Title: Degradability of Bosonic Gaussian Channels

Date: Jun 01, 2007 02:30 PM

URL: http://pirsa.org/07060004

Abstract: The notion of weak-degradability of quantum channels is introduced by generalizing the degradability definition given by Devetak and Shor. Exploiting the unitary equivalence with beam-splitter/amplifier channels we then prove that a large class of one-mode Bosonic Gaussian channels are either weakly degradable or anti-degradable. In the latter case this implies that their quantum capacity Q is null. In the former case instead, this allows us to establish the additivity of the coherent information for those maps which admit unitary representation with single-mode pure environment.

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Degradabiliy of Bosonic Gaussian Channels

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June 1st, 2007 Waterloo

NEST

Quantum Transport & Information

F. Caruso and V. Giovannetti, *Physical Review A* **74**, 062307 (2006), quant-ph/0603257
F. Caruso, V. Giovannetti, and A.S. Holevo, *New J. Phys.* **8**, 310 (2006), quant-ph/0609013

Outline

- **Open quantum systems**
  - System coupled to environment
  - Missing quantum information and decoherence
  - Weak-degradability and anti-degradability

- **Bosonic Gaussian Channels**
  - Characteristic function and Gaussian states
  - Beam-Splitter and Amplifier channel
  - Weak-degradability properties
  - A full classification
  - A better bound for maps with Q=0

- Conclusions and Outlook

Open quantum systems

The theory of open quantum systems describes the interaction of a quantum system with its environment

Quantum Mechanics

closed systems

unitary dynamics

reversible dynamics

$$i\hbar \frac{d|\Psi\rangle}{dt} = \hat{H}|\Psi\rangle$$

Schrödinger Equation

$$i\hbar \frac{d\hat{\rho}}{dt} = \left[\hat{H}, \hat{\rho}\right]$$

Liouville – von Neumann Equation

Unitary evolution condemns every closed quantum system to "purity"

Open quantum systems

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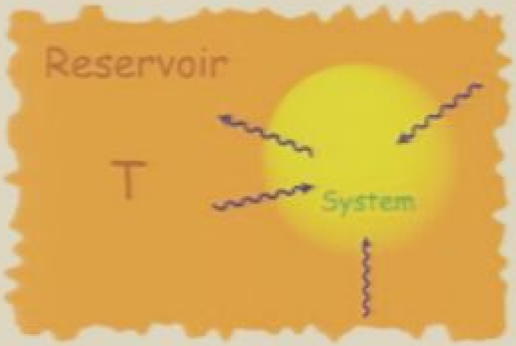

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open quantum systems

reduced density operator

$$\hat{\rho}_S(t) = Tr_E\left[\hat{\rho}_T(t)\right]$$

master equation

$$\frac{d\hat{\rho}}{dt} = L\hat{\rho}$$

non-unitary and irreversible dynamics

Open quantum systems

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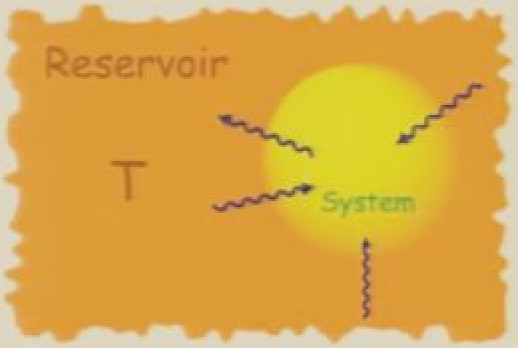

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non-unitary and
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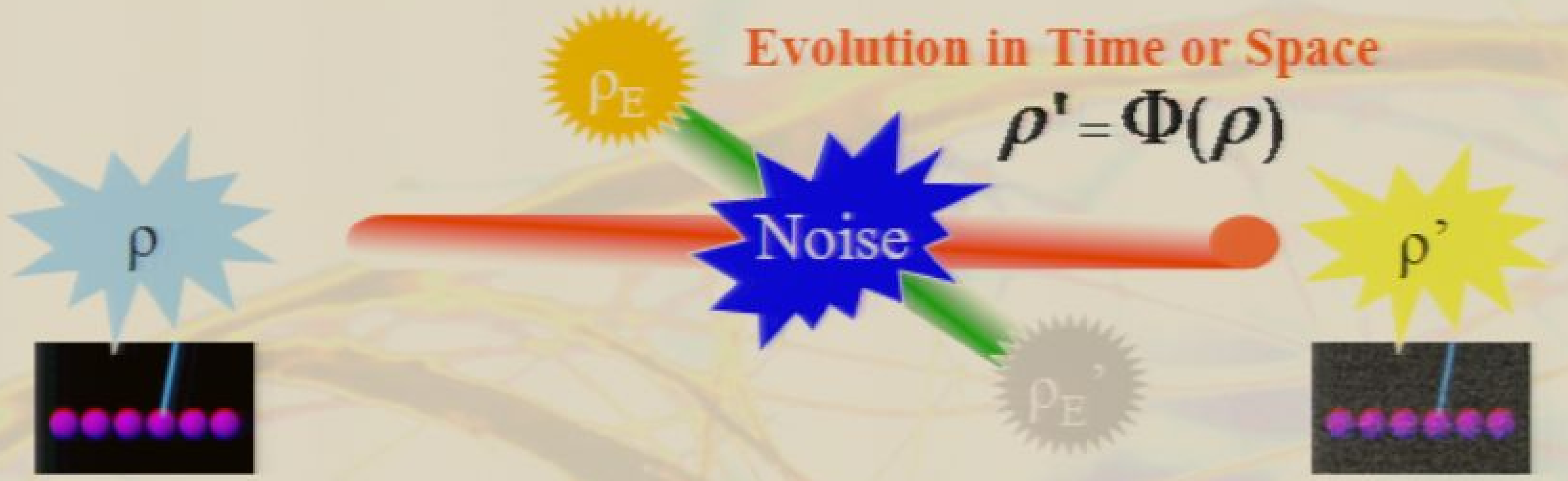
DECOHERENCE

Entanglement between the degrees of freedom of the system and those of the environment

System coupled to environment

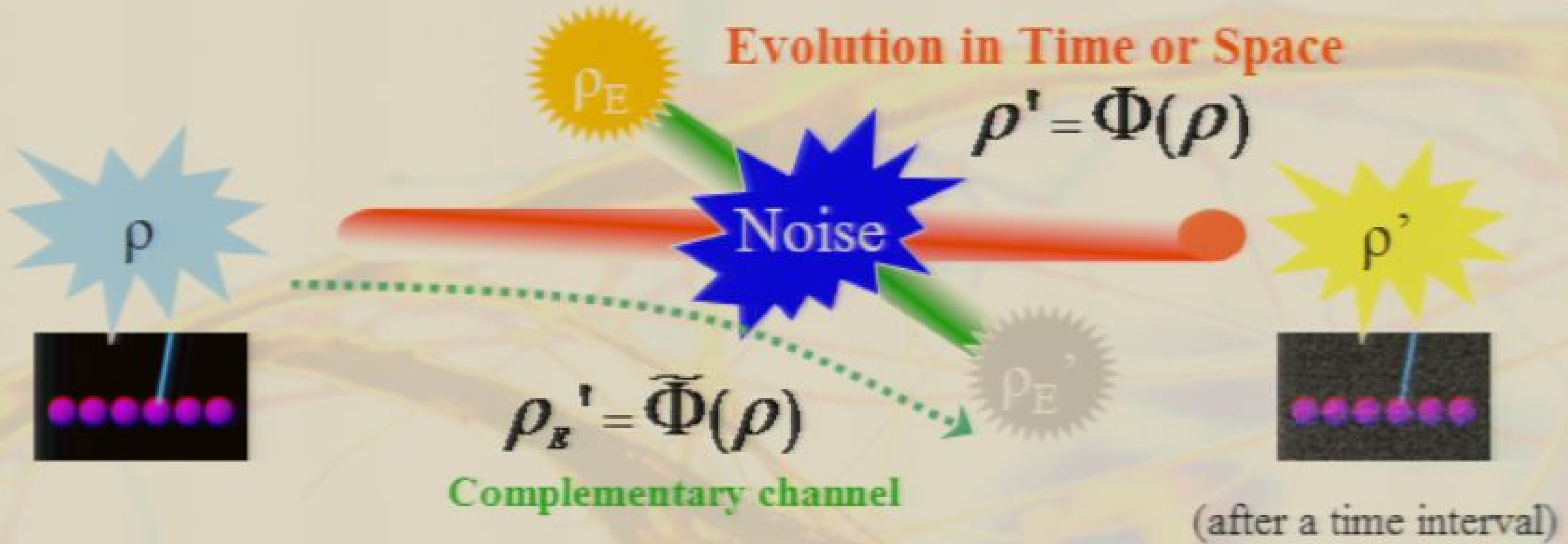


Evolution in Time or Space

$$\rho' = \Phi(\rho)$$

(after a time interval)

System coupled to environment



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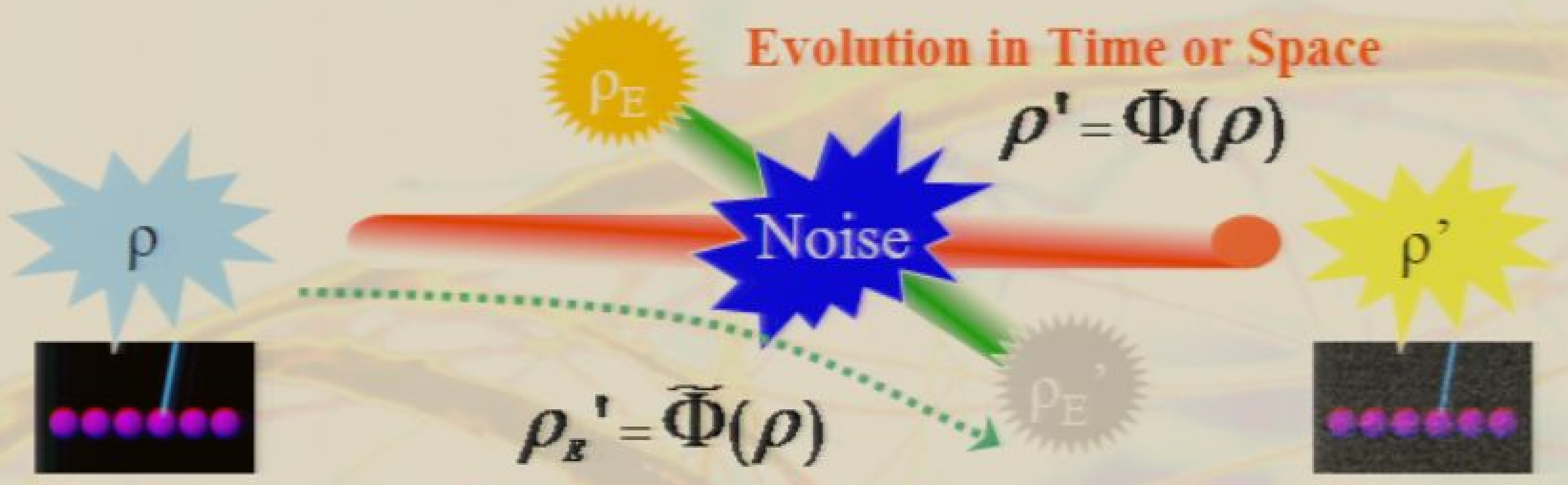


Evolution in Time or Space

$$\rho' = \Phi(\rho)$$

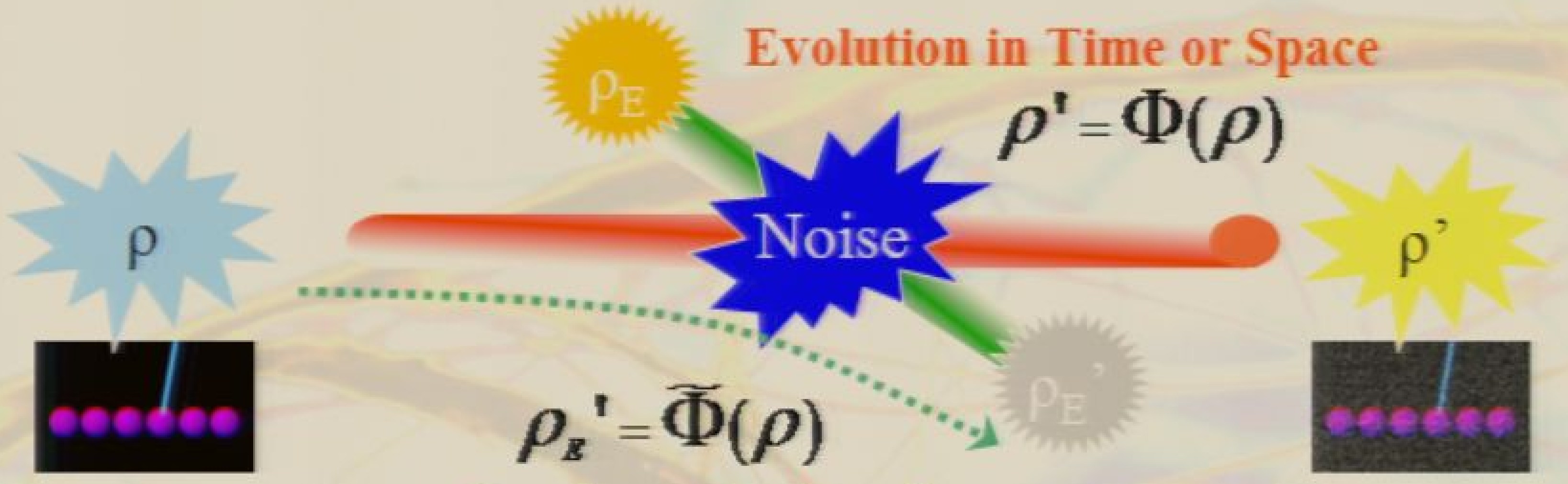
$$\rho_E' = \tilde{\Phi}(\rho)$$

Complementary channel

(after a time interval)

$$\Phi(\rho) \equiv Tr_e[U(\rho \otimes \rho_E)U^\dagger] \qquad \tilde{\Phi}(\rho) \equiv Tr_s[U(\rho \otimes \rho_E)U^\dagger]$$

System coupled to environment



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Kraus representation

$$\boxed{\Phi(\rho) = \sum_k A_k \rho A_k^\dagger}$$

Completeness relation

$$\mathrm{Tr}\ \Phi(\rho) = \mathrm{Tr}\ \rho \quad \forall\rho \iff \sum_{k=1}^{n} A_k^\dagger A_k = I$$

Missing quantum information and decoherence

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Quantum capacity measures "how good" is a channel to preserve the quantum coherence

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$$\boxed{Q(\Phi) = \lim_{n\to\infty} \frac{1}{n} \sup_{\rho} J(\rho, \Phi^{\otimes n})}$$

S. Lloyd, *PRA* 1997; H. Barnum, M.A. Nielsen, B. Schumacher, *PRA* 1998; I. Devetak, *IEEE Trans. Inf. Theory* 2005.

where

Coherent information $J(\rho,\Phi) = S(\rho') - S(\rho_E')$

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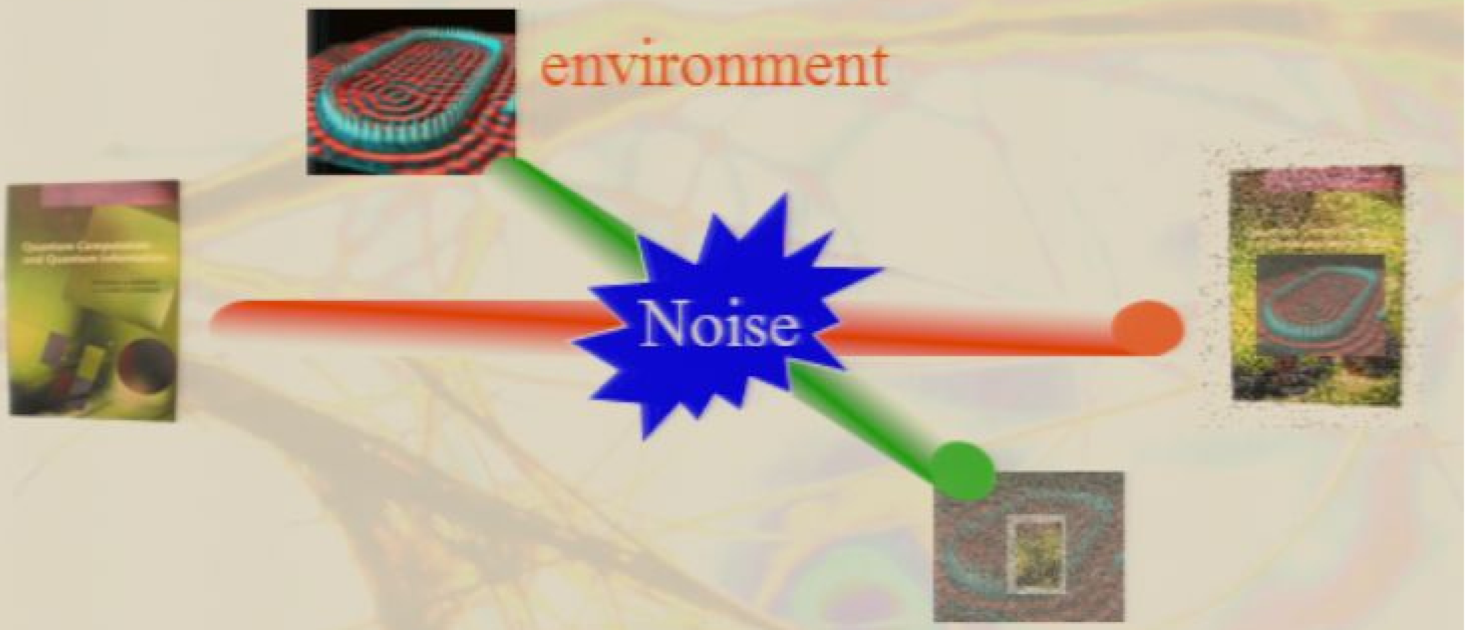
$\Phi^{\otimes n}$:

Weak-degradability and anti-degradability



environment

Noise

Weak-degradability and anti-degradability

weakly degradable

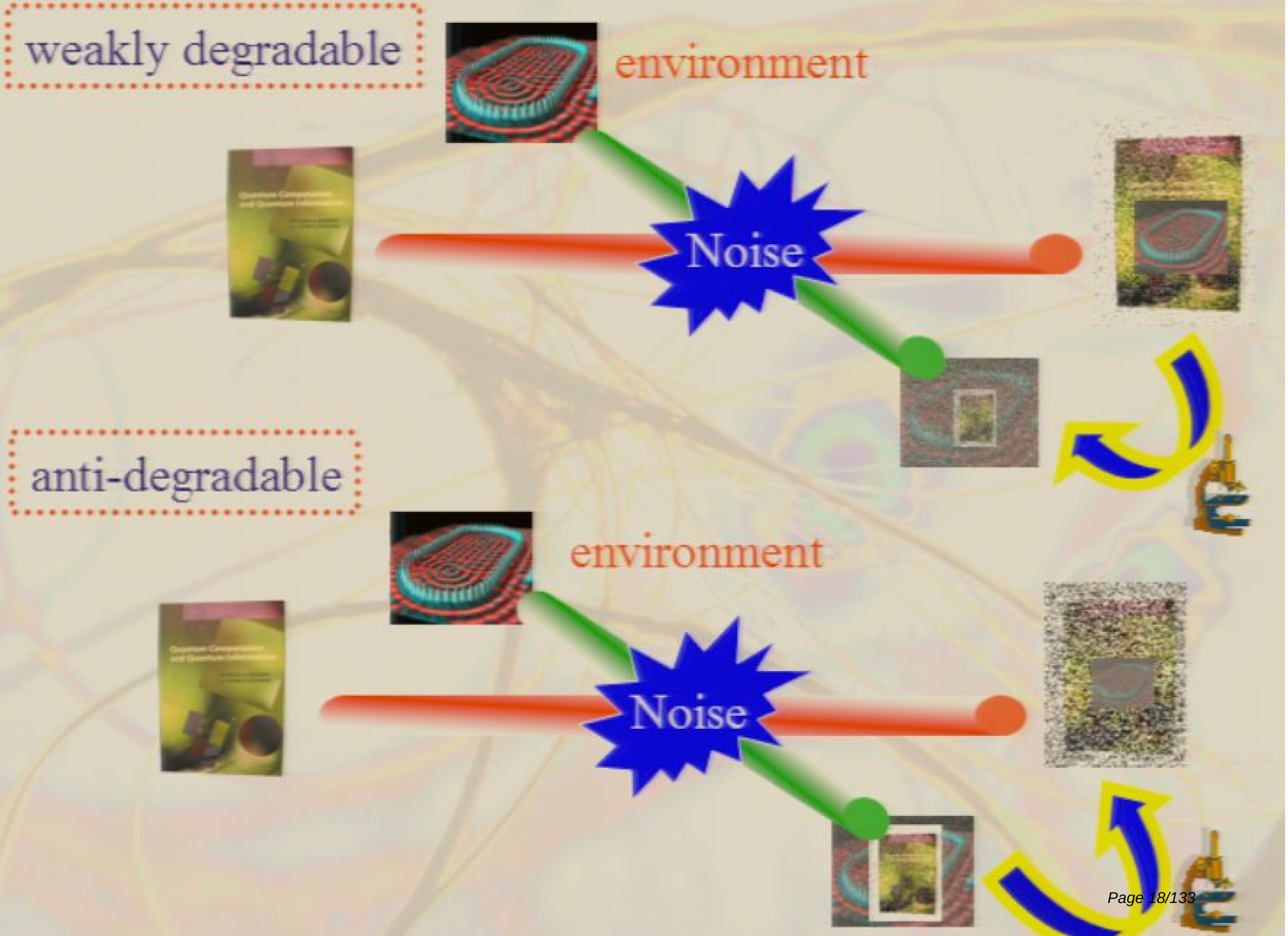
environment

Noise

Weak-degradability and anti-degradability

weakly degradable

Weak-degradability and anti-degradability



weakly degradable

environment

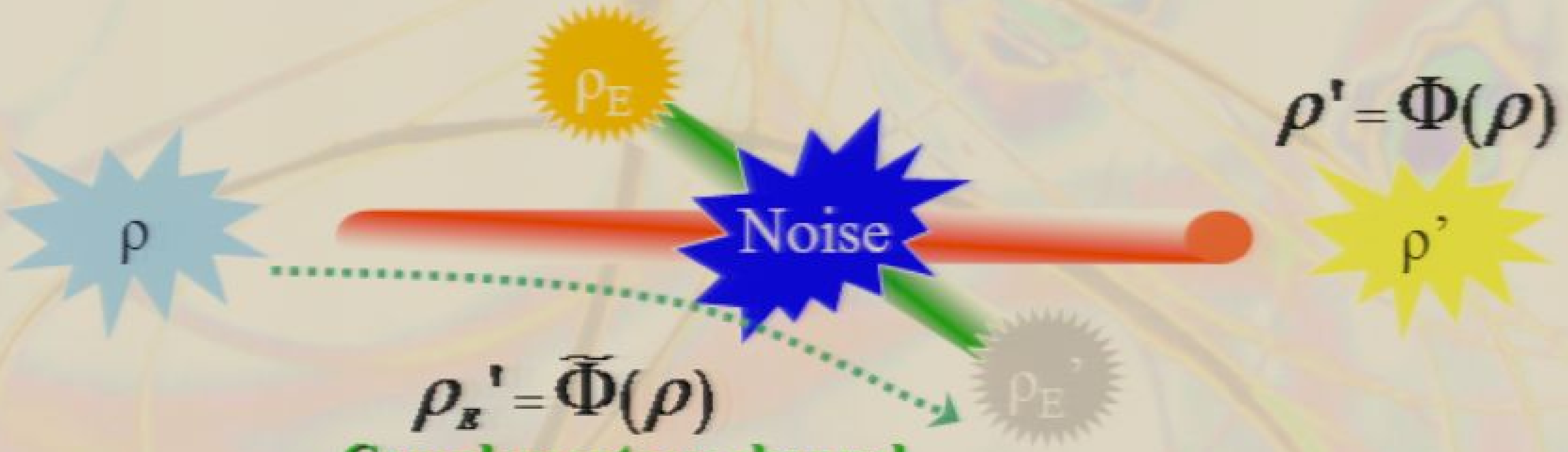
Noise

anti-degradable

environment

Noise

In a more formal way …



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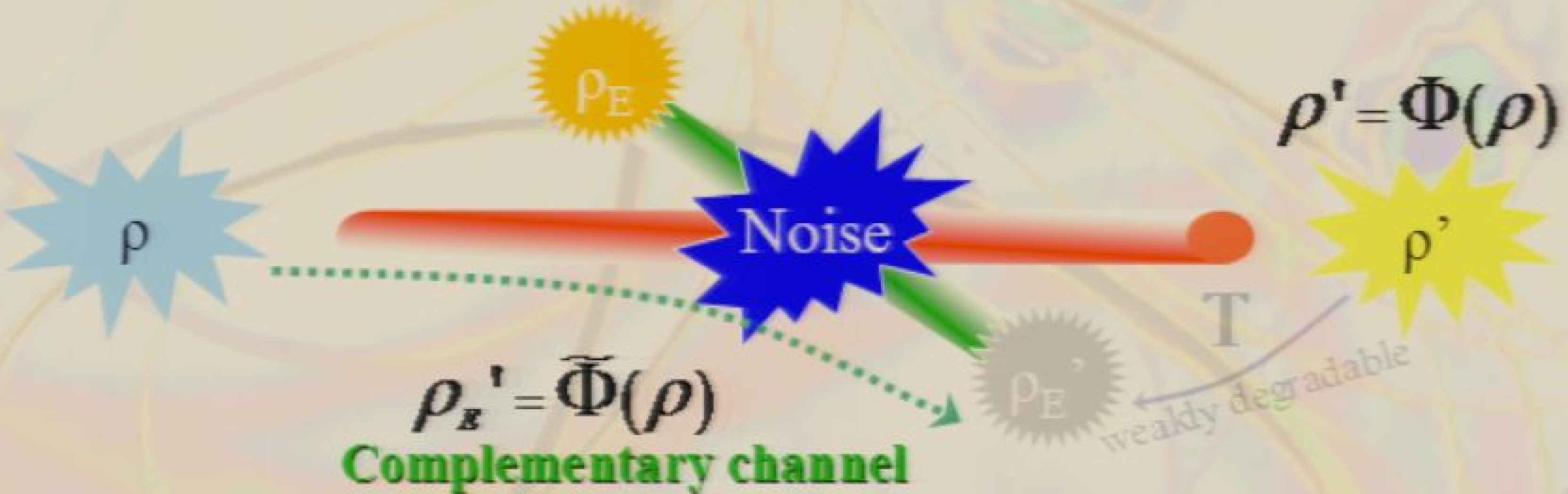
Complementary channel

In a more formal way …

Def. of weak-degradability

$$(T \circ \Phi)(\rho) = \widetilde{\Phi}(\rho)$$



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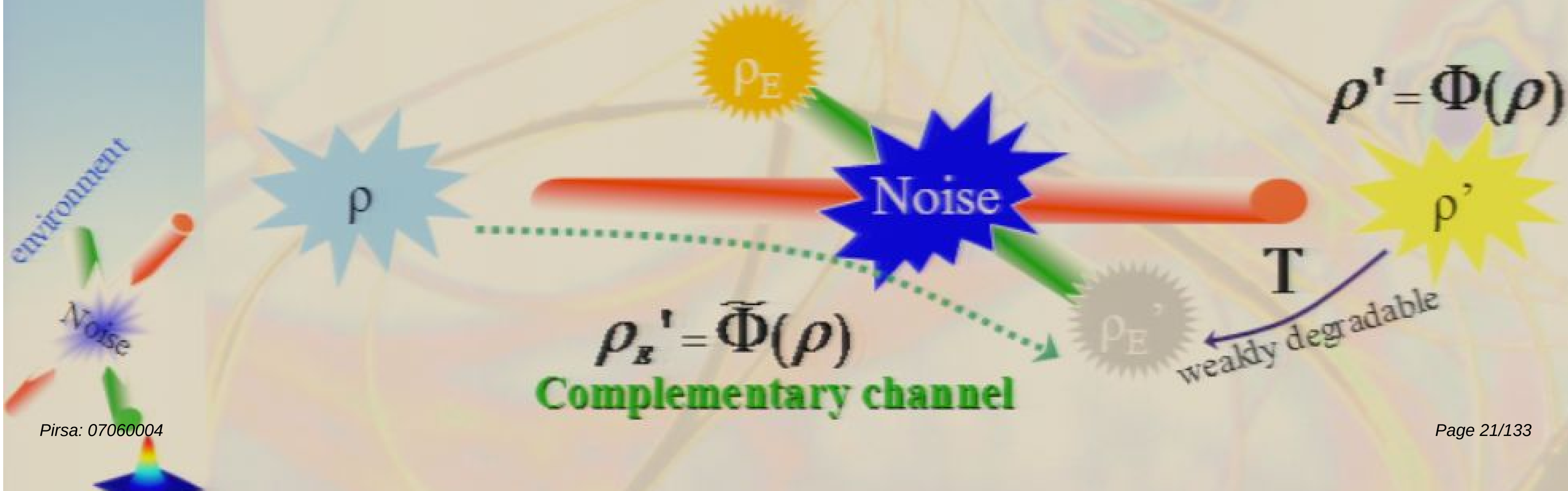
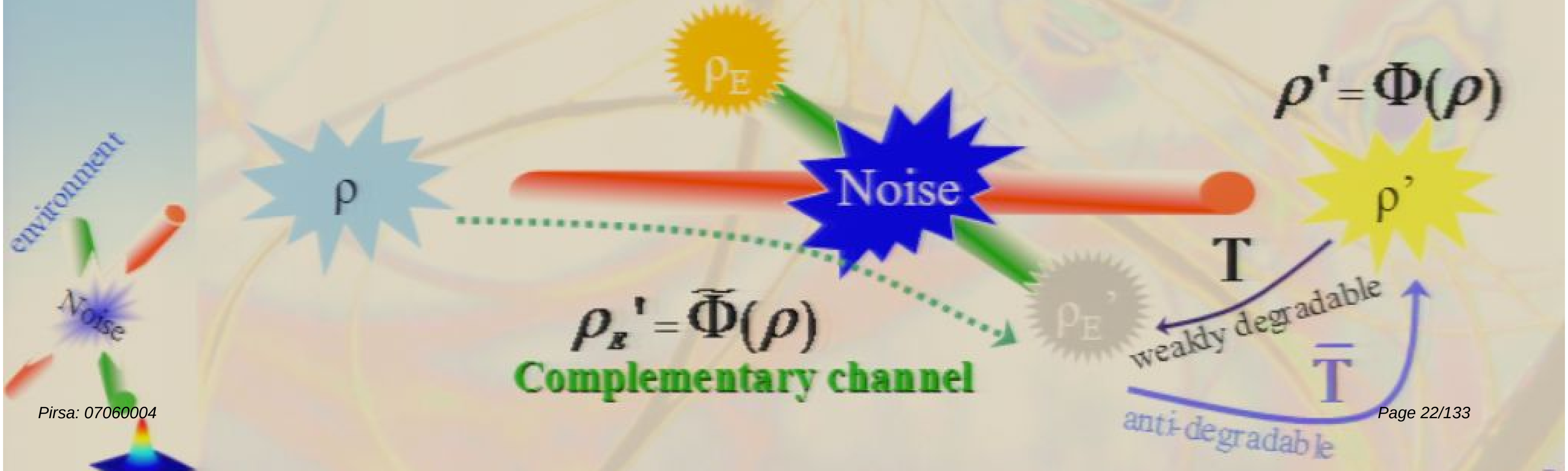


In a more formal way …

Def. of weak-degradability $$(T\circ\Phi)(\rho)=\widetilde{\Phi}(\rho)$$

Def. of anti-degradability $$(\overline{T}\circ\widetilde{\Phi})(\rho)=\Phi(\rho)$$



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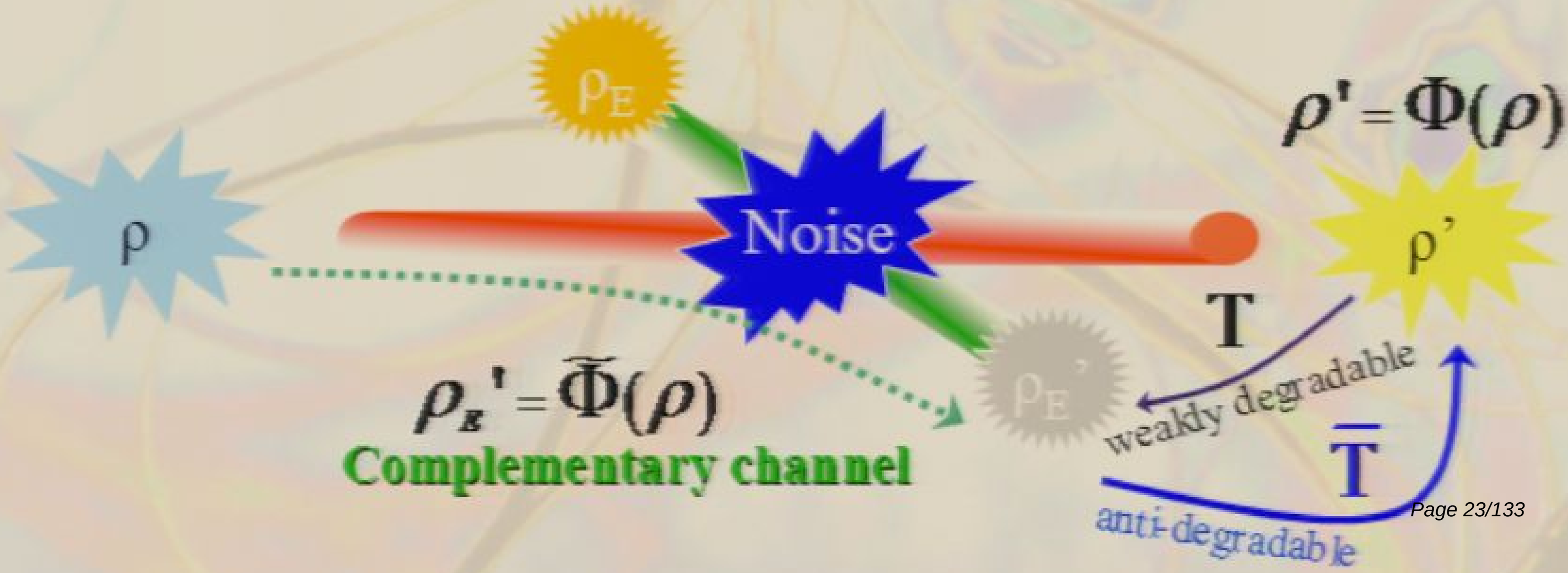


$\rho_E'=\tilde{\Phi}(\rho)$

Complementary channel

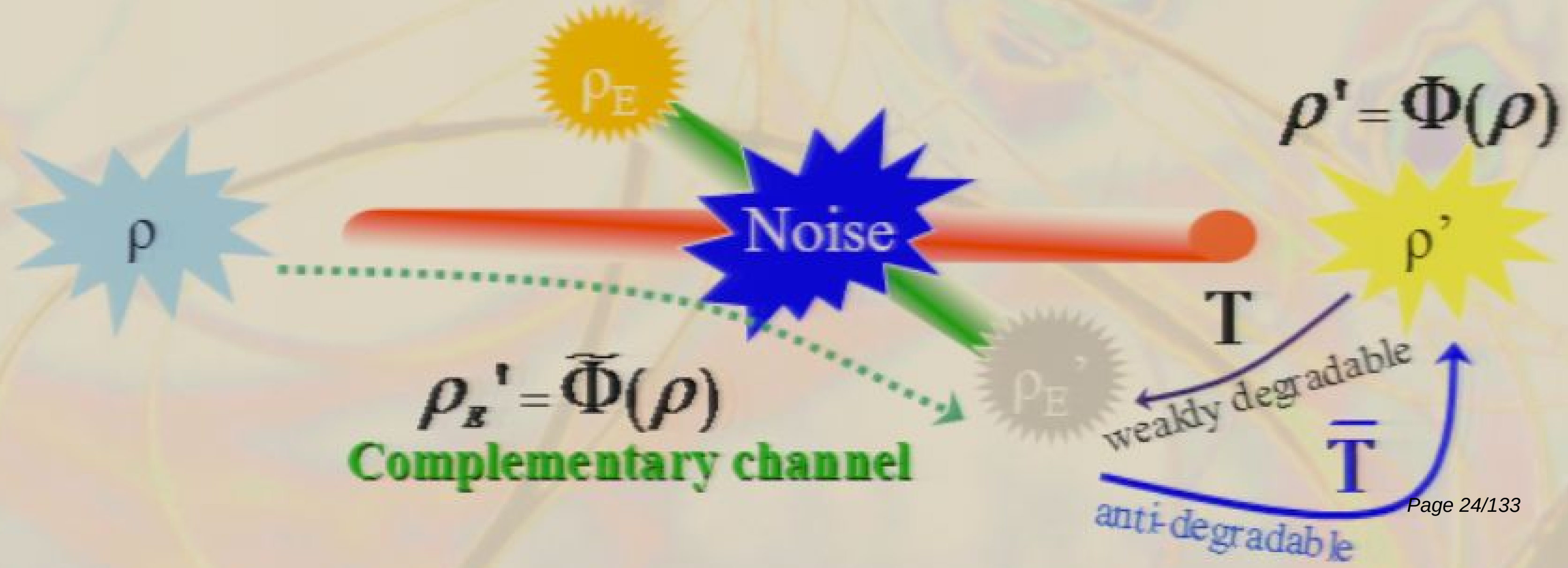
In a more formal way …

Def. of weak-degradability $(T \circ \Phi)(\rho) = \tilde{\Phi}(\rho)$

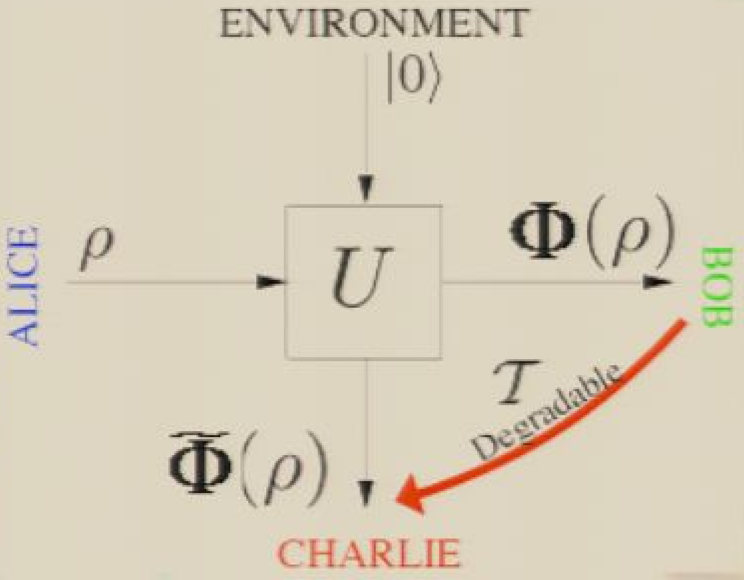
Def. of anti-degradability $(\bar{T} \circ \tilde{\Phi})(\rho) = \Phi(\rho)$

If Φ is weakly degradable (anti-degradable)

then $\tilde{\Phi}$ is anti-degradable (weakly degradable)



It implies that …



$$Q(\Phi) = \lim_{n\to\infty} \frac{1}{n} \sup_{\rho} J(\rho, \Phi^{\otimes n})$$

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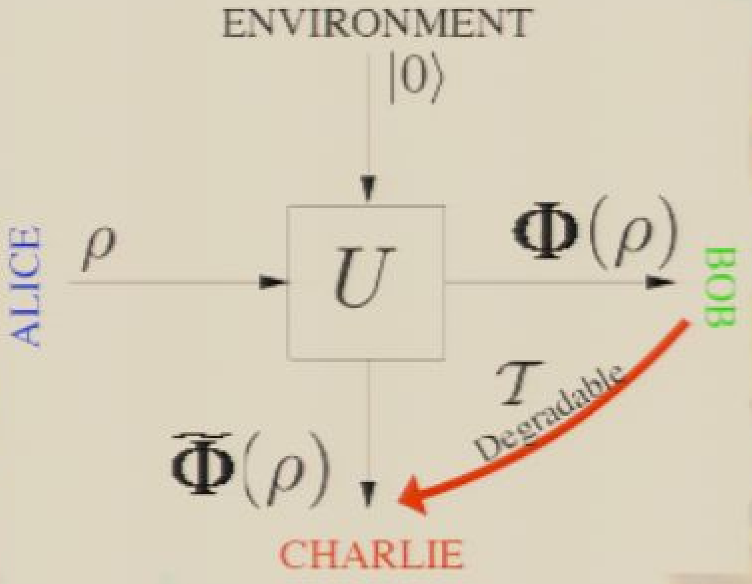

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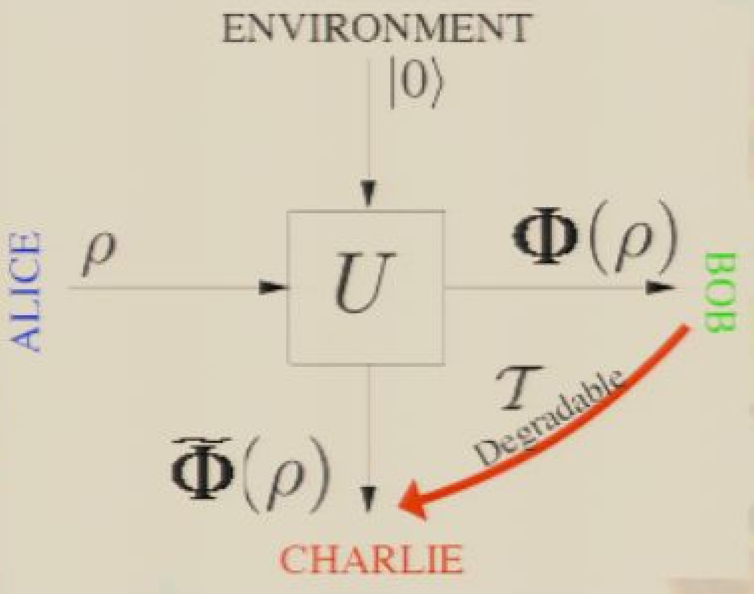

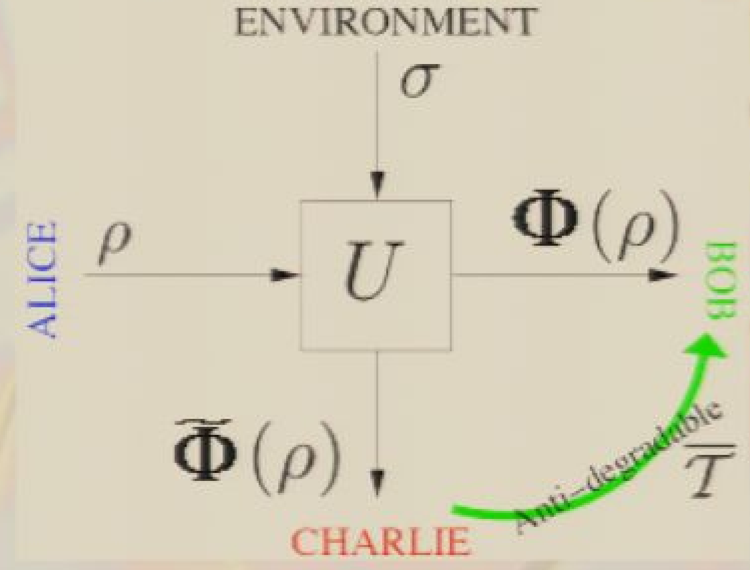

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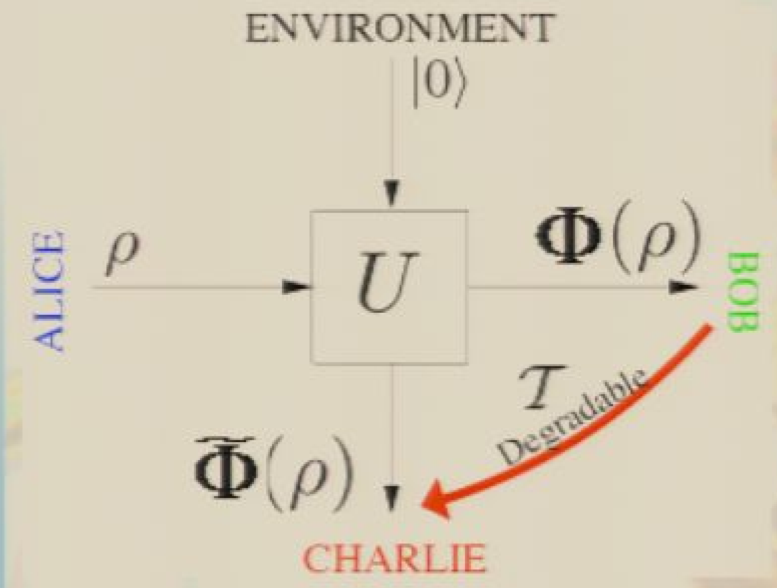

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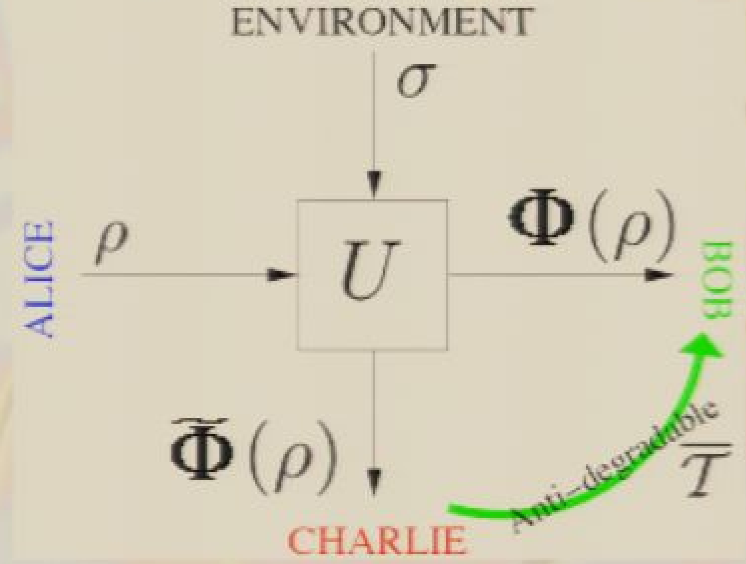

