Title: Categorizing nonclassical phenomena: the explanatory power of epistemic restrictions and contextuality

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Abstract:

Categorizing nonclassical phenomena:

the explanatory power of epistemic restrictions and contextuality

Robert Spekkens

Department of Applied Mathematics and Theoretical Physics

University of Cambridge

June 4, 2007

Operational Q physics and and the Q-C contrast, PI

Funding by: The Royal Society

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Much recent foundations work suggests (to me at least) the following foundational principle for quantum theory:

Maximal information about reality is incomplete information

Caves and Fuchs, quant-ph/9601025
Rovelli, quant-ph/9609002
Hardy, quant-ph/9906123
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Spekkens, quant-ph/0401052
Grinbaum, quant-ph/0509106

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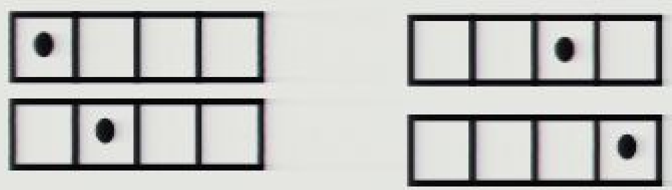
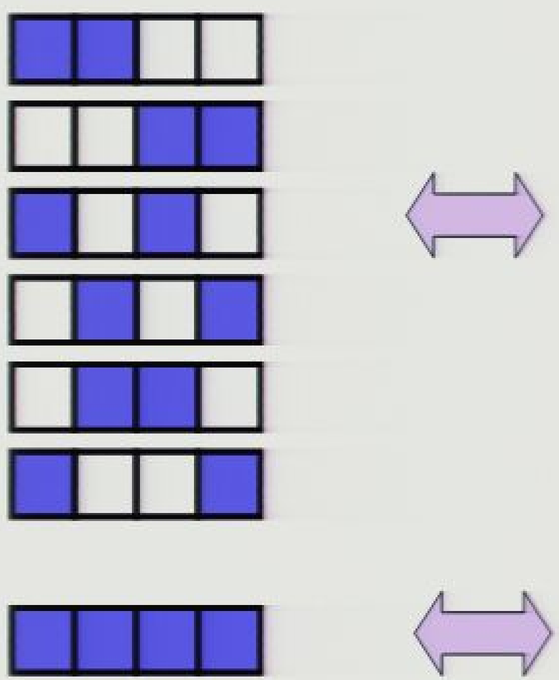
But this does not seem to be enough to derive quantum theory within a classical framework

Example: toy theory of quant-ph/0401052

Ontic states



Epistemic states



$|0\rangle$

$|1\rangle$

$|+\rangle$

$|-\rangle$

$|+i\rangle$

$|-i\rangle$

$$|\pm\rangle = |0\rangle \pm |1\rangle$$

$$|\pm i\rangle = |0\rangle \pm i|1\rangle$$

$\frac{1}{2}I$

Phenomena that can be explained (qualitatively at least) as the result of an epistemic restriction

- Coherent superposition
- Bi-partite entanglement
- tri-partite entanglement
- The monogamy of entanglement
- The ambiguity of mixtures
- No universal state inverter
- Mutually unbiased bases
- Neumark and Stinespring extension
- Choi-Jamiolkowski isomorphism
- …

- Noncommutativity
- Interference
- No-cloning
- Teleportation
- Key distribution
- Dense coding
- No bit commitment
- Interaction-free measurement
- Quantum eraser
- …

See: Spekkens quant-ph/0401052
Also Bartlett, Rudolph, and Spekkens, in preparation

What the toy theories fail to capture

- They are noncontextual (no Bell-Kochen-Specker theorem)
- They are local (no violations of Bell inequalities)
- They do not reproduce the full set of quantum states, measurements, and transformations
- Two levels of a toy qutrit do not yield a toy qubit
- There is no exponential speed-up relative to classical computation
- ...

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- ...

We can categorize nonclassical phenomena in this way

The failures help to identify the conceptual elements of quantum theory that are missing from these toy theories

Despite having no axiomatization to offer, I argue that a research program seeking a particular kind of realist axiomatization appears to be promising

The approach is:

Be very conservative. Keep almost all classical notions of reality, except:

Axiom 1. There is a restriction to how much an observer (or any system) can know about the real state of the systems with which she interacts

Axiom 2. ??? (some change to our classical notion of reality)

Contextuality is an umbrella for many missing phenomena and may therefore be our best clue for how to proceed

Phenomena that are a form of contextuality

- all variants of the Bell-Kochen-Specker theorem (algebraic, state-specific, statistical, continuous, discrete)

- all variants of Bell's theorem

- novel theorems that apply even in 2d Hilbert spaces

- The necessity of having negativity in quasiprobability representations of quantum theory

- Aspects of pre- and post-selected "paradoxes"

- Better-than-classical performance of oblivious transfer

- all variants of von Neumann's no-go theorem

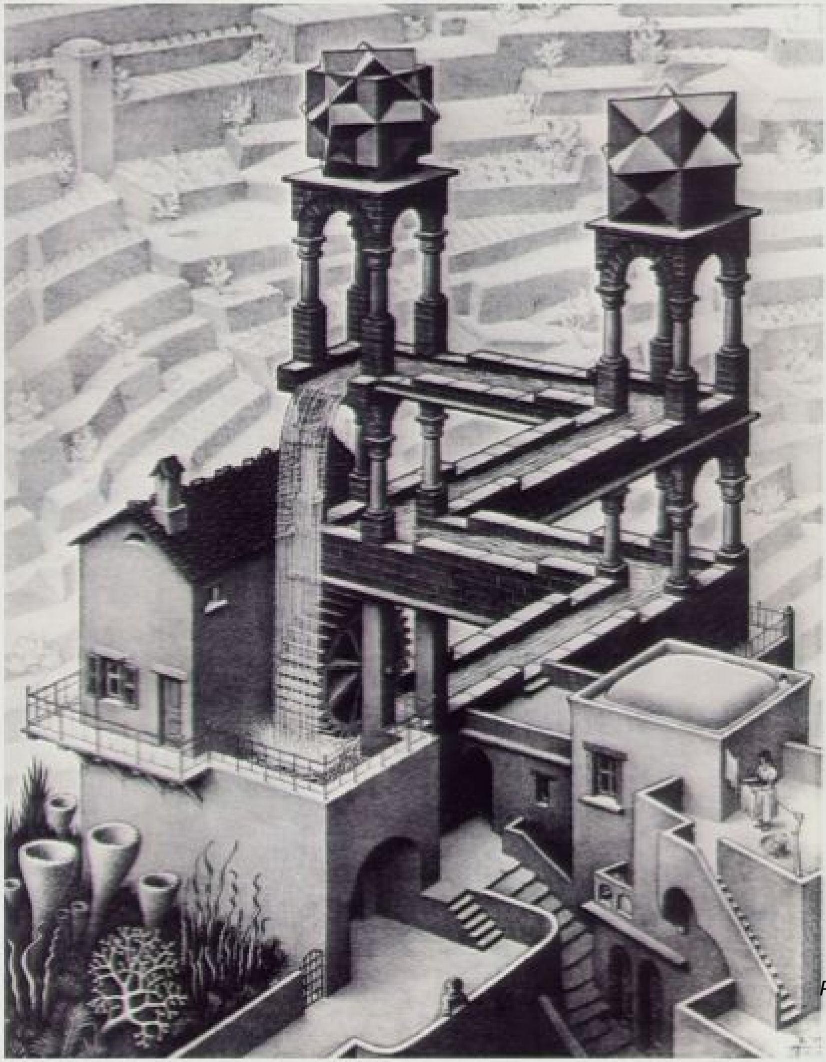
- Quantized spectra? Fermionic statistics?

Outline

- Generalizing the notion of noncontextuality to arbitrary procedures and operational theories
- Why von Neumann's no-go theorem is a proof of contextuality
- Conclusions

It was shown by Bell (1966) and Kochen and Specker (1967) that a noncontextual hidden variable model of quantum theory for Hilbert spaces of dimensionality 3 or greater is impossible. That is, quantum theory is contextual

This is the Bell-Kochen-Specker theorem



The traditional definition of contextuality does not apply to:

(1) arbitrary operational theories
(2) preparations or unsharp measurements
(3) indeterministic hidden variable models

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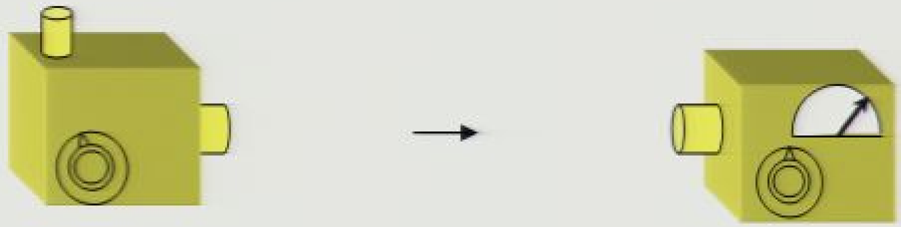
(1) arbitrary operational theories
(2) preparations or unsharp measurements
(3) indeterministic hidden variable models

Proposed new definition:
A noncontextual HV model of an operational theory is one wherein if two experimental procedures are operationally equivalent, then they have equivalent representations in the HV model.

Operational theories

Operational theories



Preparation

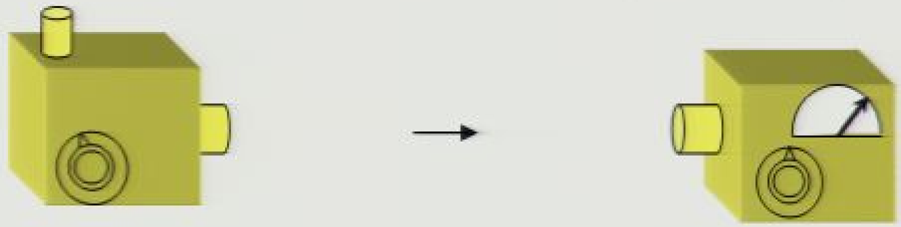
P

Measurement

M

These are defined as lists of instructions

Operational theories



Preparation

P

Measurement

M

These are defined as lists of instructions

An operational theory specifies

$p(k|\mathrm{P},\mathrm{M}) \equiv$ The probability of outcome k of M given P.

Defining operational equivalence of procedures

For preparations

$\mathsf{P} \simeq \mathsf{P}'$ if

$p(k|\mathsf{P},\mathsf{M}) = p(k|\mathsf{P}',\mathsf{M})$ for all M.

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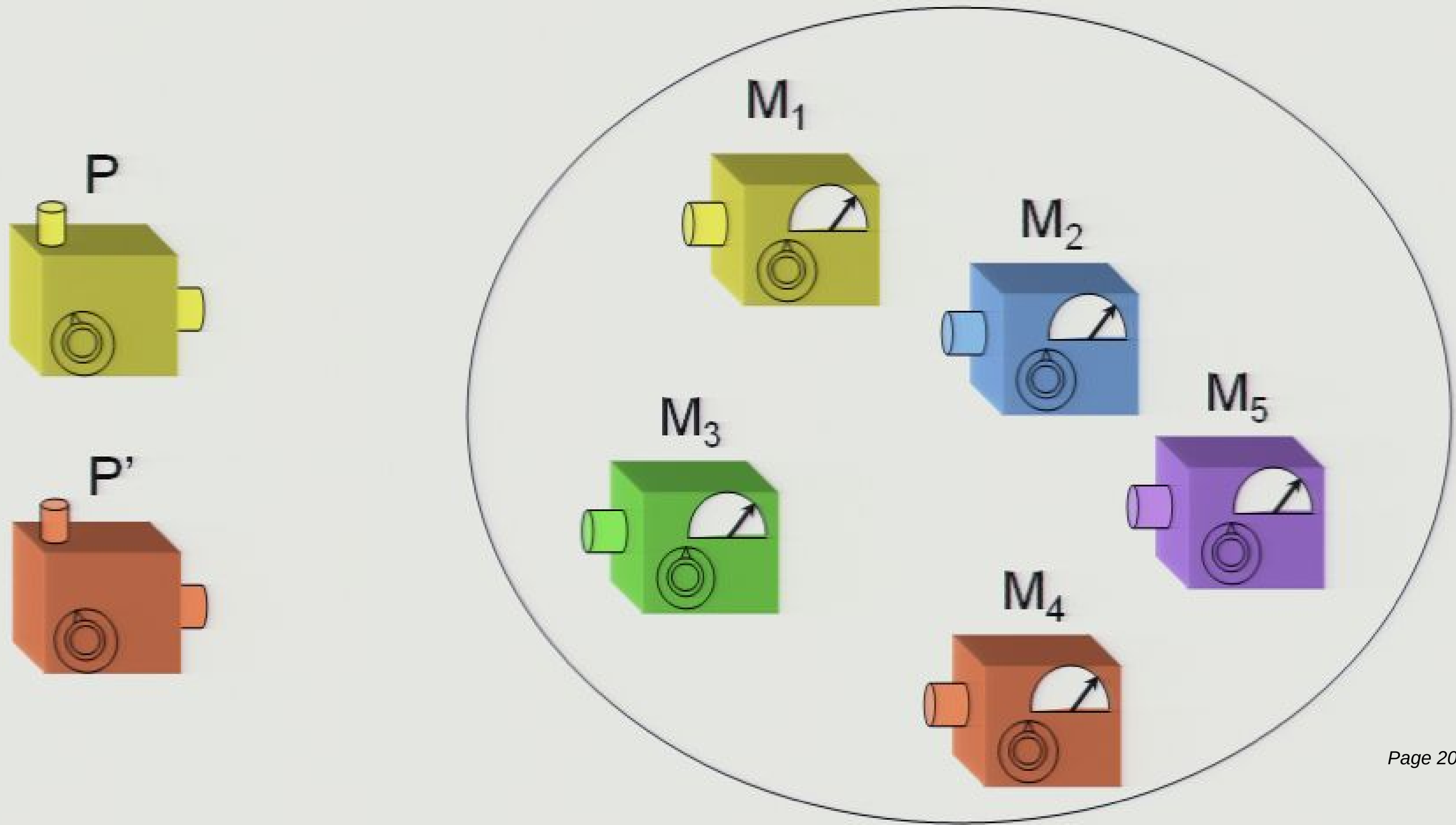

Defining operational equivalence of procedures

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Defining operational equivalence of procedures

For measurements

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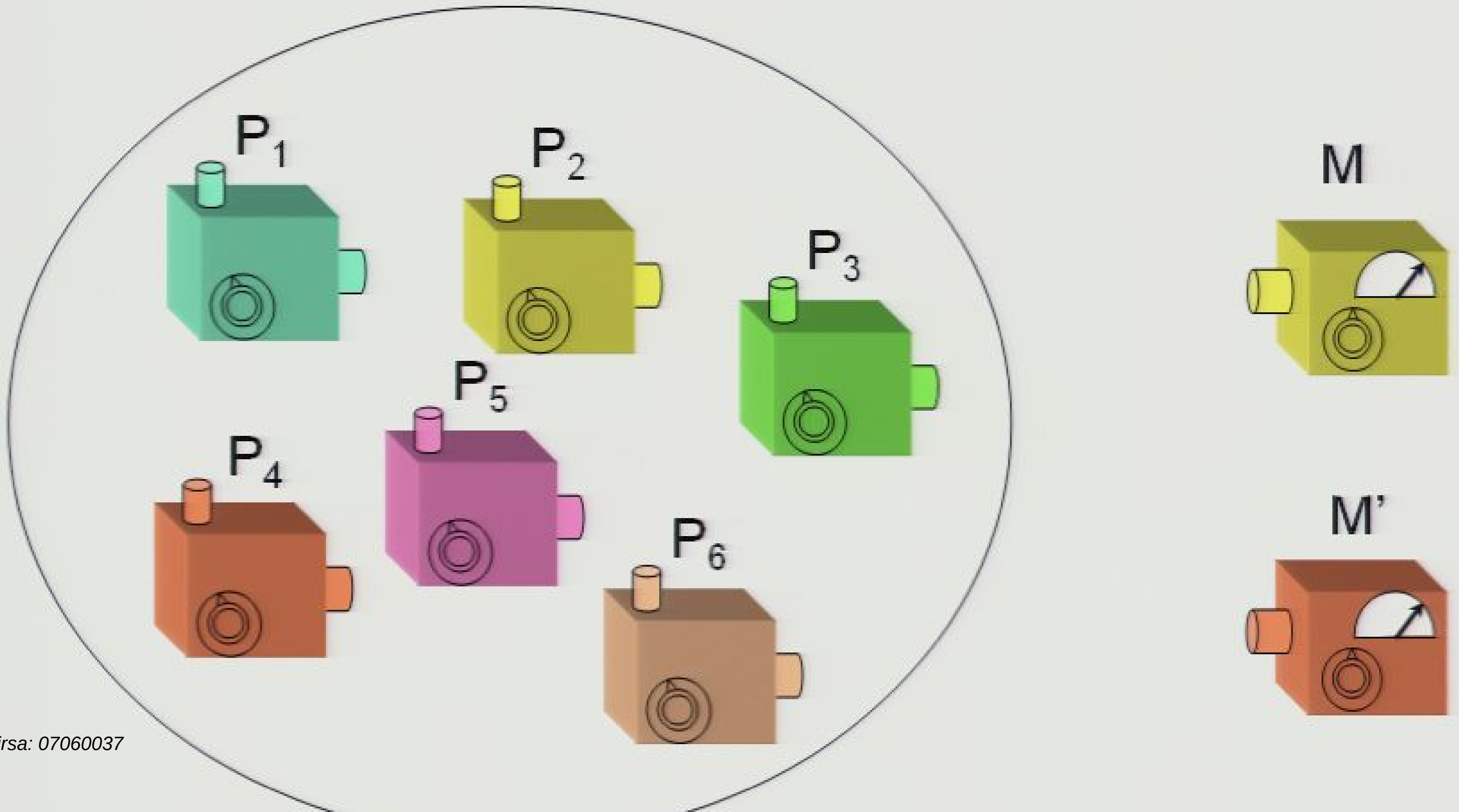
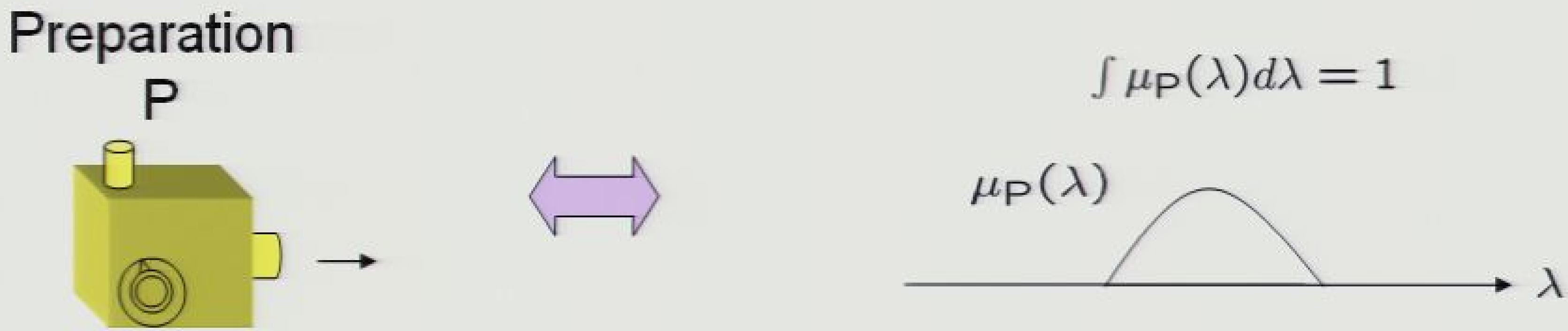


