Title: Unsharp reality and the quantum-classical contrast

Date: Jun 05, 2007 11:30 AM

URL: http://pirsa.org/07060040

Abstract:

unsharp reality

and the

quantum-classical contrast

(some observations)

Paul Busch

Workshop at Perimeter Institute for Theoretical Physics

Operational Quantum Physics and the Quantum-Classical Contrast

4-7 June, 2007

PERIMETER INSTITUTE
FOR THEORETICAL PHYSICS

unsharp reality

and the

quantum-classical contrast

(some observations)

Paul Busch

Workshop at Perimeter Institute for Theoretical Physics

Operational Quantum Physics and the Quantum-Classical Contrast

4-7 June, 2007

PERIMETER INSTITUTE
FOR THEORETICAL PHYSICS

OVERVIEW

I *Quantum-classical contrast: the problem (as I see it)*

II *Classical description for quantum system*

III *Quantum description for classical system*

DISCLAIMER

- *NOT* a *systematic* account or solution
- *patchy picture* of where (I think) we are
- some pointers

"Elucidating the role of unsharpness"

In the language of the relativity theory, the content of the relations (2) [the uncertainty relations] may be summarized in the statement that according to the quantum theory a general reciprocal relation exists between the maximum sharpness of definition of the space-time and energy-momentum vectors associated with the individuals. This circumstance may be regarded as a simple symbolical expression for the complementary nature of the space-time description and claims of causality. At the same time, however, the general character of this relation makes it possible to a certain extent to reconcile the conservation laws with the space-time co-ordination of observations, the idea of a coincidence of well-defined events in a space-time point being replaced by that of **unsharply** *defined individuals within finite space-time regions.*

Niels Bohr, 1928

I Quantum-Classical Contrast

Dilemma (for some) / Tension (for all):

(a) Quantum Theory (QT) *supersedes and contradicts* Classical Physical Theory

- hidden-variables problem, Kochen-Specker & Bell Theorems

(b) Quantum measurement *requires* Classical apparatus

- measurement problem: from *indeterminacy* to *definite outcomes*
- *concrete* quantum mechanics builds on Galilei *spacetime*

Options on offer

(1) Quantum-classical theory hierarchy
...plus "fancy" interpretation? (a); ¬(b)

(2) Theory pluralism/network? (b); ¬(a)

(3) Modification of quantum mechanics? ¬(a), ¬(b)

I Quantum-Classical Contrast

Dilemma (for some) / Tension (for all):

(a) Quantum Theory (QT) *supersedes and contradicts* Classical Physical Theory

- hidden-variables problem, Kochen-Specker & Bell Theorems

(b) Quantum measurement *requires* Classical apparatus

- measurement problem: from *indeterminacy* to *definite outcomes*
- *concrete* quantum mechanics builds on Galilei *spacetime*

Options on offer

(1) Quantum-classical theory hierarchy
...plus "fancy" interpretation? (a); ¬(b)

(2) Theory pluralism/network? (b); ¬(a)

(3) Modification of quantum mechanics? ¬(a), ¬(b)

So, *where* is that Quantum-Classical Border?!

And what about the Measurement Problem

potentiality $\rightarrow$ *actuality*

Ad (1): universality of quantum mechanics

("We live in a Quantum World.")

- Many Worlds / Many Minds / Relational / ... interpretations
- nonlocal/contextual hidden variables (e.g., Bohm)
- Consistent histories formulations
- Other no-collapse interpretations (e.g. Modal interpretations)
- Epistemic probability .... instrumentalism

Bub-Clifton Uniqueness Theorem: **Weakening realism**

J. Bub, Interpreting the Quantum World, CUP 1997

Ad (2): theory network

Against (1): *Bohr according to Ludwig*

> *Before further discussions let us again expound this sharply: The author's opinion is that the notion of quantum mechanics as the "most comprehensive" theory is wrong. The "axiomatic basis" presented in this book reflects precisely* the *conception and* the *idea espoused at the beginning of quantum mechanics with an astoundingly clear intuitive view by N. Bohr. ...... The axiomatic basis ... does not allow us to regard the objectivating description of macroscopic systems ... as some approximation to quantum mechancis.*
>
> **G. Ludwig, An Axiomatic Basis for Quantum Mechanics, Vol. 2, 1987, p. 12.**

Classical theories are *pretheories* to quantum mechanics.

Embedding of $\mathfrak{PT}_{\mathrm{m}}$ into $\mathfrak{PT}_{\mathrm{q\,exp}}$ can only be *approximate.*

Ludwig (1987): approximate embedding shown to work at the level of simple model cases...

...program is still alive and well, see, e.g.:

L. Lanz, B. Vacchini, O. Melsheimer,
Quantum theory: the role of microsystems and macrosystems
quant-ph/0701178 / J. Phys. A: Math. Theor. 40 (2007) 3123-3140

BUT NOTE: criterion for recovering dynamics comes from macro-theory → notion of *relevant observables*;
embedding is only meant to demonstrate *consistency*.

