Title: Degradation of a quantum directional reference frame

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Abstract: In this study, we are interested in the practical question of how many times a quantum directional reference frame (i.e., a spin-J system) can be used to perform a certain task with a given probability of success, under the assumption that the quantum directional reference frame evolves under a map that is covariant under rotations in SU(2). Our main theorem restricts the form of the state of the quantum reference frame as a function of how many times the covariant map was applied to it. Our results are a generalization of the paper of Bartlett el al. on the degradation of reference frames, and can be used to analyze certain types of interactions on a spin-J system.

Degradation of a quantum reference frame

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June 2, 2007

[1] Supported by the Lazaridis Fellowship.

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What's in this talk?

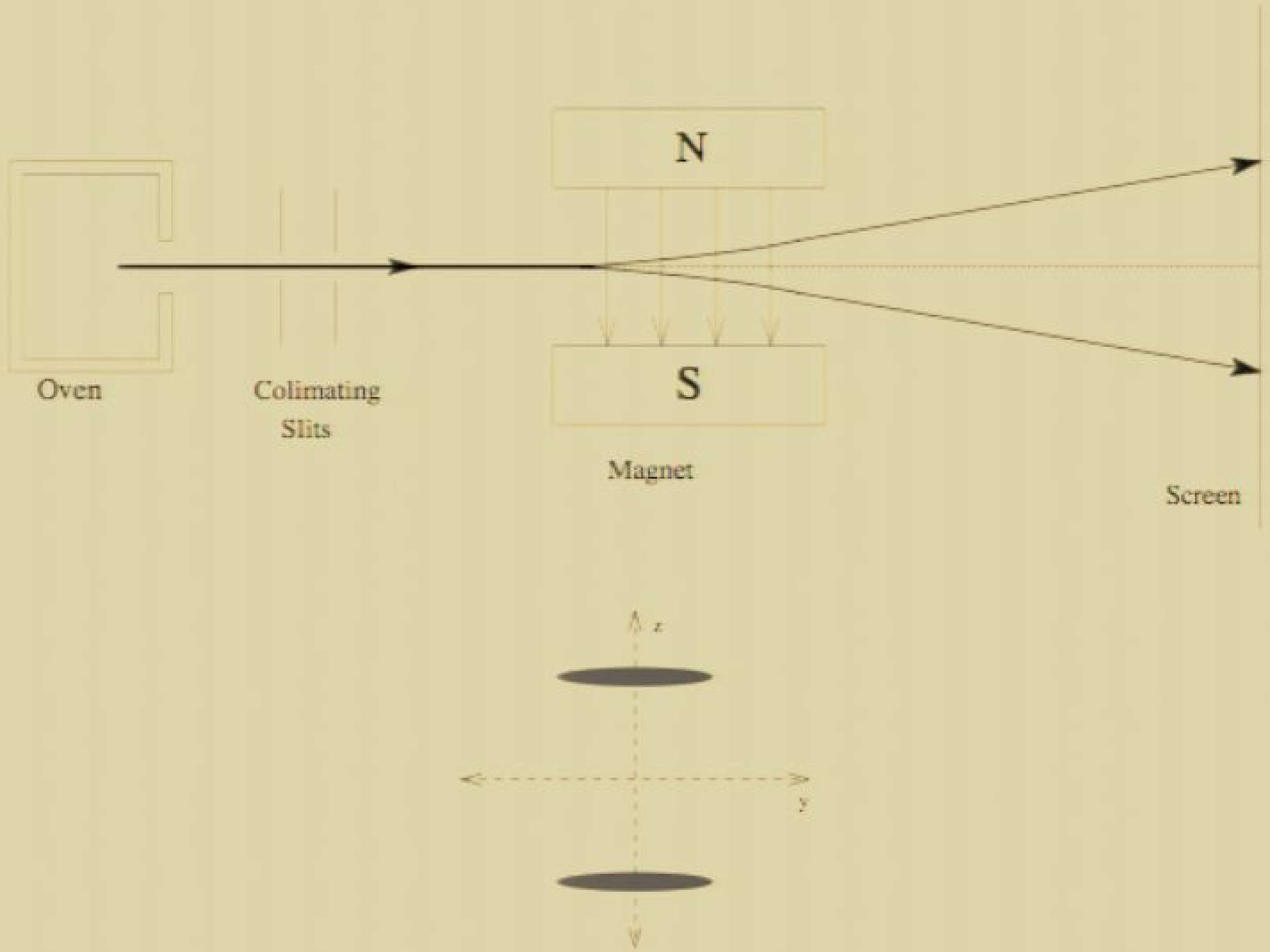
- Reminder from last year...
- Why should I care about this?
- What is the problem?
- How do we study this?
- What sorts of things can be solved with these results?

Reference frames

- Reference frames can be treated as internal or external to our system.
- If an RF is treated as internal to a quantum system it must be given degrees of freedom with associated quantum operators.
- If it is treated as external, then we merely invoke it to compare our physical system to some idealized absolute.

Reference frames: An example

A Stern-Gerlach experiment



Reference frames: An example

Suppose we wish to conserve the total angular momentum in the x-direction of a joint system:

$$\hat{\mathcal{J}}_x = \hat{J}_x + \hat{j}_x,$$

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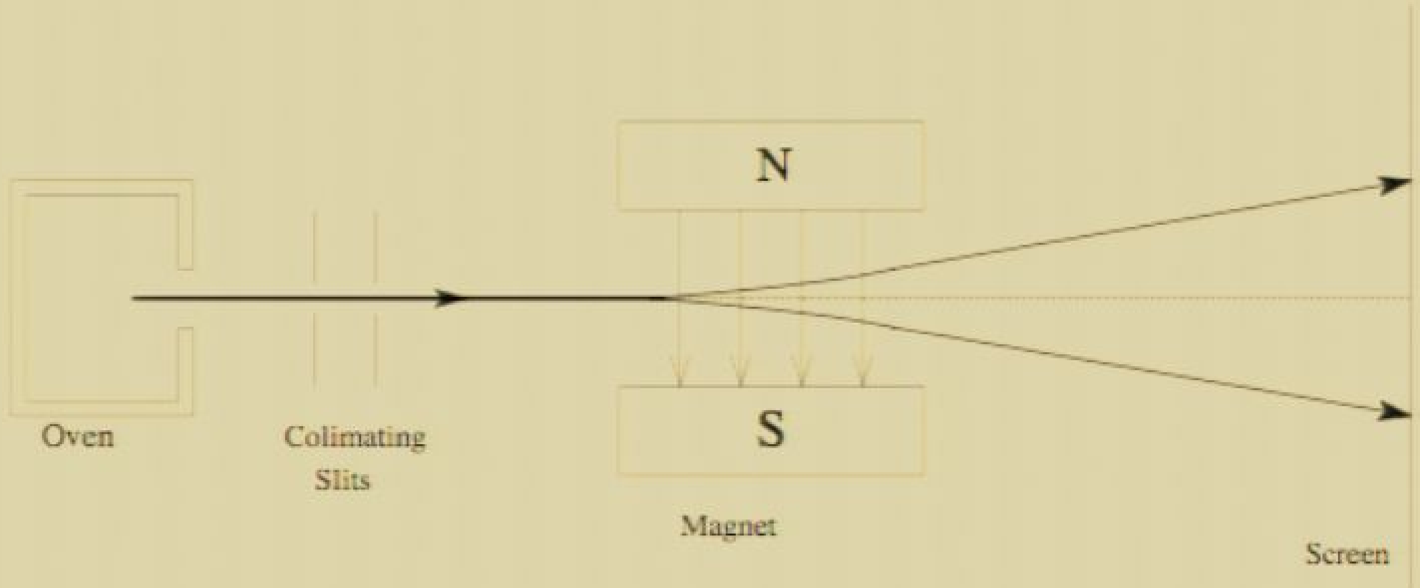

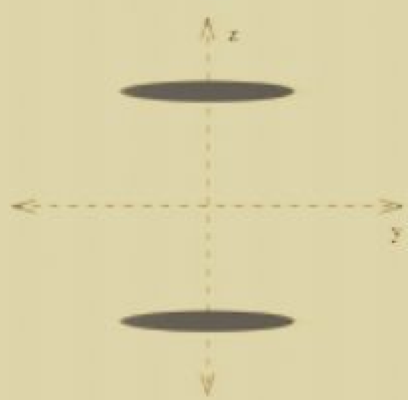

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This will not commute with j_z:

$$[\hat{\mathcal{J}}_x, \hat{j}_z] = [\hat{J}_x + \hat{j}_x, \hat{j}_z] \neq 0,$$

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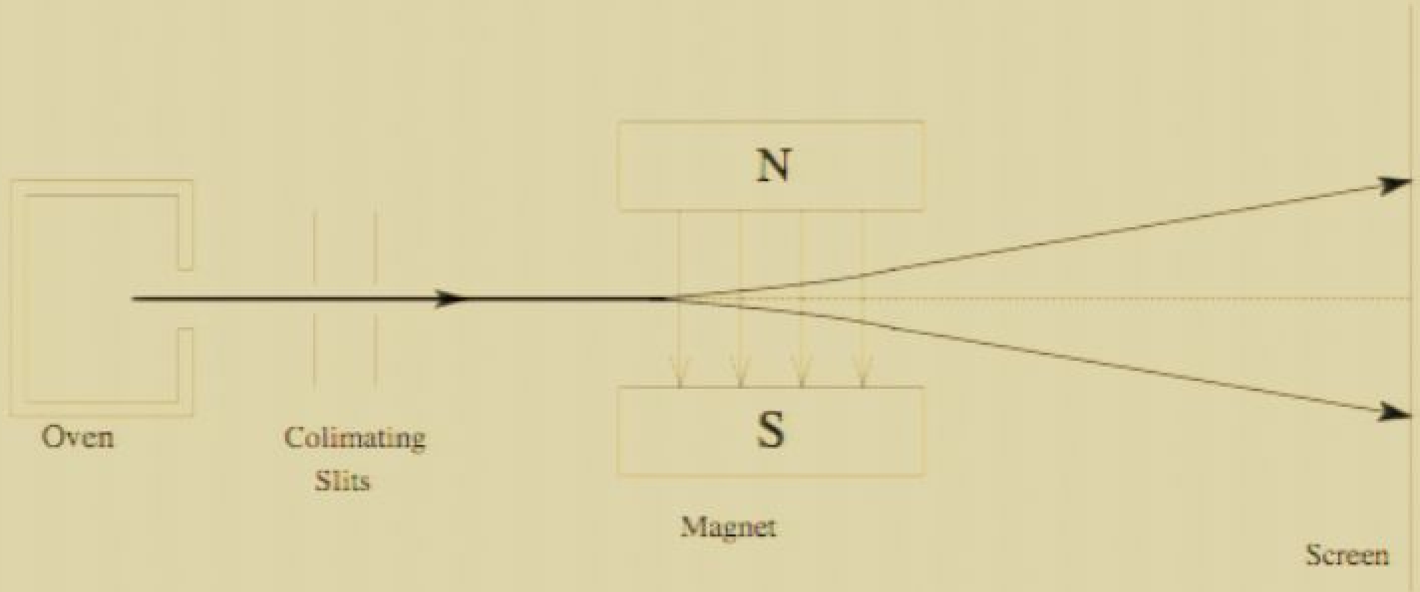

