Title: Quantum Error Correction 10B

Date: Mar 13, 2007 05:00 PM

URL: http://pirsa.org/07030023

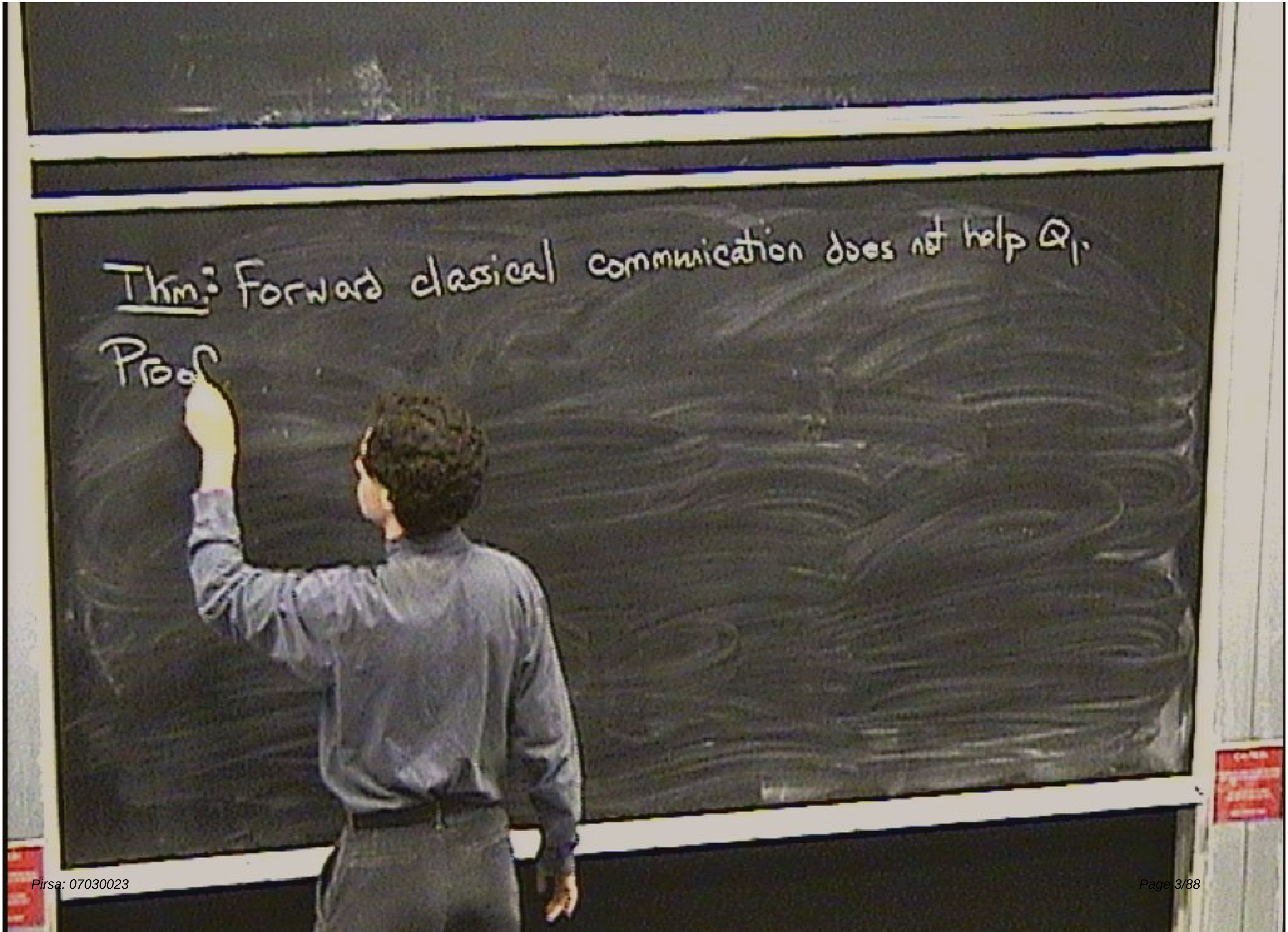
Abstract: One- and two-way quantum capacities for the erasure channel, upper and lower bounds on the quantum capacities for the depolarizing channel, coherent information

