Title: Lower bounds for Generalized Quantum Finite Automata

Date: Jun 04, 2007 05:40 PM

URL: http://pirsa.org/07060028

Abstract: For most variations of Quantum finite automata (QFA), it is an open question to characterize the language recognition power of these machines. We extend several techniques used to obtain lower bounds on Kondacs and Watrous' 1-way Quantum Finite Automata to the case of Nayak's Generalized Quantum Finite Automata (GQFA). A consequence of these results is that the class of languages recognized by GQFAs is not closed under union.

Lower Bounds for Generalized Quantum Finite Automata

Mark Mercer

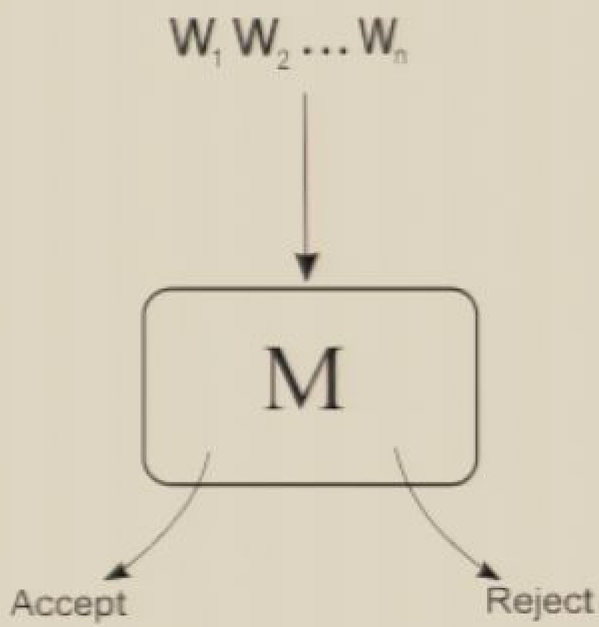
4 June 2007

1. Overview
2. Definitions
3. KWQFA lower bound
4. Extension to GQFA

1. Overview
2. Definitions
3. KWQFA lower bound
4. Extension to GQFA

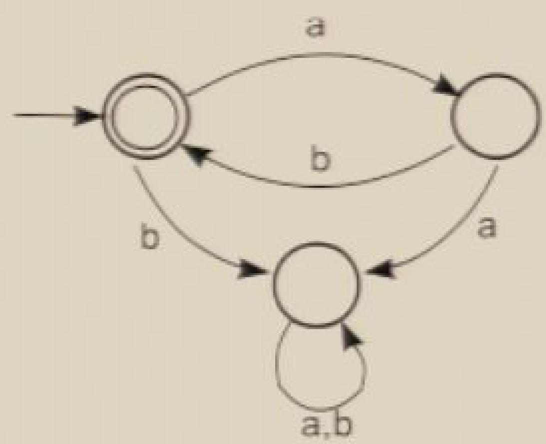
Decision problems and Languages

Σ	finite alphabet
Σ^*	set of finite strings (words) over Σ
$L \subseteq \Sigma^*$	language



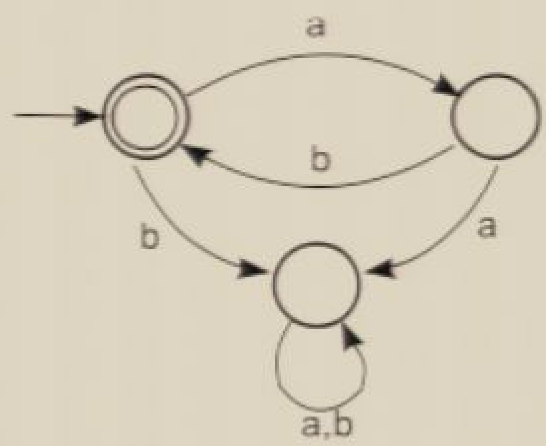
Finite Automata

When the number of states of M is finite and fixed, we call M a *finite automata*



Finite Automata

When the number of states of M is finite and fixed, we call M a *finite automata*



The computational power of finite automata is well understood.

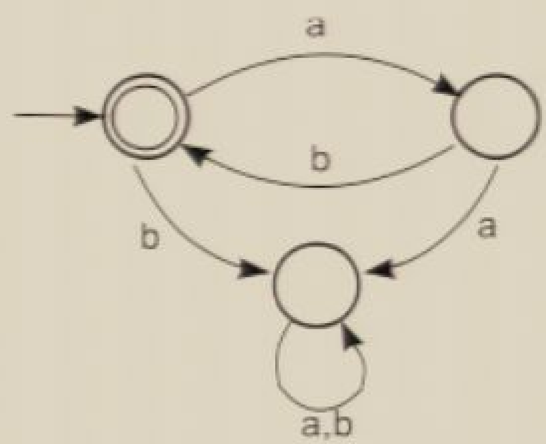
Quantum Finite Automata

Basic idea:

- Machine state: $|\psi\rangle \in \mathbb{C}^n$
- Input: $w \in \Sigma^*$
- Each letter $\sigma \in \Sigma$ induces some operation on the state
- State is measured to determine output

Finite Automata

When the number of states of M is finite and fixed, we call M a *finite automata*

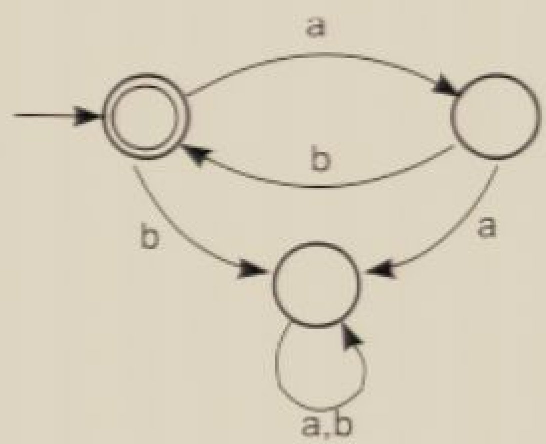


The computational power of finite automata is well understood.

What about the quantum case?

Finite Automata

When the number of states of M is finite and fixed, we call M a *finite automata*



The computational power of finite automata is well understood.

What about the quantum case?

Quantum Finite Automata

Basic idea:

- Machine state: $|\psi\rangle \in \mathbb{C}^n$
- Input: $w \in \Sigma^*$
- Each letter $\sigma \in \Sigma$ induces some operation on the state
- State is measured to determine output

Quantum Finite Automata

Basic idea:

- Machine state: $|\psi\rangle \in \mathbb{C}^n$
- Input: $w \in \Sigma^*$
- Each letter $\sigma \in \Sigma$ induces some operation on the state
- State is measured to determine output

QFAs are parameterized by the class of permissible operations

Central Question: What classes of languages can be recognized by certain restricted classes of operations?

Results

For Generalized QFA (GQFA):

- new necessary conditions for recognition
- the class of languages recognized by GQFA is not closed under union
- there is no way a priori way to boost the acceptance probability of a GQFA.

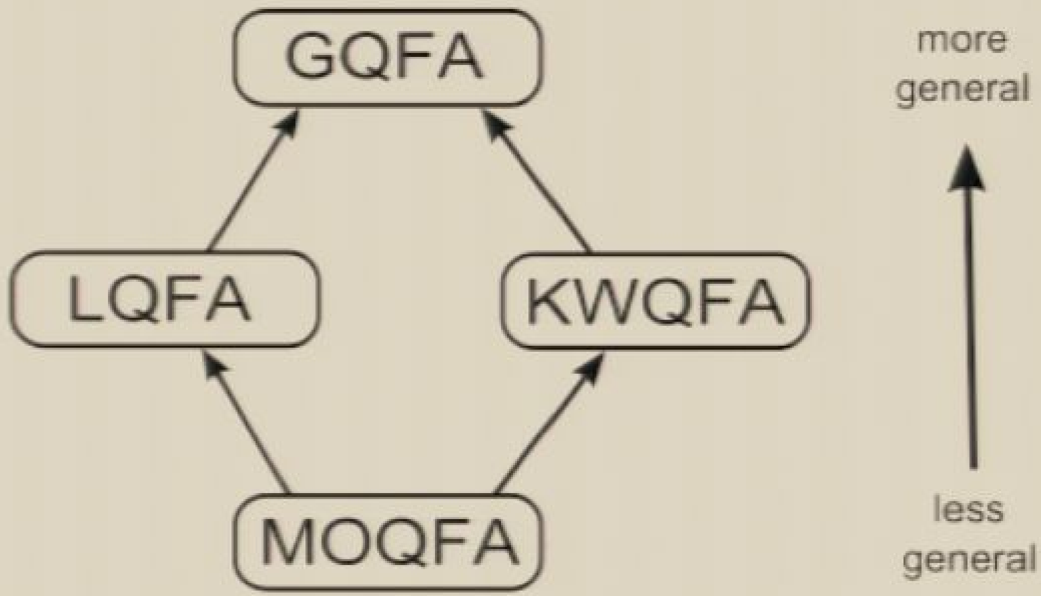
Results

For Generalized QFA (GQFA):

- new necessary conditions for recognition
- the class of languages recognized by GQFA is not closed under union
- there is no way a priori way to boost the acceptance probability of a GQFA.