Ad (2): theory network

Against (1): *Bohr according to Ludwig*

> *Before further discussions let us again expound this sharply: The author's opinion is that the notion of quantum mechanics as the "most comprehensive" theory is wrong. The "axiomatic basis" presented in this book reflects precisely* the *conception and* the *idea espoused at the beginning of quantum mechanics with an astoundingly clear intuitive view by N. Bohr. ...... The axiomatic basis ... does not allow us to regard the objectivating description of macroscopic systems ... as some approximation to quantum mechancis.*
>
> **G. Ludwig, An Axiomatic Basis for Quantum Mechanics, Vol. 2, 1987, p. 12.**

Classical theories are *pretheories* to quantum mechanics.

Embedding of $\mathfrak{PT}_{\mathrm{m}}$ into $\mathfrak{PT}_{\mathrm{q\,exp}}$ can only be *approximate.*

Ludwig (1987): approximate embedding shown to work at the level of simple model cases...

...program is still alive and well, see, e.g.:

L. Lanz, B. Vacchini, O. Melsheimer,
Quantum theory: the role of microsystems and macrosystems
quant-ph/0701178 / J. Phys. A: Math. Theor. 40 (2007) 3123-3140

BUT NOTE: criterion for recovering dynamics comes from macro-theory → notion of *relevant observables*;
embedding is only meant to demonstrate *consistency*.

Ad (2): theory network

Against (1): *Bohr according to Ludwig*

> *Before further discussions let us again expound this sharply: The author's opinion is that the notion of quantum mechanics as the "most comprehensive" theory is wrong. The "axiomatic basis" presented in this book reflects precisely* the *conception and* the *idea espoused at the beginning of quantum mechanics with an astoundingly clear intuitive view by N. Bohr. ...... The axiomatic basis ... does not allow us to regard the objectivating description of macroscopic systems ... as some approximation to quantum mechancis.*
>
> **G. Ludwig, An Axiomatic Basis for Quantum Mechanics, Vol. 2, 1987, p. 12.**

Classical theories are *pretheories* to quantum mechanics.

Embedding of $\mathfrak{PT}_{\mathrm{m}}$ into $\mathfrak{PT}_{\mathrm{q\,exp}}$ can only be *approximate.*

Ludwig (1987): approximate embedding shown to work at the level of simple model cases...

...program is still alive and well, see, e.g.:

L. Lanz, B. Vacchini, O. Melsheimer,
Quantum theory: the role of microsystems and macrosystems
quant-ph/0701178 / J. Phys. A: Math. Theor. 40 (2007) 3123-3140

BUT NOTE: criterion for recovering dynamics comes from macro-theory → notion of *relevant observables*;
embedding is only meant to demonstrate *consistency*.

Ad (3): modifying quantum mechanics

- spontaneous collapse / dynamical reduction models
- gravity-induced collapse conjecture
- some *histories* extensions of quantum theory

Maintaining **STRONG REALISM** and **UNIVERSALITY** whilst incorporating new modality of **POTENTIALITY**

Game still wide open …

… so where to take it?

→ *structural/conceptual comparison*

→ *the role of unsharpness*

Quantumness vs Classicality

Focus here:

operational		*formal*
disturbance/ limit of joint measurability	$\longleftrightarrow$	noncommutativity
indeterminacy	$\longleftrightarrow$	superposition

II Classical description of a quantum system

Uniqueness Theorem:
There is (essentially) only one "good" classical representation of a quantum probabilistic theory.

Operational point of view

quantum and classical statistical models (dualities)

$$\left.\begin{array}{l} \mathcal{S} \ - \ \text{set of } \textit{states} \\ \mathcal{E} \ - \ \text{set of } \textit{effects} \end{array}\right\} \quad \mathcal{S} \times \mathcal{E} \ni (\rho, E) \mapsto \mathsf{p}_\rho(E)$$

classical:

$$\mathcal{S}_c = M(\Omega, \mathcal{B}(\Omega))_1^+$$

$$\mathcal{E}_c = \{f : \Omega \to [0,1] \, : \, f \text{ measurable}\}$$

$$\mathsf{p}_\mu(f) = \int_\Omega f \, d\mu$$

– follows from compatibility of all sharp effects

quantum:

$$\mathcal{S}_q = \{\text{density operators}\}$$

$$\mathcal{E}_q = \{\text{quantum effects}\}$$

$$\mathsf{p}_\rho(E) = \mathrm{tr}\,[\rho\, E]$$

– follows from complementarity postulate and existence of ideal measurements.