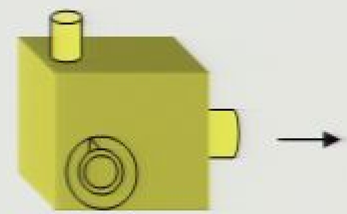
A hidden variable model of an operational theory assumes primitives of systems and properties

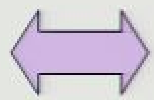


$\int \mu_{\mathsf{P}}(\lambda) d\lambda = 1$

A hidden variable model of an operational theory assumes primitives of systems and properties

Preparation
P





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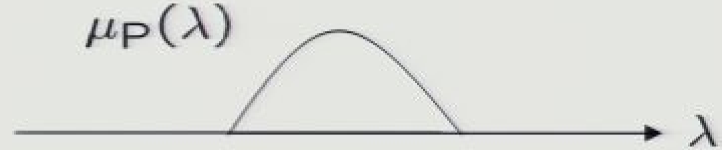
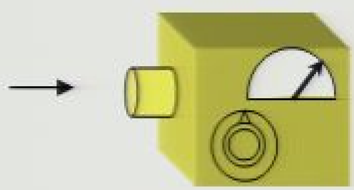
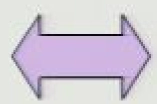


Measurement
M



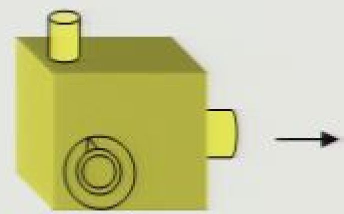


$0 \leq \xi_{\mathrm{M},k} \leq 1$

$\sum_k \xi_{\mathrm{M},k}(\lambda) = 1$ for all λ

A hidden variable model of an operational theory assumes primitives of systems and properties

Preparation
P



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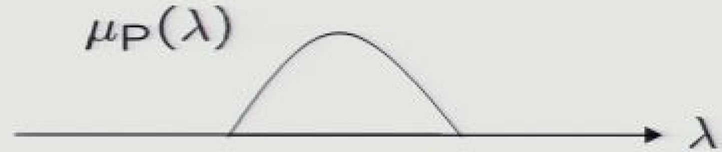

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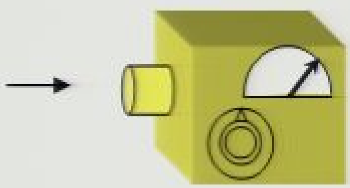

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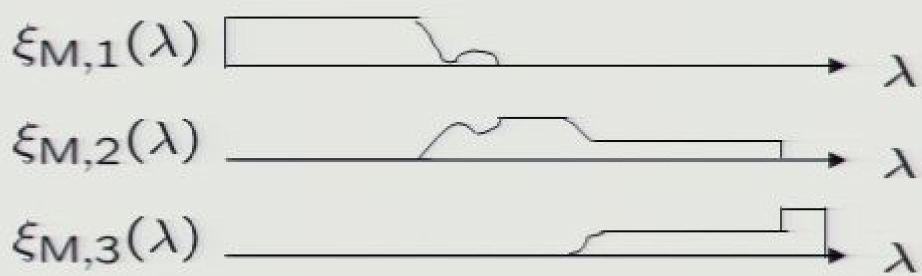

$$p(k|\mathsf{P},\mathsf{M}) = \int d\lambda\, \xi_{\mathsf{M},k}(\lambda)\, \mu_{\mathsf{P}}(\lambda)$$

Defining noncontextuality in operational theories

Preparation Noncontextuality

if $\mathrm{P} \simeq \mathrm{P}'$ then $\mu_{\mathrm{P}}(\lambda) = \mu_{\mathrm{P}'}(\lambda)$

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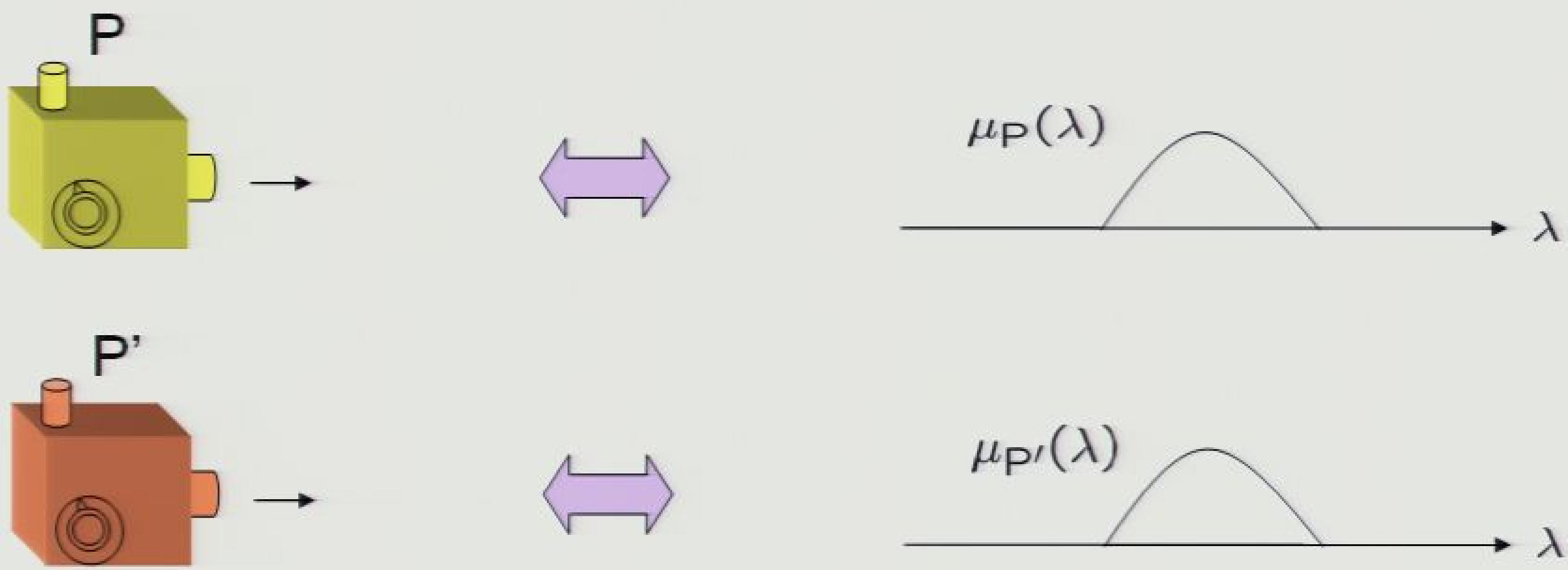
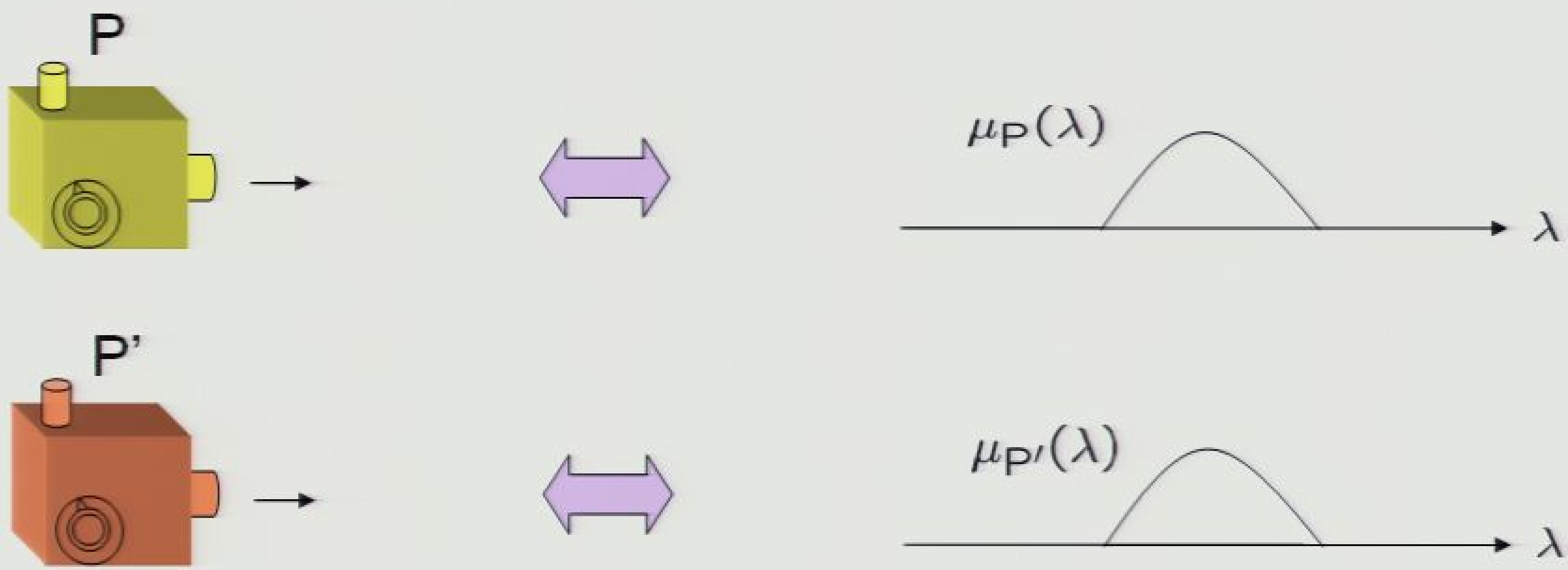


Defining noncontextuality in operational theories

Preparation Noncontextuality

$$\text{if } P \simeq P' \text{ then } \mu_P(\lambda) = \mu_{P'}(\lambda)$$



Differences between P and P' are differences of context

Defining noncontextuality in operational theories

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if $\mathsf{M} \simeq \mathsf{M}'$ then $\xi_{\mathsf{M},k}(\lambda) = \xi_{\mathsf{M}',k}(\lambda)$

Defining noncontextuality in operational theories

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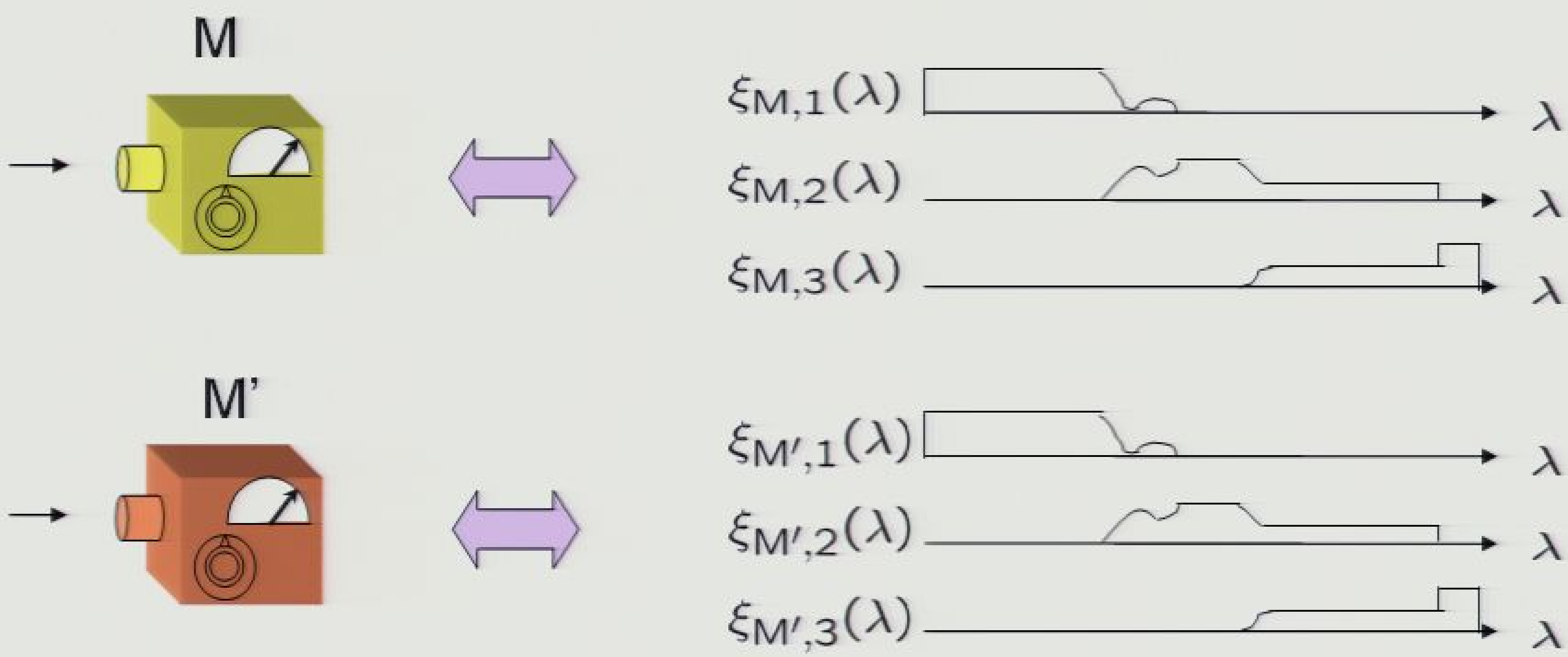

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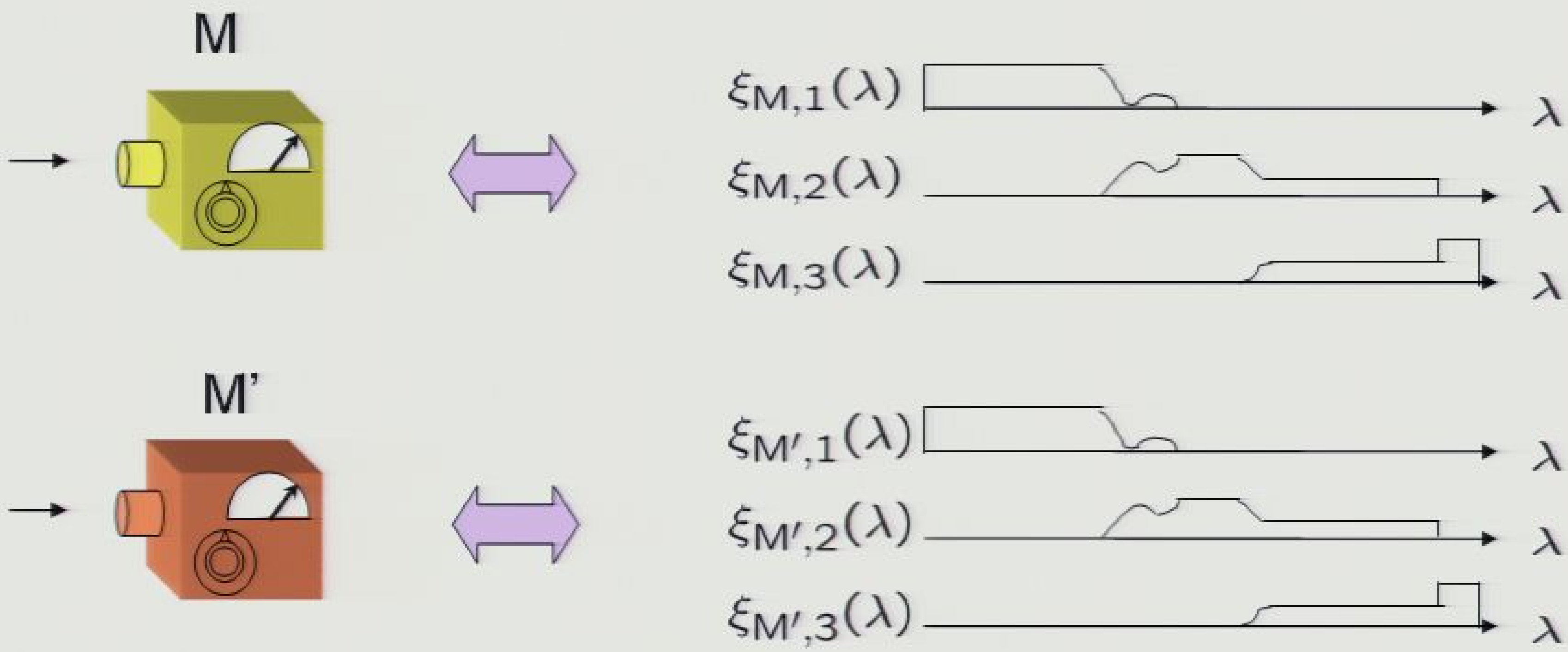


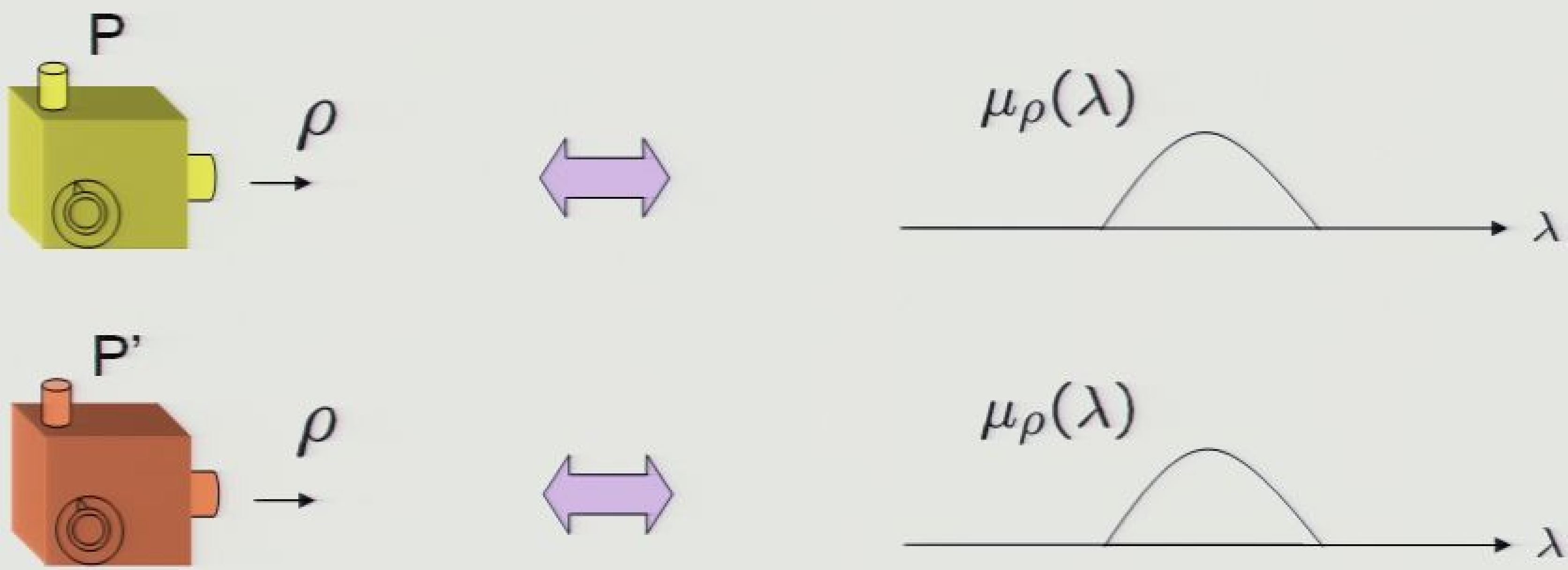
Differences between M and M' are differences of context