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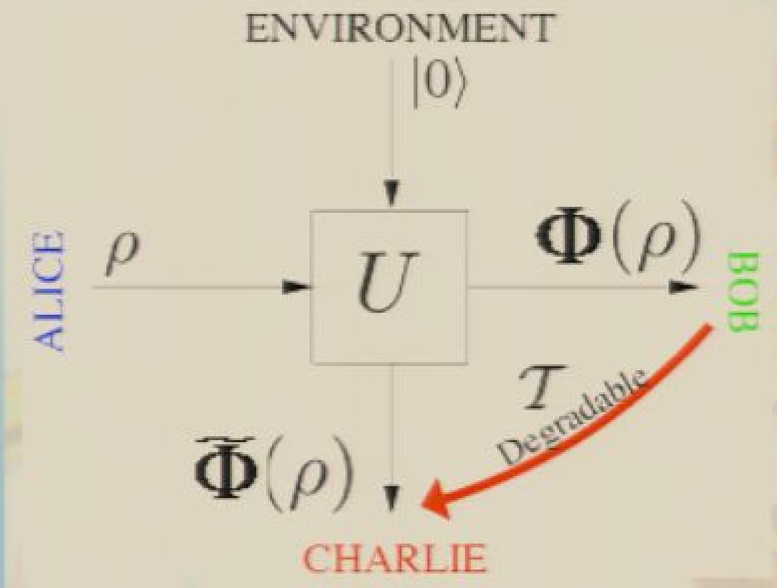

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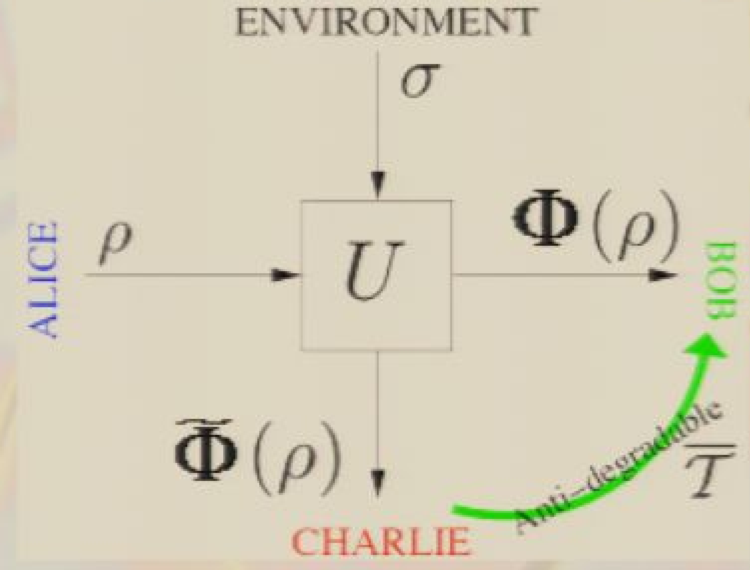

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No-Cloning Theorem → Q=0

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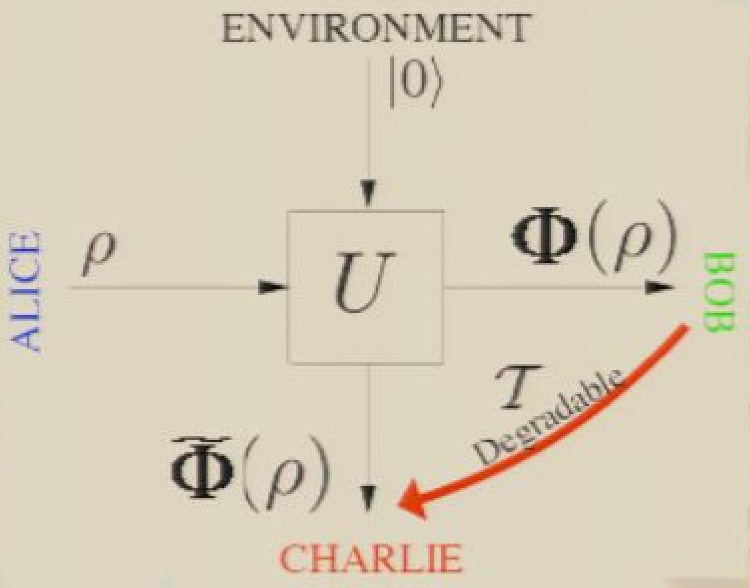

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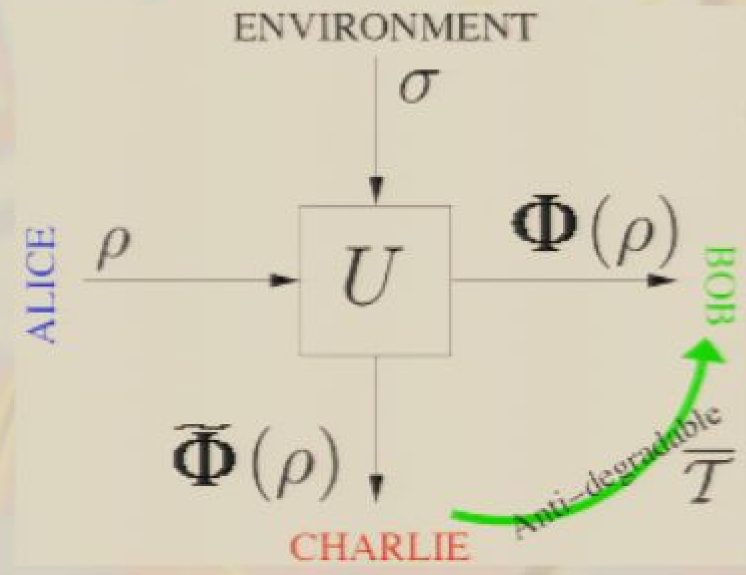
This property is invariant under unitary transformation



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  - Missing quantum information and decoherence
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  - Characteristic function and Gaussian states
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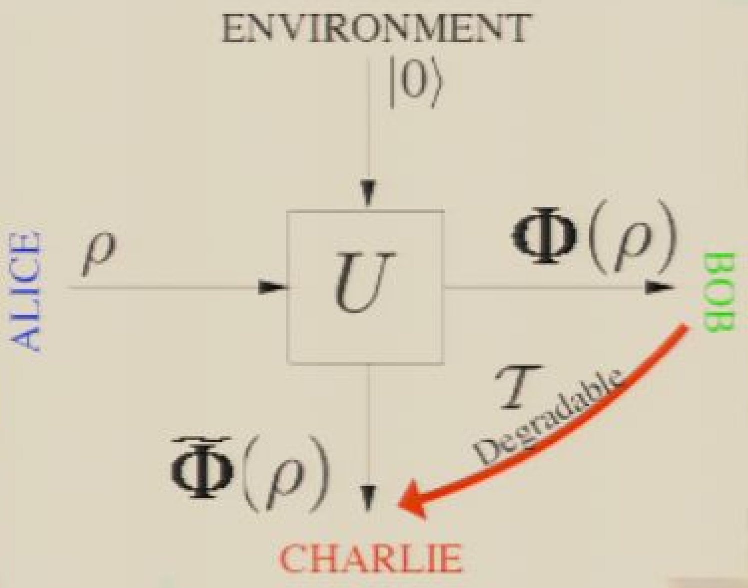

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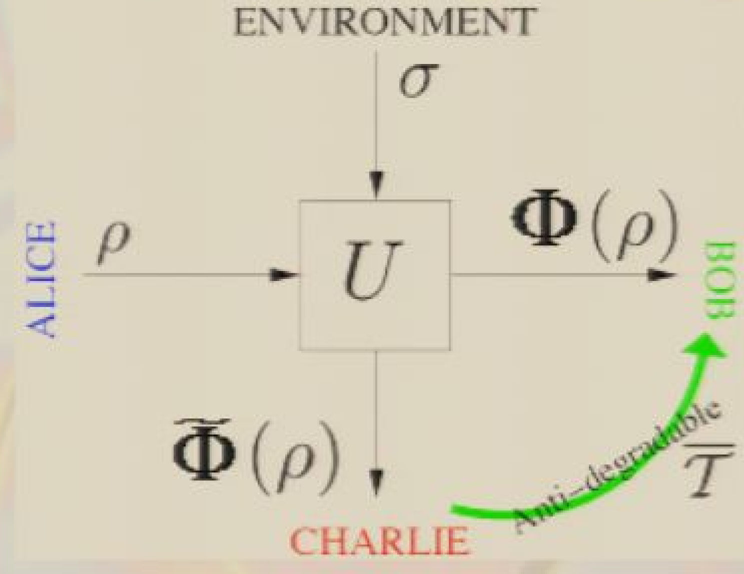

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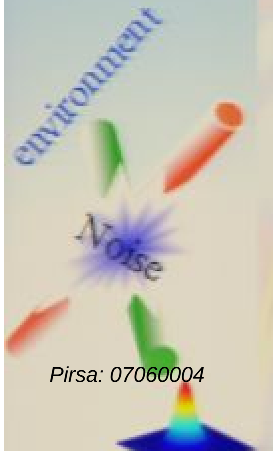
Characteristic functions and Gaussian states

- **Characteristic function** $$\chi(\mu) = \mathrm{Tr}[\rho\,\underbrace{\exp(\mu a^\dagger - \mu^* a)}_{\text{Displacement operator}}]$$



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$$W(x, p) = \frac{1}{2\pi} \int \exp(\eta^{*}\alpha - \eta\alpha^{*})\chi(\eta)\,\mathrm{d}^{2}\eta$$

Characteristic functions and Gaussian states

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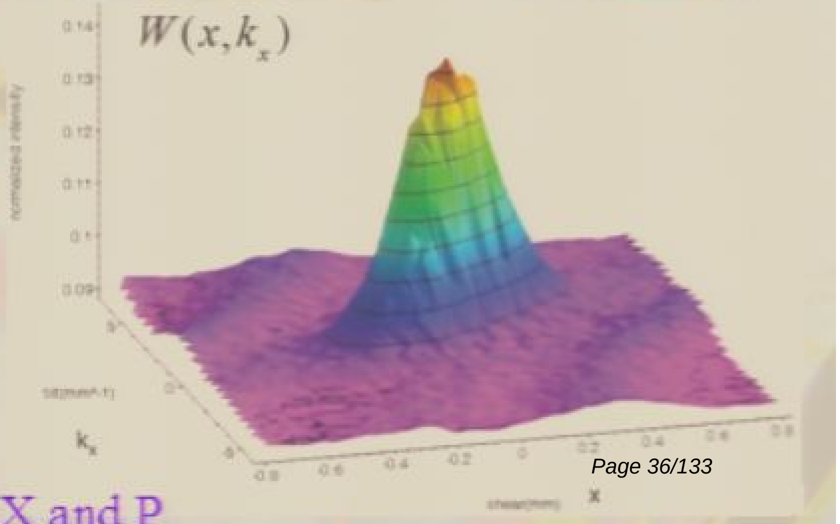
$$W(x, p) = \frac{1}{2\pi} \int \exp(\eta^* \alpha - \eta \alpha^*) \chi(\eta) \, \mathrm{d}^2 \eta$$

- ☐ A state is called **Gaussian**, if and only if its characteristic function (or its Wigner function) is a Gaussian

$$\chi(\mu) = \exp\left[-\zeta_0 \cdot \zeta^\dagger - \frac{1}{2} \zeta \cdot \Gamma \cdot \zeta^\dagger \right], \quad \zeta \equiv (\mu^*, -\mu)$$

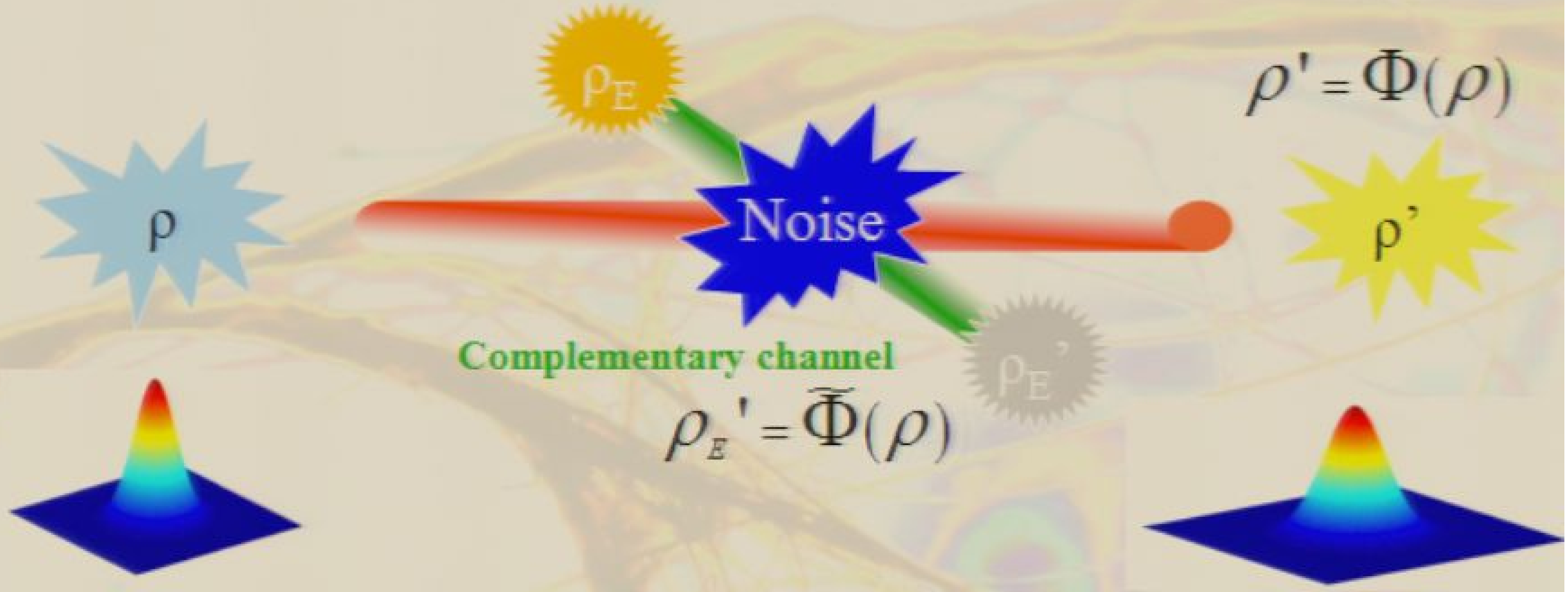
$$\Gamma \equiv \begin{bmatrix} \langle \{\Delta a, \Delta a^\dagger\} \rangle / 2 & \langle (\Delta a^\dagger)^2 \rangle \\ \langle (\Delta a)^2 \rangle & \langle \{\Delta a, \Delta a^\dagger\} \rangle / 2 \end{bmatrix}$$



Examples:

- vacuum state
- coherent state
- thermal state
- squeezed state
- ground state of Hamiltonian quadratic in X and P

Gaussian Channels



$$\rho' = \Phi(\rho)$$

Complementary channel

$$\rho_E' = \tilde{\Phi}(\rho)$$

Gaussian quantum channels play a quite central role because they describe:

- good models for the transmission of light through optical fibers
- linear couplings between bosonic systems with quadratic Hamiltonians
- random classical noise, introduced by Gaussian random displacements in phase space
- losses modelled as a beam splitter like interaction with the vacuum or a thermal mode
- amplification and/or squeezing transformations

Bosonic Gaussian Channels

$\{\mathbf{b},\mathbf{b}^\dagger\}$

$\{\mathbf{a},\mathbf{a}^\dagger\}$

ρ_E

$\rho' = \Phi(\rho)$

ρ

Noise

ρ'

ρ_E'

Complementary channel

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Bosonic Channel

Bosonic Gaussian Channels

$$\rho' = \Phi(\rho)$$

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Bosonic Gaussian Channel

and

$$\boxed{\chi'(z) = \chi(K^T z)\,\xi(K_E^T z)}$$

where

K, K_E are linear maps between the phase spaces and $z = \sqrt{2}\begin{pmatrix}\mathrm{Im}\,\mu \\ \mathrm{Re}\,\mu\end{pmatrix}$

A.S. Holevo, R.F. Werner, *PRA* 2001

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Displacement evolution $$\boxed{D'(z) = D(K^T z)\,\xi(K_E^T z)}$$

A.S. Holevo, R.F. Werner, *PRA* 2001

One-parameter family of unitaries

- ❑ **Bogoliubov transformation**

$$\vec{v}^T = (a, a^\dagger, b, b^\dagger)$$

$$U \vec{v}\, U^\dagger = A \cdot \vec{v}$$

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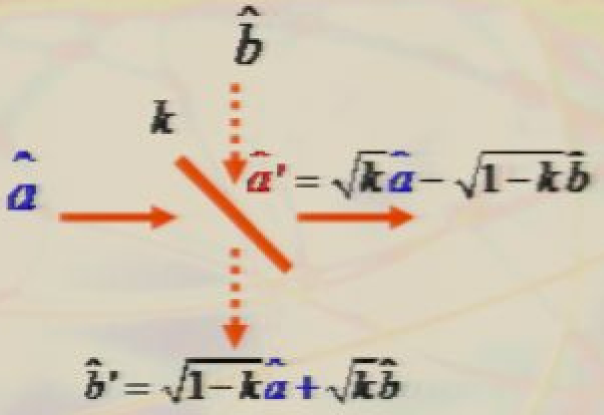
$$\boxed{0 \le k \le 1}$$

$$A^{(k)} = \begin{pmatrix} \sqrt{k} & 0 & -\sqrt{1-k} & 0 \\ 0 & \sqrt{k} & 0 & -\sqrt{1-k} \\ \sqrt{1-k} & 0 & \sqrt{k} & 0 \\ 0 & \sqrt{1-k} & 0 & \sqrt{k} \end{pmatrix}$$



$$\hat{a}' = \sqrt{k}\,\hat{a} - \sqrt{1-k}\,\hat{b}$$

$$\hat{b}' = \sqrt{1-k}\,\hat{a} + \sqrt{k}\,\hat{b}$$

One-parameter family of unitaries

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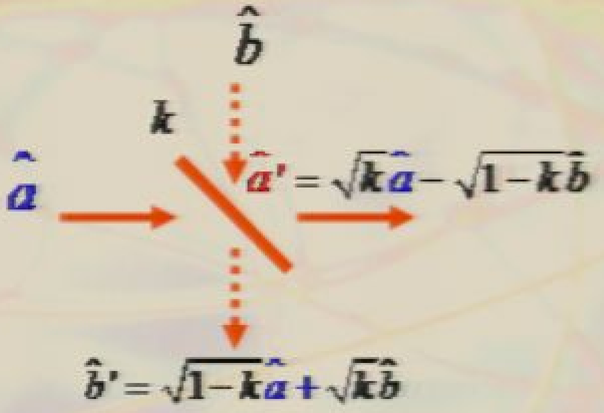

$\boxed{k\ge 1}$

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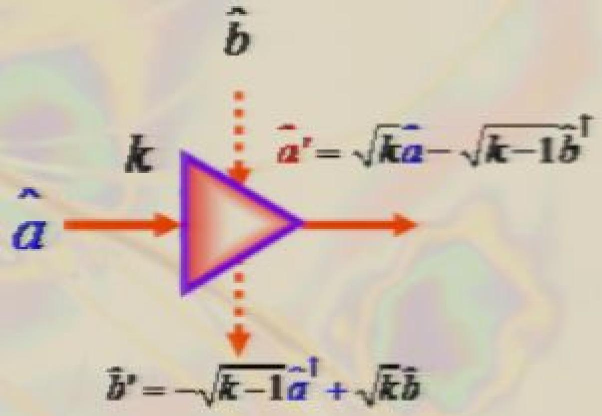

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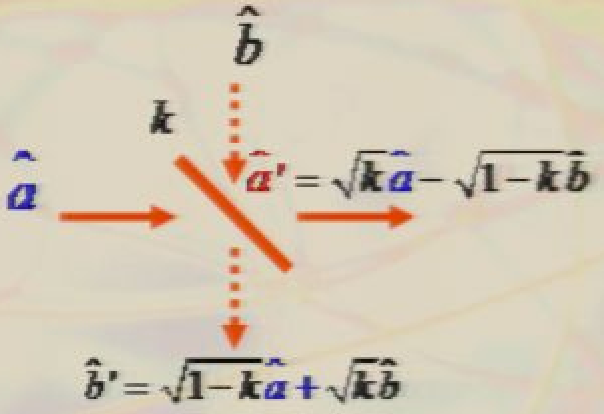
$$A^{(k)} = \begin{pmatrix} \sqrt{k} & 0 & -\sqrt{1-k} & 0 \\ 0 & \sqrt{k} & 0 & -\sqrt{1-k} \\ \sqrt{1-k} & 0 & \sqrt{k} & 0 \\ 0 & \sqrt{1-k} & 0 & \sqrt{k} \end{pmatrix}$$

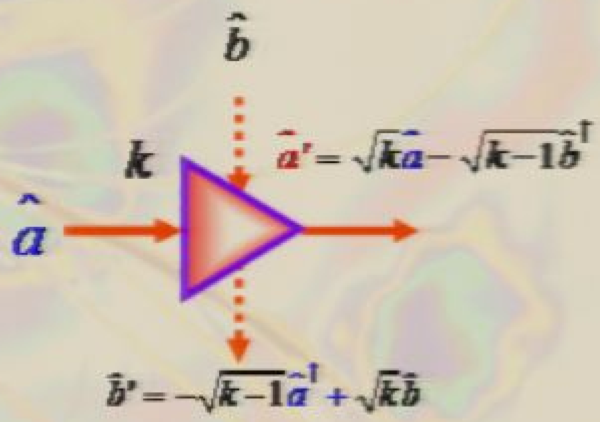


$\hat{a}' = \sqrt{k}\,\hat{a} - \sqrt{1-k}\,\hat{b}$

$\hat{b}' = \sqrt{1-k}\,\hat{a} + \sqrt{k}\,\hat{b}$

$\boxed{k \ge 1}$

$$A^{(k)} = \begin{pmatrix} \sqrt{k} & 0 & 0 & -\sqrt{k-1} \\ 0 & \sqrt{k} & -\sqrt{k-1} & 0 \\ 0 & -\sqrt{k-1} & \sqrt{k} & 0 \\ -\sqrt{k-1} & 0 & 0 & \sqrt{k} \end{pmatrix}$$



$\hat{a}' = \sqrt{k}\,\hat{a} - \sqrt{k-1}\,\hat{b}^\dagger$

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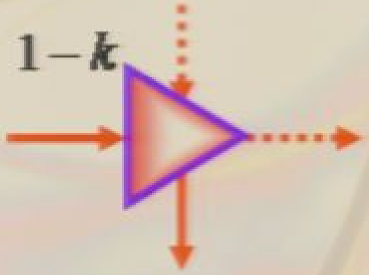
$\boxed{k < 0}$



