Title: Multiplicativity and Strong Multiplicativity of Norms on Transformations

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Abstract: Proving the additivity of the classical capacity of quantum channels is a major open problem in quantum information. This problem is related to the multiplicativity of certain norms with respect to the tensor product. These problems are introduced and some approached to resolving them are discussed. Several special cases that have been solved are also mentioned.

Multiplicativity Conjectures

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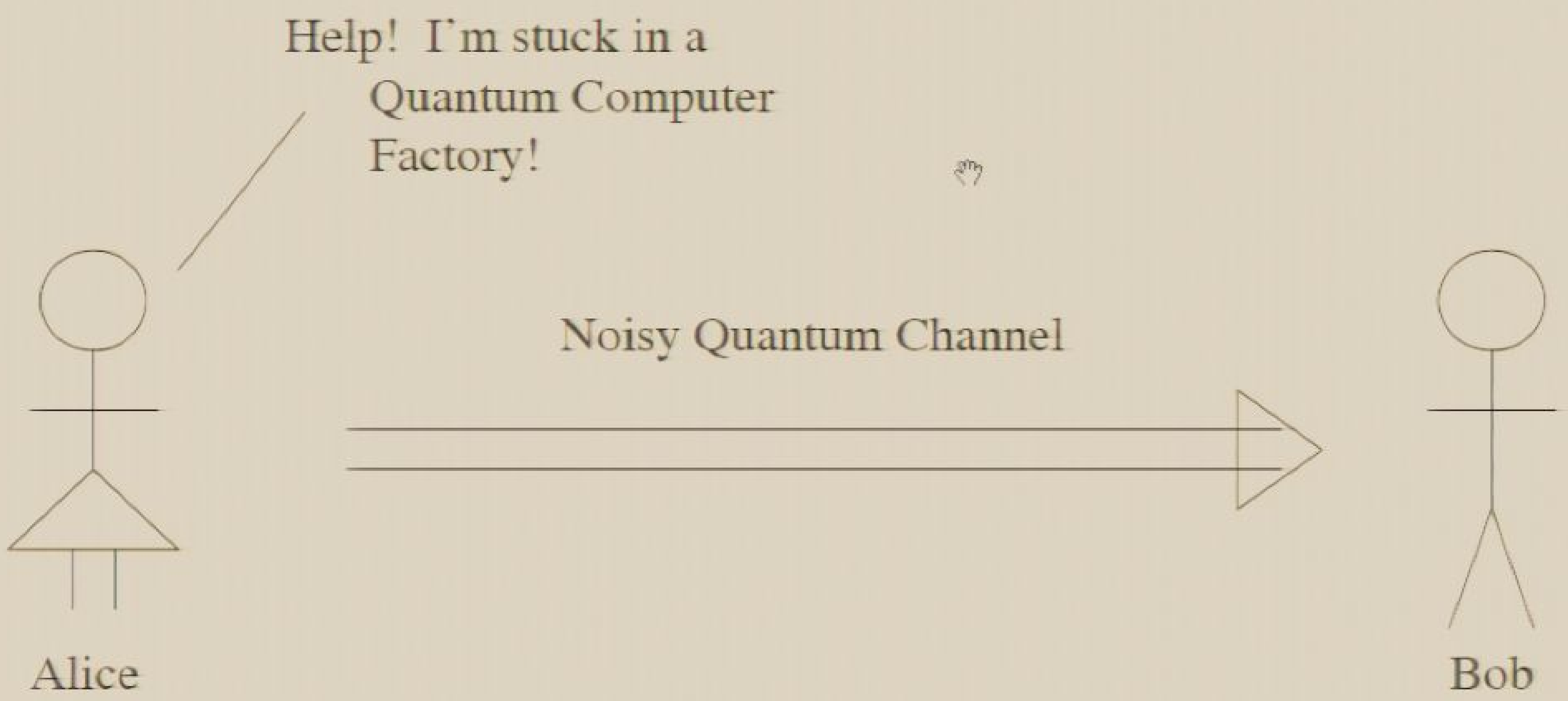
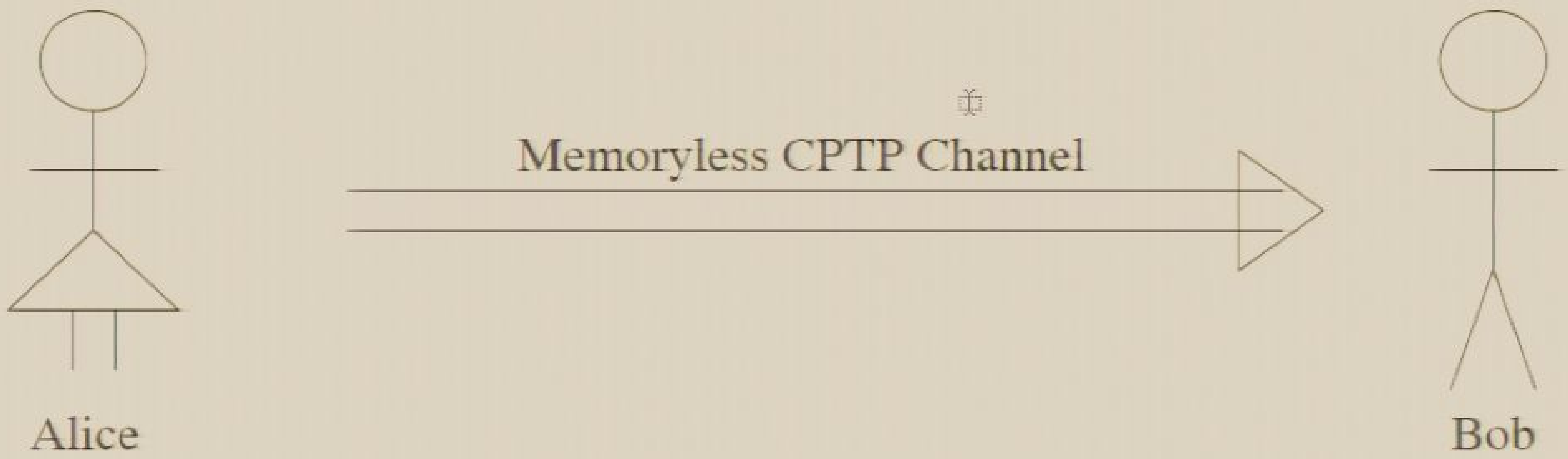
Alice and Bob