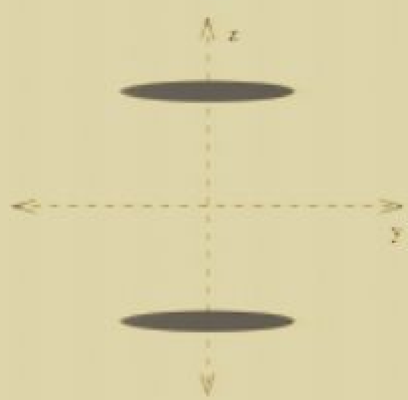

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but the measurement we are actually performing is $\hat{\mathcal{J}}^2$. This *does* commute with the total angular momentum along the x-direction:

$$\begin{aligned} [\hat{\mathcal{J}}_x, \hat{\mathcal{J}}^2] &= [\hat{\mathcal{J}}_x, (\hat{\mathcal{J}}_x^2 + \hat{\mathcal{J}}_y^2 + \hat{\mathcal{J}}_z^2)] \\ &= i\hbar(\hat{\mathcal{J}}_y\hat{\mathcal{J}}_z + \hat{\mathcal{J}}_z\hat{\mathcal{J}}_y - \hat{\mathcal{J}}_z\hat{\mathcal{J}}_y - \hat{\mathcal{J}}_y\hat{\mathcal{J}}_z) \\ &= 0 \end{aligned}$$

Previous work on the problem

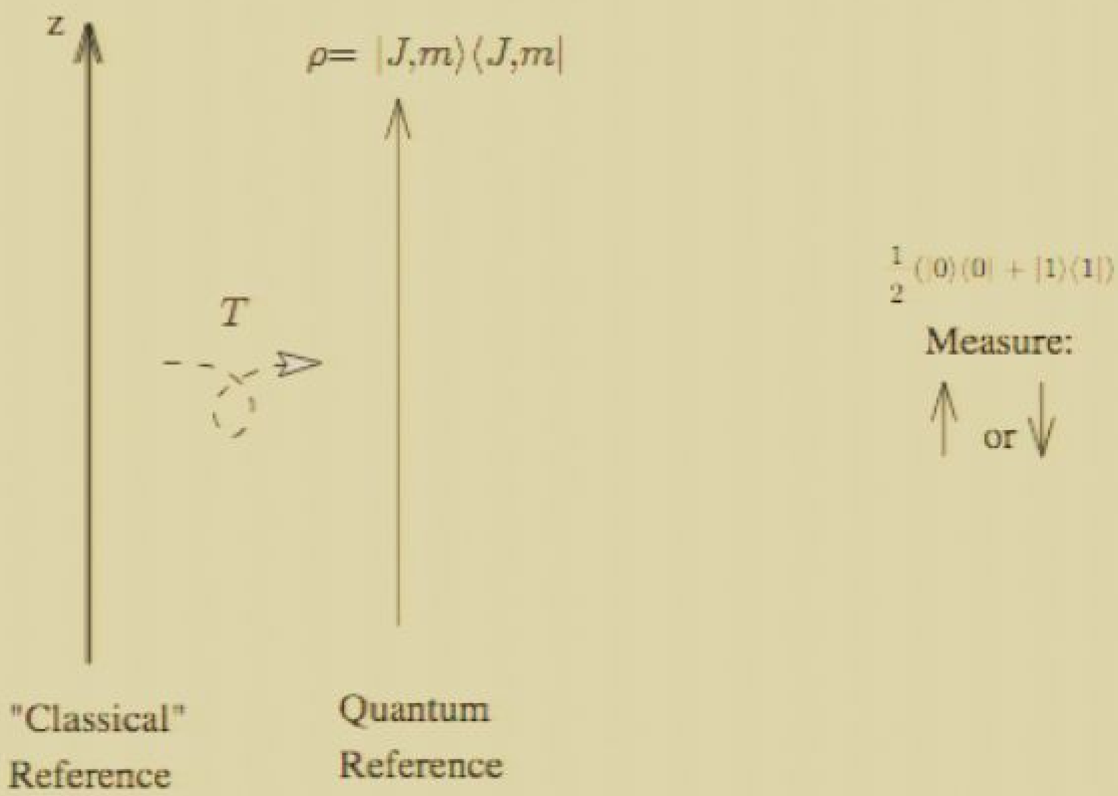

Figure: The scenario studied by [BRST06]. The quantum reference spin is used to measure the direction of a series of spin-$\frac{1}{2}$ particles in the completely mixed state by means of a projection onto the $J - \frac{1}{2}$ or $J + \frac{1}{2}$ subspaces.

Previous work on the problem

- The fidelity function is defined to be the probability that when the reference is used to measure another system it returns the correct result.

$$F = \mathrm{Tr}[\sum_i \Pi_{J+i}(|\, Jm\rangle \langle Jm \,| \otimes |\, i\rangle \langle i \,|)]$$

  The index i counts over the various possible outcomes of the measurement. Π_{J+i} is a projector onto the subspace where the new total angular momentum is $\mathcal{J} = J + i$.

- In the case of $j = \frac{1}{2}$, fidelity was found to vary with n, the number of iterations of the map, as

$$F = \frac{1}{2} + \frac{J}{2J+1}\left(1 - \frac{2}{(2J+1)^2}\right)^n$$

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- Fidelity can also be expressed as a function of the moments of the z-component of angular momentum:

$$F(\langle m \rangle) = \frac{1}{2} + \frac{J}{2J+1}\langle m \rangle$$

where $\langle m \rangle = \mathrm{Tr}[\xi(\rho) J_z]$

What we do

- We consider a much more general set of transformations ξ, where ξ is a *covariant map* under rotations. Such a map obeys the relation

$$R\xi(\rho)R^{-1} = \xi(R\rho R^{-1}), \quad \forall R \in SU(2).$$

- In general,

$$\begin{aligned}\xi(\rho) &= q_0\rho + \frac{q_1}{J(J+1)}\sum_i J_i\rho J_i^\dagger + \frac{q_2}{(J(J+1))^2}\sum_{i,j} J_iJ_j\rho J_i^\dagger J_j^\dagger \\ &+ \cdots + \frac{q_{2J}}{(J(J+1))^{2J}}\sum_{i,j,..k} J_iJ_j...J_k\rho J_i^\dagger J_j^\dagger ...J_k^\dagger\end{aligned}$$

A map of this form is covariant and all covariant maps can be expressed in this way, for appropriate values of $q_0, q_1, \ldots q_{2J}$.

