Title: An Introduction to Decoherence-Free Subspaces

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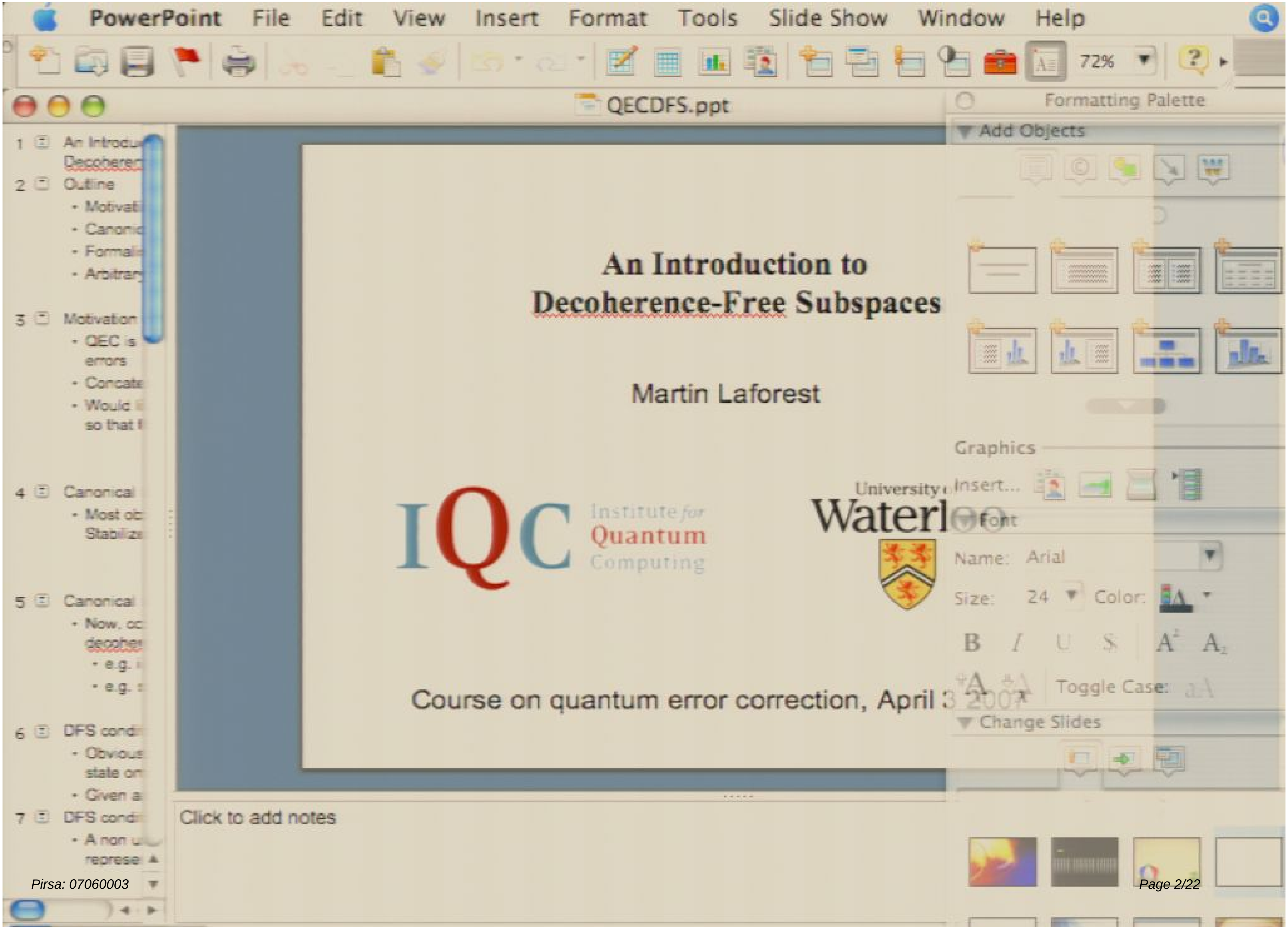
Abstract: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Martin Laforest
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Outline
Motivation
Canonical
DFS cond

An Introduction to Decoherence-Free Subspaces

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Course on quantum error correction, April 3 2007

Outline

- Motivation
- Canonical examples
- Formalism for the existence of DFS
- Arbitrary collective rotations DFS

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Motivation

- QEC is good, very good, but it only *reduce* the errors
- Concatenation of QECC takes many qubits
- Would like to find a way to encode information so that the noise has no effect on it