Quantum theory

Defining noncontextuality in quantum theory

Preparation Noncontextuality in QT

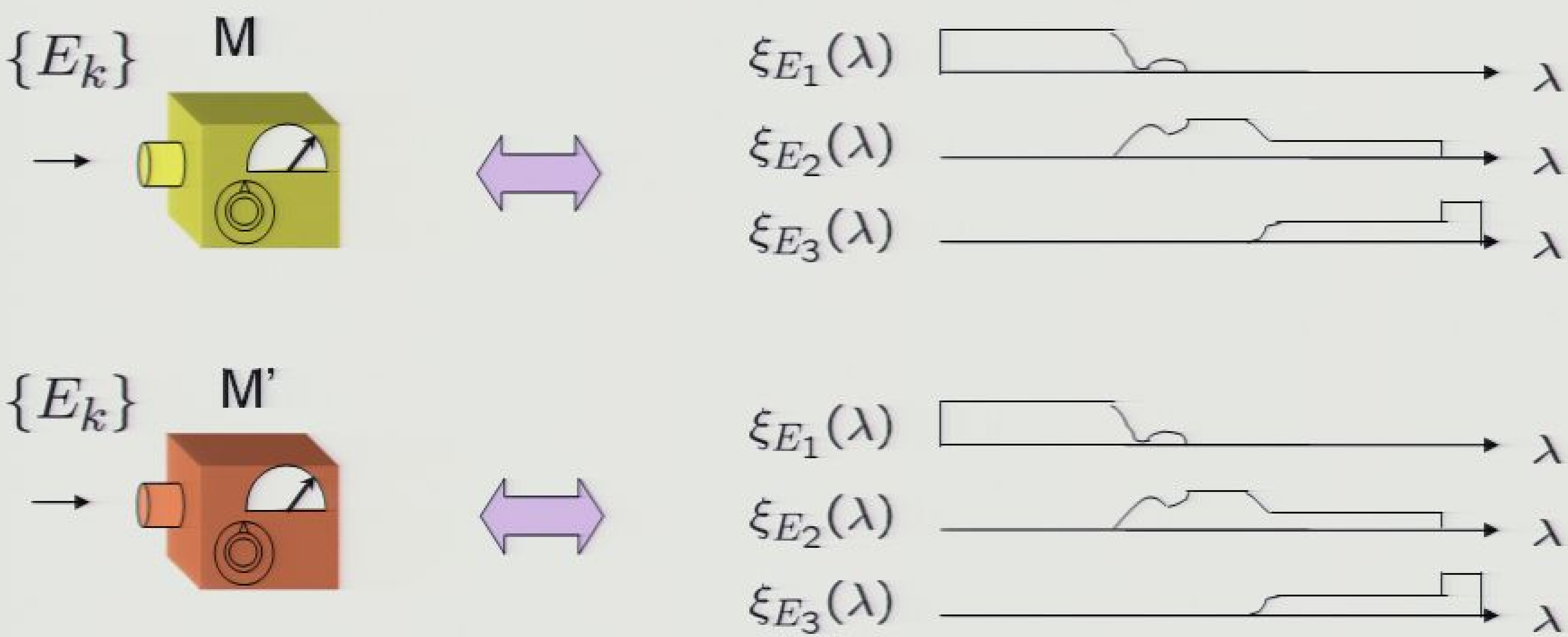
if P, P'$\rightarrow \rho$ then $\mu_{\mathrm{P}}(\lambda) = \mu_{\mathrm{P'}}(\lambda) = \mu_\rho(\lambda)$



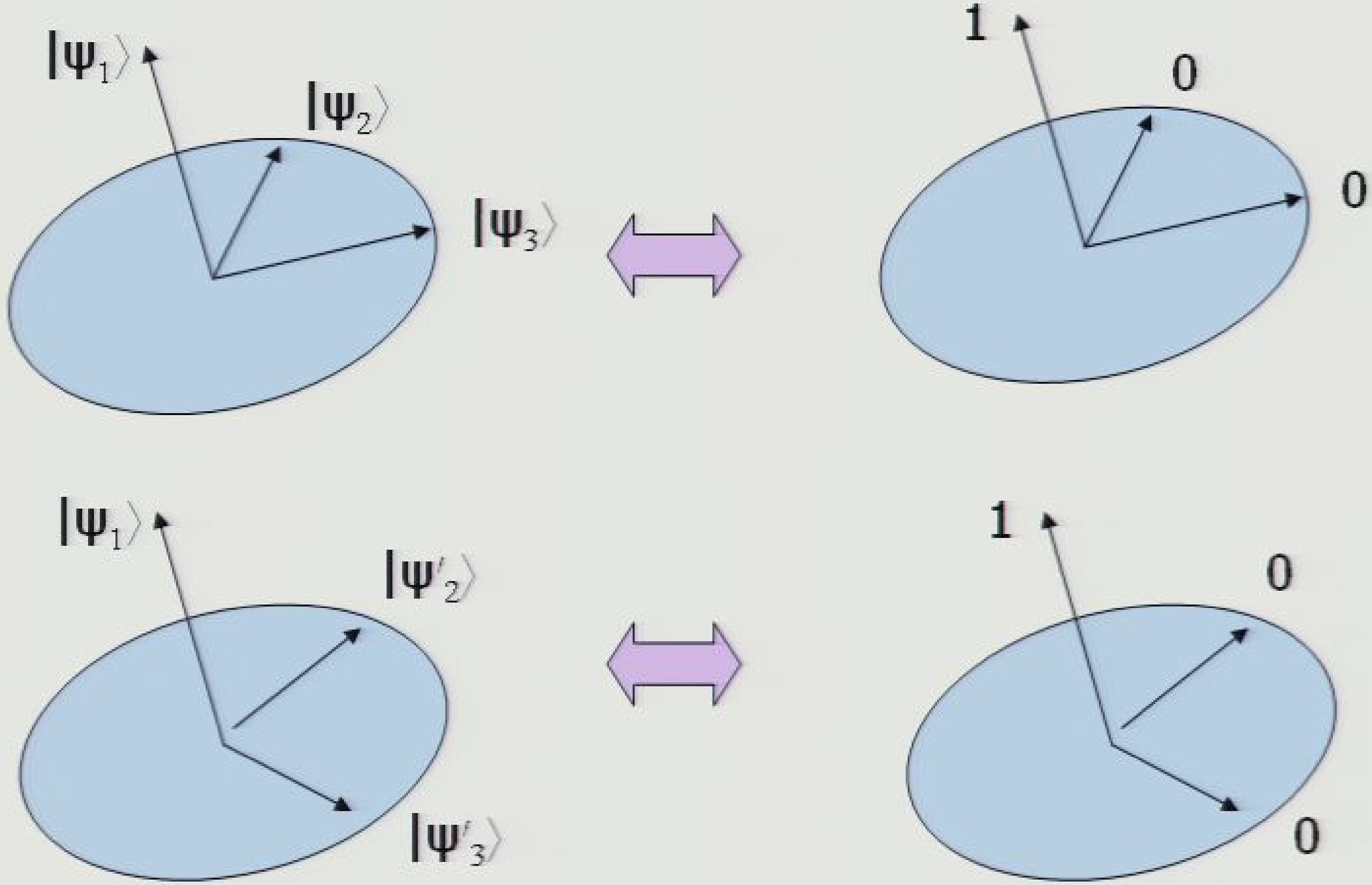
Defining noncontextuality in quantum theory

Measurement Noncontextuality in QT

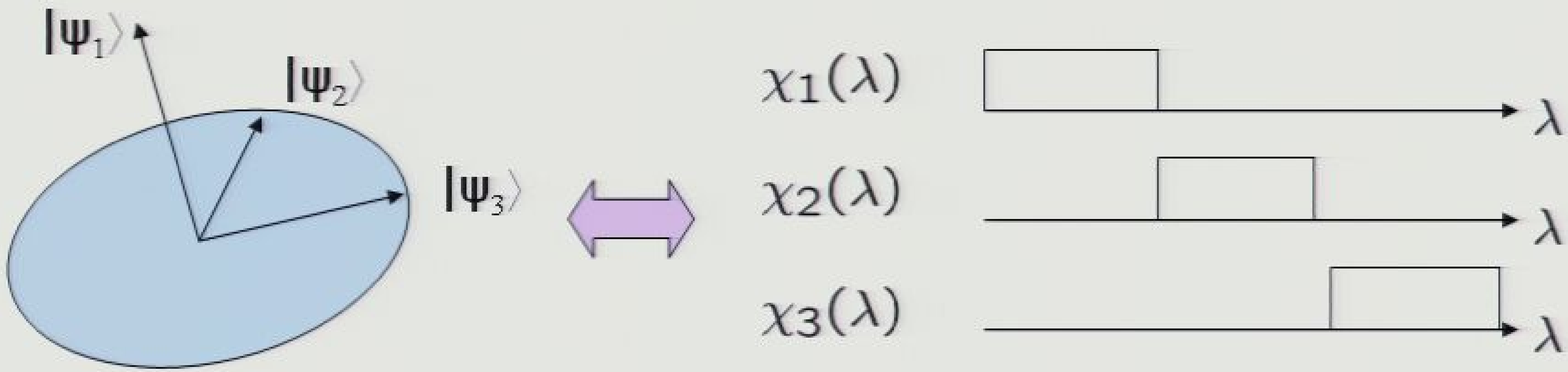
$$\text{if } \mathsf{M},\mathsf{M}' \rightarrow \{E_k\} \text{ then } \xi_{\mathsf{M},k}(\lambda) = \xi_{\mathsf{M}',k}(\lambda) = \xi_{E_k}(\lambda)$$

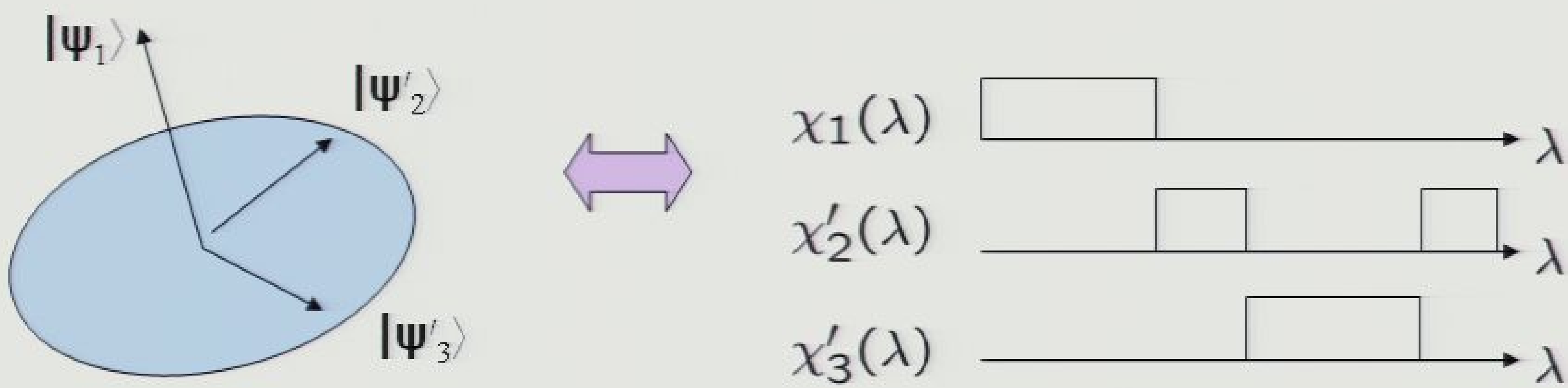


The traditional notion of noncontextuality

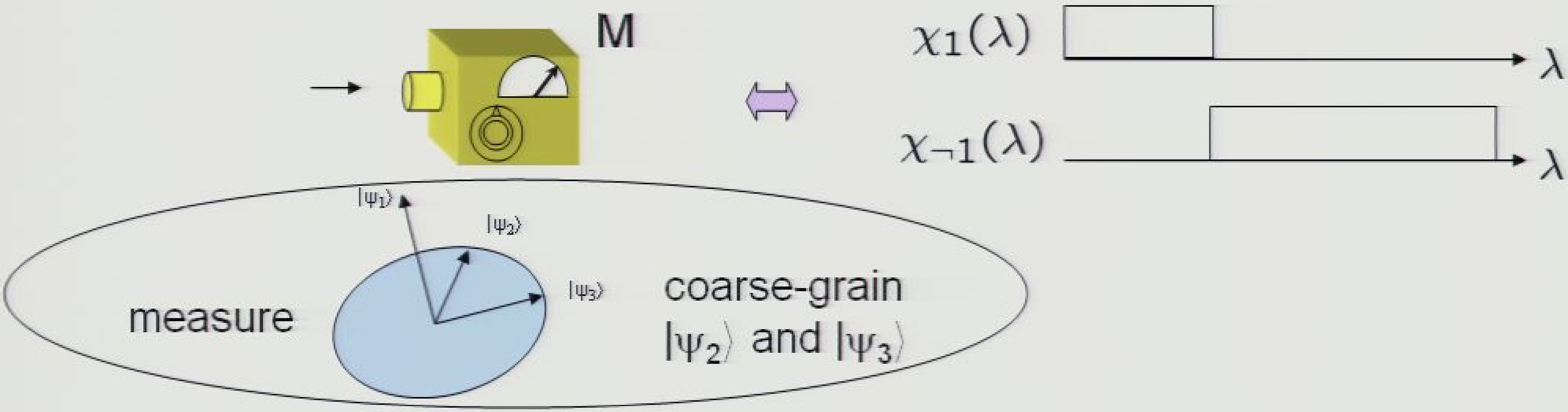


How to formulate the traditional notion of noncontextuality:

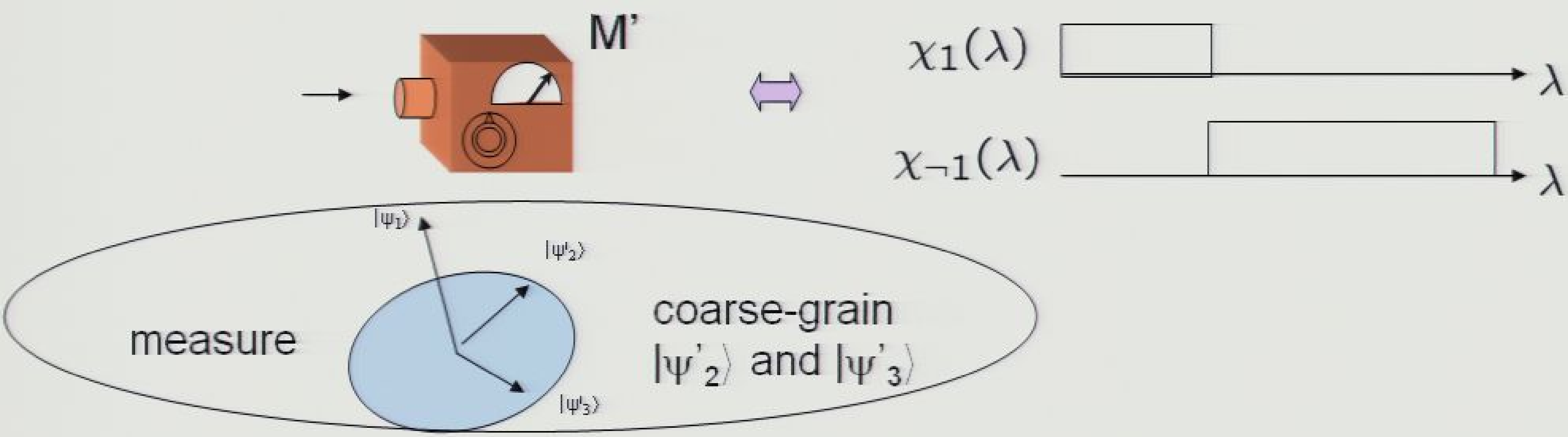




This is equivalent to assuming:



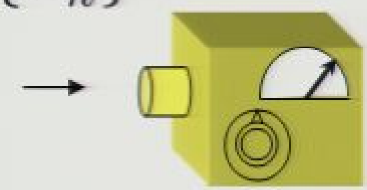
$$\{|\psi_1\rangle\langle\psi_1|,\ I - |\psi_1\rangle\langle\psi_1|\}$$



$$\{|\psi_1\rangle\langle\psi_1|,\ I - |\psi_1\rangle\langle\psi_1|\}$$

But recall that the most general representation was

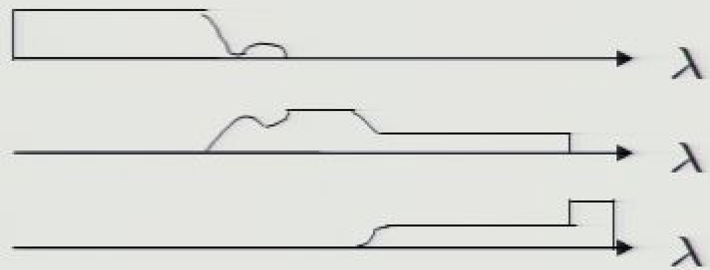
$\{P_k\}$ M



$\xi_{P_1}(\lambda)$

$\xi_{P_2}(\lambda)$

$\xi_{P_3}(\lambda)$



Therefore:

traditional notion of noncontextuality = revised notion of noncontextuality for sharp measurements

and

outcome determinism for sharp measurements

So, the proposed definition of noncontextuality is not simply a generalization of the traditional notion

For sharp measurements, it is a revision of the traditional notion

Local determinism:

We ask: Does the outcome depend on space-like separated events (in addition to local settings and λ)?

Bell's local causality:

We ask: Does the probability of the outcome depend on space-like separated events (in addition to local settings and λ)?

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Traditional notion of measurement noncontextuality:

We ask: Does the outcome depend on the measurement context (in addition to the observable and λ)?

The proposed revised notion of measurement noncontextuality:

We ask: Does the probability of the outcome depend on the measurement context (in addition to the observable and λ)?

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Noncontextuality and determinism are separate issues traditional notion of noncontextuality = revised notion of noncontextuality for sharp measurements

and

outcome determinism for sharp measurements

No-go theorems for previous notion are not necessarily no-go theorems for the new notion!

