joint discr.) | $\dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S}^{\times 2}) - 1$ | (D4) |
| (D2)+(D4) | $\dim(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S}^{\times 2})^2 - 1$ | (D24) |
| (D24) | $\dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S})^2 - 1$ | (D24b) |
| (D4+D24b) | $\dim_{\#}(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S})^2$ | $(\otimes)$ |
| P2 (faith.) | $\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1$ | $(\mathfrak{T})$ |
| (D1)+(D24b) | $\dim(\mathfrak{P}_{\mathbb{R}}) = \dim_{\#}(\mathfrak{S})^2$ | $(\mathfrak{P})$ |

$$\dim(\mathfrak{P}_{\mathbb{R}}) = \dim_{\#}(\mathfrak{S})^2$$

# Positive bilinear form

***Positive form over generalized effects:*** from $\Phi$ real symmetric form over effects obtain the positive form (for finite dimensions)

$$|\Phi| := \Phi_+ - \Phi_-$$

$$|\Phi|(\underline{\mathscr{A}}, \underline{\mathscr{B}}) = \Phi(\underline{\mathscr{A}}, \varsigma(\underline{\mathscr{B}})), \quad \varsigma(\underline{\mathscr{A}}) = (\mathscr{P}_+ - \mathscr{P}_-)(\underline{\mathscr{A}})$$

$$\varsigma^2 = \mathscr{I}$$

# The complex Hilbert space formulation

For finite dimensions the real Hilbert space $\mathfrak{P}_{\mathbb{R}}$ is isomorphic to the real Hilbert space of Hermitian complex matrices representing selfadjoint operators over a complex Hilbert space $\mathsf{H}$ of dimensions $\dim(\mathsf{H}) = \dim_{\#}(\mathfrak{S})$.

**This is the Hilbert space formulation of Quantum Mechanics**

# The complex Hilbert space formulation

If the state is also preparationally faithful then one can make every state correspond to an effect

Then one can write the probability rule in terms of a real scalar product pairing between states and effects, with the convex cones of effects and states corresponding to the convex cone of positive matrices.

# The complex Hilbert space formulation

If the state is also preparationally faithful then one can make every state correspond to an effect

Then one can write the probability rule in terms of a real scalar product pairing between states and effects, with the convex cones of effects and states corresponding to the convex cone of positive matrices.

# The complex Hilbert space formulation

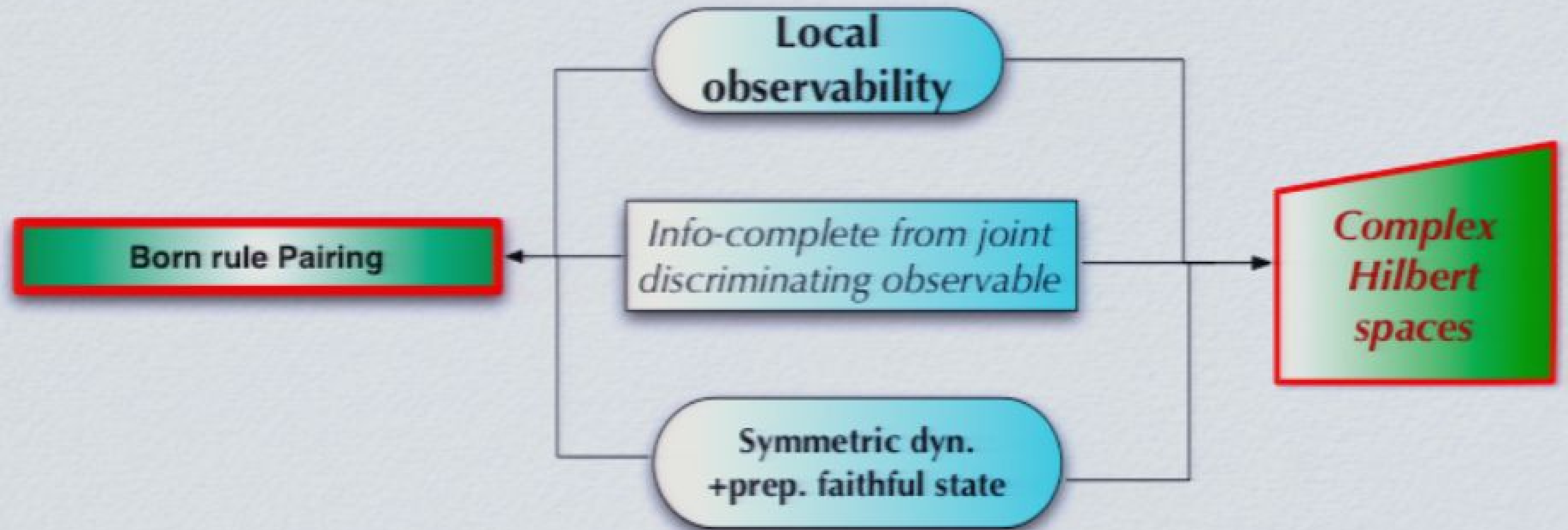
Since $\Phi$ is *preparationally* faithful, then for every state $\omega$ there exists a suitable transformation $\mathscr{T}_\omega$ such that $\omega = \Phi_{\mathscr{I},\mathscr{T}_\omega}|_1$ with probability $\Phi(\mathscr{I},\mathscr{T}_\omega) > 0$

Then we can write the probability rule in terms of the pairing between states and effects:

$$\omega(\underline{\mathscr{C}}) = \Phi_{\mathscr{I},\mathscr{T}_\omega}|_1(\underline{\mathscr{C}}) = |\Phi|(\underline{\mathscr{C}}, \underline{\widetilde{\mathscr{T}}}_\omega),$$

$$\underline{\widetilde{\mathscr{T}}}_\omega = \frac{\varsigma(\underline{\mathscr{T}}_\omega)}{\Phi(\underline{\mathscr{I}}, \underline{\mathscr{T}}_\omega)}$$

# The complex Hilbert space formulation



# The complex Hilbert space formulation

***Rest of the construction:***

- *construct complex operators by complex linear combination of effects*

# The complex Hilbert space formulation

***Rest of the construction:***

- ***construct complex operators by complex linear combination of effects***

- *physical transformations are described by CP trace-decreasing maps*

# The complex Hilbert space formulation

*Rest of the construction:*

- *construct complex operators by complex linear combination of effects*
- *physical transformations are described by CP trace-decreasing maps*
- ...

# Construction of a C*-algebra of transformations

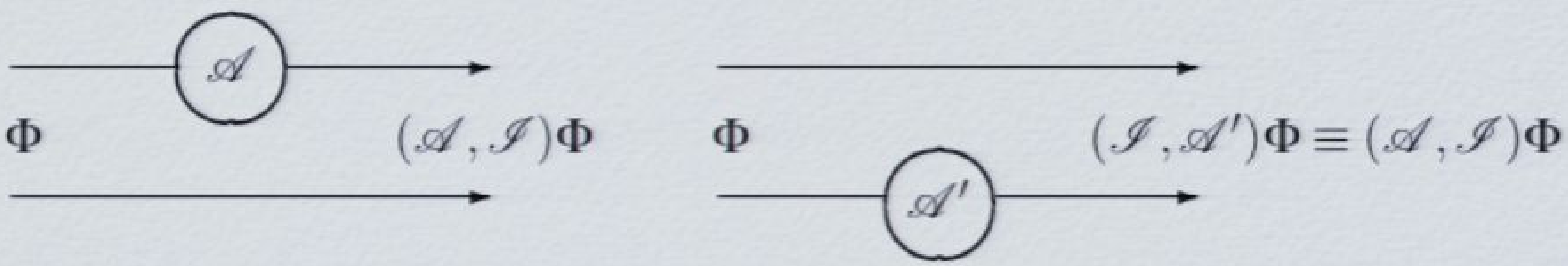
# Operational definition of *transposed*