Canonical examples

- Most obvious example for this class: Stabilizer code [[n,k,d]].

$$\mathcal{S} = \langle M_1, \ldots, M_k \rangle$$

$$T(\mathcal{S}) = \{|\psi\rangle \in \mathcal{H}^{\otimes n} |\, M|\psi\rangle = |\psi\rangle, \forall M \in \mathcal{S}\}$$

- E.g. the 3 qubit bit flip code can be used as a pair wise ZZ coupling DFS

$$S|\psi\rangle = |\psi\rangle$$

Canonical examples

- Now, consider "collective phase decoherence".
  - e.g. invariant under qubit permutation
  - e.g. space invariant

$$\rho_n \rightarrow \int_0^{2\pi} R_z^1(\theta)\ldots R_z^n(\theta)\rho_n R_z^{1\dagger}(\theta)\ldots R_z^{n\dagger}(\theta)p(\theta)d\theta$$

- Consider 2 qubits

$$|00\rangle \rightarrow e^{-i\theta}|00\rangle \qquad |01\rangle \rightarrow |01\rangle$$
$$|11\rangle \rightarrow e^{i\theta}|11\rangle \qquad |10\rangle \rightarrow |10\rangle$$

- More generally, for n qubit, any state with the same amount of zeros and ones will create a DFS of dimension

$$d = \binom{n}{k}$$

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DFS conditions (OSR)

- Obviously, we want all the errors to affect any state on the DFS in the same way.
- Given a noise process with kraus operator E_a

$$\rho \to \sum_{a=1}^{2^{2n}} E_a \rho E_a^\dagger, \ \sum_{a=1}^{2^{2n}} E_a^\dagger E_a = \mathbb{1}$$

- We want the DFS basis to behave as

$$E_a|k\rangle = c_a U|k\rangle$$

where U is unitary and act irreducibly on the DFS, i.e.

$$E_a = \begin{pmatrix} c_a U & 0 \\ 0 & E_a^\perp \end{pmatrix}$$

$$\therefore |\psi\rangle\langle\psi| \to \sum E_a|\psi\rangle\langle\psi|E_a^\dagger = \sum c_a c_a^* U|\psi\rangle\langle\psi|U^\dagger = U|\psi\rangle\langle\psi|U^\dagger$$

DFS conditions (Lindblad)

- A non unitary Markovian process can be represented in the Lindblad form

$$\dot{\rho} = -i[H, \rho] + \sum_a \mathcal{D}[F_a]\rho$$

$$\mathcal{D}[\mathcal{O}]\rho = \mathcal{O}\rho\mathcal{O}^\dagger - \frac{1}{2}\{\mathcal{O}^\dagger\mathcal{O}, \rho\}$$

- Two conditions become obvious:
  – $D[F_a]\rho=0$ for all state in the DFS

$$\therefore F_a|k\rangle = c_a|k\rangle, \forall |k\rangle \in DFS$$

  – The action of *H* must be irreducible on the DFS

- Other possible formulation:
  – Using QEC conditions
  – Using Hamiltonian formalism

DFS conditions (Irrep)

- This is an attempt at making it understandable!
- Take the Kraus operator and construct an algebra *A*
- Consider the commutant *A'* (enough to verify for the generator of *A)*
- Consider a common eigenspace B of all the operators in A'. The eigenvalue could be different for each operator.

$$If\, M_a \in \mathcal{A}' \; and \; M_a|\psi\rangle = c_a|\psi\rangle,\ \forall|\psi\rangle \in B$$

$$M_a A|\psi\rangle = AM_a|\psi\rangle = c_a A|\psi\rangle,\ \forall|\psi\rangle \in B,\ A \in \mathcal{A}$$

$$\implies A|\psi\rangle \in B,\ \forall A \in \mathcal{A}$$

$$\implies B \text{ is invariant under the action of } \mathcal{A}$$

- Finding the irreducible representation of *A* will give us information about existence of a DFS
- DFS exist iff there exist symmetries in the noise, e.g. *A'* contains at least one elements else then the identity

DFS conditions (Irrep)

- If we have :
  - A 1-d irrep ⇒ 1-d DFS
  - 2 equivalent 1-d irrep ⇒ 2-d DFS
  - A 2-d irrep ⇒ 2-d DFS with unitary evolution
  - 2 unitarily equivalent 2-d irrep ⇒ 2-d noiseless subsystem
- Suppose the error algebra *A* is a group algebra of group *G*, i.e.

$$M_a = \sum_n b_{an} G_n, \ \forall M_a \in \mathcal{A}$$

- Then there exist a N_1 dimensional DFS where N_1 is the number of 1d irrep of *G*.