In face of contradiction, could give up ODSM

However, one can prove that

preparation noncontextuality $\longrightarrow$ outcome determinism for sharp measurements

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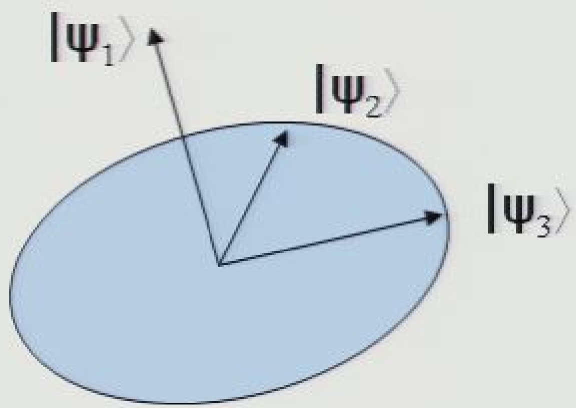
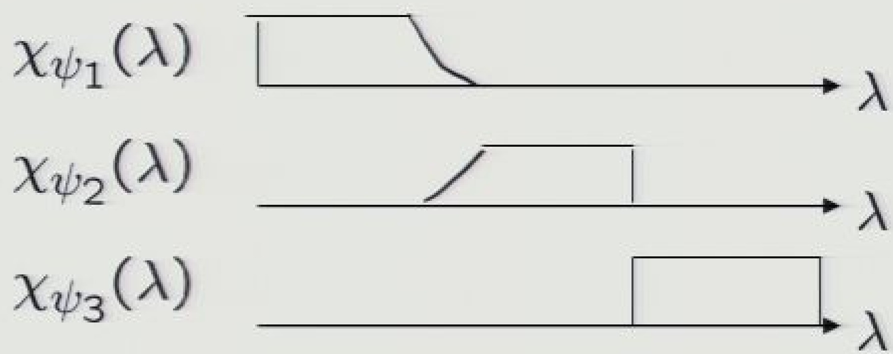
preparation noncontextuality $\longrightarrow$ outcome determinism for sharp measurements

Proof





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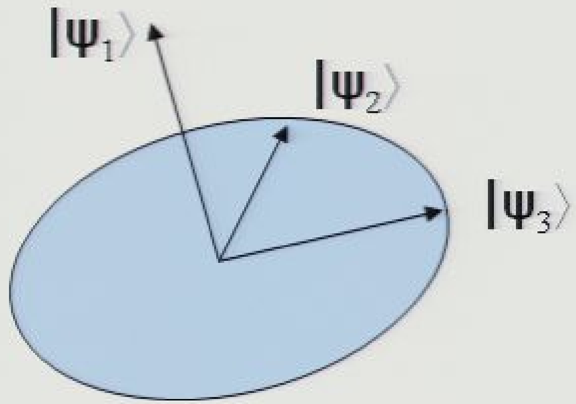

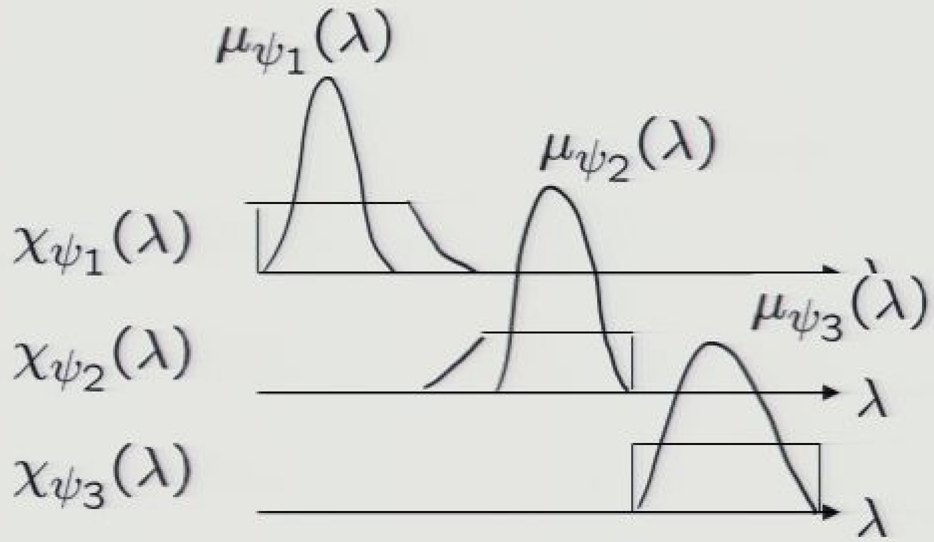

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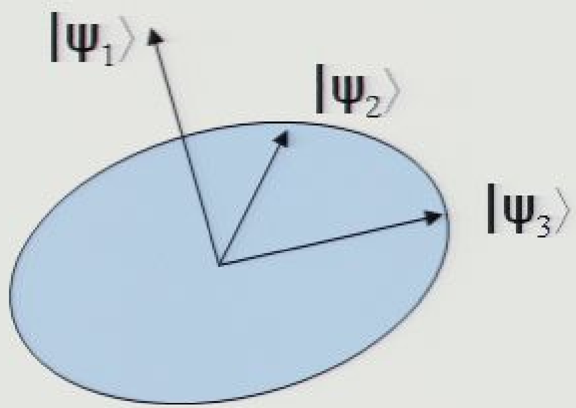
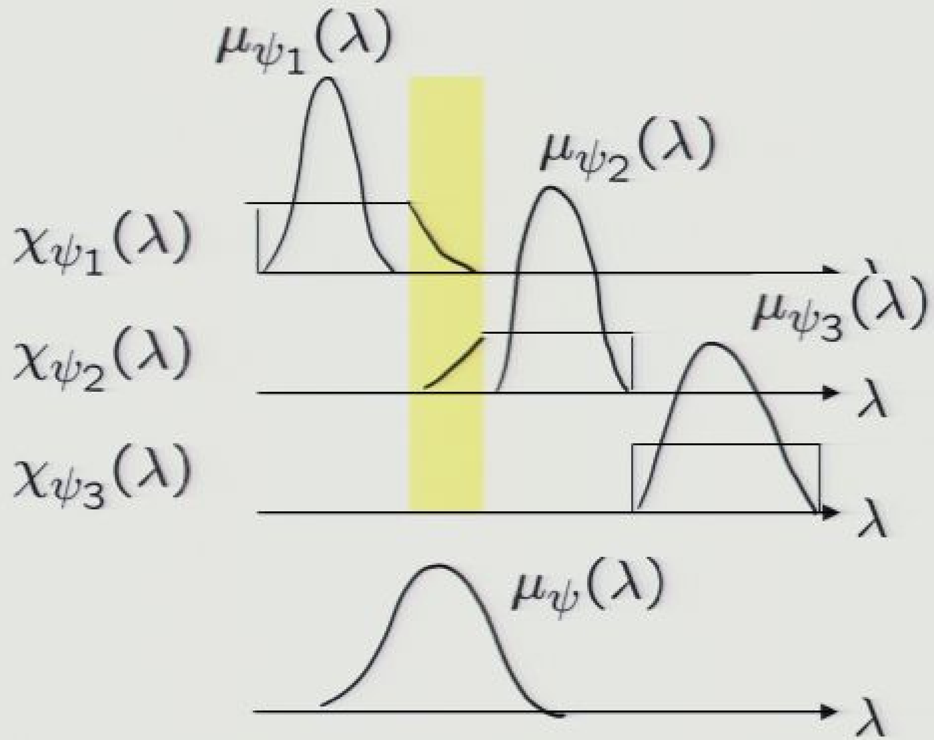




$$\mu_{I/3}(\lambda) = \tfrac{1}{3}\mu_{\psi_1}(\lambda) + \tfrac{1}{3}\mu_{\psi_2}(\lambda) + \tfrac{1}{3}\mu_{\psi_3}(\lambda)$$

$$\mu_{I/3}(\lambda) = p\mu_{\psi}(\lambda) + \ldots$$

We've established that

preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

measurement noncontextuality and preparation noncontextuality → measurement noncontextuality and outcome determinism for sharp measurements

We've established that

preparation noncontextuality $\longrightarrow$ outcome determinism for sharp measurements

Therefore:

measurement noncontextuality and preparation noncontextuality $\longrightarrow$ Traditional notion of noncontextuality

We've established that

preparation noncontextuality → outcome determinism for sharp measurements

Therefore:

measurement noncontextuality
and
preparation noncontextuality → Traditional notion of noncontextuality

no-go theorems for the traditional notion of noncontextuality can be salvaged as no-go theorems for the generalized notion

… and there are many new proofs

Phenomena that are a form of contextuality

- all variants of the Bell-Kochen-Specker theorem (algebraic, state-specific, statistical, continuous, discrete)

- all variants of Bell's theorem

- novel no-go theorems, including many in 2d Hilbert spaces (see PRA **71**, 052108)

-The necessity of having negativity in quasiprobability representations of quantum theory

- Aspects of pre- and post-selected "paradoxes" (joint work with M. Leifer, PRL **95,** 200405)

- Better-than-classical performance of oblivious transfer (joint work with B. Toner)

- all variants of von Neumann's no-go theorem (rest of talk)

Von Neumann's no-go theorem for hidden variables is a proof of contextuality

von Neumann's assumptions about HV models of QT

- $A \;\rightarrow\; f_A(\lambda)$
- $f_A(\lambda) \in \mathrm{spec}(A)$

$$\begin{pmatrix} f_P(\lambda) = 0 \text{ or } 1 \\ f_I(\lambda) = 1 \end{pmatrix}$$

"Dipersion-free ensemble"

- if $A = B + C$ then $f_A(\lambda) = f_B(\lambda) + f_C(\lambda)$

even if A, B, and C do not commute

The latter goes beyond traditional noncontextuality

Theorem: Such a HV model of quantum theory does not exist.

Von Neumann's proof

if $A = B + C$ then $f_A(\lambda) = f_B(\lambda) + f_C(\lambda)$
or equivalently, $f_{B+C}(\lambda) = f_B(\lambda) + f_C(\lambda)$

Lemma: Any function g that is a linear function over the Hermitian operators has the form

$$g(A) = Tr(\omega A)$$

for some Hermitian operator ω.

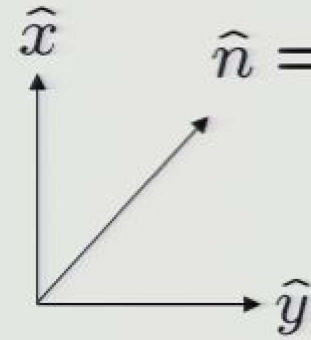
$\rightarrow f_A(\lambda) = \mathrm{Tr}(\omega(\lambda) A)$

$f_P(\lambda) \geq 0$ for all $P \quad \rightarrow \quad \omega(\lambda) \geq 0$
$f_I(\lambda) = 1 \quad \rightarrow \quad \mathrm{Tr}(\omega(\lambda)) = 1$

$\omega(\lambda)$ is a density operator

But $\quad f_P(\lambda) = 0$ or $1 \quad$ for all P CONTRADICTION

A simpler proof (Belifante, Ballentine)



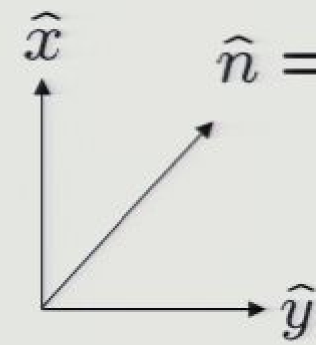
$$\widehat{n} = \frac{\widehat{x}+\widehat{y}}{\sqrt{2}}$$

$$S_n = \frac{1}{\sqrt{2}} S_x + \frac{1}{\sqrt{2}} S_y$$

$$\underbrace{f_n(\lambda)}_{\in \{-\frac{1}{2}, \frac{1}{2}\}} = \underbrace{\frac{1}{\sqrt{2}} f_x(\lambda) + \frac{1}{\sqrt{2}} f_y(\lambda)}_{\in \{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\}}$$

CONTRADICTION

A simpler proof (Belifante, Ballentine)



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CONTRADICTION

Note: The solution of Horn's problem constrains the spectra of A, B, C when A=B+C. This may yield insights into such no-go theorems (joint work with J. Emerson and M. Christandl)

We argue that

Noncontextuality for preparations and measurements $\longrightarrow$ von Neumann's assumptions

Therefore, no-go theorems based on vN's assumptions can be salvaged as no-go theorems for the generalized notion of NC

von Neumann's assumptions about HV models of QT

- $A \;\rightarrow\; f_A(\lambda)$

- $f_A(\lambda) \in \mathrm{spec}(A)$

$$\begin{pmatrix} f_P(\lambda) = 0 \text{ or } 1 \\ f_I(\lambda) = 1 \end{pmatrix}$$ "Dispersion-free ensemble"

- if $A = B + C$ then $f_A(\lambda) = f_B(\lambda) + f_C(\lambda)$

even if A, B, and C do not commute

The latter goes beyond traditional noncontextuality

von Neumann's assumptions about HV models of QT

- $A \;\rightarrow\; f_A(\lambda)$ justified by noncontextuality for sharp mmts

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$$A = B + C$$

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Sort the terms by the sign of their eigenvalues

$$\sum_{a_+} a_+ P_{a_+} + \sum_{b_-} |b_-| P_{b_-} + \sum_{c_-} |c_-| P_{c_-} = \sum_{a_-} |a_-| P_{a_-} + \sum_{b_+} b_+ P_{b_+} + \sum_{c_+} c_+ P_{c_+}$$

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This defines a positive operator. Let r = maximum coefficient.
Divide by 3r.

$$\sum_{a_+} \frac{|a_+|}{3r} P_{a_+} + \sum_{b_-} \frac{|b_-|}{3r} P_{b_-} + \sum_{c_-} \frac{|c_-|}{3r} P_{c_-} = \sum_{a_-} \frac{|a_-|}{3r} P_{a_-} + \sum_{b_+} \frac{|b_+|}{3r} P_{b_+} + \sum_{c_+} \frac{|c_+|}{3r} P_{c_+}$$

This defines an effect that can be decomposed in two ways.

$$A = B + C$$

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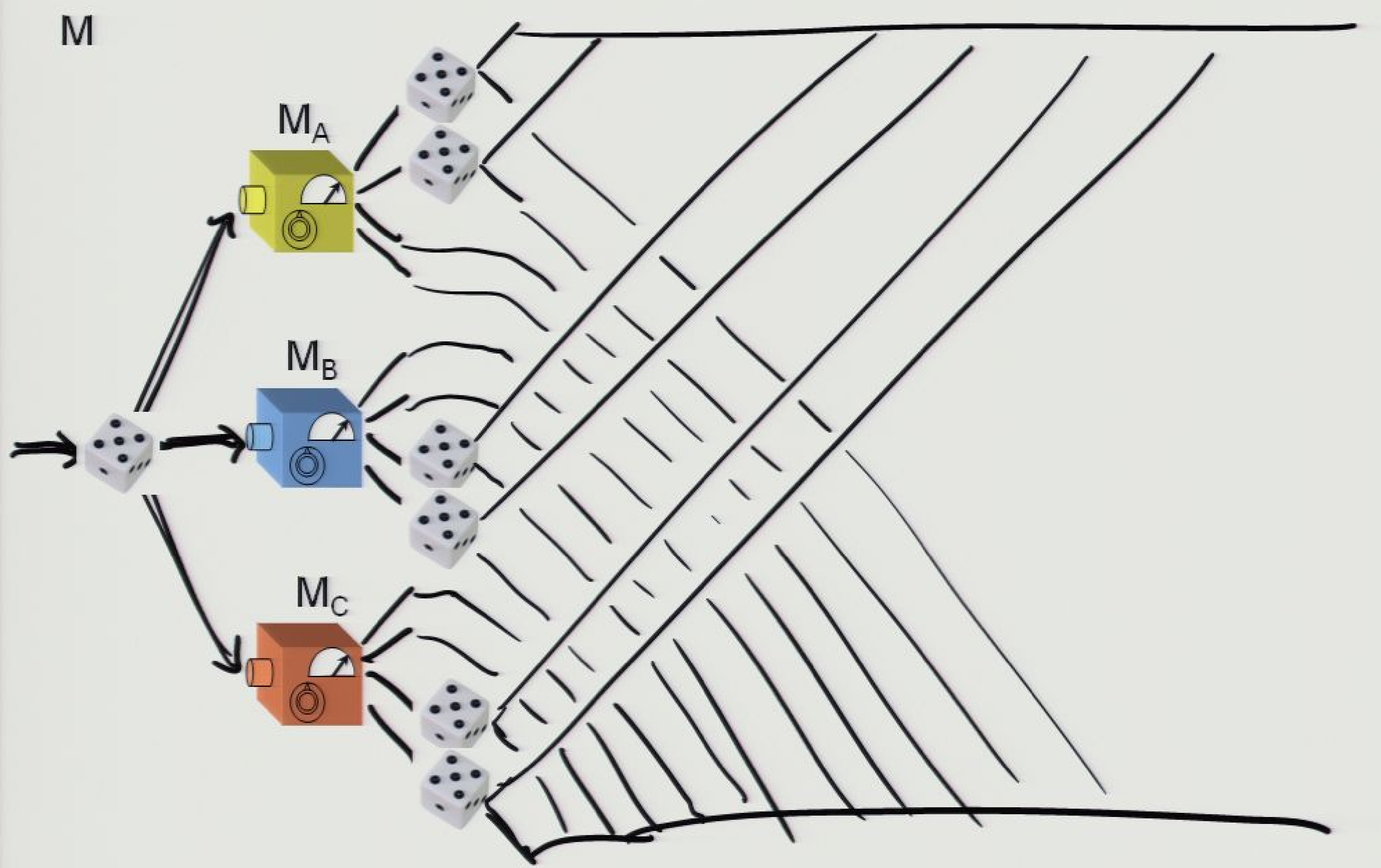
This defines an effect that can be decomposed in two ways.