Alice and Bob, More Formally

- Alice and Bob do *not* share entanglement



- Does Alice need to send an entangled state through $\Phi^{\otimes n}$ or have Bob perform a complicated measurement to optimally send *classical* information?

Schatten p-norm of a State

- The p-norm of a density matrix σ is given by

$$\|\sigma\|_p = \left(\sum_i \lambda_i(\sigma)^p\right)^{\frac{1}{p}}$$

where $\lambda_i(\sigma)$ is the ith eigenvalue of σ.

- These norms generalize some popular norms

$$\|\sigma\|_1 = \|\sigma\|_{\mathrm{tr}}, \quad \|\sigma\|_2 = \|\sigma\|_{\mathrm{F}}, \quad \|\sigma\|_\infty = \|\sigma\|$$

Properties

- Unitary invariance: $\|\sigma\|_p = \|U\sigma\|_p = \|\sigma U\|_p$
- $\|\sigma\|_p \leq 1$
- $\|\sigma \otimes \gamma\|_p = \|\sigma\|_p \|\gamma\|_p$
- These norms measure the purity of a state

$$\| |\psi\rangle\langle\psi| \|_p = 1$$

$$\frac{1}{2} \| |0\rangle\langle 0| + |1\rangle\langle 1| \|_p = \frac{1}{2^{1-1/p}}$$

$$\frac{1}{d} \| I \|_p = \frac{1}{d^{1-1/p}}$$

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Schatten p-norm of a Transformation

- For Φ a CPTP map we define

$$\|\Phi\|_p = \max_{\sigma} \|\Phi(\sigma)\|_p = \max_{|\psi\rangle} \|\Phi(|\psi\rangle\langle\psi|)\|_p$$

- Alternately: how *pure* can Alice make the output of the channel?

