Title: A Hamiltonian framework for cosmological perturbations to any order

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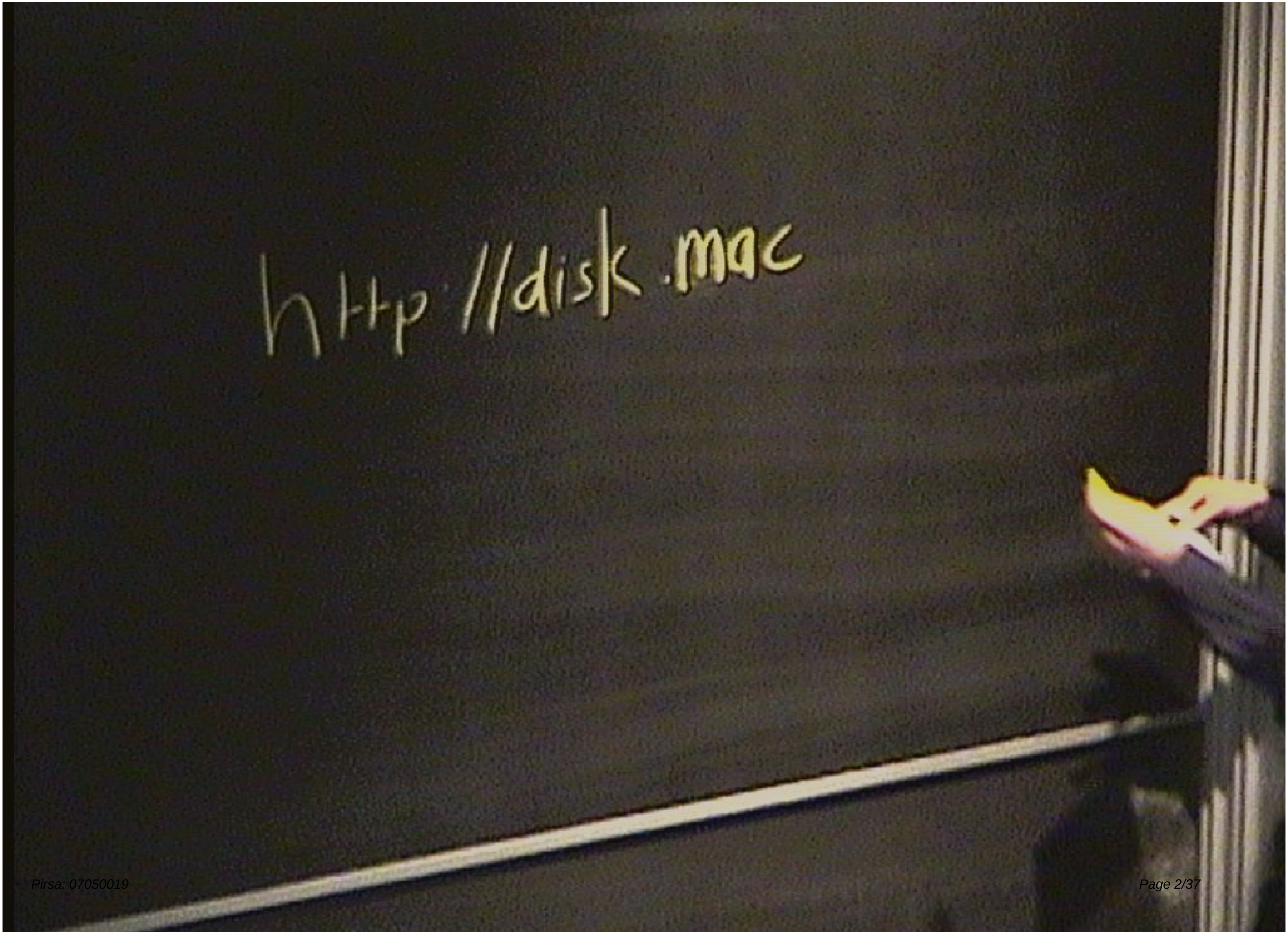
Abstract: We introduce a framework that allows to calculate cosmological perturbations in a gauge invariant manner to any order. The