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- Eg. The depolarizing channel on a qubit is a covariant map:

$$\xi(\rho) = (1-p)\rho + \frac{p}{3}\sum_{i=1}^{3} J_i\rho J_i = (1-p')\rho + \frac{p'I}{2}$$

Proof sketch

- The covariance condition can be expressed in the Louiville representation as

$$(R \otimes R^*)\bar{\xi}v = \bar{\xi}(R \otimes R^*)v,$$

  so $\bar{\xi}$ must commute with $R \otimes R^*$.
- The commutant of the group of all operations $R \otimes R^*, R \in SU(2)$, is of the form

$$C = \sum_{m'=0}^{2J} w_{m'} \Pi_{m'}$$

  where $\Pi_{m'}$ is the projector onto the state space $J_z = m' - J = m$ and the $w_{m'}$ are complex coefficients.
- There are $2J + 1$ independent projectors in C. To characterize every possible covariant mapping $\bar{\xi}$, we will require $2J + 1$ independent operators.

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Proof sketch

- Again in the Louiville representation, the density matrix ρ is expressed as the vector, v, and the map becomes

$$\bar{\xi}(v) \;=\; \sum_{n=0}^{2J} \frac{q_n}{(J(J+1))^n} \left(\sum_i J_i \otimes J_i \right)^n v.$$

- The eigenvalues of the operator in the nth term of the expansion of the map can be expressed as λ_K^n where the λ_K are the eigenvalues of the operator in the first term, that is of $\sum_i J_i \otimes J_i$.

Proof sketch

- What is now required is to show that these $2J+1$ operators are independent. This will be the case if the λ_K^n are non-degenerate.
- To find the λ_K, expand $\sum_i (J_i \otimes I + I \otimes J_i)^2$ to get a new expression for $\sum_i J_i \otimes J_i$:

$$\begin{aligned}\sum_i J_i \otimes J_i &= \frac{1}{2}\left(\sum_i (J_i \otimes I + I \otimes J_i)^2 - \sum_i J_i^2 \otimes I - I \otimes \sum_i (J_i)^2\right) \\ &= \frac{1}{2}\left(\mathcal{J}^2 - \mathbf{J}^2 \otimes I - I \otimes \mathbf{J}^2\right)\end{aligned}$$

where $\mathcal{J}$ is the total angular momentum of a joint system composed of two spin-J systems.

Proof sketch

- The λ_K, which are eigenvalues of the expression:

$$\lambda_K = \frac{1}{2}\left(c_K K(K+1) - J(J+1) - J(J+1)\right) \quad (1)$$

$K(K+1)$ is the eigenvalue resulting from the vector addition of the two spin-J systems; K can range from 0 to $2J$, and the c_K are the appropriate Clebsch Gordon coefficients.

- There will be $2J+1$ different values of K, so there will be $2J+1$ different eigenvalues. They will be the $2J+1$ roots of the polynomials

$$P_k(x) \;=\; \sum_{n=0}^{2J} q_n x^n \quad (2)$$

and thus all the roots are shown to be unique.

- It demonstrated that in setting the $2J+1$ q_n parameters, we have all the freedom required to describe an arbitrary covariant map.

What we do

So, any covariant map, $\xi(\rho)$, can be written in the form

$$\xi(\rho) = q_0\rho + \frac{q_1}{J(J+1)}\sum_i J_i\rho J_i^\dagger + \frac{q_2}{(J(J+1))^2}\sum_{i,j} J_iJ_j\rho J_i^\dagger J_j^\dagger + \cdots + \frac{q_{2J}}{(J(J+1))^{2J}}\sum_{i,j,..k} J_iJ_j...J_k\rho J_i^\dagger J_j^\dagger ...J_k^\dagger$$

where by *covariant*, we mean

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How do the moments $\langle m^\ell \rangle$ of the J_z operator evolve under such map?

- By the cyclic property of the trace

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- For the complete map, where J_z is the measurement of the z-projection of the spin-J system:

$$\mathrm{Tr}[\xi(\rho)J_z^\ell] = \begin{cases} \sum_{k=0}^{\ell/2} A_{2k}^{(\ell)} \mathrm{Tr}[\rho J_z^{2k}], & \text{if } \ell \text{ is even} \\ \sum_{k=1}^{(\ell+1)/2} A_{2k-1}^{(\ell)} \mathrm{Tr}[\rho J_z^{2k-1}], & \text{if } \ell \text{ is odd,} \end{cases}$$

The scaling

For the $j = \frac{1}{2}$ case studied by Bartlett *et al.*, we can derive the same scaling analytically:

$$\sum_{i=1}^{3} J_i J_z J_i = (J(J+1) - 1) J_z$$

(using [R05]). So,

$$\begin{aligned} \mathrm{Tr}[\xi(\rho) J_z] &= A_1^{(1)} \, \mathrm{Tr}[\rho J_z] \\ &= \left(q_0 + \frac{(J(J+1)-1)}{J(J+1)} q_1 \right) \mathrm{Tr}[\rho J_z] \end{aligned}$$

For a map repeated n times we have

$$\mathrm{Tr}[\xi'(\rho) J_z] = \left(1 - \frac{q_1}{J(J+1)} \right)^n \mathrm{Tr}[\rho J_z]$$

So $\quad A_1^{(1)}(n) = 1 - \frac{n q_1}{J^2} + O(\frac{1}{J^3}).$

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The j=1 case.

$$F = \frac{1}{6} + \frac{[(2J+1)^2 - 2]}{6J(J+1)(2J+1)} \mathrm{Tr}[\xi(\rho)J_z] + \frac{1}{2J(J+1)} \mathrm{Tr}[\xi(\rho)J_z^2].$$

Note that the fidelity in this case depends on the second moment as well as the first. From our previous analysis, we have:

$$\begin{aligned} \mathrm{Tr}[\xi(\rho)J_z] &= A_1^{(1)} \mathrm{Tr}[\rho J_z], \\ \mathrm{Tr}[\xi(\rho)J_z^2] &= A_0^{(2)} + A_2^{(2)} \mathrm{Tr}[\rho J_z^2]. \end{aligned}$$