One-parameter family of unitaries

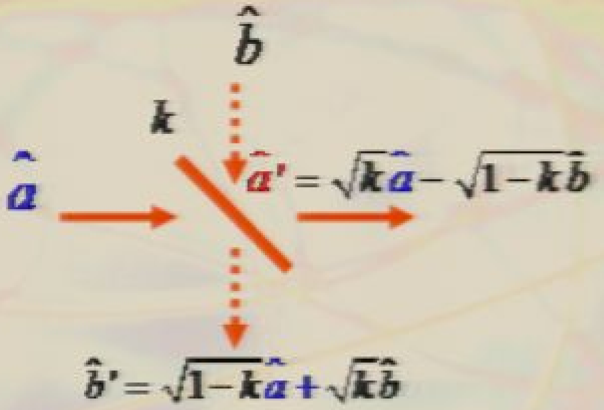
- ☐ **Bogoliubov transformation**

$$U\vec{v}\,U^\dagger = A\cdot\vec{v}$$

$$\vec{v}^T = (a, a^\dagger, b, b^\dagger)$$

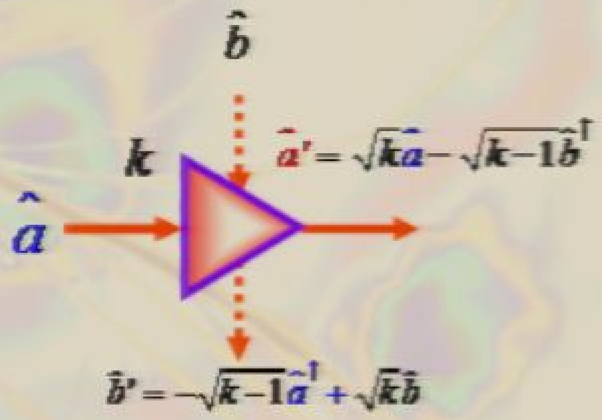
$\boxed{0 \le k \le 1}$

$$A^{(k)} = \begin{pmatrix} \sqrt{k} & 0 & -\sqrt{1-k} & 0 \\ 0 & \sqrt{k} & 0 & -\sqrt{1-k} \\ \sqrt{1-k} & 0 & \sqrt{k} & 0 \\ 0 & \sqrt{1-k} & 0 & \sqrt{k} \end{pmatrix}$$



$\boxed{k \ge 1}$

$$A^{(k)} = \begin{pmatrix} \sqrt{k} & 0 & 0 & -\sqrt{k-1} \\ 0 & \sqrt{k} & -\sqrt{k-1} & 0 \\ 0 & -\sqrt{k-1} & \sqrt{k} & 0 \\ -\sqrt{k-1} & 0 & 0 & \sqrt{k} \end{pmatrix}$$



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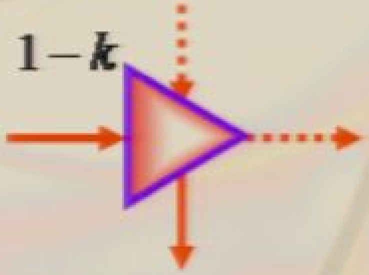


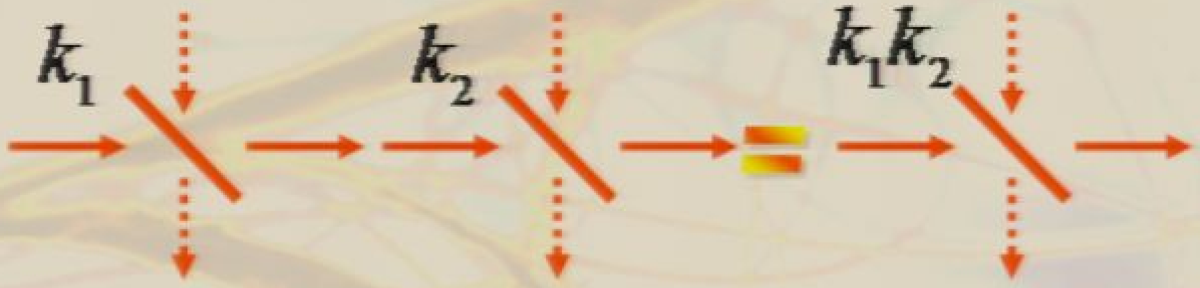
$$\chi(\mu) \rightarrow \chi'(\mu) = \begin{cases} \chi(\sqrt{k}\mu)\,\xi(\sqrt{1-k}\mu) & k \in [0,1] \\ \chi(\sqrt{k}\mu)\,\xi(-\sqrt{k-1}\mu^*) & k \ge 1 \end{cases}$$

Beam-splitter and amplifier channel

Composition rules

Beam-splitter and amplifier channel

Composition rules



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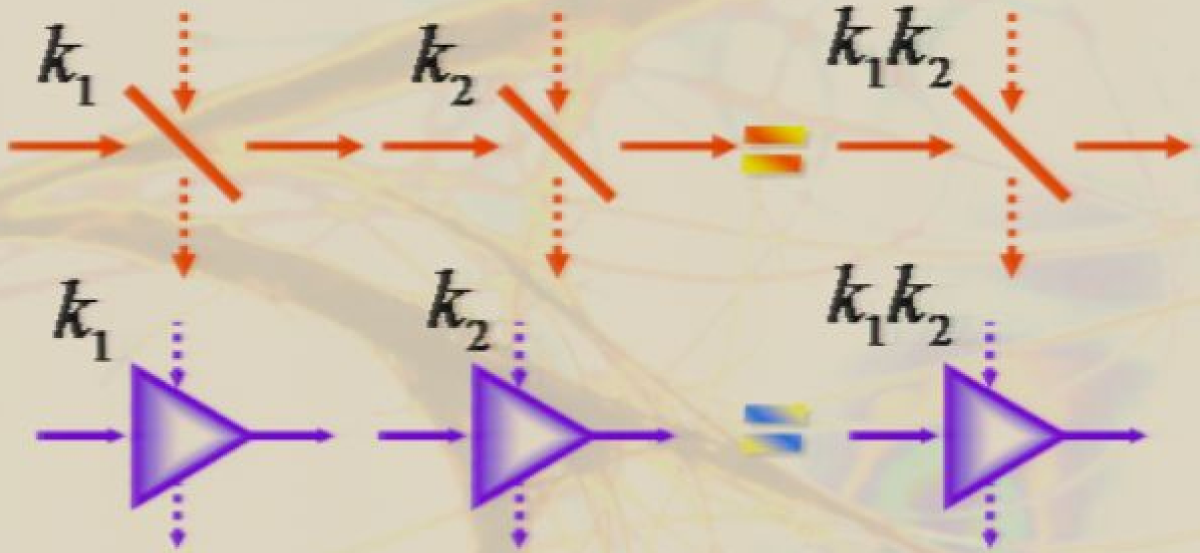

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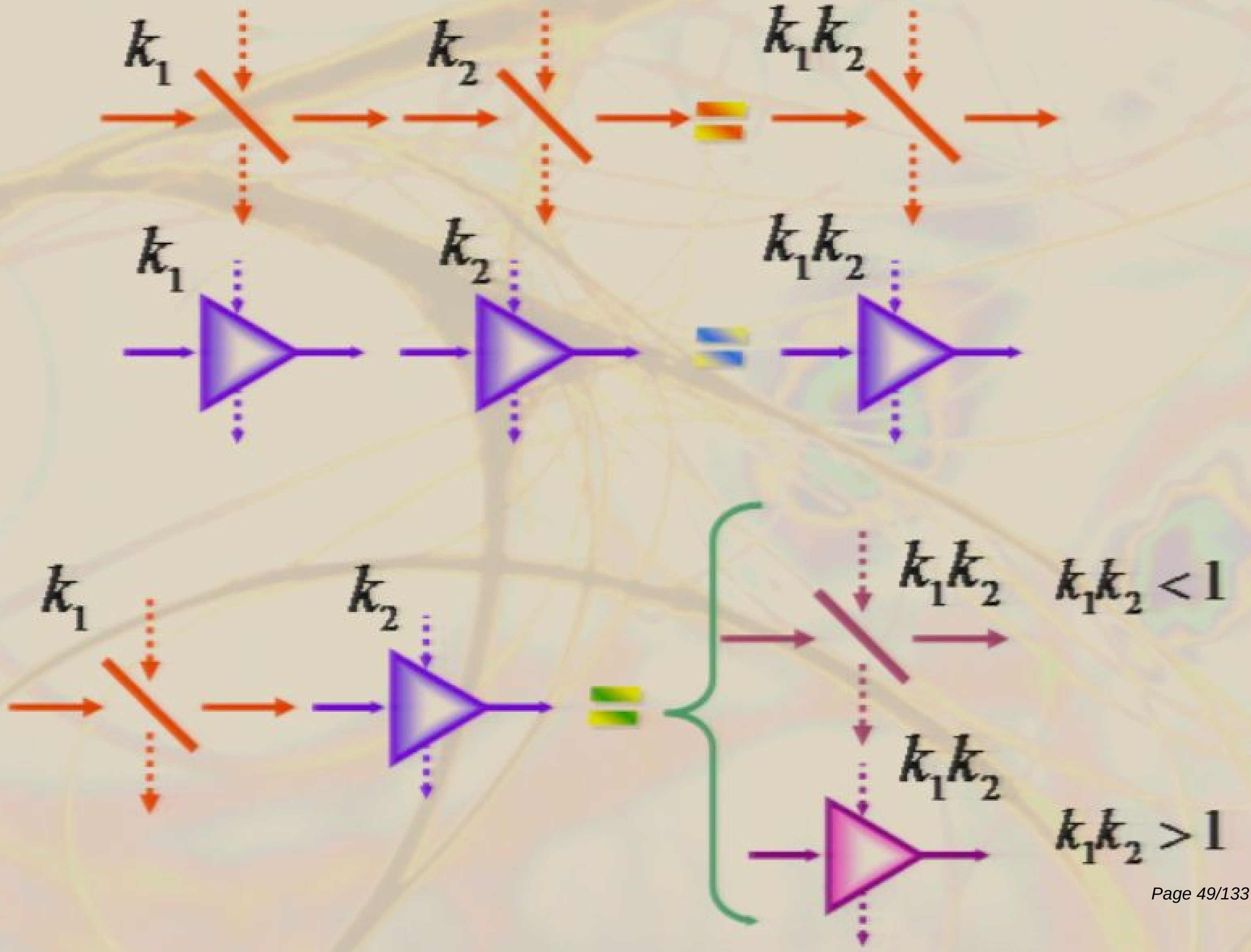
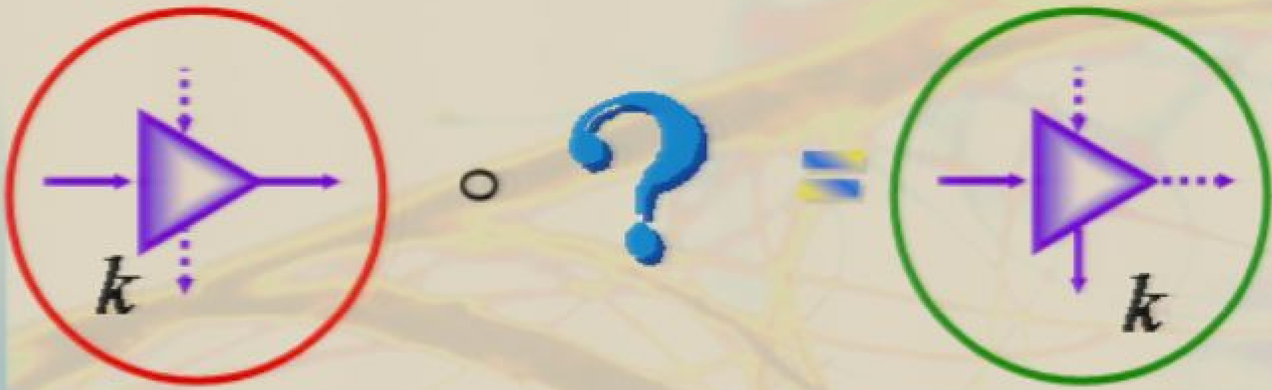


Degradability properties



$k \geq 1$

$(\mathrm{T} \circ \mathcal{E})(\rho) = \tilde{\mathcal{E}}(\rho)$

Degradability properties

$$k' = (2k-1)/k$$

$$k \geq 1$$

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weak-degradable

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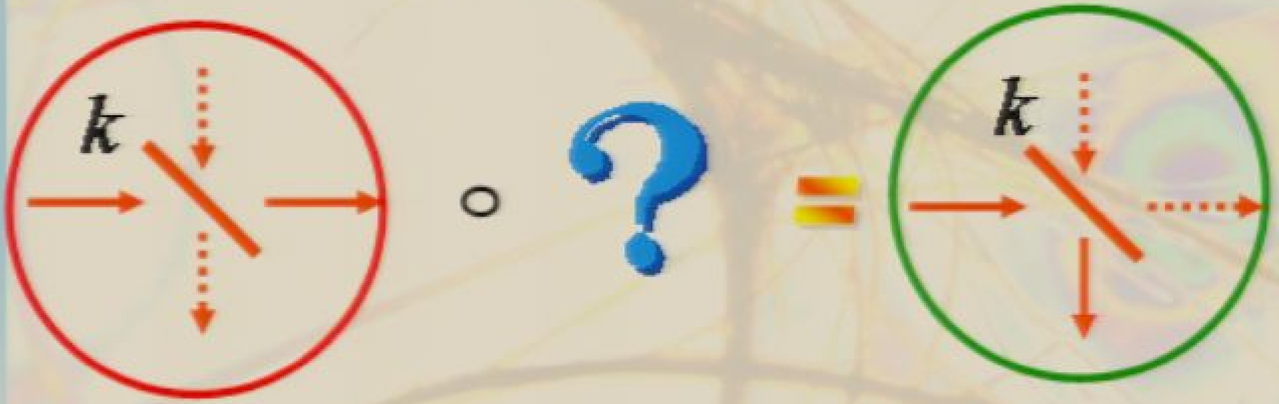

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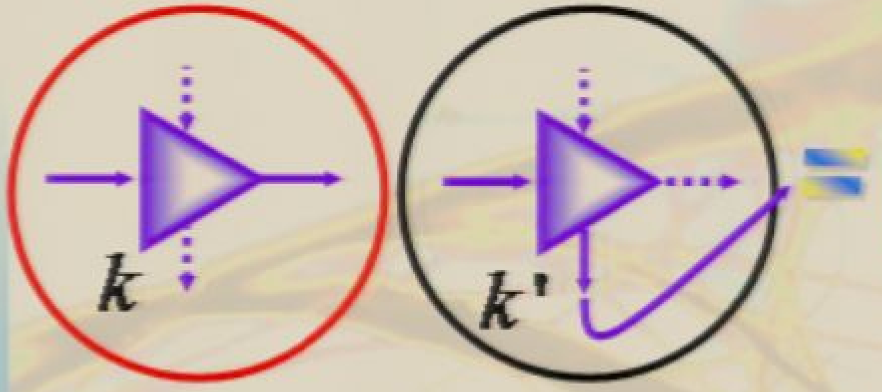

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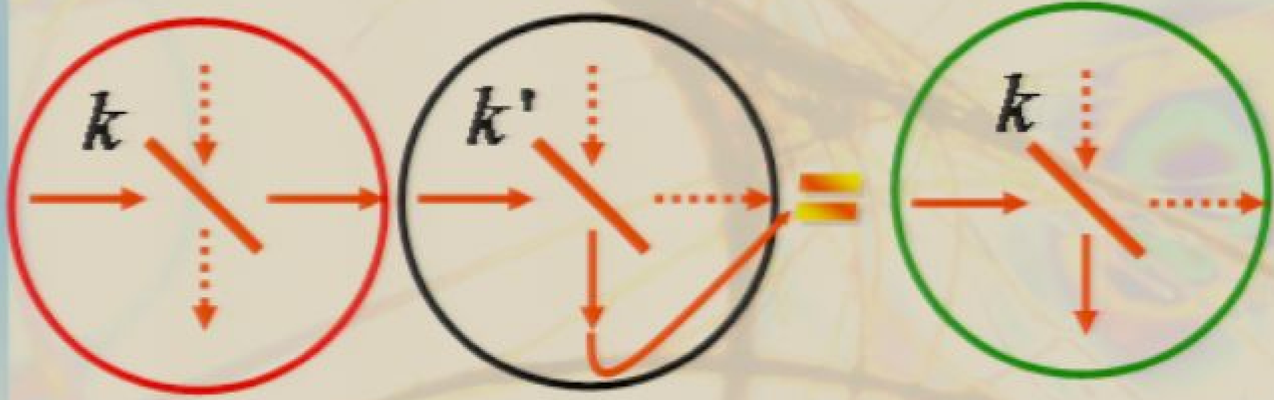

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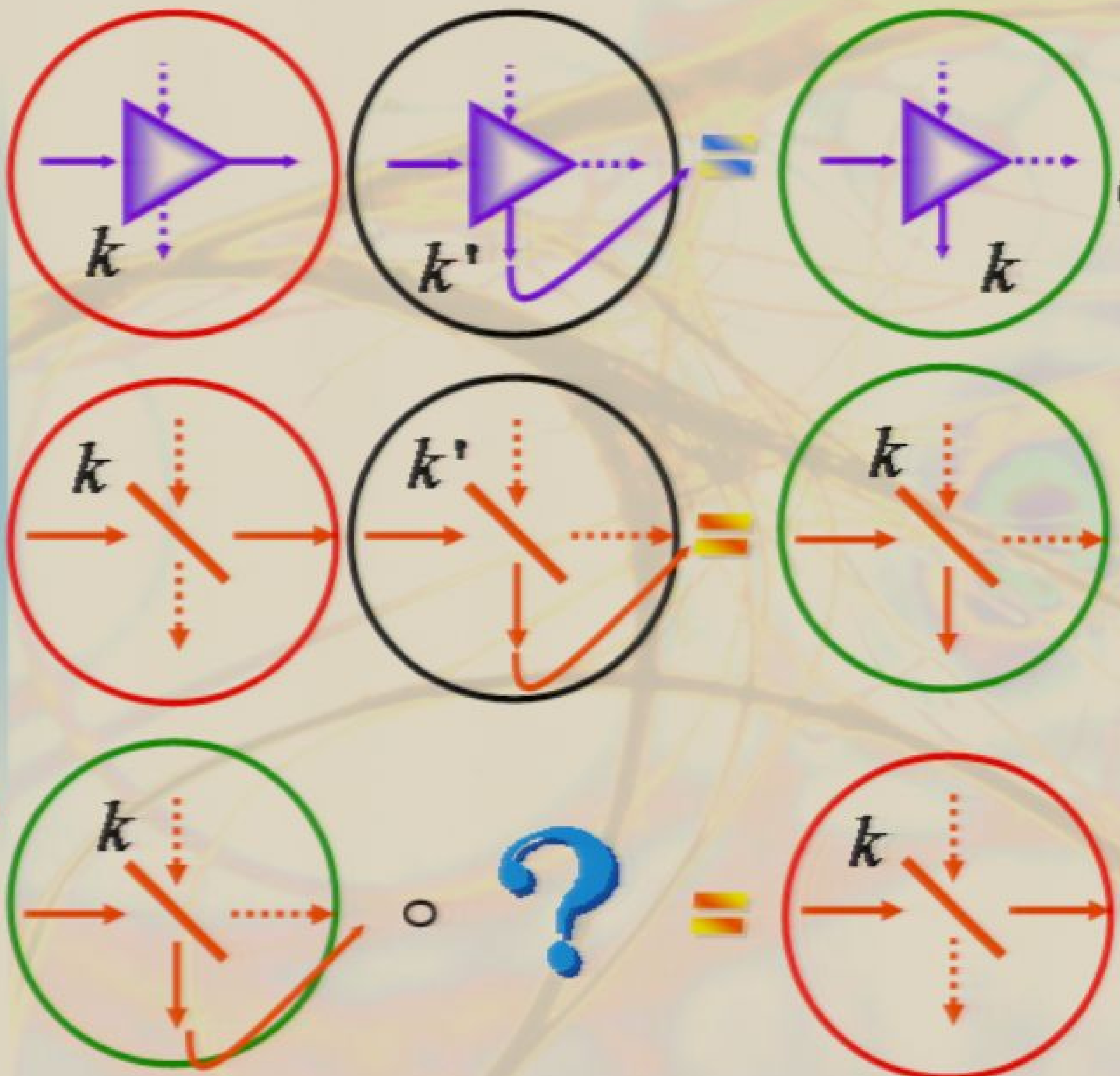

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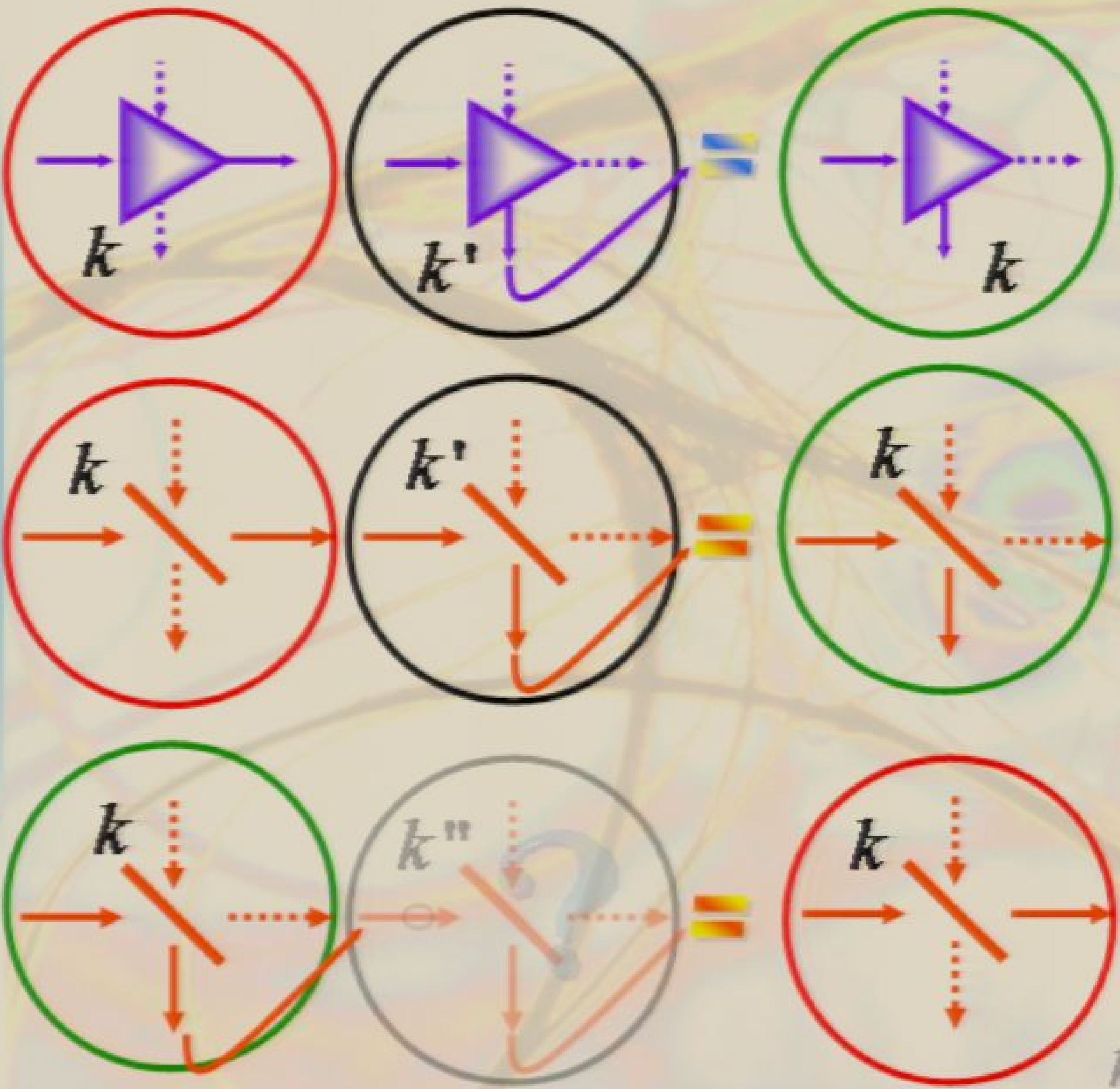
Degradability properties