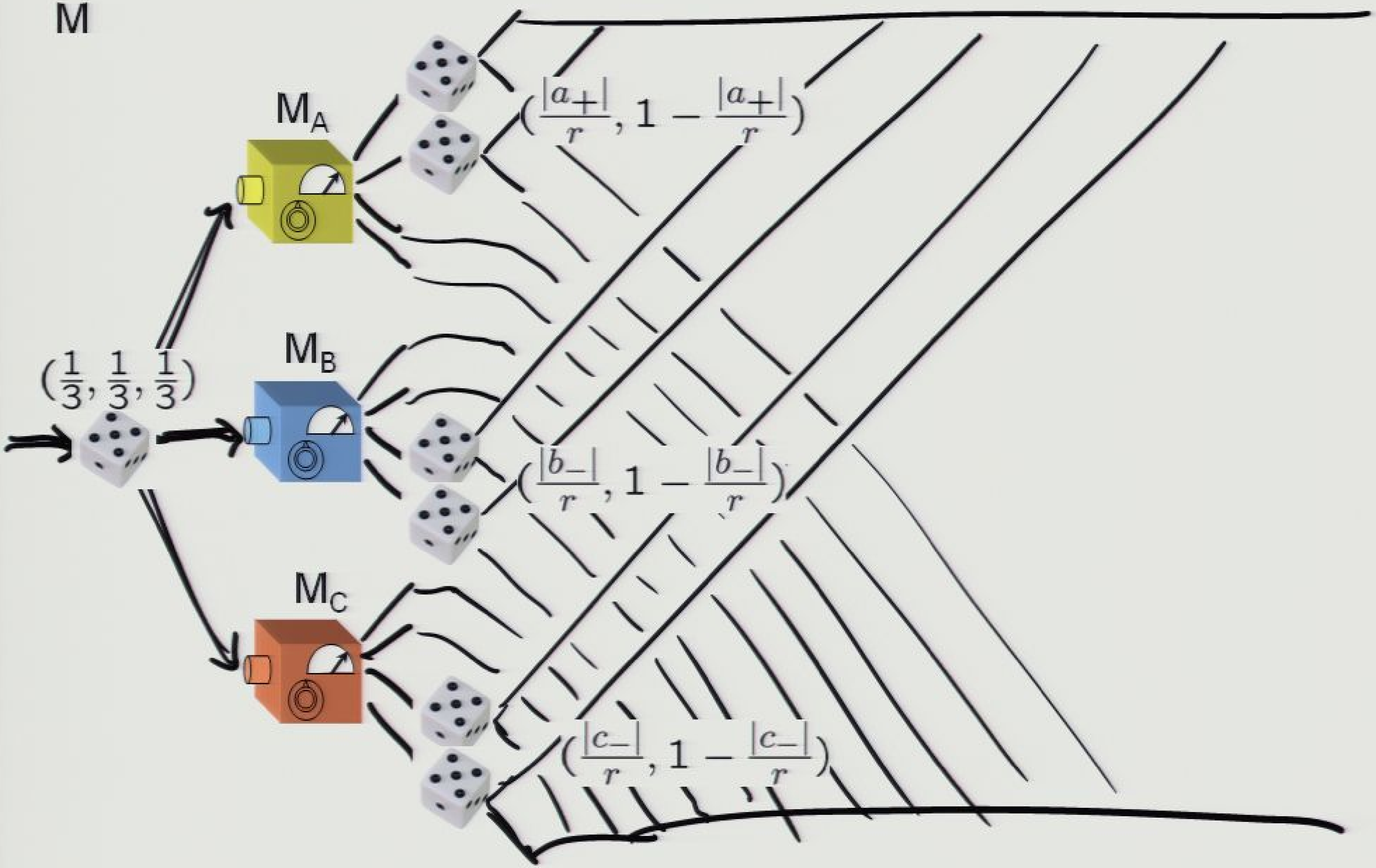
One can deduce that

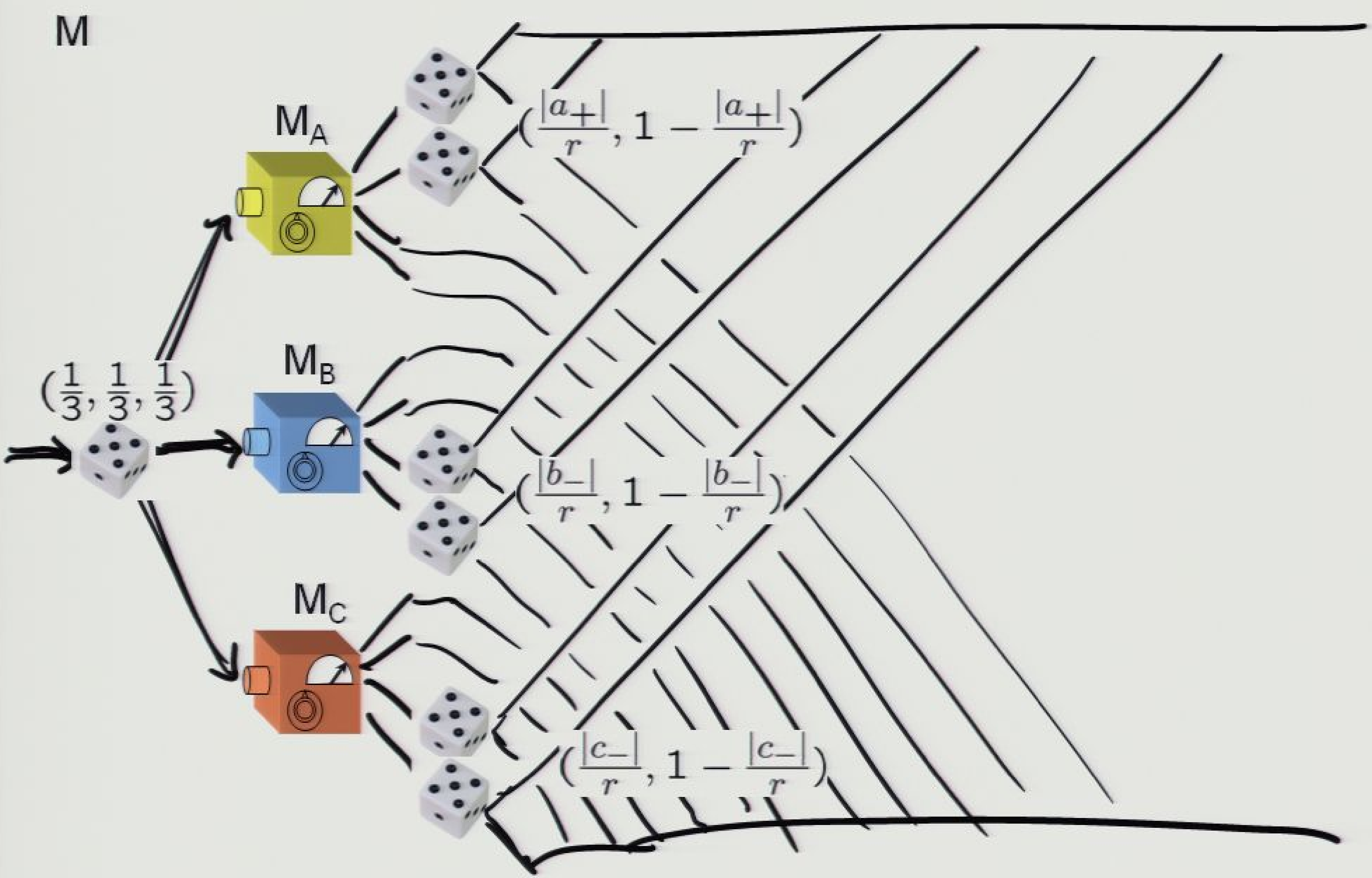
$$\sum_{a_+} \frac{|a_+|}{3r} \chi_{a_+}(\lambda) + \sum_{b_-} \frac{|b_-|}{3r} \chi_{b_-}(\lambda) + \sum_{c_-} \frac{|c_-|}{3r} \chi_{c_-}(\lambda) = \sum_{a_-} \frac{|a_-|}{3r} \chi_{a_-}(\lambda) + \sum_{b_+} \frac{|b_+|}{3r} \chi_{b_+}(\lambda) + \sum_{c_+} \frac{|c_+|}{3r} \chi_{c_+}(\lambda)$$



M
M_A
M_B
M_C

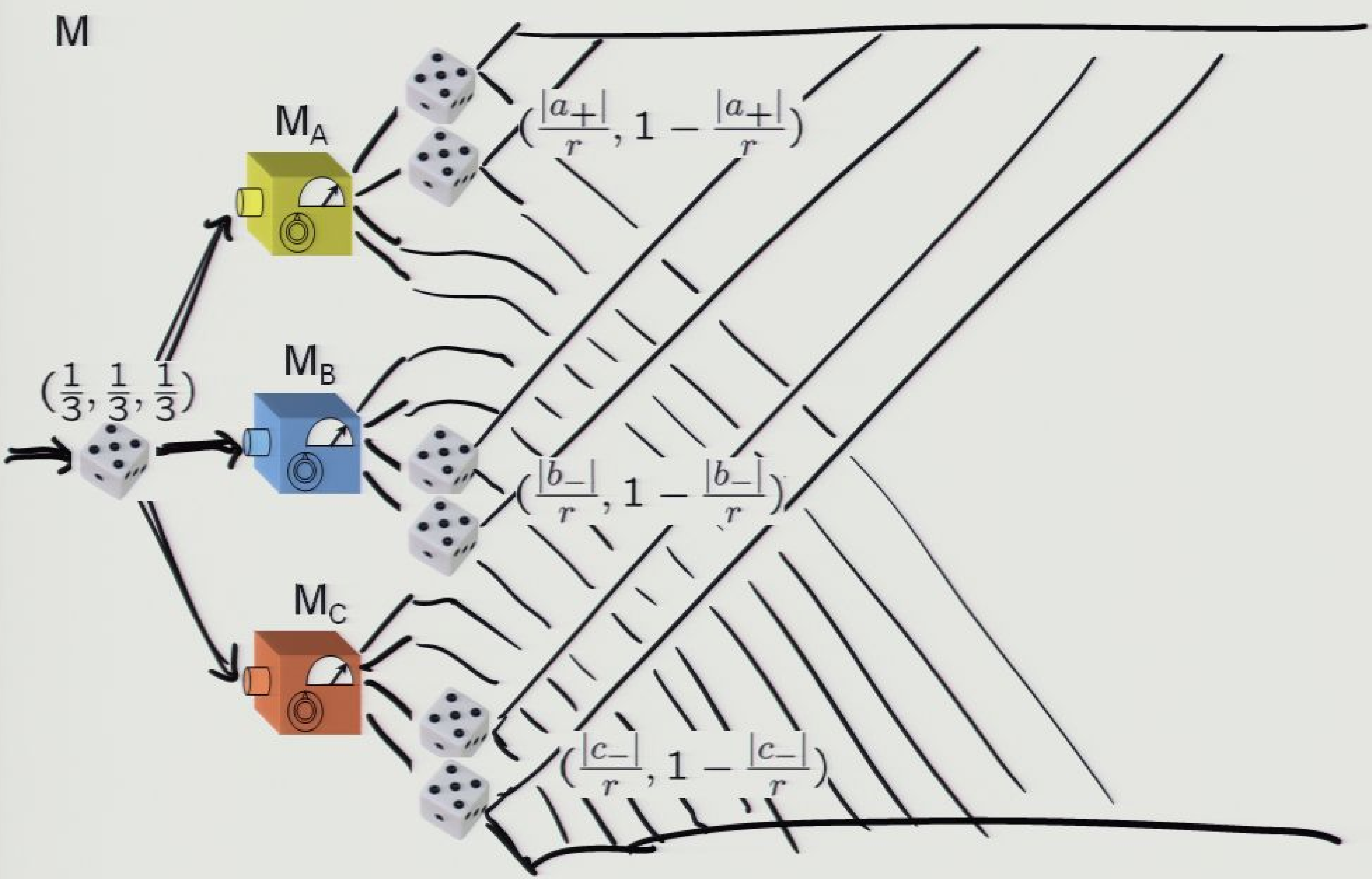
M


M_A
$(\frac{|a_+|}{r}, 1 - \frac{|a_+|}{r})$
$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
M_B
$(\frac{|b_-|}{r}, 1 - \frac{|b_-|}{r})$
M_C
$(\frac{|c_-|}{r}, 1 - \frac{|c_-|}{r})$



probability of top branch outcome of M given preparation P

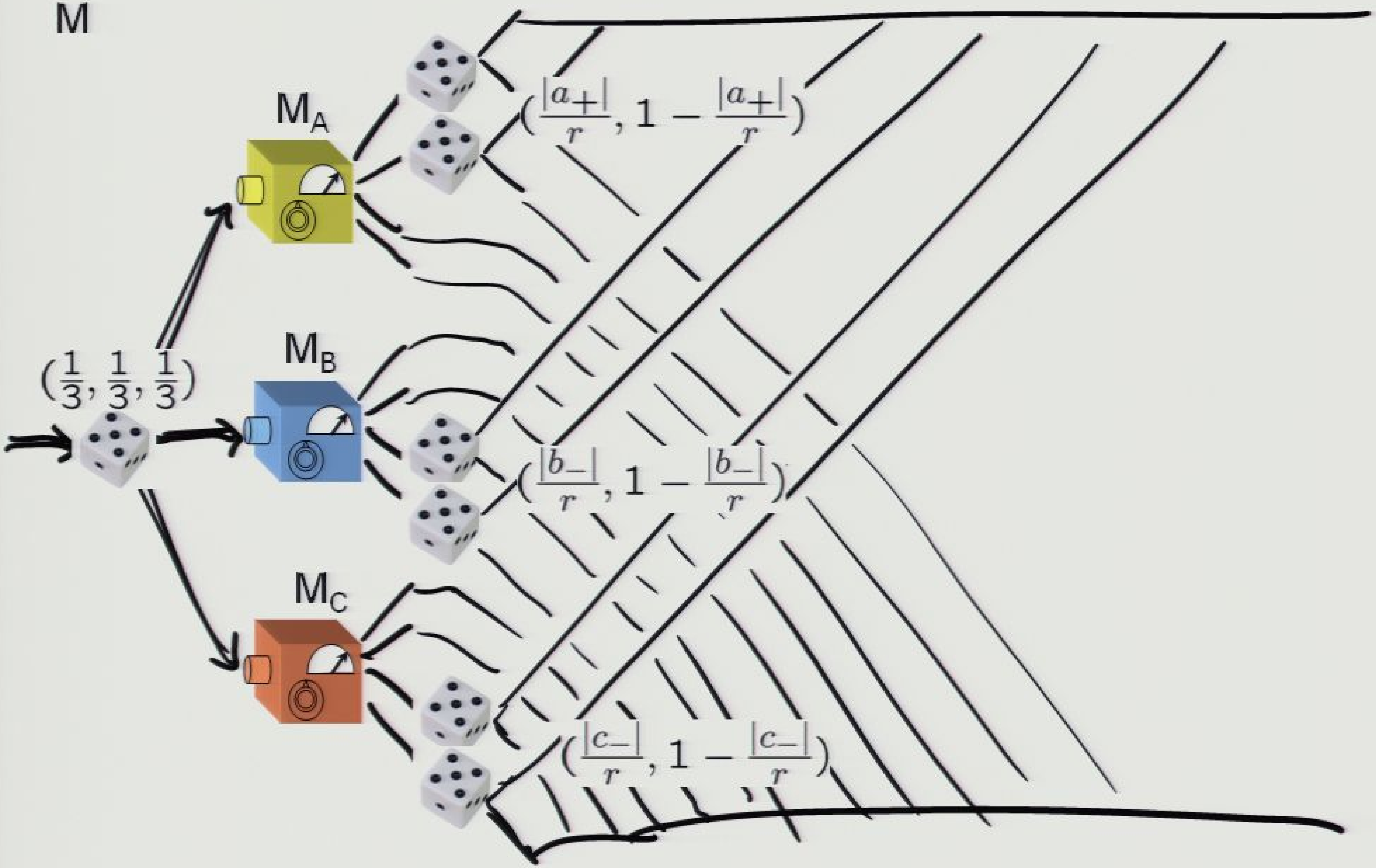
$$\frac{1}{3} \times \Pr(a_+|\mathrm{M}_A, \mathrm{P}) \times \frac{|a_+|}{r}$$



probability of top branch outcome of M given preparation P

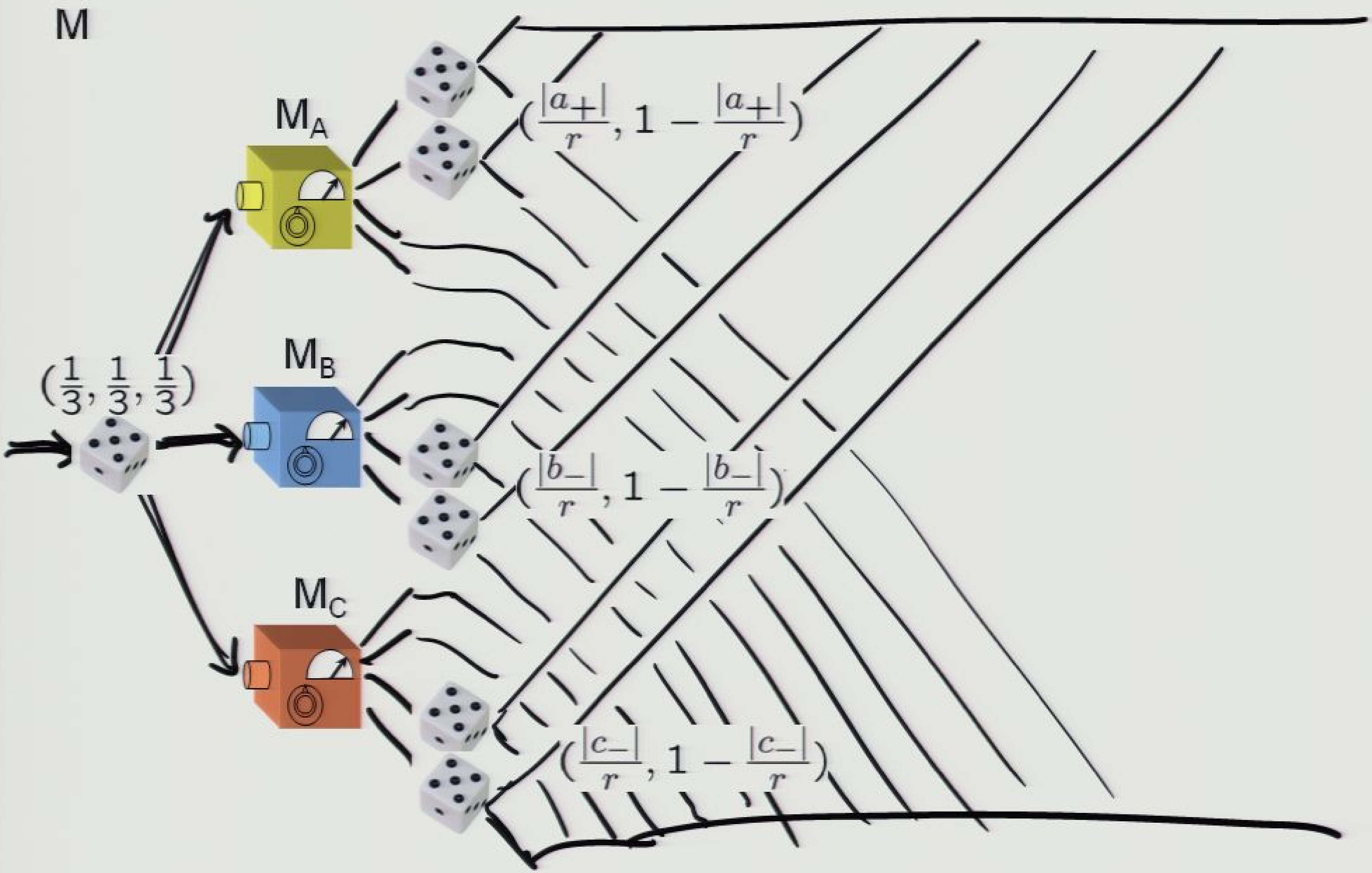
$$\frac{1}{3}\times \mathrm{Pr}(a_+|\mathrm{M}_A,\mathrm{P})\times\frac{|a_+|}{r} \quad = \frac{1}{3}\times \mathrm{Tr}(P_{a_+}\rho)\times\frac{|a_+|}{r}$$