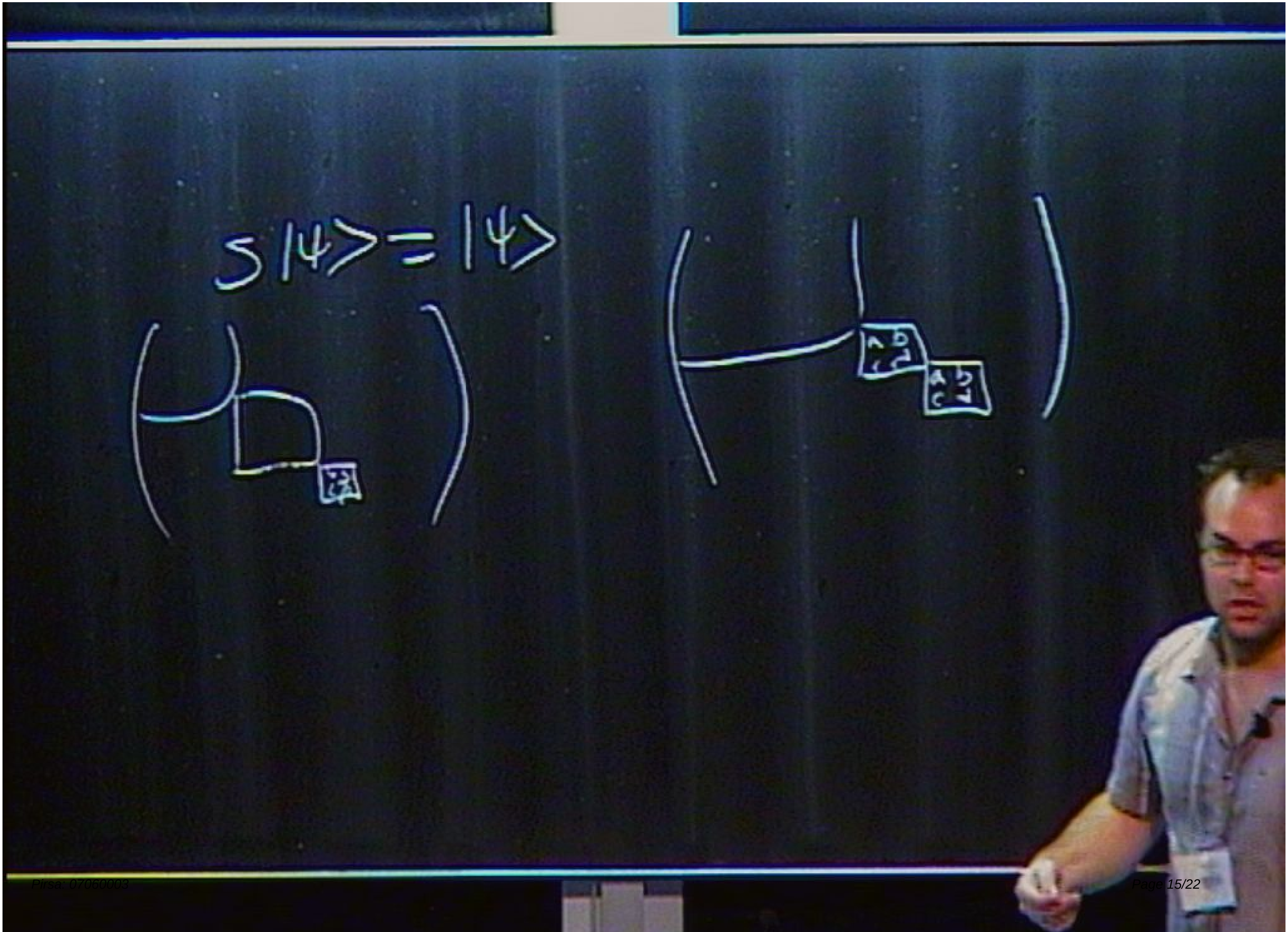
$$\implies G_n = \{0, 1\}$$

$$\implies M_a|k\rangle = \sum_n b_{an}|k\rangle = b_a|k\rangle$$

where $|k\rangle$ belongs to the k^{th} 1d irrep