Existence of symmetric faithful states

⇓

"transposition" over the real algebra $\mathcal{A}$ of (generalized) transformations

$$\mathscr{A} \Longleftrightarrow \mathscr{A}'$$

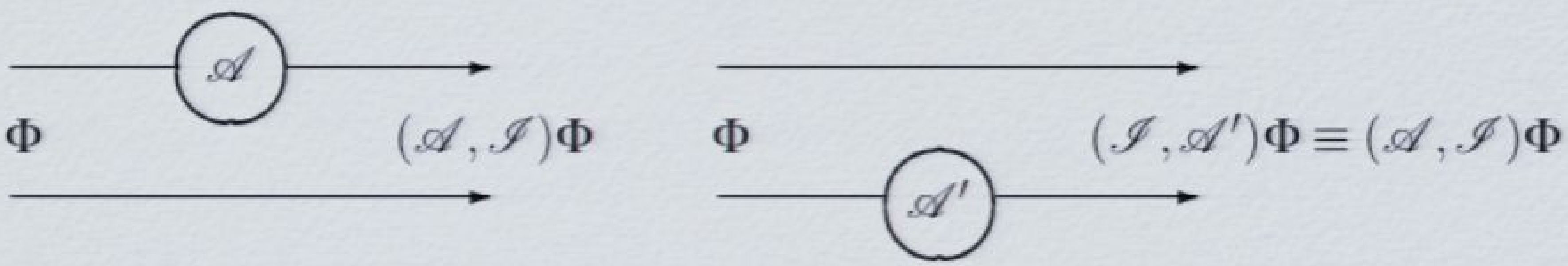


# Operational definition of *transposed*

Existence of symmetric faithful states

"transposition" over the real algebra $\mathcal{A}$ of (generalized) transformations

$$\mathscr{A} \Longleftrightarrow \mathscr{A}'$$




$$\Phi(\mathscr{B}\circ\mathscr{A},\mathscr{C}) = \Phi(\mathscr{B},\mathscr{C}\circ\mathscr{A}')$$

# The complex conjugation

The involution $\varsigma$ corresponds to a generalized transformation

$$\varsigma(\underline{\mathscr{A}}) = \underline{\mathscr{A}} \circ \mathscr{Z}$$

Correspondingly the involution over transformations reads

$$\varsigma(\mathscr{A}) = \mathscr{Z} \circ \mathscr{A} \circ \mathscr{Z}$$

which is composition preserving, namely

$$\varsigma(\mathscr{B} \circ \mathscr{A}) = \varsigma(\mathscr{B}) \circ \varsigma(\mathscr{A}).$$

# The adjoint

Scalar product over $\mathfrak{P}_{\mathbb{R}}$ :

$$_{\Phi}\langle \underline{\mathscr{B}} | \underline{\mathscr{A}} \rangle_{\Phi} := \Phi(\varsigma(\underline{\mathscr{B}'}), \underline{\mathscr{A}'}) = \Phi|_1(\mathscr{B}^\dagger \circ \mathscr{A})$$

# The adjoint

Scalar product over $\mathfrak{P}_{\mathbb{R}}$ :

$$_{\Phi}\langle \underline{\mathscr{B}} | \underline{\mathscr{A}} \rangle_{\Phi} := \Phi(\varsigma(\underline{\mathscr{B}}'), \underline{\mathscr{A}}') = \Phi|_1(\mathscr{B}^\dagger \circ \mathscr{A})$$

$\varsigma$ works as a complex-conjugation in the sense that

$\mathscr{A}^\dagger := \varsigma(\mathscr{A}')$ works as an adjoint, namely

$$_{\Phi}\langle \mathscr{C}^\dagger \circ \underline{\mathscr{A}} | \underline{\mathscr{B}} \rangle_{\Phi} = {}_{\Phi}\langle \underline{\mathscr{A}} | \mathscr{C} \circ \underline{\mathscr{B}} \rangle_{\Phi}$$

# GNS construction for representing transformations

Representations $\pi_\Phi$ of transformations $\mathscr{A} \in \mathcal{A}$ over effects $\mathcal{A}/\mathcal{J}$

$$\pi_\Phi(\mathscr{A})|\underline{\mathscr{B}}\rangle_\Phi \doteq |\underline{\mathscr{A} \circ \mathscr{B}}\rangle_\Phi$$

The Born rule rewrites in the form of pairing:

$$\omega(\underline{\mathscr{A}}) = {}_\Phi\langle\underline{\mathscr{A}^\dagger}|\varrho\rangle_\Phi$$

# GNS construction for representing transformations

Representations $\pi_\Phi$ of transformations $\mathscr{A} \in \mathcal{A}$ over effects $\mathcal{A}/\mathfrak{I}$

$$\pi_\Phi(\mathscr{A})|\underline{\mathscr{B}}\rangle_\Phi \doteq |\underline{\mathscr{A} \circ \mathscr{B}}\rangle_\Phi$$

The Born rule rewrites in the form of pairing:

$$\omega(\underline{\mathscr{A}}) = {}_\Phi\langle \underline{\mathscr{A}^\dagger}|\varrho\rangle_\Phi$$

with representation of states given by

$$\varrho = \underline{\mathscr{T}'}_\omega/\Phi(\underline{\mathscr{T}}_\omega, \mathscr{I})$$

# GNS construction for representing transformations

Representations $\pi_\Phi$ of transformations $\mathscr{A} \in \mathcal{A}$ over effects $\mathcal{A}/\mathfrak{I}$

$$\pi_\Phi(\mathscr{A})|\underline{\mathscr{B}}\rangle_\Phi \doteq |\underline{\mathscr{A} \circ \mathscr{B}}\rangle_\Phi$$

The Born rule rewrites in the form of pairing:

$$\omega(\underline{\mathscr{A}}) = {}_\Phi\langle\underline{\mathscr{A}^\dagger}|\varrho\rangle_\Phi$$

with representation of states given by

$$\varrho = \underline{\mathscr{T}'_\omega}/\Phi(\underline{\mathscr{T}_\omega}, \mathscr{I})$$

The representation of transformations is given by

$$\omega(\underline{\mathscr{B}} \circ \mathscr{A}) = {}_\Phi\langle\underline{\mathscr{B}^\dagger}|\mathscr{A}|\rho\rangle_\Phi :=$$
$${}_\Phi\langle\underline{\mathscr{B}^\dagger}|\mathscr{A} \circ \rho\rangle_\Phi \equiv {}_\Phi\langle\mathscr{A}^\dagger \circ \underline{\mathscr{B}^\dagger}|\rho\rangle_\Phi$$

# GNS construction for representing transformations

Representations $\pi_\Phi$ of transformations $\mathscr{A} \in \mathcal{A}$ over effects $\mathcal{A}/\mathfrak{I}$

$$\pi_\Phi(\mathscr{A})|\underline{\mathscr{B}}\rangle_\Phi \doteq |\underline{\mathscr{A} \circ \mathscr{B}}\rangle_\Phi$$

The Born rule rewrites in the form of pairing:

$$\omega(\underline{\mathscr{A}}) = {}_\Phi\langle\underline{\mathscr{A}^\dagger}|\varrho\rangle_\Phi$$

with representation of states given by

$$\varrho = \underline{\mathscr{T}'_\omega}/\Phi(\underline{\mathscr{T}_\omega}, \mathscr{I})$$

The representation of transformations is given by

$$\omega(\underline{\mathscr{B}} \circ \mathscr{A}) = {}_\Phi\langle\underline{\mathscr{B}^\dagger}|\mathscr{A}|\rho\rangle_\Phi :=$$

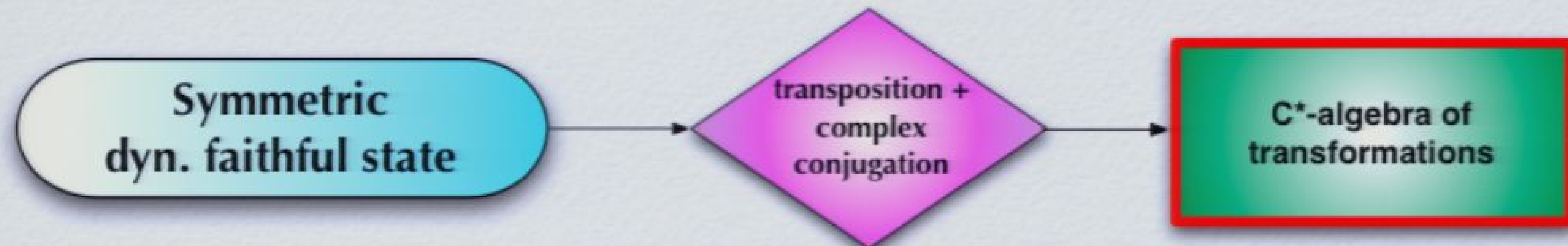
$${}_\Phi\langle\underline{\mathscr{B}^\dagger}|\mathscr{A} \circ \rho\rangle_\Phi \equiv {}_\Phi\langle\mathscr{A}^\dagger \circ \underline{\mathscr{B}^\dagger}|\rho\rangle_\Phi$$