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$$k'' = (1-2k)/(1-k)$$

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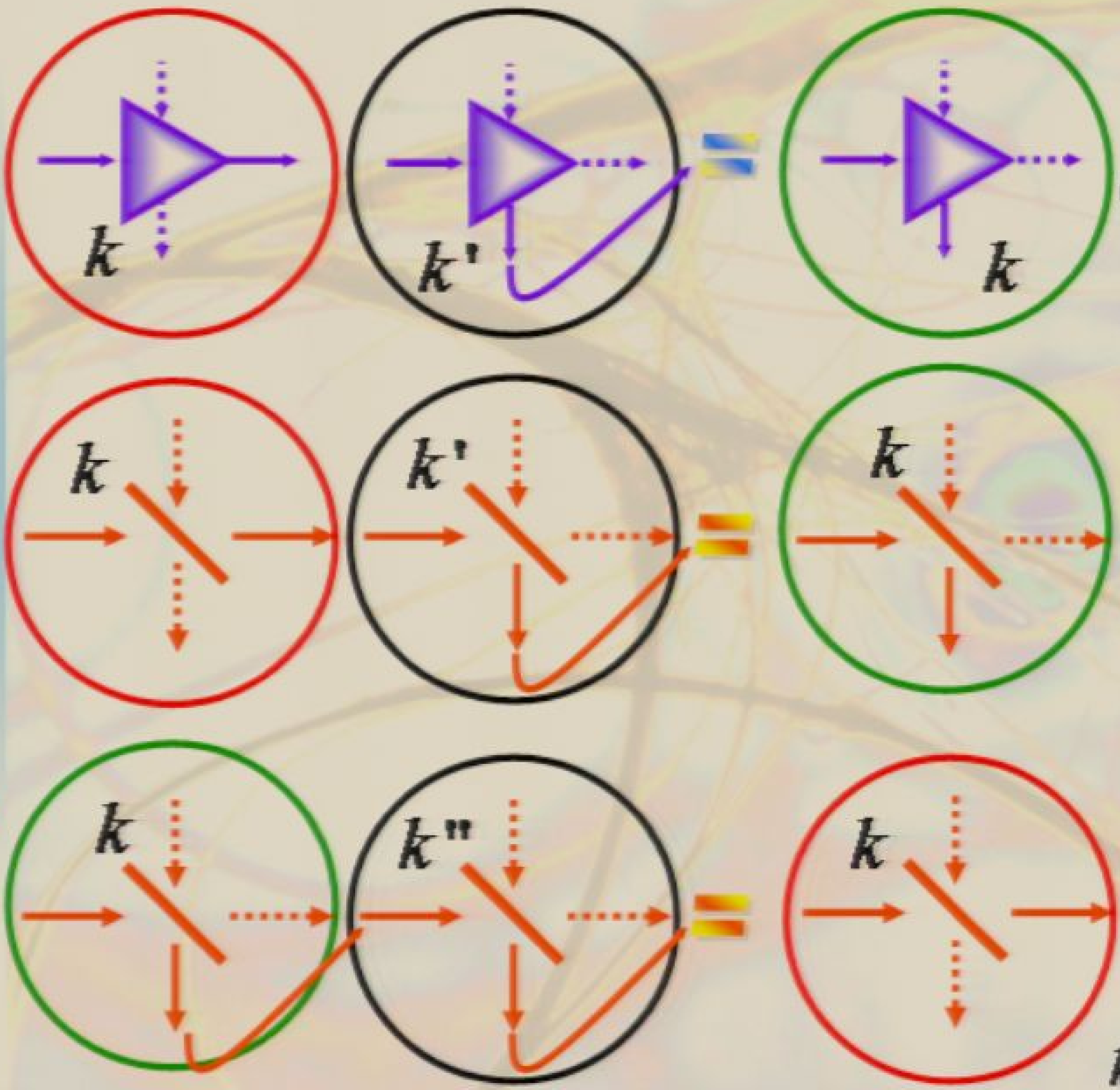

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Degradability of BS and Amplifier channels

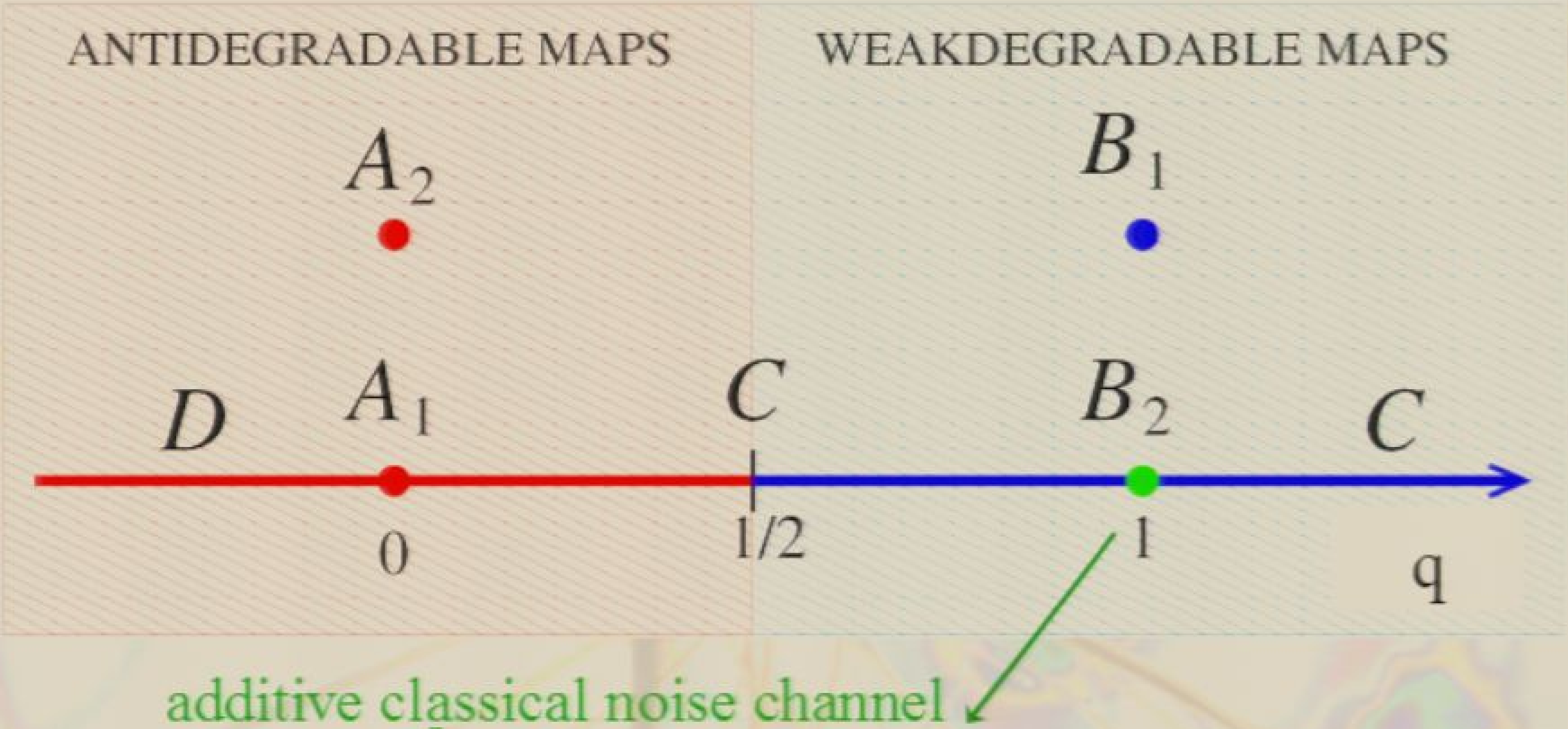
Value of q	Equivalent map	
$q < 0$	$\tilde{\mathcal{E}}[1-q, \sigma'_b]$ conjugate amplifier	Anti-degradable $(Q = 0)$
$0 < q \leqslant 1/2$	$\mathcal{E}[q, \sigma'_b]$ BS of transmissivity q	Anti-degradable $(Q = 0)$
$1/2 \leqslant q < 1$	BS of transmissivity q $\mathcal{E}[q, \sigma'_b]$	Weakly degradable (degradable for σ'_b pure)
$1 < q$	amplifier	

$$q \equiv |A_{11}|^2 - |A_{12}|^2 = \mathrm{k}$$

Bosonic Gaussian Channels

≡ Squeezing

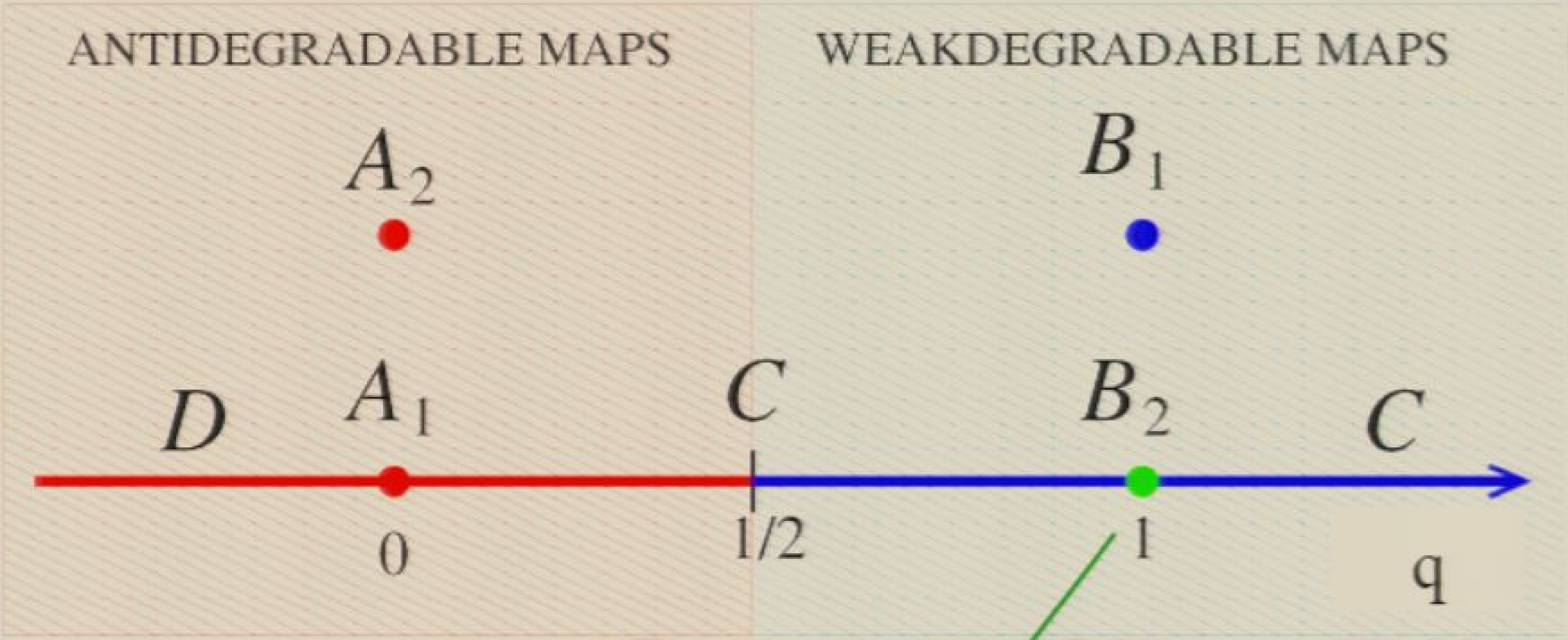
A full classification …



additive classical noise channel

$$\Phi(\rho) = \int d^2z \, p(z) \, D(z) \, \rho \, D(-z)$$

A full classification …



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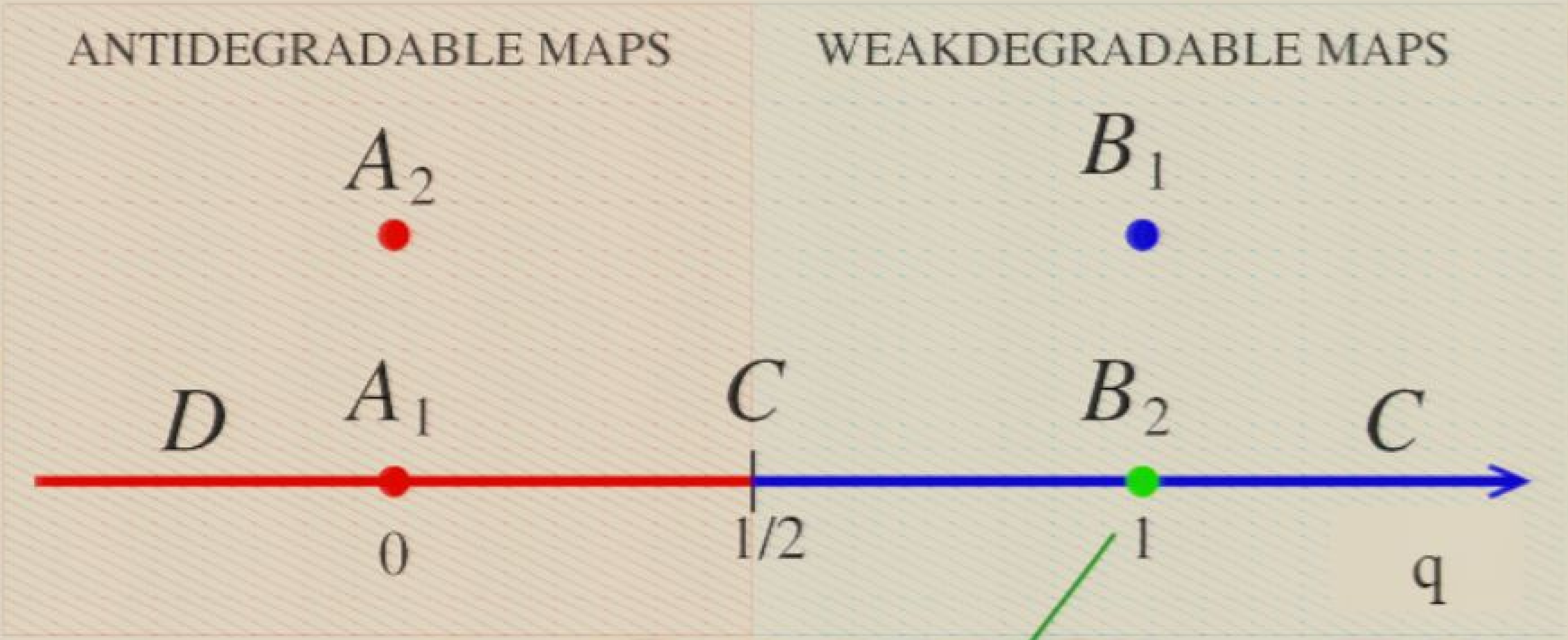

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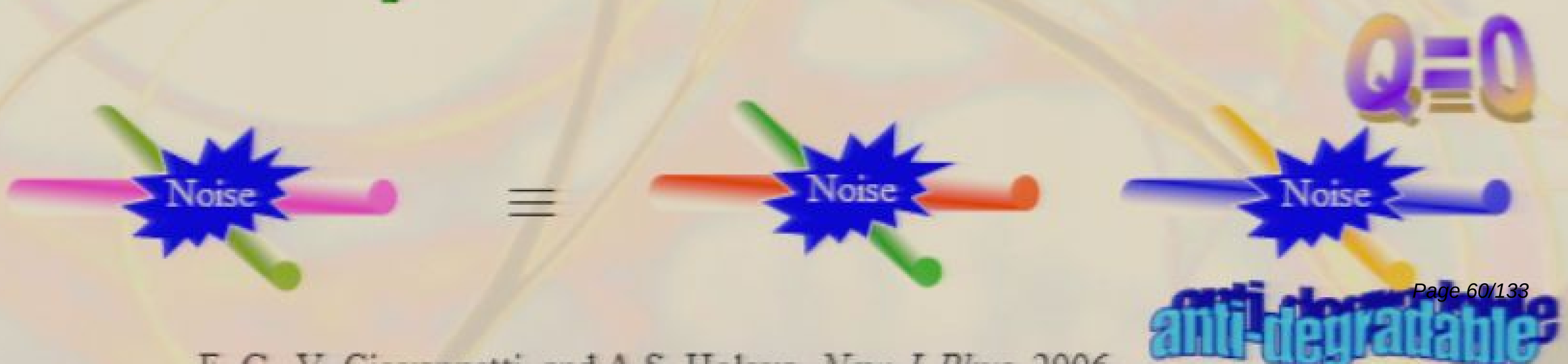

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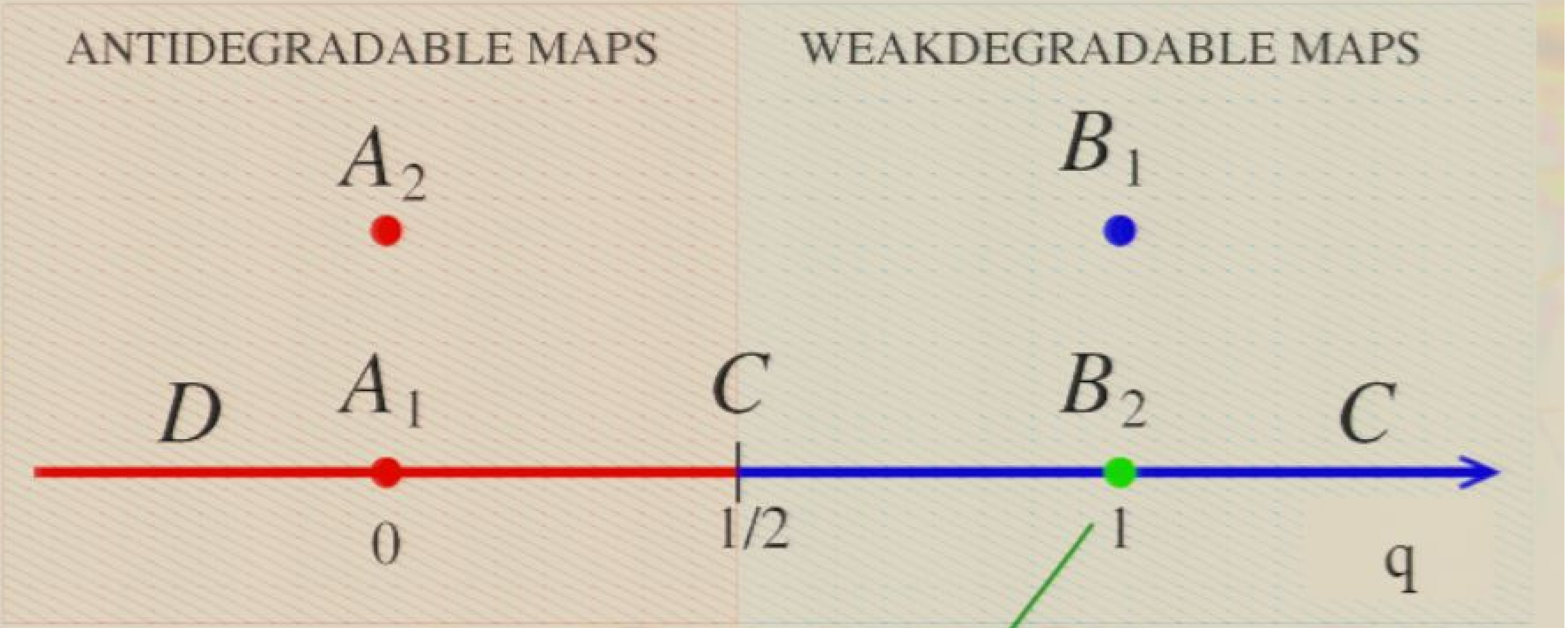

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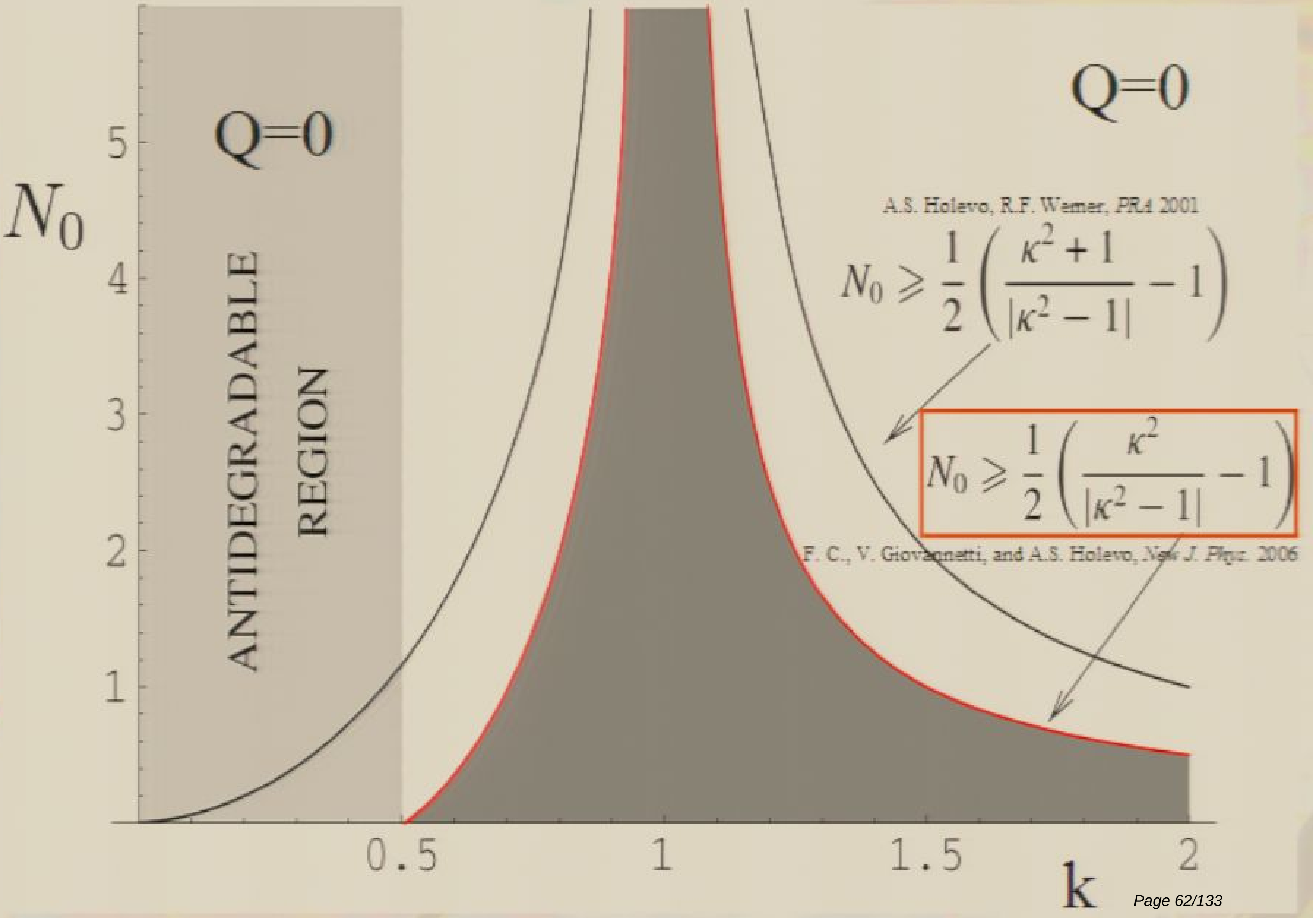
F. C., V. Giovannetti and A.S. Holevo, *New J. Phys.* 2006

A better bound for maps with Q=0



A.S. Holevo, R.F. Werner, *PRA* 2001

$$N_0 \geqslant \frac{1}{2}\left(\frac{\kappa^2+1}{|\kappa^2-1|}-1\right)$$

$$\boxed{N_0 \geqslant \frac{1}{2}\left(\frac{\kappa^2}{|\kappa^2-1|}-1\right)}$$

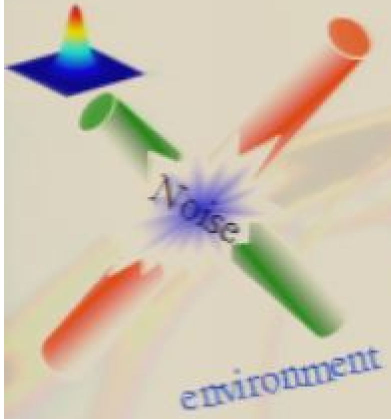
F. C., V. Giovannetti, and A.S. Holevo, *New J. Phys.* 2006

(N_0 is the average photon number of the single environmental mode in a thermal state)

Conclusions and Outlook

F. Caruso and V. Giovannetti, *Physical Review A* **74**, 062307 (2006), quant-ph/0603257

Conclusions and Outlook



- The notion of weak-degradability of quantum channels is introduced by generalizing the degradability definition given by Devetak and Shor. We consider the physical picture of the noise evolution of an open quantum system interacting unitarily with an environment prepared in a *mixed* state.