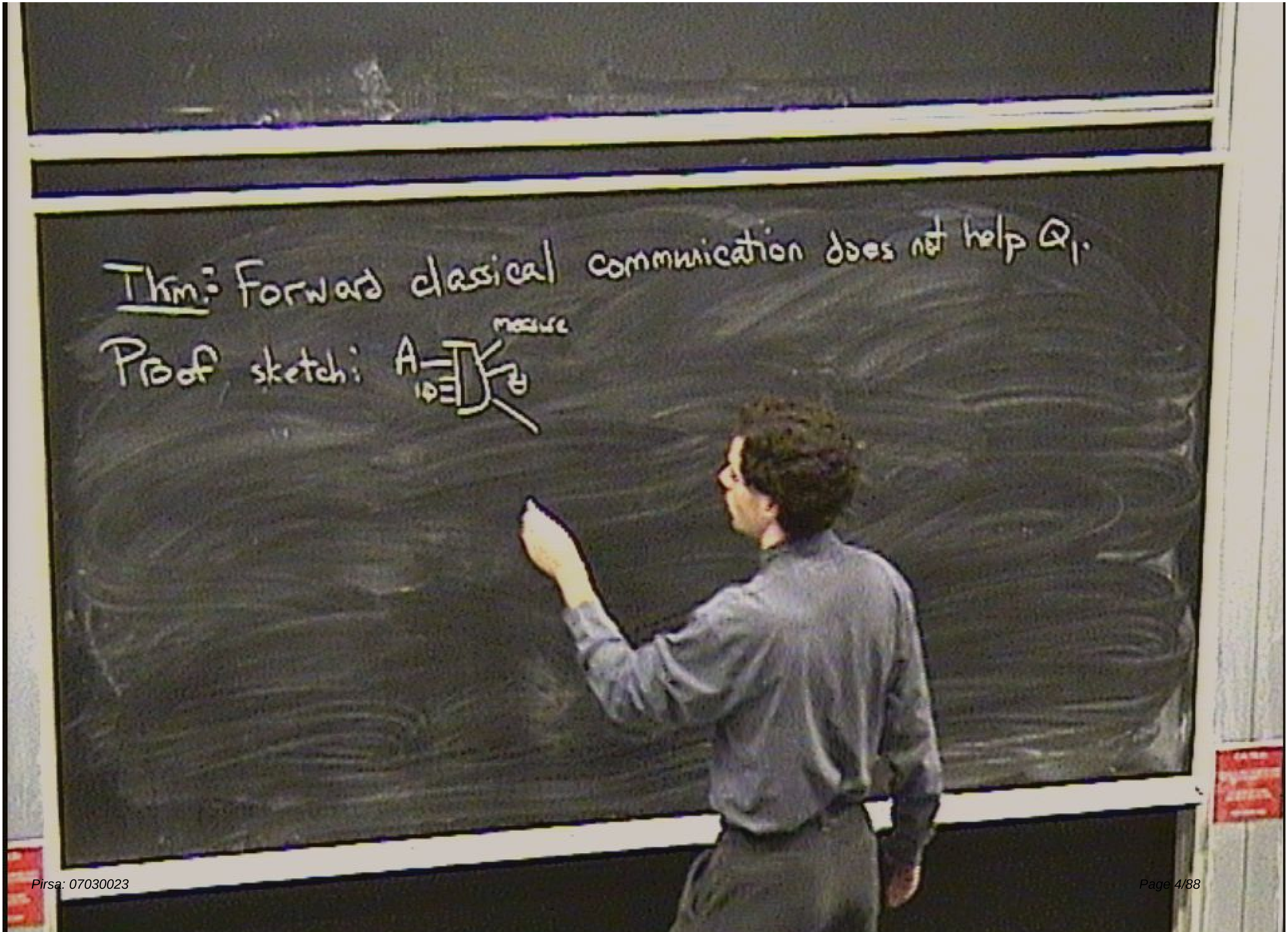
Alice wishes to send k qubits to Bob with n physical qubits transmitted. Channel is $\mathcal{N}^{\otimes n}$. She wants fidelity $\to 1$ as $n \to \infty$.