two main features of this framework are to take physical observables as basic objects and to treat the variables describing the background geometry as fully dynamical. Backreaction effects can therefore naturally adressed. At the end I will mention applications to Loop Quantum Cosmology.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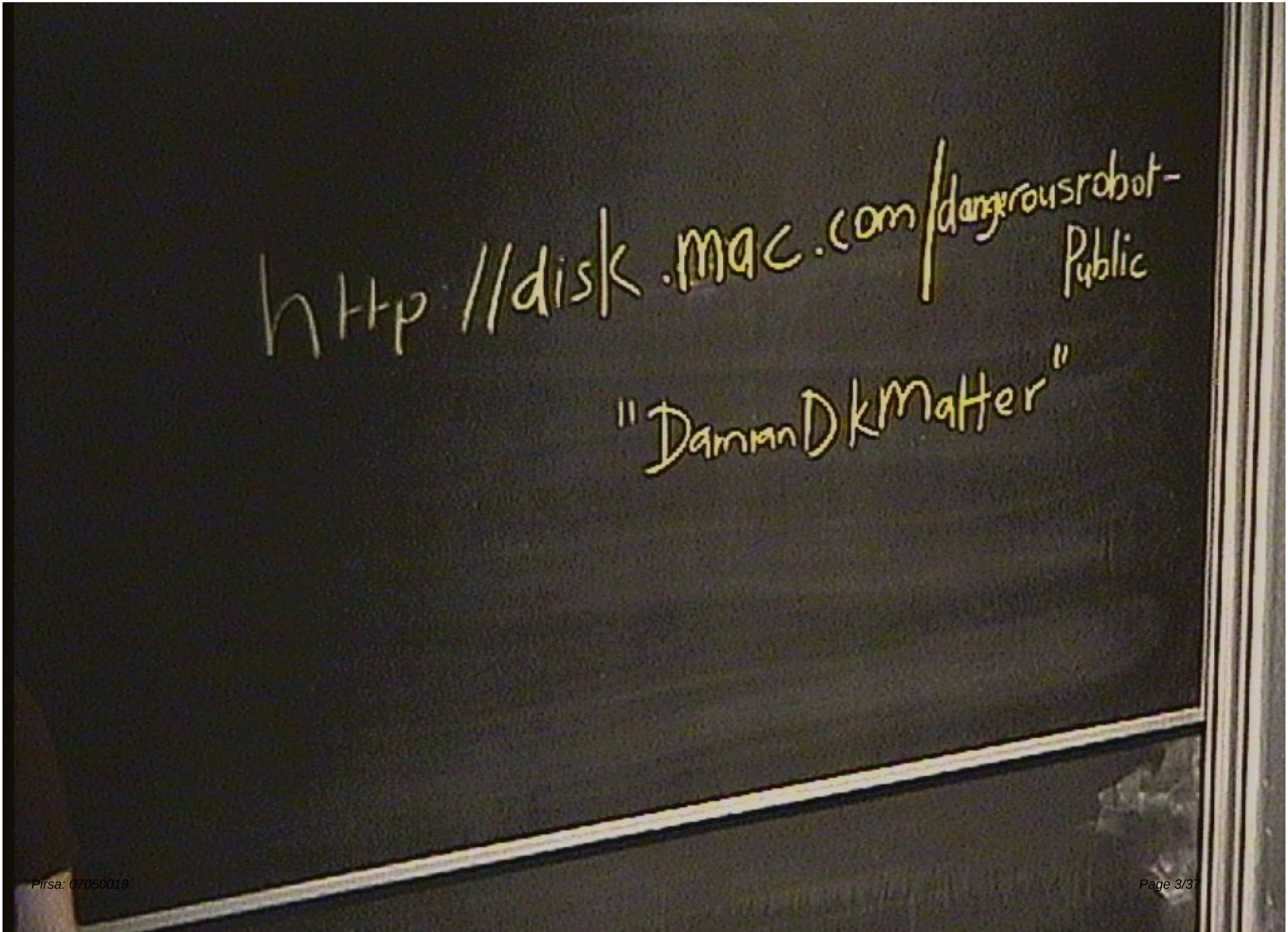
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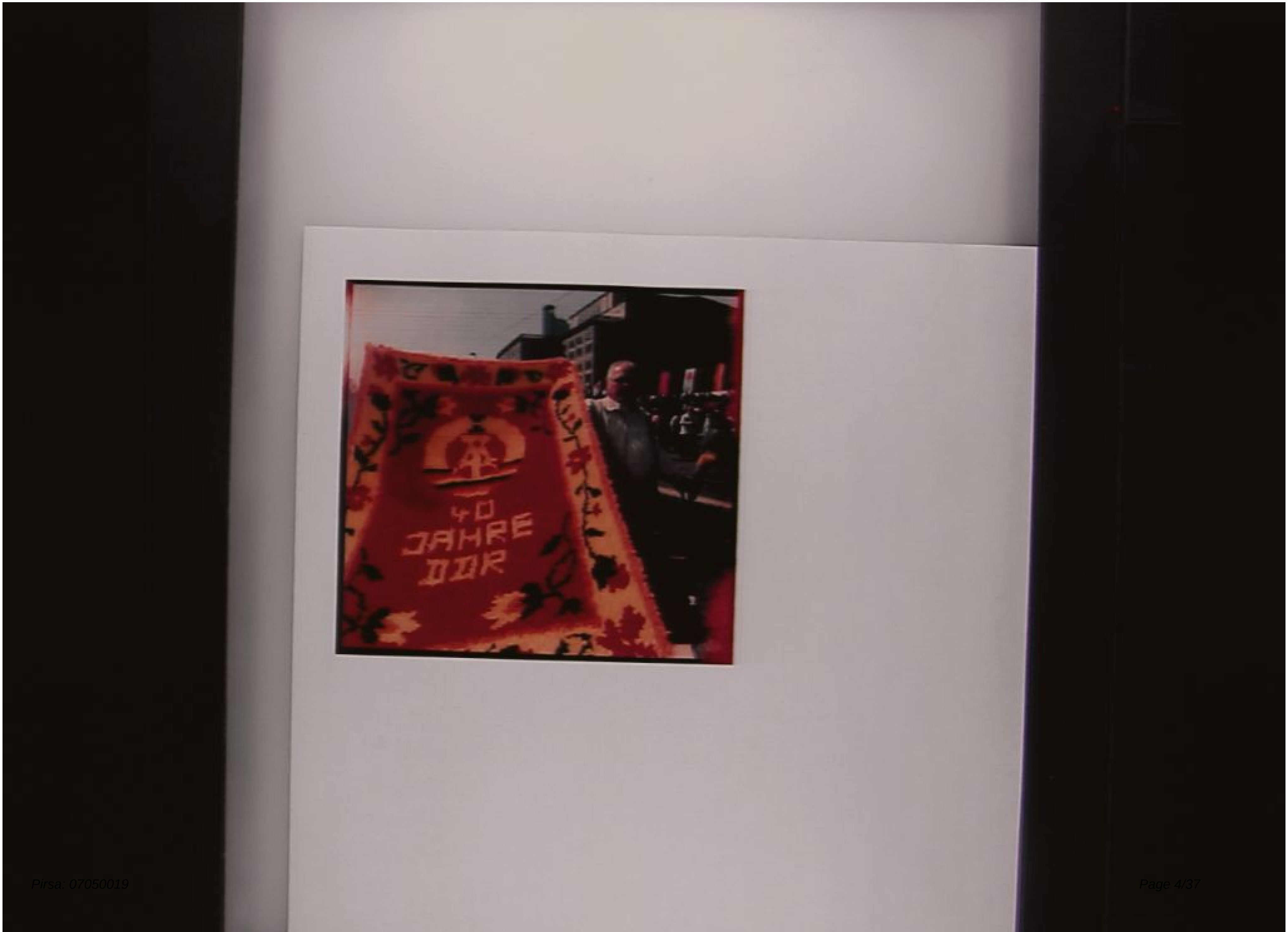
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A HAMILTONIAN FRAMEWORK FOR COSMOLOGICAL PERTURBATIONS TO ANY ORDER