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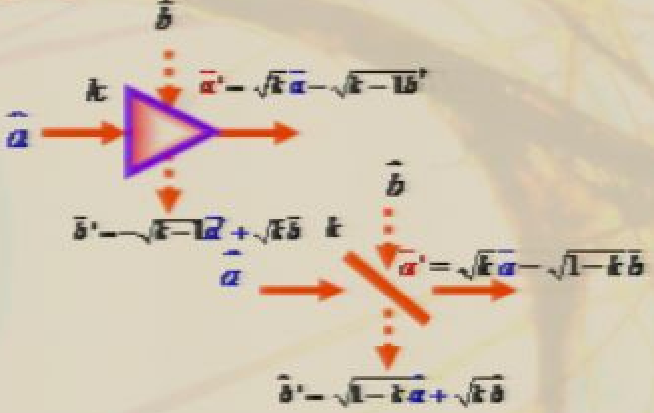

- We prove that the Bosonic Gaussian channels are either weakly degradable or anti-degradable, i.e. either $Q=Q^1$ (additivity) or $Q=0$, respectively, exploiting the fact that these maps are unitarily equivalent to Beam-Splitter/Amplifier channel.

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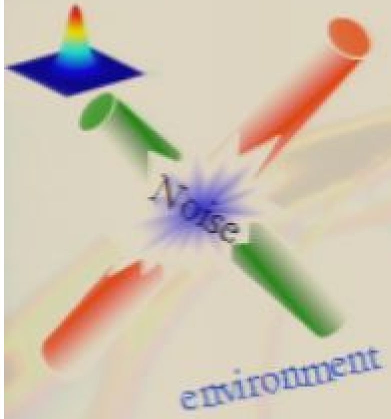

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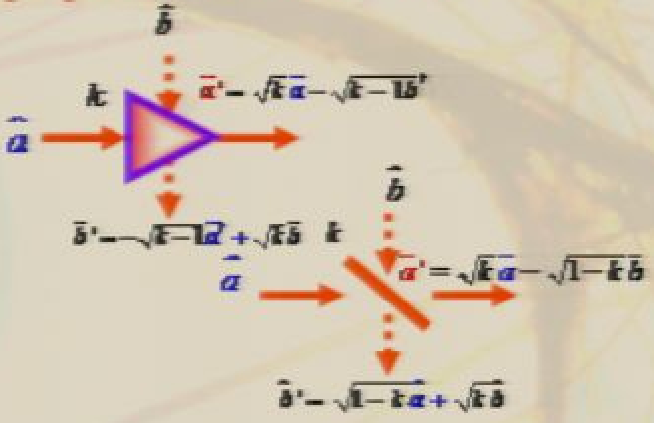

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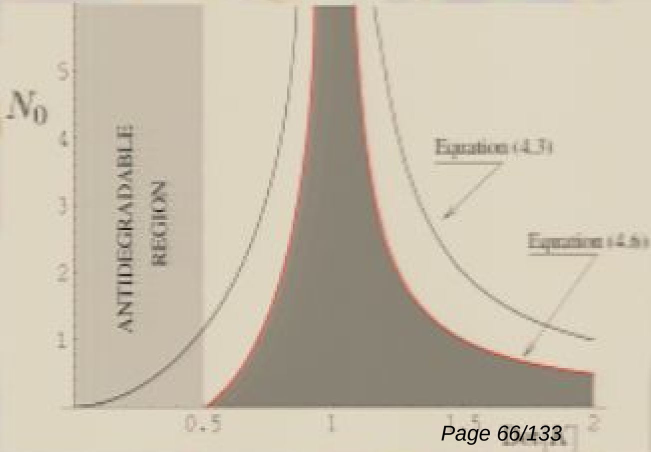
- A new set of channels which have null quantum capacity is identified. This is done by exploiting the composition rules of one-mode Gaussian maps and the fact that anti-degradable channels cannot be used to transfer quantum information (i.e., $Q=0$).



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Motivations

- Up to now, we have analyzed **continuous variable systems**

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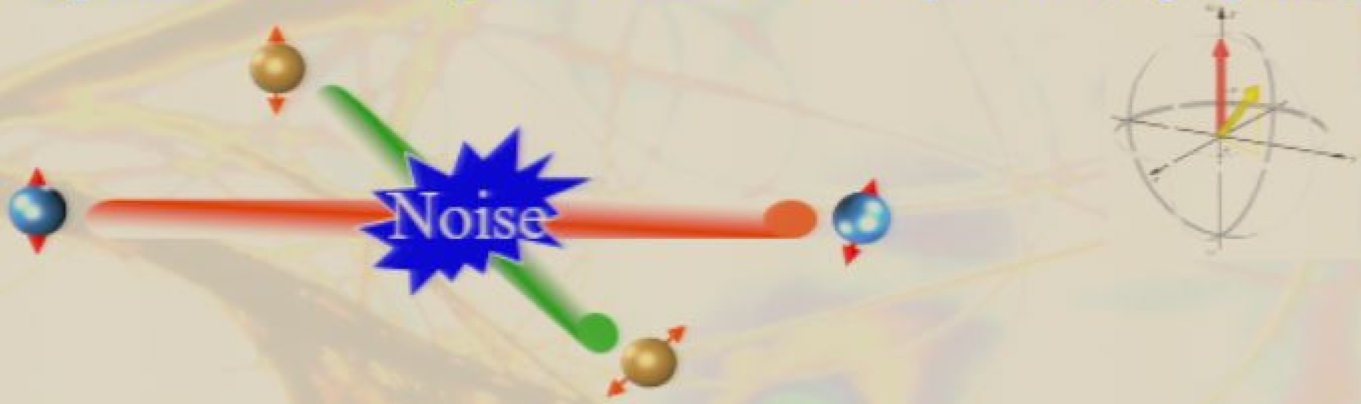

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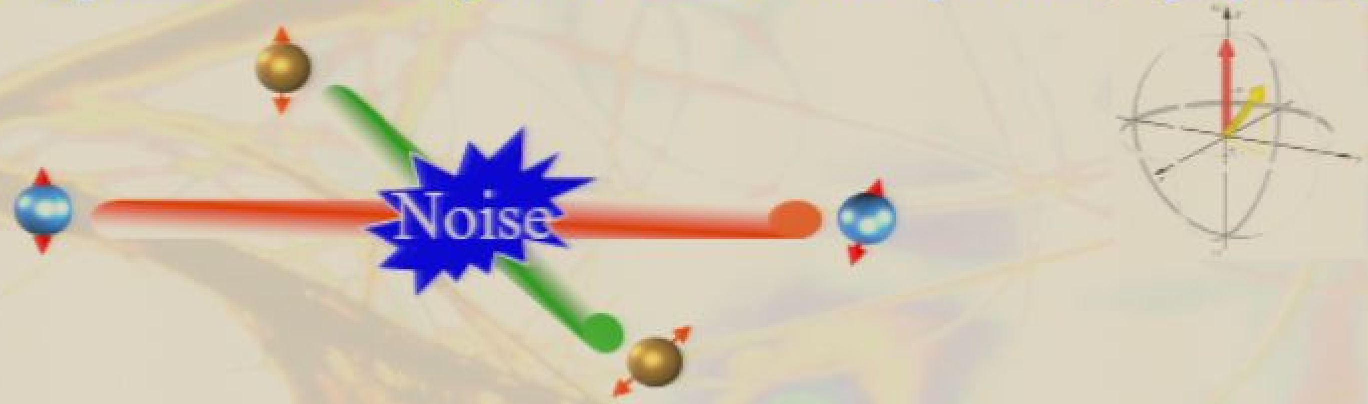

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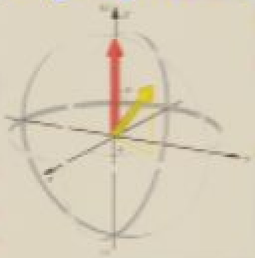

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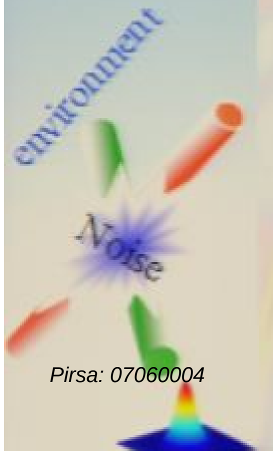
- Is it possible to perform this analysis for finite-dim. systems (e.g. qubits)?

- "Gaussianity implies weak-degradability": is it true for finite-d systems?
- The answer is YES !!!
- We need:
  - a definition of Gaussian maps
  - a "phase space representation" of a two-level quantum system
  - the Grassmann algebra
  - the mapping $\hat{a} = |0\rangle\langle 1|$



Characteristic functions

Characteristic function

$$\boxed{\chi(\boldsymbol{\xi}) = \mathrm{Tr}\left[\rho\, D(\boldsymbol{\xi})\right] = \mathrm{Tr}\left[\rho \exp\left(\sum_n \left(\xi_n a_n^\dagger - a_n \xi_n^*\right)\right)\right]}$$

Displacement operator

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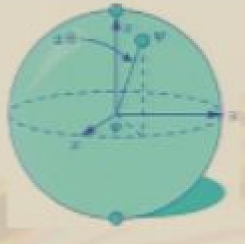
$$G(\xi,\eta) = \mathrm{Tr}\left[\Phi\left(\sigma_z D(-\xi)\right) D(\eta)\right]$$

Qubit quantum operations

Bloch representation

$$\boxed{\rho = \frac{1}{2}[I + \mathbf{r}\cdot\sigma]}$$



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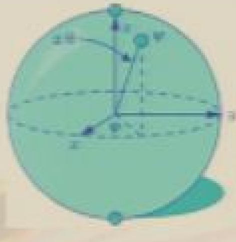

$$\boxed{\Phi\left(\frac{1}{2}[I + \mathbf{r}\cdot\sigma]\right) = \frac{1}{2}[I + (\mathbf{t} + \mathrm{T}\mathbf{r})\cdot\sigma]}$$

M.B. Ruskai, S. Szarek, E. Werner, *Lin. Alg. Appl.* 2002

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$$\mathbb{T} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ t_1 & \lambda_1 & 0 & 0 \\ t_2 & 0 & \lambda_1 & 0 \\ t_3 & 0 & 0 & \lambda_3 \end{pmatrix}$$

t T

Qubit quantum operations

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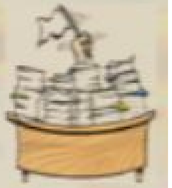
t T

Qubit channels with qubit environment in a pure state

$$\lambda_1\lambda_2 = \lambda_3 \qquad t_1 = t_2 = 0 \qquad t_3^2 = (1-\lambda_1^2)(1-\lambda_2^2)$$

Qubit Gaussian channels



$$\Phi(\rho) = \sum_k A_k \rho A_k^\dagger \xrightarrow[\text{dual channel}]{} \boxed{\Phi_H(\Theta) = \sum_k A_k^\dagger \Theta A_k}$$

Schroedinger picture $\mathrm{Tr}[\Phi(\rho)\ \Theta] = \mathrm{Tr}[\rho\ \Phi_H(\Theta)]$ Heisenberg picture

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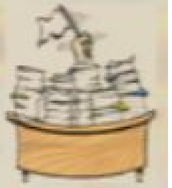

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Heisenberg picture

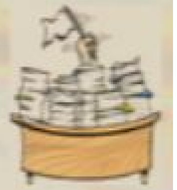
Qubit Gaussian channels

$$\boxed{\Phi_H(D(\xi)) = D(a\xi + b\xi^*) f(\xi)}$$

Bosonic Gaussian channels

$$D'(z) = D(K^T z)\, \xi(K_g^T z)$$

Qubit Gaussian channels



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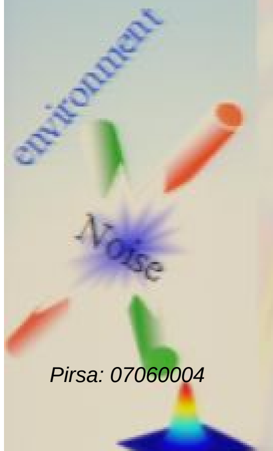

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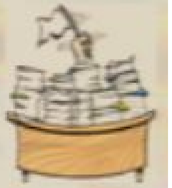

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If $\quad \lambda_1\lambda_2 = \lambda_3 \quad$ & $\quad t_1 = t_2 = 0$

$$\chi'(\xi) \;=\; \chi\left(\frac{\lambda_1+\lambda_2}{2}\xi - \frac{\lambda_1-\lambda_2}{2}\xi^*\right)\left[1 + \frac{t_3}{2}\xi\xi^*\right]$$

Some examples of qubit channels

- Bit flip or dephasing **channel**
- Bit-Phase flip **channel**
- Amplitude damping or BS **channel**
- Generalized amplitude damping or BS **channel**

Gaussian

- Depolarizing **channel**
- Phase flip **channel**

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Bit flip or dephasing channel: it flips |0> to |1> with probability 1-p

$$A_0 = \sqrt{p}\, I = \sqrt{p} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

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$$t_1 = t_2 = t_3 = 0 \quad \text{(unital map)}$$

$$\lambda_1 = 1\,. \quad \lambda_2 = \lambda_3 = 2p - 1$$

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$$\Phi_H(D(\xi)) = D_e(\xi) + D_o((2p-1)\xi)$$

$\equiv$ **Phase damping channel**

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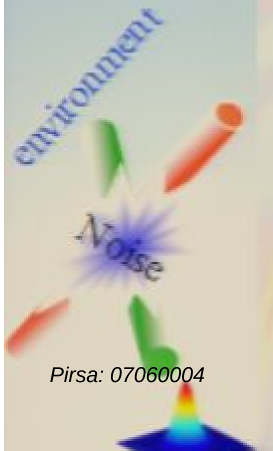

Qubit-qubit channels (pure environment)

Qubit channels with qubit environment in a pure state $|0\rangle$

$\boxed{\lambda_1\lambda_2 = \lambda_3}$ $t_3^2 = (1-\lambda_1^2)(1-\lambda_2^2)$

$\boxed{t_1 = t_2 = 0}$ all Gaussian channels!

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$$A_1 = \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\phi \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \sin\phi \\ \sin\theta & 0 \end{pmatrix}$$

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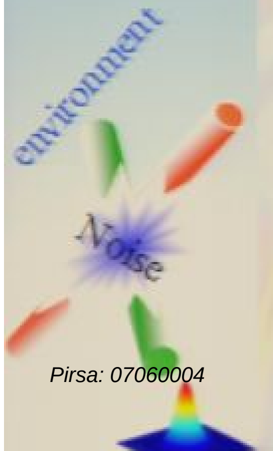
$\theta = \phi$ **Bit-Phase flip channel**

$\theta = 0$ **Amplitude damping or BS channel**



Qubit-qubit channels (pure environment)

$$\Phi_{qubit}(\theta,\varphi)$$

Qubit-qubit channel

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Qubit-qubit complementary channel

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$\Phi_{qubit}(\theta,\varphi)$

Qubit-qubit channel

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Qubit-qubit complementary channel

$$\cos(\varphi) \rightleftharpoons \sin(\varphi)$$

$$\tilde{\chi}'(\xi) = \chi\left(\xi\cos\theta\sin\phi - \xi^*\sin\theta\cos\phi\right)\left[1 + \frac{\cos(2\theta)+\cos(2\phi)}{4}\xi\xi^*\right]$$

$$\tilde{\Phi}_H(D(\xi)) = D\left(\xi\cos\theta\sin\phi - \xi^*\sin\theta\cos\phi\right)\left[1 + \frac{\cos(2\theta)+\cos(2\phi)}{4}\xi\xi^*\right]$$

the complementary map is a Gaussian qubit-qubit map!

Weak-degradability of qubit-qubit maps (pure env.)

$$(\Psi \circ \Phi_{qubit})(\rho) = \tilde{\Phi}_{qubit}(\rho)$$

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$\equiv$

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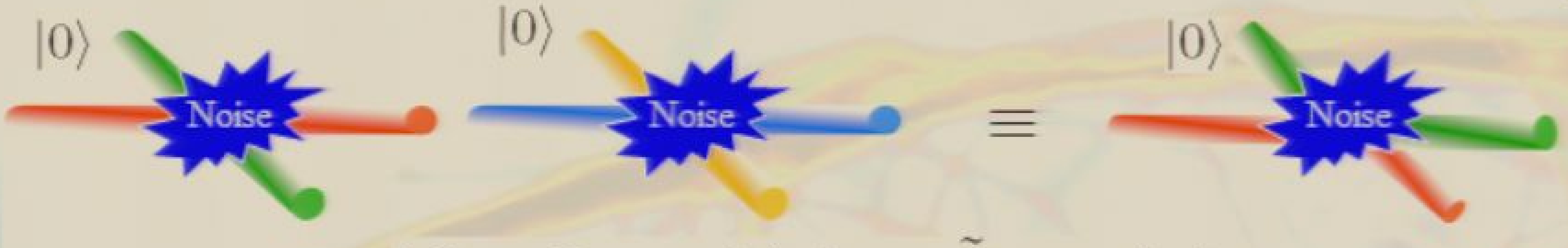
$\Psi \equiv \Phi_{qubit}(\theta_x, \phi_x)$ $\quad (\Phi_{qubit}(\theta_x, \phi_x) \circ \Phi_{qubit})(\rho) = \tilde{\Phi}_{qubit}(\rho)$

Weak-degradability of qubit-qubit maps (pure env.)



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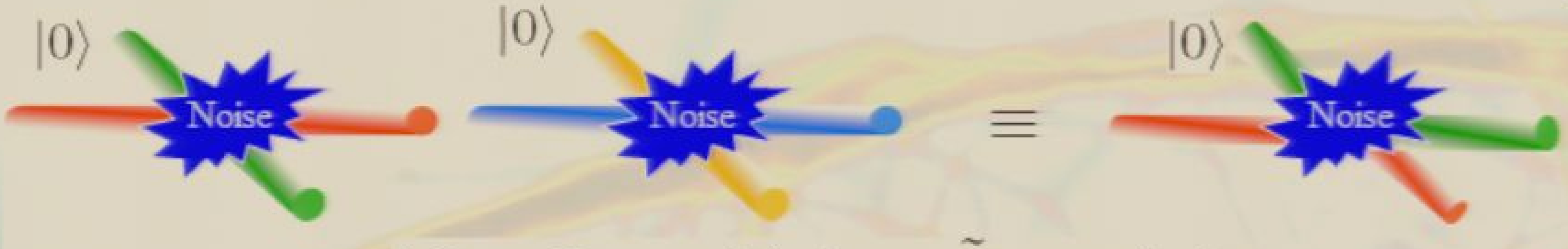

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degradable channels (i.e., Q=Q¹)

$$\cos(2\theta_x) = \frac{\cos(2\theta) - \cos(2\phi) + 2\cos(2\theta)\cos(2\phi)}{\cos(2\theta) + \cos(2\phi)}$$

$$\cos(2\phi_x) = \frac{\cos(2\theta) - \cos(2\phi) - 2\cos(2\theta)\cos(2\phi)}{\cos(2\theta) + \cos(2\phi)}$$

for $\cos(2\theta)/\cos(2\phi) \leq 0$

M.M. Wolf, D. Perez-Garcia, quant-ph

$(\Phi_{qubit}(\theta_x, \phi_x) \circ \tilde{\Phi}_{qubit})(\rho) = \Phi_{qubit}(\rho)$

anti-degradable channels (i.e. Q=0)

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M.M. Wolf, D. Perez-Garcia, quant-ph/0607070 (2006)

anti-degradable channels (i.e. $Q=0$)

$$(\Phi_{qubit}(\theta_x, \phi_x) \circ \tilde{\Phi}_{qubit})(\rho) = \Phi_{qubit}(\rho)$$

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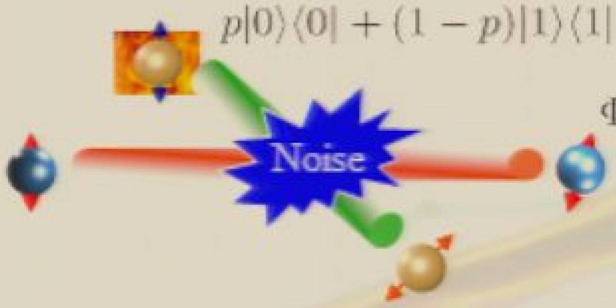
Qubit-qubit channels (mixed environment)

$p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$



$$\Phi^{m}_{qubit}(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes (p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|))\,U^{\dagger}]$$