(Bugajski & Lahti 1985)

classical embedding

$\Phi : \mathcal{S}_q \to \mathcal{S}_c$ (affine); $\Phi' : \mathcal{E}_c \to \mathcal{E}_q$ ("quantization")

Want: coverage of *all* quantum effects $\Rightarrow$ need: Φ injective.

Unique family of solutions: **informationally complete observables**

$$A : \mathcal{B}(\Omega) \to \mathcal{E}_q$$

$$\mathcal{S}_q \ni \rho \mapsto \Phi_A(\rho) \equiv \mathsf{p}_\rho^A \in M(\Omega, \mathcal{B})_1^+ = \mathcal{S}_c$$

$$\mathsf{p}_\rho^A(X) = \int_\Omega \chi_X(\omega) d\mathsf{p}_\rho^A(\omega) = \mathrm{tr}\,[\rho A(X)]$$

$$\mathcal{E}_c \ni \chi_X \mapsto \Phi'(\chi_X) := A(X) \in \mathcal{E}_q$$

However: representation is partial: Φ' is *not* surjective.

(Cf. Wigner function.)

classical extension

$\Psi : \mathcal{S}_c \to \mathcal{S}_q$ (affine "reduction" map); $\Psi' : \mathcal{E}_q \to \mathcal{E}_c$

Want: coverage of *all* quantum states $\Rightarrow$ need: Ψ surjective.

Solution: **Misra-Bugajski map**

$\Omega =$ subset of pure states ω of $\mathcal{S}_q$

$\mathcal{S}_c = M(\Omega, \mathcal{B}(\Omega))_1^+$

$$M(\Omega, \mathcal{B}(\Omega))_1^+ \ni \mu = \int_\Omega \delta_\omega \, d\mu(\omega) \mapsto \Psi(\mu) := \int_\Omega \omega \, d\mu(\omega) \equiv \rho_\mu \in \mathcal{S}_q$$

$$\mathrm{tr}\,[\rho_\mu E] = \int_\Omega \mathrm{tr}\,[\omega E] \, d\mu(\omega) = \int_\Omega f_E(\omega) \, d\mu(\omega)$$

$$\mathcal{E}_q \ni E \mapsto \Psi'(E) = f_E \in \mathcal{E}_c$$

This is the (essentially) unique *non-redundant* solution.

classical:

$$\left.\begin{array}{l} \mathcal{S}_c = M(\Omega, \mathcal{B}(\Omega))_1^+ \\ \mathcal{E}_c = \{f : \Omega \to [0,1] \ : \ f \text{ measurable}\} \end{array}\right\} \quad \mathsf{p}_\mu(f) = \int_\Omega f \, d\mu$$

– follows from compatibility of all sharp effects

quantum:

$$\left.\begin{array}{l} \mathcal{S}_q = \{\text{density operators}\} \\ \mathcal{E}_q = \{\text{quantum effects}\} \end{array}\right\} \quad \mathsf{p}_\rho(E) = \mathrm{tr}\,[\rho\, E]$$

– follows from complementarity postulate and existence of ideal measurements.

(Bugajski & Lahti 1985)

classical embedding

$$\Phi : \mathcal{S}_q \to \mathcal{S}_c \text{ (affine)}; \; \Phi' : \mathcal{E}_c \to \mathcal{E}_q \text{ ("quantization")}$$

Want: coverage of *all* quantum effects ⇒ need: Φ injective.