Variations

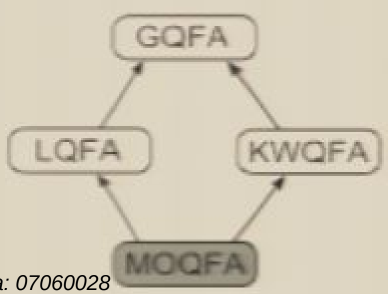
We will consider four variations of QFA:



Measure Once QFA (MOQFA)

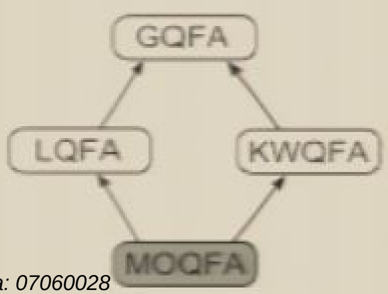
Defined by:

- input alphabet Σ
- dimension n
- initial state $|q_0\rangle \in \mathcal{H}^n$
- a unitary $U_\sigma : \mathcal{H}^n \rightarrow \mathcal{H}^n, \forall \sigma \in \Sigma$
- projective measurement $\{P_{acc}, P_{rej}\}$



Measure Once QFA (MOQFA)

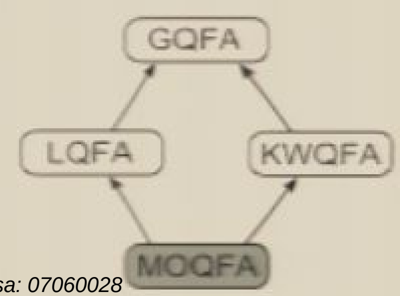
- initial state is $|q_0\rangle$
- on input $w_1 \dots w_k \in \Sigma^*$, we apply $|q_0\rangle \mapsto U_{w_k} \dots U_{w_1}|q_0\rangle$ and we apply the measurement $\{P_{acc}, P_{rej}\}$
- recognition criteria: bounded probability



Measure Once QFA (MOQFA)

- initial state is $|q_0\rangle$
- on input $w_1 \ldots w_k \in \Sigma^*$, we apply $|q_0\rangle \mapsto U_{w_k} \ldots U_{w_1}|q_0\rangle$ and we apply the measurement $\{P_{acc}, P_{rej}\}$
- recognition criteria: bounded probability

MOQFA recognize exactly those languages which can be recognized by permutation automata – but can do so efficiently.

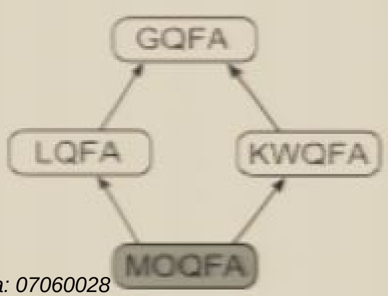


Measure Once QFA (MOQFA)

- initial state is $|q_0\rangle$
- on input $w_1 \ldots w_k \in \Sigma^*$, we apply $|q_0\rangle \mapsto U_{w_k} \ldots U_{w_1}|q_0\rangle$ and we apply the measurement $\{P_{acc}, P_{rej}\}$
- recognition criteria: bounded probability

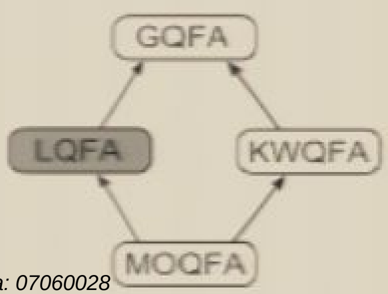
MOQFA recognize exactly those languages which can be recognized by permutation automata – but can do so efficiently.

The associated language class has many natural closure properties.



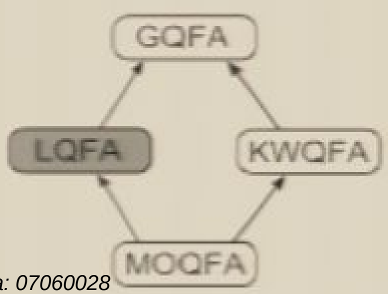
Latvian QFA (LQFA)

- Same as in MOQFA, except that each σ induces a unitary operation U_σ followed by a projective measurement $\{P_{\sigma,i}\}_{i<m_\sigma}$



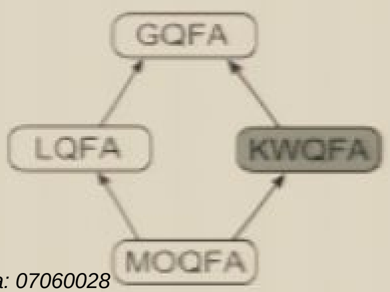
Latvian QFA (LQFA)

- Same as in MOQFA, except that each σ induces a unitary operation U_σ followed by a projective measurement $\{P_{\sigma,i}\}_{i<m_\sigma}$
- State of the machine, taken over probabilistic choices, is a mixed state
- The class of recognized languages is completely classified, and recognition by LQFA is decidable
- Key lower bounds: $\Sigma^* a$ and $a\Sigma^*$