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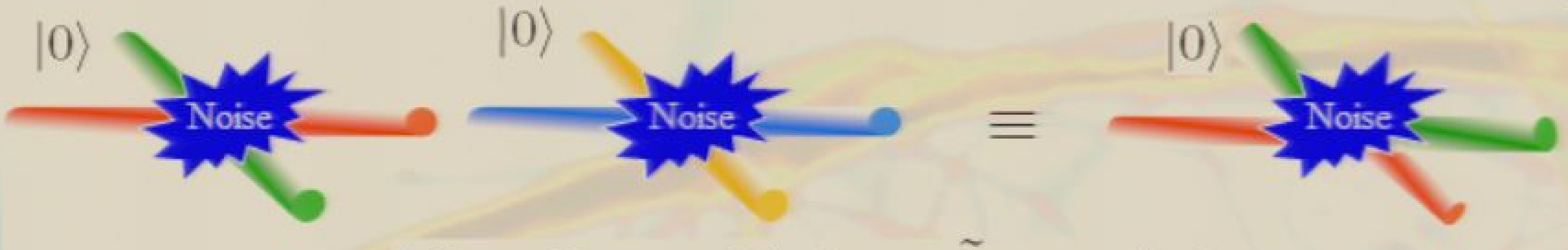

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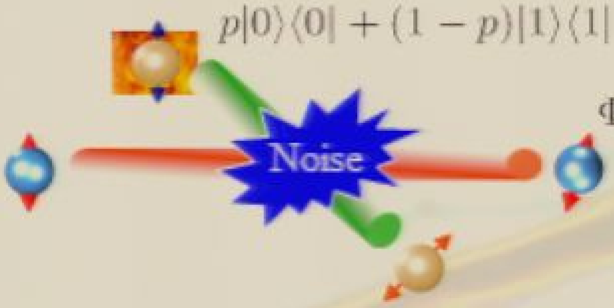

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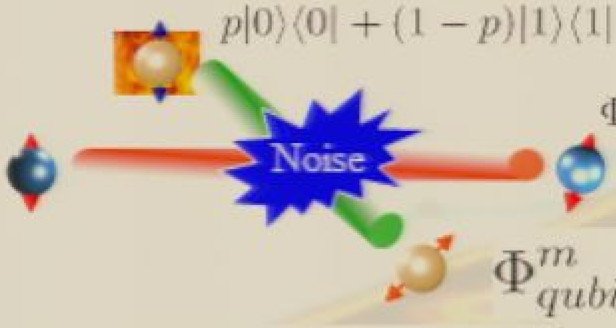

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$$\Phi^{m}_{qubit}(\rho) = B_0\rho B_0^{\dagger} + B_1\rho B_2^{\dagger} + B_2\rho B_2^{\dagger} + B_3\rho B_3^{\dagger}$$

$$B_0 = \sqrt{p}A_0 = \sqrt{p}\begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\phi \end{pmatrix} \qquad B_2 = \sqrt{1-p}\,\sigma_x A_0 \sigma_x$$

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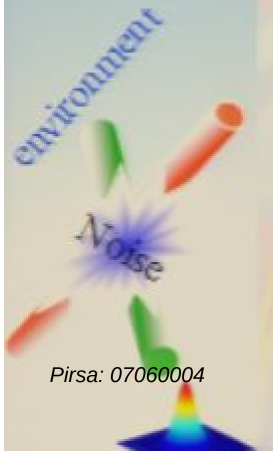
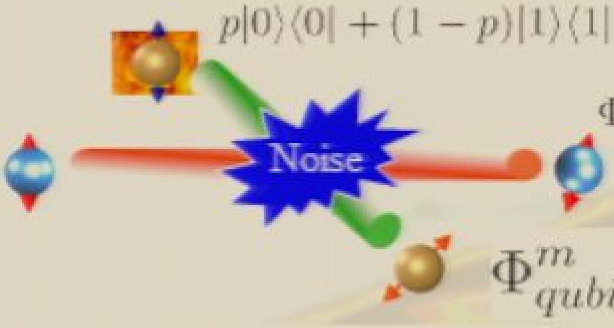


Qubit-qubit channels (mixed environment)



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all Gaussian channels!

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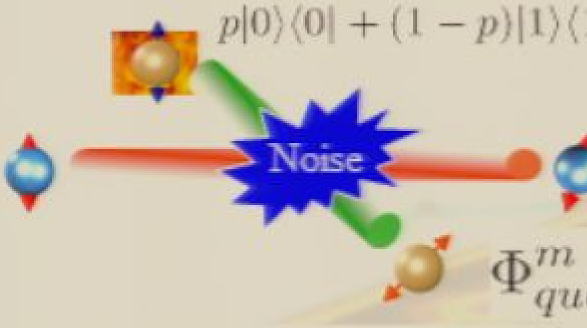

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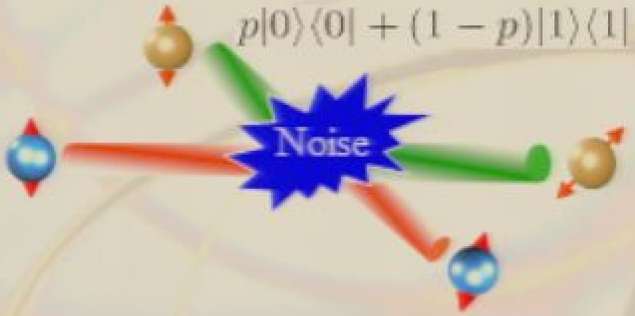
all Gaussian channels!



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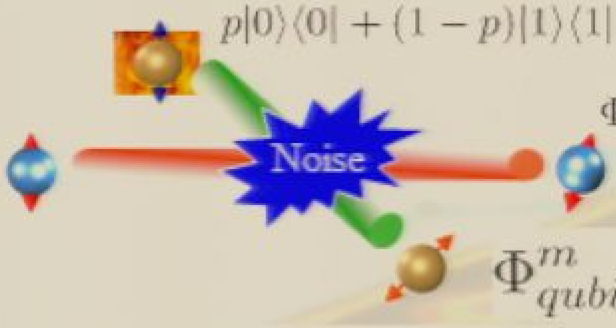
Qubit-qubit channels (mixed environment)



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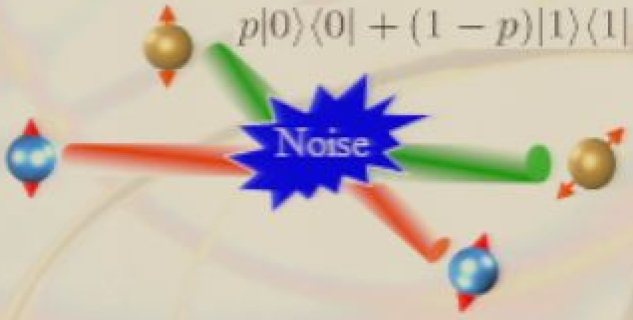

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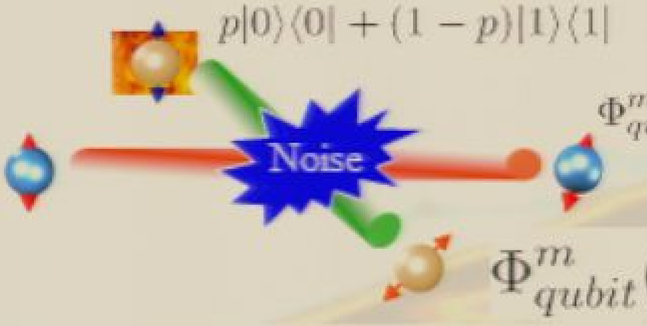
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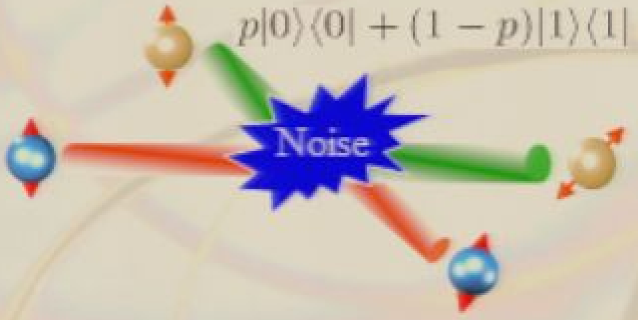
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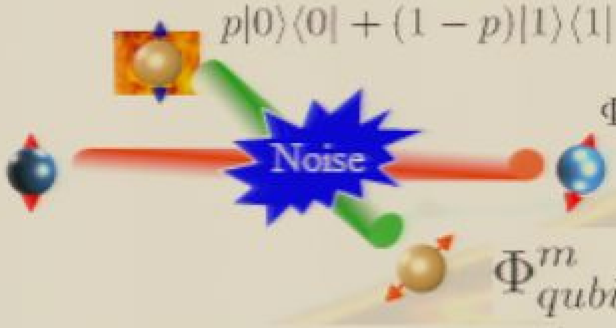
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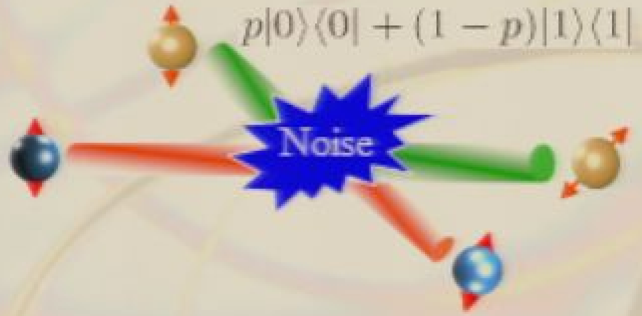
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the complementary map is not a qubit-qubit map with mixed environment!

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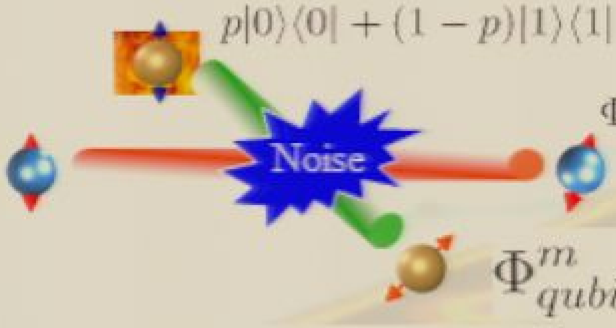

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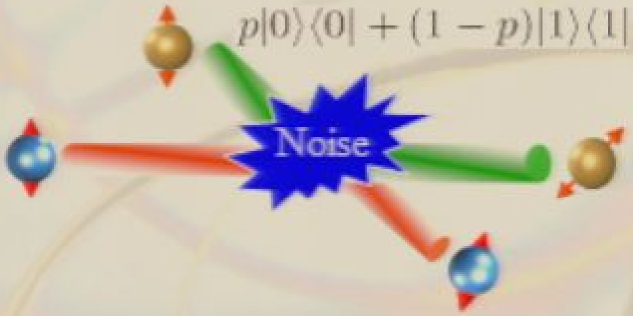

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the complementary map is NOT Gaussian!

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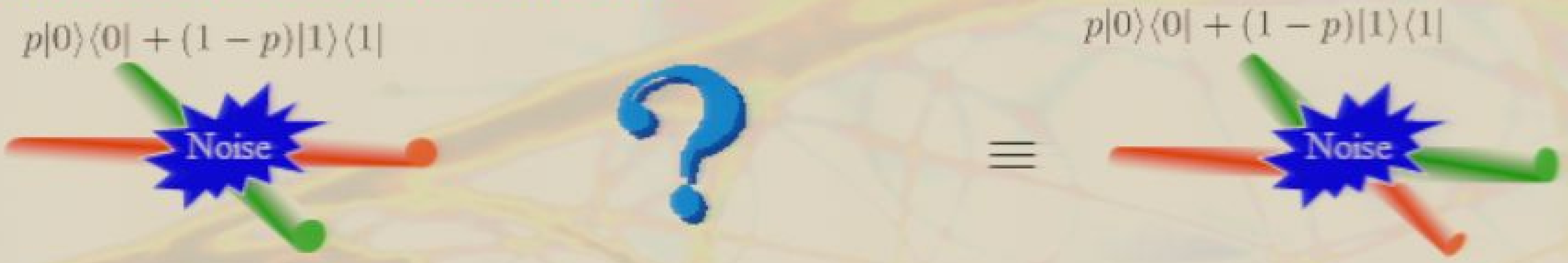
Weak-deg. of qubit-qubit maps (mixed env.)

$\boxed{\text{for } \cos(2\theta)/\cos(2\phi) \geq 0}$ $(\Psi \circ \Phi^{m}_{qubit})(\rho) = \tilde{\Phi}^{m}_{qubit}(\rho)$



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weakly degradable channels

$$\cos(2\theta_x) = \frac{\cos(2\theta) - \cos(2\phi) + 2\cos(2\theta)\cos(2\phi)}{\cos(2\theta) + \cos(2\phi)}$$

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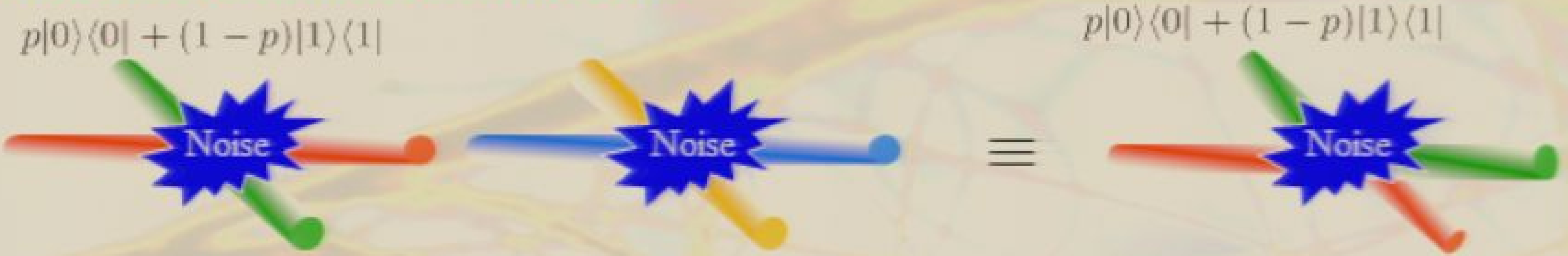

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Phase flip channel

for $\cos(2\theta)/\cos(2\phi) \leq 0$

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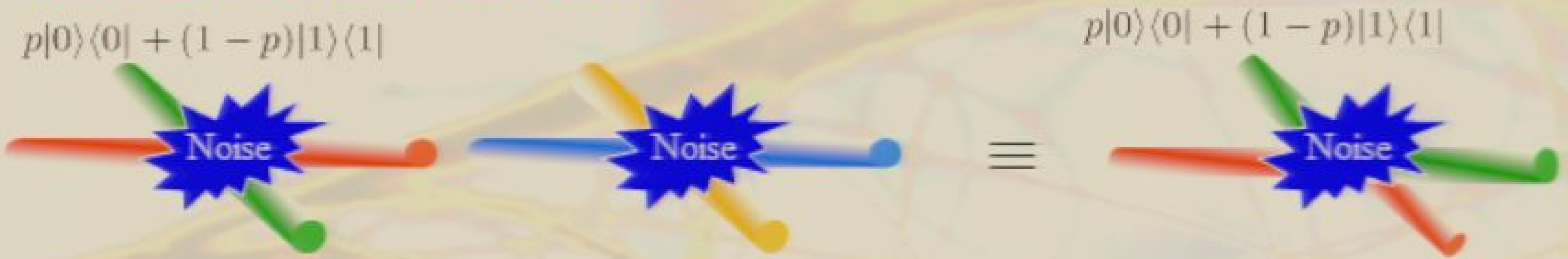

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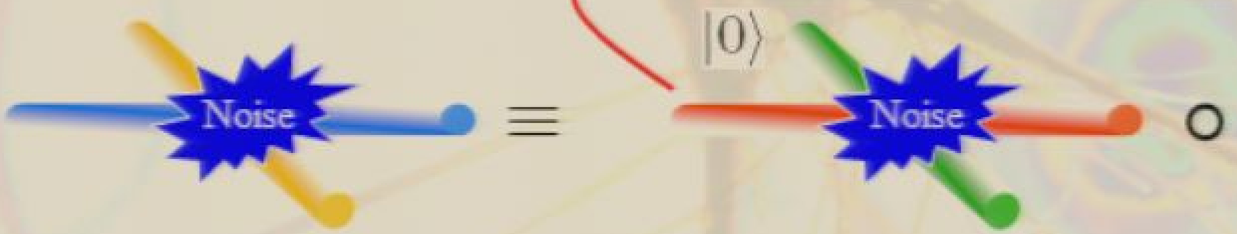

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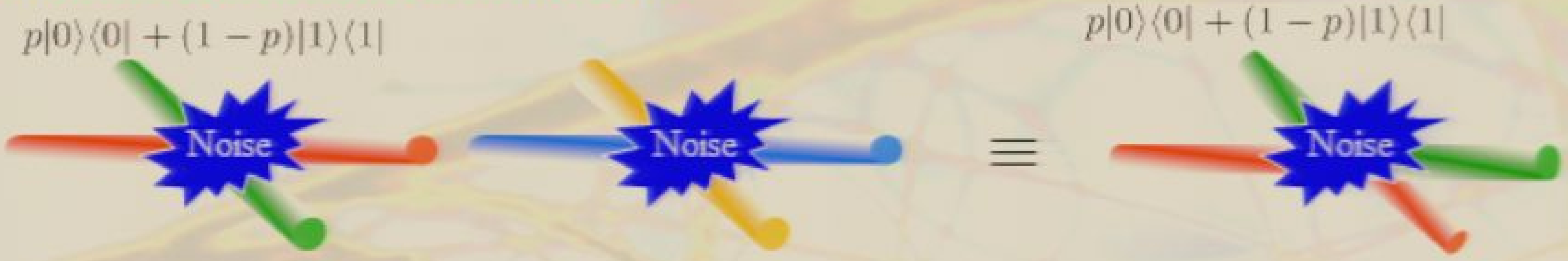

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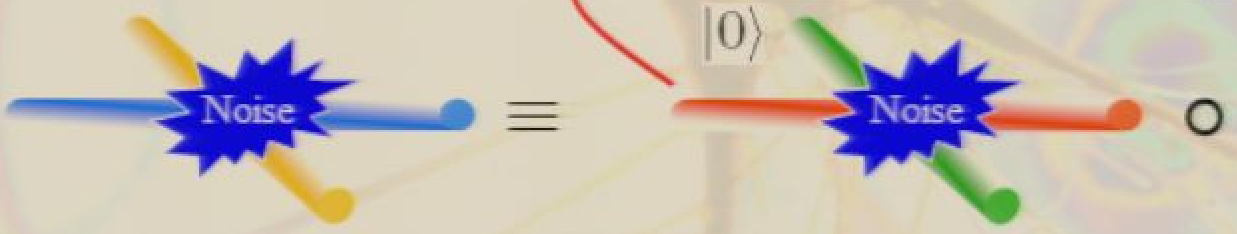

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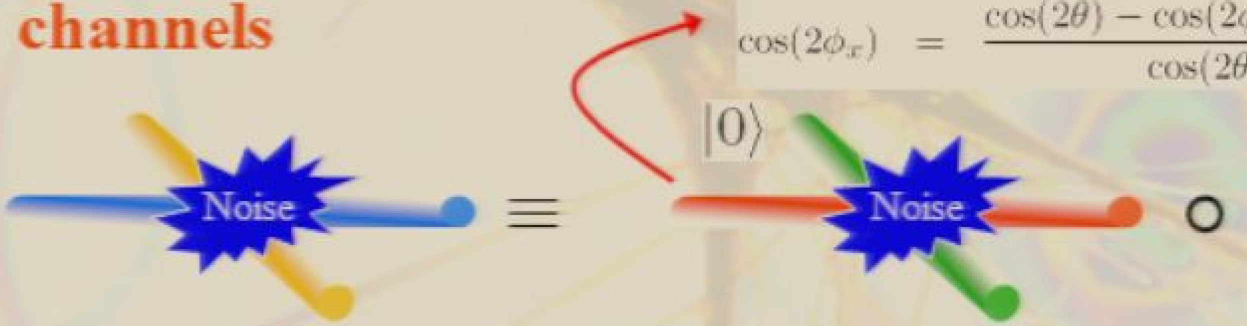

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by using a "bottleneck" inequality $Q(\Phi^{m}_{qubit}) = 0$

These qubit-qubit maps with mixed environment cannot transfer quantum information!

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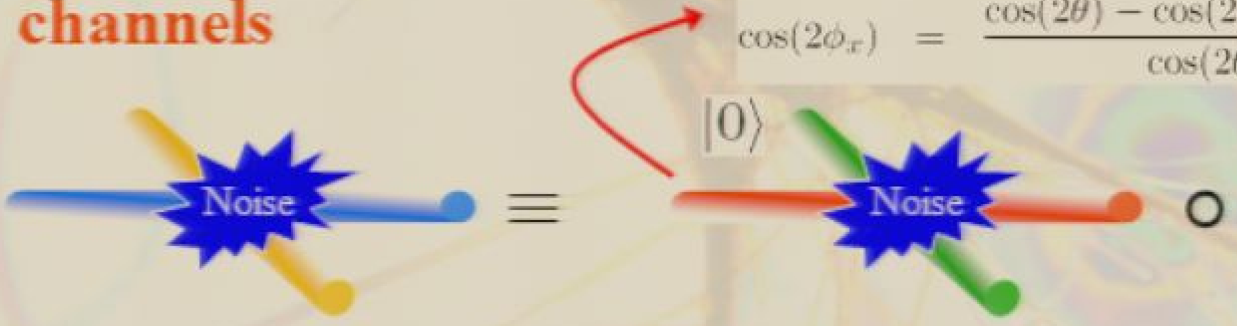

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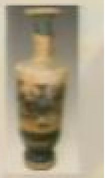
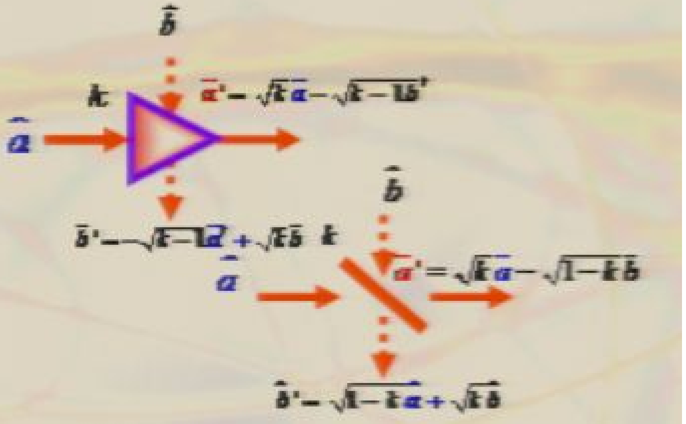


Conclusions and Outlook

F. Caruso and V. Giovannetti, *Physical Review A* **74**, 062307 (2006), quant-ph/0603257
F. Caruso, V. Giovannetti, and A.S. Holevo, *New J. Phys.* **8**, 310 (2006), quant-ph/0609013

Conclusions and Outlook

- We prove that the Bosonic Gaussian channels are either weakly degradable or anti-degradable, i.e. either $Q=Q^1$ (additivity) or $Q=0$, respectively, exploiting the fact that these maps are unitarily equivalent to Beam-Splitter/Amplifier channel.