Unique family of solutions: **informationally complete observables**

$$A : \mathcal{B}(\Omega) \to \mathcal{E}_q$$

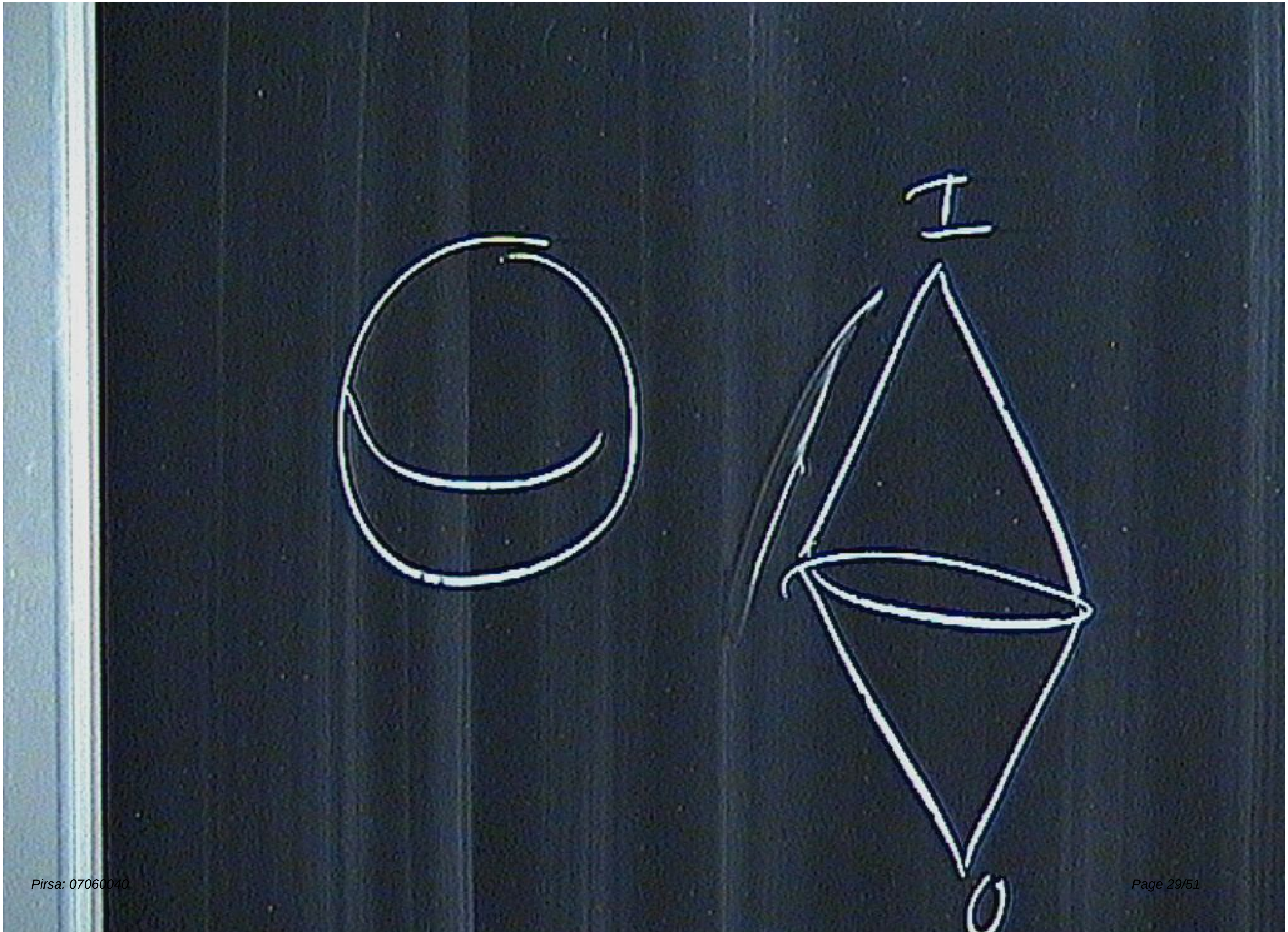
$$\mathcal{S}_q \ni \rho \mapsto \Phi_A(\rho) \equiv \mathsf{p}_\rho^A \in M(\Omega, \mathcal{B})_1^+ = \mathcal{S}_c$$

$$\mathsf{p}_\rho^A(X) = \int_\Omega \chi_X(\omega) d\mathsf{p}_\rho^A(\omega) = \mathrm{tr}\,[\rho A(X)]$$

$$\mathcal{E}_c \ni \chi_X \mapsto \Phi'(\chi_X) := A(X) \in \mathcal{E}_q$$

However: representation is partial: Φ' is *not* surjective.

(Cf. Wigner function.)


I
0

classical embedding

$\Phi : \mathcal{S}_q \to \mathcal{S}_c$ (affine); $\Phi' : \mathcal{E}_c \to \mathcal{E}_q$ ("quantization")

Want: coverage of *all* quantum effects $\Rightarrow$ need: Φ injective.
Unique family of solutions: **informationally complete observables**