# C*-algebra of transformations

*Postulates*
*Axioms*
*Theorems*

# C*-algebra of transformations

*Postulates*
*Axioms*
*Theorems*



# C*-algebra of transformations

| | | |
|---|---|---|
| state-effect duality | $\dim(\mathfrak{P}) = \dim(\mathfrak{S}) + 1$ | (D1) |
| P2 (prep. faith.) | $\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1$ | ($\mathfrak{T}$) |
| ($\mathfrak{T}$)+GNS | $\dim(\mathfrak{S}^{\times 2}) + 1 = (\dim(\mathfrak{S}) + 1)^2$ | ($\mathfrak{T}_4$)≡(D2) |
| P3 (loc. observability) | $\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$ | (D2) |

# C*-algebra of transformations

| | | |
|---|---|---|
| state-effect duality | $\dim(\mathfrak{P}) = \dim(\mathfrak{S}) + 1$ | (D1) |
| P2 (prep. faith.) | $\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1$ | $(\mathfrak{T})$ |
| $(\mathfrak{T})$+GNS | $\dim(\mathfrak{S}^{\times 2}) + 1 = (\dim(\mathfrak{S}) + 1)^2$ | $(\mathfrak{T}_4)\equiv$(D2) |
| P3 (loc. observability) | $\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$ | (D2) |

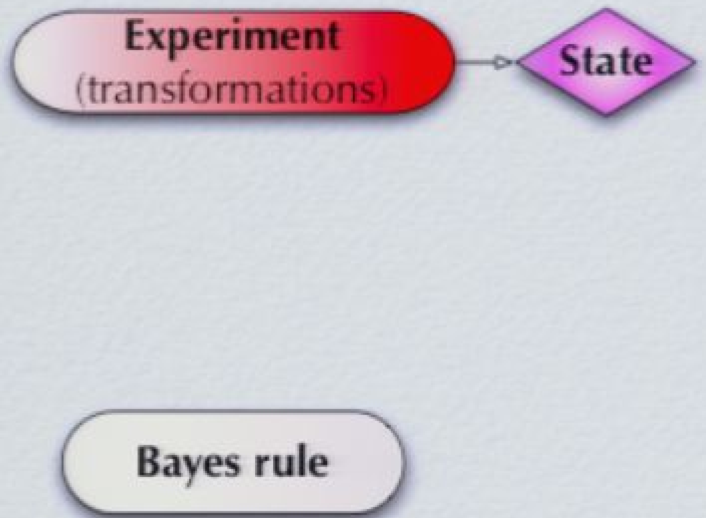
# C*-algebra of transformations

| | | |
|---|---|---|
| state-effect duality | $\dim(\mathfrak{P}) = \dim(\mathfrak{S}) + 1$ | (D1) |
| P2 (prep. faith.) | $\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1$ | ($\mathfrak{T}$) |
| ($\mathfrak{T}$)+GNS | $\dim(\mathfrak{S}^{\times 2}) + 1 = (\dim(\mathfrak{S}) + 1)^2$ | ($\mathfrak{T}_4$)≡(D2) |
| P3 (loc. observability) | $\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$ | (D2) |