Bianca Dittrich, PI

PI, May 1st 2007

B.D., Johannes Tambornino gr-qc/0610060
(CQG 2007)

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Introduction

- (linear) perturbation theory:
  standard tool in cosmology $\rightsquigarrow$ QFT on curved background

- predictions from Quantum Gravity ?

Quantum Gravity

background
$\langle vac|\, \widehat{a(t)}\, |vac\rangle$

perturbations
$\langle vac|\, \widehat{pert(x)}\, \widehat{pert(y)}\, |vac\rangle$

have to be gauge invariant

mini super space models

?

"effective" approach
- gauge fixed

- How can we describe evolution of perturbations and background variables in a gauge invariant manner ?

Gauge invariant local observables ?

- Hamiltonian framework ?

  [Langlois '93 : · non-extendable to higher order
  · meaning of Hamiltonian ?]

- gauge invariant framework beyond linear order ?

  - backreaction effects

    [Kolb et al., Brandenberger et al., Unruh , ...]

  - second order : gauge invariants ?

    [Bruni et al '96 , ...]

A new framework based on Observables

[B. D., J. Tambornino '07]

- Physical observables do not depend on choice of coord's
  - value of a scalar field at a point where four other scalar fields vanish

- Change of view point:
  - do not perturb around background mfd. rather

    perturb around a (phase space) sector describing solutions with huge symmetry

  - parameters describing the homogeneous feature of the universe arise through averaging over all degrees of freedom

⇝ Backreaction effects can be studied naturally

- applicable to:
  - Cosmology
  - Black Holes
  - Midisuperspace models

Results:

- first gauge invariant canonical framework
- defines the notion of gauge invariant observables of order n
- allows computation of perturbations in a gauge invariant manner to arbitrary high order

- explicit formulas in a "feynman graph" like language:

  evolution is described using

  - "free" propagation
  - terms describing interaction processes

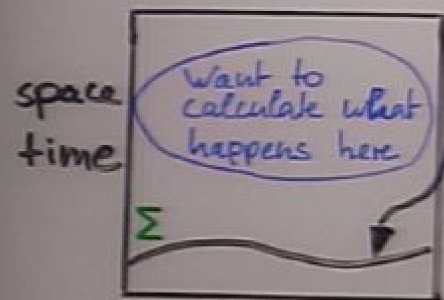
$$\int^{\tau} d\tau' \, \underline{\text{free prop.}}^{(\tau-\tau')} [\{ \underline{\text{free prop.}}^{(\tau)}[\text{Field}], {}^{(3)}H\}]$$

  - terms taking care of gauge invariance

Overview

- Gauge invariance canonical
- Introduce perturbations
- Complete observables and their approximations
- Relation to covariant picture
- Conclusion

Canonical / initial value formulation

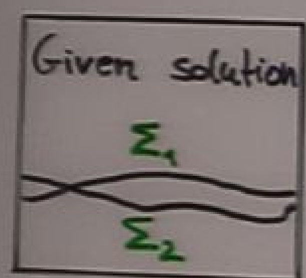


phase space = space of initial data on spatial 3D mfd. Σ

(3-metric q_{ab}, extrins. curv. K_{ab})

= parametrization of space of solutions

Gauge equivalence



A given solution induces different sets of initial data on different hypersurfaces Σ_1, Σ_2.

⇒ These initial data are **gauge equivalent**.

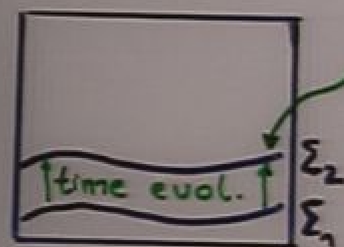
⇒ **time evolution** leads to gauge equivalent sets of ini. data

⇒ deformation of hypersurface generated by **gauge generators (constraints)**

Gauge invariant observables

→ Functions of initial data, that give the same result if evaluated on gauge equivalent data.

→ need to be constant in "time".



function evaluated on initial data on Σ_2

=

function evaluated ... Σ_1

How do we describe time evolution with observables?

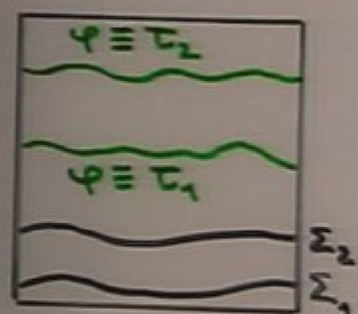
→ Relational (complete) observables

Complete observables

[... many, C. Rovelli '90s, B.D. 04, 05]

- Define "position" and shape of hypersurface using physical fields ($\boxed{\text{clocks}}$) instead of coordinates.
- Express evolution of other fields in this $\boxed{\text{clock time}}$. $\rightsquigarrow$ clock time generator

[for explicite formulas see B.D. 04, 05]



$Vol[\{\varphi \equiv \tau\}]$ does not depend whether one calculates it from the initial data on Σ_1 or Σ_2.

$$\frac{\partial \, Vol[\{\varphi \equiv \tau\}]}{\partial \tau} = \{ Vol[\{\varphi \equiv \tau\}], \text{clock time generator} \}$$