The Multiplicativity Conjecture

Conjecture (AHW2000)

For all $1 \leq p \leq \infty$ *and all CPTP maps* Φ, Ψ

$$\|\Phi \otimes \Psi\|_p = \|\Phi\|_p \|\Psi\|_p$$

- A channel Φ is *multiplicative* if this holds for all Ψ.
- If true for $p \in [1, 1+\varepsilon)$ than Alice gains nothing from entangling her messages. [S2004]

The Easy Direction

$$\begin{aligned} \|\Phi \otimes \Psi\|_p &= \max_{\sigma} \|(\Phi \otimes \Psi)(\sigma)\|_p \\ &\geq \max_{\sigma \otimes \gamma} \|\Phi(\sigma) \otimes \Psi(\gamma)\|_p \\ &= \|\Phi\|_p \|\Psi\|_p \end{aligned}$$

- Informally: Alice can always restrict her strategy to separable states.

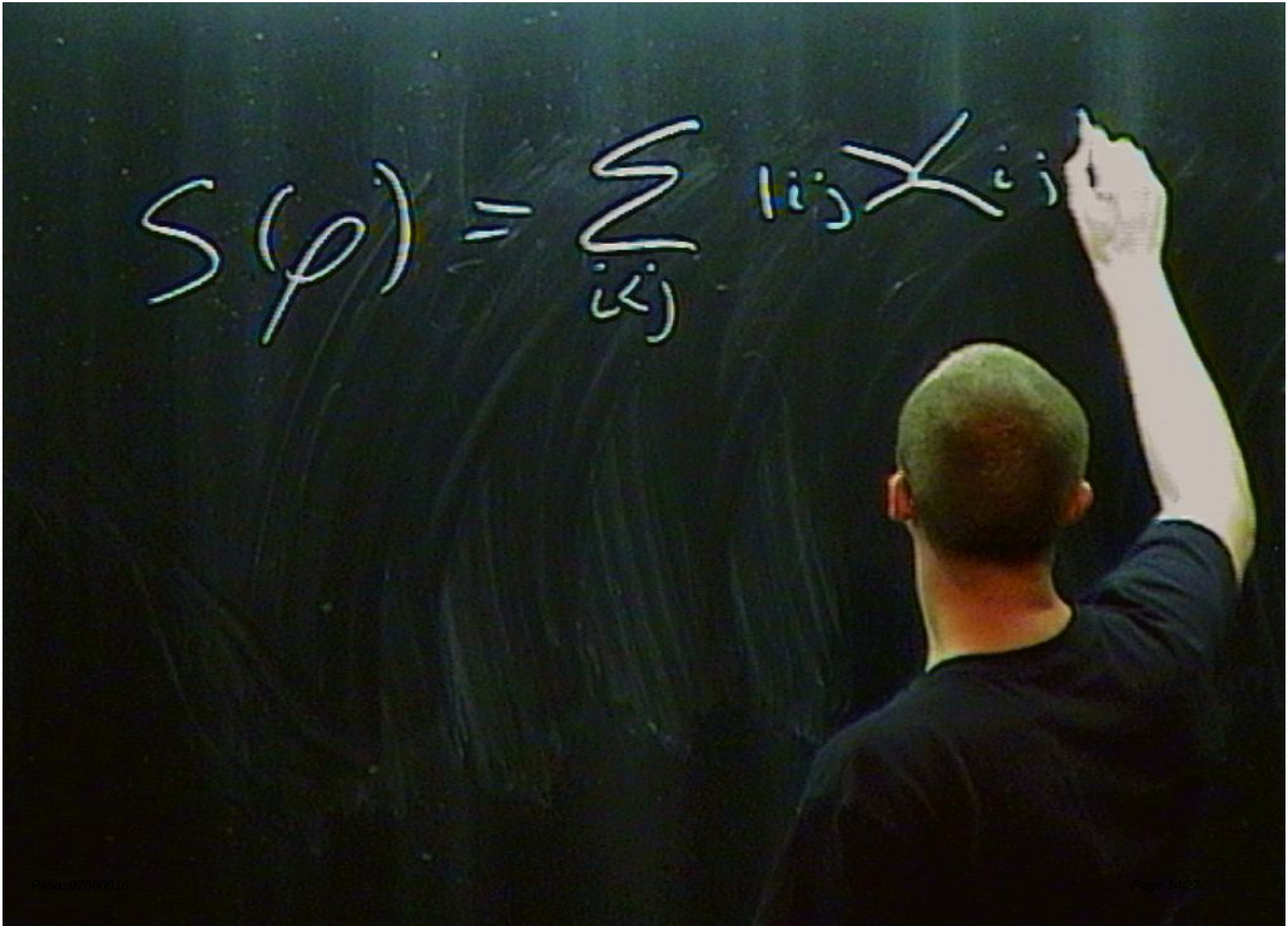
The Werner-Holevo Counterexample

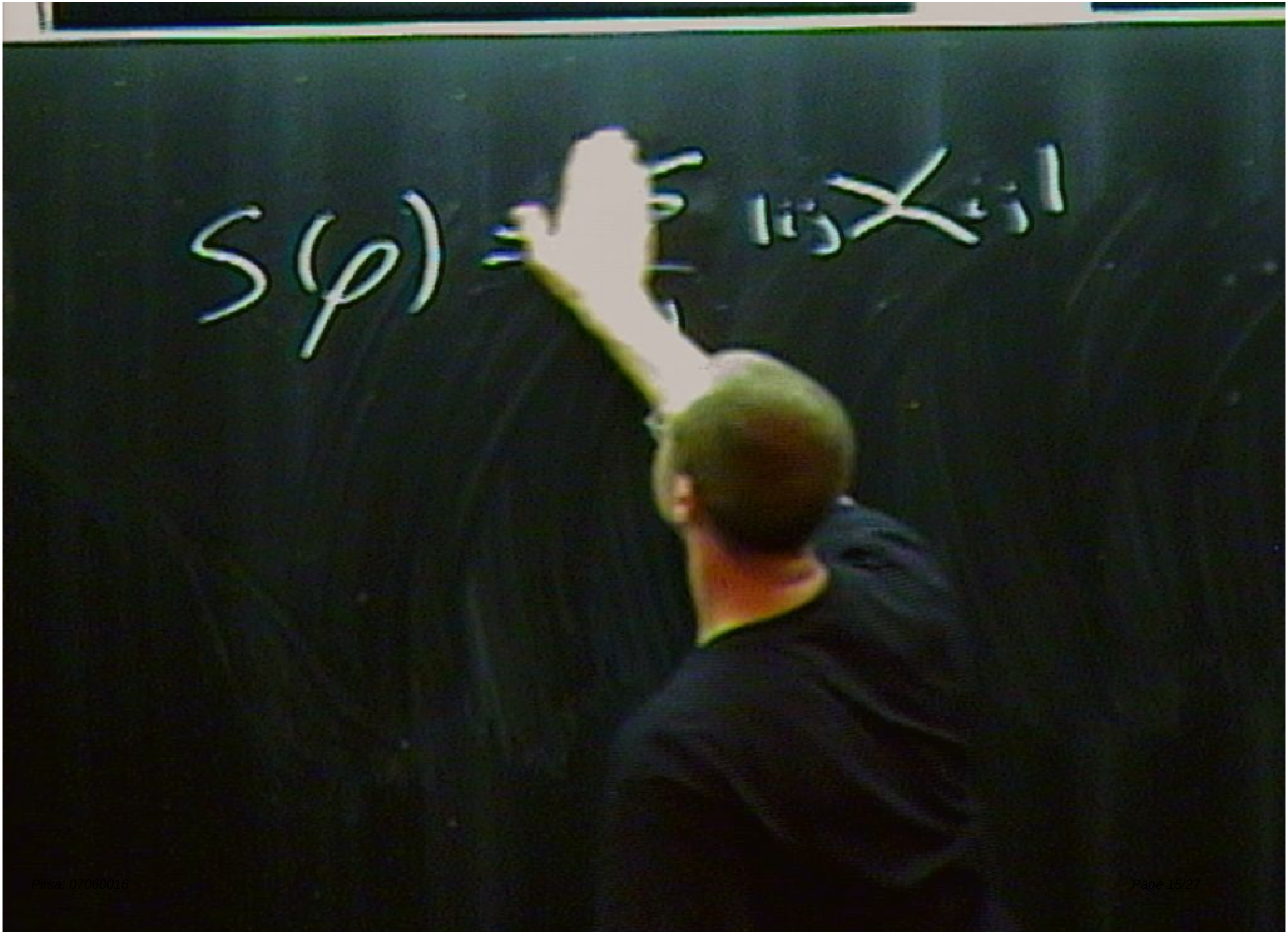
$$S(\sigma) = \frac{1}{d-1}\Big((\operatorname{tr}\sigma)I - \sigma^{\top}\Big)$$