M



Associated effect

$$\frac{|a_+|}{3r}P_{a_+}$$

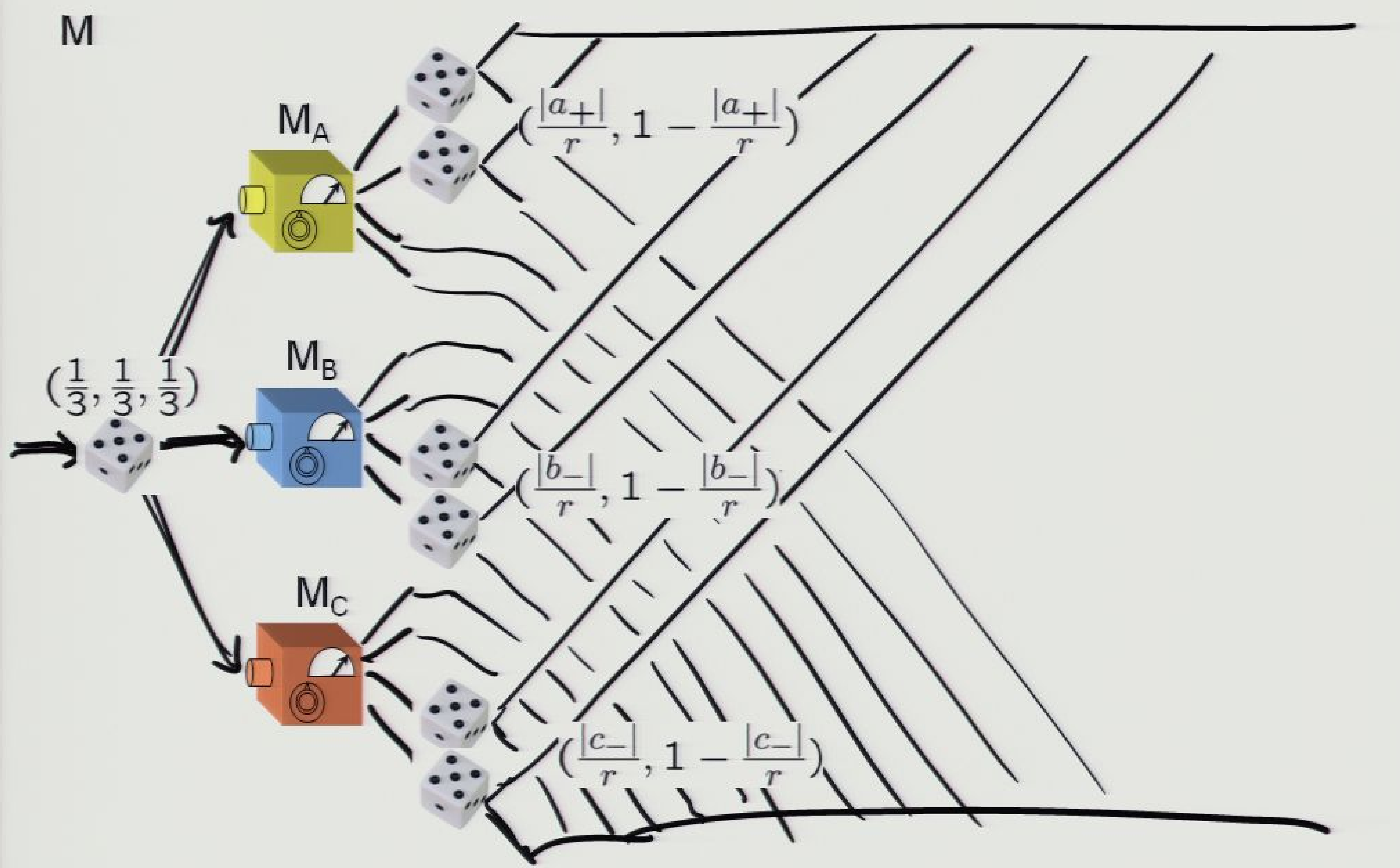


Associated effect

$$\frac{|a_+|}{3r} P_{a_+}$$

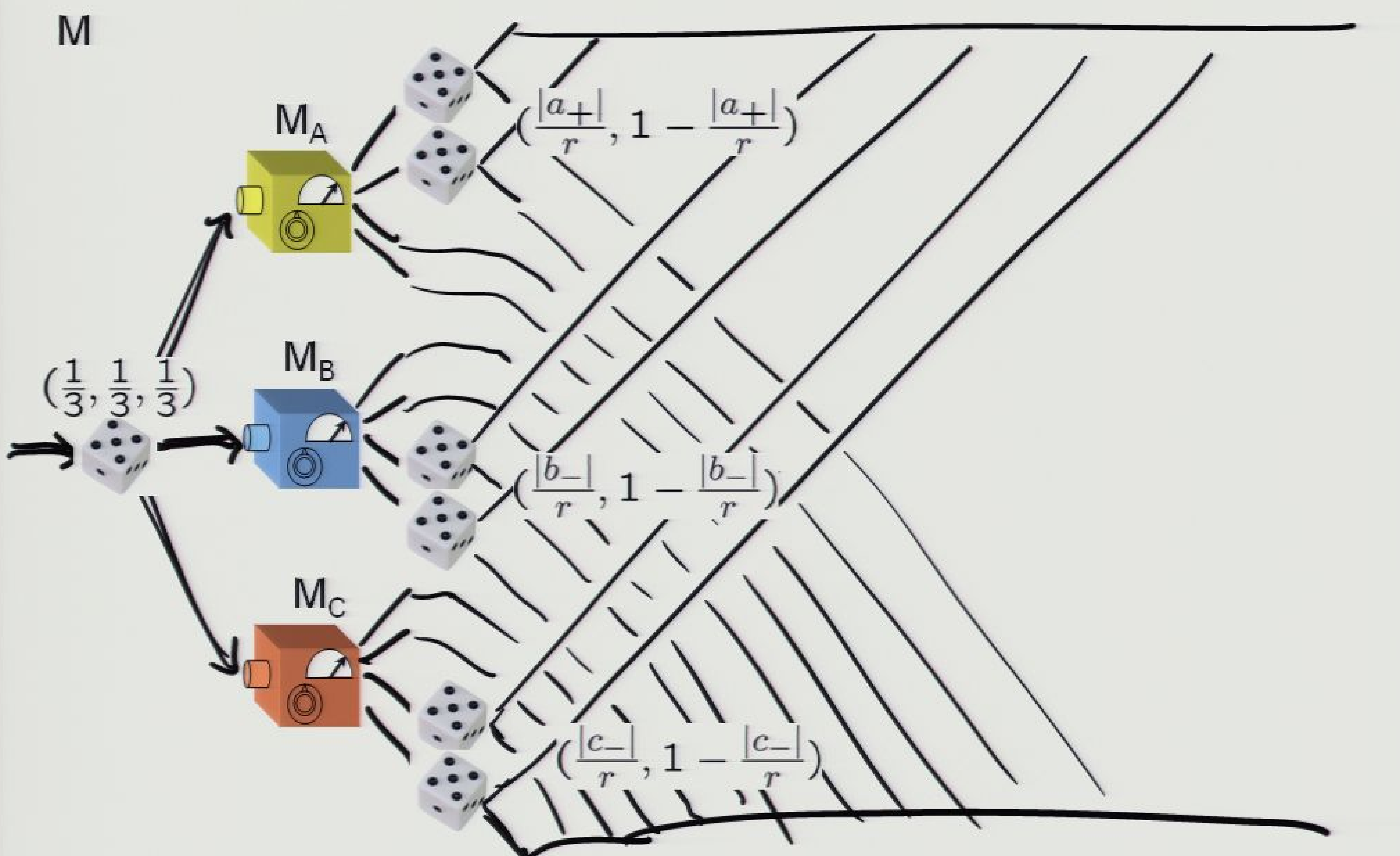
Associated indicator function

$$\frac{|a_+|}{3r} \chi_{a_+}(\lambda)$$



probability of *some* upward branch outcome of M given preparation P

$$\sum \frac{|a_+|}{3r}\mathrm{Pr}(a_+|\mathrm{M}_A, \mathrm{P}) + \sum \frac{|b_-|}{3r}\mathrm{Pr}(b_-|\mathrm{M}_B, \mathrm{P}) + \sum \frac{|c_-|}{3r}\mathrm{Pr}(c_-|\mathrm{M}_C, \mathrm{P})$$

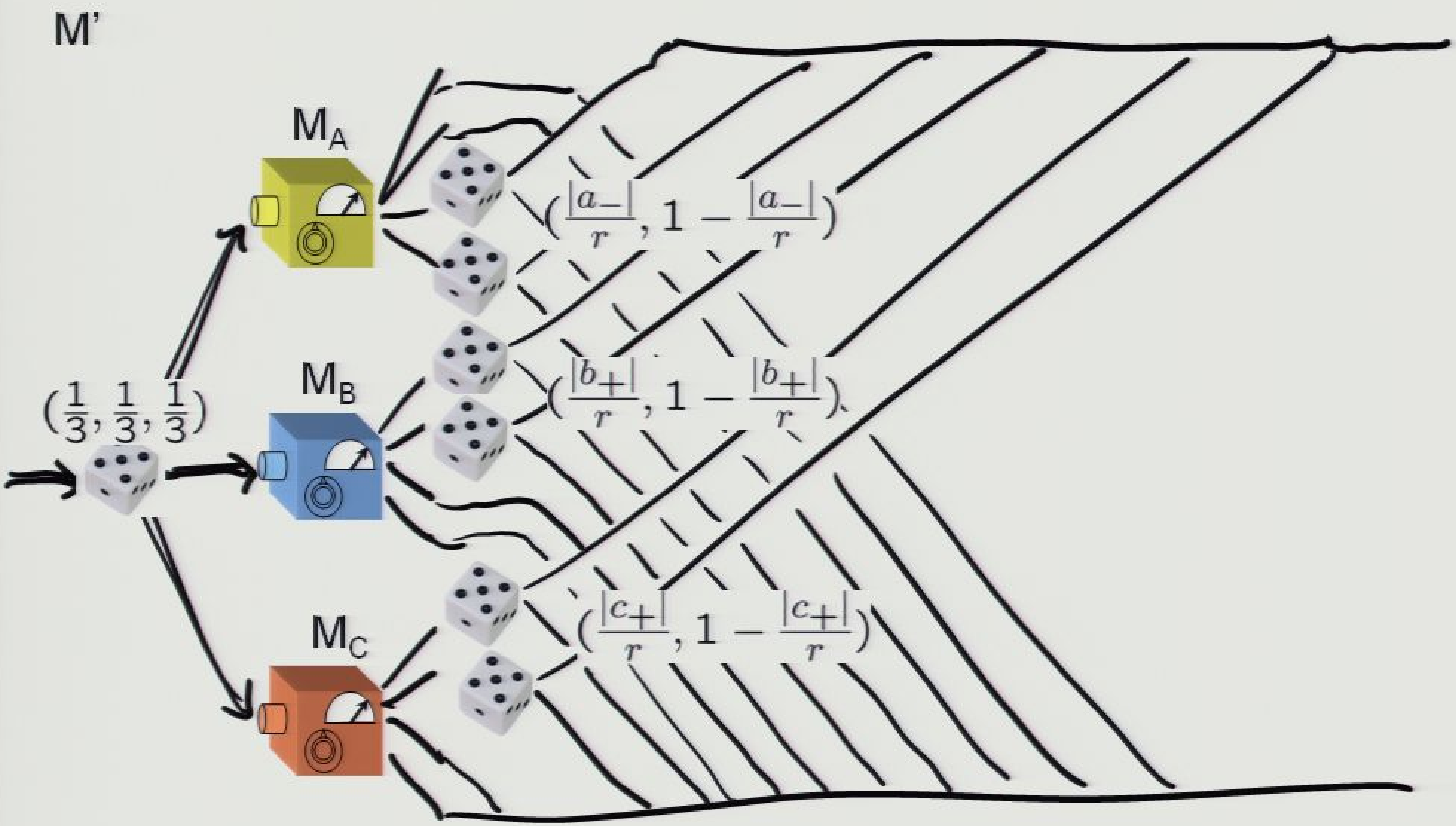


Associated effect

$$\sum_{a_+} \frac{|a_+|}{3r} P_{a_+} + \sum_{b_-} \frac{|b_-|}{3r} P_{b_-} + \sum_{c_-} \frac{|c_-|}{3r} P_{c_-}$$

Associated indicator function

$$\sum_{a_+} \frac{|a_+|}{3r} \chi_{a_+}(\lambda) + \sum_{b_-} \frac{|b_-|}{3r} \chi_{b_-}(\lambda) + \sum_{c_-} \frac{|c_-|}{3r} \chi_{c_-}(\lambda)$$



For the outcome corresponding to some upward branch of M'

Associated effect

$$\sum_{a_-}\frac{|a_-|}{3r}P_{a_-}+\sum_{b_+}\frac{|b_+|}{3r}P_{b_+}+\sum_{c_+}\frac{|c_+|}{3r}P_{c_+}$$

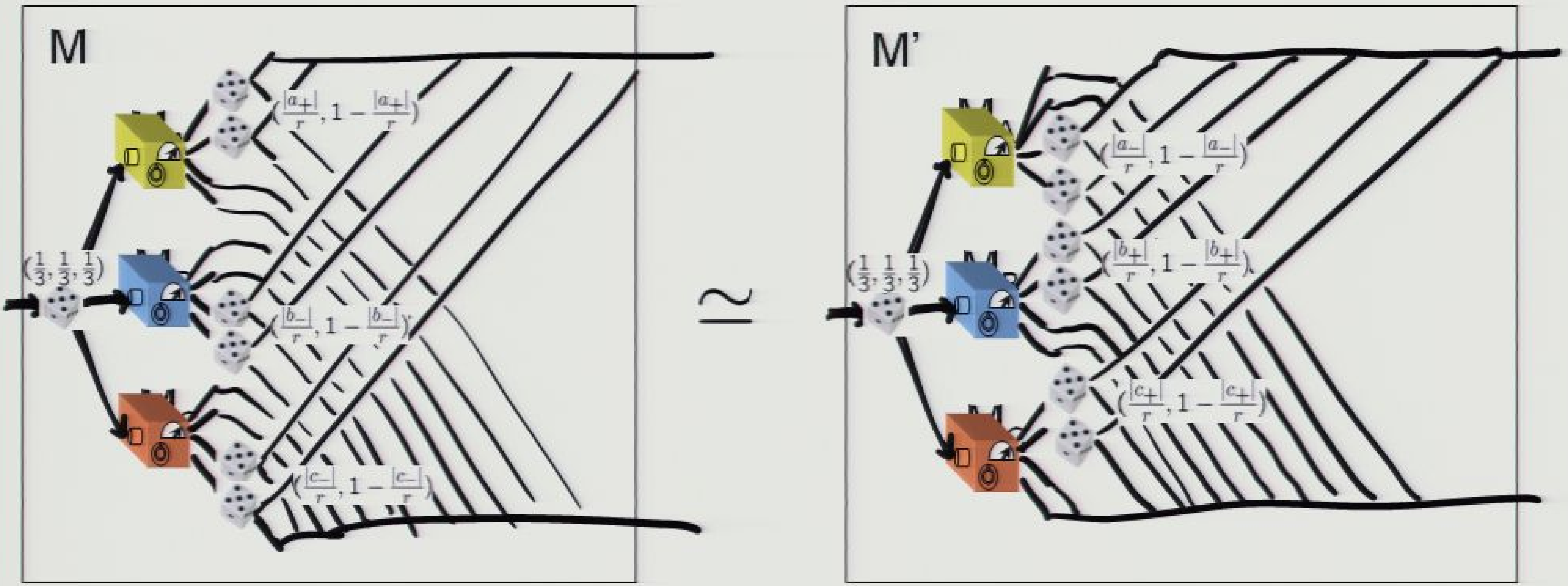
Associated indicator function

$$\sum_{a_-}\frac{|a_-|}{3r}\chi_{a_-}(\lambda)+\sum_{b_+}\frac{|b_+|}{3r}\chi_{b_+}(\lambda)+\sum_{c_+}\frac{|c_+|}{3r}\chi_{c_+}(\lambda)$$

By assumption

$$\sum_{a_+}\frac{|a_+|}{3r}P_{a_+}+\sum_{b_-}\frac{|b_-|}{3r}P_{b_-}+\sum_{c_-}\frac{|c_-|}{3r}P_{c_-}=\sum_{a_-}\frac{|a_-|}{3r}P_{a_-}+\sum_{b_+}\frac{|b_+|}{3r}P_{b_+}+\sum_{c_+}\frac{|c_+|}{3r}P_{c_+}$$

Consequently, M and M' are operationally equivalent



By assumption

$$\sum_{a_+} \frac{|a_+|}{3r} P_{a_+} + \sum_{b_-} \frac{|b_-|}{3r} P_{b_-} + \sum_{c_-} \frac{|c_-|}{3r} P_{c_-} = \sum_{a_-} \frac{|a_-|}{3r} P_{a_-} + \sum_{b_+} \frac{|b_+|}{3r} P_{b_+} + \sum_{c_+} \frac{|c_+|}{3r} P_{c_+}$$

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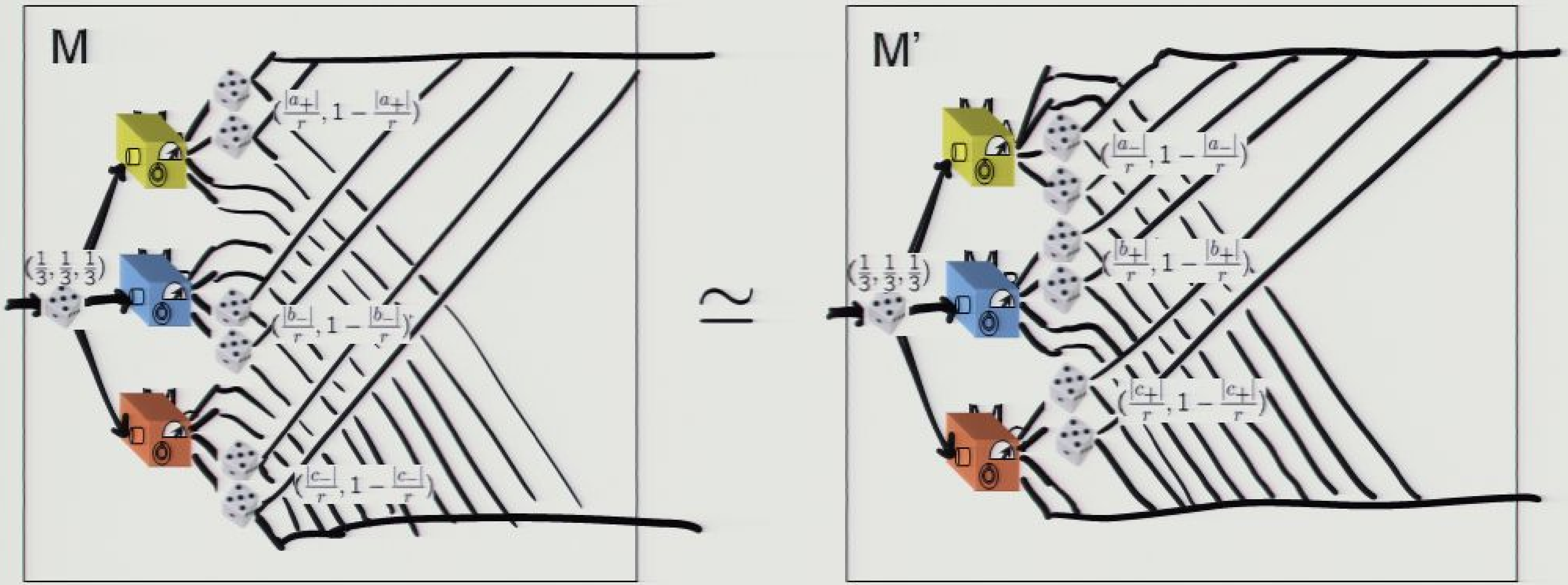

But then, by noncontextuality for unsharp mmts

$$\sum \frac{|a_+|}{3r} \chi_{a_+}(\lambda) + \sum \frac{|b_-|}{3r} \chi_{b_-}(\lambda) + \sum \frac{|c_-|}{3r} \chi_{c_-}(\lambda) = \sum \frac{|a_-|}{3r} \chi_{a_-}(\lambda) + \sum \frac{|b_+|}{3r} \chi_{b_+}(\lambda) + \sum \frac{|c_+|}{3r} \chi_{c_+}(\lambda)$$

$$\sum_{a_+}\frac{|a_+|}{3r}\chi_{a_+}(\lambda)+\sum_{b_-}\frac{|b_-|}{3r}\chi_{b_-}(\lambda)+\sum_{c_-}\frac{|c_-|}{3r}\chi_{c_-}(\lambda)=\sum_{a_-}\frac{|a_-|}{3r}\chi_{a_-}(\lambda)+\sum_{b_+}\frac{|b_+|}{3r}\chi_{b_+}(\lambda)+\sum_{c_+}\frac{|c_+|}{3r}\chi_{c_+}(\lambda)$$

$$\sum_{a_+} \frac{|a_+|}{3r}\chi_{a_+}(\lambda) + \sum_{b_-} \frac{|b_-|}{3r}\chi_{b_-}(\lambda) + \sum_{c_-} \frac{|c_-|}{3r}\chi_{c_-}(\lambda) = \sum_{a_-} \frac{|a_-|}{3r}\chi_{a_-}(\lambda) + \sum_{b_+} \frac{|b_+|}{3r}\chi_{b_+}(\lambda) + \sum_{c_+} \frac{|c_+|}{3r}\chi_{c_+}(\lambda)$$

Multiplying by 3r and rearranging terms, we have

$$\sum_a a\chi_a(\lambda) = \sum_b b\chi_b(\lambda) + \sum_c c\chi_c(\lambda)$$

$$\sum_{a_+} \frac{|a_+|}{3r} \chi_{a_+}(\lambda) + \sum_{b_-} \frac{|b_-|}{3r} \chi_{b_-}(\lambda) + \sum_{c_-} \frac{|c_-|}{3r} \chi_{c_-}(\lambda) = \sum_{a_-} \frac{|a_-|}{3r} \chi_{a_-}(\lambda) + \sum_{b_+} \frac{|b_+|}{3r} \chi_{b_+}(\lambda) + \sum_{c_+} \frac{|c_+|}{3r} \chi_{c_+}(\lambda)$$

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$$\sum_a a\chi_a(\lambda) = \sum_b b\chi_b(\lambda) + \sum_c c\chi_c(\lambda)$$

$$f_A(\lambda) = f_B(\lambda) + f_C(\lambda)$$

$$\sum_{a_+}\frac{|a_+|}{3r}\chi_{a_+}(\lambda)+\sum_{b_-}\frac{|b_-|}{3r}\chi_{b_-}(\lambda)+\sum_{c_-}\frac{|c_-|}{3r}\chi_{c_-}(\lambda)=\sum_{a_-}\frac{|a_-|}{3r}\chi_{a_-}(\lambda)+\sum_{b_+}\frac{|b_+|}{3r}\chi_{b_+}(\lambda)+\sum_{c_+}\frac{|c_+|}{3r}\chi_{c_+}(\lambda)$$

Multiplying by 3r and rearranging terms, we have

$$\sum_a a\chi_a(\lambda)=\sum_b b\chi_b(\lambda)+\sum_c c\chi_c(\lambda)$$

$$f_A(\lambda)=f_B(\lambda)+f_C(\lambda)$$

So we have rederived von Neumann's assumption!