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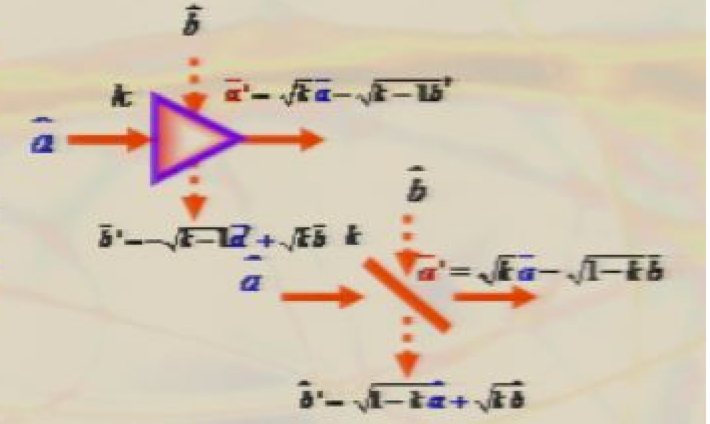

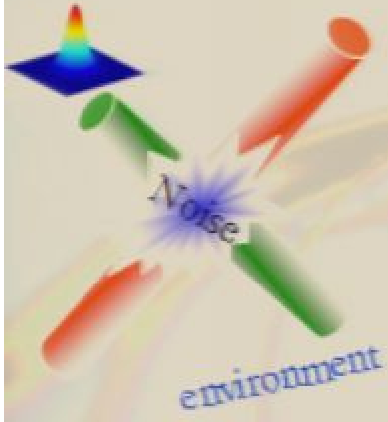

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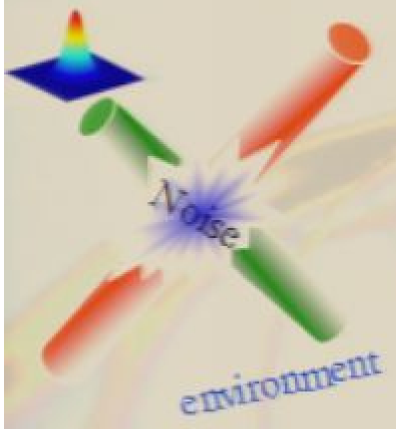

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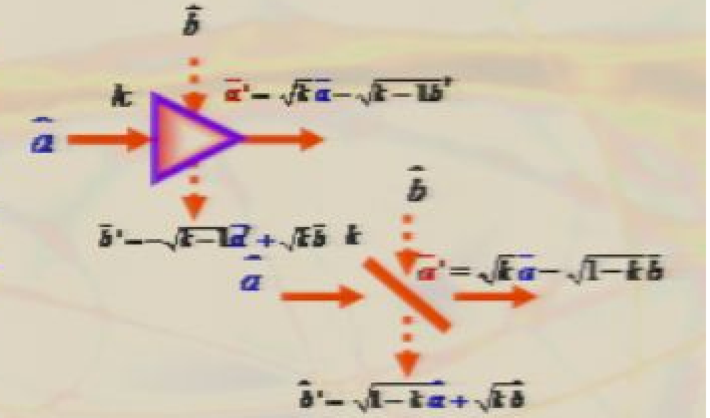

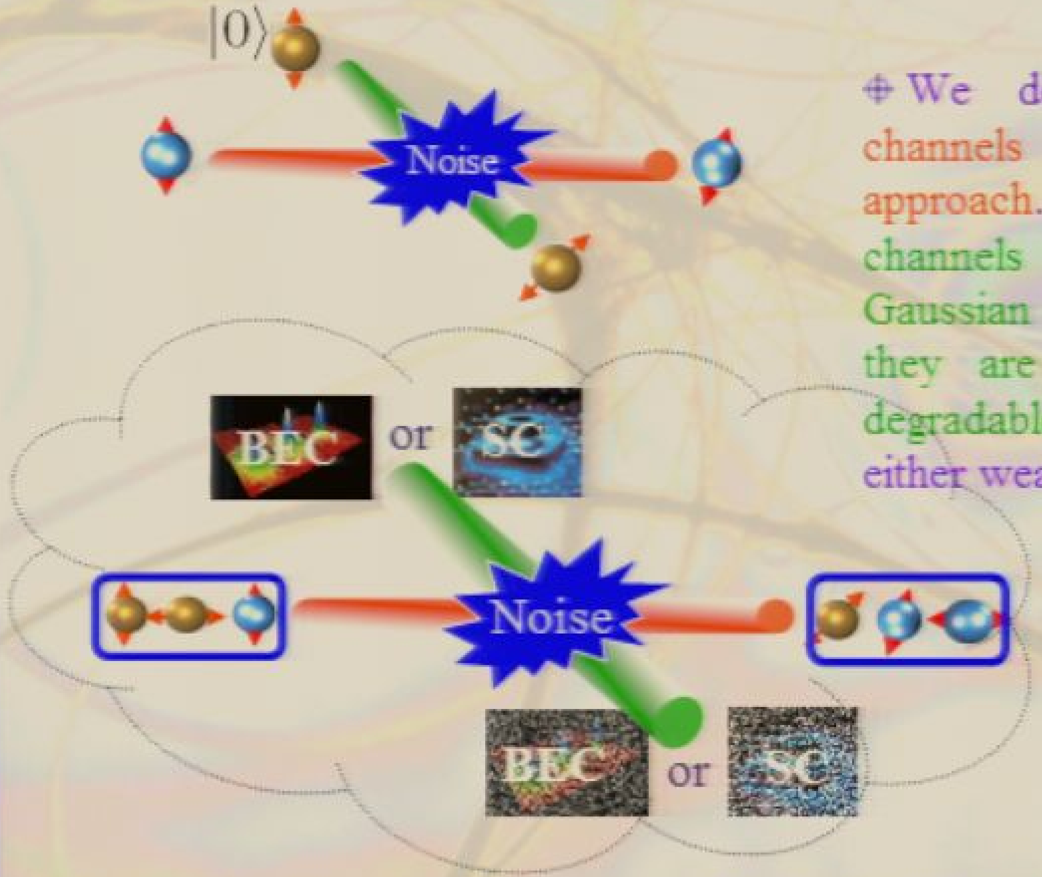

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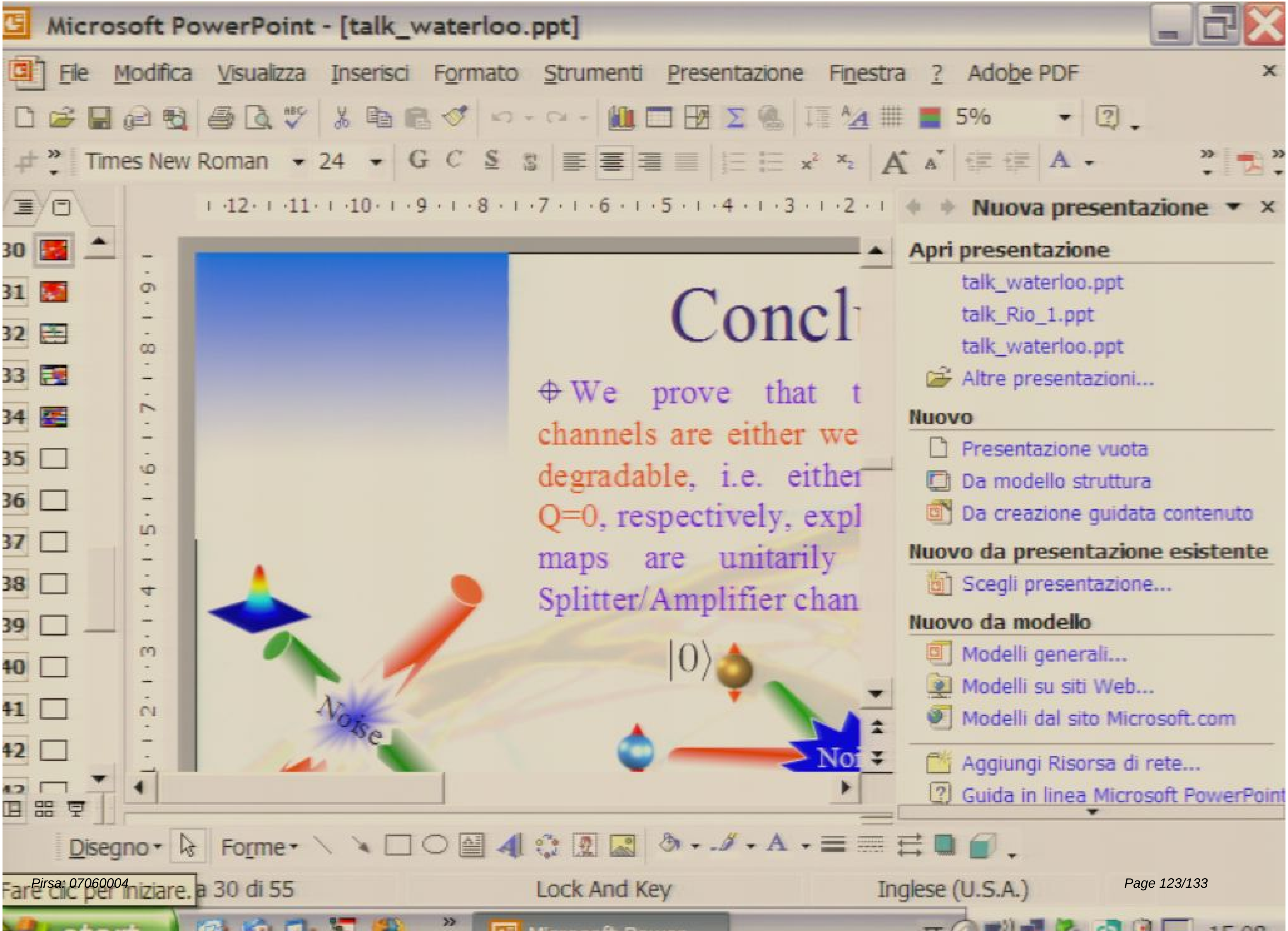

Concl

- We prove that t channels are either we degradable, i.e. either Q=0, respectively, expl maps are unitarily Splitter/Amplifier chan

Grassmann Algebra

Fermionic operator

$$\begin{aligned}\{a_n, a_m^\dagger\} &= \delta_{nm} \\ \{a_n, a_m\} &= 0 \\ \{a_n^\dagger, a_m^\dagger\} &= 0 \\ a_n|0\rangle &= 0\end{aligned}$$

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$\Longrightarrow$

Grassmann variables

$$\begin{aligned}\{\gamma_n, \gamma_m\} &= 0\\ \{\gamma_n^*, \gamma_m\} &= 0\\ \{\gamma_n^*, \gamma_m^*\} &= 0\end{aligned}$$

$$\left(a_1\beta_2 a_3^\dagger \gamma_4^*\right)^\dagger = \gamma_4 a_3 \beta_2^* a_1^\dagger$$

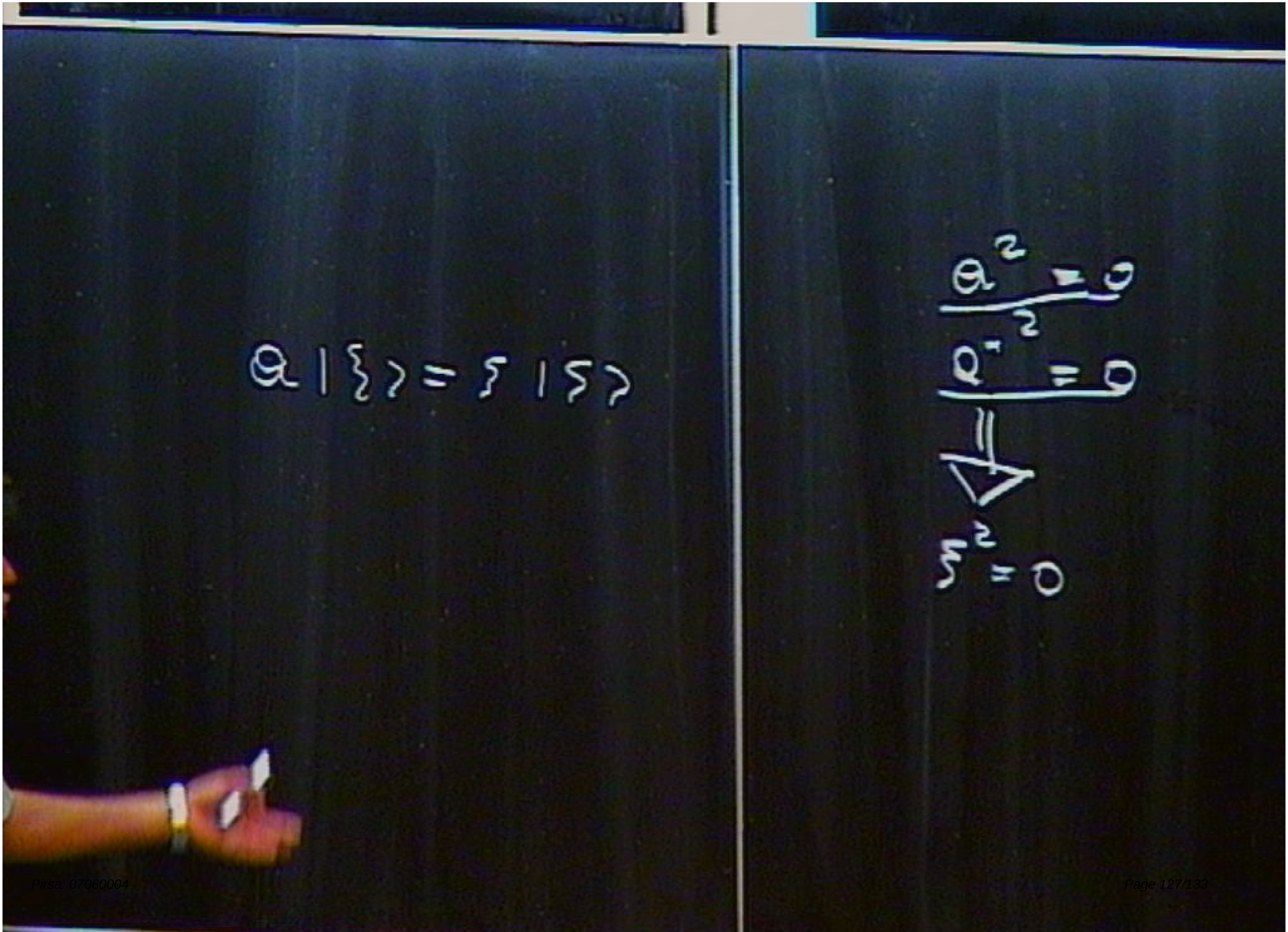
$$\begin{aligned}\gamma_n^2 &= 0\\ \gamma_n^{*2} &= 0\end{aligned}$$

$$\{\gamma_n, a_m\} = 0$$

K.E. Cahill, R.J. Glauber, *PRA* 1999

$$Q^2 = 0$$

$$Q^{\dagger 2} = 0$$



$$Q|\xi\rangle = \xi|\xi\rangle$$

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$$\Downarrow$$

$$\xi^2 = 0$$

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$$\begin{aligned} \{a_n, a_m^\dagger\} &= \delta_{nm} \\ \{a_n, a_m\} &= 0 \\ \{a_n^\dagger, a_m^\dagger\} &= 0 \\ a_n|0\rangle &= 0 \end{aligned}$$

Grassmann variables

$$\begin{aligned} \{\gamma_n, \gamma_m\} &= 0 \\ \{\gamma_n^*, \gamma_m\} &= 0 \\ \{\gamma_n^*, \gamma_m^*\} &= 0 \end{aligned}$$

$$\left(a_1\beta_2 a_3^\dagger \gamma_4^*\right)^\dagger = \gamma_4 a_3 \beta_2^* a_1^\dagger$$

$$\begin{aligned} \gamma_n^2 &= 0 \\ \gamma_n^{*2} &= 0 \end{aligned}$$

$$\{\gamma_n, a_m\} = 0$$

Function of Grassmann variables

$$f(\xi) = u + \xi t$$

Derivative and integration

$$\frac{df(\xi)}{d\xi} = t$$

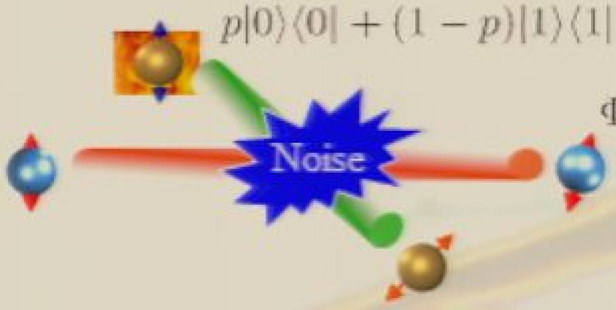
$$\begin{aligned} \int d\xi_n &= \int d\xi_n^* = 0 \\ \int d\xi_n\, \xi_m &= \delta_{nm} \\ \int d\xi_n^*\, \xi_m^* &= \delta_{nm}. \end{aligned}$$

$$\int d\xi f(\xi) = \int d\xi (u + \xi t) = t = \frac{df(\xi)}{d\xi}$$

$$\int d^2\xi_n = \int d\xi_n^*\, d\xi_n \qquad d\xi_n\, d\xi_n^* = -d\xi_n^*\, d\xi_n$$

K.E. Cahill, R.J. Glauber, *PRA* 1999

Qubit-qubit channels (mixed environment)

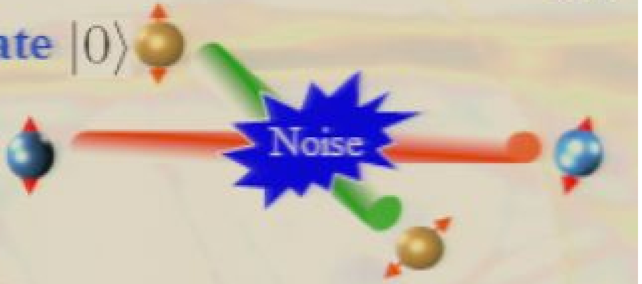


$p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$

$$\Phi^{m}_{qubit}(\rho) = \mathrm{Tr}_E[U\,(\rho \otimes (p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|))\,U^{\dagger}]$$

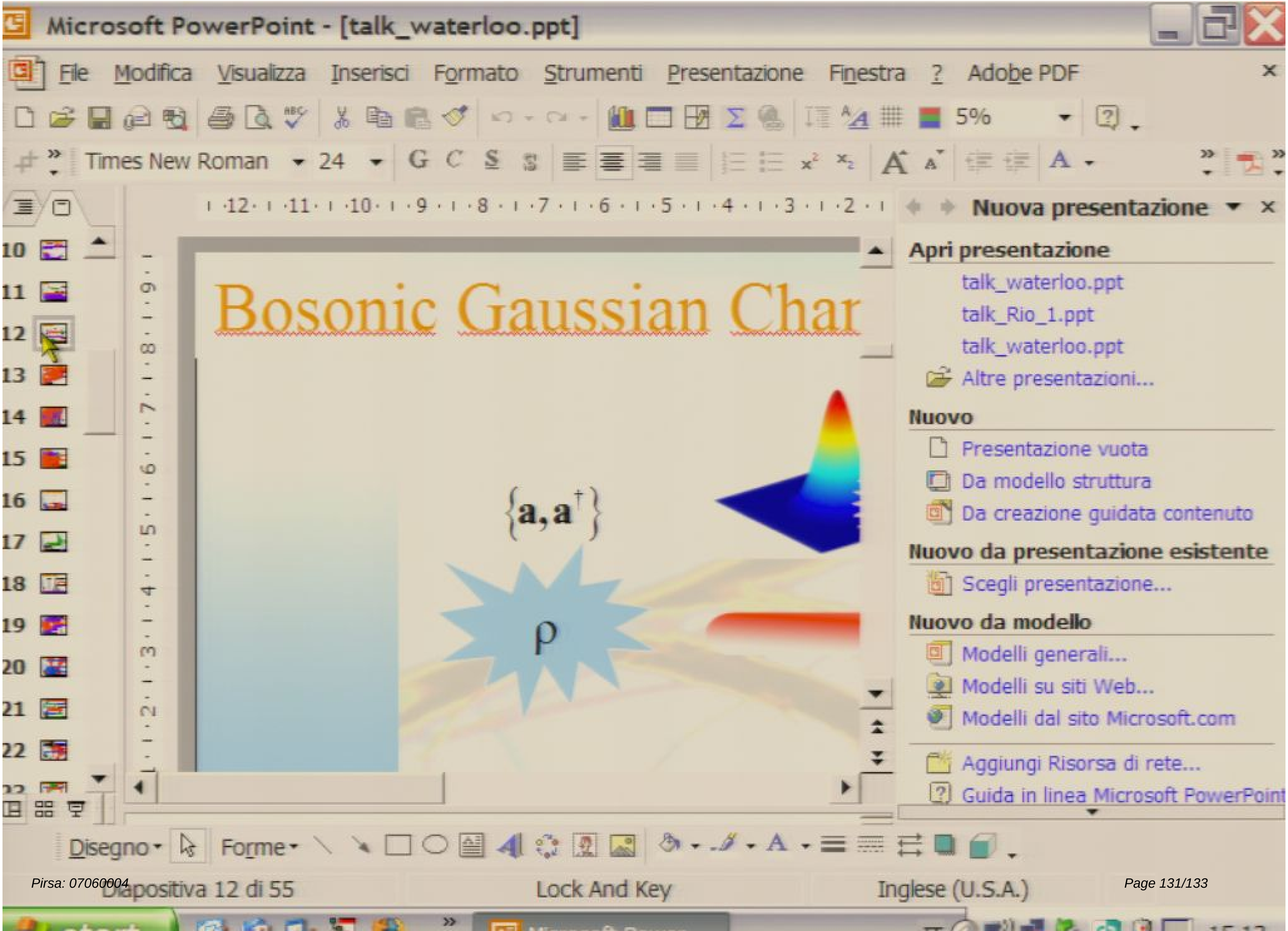
Qubit-qubit channels (pure environment)

Qubit channels with qubit environment in a pure state $|0\rangle$



$$\boxed{\lambda_1 \lambda_2 = \lambda_3} \qquad t_3^2 = (1-\lambda_1^2)(1-\lambda_2^2)$$

$$\boxed{t_1 = t_2 = 0} \qquad \text{all Gaussian channels!}$$



Bosonic Gaussian Char

$\{\mathbf{a}, \mathbf{a}^{\dagger}\}$

ρ

Bosonic Gaussian Channels

$$\rho' = \Phi(\rho)$$

$$\rho_E' = \tilde{\Phi}(\rho)$$

Bosonic Gaussian Channel

and

$$\boxed{\chi'(z) = \chi(K^T z)\,\xi(K_E^T z)}$$

where

K, K_E are linear maps between the phase spaces and $z = \sqrt{2}\begin{pmatrix}\mathrm{Im}\,\mu \\ \mathrm{Re}\,\mu\end{pmatrix}$

Displacement evolution $D'(z) = D(K^T z)\,\xi(K_E^T z)$

A.S. Holevo, R.F. Werner, *PRA* 2001

Bosonic Gaussian Channels

$\{\mathbf{a},\mathbf{a}^\dagger\}$

$\{\mathbf{b},\mathbf{b}^\dagger\}$

$\rho' = \Phi(\rho)$

Complementary channel

$\rho_E' = \tilde{\Phi}(\rho)$

Bosonic Gaussian Channel

and

$$\boxed{\chi'(z) = \chi(K^T z)\,\xi(K_E^T z)}$$

where

K, K_E are linear maps between the phase spaces and $z = \sqrt{2}\begin{pmatrix} \mathrm{Im}\,\mu \\ \mathrm{Re}\,\mu \end{pmatrix}$

A.S. Holevo, R.F. Werner, *PRA* 2001