$\implies$ This is the OSR condition

- Can be generalize for non-group algebra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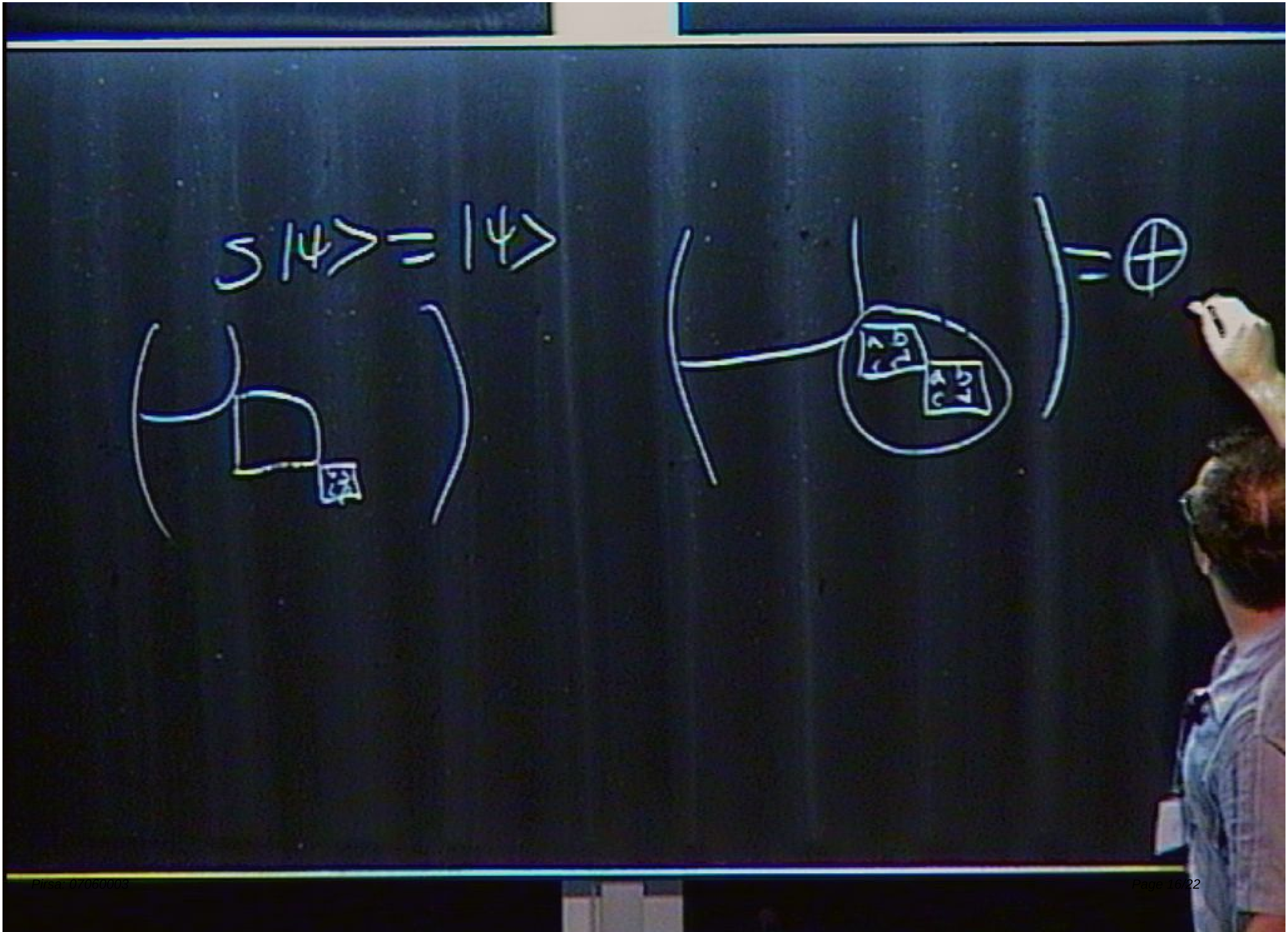