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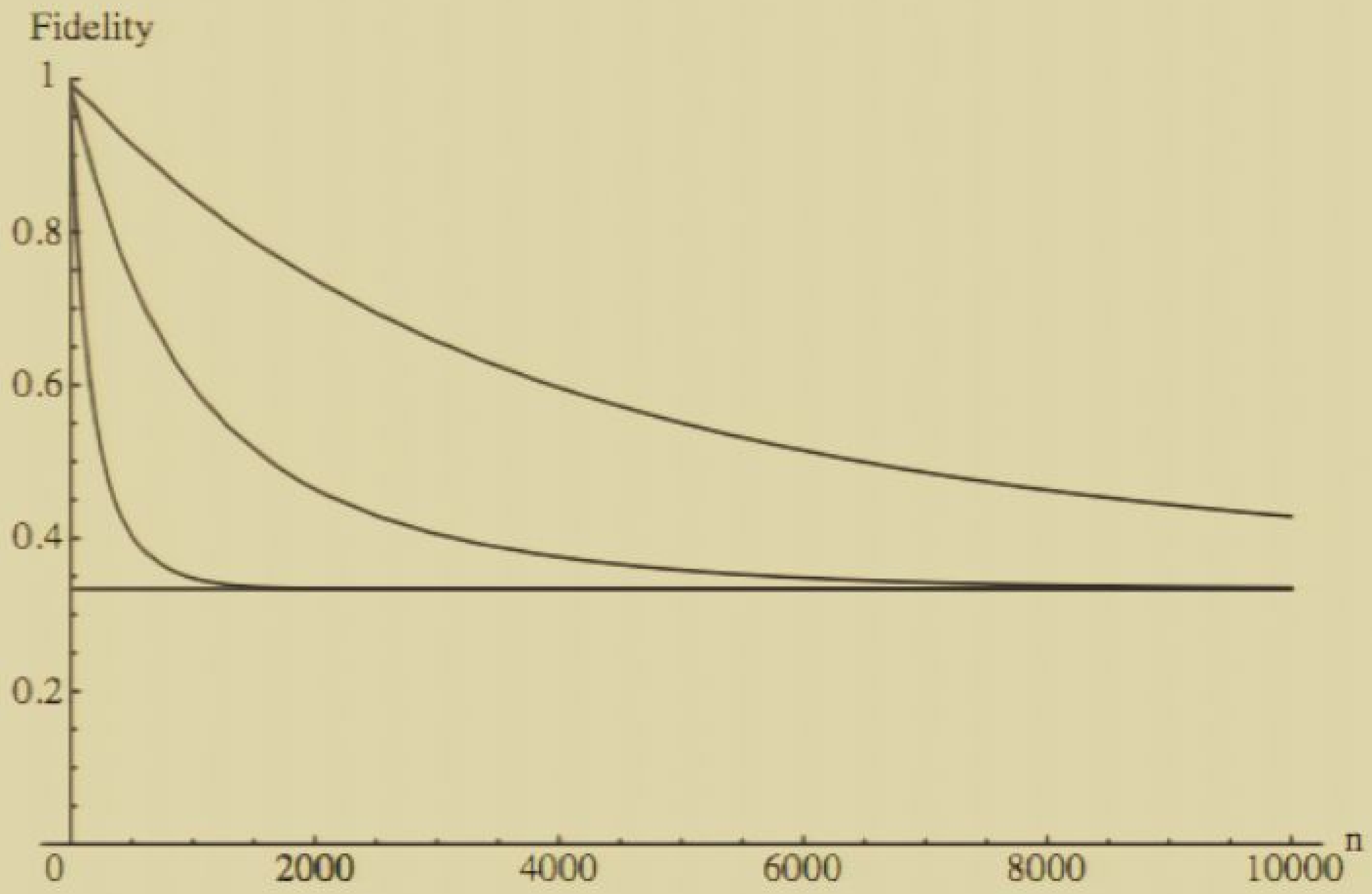

Figure: A plot of Fidelity, F, against the number of repetitions of the map n for reference systems of total angular momentum $J = 20, J = 50$, and $J = 100$.