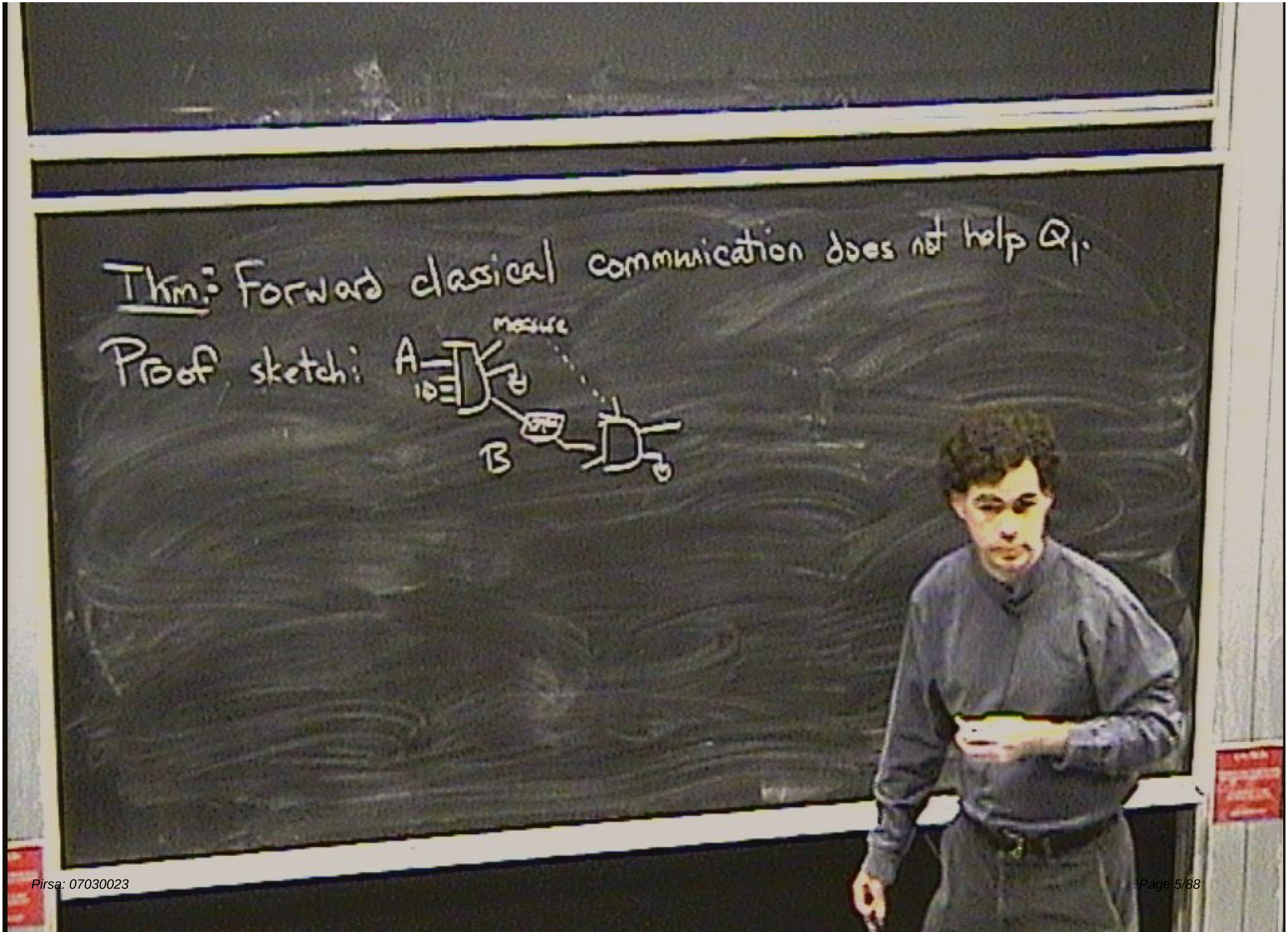
Def: The one-way quantum channel capacity $Q_1(\mathcal{N}) = \sup k/n$, with sup taken over QECCs with fidelity $\to 1$ as $n \to \infty$.