quantum information theory group

# Summary

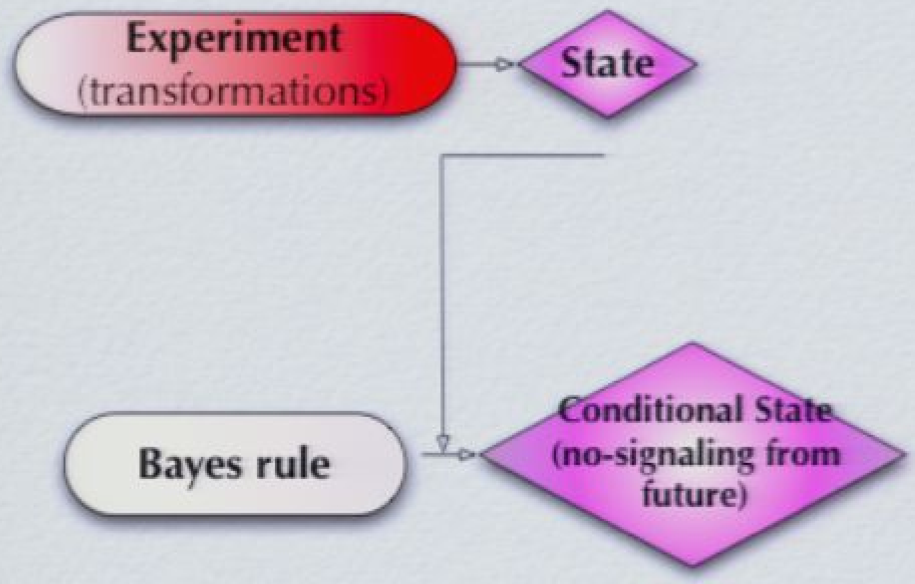
*Postulates*
*Axioms*
*Theorems*




*quant-ph 0611094, 0612162, 0701217, 0701219*

# Summary

*Postulates*
*Axioms*
*Theorems*

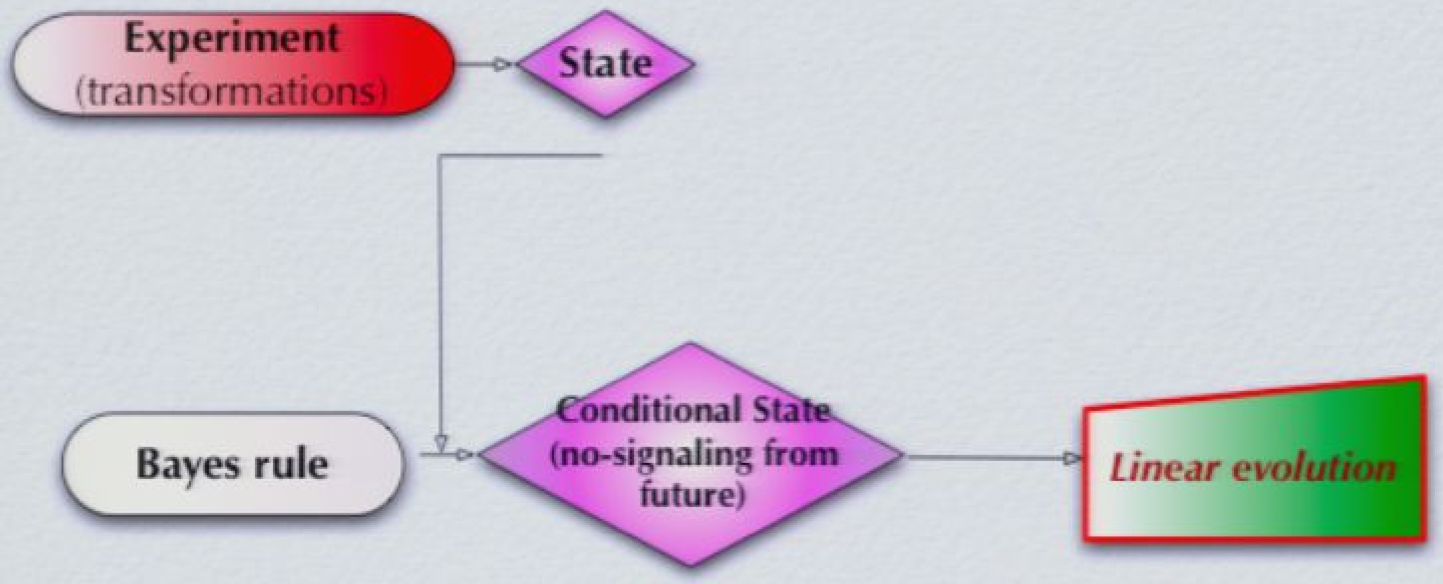



*quant-ph 0611094, 0612162, 0701217, 0701219*

www.qubit.it

# Summary

*Postulates*
*Axioms*
*Theorems*

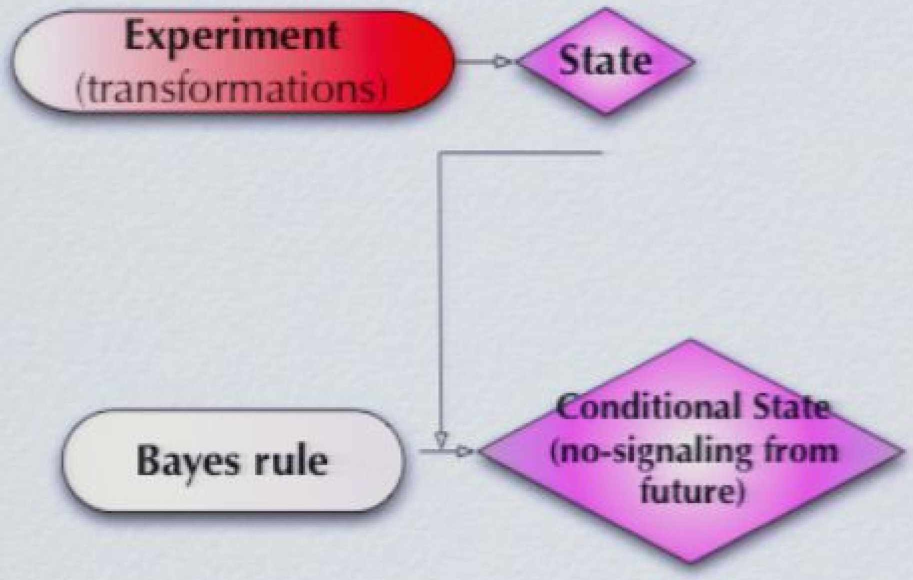



*quant-ph 0611094, 0612162, 0701217, 0701219*

*www.qubit.it*

# Summary

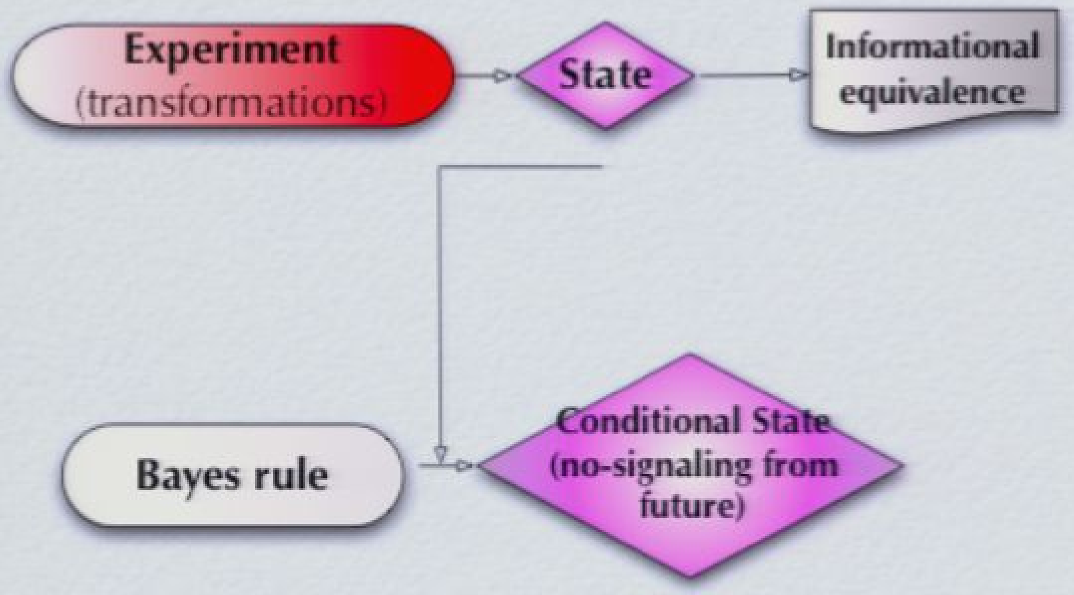
*Postulates*
*Axioms*
*Theorems*




*quant-ph 0611094, 0612162, 0701217, 0701219*

# Summary

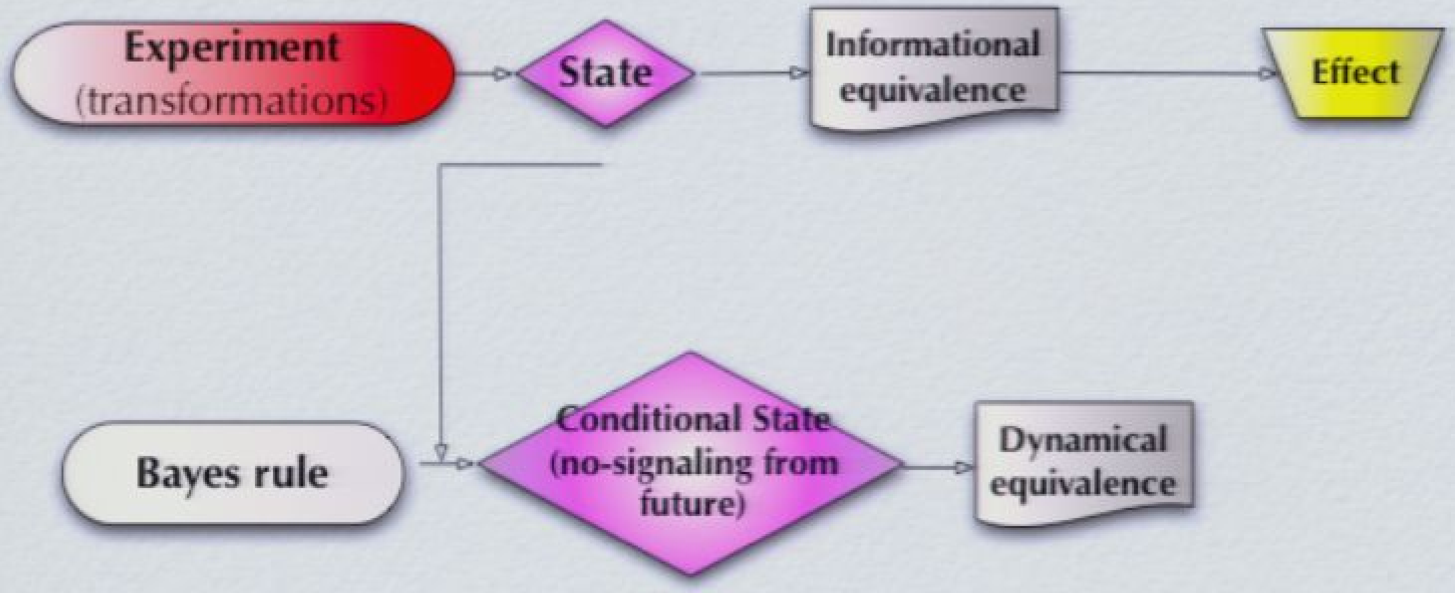
*Postulates*
*Axioms*
*Theorems*




*quant-ph 0611094, 0612162, 0701217, 0701219*

# Summary

*Postulates*
*Axioms*
*Theorems*

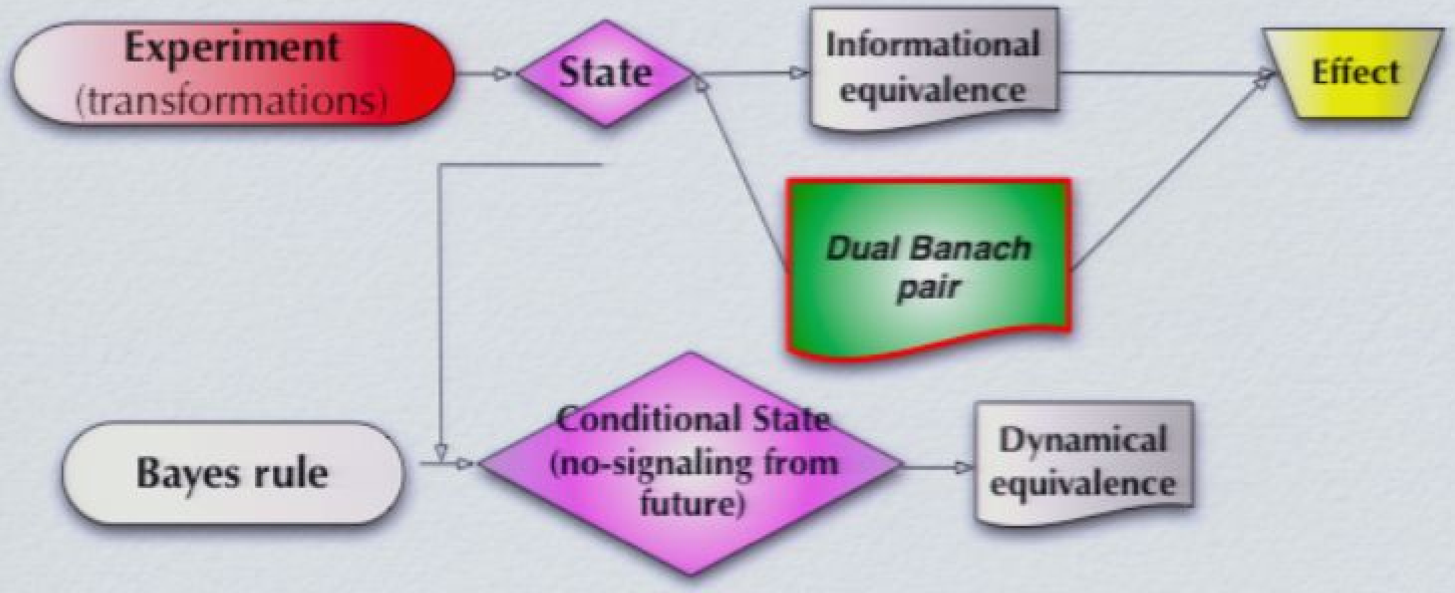



*quant-ph 0611094, 0612162, 0701217, 0701219*

*www.qubit.it*

# Summary

*Postulates*
*Axioms*
*Theorems*

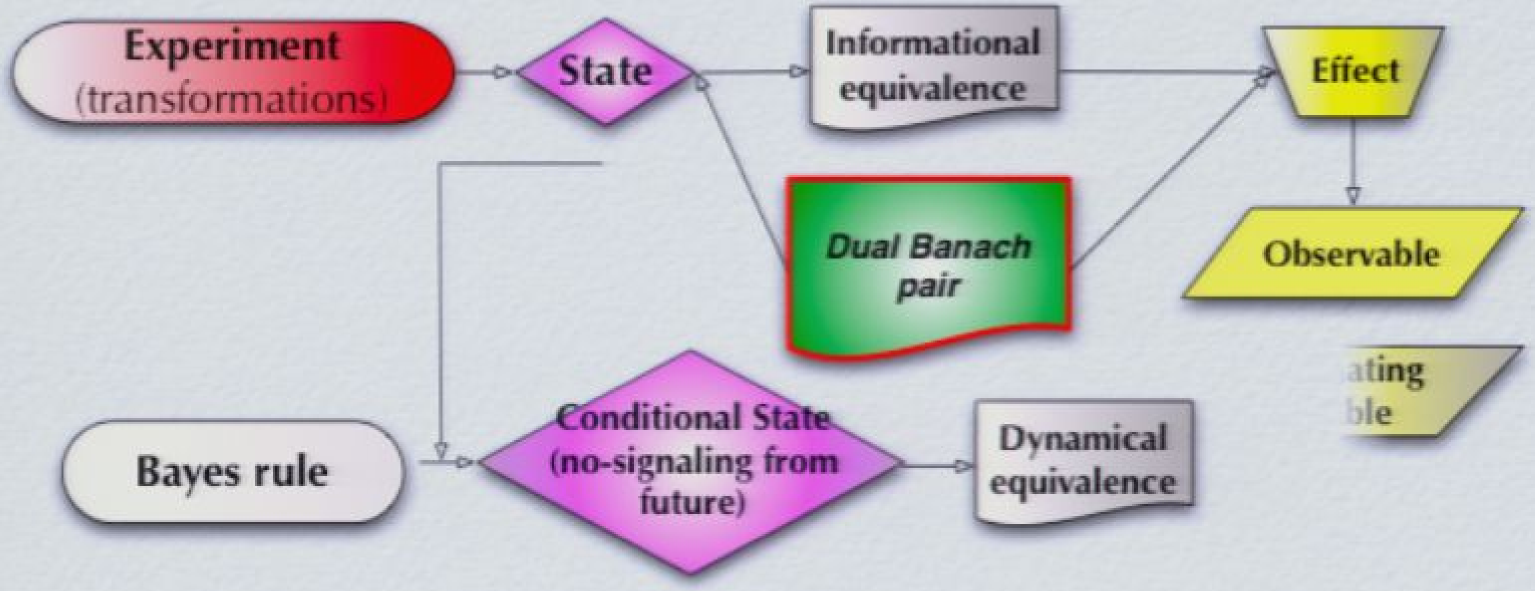



*quant-ph 0611094, 0612162, 0701217, 0701219*

QUit quantum information theory group

# Summary

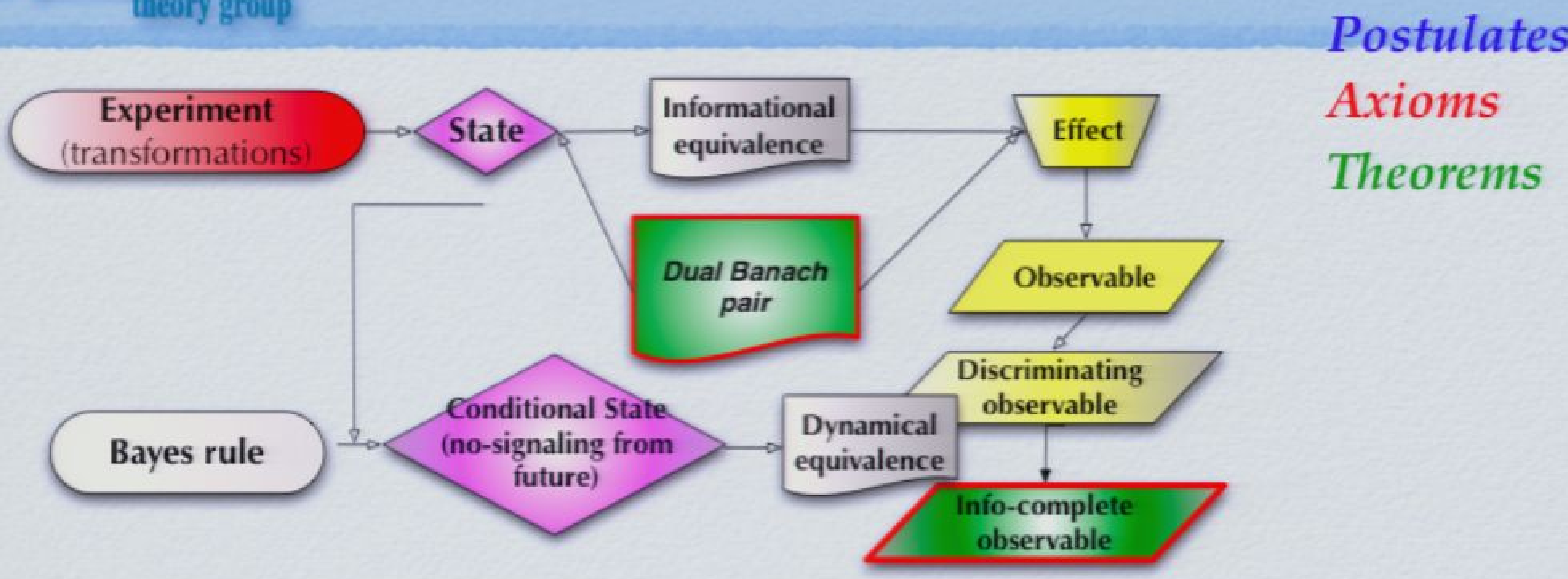
*Postulates*
*Axioms*
*Theorems*




*quant-ph 0611094, 0612162, 0701217, 0701219*

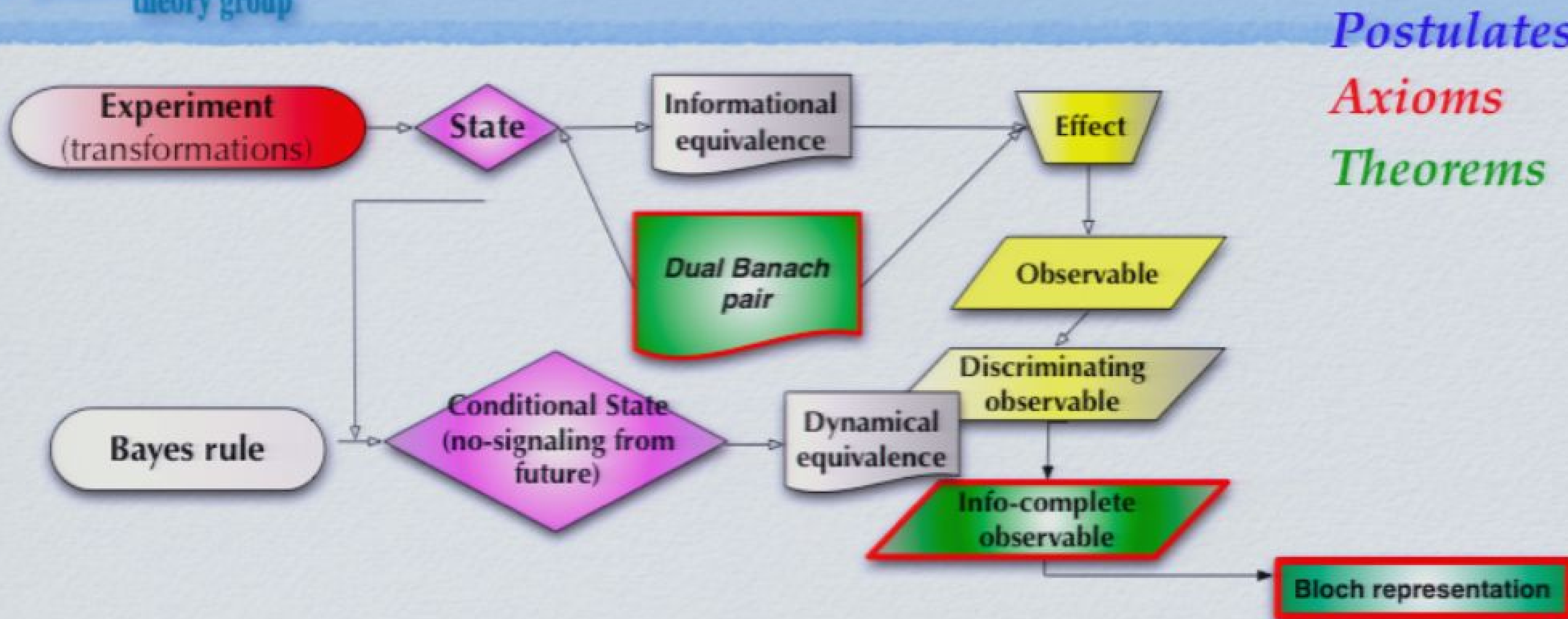
*www.qubit.it*

# Summary




quant-ph 0611094, 0612162, 0701217, 0701219

# Summary

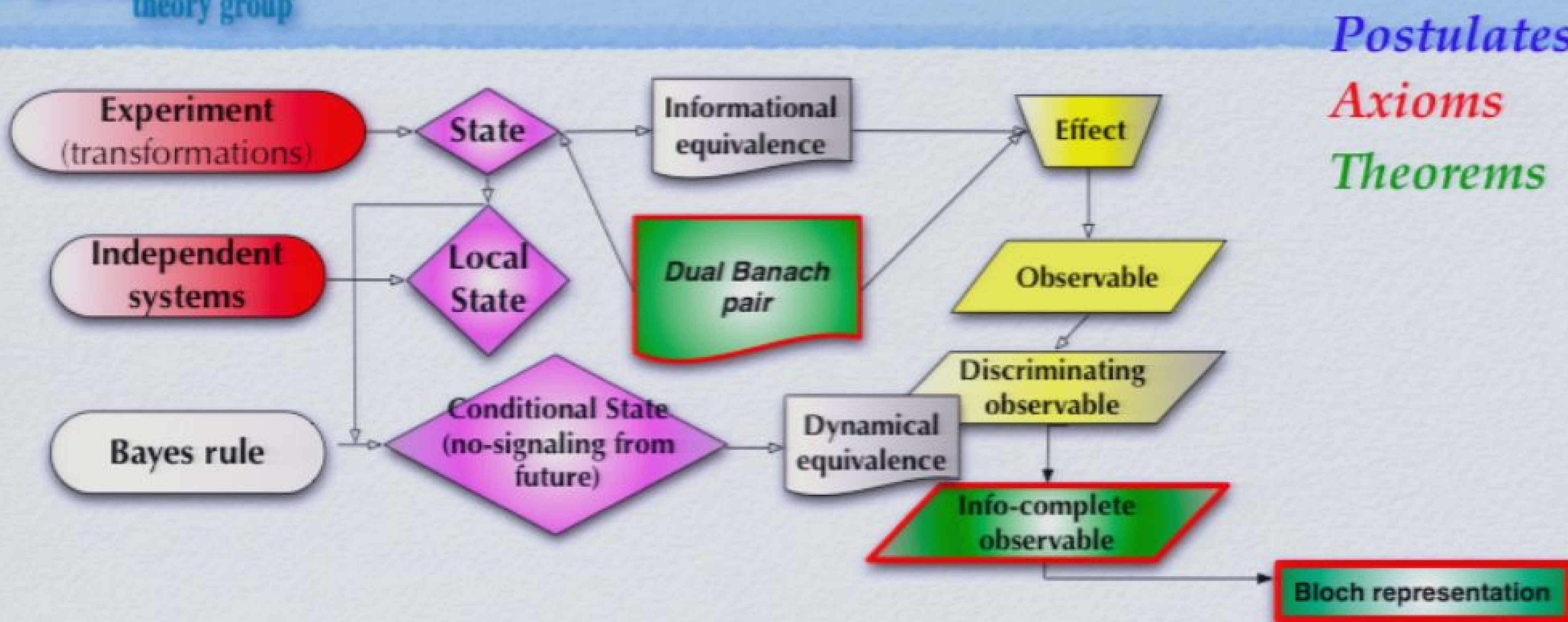