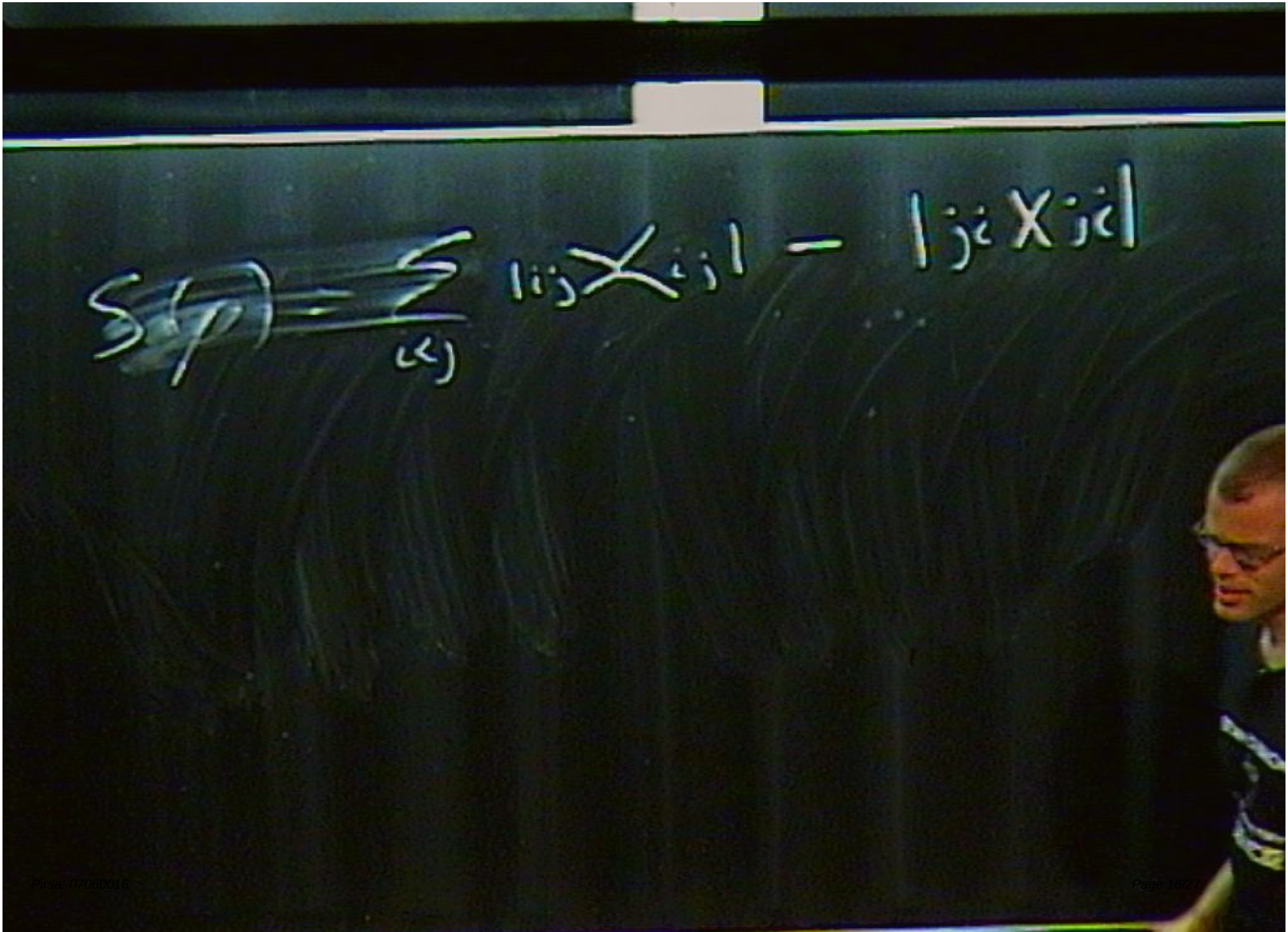
- This channel is a counterexample for $p > 4.79$ [WH2002]
- On any pure state $|\psi\rangle$