Introduce

Chop up

$$\boxed{\begin{array}{c}\text{phase space} = \\ \text{space of fields on } \Sigma\end{array}}$$

$\mathcal{P}$: projection acting on space of phase space functions

$\mathcal{P}$[fcts]	($\mathbb{I}$d $-\ \mathcal{P}$) [fcts]
- parameters describing homogeneous feature	- Perturbations
▪ $Q := \frac{1}{3\int_\Sigma d^3\sigma} \int_\Sigma q_{ab}\,\delta^{ab}\,d^3\sigma =: \mathcal{P}[q_{ab}]$	$h_{ab} = q_{ab} - Q\,\delta_{ab}$
▪ $K := \frac{1}{3\int_\Sigma d^3\sigma} \int_\Sigma K_{ab}\,\delta^{ab}\,d^3\sigma =: \mathcal{P}[K_{ab}]$	$k_{ab} = K_{ab} - K\,\delta_{ab}$

$\mathcal{P}$: averaging

- division consistent with kinematical structure
- (Q, K, h_{ab}, k_{ab}): slightly over-complete description of phase space

Can expand arbitrary phase space functions in terms of (Q, K, h_{ab}, k_{ab}):

$$\mathrm{Vol}(\Sigma) = \int_\Sigma \sqrt{\det q}\, d^3\sigma = \int_\Sigma \sqrt{\det(Q\,\delta_{ab} + h_{ab})}\, d^3\sigma$$

$$= Q^{3/2}\int d^3\sigma + \frac{1}{2}\underbrace{\int Q^{1/2}\, h^b{}_b\, d^3\sigma}_{=0} + \frac{1}{8}\int Q^{-1/2}\left(h^b{}_b h^a{}_a - h^a{}_b h^b{}_a\right) d^3\sigma$$

= zeroth order term + second order term

<u>Define</u>:

- zeroth order quantities: Q, K
- first order quantities: h_{ab}, k_{ab}
- n^{th}-order quantities: $f(Q,K)\,[h_{ab}]^n$, $f(Q,K)\,[k_{ab}]^n, \ldots$

- "Background" variables emerge from full phase space.
- Do not introduce second or higher order "variations" as in covariant theory.

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Gauge invariant observables of order m

are invariant under infinit.'mal gauge trafo's up to terms of order m.

(gauge trafo = deformation of initial data hypersurface)

⇒ Do not need to find (closed) m-order gauge generators / constraints.

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$$\boxed{\begin{array}{l}\text{• "Background" variables emerge from full phase space.} \\ \text{• Do not introduce second or higher order "variations" as in covariant theory.}\end{array}}$$

Gauge invariant observables of order m

Zeroth order gauge invariant observables (of zeroth order) coincide with the gauge invariant observables of the symmetry reduced model

Not all zeroth order variables are gauge invariants to zeroth order.

First order gauge invariant observables coincide with usual linearly gauge invariant variables (▪ Bardeen potential, tensor modes).

First order complete observables are additionally invariant under "global time translations".

Second and higher gauge invariants

- products of first order invariants
- higher order extensions of zeroth and first order complete observables

Here: taking "global time translation" into account is crucial !

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How can we find gauge invariants to order m?

→ Use relational / complete observables

- Explicit clock time evolution equations and power series expansion available [B.D. '04]
- Using these compute complete observables up to order m.
- Technically: (weakly) Abelianized gauge generators

⇝ Feynman-graph like expansion
⊙ gauge invariant extension terms.

Choose

= phase space functions

- A "global clock" T

(zeroth + 2nd + ...) order

- scale factor, averaged scalar field, ...

- describe global time translation

- global clock time generator

$\tilde{C}$ = 0th + 2nd + higher order

- "inhomogeneous clocks" ($4 \times \infty$)

(1st + 2nd + ...) order

- scalar fields (inhomogen.)
- longitudinal clocks

$$T^0 \sim \Delta^{-1}\left({}^{LL}k - \tfrac{1}{2}\,{}^{T}k\right)$$

$$T^a \sim \Delta^{-1}\left(\partial^a\,{}^{LL}h - \tfrac{1}{2}\,\partial^a\,{}^{T}h + \partial_b\,{}^{LT}h^{ba} + \partial_b\,{}^{TL}h^{(ab)}\right)$$

- nonlocal, can be used for calcul.'s to higher order

- describe shape of hypersurface

- gauge generators

$\tilde{C}_K(k)$ = 1st + higher order

k: Fourier mode

adapted to T's

How can we find gauge invariants to order n?