*quant-ph 0611094, 0612162, 0701217, 0701219*

*www.qubit.it*

quantum information theory group

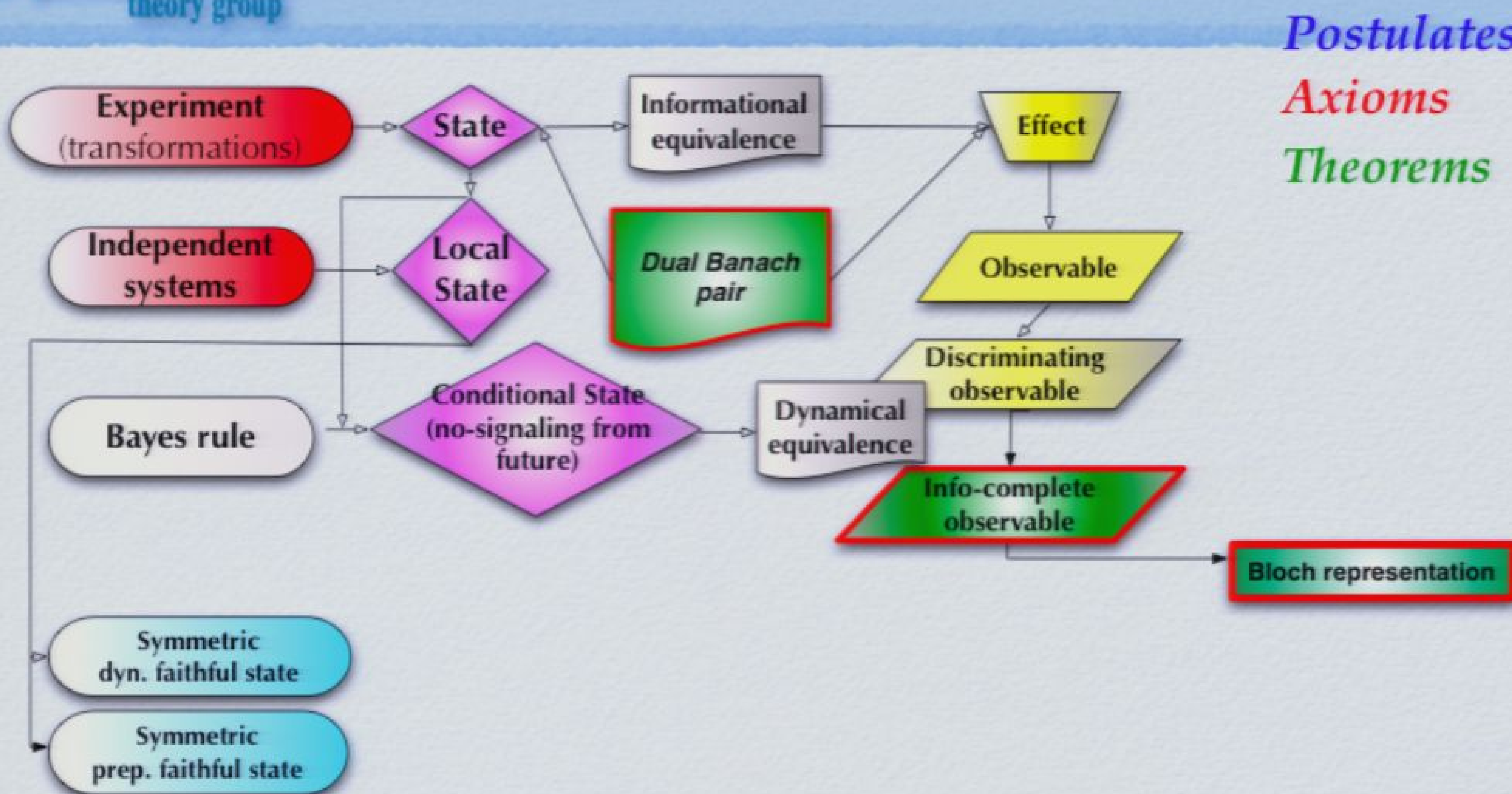
# Summary



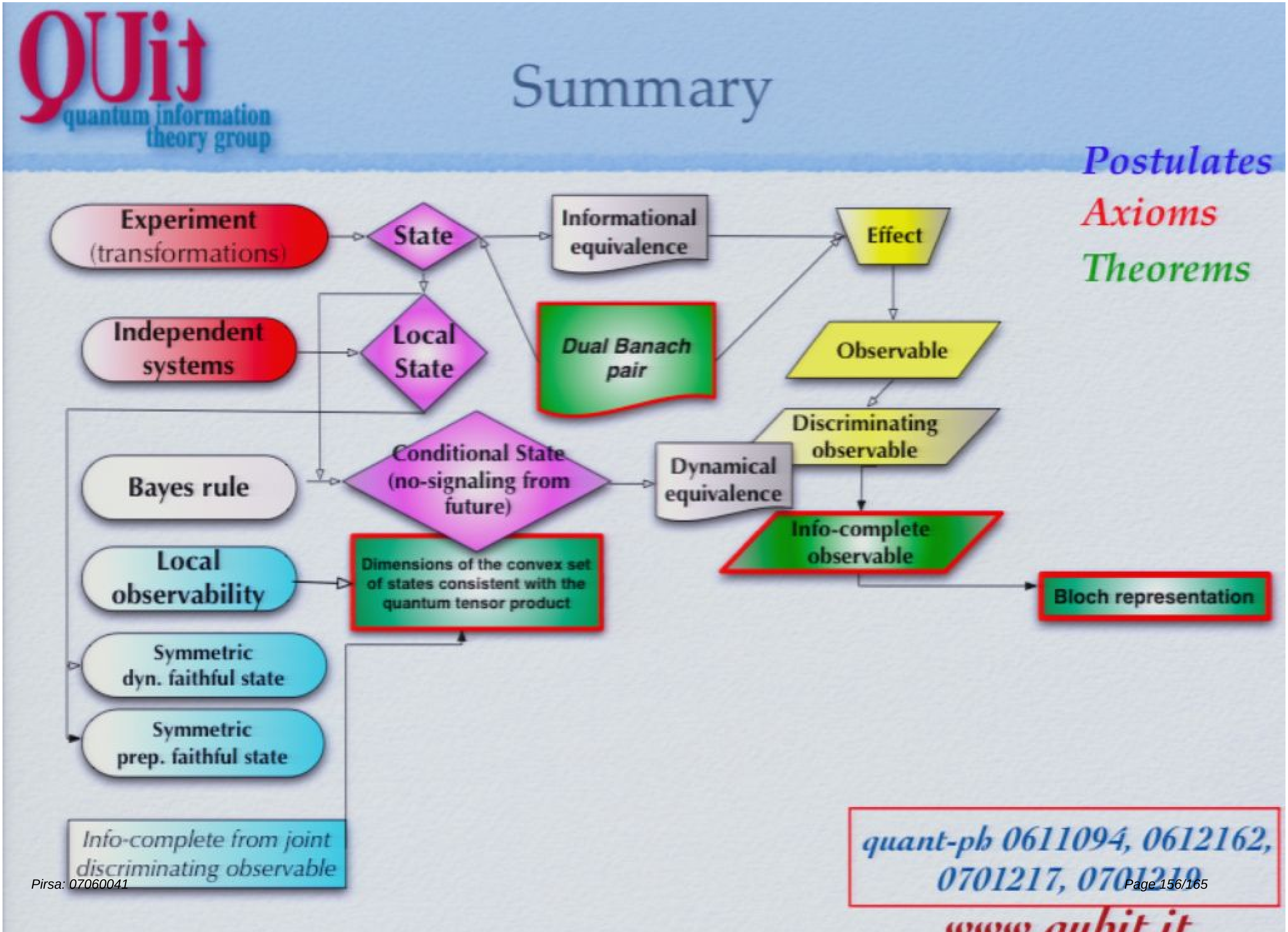

*quant-ph 0611094, 0612162, 0701217, 0701219*

www.qubit.it

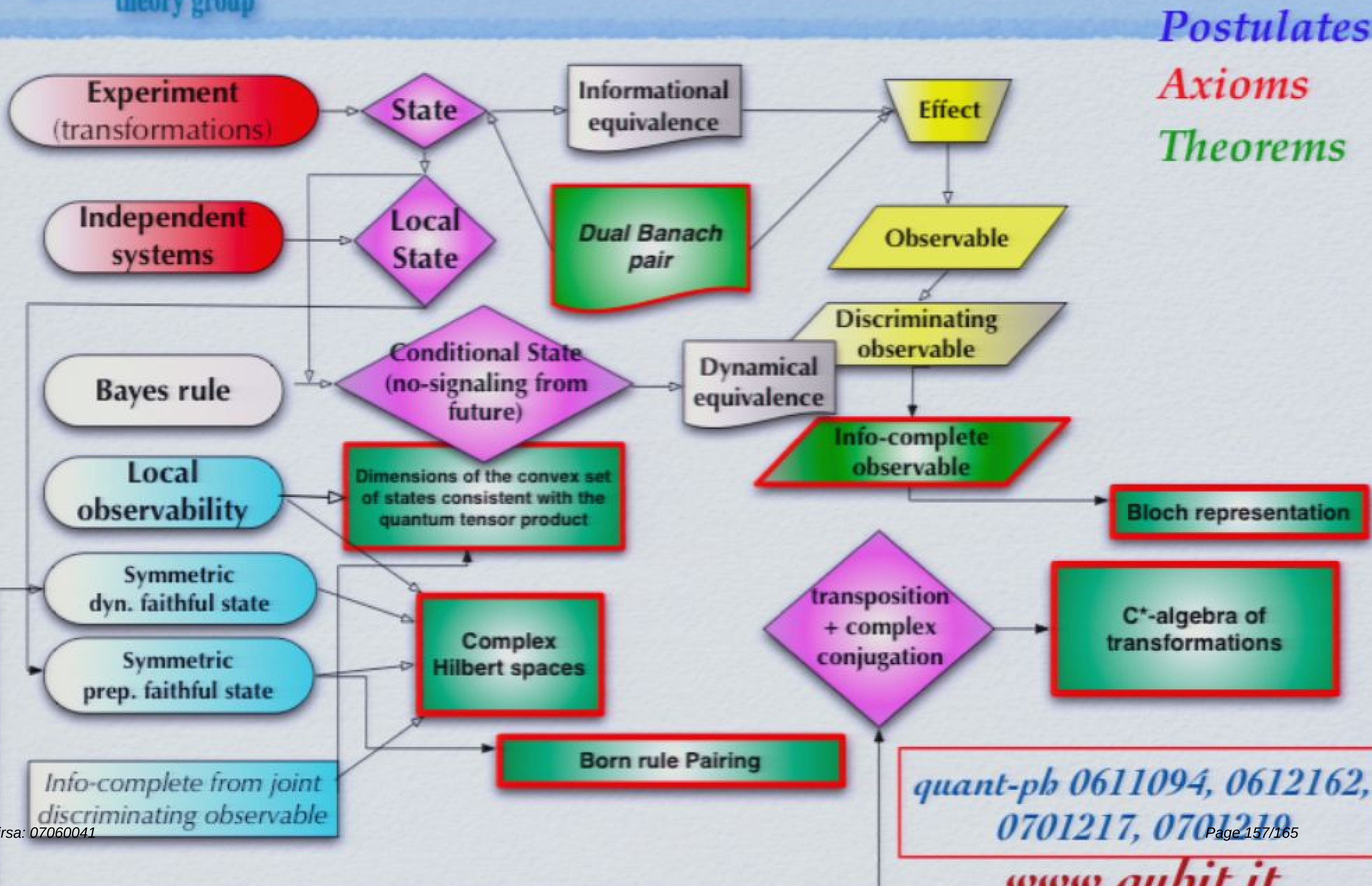
# Summary




*quant-ph 0611094, 0612162, 0701217, 0701219*


QUit
quantum information theory group
Summary
Postulates
Axioms
Theorems
Experiment (transformations)
State
Informational equivalence
Effect
Independent systems
Local State
Dual Banach pair
Observable
Discriminating observable
Conditional State (no-signaling from future)
Bayes rule
Dynamical equivalence
Info-complete observable
Local observability
Dimensions of the convex set of states consistent with the quantum tensor product
Bloch representation
Symmetric dyn. faithful state
Symmetric prep. faithful state
Info-complete from joint discriminating observable
quant-ph 0611094, 0612162, 0701217, 0701219

# Summary



Postulates
Axioms
Theorems
Experiment (transformations)
State
Informational equivalence
Effect
Independent systems
Local State
Dual Banach pair
Observable
Discriminating observable
Bayes rule
Conditional State (no-signaling from future)
Dynamical equivalence
Info-complete observable
Local observability
Dimensions of the convex set of states consistent with the quantum tensor product
Bloch representation