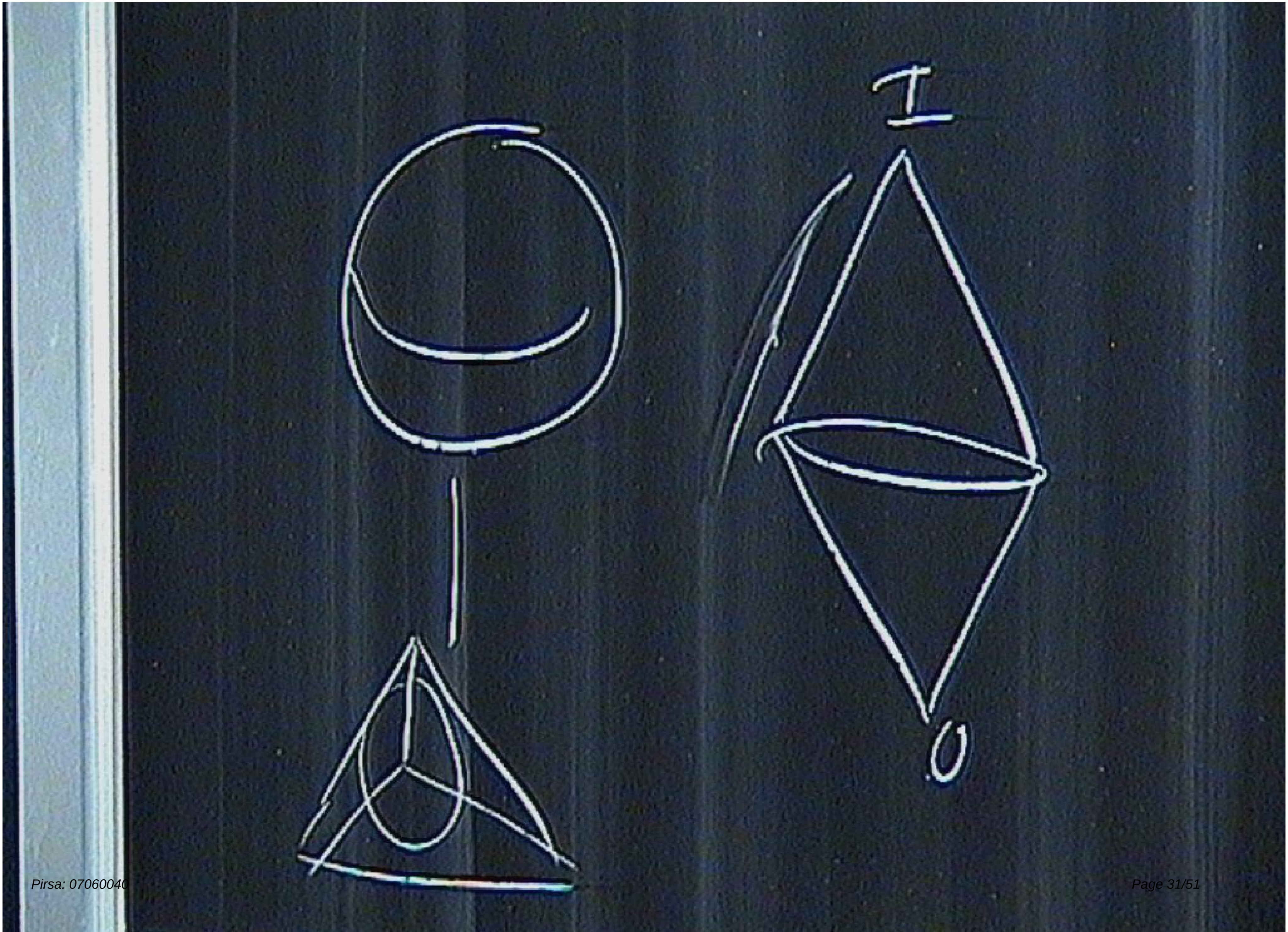
$$A : \mathcal{B}(\Omega) \to \mathcal{E}_q$$