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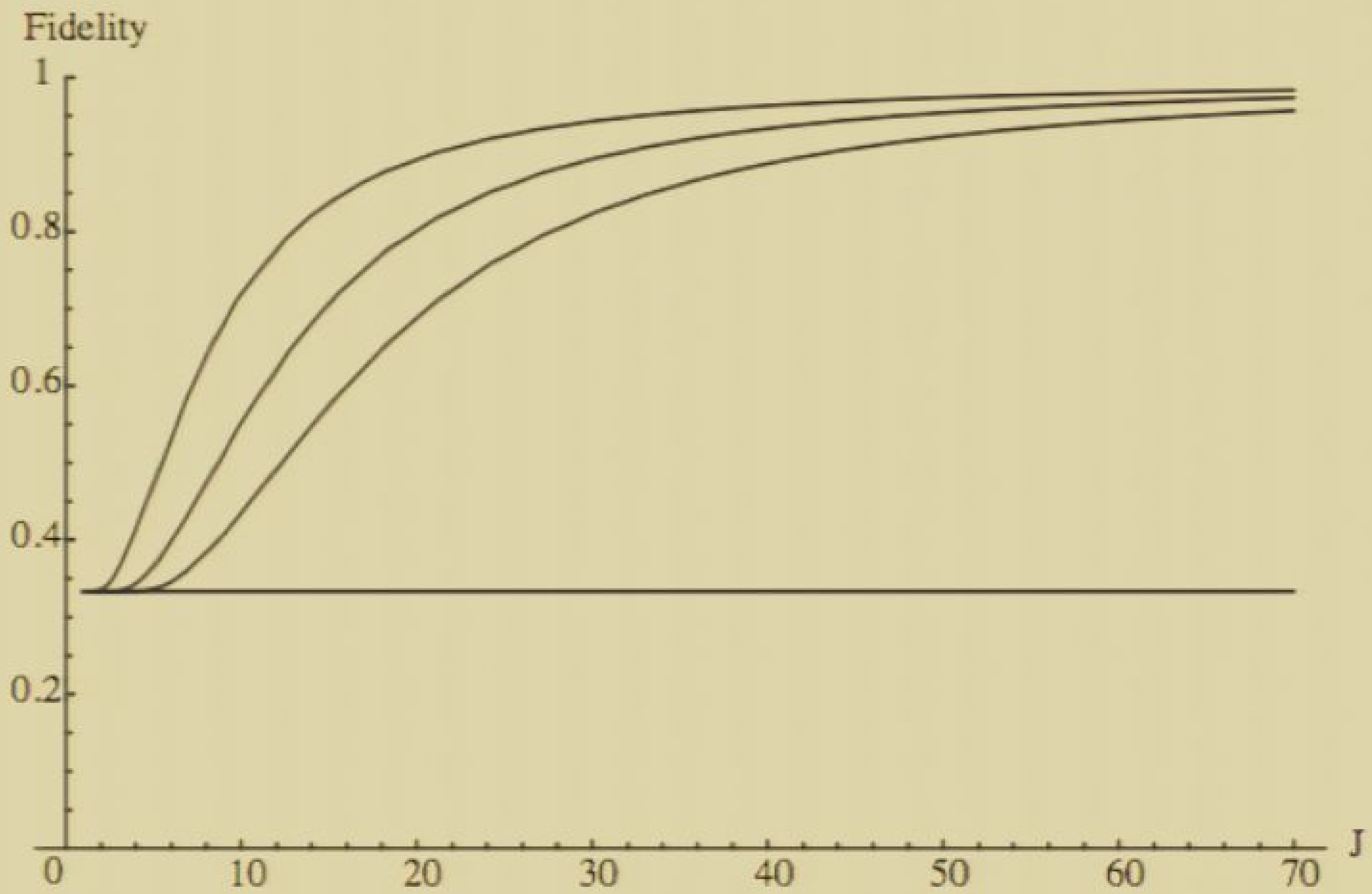

Figure: A plot of Fidelity, F, against the size of the reference system J for $n = 20, n = 50$, and $n = 100$.

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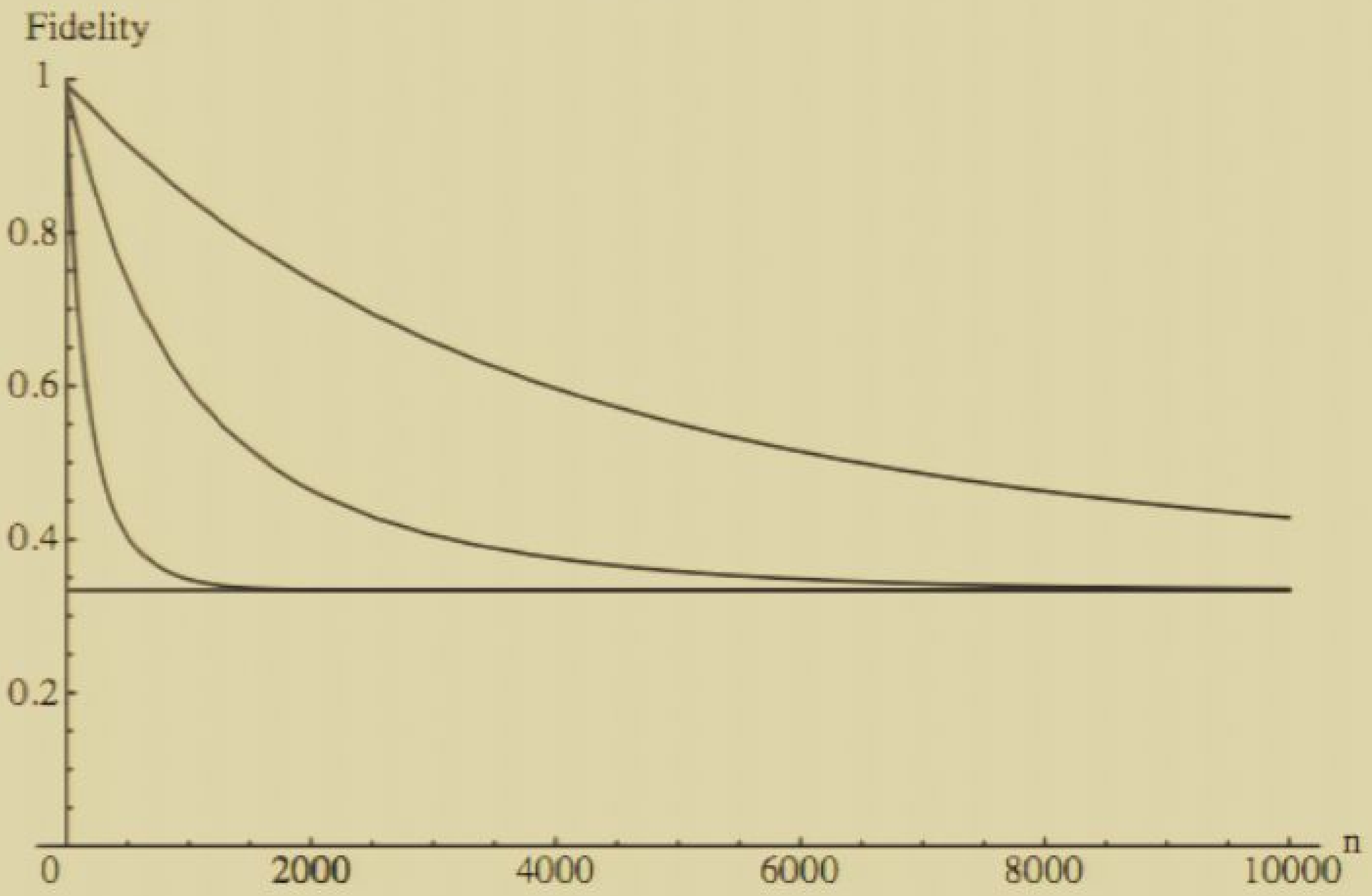