$$\|S(|\psi\rangle\langle\psi|)\|_{\infty} = \frac{1}{d-1}$$
$$\|S\|_{\infty}^{2} = \frac{1}{(d-1)^2}$$
$$\|S \otimes S\|_{\infty} = \frac{2}{d(d-1)}$$

- Multiplicativity (of the ∞-norm) is violated for any $d > 2$


$S(\varphi) = \sum_{i<j} l_{ij} X_{ij}$


S(φ) =



$\sum_{i<j} |ij\rangle\langle ij| - |ji\rangle\langle ji|$

$$S(\rho) = \sum_{i<j} |ij\rangle\langle ij| - |ji\rangle\langle ji|$$

Where the Conjecture is True

$$\|\Phi \otimes \Psi\|_p = \|\Phi\|_p \|\Psi\|_p$$

if Ψ is an arbitrary channel and Φ:

- is the identity channel
- has a pure output state:

$$\begin{aligned}\|\Phi \otimes \Psi\|_p &= \max_{\sigma} \|(\Phi \otimes I)(I \otimes \Psi)(\sigma)\|_p \\ &= \max_{\sigma \in \operatorname{Im} I \otimes \Psi} \|(\Phi \otimes I)(\sigma)\|_p \\ &\leq \|\Phi \otimes I\|_p \\ &= \|\Phi\|_p = \|\Phi\|_p \|\Psi\|_p\end{aligned}$$

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More Interesting Channels

Multiplicativity holds for

- unital qubit channels [K2002]

$$\Phi(I) = I$$

- entanglement breaking channels [K2003]

$$\Phi(\sigma) = \sum_i Tr(M_i \sigma)\xi_i$$

- diagonal channels [K2004, H2005]

$$\Phi(\sigma) = \sum_i A_i \sigma A_i^*, \quad A_i \text{ diagonal}$$

- entrywise positive channels (for integer p) [KR2004]

Random Unitary Channels

Definition

Φ is random unitary if there exist unitaries U_i and $c_i > 0$ with $\sum_i c_i = 1$ such that

$$\Phi(\sigma) = \sum_i c_i U_i \sigma U_i^*$$

- Unital qubit channels are random unitary.

Strong Multiplicativity

Definition

Φ is *strongly multiplicative* if

1. Φ is multiplicative
2. $q\Phi + (1-q)\Psi$ is multiplicative for any multiplicative Ψ

- The channel $N(\sigma) = I/d$ is strongly multiplicative for $p = \infty$.
- There is a channel that is multiplicative but not strongly so.

Is the Identity Strongly Multiplicative?

- $\Phi(\sigma) = U\sigma U^*$ is multiplicative.
- If I were strongly multiplicative, then

$$\Phi(\sigma) = qV^* U\sigma U^* V + (1-q)\sigma$$

  is multiplicative.
- By unitary invariance, we have the multiplicativity of

$$\Phi(\sigma) = qU\sigma U^* + (1-q)V\sigma V^*$$

No.

Consider the Werner-Holevo channel S and related R:

$$S_d(\sigma) = \frac{1}{d-1}\left((\operatorname{tr}\sigma)I - \sigma^\top\right)$$
$$R_d(\sigma) = \frac{1}{d+1}\left((\operatorname{tr}\sigma)I + \sigma^\top\right)$$

- S_4 is both not multiplicative and random unitary

$$S_4 = \frac{S_2 \otimes R_2}{2} + \frac{R_2 \otimes S_2}{2}$$

Open Problem

Resolve the AHW conjecture for p $\in [1, 1+\varepsilon)$.

Open Problem

Find counterexamples for p < 4.78*?*

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