Can we just verify that A=B+C rather than the implementing the two measurements just described?

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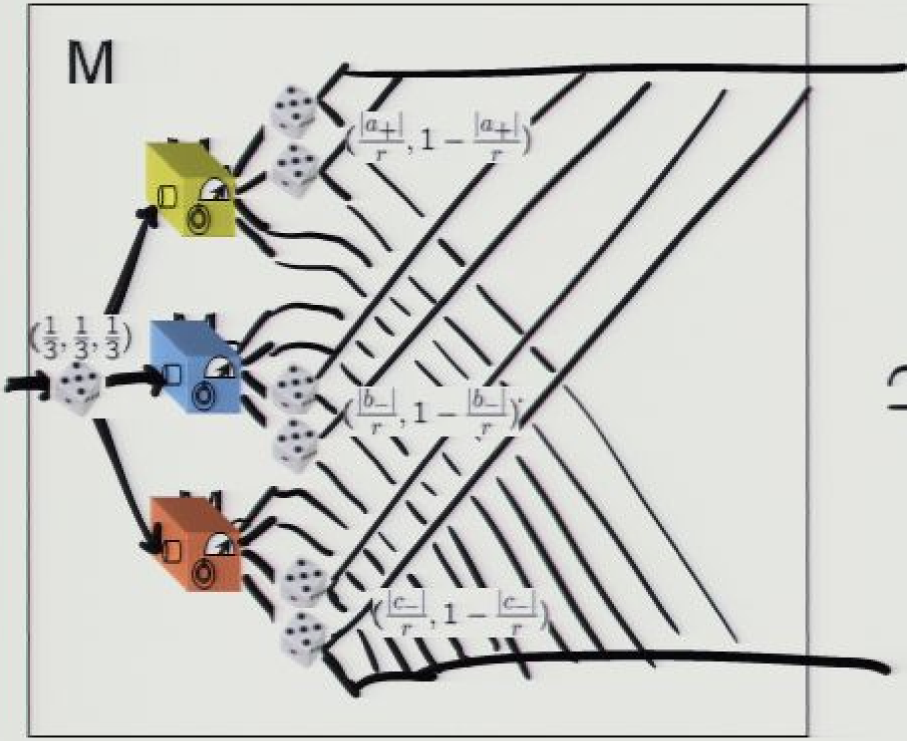
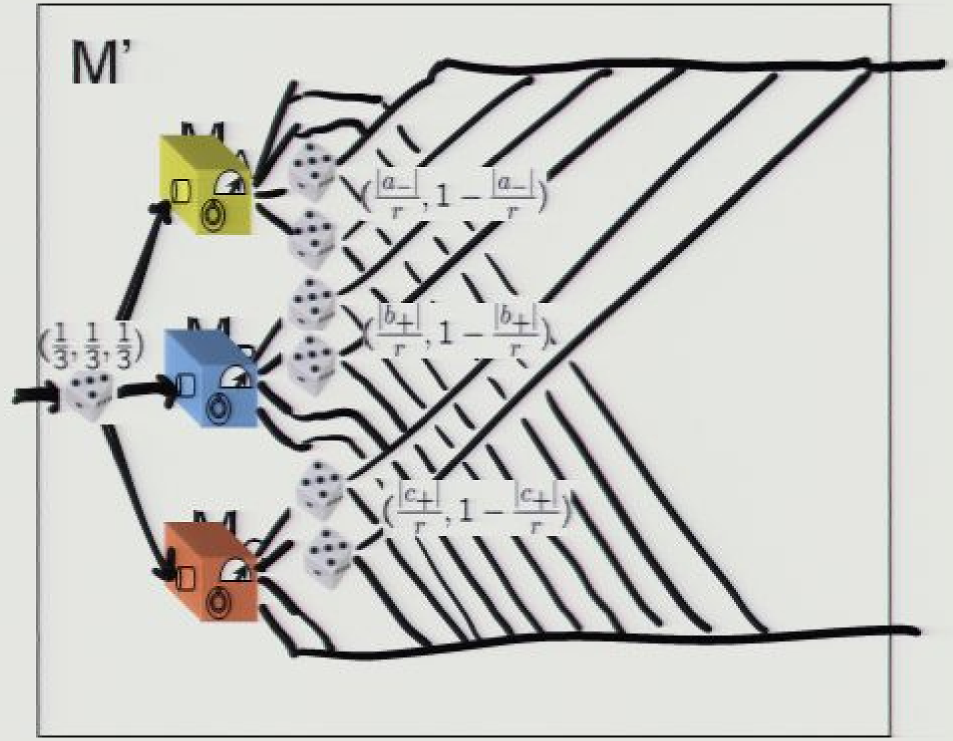
Yes.



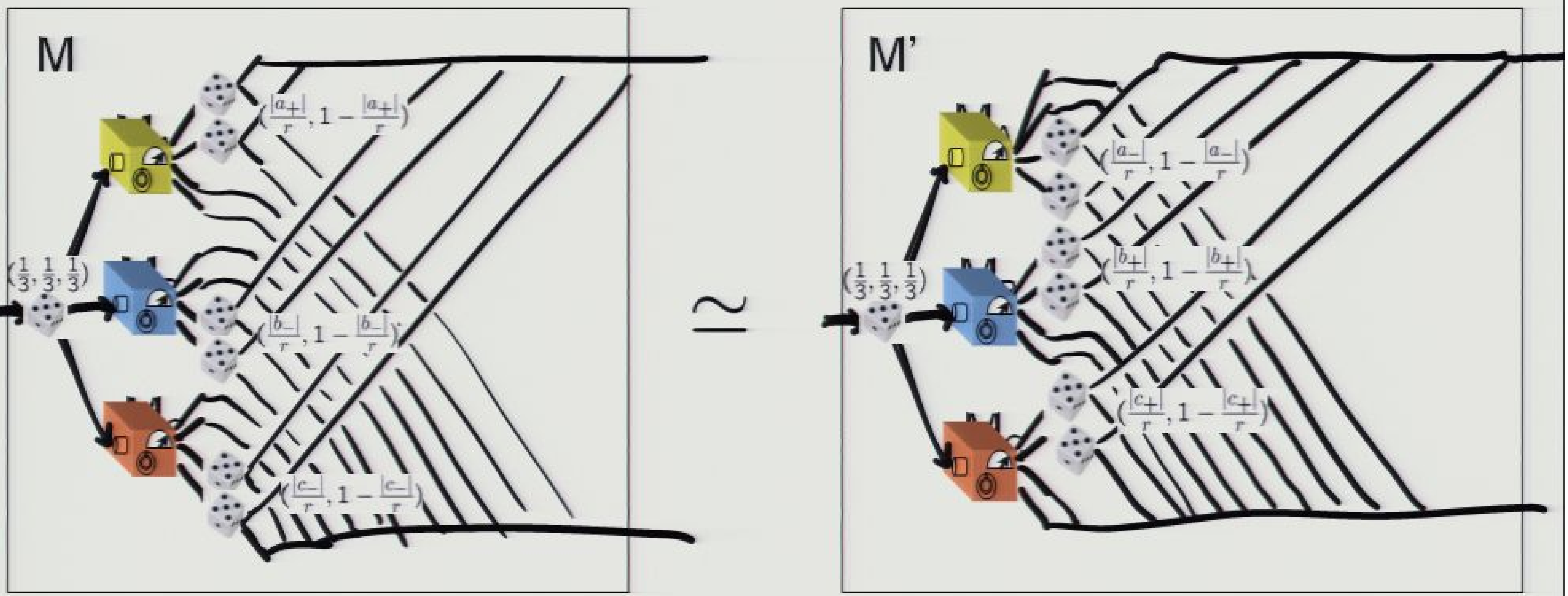
$\simeq$



The empirical content of M $\simeq$ M' is that

$$\sum_{a_+}\frac{|a_+|}{3r}\Pr(a_+|\mathsf{M}_A,\mathsf{P})+\sum_{b}\frac{|b_-|}{3r}\Pr(b_-|\mathsf{M}_B,\mathsf{P})+\sum_{c_-}\frac{|c_-|}{3r}\Pr(c_-|\mathsf{M}_C,\mathsf{P})$$
$$=\sum_{a_-}\frac{|a_-|}{3r}\Pr(a_-|\mathsf{M}_A,\mathsf{P})+\sum_{b_+}\frac{|b_+|}{3r}\Pr(b_+|\mathsf{M}_B,\mathsf{P})+\sum_{c_+}\frac{|c_+|}{3r}\Pr(c_+|\mathsf{M}_C,\mathsf{P})$$

for all preparations P



But by noncontextuality, the rolling of the dice cannot be important

Instead, just determine $\mathrm{Pr}(a|\mathrm{M}_A,\mathrm{P}),\ \mathrm{Pr}(b|\mathrm{M}_B,\mathrm{P}),\ \mathrm{Pr}(c|\mathrm{M}_C,\mathrm{P})\ \forall \mathrm{P}$

Then numerically verify that

$$\sum_{a_+}\frac{|a_+|}{3r}\mathrm{Pr}(a_+|\mathrm{M}_A,\mathrm{P})+\sum_{b}\frac{|b_-|}{3r}\mathrm{Pr}(b_-|\mathrm{M}_B,\mathrm{P})+\sum_{c_-}\frac{|c_-|}{3r}\mathrm{Pr}(c_-|\mathrm{M}_C,\mathrm{P})$$
$$=\sum_{a_-}\frac{|a_-|}{3r}\mathrm{Pr}(a_-|\mathrm{M}_A,\mathrm{P})+\sum_{b_+}\frac{|b_+|}{3r}\mathrm{Pr}(b_+|\mathrm{M}_B,\mathrm{P})+\sum_{c_+}\frac{|c_+|}{3r}\mathrm{Pr}(c_+|\mathrm{M}_C,\mathrm{P})$$

for all preparations P

But this is equivalent to numerically verifying that

$$\sum_a a \Pr(a|\mathsf{M}_A, \mathsf{P}) = \sum_b b \Pr(b|\mathsf{M}_B, \mathsf{P}) + \sum_c c \Pr(c|\mathsf{M}_C, \mathsf{P}) \ \ \forall \mathsf{P}$$

which is precisely the empirical content of

$$A = B + C$$

Faster proof:

Lemma: Any function g over positive operators satisfying

$$g(\sum_k r_k E_k) = \sum_k r_k g(E_k)$$

where $r_k \geq 0$, can be extended uniquely to a linear function over the Hermitian operators

$$g(\sum_j a_j A_j) = \sum_j a_j g(A_j)$$

where the a_j are real.

See: Busch, Phys. Rev. Lett. **91**, 120403 (2003)
Caves, Fuchs, Manne, and Renes, Found.Phys. **34**, 193 (2004)

Noncontextuality for preparations and measurements $\longrightarrow$ von Neumann's assumptions

Were von Neumann's assumptions "silly"?

Mermin on von Neumann:

"...to require that $v(A+B)=v(A)+v(B)$ in each individual system of the ensemble is to ensure that a relation holds in the mean by imposing it case by case ---a sufficient, but hardly a necessary condition. Silly!"

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Mermin on Bell-Kochen-Specker:

"If we do the experiment to measure A with B,C,... on an ensemble of systems prepared in the state and ignore the results of the other observables, we get exactly the same statistics for A as we would have obtained had we instead done the quite different experiment to measure A with L,M,... on that same ensemble. The obvious way to account for this, particularly when entertaining the possibility of a hidden-variables theory, is to propose that both experiments reveal a set of values for A in the individual systems that is the same, regardless of which experiment we choose to extract them from."

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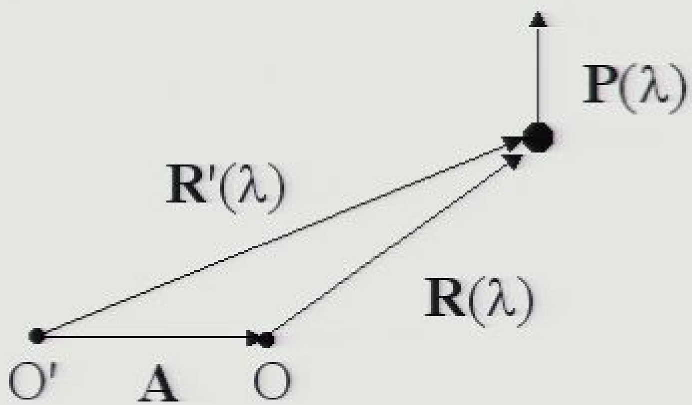
The obvious way is not the only way – it is a sufficient but not a necessary condition.

Either both proofs are silly or neither is!