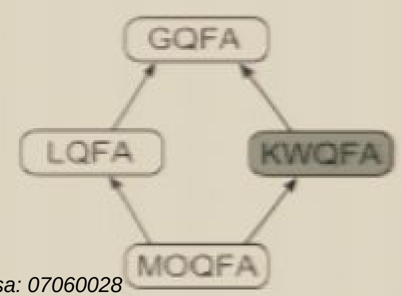
Kondacs-Watrous QFA (KWQFA)

- Same as MOQFA, except after each U_a we also apply the operation $\{P_{acc}, P_{rej}, P_{non}\}$, and halt and accept (reject) if outcome is *acc* (*rej*)



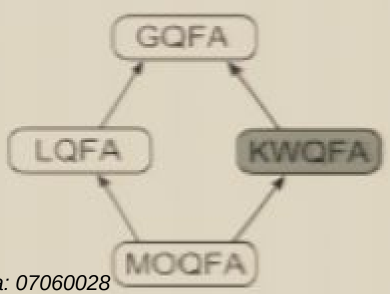
Kondacs-Watrous QFA (KWQFA)

- Same as MOQFA, except after each U_a we also apply the operation $\{P_{acc}, P_{rej}, P_{non}\}$, and halt and accept (reject) if outcome is *acc* (*rej*)
- Can recognize e.g. $\Sigma^* a \Sigma^*$ and $\Sigma^* a_1 \Sigma^* a_2 \Sigma^* a_3 \Sigma^*$
- Cannot recognize $\Sigma^* a$ but can recognize $a\Sigma^*$



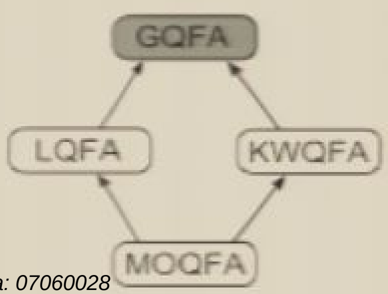
Kondacs-Watrous QFA (KWQFA)

- Same as MOQFA, except after each U_a we also apply the operation $\{P_{acc}, P_{rej}, P_{non}\}$, and halt and accept (reject) if outcome is *acc* (*rej*)
- Can recognize e.g. $\Sigma^* a \Sigma^*$ and $\Sigma^* a_1 \Sigma^* a_2 \Sigma^* a_3 \Sigma^*$
- Cannot recognize $\Sigma^* a$ but can recognize $a\Sigma^*$
- Not closed under union



Generalized QFA (GQFA)

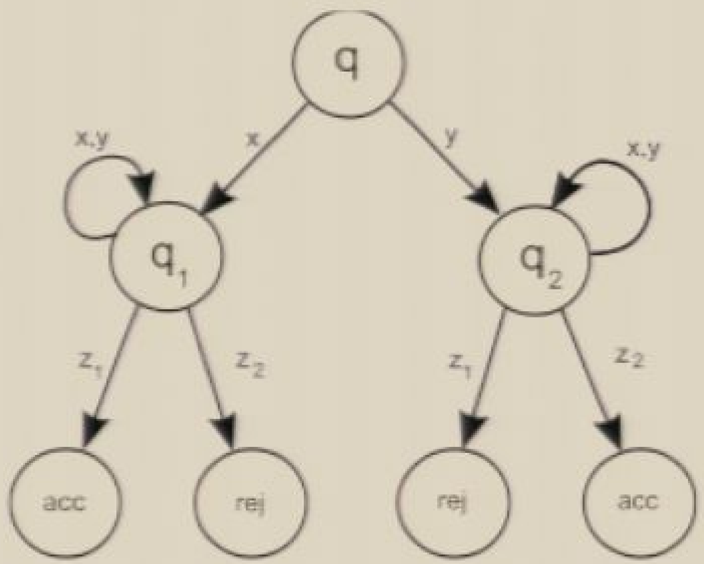
- Same as KWQFA, but each letter induces a unitary transformation and a measurement
- State of the machine, taken over probabilistic choices, is a mixed state
- Can recognize $a\Sigma^*$ but not $\Sigma^* a$



KWQFA lower bound

Theorem

If the minimal automaton for L contains:



Then L cannot be recognized by KWQFA.

KWQFA lower bound: key lemma

For $w \in \Sigma^*$, let A_w be the action on reading w, conditioned on not halting. Let $|\psi\rangle$ represent unnormalized state.