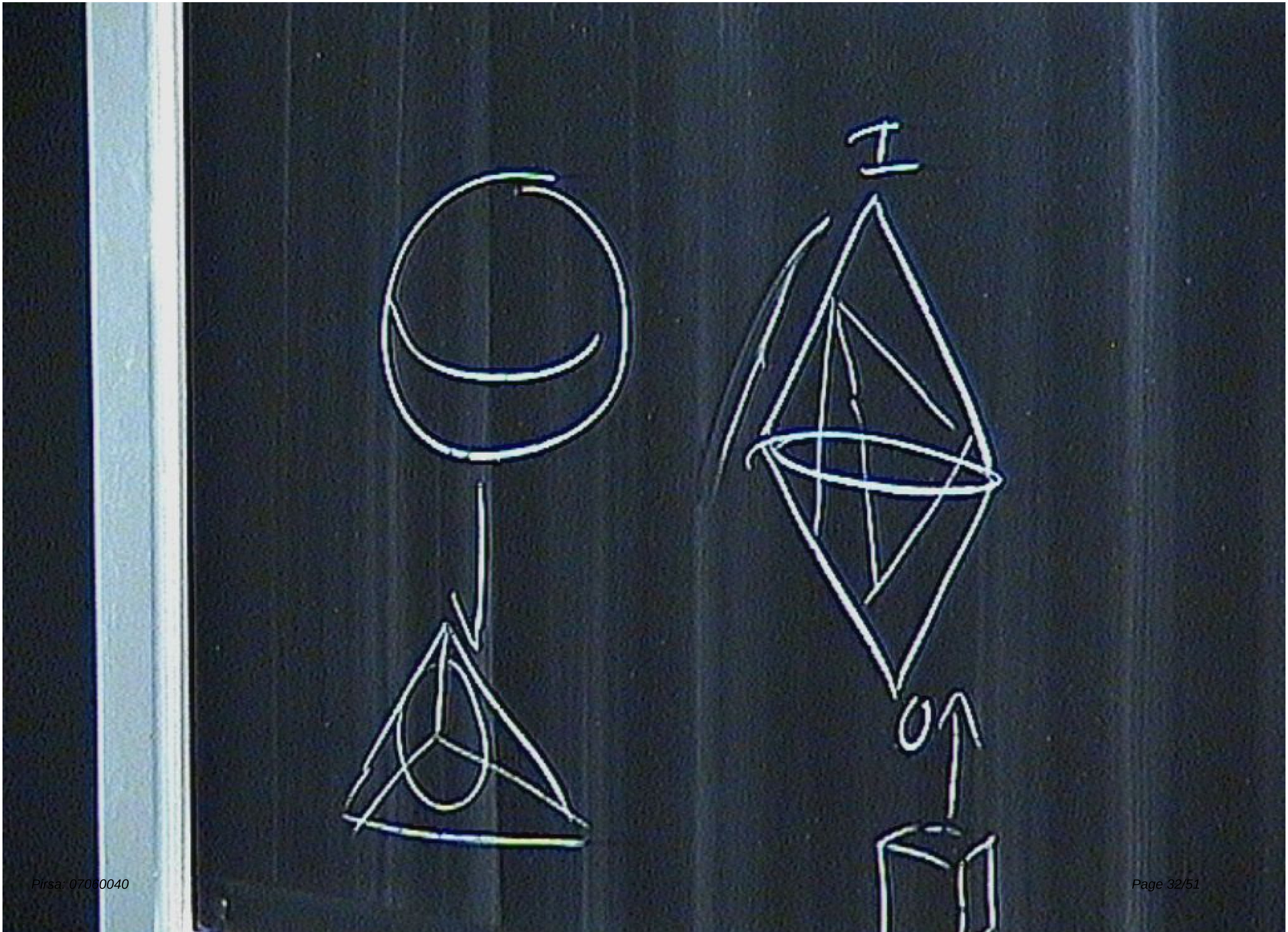
$$\mathcal{S}_q \ni \rho \mapsto \Phi_A(\rho) \equiv \mathsf{p}_\rho^A \in M(\Omega, \mathcal{B})_1^+ = \mathcal{S}_c$$

$$\mathsf{p}_\rho^A(X) = \int_\Omega \chi_X(\omega) d\mathsf{p}_\rho^A(\omega) = \mathrm{tr}\,[\rho A(X)]$$

$$\mathcal{E}_c \ni \chi_X \mapsto \Phi'(\chi_X) := A(X) \in \mathcal{E}_q$$

However: representation is partial: Φ' is *not* surjective.
(Cf. Wigner function.)


I
0


I
0

classical embedding

$\Phi : \mathcal{S}_q \to \mathcal{S}_c$ (affine); $\Phi' : \mathcal{E}_c \to \mathcal{E}_q$ ("quantization")

Want: coverage of *all* quantum effects $\Rightarrow$ need: Φ injective.

Unique family of solutions: **informationally complete observables**

$$A : \mathcal{B}(\Omega) \to \mathcal{E}_q$$

$$\mathcal{S}_q \ni \rho \mapsto \Phi_A(\rho) \equiv \mathsf{p}_\rho^A \in M(\Omega, \mathcal{B})_1^+ = \mathcal{S}_c$$

$$\mathsf{p}_\rho^A(X) = \int_\Omega \chi_X(\omega) d\mathsf{p}_\rho^A(\omega) = \mathrm{tr}\,[\rho A(X)]$$

$$\mathcal{E}_c \ni \chi_X \mapsto \Phi'(\chi_X) := A(X) \in \mathcal{E}_q$$

However: representation is partial: Φ' is *not* surjective.

(Cf. Wigner function.)

classical extension

$\Psi : \mathcal{S}_c \to \mathcal{S}_q$ (affine "reduction" map); $\Psi' : \mathcal{E}_q \to \mathcal{E}_c$

Want: coverage of *all* quantum states $\Rightarrow$ need: Ψ surjective.

Solution: Misra-Bugajski map

Ω= subset of pure states ω of $\mathcal{S}_q$
$\mathcal{S}_c = M(\Omega, \mathcal{B}(\Omega))_1^+$

$$M(\Omega, \mathcal{B}(\Omega))_1^+ \ni \mu = \int_\Omega \delta_\omega \, d\mu(\omega) \mapsto \Psi(\mu) := \int_\Omega \omega \, d\mu(\omega) \equiv \rho_\mu \in \mathcal{S}_q$$

$$\mathrm{tr}\,[\rho_\mu E] = \int_\Omega \mathrm{tr}\,[\omega E] \, d\mu(\omega) = \int_\Omega f_E(\omega) \, d\mu(\omega)$$

$$\mathcal{E}_q \ni E \mapsto \Psi'(E) = f_E \in \mathcal{E}_c$$

This is the (essentially) unique *non-redundant* solution.