More variants of von Neumann's no-go theorem

Schrödinger's example

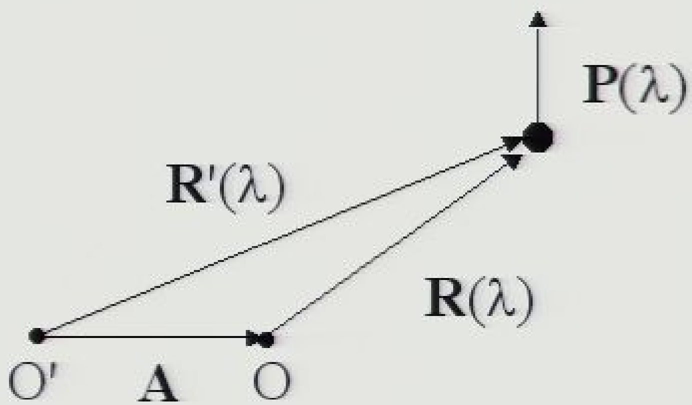


$$\vec{L} = \vec{R} \times \vec{P}$$

$$\vec{L}(\lambda) = \vec{R}(\lambda) \times \vec{P}(\lambda)$$

More variants of von Neumann's no-go theorem

Schrödinger's example



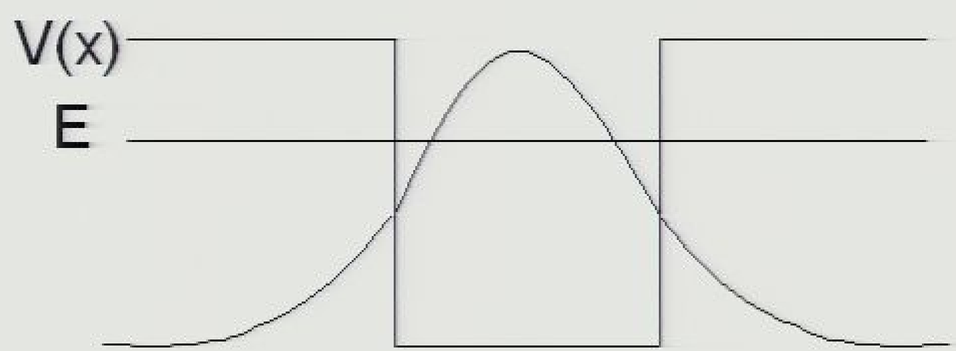
$$\vec{L} = \vec{R} \times \vec{P}$$

$$\vec{L}(\lambda) = \vec{R}(\lambda) \times \vec{P}(\lambda)$$

The tunneling example

$$H = \frac{P^2}{2m} + V(X)$$

$$H(\lambda) = \frac{P(\lambda)^2}{2m} + V(X(\lambda))$$



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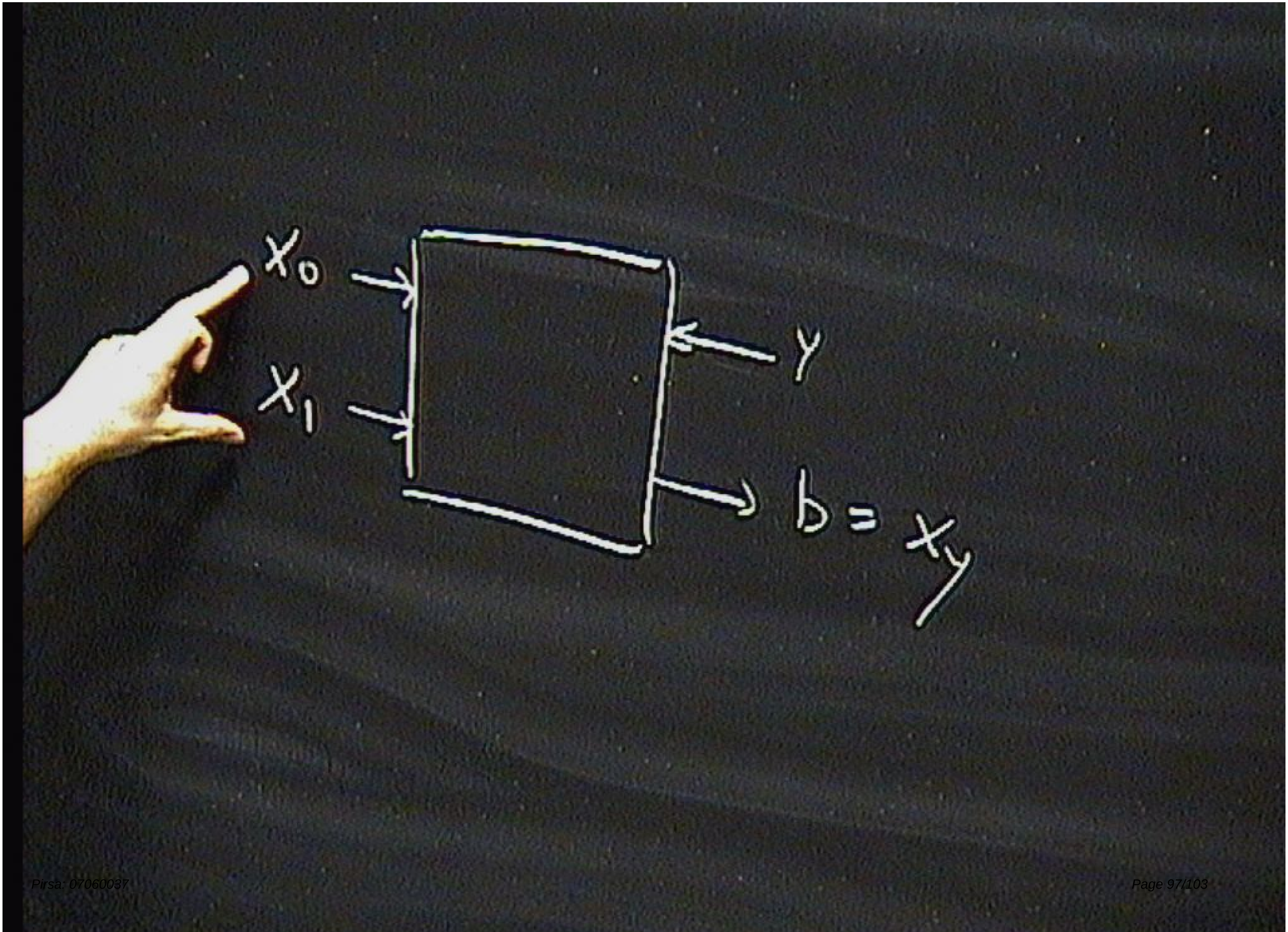
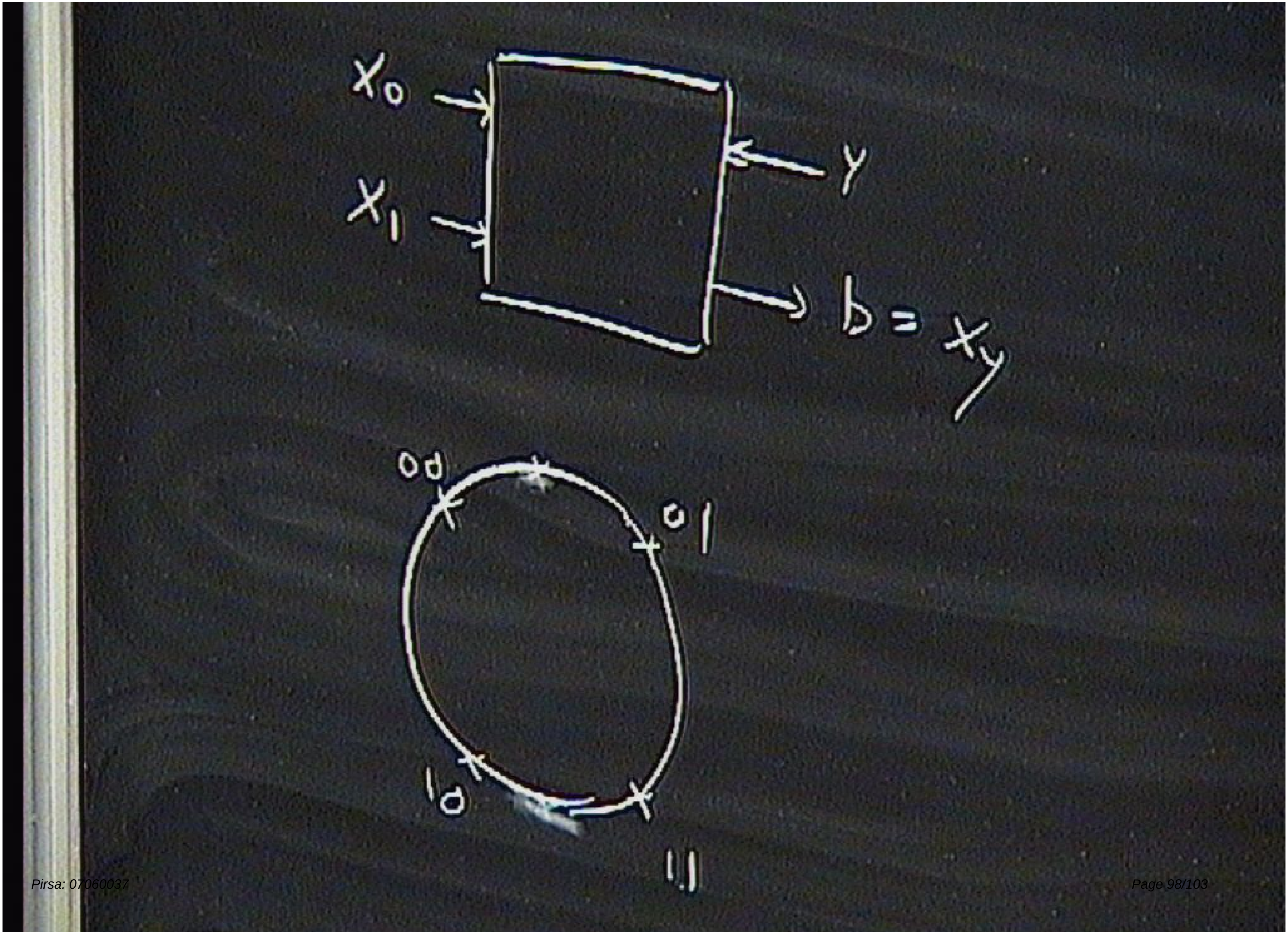
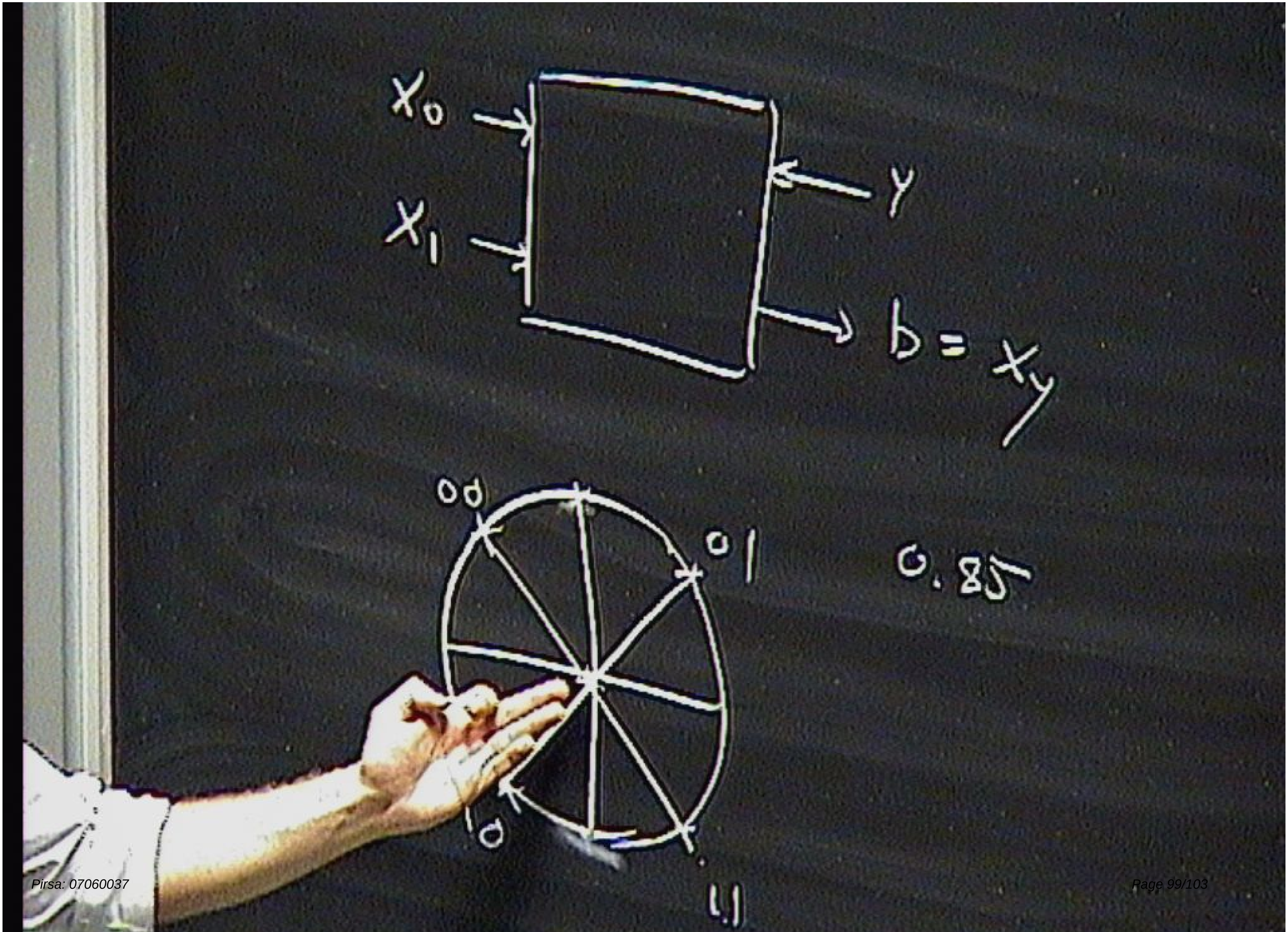
Most notions of nonclassicality can be understood as either:

- The result of an epistemic restriction
- An instance of the generalized notion of contextuality


x_0
x_1
y
$b = x_y$

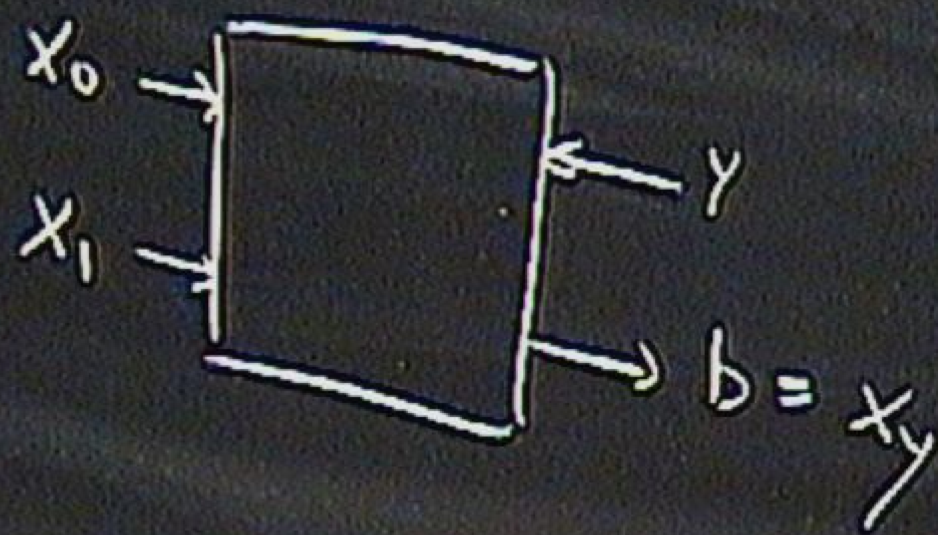


x_0
x_1
y
$b = x_y$
00
01
10
11

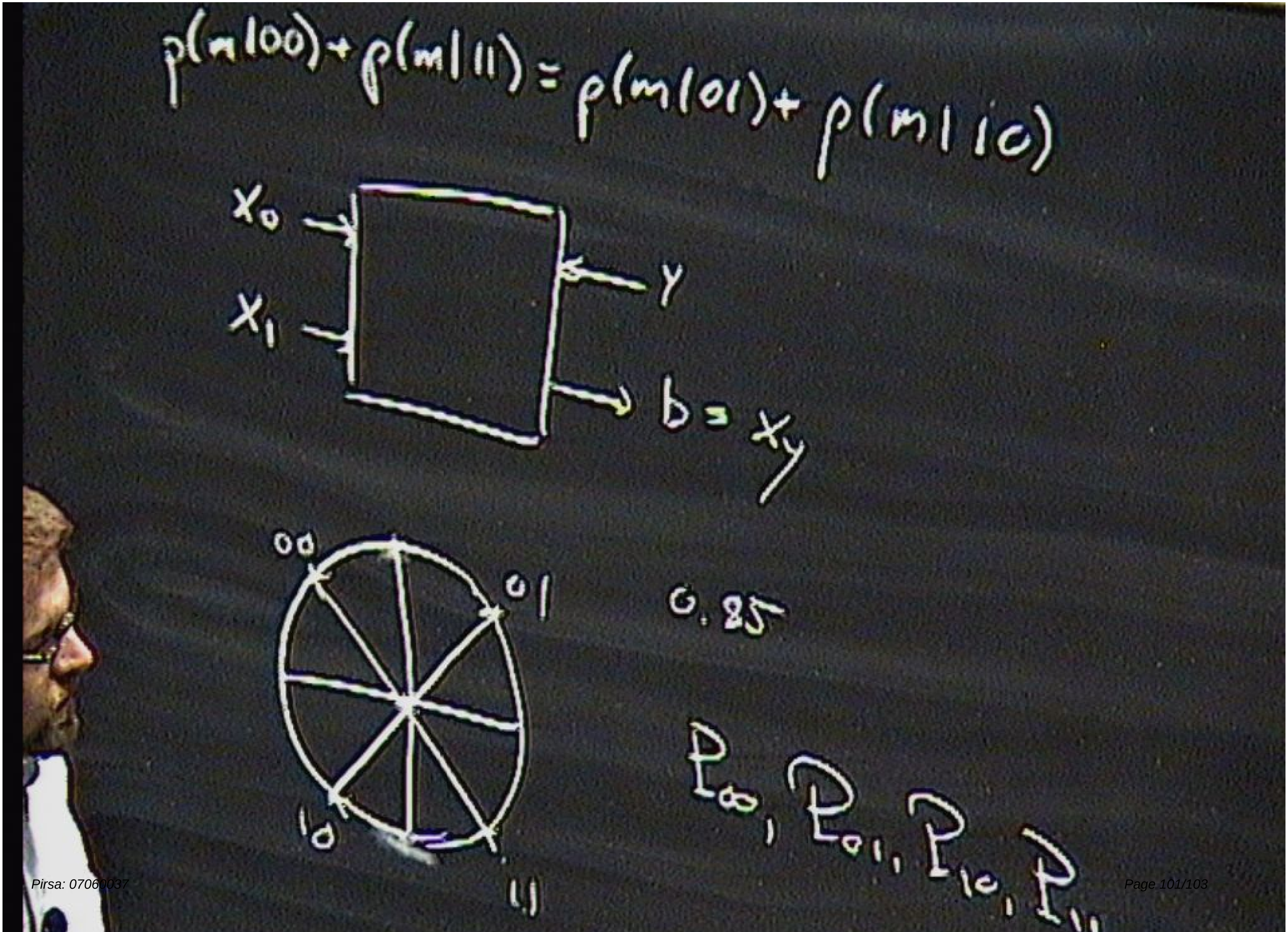


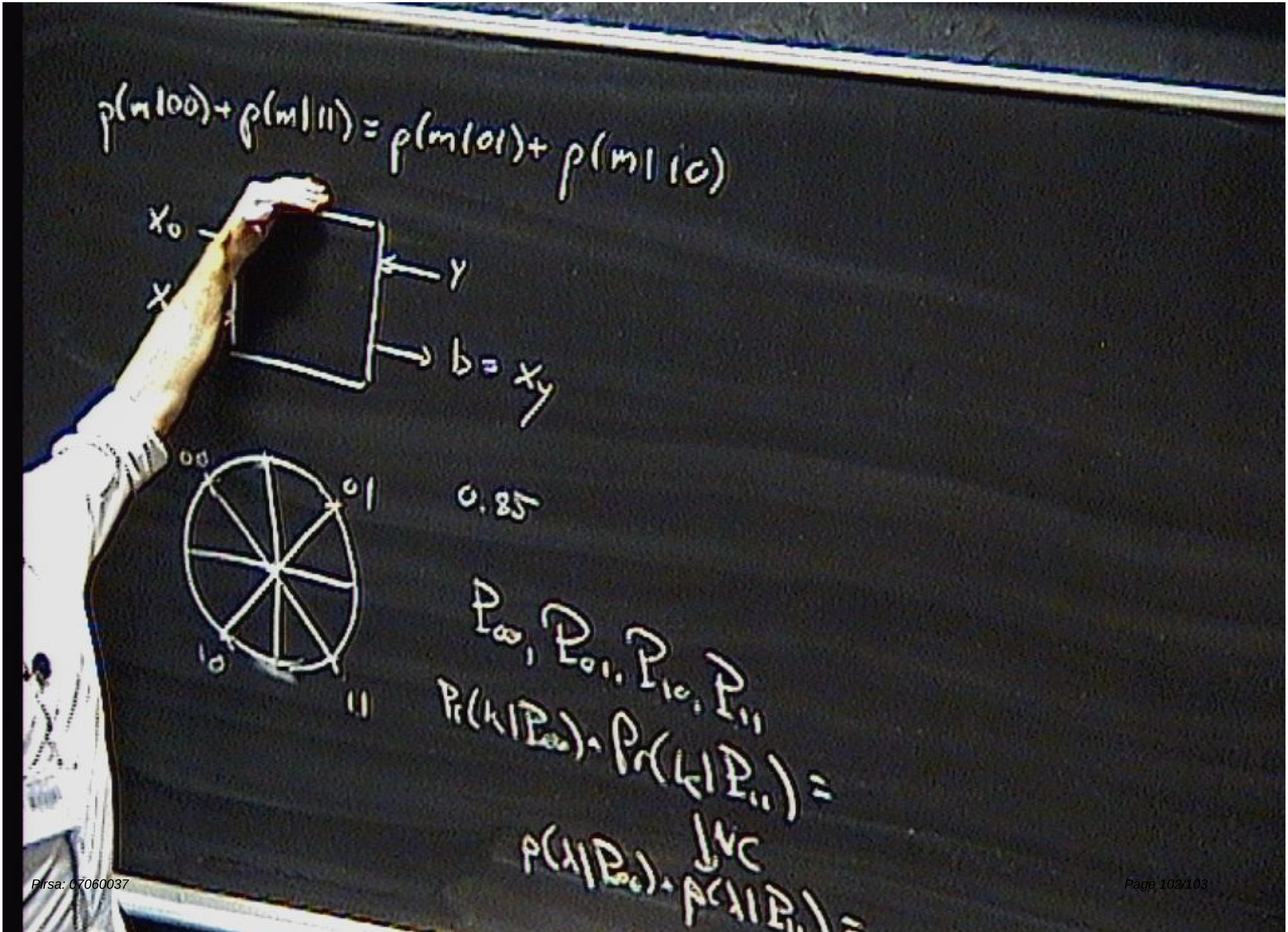
X0
X1
Y
b = xy
00
01
0.85
0
11

$$p(m|00)+p(m|11)=p(m|01)+p(m|10)$$



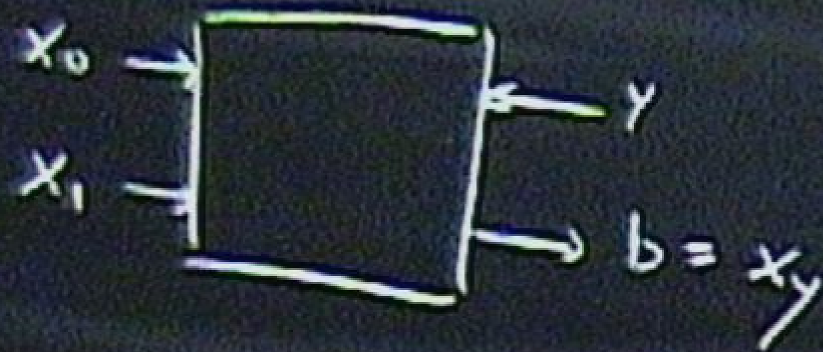
0.85


p(m|00) + p(m|11) = p(m|01) + p(m|10)
X0
X1
y
b = xy
00
01
10
11
0.85
P00, P01, P10, P11



$p(m|00) + p(m|11) = p(m|01) + p(m|10)$
x_0
x_1
y
$b = x_y$
00
01
10
11
0.85
$P_{00}, P_{01}, P_{10}, P_{11}$
$Pr(\lambda|P_{00}) \cdot Pr(\lambda|P_{11}) =$
NC
$p(\lambda|P_{00}) \cdot p(\lambda|P_{11}) =$

$$p(m|00)+p(m|11) = p(m|01)+p(m|10)$$



0.85

$$P_{00}, P_{01}, P_{10}, P_{11}$$

$$Pr(k|P_{00}) + Pr(k|P_{11}) =$$

$$\Big\downarrow NC$$

$$p(\lambda|P_{00}) + p(\lambda|P_{11}) =$$