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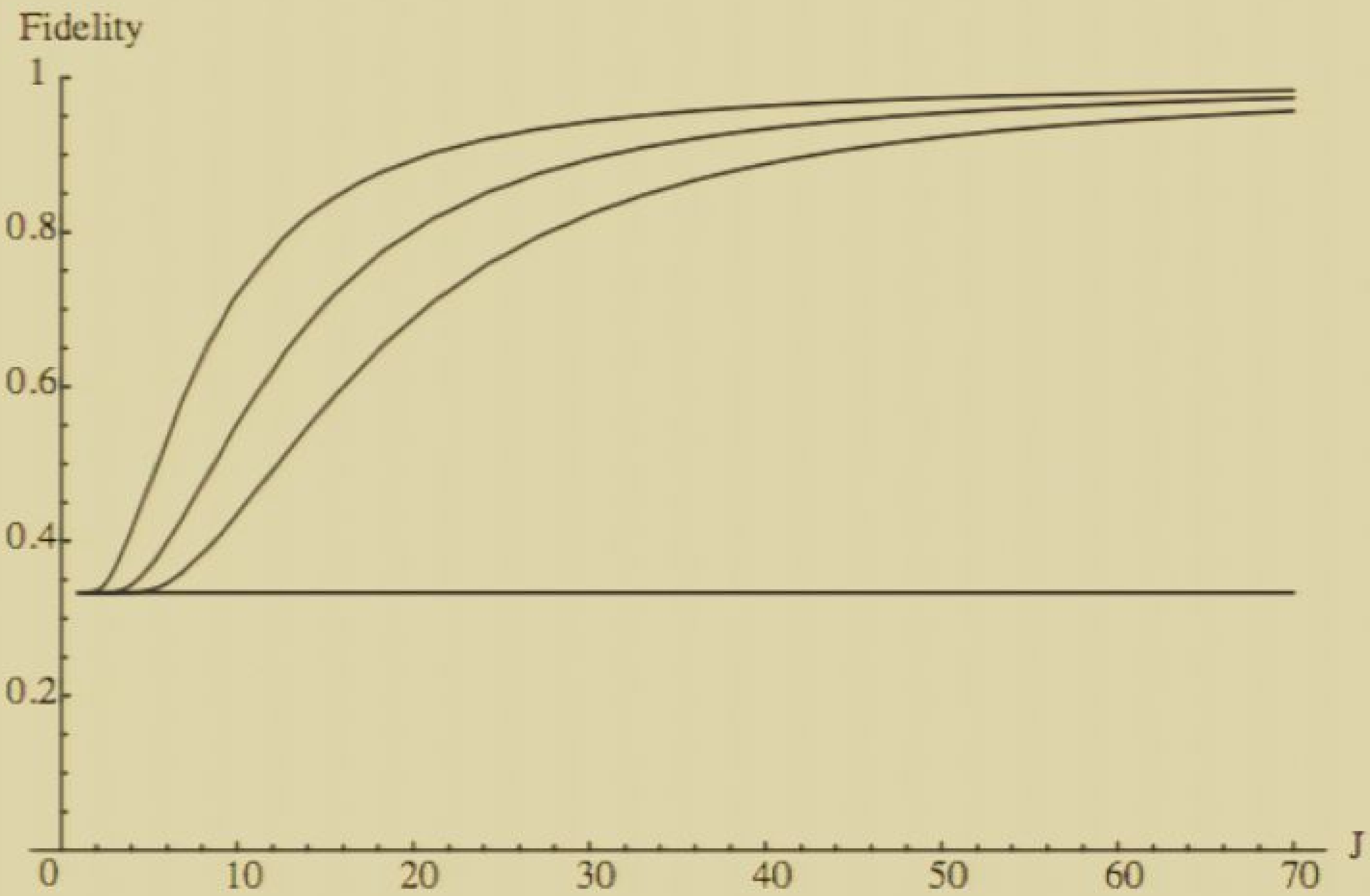

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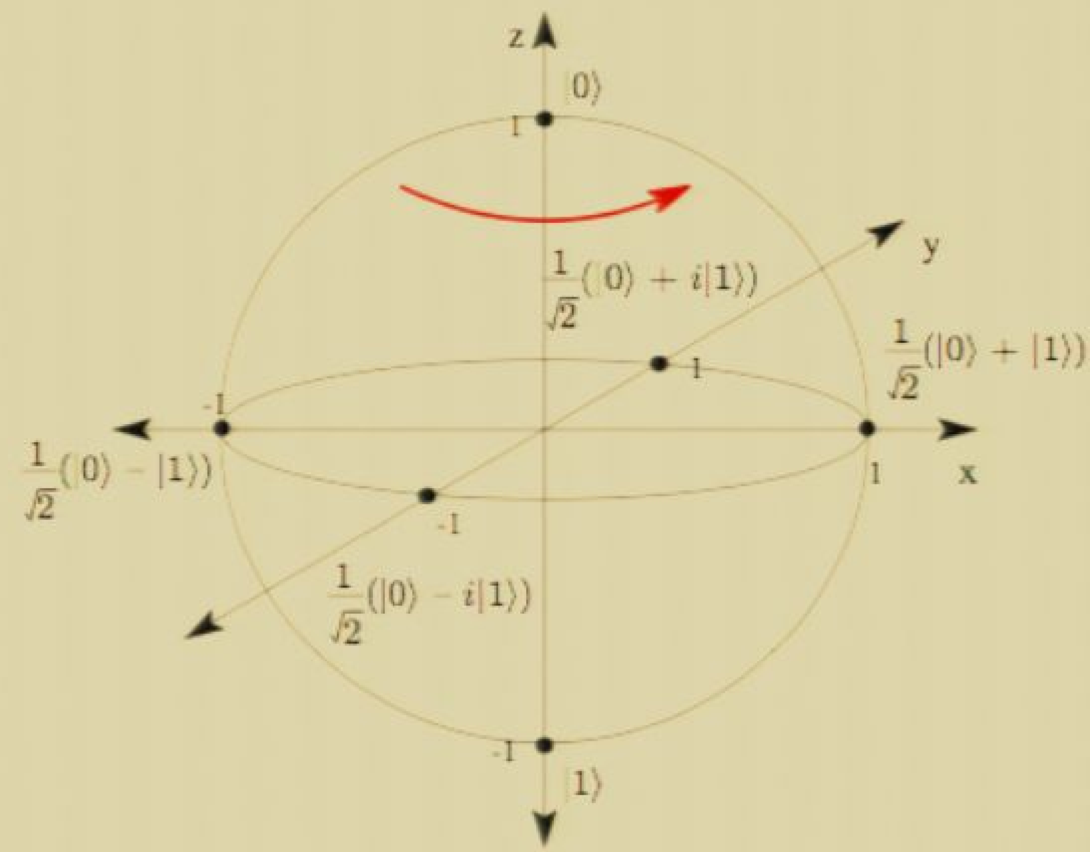
The unitary gate case.