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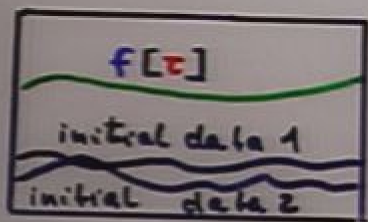
Compute complete observables

$f[\tau]$ gives value of f on hypersurface on which $T = \tau$, $T^K(k) = 0$.



evolve with $\tilde{C}$ $\Leftarrow$ gauge invariant extension terms $\tilde{C}_K(k)$

exact complete observable

$$f[\tau] = \sum_{p=0}^{\infty} \sum_{k's} \frac{1}{p!} \cdot$$

$$\{ \cdots \{ \alpha_{\tilde{C}}^{\tau - T}(f), \tilde{C}_{K_1}(k_1)\} \cdots$$

$$\cdots \}, \tilde{C}_{K_p}(k_p)\} (-T^{K_1}(k_1)) \cdots (-T^{K_p}(k_p))$$

where $\alpha_{\tilde{C}}^{\tau - T}(f)$ evolution of f with $\tilde{C}$ for a time $\tau - T$

Backreac... order

$$ {}^{[2]}f[\tau] = $$

$$ \alpha_{free}^{\tau - T}(f) $$

0th order term

$$ + \sum_k \{ \alpha_{free}^{\tau - T}(f), {}^{(1)}\tilde{C}_k(k) \} (-T^k(k)) $$

$$ + \frac{1}{2} \sum_{k_1 k_2} {}^{(0)}\{\{ \alpha_{free}^{\tau - T}(f), {}^{(1)}\tilde{C}_{k_1}(k_1)\}, {}^{(1)}\tilde{C}_{k_2}(k_2)\} \, T^{k_1}(k_1)\, T^{k_2}(k_2) $$

2nd order gauge invariant extension of 0th order term

$$ + \int_0^{\tau - T} ds \; \alpha_{free}^{\tau - T - s} [\underbrace{\{ \alpha_{free}^{s}(f), {}^{(2)}\tilde{C} \}}_{\text{coupling of homogeneous and inhomogeneous modes}}] $$