Lemma

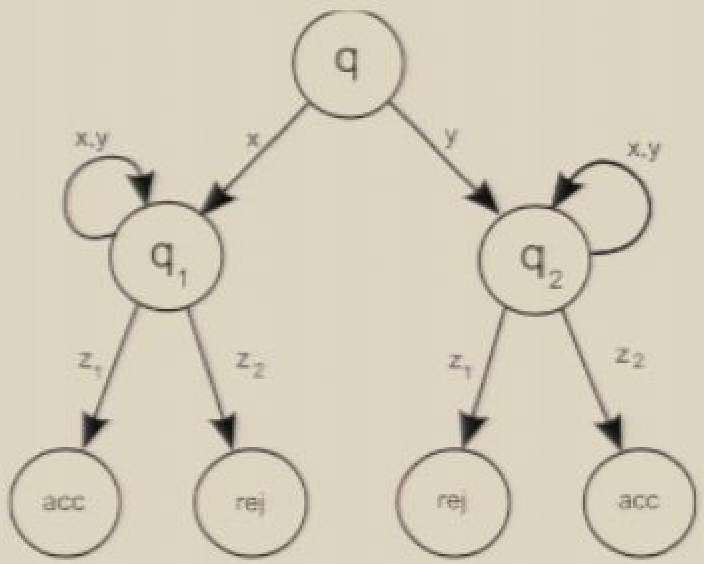
Let M be a KWQFA. For all $w \in \Sigma^*$ there exists a partition of the state space into orthogonal subspaces $E_1^w \oplus E_2^w$ such that

1. $|\psi\rangle \in E_1^w$ *implies* $A_w|\psi\rangle \in E_1^w$ *and* $\||\psi\rangle\| = \|A_w|\psi\rangle\|$, *and*
2. $|\psi\rangle \in E_2^w$ *implies that* $\lim_{k\to\infty} \|(A_w)^k|\psi\rangle\| = 0$.

KWQFA lower bound

Theorem

If the minimal automaton for L contains:



Then L cannot be recognized by KWQFA.

KWQFA lower bound: key lemma

For $w \in \Sigma^*$, let A_w be the action on reading w, conditioned on not halting. Let $|\psi\rangle$ represent unnormalized state.

Lemma

Let M be a KWQFA. For all $w \in \Sigma^*$ there exists a partition of the state space into orthogonal subspaces $E_1^w \oplus E_2^w$ such that

1. $|\psi\rangle \in E_1^w$ *implies* $A_w|\psi\rangle \in E_1^w$ *and* $\||\psi\rangle\| = \|A_w|\psi\rangle\|$, *and*
2. $|\psi\rangle \in E_2^w$ *implies that* $\lim_{k\to\infty} \|(A_w)^k|\psi\rangle\| = 0$.

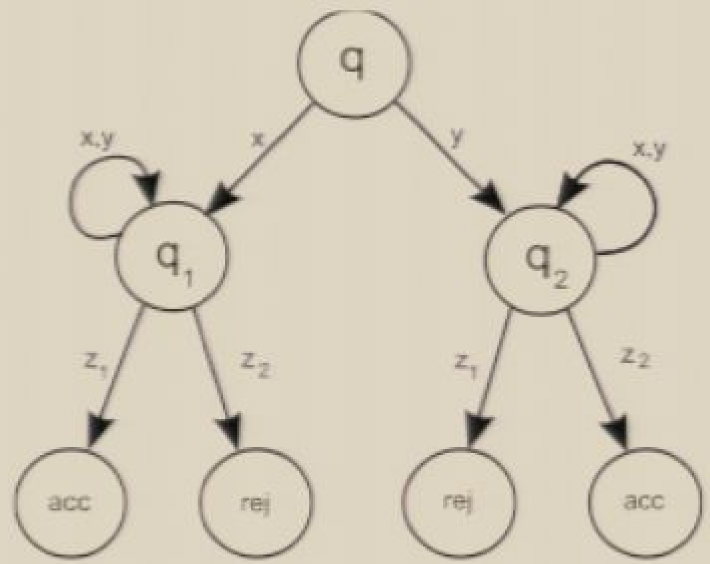
KWQFA lower bound: Proof sketch

- The lemma can be extended so that it works simultaneously for all $w \in (x \cup y)^*$
- We can find $w_x, w_y \in (x \cup y)^*$ such that the E_2 component after reading xw_x or yw_y is small.
- The E_1 and E_2 part of M act independently, and the part of the state in E_1 acts exactly as an MOQFA.
- The behavior of the MOQFA component dominates, and it cannot distinguish between $xw_xw'_x$ and $yw_yw'_y$ for general w'_x, w'_y.

KWQFA lower bound

Theorem

If the minimal automaton for L contains:



Then L cannot be recognized by KWQFA.

KWQFA lower bound: key lemma

For $w \in \Sigma^*$, let A_w be the action on reading w, conditioned on not halting. Let $|\psi\rangle$ represent unnormalized state.