THEOREM (PB & W. Stulpe 2007):
Any reduction map Ψ with the property $\mathcal{S}_{q,\text{pure}} = \{\Psi\delta_\omega : \omega \in \Omega\}$ can be represented according to

$$\operatorname{tr}[\Psi(\mu)E] = \int_\Omega \operatorname{tr}[\omega E] \ (\mu \circ i^{-1})(d\omega)$$

where $\mu \in \mathcal{S}_c$, $E \in \mathcal{E}_q$, $i : \Omega \to \mathcal{S}_{q,\text{pure}}$ is the mapping $\omega \mapsto i(\omega) = \Psi\delta_\omega$, and $\mu \circ i^{-1}$ the image measure.

Implication:

All quantum effects *are* fuzzy classical effects.

THEOREM (PB & W. Stulpe 2007):
Any reduction map Ψ with the property $\mathcal{S}_{q,\text{pure}} = \{\Psi\delta_\omega : \omega \in \Omega\}$ can be represented according to

$$\text{tr}\,[\Psi(\mu)E] = \int_\Omega \text{tr}\,[\omega E]\;\;(\mu \circ i^{-1})(d\omega)$$

where $\mu \in \mathcal{S}_c$, $E \in \mathcal{E}_q$, $i : \Omega \to \mathcal{S}_{q,\text{pure}}$ is the mapping $\omega \mapsto i(\omega) = \Psi\delta_\omega$, and $\mu \circ i^{-1}$ the image measure.

Implication:

All quantum effects *are* fuzzy classical effects.

III Quantum description of a classical system

Unsharpness helps to restore (some – but not all) classicality.

(1) (approximate) joint measurability

(2) noninvasive measurability

Ad (1): joint measurability

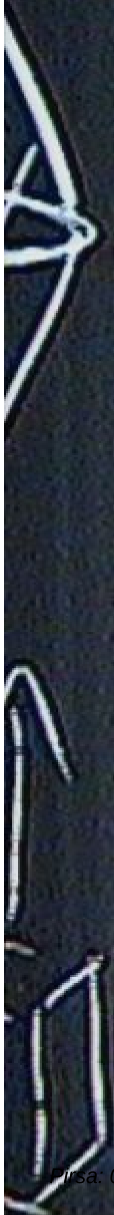
Example: position Q and momentum P
minimal operational requirement: *joint probability*

$$\mathcal{B}(\mathbb{R}^2) \ni X \times Y \mapsto \mathsf{p}_\rho(X \times Y) = \mathrm{tr}\,[\rho G(X \times Y)]$$

von Neumann (1932): *joint measurability* $\Leftrightarrow$ *commutativity*; discrete approximate solution: phase space lattice of coherent states

Wigner (1932): requires $G(X \times \mathbb{R}) = \mathsf{Q}(X)$, $G(\mathbb{R} \times Y) = \mathsf{P}(Y)$; finds "Wigner function", i.e., $G(X \times Y)$ *not* positive.

Husimi (1939): discovers coherent-state based phase space observable



$$(q,p) \overset{\rho}{\mapsto} \frac{\mathrm{tr}\left[\rho\, |\eta_{qp}\rangle\langle\eta_{qp}|\right]}{2\pi\hbar}$$

Ad (1): joint measurability

Example: position Q and momentum P
minimal operational requirement: *joint probability*

$$\mathcal{B}(\mathbb{R}^2) \ni X \times Y \mapsto \mathsf{p}_\rho(X \times Y) = \mathsf{tr}\,[\rho G(X \times Y)]$$

von Neumann (1932): *joint measurability* $\Leftrightarrow$ *commutativity*; discrete approximate solution: phase space lattice of coherent states

Wigner (1932): requires $G(X \times \mathbb{R}) = \mathsf{Q}(X)$, $G(\mathbb{R} \times Y) = \mathsf{P}(Y)$; finds "Wigner function", i.e., $G(X \times Y)$ *not* positive.

Husimi (1939): discovers coherent-state based phase space observable

General phase space observable

Requirements: $G : \mathcal{B}(\mathbb{R}^2) \to \mathcal{E}_q$ is an *observable* (POVM) *covariant* under translations and boosts:

$$W(q,p)G(Z)W(q,p)^* = G(Z + (q,p))$$

Unique class of solutions:

$$\mathcal{B}(\mathbb{R}^2) \ni Z \mapsto G^T(Z) = \frac{1}{2\pi\hbar} \int_Z W(q,p)TW(q,p)^* \, dqdp,$$

($T =$ a positive operator of trace 1)

Davies 1976; Ali, Prugovecki 1978; Holevo 1982; Werner 1984; Cassinelli et al 2004; Kiukas, Lahti, Ylinen 2006

Marginal observables

$$G^T(X\times\mathbb{R}) = \mathsf{Q}_\mu(X) = (\mu*\mathsf{Q})(X), \quad G^T(\mathbb{R}\times Y) = \mathsf{P}_\nu(Y) = (\nu*\mathsf{P})(Y).$$

$\mu = \mu_T = \mathsf{p}^{\mathsf{Q}}_{\Pi T \Pi^*}$ $\quad \nu = \nu_T = \mathsf{p}^{\mathsf{P}}_{\Pi T \Pi^*}$ $\quad \Pi$=parity operator

joint measurability $\Rightarrow$ unsharpness

Ad (2): noninvasive measurability

We are just *learning* to quantify the *disturbance vs inaccuracy* trade-off!