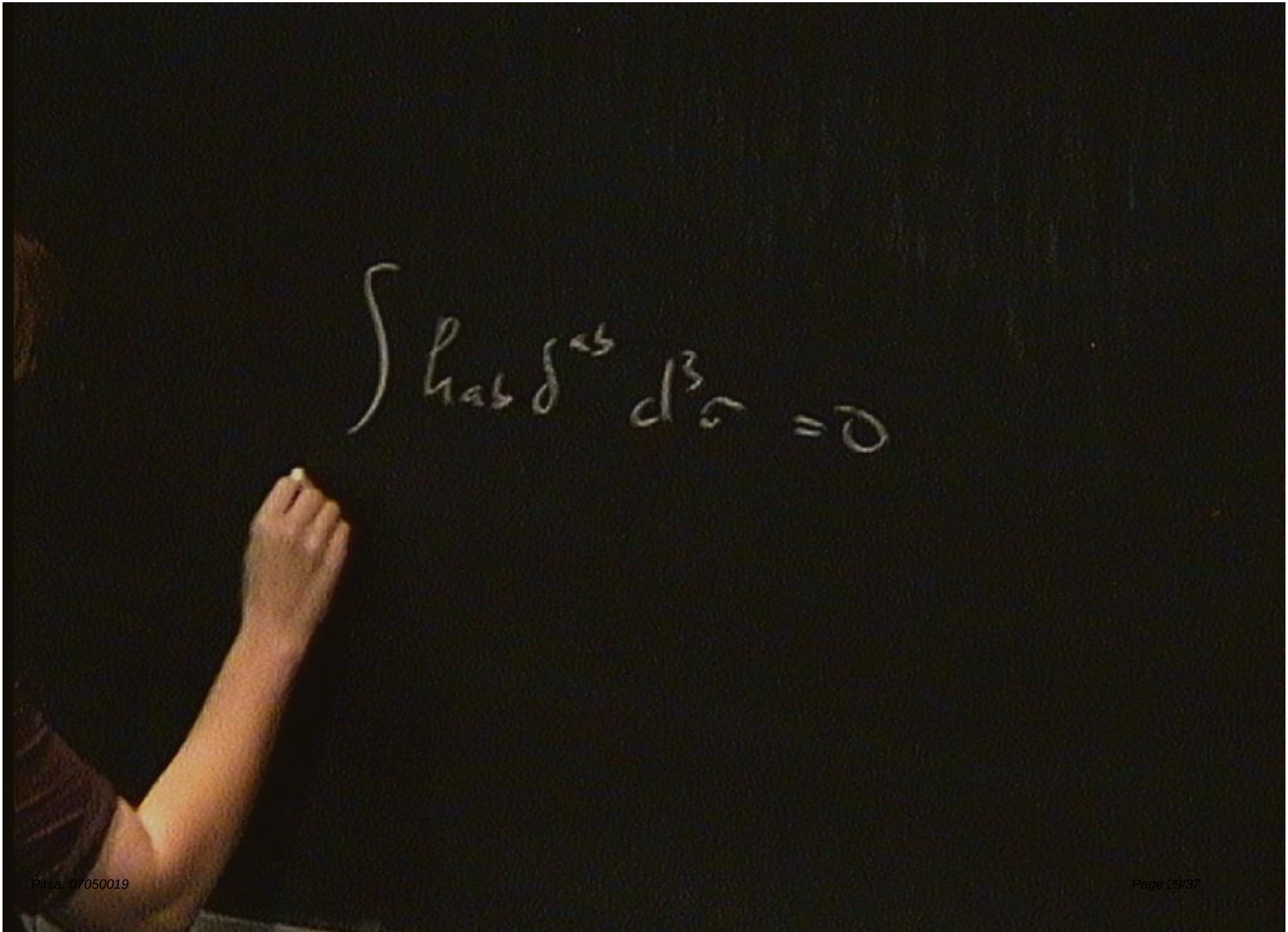
Interaction term

+ 2nd order gauge inv. extension of Interaction

= 0th order term + 2nd order term

0th order term ↓ coincides with symmetry reduced model

2nd order term ↓ corrections coming from
* interaction
* gauge invariance


$\int h_{ab} \delta^{ab} \, d^3\sigma = 0$

$$\int h_{ab}\delta^{ab}\, d^3\sigma = 0$$

$$g_{ab} = Q\,\delta_{ab} + h_{ab}$$

$$Q = \int h_{ab}\delta^{ab}\, d^3\sigma$$

$$\int h_{ab}\,\delta^{ab}\,d^3\sigma = 0$$

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$$Q = \int h_{ab}\,\delta^{ab}\,d^3\sigma$$

$^{[2]}f\,[$

$$\alpha_{free}^{\tau-T}(f)$$

0th order term

$$+\sum_k \{\alpha_{free}^{\tau-T}(f),\ {}^{(1)}\tilde{C}_k(k)\}(-T^k(k))$$

2nd order gauge invariant extension of 0th order term

$$+\frac{1}{2}\sum_{k_1k_2} {}^{(0)}\{\{\alpha_{free}^{\tau-T}(f),\ {}^{(1)}\tilde{C}_{k_1}(k_1)\},\ {}^{(1)}\tilde{C}_{k_2}(k_2)\}\ T^{k_1}(k_1)\,T^{k_2}(k_2)$$

$$+\int_0^{\tau-T} ds\ \alpha_{free}^{\tau-T-s}\Big[\underbrace{\{\alpha_{free}^{s}(f),\ {}^{(2)}\tilde{C}\}}_{\text{coupling of homogeneous and inhomogeneous modes}}\Big]$$

Interaction term

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= 0th order term + 2nd order term

↓ ... with ...