Lemma

Let M be a KWQFA. For all $w \in \Sigma^*$ there exists a partition of the state space into orthogonal subspaces $E_1^w \oplus E_2^w$ such that

1. $|\psi\rangle \in E_1^w$ *implies* $A_w|\psi\rangle \in E_1^w$ *and* $\||\psi\rangle\| = \|A_w|\psi\rangle\|$, *and*
2. $|\psi\rangle \in E_2^w$ *implies that* $\lim_{k\to\infty} \|(A_w)^k|\psi\rangle\| = 0$.

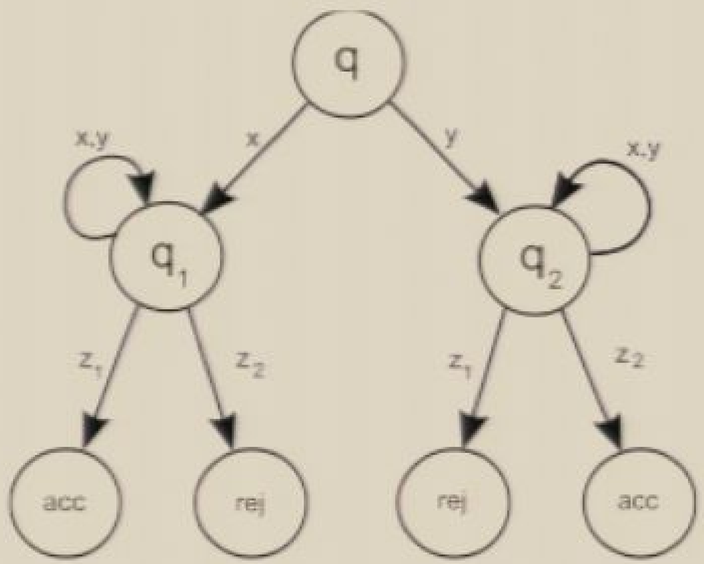
KWQFA lower bound: Proof sketch

- The lemma can be extended so that it works simultaneously for all $w \in (x \cup y)^*$
- We can find $w_x, w_y \in (x \cup y)^*$ such that the E_2 component after reading xw_x or yw_y is small.
- The E_1 and E_2 part of M act independently, and the part of the state in E_1 acts exactly as an MOQFA.
- The behavior of the MOQFA component dominates, and it cannot distinguish between $xw_x w'_x$ and $yw_y w'_y$ for general w'_x, w'_y.

GQFA lower bound

Theorem

If the minimal automaton for L contains:



Then L cannot be recognized by a GQFA.

GQFA lower bound: key lemma

For $w \in \Sigma^*$, let A_w be the action on reading w, conditioned on not halting.

Lemma

For all $w \in \Sigma^*$ there exists a partition of $\mathcal{H}^n$ into orthogonal subspaces $E_1 \oplus E_2$ such that for all density matrices ρ

1. *If $supp(\rho) \subseteq E_1$, then $supp(A_w \rho) \subseteq E_1$ and $Tr(A_w \rho) = Tr(\rho)$*
2. *If $supp(\rho) \subseteq E_2$, then for all $\varepsilon > 0$, $\lim_{k\to\infty} Tr((A_w)^k \rho) = 0$.*

GQFA lower bound: sketch of lemma proof

- Define E_1^1 to be the span of pure states $|\psi\rangle$ such that $span(A_w|\psi\rangle\langle\psi|) \in S_{non}$
- Define inductively E_1^i to be the span of pure state $|\psi\rangle$ such that for all $i' < i$, $supp((A_w)^{i'}|\psi\rangle\langle\psi|) \in E_1^{i-i'}$, and $Tr(A_w|\psi\rangle\langle\psi|) = Tr(|\psi\rangle\langle\psi|)$

GQFA lower bound: sketch of lemma proof

- Define E_1^1 to be the span of pure states $|\psi\rangle$ such that $span(A_w|\psi\rangle\langle\psi|) \in S_{non}$
- Define inductively E_1^i to be the span of pure state $|\psi\rangle$ such that for all $i' < i$, $supp((A_w)^{i'}|\psi\rangle\langle\psi|) \in E_1^{i-i'}$, and $Tr(A_w|\psi\rangle\langle\psi|) = Tr(|\psi\rangle\langle\psi|)$
- These subspaces satisfy $E_{i+1} \subseteq E_i$, thus there is an i_0 such that $E_1^{i_0} = E_1^{i_0+j}$ for all $j > 0$
- $supp(\rho) \subseteq E_1$ implies that $supp(A_w\rho) \subseteq E_1$ and $supp(\rho) \subseteq E_2$ implies that $supp(A_w\rho) \subseteq E_2$
- There is a j such that for all ρ with $supp(\rho) \in E_2$ $(A_w)^j$ decreases the trace of ρ by at least a constant fraction

GQFA lower bound: key lemma

For $w \in \Sigma^*$, let A_w be the action on reading w, conditioned on not halting.