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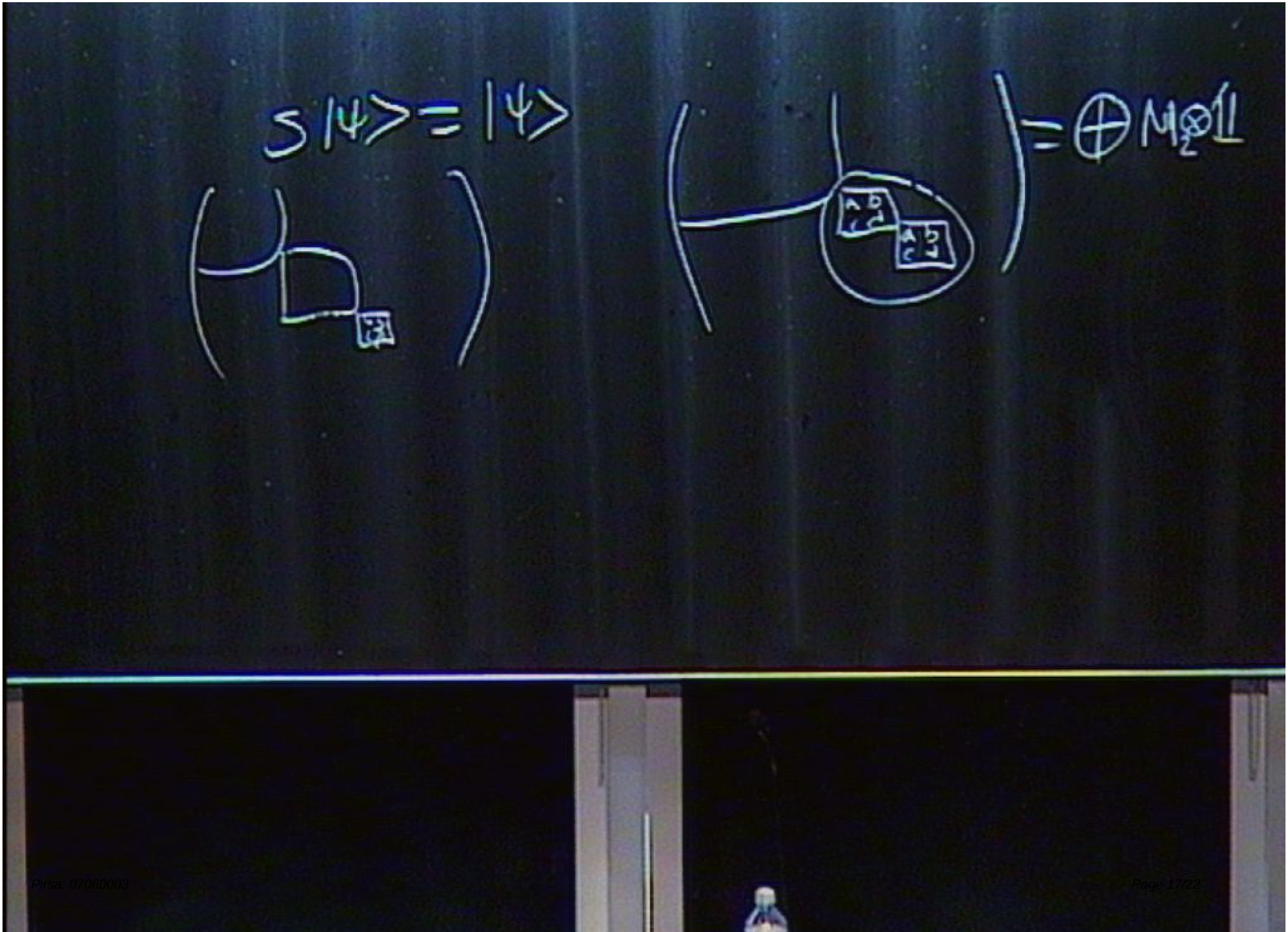
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- Can be generalize for non-group algebra