↓ corrections coming from
* interaction
* gauge invariance

$$
\begin{aligned}
{}^{[2]}f[\;\;] &= \alpha_{\text{free}}^{\tau-T}(f) \\
&+ \sum_k \{\alpha_{\text{free}}^{\tau-T}(f), {}^{(1)}\tilde{C}_k(k)\}(-T^k(k)) \\
&+ \frac{1}{2}\sum_{k_1k_2} {}^{(0)}\{\{\alpha_{\text{free}}^{\tau-T}(f), {}^{(1)}\tilde{C}_{k_1}(k_1)\}, {}^{(1)}\tilde{C}_{k_2}(k_2)\} T^{k_1}(k_1)\, T^{k_2}(k_2) \\
&+ \int_0^{\tau-T} ds\; \alpha_{\text{free}}^{\tau-T-s}\Big[\underbrace{\{\alpha_{\text{free}}^{s}(f), {}^{(2)}\tilde{C}\}}_{\text{coupling of homogeneous and inhomogeneous modes}}\Big]
\end{aligned}
$$

0th order term

2nd order gauge invariant extension of 0th order term

Interaction term

+ 2nd order gauge inv. extension of Interaction

= 0th order term + 2nd order term

0th order term ↓ coincides with symmetry reduced model

2nd order term ↓ corrections coming from
* interaction
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- Can be generalized to arbitrary high order
- Similar formulas can be obtained for evolution of perturbations

As interactions one will get

* similar interactions as in flat space time:
  - between inhomogeneous modes
  - mode coupling: encoded in $\tilde{G}$
    [flat space example in B.D., J.Tambornino '06]

* coupling between homogeneous and inhomogeneous modes

Example: Anisotropies as perturbations of a homogeneous and isotropic universe

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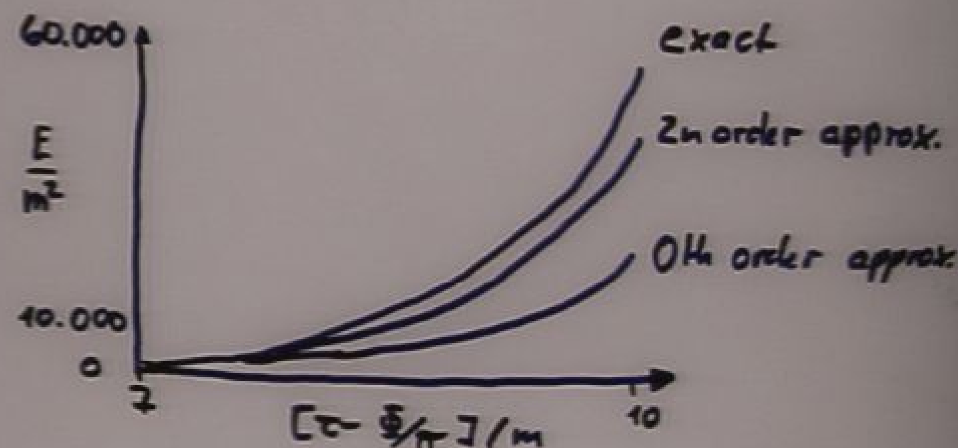
Bianchi I : $$q_{ab} = Q\delta_{ab} + \begin{pmatrix} h_1 & & 0 \\ & h_2 & \\ 0 & & h_3 \end{pmatrix}$$

coupled to homogeneous scalar field → clock



initial values:

$A = 1$ $\qquad a_1 = a_2 = 0.1 = -1/2\, a_3$

$E = 1\, m^2$ $\qquad e_1 = e_2 = 0.1\, m^2 = -1/2\, e_3$

Relation to covariant picture

$$\tilde{G} = N^{\perp} C_{\perp} + N^a C_a$$

clock time generator — phase space dep. lapse — phase space dep. shift

$$N^j(k) := (\mathcal{A}^{-1})^j_{\ 0}(-k,0) \quad \text{with} \quad \mathcal{A}^k_{\ j}(k,k') = \{T^k(k), G_j(k')\}$$

$\Rightarrow$ 4D metric adapted to clocks

$$g_{\mu\nu} \sim \begin{pmatrix} -(N^{\perp})^2 + q_{ab}N^aN^b & N_a \\ N_b & q_{ab} \end{pmatrix}$$

- longitudinal clocks $\rightsquigarrow$ 4D metric in longit. gauge
- – use clocks also to higher order: $N_a = \mathcal{O}(2)$
- – or add higher order terms to clocks such that $N_a = \mathcal{O}(n)$
- – Bardeen potential: complete observable to $f \sim h^a{}_a$

Sum

- and things I did not mention

- gauge invariant perturbative scheme to any order based on observables
- "background" variables are fully included
  → needed to define clock time
- backreaction terms arise naturally
- recover results from linear perturbation theory

- physical Hamiltonians
- transformations between different clocks
- linearization instabilities
- causality properties of observables:
  - non-local terms at higher order