Symmetric dyn. faithful state
Complex Hilbert spaces
transposition + complex conjugation
C*-algebra of transformations
Symmetric prep. faithful state
Born rule Pairing
Info-complete from joint discriminating observable
quant-ph 0611094, 0612162, 0701217, 0701219

# Open problems

$\dim(\mathfrak{P}) = \infty$ Existence of $\varsigma$ (i.e. existence of the decomposition of the Banach space $\mathfrak{P}_{\mathbb{R}}$ into positive and negative parts for the symmetric real form $\Phi$

*quant-ph 0611094, 0612162, 0701217, 0701219*

www.qubit.it

# Open problems

$\dim(\mathfrak{P}) = \infty$ Existence of $\varsigma$ (i.e. existence of the decomposition of the Banach space $\mathfrak{P}_{\mathbb{R}}$ into positive and negative parts for the symmetric real form $\Phi$

$\dim(\mathfrak{P}) \leqslant \infty$ Extrapolation:

$$\dim(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S}^{\times 2})^2 - 1 \implies \dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S})^2 - 1$$

Find a simple postulate discriminating the quantum from the classical C*-algebras

*quant-ph 0611094, 0612162, 0701217, 0701219*

www.qubit.it

# Open problems

$\dim(\mathfrak{P}) = \infty$ Existence of $\varsigma$ (i.e. existence of the decomposition of the Banach space $\mathfrak{P}_{\mathbb{R}}$ into positive and negative parts for the symmetric real form $\Phi$

$\dim(\mathfrak{P}) \leqslant \infty$ Extrapolation:

$$\dim(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S}^{\times 2})^2 - 1 \implies \dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S})^2 - 1$$

Find a simple postulate discriminating the quantum from the classical C*-algebras

Exploit purity of $\Phi$

*quant-ph 0611094, 0612162, 0701217, 0701219*

www.qubit.it

# Postulates (in progress)

- **Postulate 1 (Independent systems)** *There exist **independent** systems.*
- **Postulate 2 (Symmetric faithful state)** *For every composite system made of two identical physical systems there exists a **symmetric joint state** that is both **dynamically** and **preparationally faithful**.*
- **Postulate 3 (Pure symmetric faithful state)** *If there exists a pure symmetric faithful state then we have Quantum Mechanics*

# The complex conjugation

The involution $\varsigma$ corresponds to a generalized transformation

$$\varsigma(\underline{\mathscr{A}}) = \underline{\mathscr{A}} \circ \mathscr{Z}$$

Correspondingly the involution over transformations reads

$$\varsigma(\mathscr{A}) = \mathscr{Z} \circ \mathscr{A} \circ \mathscr{Z}$$

which is composition preserving, namely

$$\varsigma(\mathscr{B} \circ \mathscr{A}) = \varsigma(\mathscr{B}) \circ \varsigma(\mathscr{A}).$$

# The C*-algebra of generalized transformations

Take complex linear combinations of generalized transformations and define $\varsigma(c\mathscr{A}) = c^* \varsigma(\mathscr{A})$ for $c \in \mathbb{C}$