Arbitrary collective rotation

- Consider an error model of N qubit collective rotation

$$R_{\vec{n}\cdot\vec{\sigma}}(\theta) = e^{-i\frac{\theta}{2}(n_x X + n_y Y + n_z Z)} \qquad A = \sum_{i=1}^{n} A_i$$

- This is simply the addition of spin angular momentum problem

$$\mathcal{A} = \langle X, Z \rangle,\ \mathcal{A}' = \{J^2 = X^2 + Y^2 + Z^2\}$$

- So the DFS are thus the eigenspace of J^2, e.g.

$$H_j^{n_j} = span\{|j,m\rangle, -j \le m \le j\},\ \{0, \frac{1}{2}\} \le j \le \frac{N}{2}$$

- Since *A'* ⊂ *A*, the irrep of *A'* will have the same structure as that of J^2
  - Clebsch-Gordan decomposition

$$A = \bigoplus_j n_j D_j(A)$$

$$D_0(A) = 0,\ \forall A$$

Arbitrary collective rotation

- 1d irrep can only happen for even N
- If the 1d multiplicity $n_1=2^k$, we have a DFS encoding k qubits
- Using correspondence between Young tableau and representation theory, one can calculate that

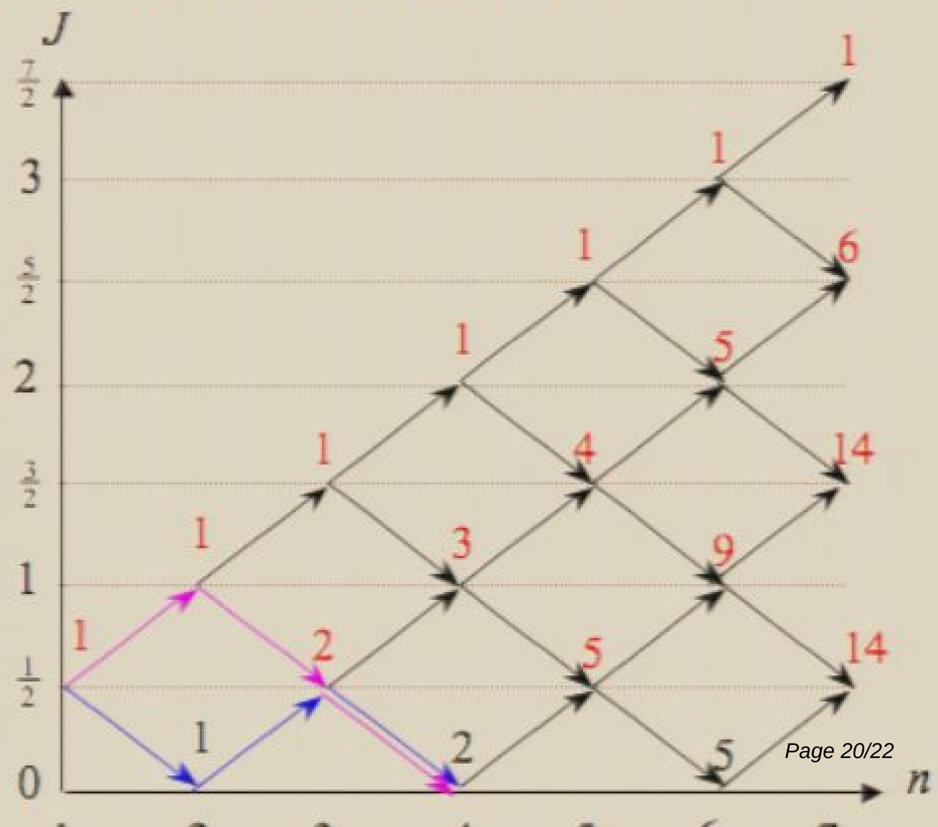
$$n_1(N) = \frac{N!}{(\frac{n}{2}+1)!\,(\frac{n}{2})!} \implies n_1(4) = 2$$

$$|0_L\rangle = (|01\rangle - |10\rangle) \otimes (|01\rangle - |10\rangle)$$

$$|1_L\rangle = |\Psi_+\rangle|\Psi_-\rangle - |\Psi_0\rangle|\Psi_0\rangle + |\Psi_-\rangle|\Psi_+\rangle$$

$$|\Psi_+\rangle = |00\rangle, |\Psi_-\rangle = |11\rangle$$

$$|\Psi_0\rangle = |01\rangle + |10\rangle$$



$$
X^4 = \left(\begin{array}{c|c|c|c|c|c}
\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{matrix} & 0 & 0 & 0 & 0 & 0 \\
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0 & \begin{matrix} 0 & \sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & 0 \end{matrix} & 0 & 0 & 0 & 0 \\
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0 & 0 & \begin{matrix} 0 & \sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & 0 \end{matrix} & 0 & 0 & 0 \\
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0 & 0 & 0 & \begin{matrix} 0 & \sqrt{\frac{1}{2}} & 0 \\ \sqrt{\frac{1}{2}} & 0 & \sqrt{\frac{1}{2}} \\ 0 & \sqrt{\frac{1}{2}} & 0 \end{matrix} & 0 & \\
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0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
$$

Conclusion

- The existence of a DFS necessarily requires symmetry in the noise (I.e. coupling to the environment)
- If no symmetry exist, we can find approximated DFS
- Or concatenate DFS with QECC/dynamical decoupling
- DFS can be generalize to noiseless subsystems
- NS allows FT computation to take a nicer form.