Example of particle: model of phase-space measurement

von Neumann 1932; Arthurs & Kelly 1965

$$U = \exp\left(-\frac{i\lambda}{\hbar}\hat{Q}\otimes\hat{P}_1\otimes\mathbb{1}_2 + \frac{i\kappa}{\hbar}\hat{P}\otimes\mathbb{1}_1\otimes\hat{Q}_2\right).$$

$$\langle\psi|G^T(X\times Y)|\psi\rangle := \langle U\psi\otimes\Psi_1\otimes\Psi_2|\mathbb{1}\otimes \mathsf{Q}_1(\lambda X)\otimes\mathsf{P}_2(\kappa Y)|U\psi\otimes\Psi_1\otimes\Psi_2\rangle.$$

$$\Delta(\mu)^2 = \frac{1}{\lambda^2}\Delta(\hat{Q}_1,\Psi_1)^2 + \frac{\kappa^2}{4}\Delta(\hat{Q}_2,\Psi_2)^2,$$

$$\Delta(\nu)^2 = \frac{1}{\kappa^2}\Delta(\hat{P}_2,\Psi_2)^2 + \frac{\lambda^2}{4}\Delta(\hat{P}_1,\Psi_1)^2.$$

$$\mathcal{I}_X(P[\psi]) = \mathrm{tr}_{1,2}[\mathbb{1}\otimes\mathsf{Q}_1(\lambda X)\otimes\mathsf{P}_2(\kappa Y)P[U\psi\otimes\Psi_1\otimes\Psi_2]]$$

nondisturbance $\Rightarrow$ *macroscopic inaccuracy*

Objective description – definite values

criterion of physical reality:

If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to that physical quantity.

Einstein, Podolsky, Rosen (1935)

$$\exists \mathcal{I} : \; \mathrm{tr}\,[\mathcal{I}_X(\rho)] = 1 \;\Rightarrow\; \mathcal{I}_X(\rho) = \rho$$

Realization for sharp observable ($E(X) = P$ projection): $\mathcal{I}_X(\rho) = P\rho P$
→ ideal measurements; these are also *repeatable* (undesirable!)

Approximate realization: $E(X) = E$ effect, $\mathcal{I}_X(\rho) = E^{1/2}\rho E^{1/2}$

→ *approximate* ideality; lose repeatability!

more unsharpness ⇒ less disturbance (and less information)

So, what's missing?

"definite values":

- rules out 'lots of' states → loss of *superposition principle*
- can only be realized for 'a few' (relevant macroscopic) observables at a time
- will, in any case, have to be 'unsharp' for typical observables (phase space)

measurement – actualization of potentialities:

→ insolubility of the measurement problem (Wigner, Shimony, Fine) extends to unsharp object and pointer observables (PB & A. Shimony 1996; Bassi & Ghirardi 2000/2003; Grubl 2003)

Conclusion

- understanding of *quantum-classical contrast* and *border* is inextricably linked with decision on interpretational stance on quantum mechanics
- quantum probabilistic theory can be presented in only one *non-redundant* way as a *restricted classical theory*; the restriction being that all quantum observables *are* fuzzy classical observables
- *kinematic* aspects of a classical system can be approximated in quantum mechanical terms; but this requires the representation of classical observables by *unsharp* quantum observables
- problem of quantum modeling of classical *dynamics* not addressed; but see the *Insolubility Theorem* of quantum measurement theory: unsharpness is *not* sufficient

$$0 \leq E \leq \mathbb{1}$$

$$S(E) = W(E) - W(EE')$$

$$E' = \mathbb{1} - E$$

$$0 \le E \le \mathbb{1}$$

$$S(E) = W(E) - W(EE')$$

$$E' = \mathbb{1} - E$$

$$EE' \quad E, E'$$

$$0 \le E \le \mathbb{1} \qquad \in \{0,1\}$$

$$S(E) = W(E) - W(EE')$$

$$E' = \mathbb{1} - E$$

$$EE' \le E, E'$$

$$S(E) = 1 \iff E = E^2$$

$$S(E) = 0 \iff E = \lambda \mathbb{1} \qquad \lambda > 0$$