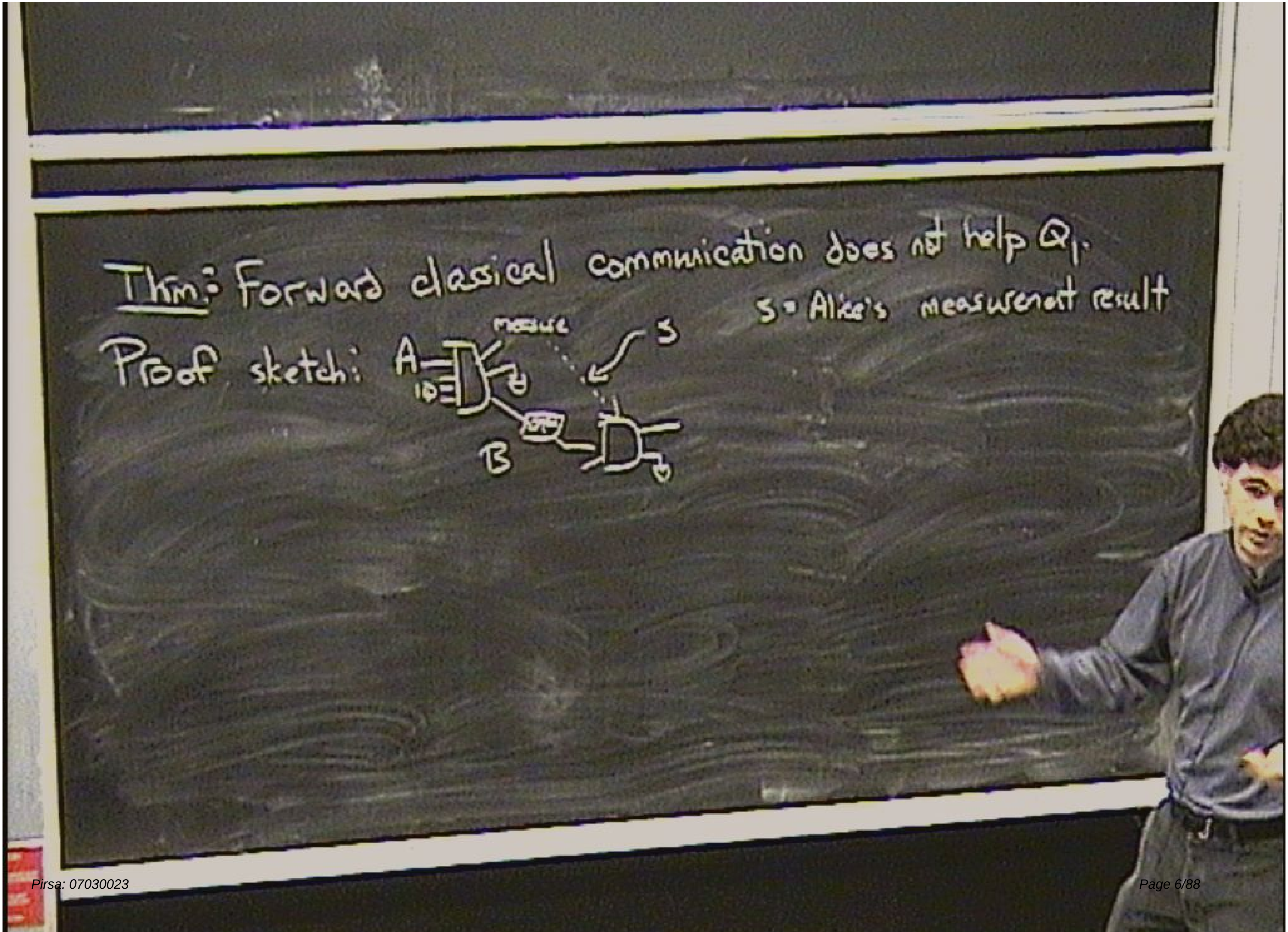
The two-way quantum capacity $Q_2(\mathcal{N}) = \sup k/n$, with sup over all procedures involving n quantum transmissions, followed by A&B doing local gates & measurements and classical communications. (A encodes before transmission)