What kind of operations can we do on a qubit with a spin-J reference system?



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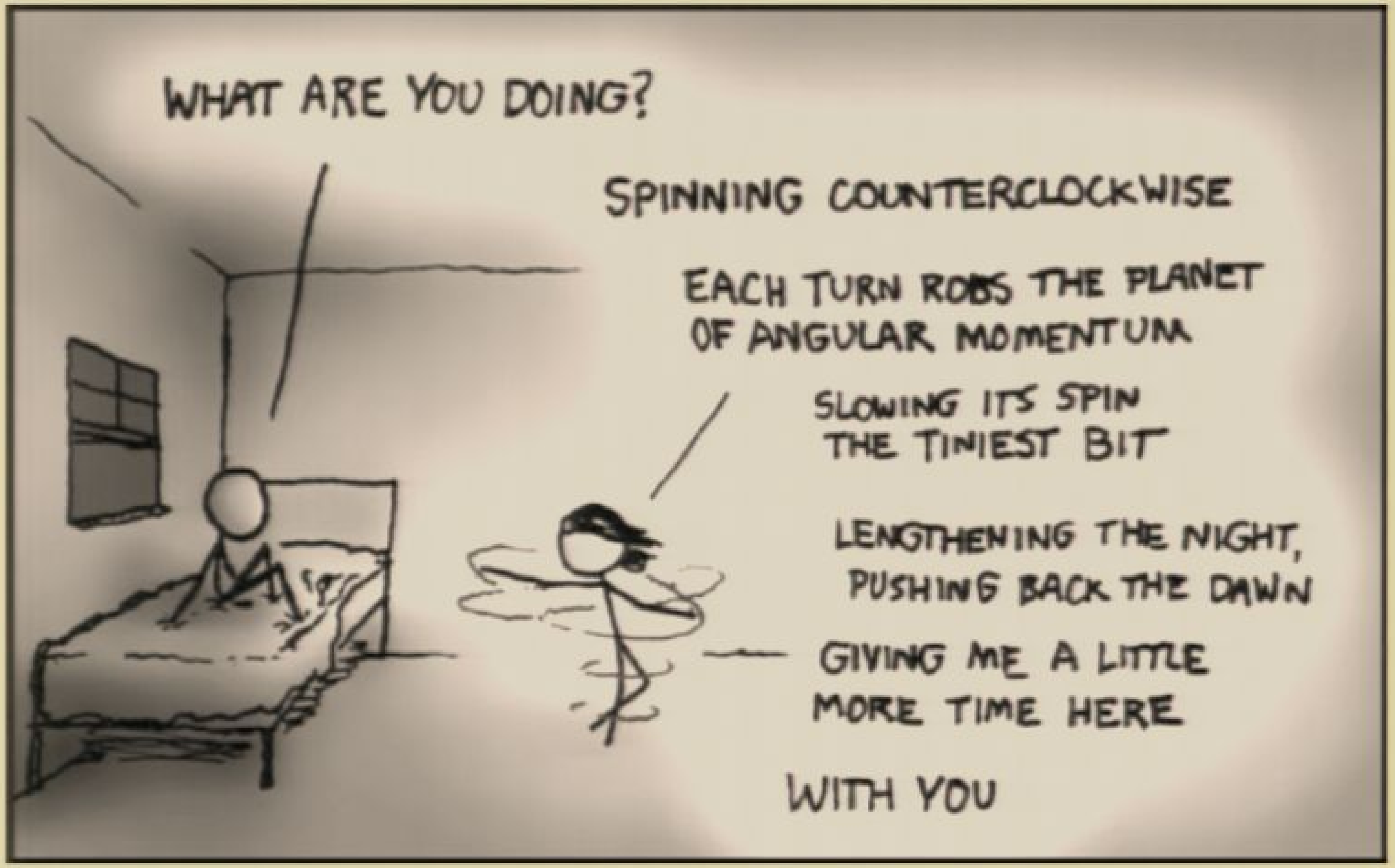
The xkcd nerd-romance understanding:



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- Fidelity for this map

$$F(Z,\xi) = \frac{1}{3}\left(\frac{4\langle m^2\rangle}{J(J+1)(2J+1)} + 2\right)$$

which is a function of the second moment.

Summary & Future work

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- Reference frames of size comparable to that of the quantum system should be treated quantumly. These interactions lead to decoherence effects.
- We have a generic way of expressing any covariant map in terms of angular momentum operators.
- Using this form of the covariant map allows for the characterization of the dynamics of the reference frame under the action of the map in terms of the parameters of the map. This can be done for any covariant map.

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Future Work

- Look for nice forms of other classes of map.
- Consider the case where a record is kept of the measurement results.

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Reference frames, superselection rules, and quantum information.
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