→ c-generalized transformations: $\mathfrak{T}_{\mathbb{C}}$
c-generalized effects: $\mathfrak{P}_{\mathbb{C}}$

complex Banach spaces

complex C*-algebra

# C*-algebra of transformations

| | | |
|---|---|---|
| state-effect duality | $\dim(\mathfrak{P}) = \dim(\mathfrak{S}) + 1$ | (D1) |
| P2 (prep. faith.) | $\dim(\mathfrak{T}) = \dim(\mathfrak{S}^{\times 2}) + 1$ | ($\mathfrak{T}$) |
| ($\mathfrak{T}$)+GNS | $\dim(\mathfrak{S}^{\times 2}) + 1 = (\dim(\mathfrak{S}) + 1)^2$ | ($\mathfrak{T}_4$)≡(D2) |
| P3 (loc. observability) | $\dim(\mathfrak{S}_{12}) = \dim(\mathfrak{S}_1)\dim(\mathfrak{S}_2) + \dim(\mathfrak{S}_1) + \dim(\mathfrak{S}_2)$ | (D2) |

quantum information theory group

# Open problems

$\dim(\mathfrak{P}) = \infty$ Existence of $\varsigma$ (i.e. existence of the decomposition of the Banach space $\mathfrak{P}_{\mathbb{R}}$ into positive and negative parts for the symmetric real form $\Phi$

$\dim(\mathfrak{P}) \leqslant \infty$ Extrapolation:

$$\dim(\mathfrak{S}^{\times 2}) = \dim_{\#}(\mathfrak{S}^{\times 2})^2 - 1 \quad \Longrightarrow \quad \dim(\mathfrak{S}) = \dim_{\#}(\mathfrak{S})^2 - 1$$

Find a simple postulate discriminating the quantum from the classical C*-algebras

Exploit purity of $\Phi$

*quant-ph 0611094, 0612162, 0701217, 0701219*

www.qubit.it