Lemma

For all $w \in \Sigma^*$ there exists a partition of $\mathcal{H}^n$ into orthogonal subspaces $E_1 \oplus E_2$ such that for all density matrices ρ

1. *If $supp(\rho) \subseteq E_1$, then $supp(A_w \rho) \subseteq E_1$ and $Tr(A_w \rho) = Tr(\rho)$*
2. *If $supp(\rho) \subseteq E_2$, then for all $\varepsilon > 0$, $\lim_{k \to \infty} Tr((A_w)^k \rho) = 0$.*

Again, the E_1 part will dominate.

GQFA lower bound: sketch of lemma proof

- Define E_1^1 to be the span of pure states $|\psi\rangle$ such that $span(A_w|\psi\rangle\langle\psi|) \in S_{non}$
- Define inductively E_1^i to be the span of pure state $|\psi\rangle$ such that for all $i' < i$, $supp((A_w)^{i'}|\psi\rangle\langle\psi|) \in E_1^{i-i'}$, and $Tr(A_w|\psi\rangle\langle\psi|) = Tr(|\psi\rangle\langle\psi|)$

GQFA lower bound: sketch of lemma proof

- Define E_1^1 to be the span of pure states $|\psi\rangle$ such that $span(A_w|\psi\rangle\langle\psi|) \in S_{non}$
- Define inductively E_1^i to be the span of pure state $|\psi\rangle$ such that for all $i' < i$, $supp((A_w)^{i'}|\psi\rangle\langle\psi|) \in E_1^{i-i'}$, and $Tr(A_w|\psi\rangle\langle\psi|) = Tr(|\psi\rangle\langle\psi|)$
- These subspaces satisfy $E_{i+1} \subseteq E_i$, thus there is an i_0 such that $E_1^{i_0} = E_1^{i_0+j}$ for all $j > 0$
- $supp(\rho) \subseteq E_1$ implies that $supp(A_w\rho) \subseteq E_1$ and $supp(\rho) \subseteq E_2$ implies that $supp(A_w\rho) \subseteq E_2$

GQFA lower bound: sketch of lemma proof

- Define E_1^1 to be the span of pure states $|\psi\rangle$ such that $span(A_w|\psi\rangle\langle\psi|) \in S_{non}$
- Define inductively E_1^i to be the span of pure state $|\psi\rangle$ such that for all $i' < i$, $supp((A_w)^{i'}|\psi\rangle\langle\psi|) \in E_1^{i-i'}$, and $Tr(A_w|\psi\rangle\langle\psi|) = Tr(|\psi\rangle\langle\psi|)$
- These subspaces satisfy $E_{i+1} \subseteq E_i$, thus there is an i_0 such that $E_1^{i_0} = E_1^{i_0+j}$ for all $j > 0$
- $supp(\rho) \subseteq E_1$ implies that $supp(A_w\rho) \subseteq E_1$ and $supp(\rho) \subseteq E_2$ implies that $supp(A_w\rho) \subseteq E_2$
- There is a j such that for all ρ with $supp(\rho) \in E_2$ $(A_w)^j$ decreases the trace of ρ by at least a constant fraction

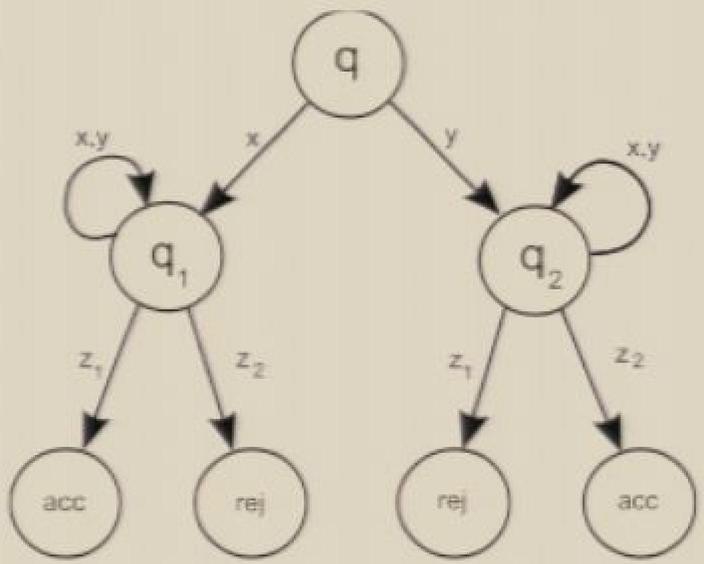
GQFA lower bound: Proof sketch

- Extend lemma so that it works simultaneously for all $w \in (x \cup y)^*$
- Find $w_x, w_y \in (x \cup y)^*$ such that the E_2 component after reading xw_x or yw_y is small
- The part of the state in E_1 acts exactly as an LQFA
- The behavior of the LQFA component dominates, and so by previous results it cannot distinguish between $xw_x w'_x$ and $yw_y w'_y$ for general w'_x, w'_y

GQFA lower bound

Theorem

If the minimal automaton for L contains:



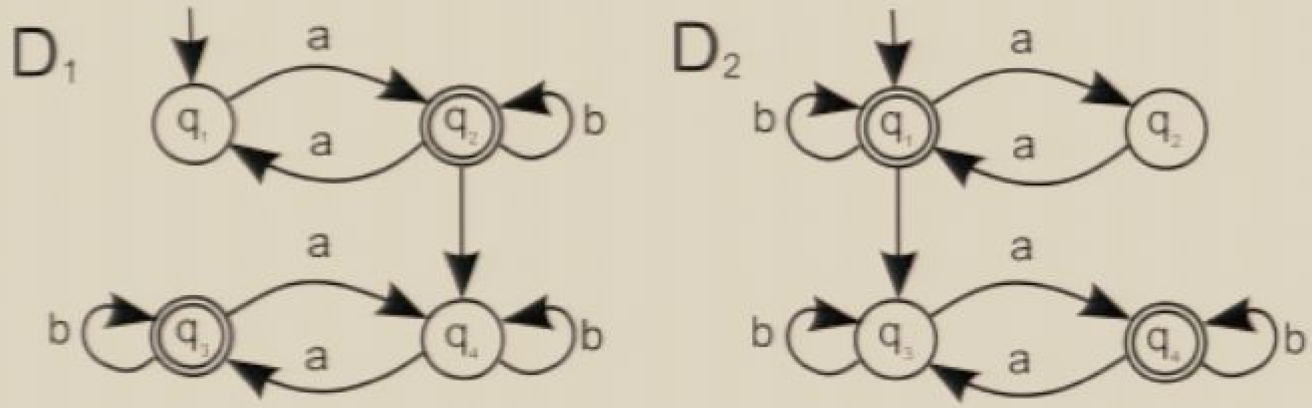
Then L cannot be recognized by a GQFA.

GQFA lower bound: Proof sketch

- Extend lemma so that it works simultaneously for all $w \in (x \cup y)^*$
- Find $w_x, w_y \in (x \cup y)^*$ such that the E_2 component after reading xw_x or yw_y is small
- The part of the state in E_1 acts exactly as an LQFA
- The behavior of the LQFA component dominates, and so by previous results it cannot distinguish between $xw_xw'_x$ and $yw_yw'_y$ for general w'_x, w'_y

GQFA:nonclosure under union

If we consider:



The languages L_1 and L_2 recognized by these automata can be recognized by GQFA, but the minimal automaton for $L_1 \cup L_2$ contains the forbidden construction.

Closing Remarks

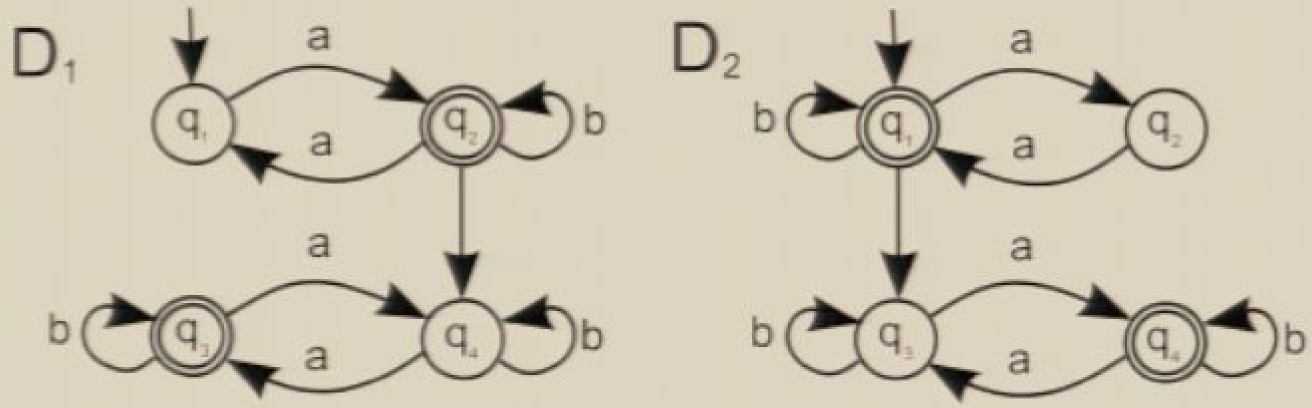
We have been able to lift lower bounds on LQFA and KWQFA to the case of GQFA.

The results show a close connection between the four QFA variations.

KWQFA and GQFA are hard to characterize exactly, but both variations contain a substructure which is well-behaved.

GQFA:nonclosure under union

If we consider:



The languages L_1 and L_2 recognized by these automata can be recognized by GQFA, but the minimal automaton for $L_1 \cup L_2$ contains the forbidden construction.