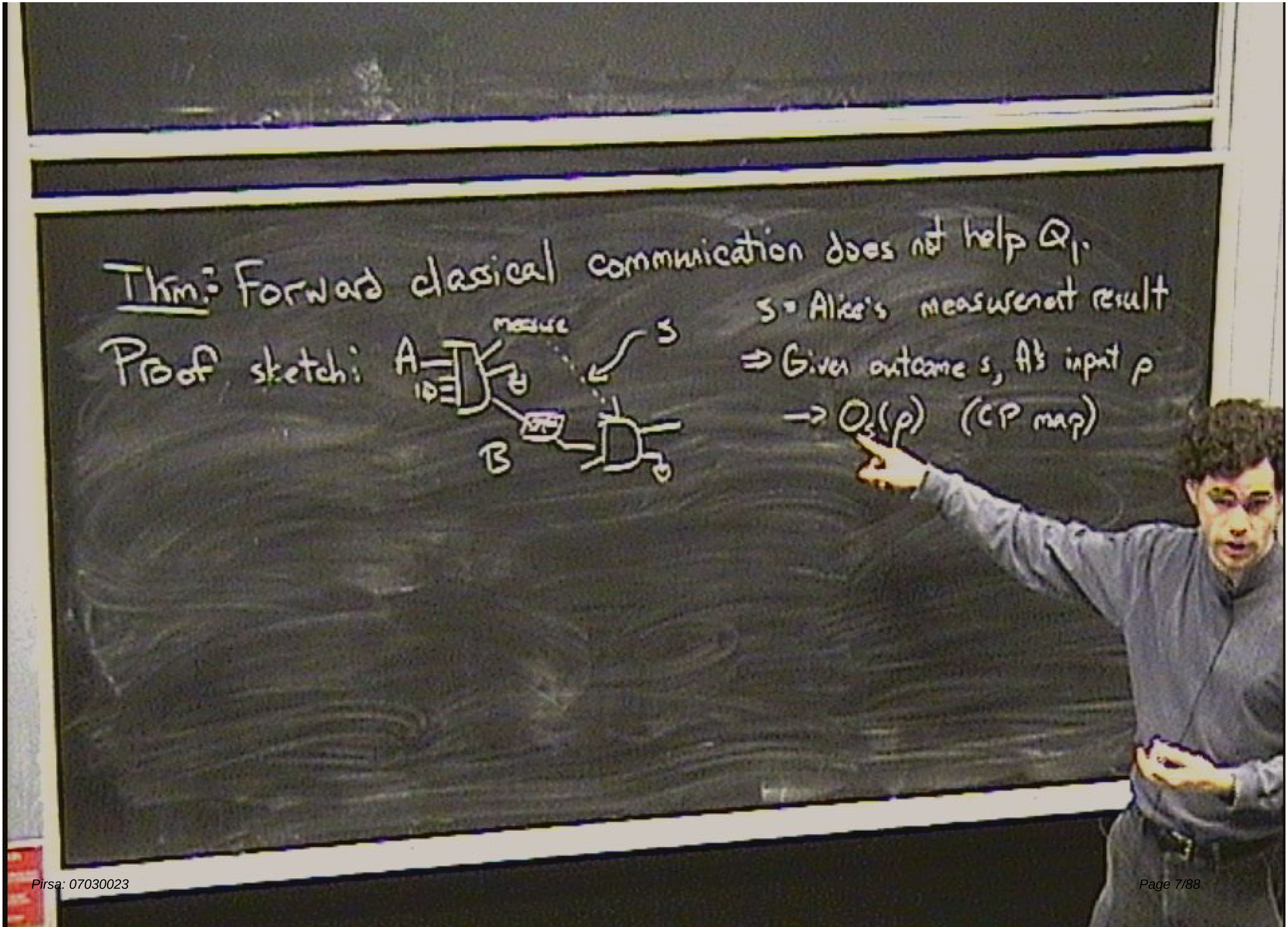
Distillable entanglement $D_2(\rho)$ is $\sup k/n$, over 2-EDPs that take $\rho^{\otimes n}$ to $|\Phi^+\rangle^{\otimes k}$ with fidelity $\to 1$ as $n \to \infty$


Thm: Forward classical communication does not help Q1.
Proof


Thm: Forward classical communication does not help Q1.
Proof sketch:
measure
A


Thm: Forward classical communication does not help Q_1.
measure
Proof sketch: A
B


Thm: Forward classical communication does not help Q1.
s = Alice's measurement result
Proof sketch:
A
|0⟩
measure
s
B


Thm: Forward classical communication does not help Q_1.
s = Alice's measurement result
Proof sketch:
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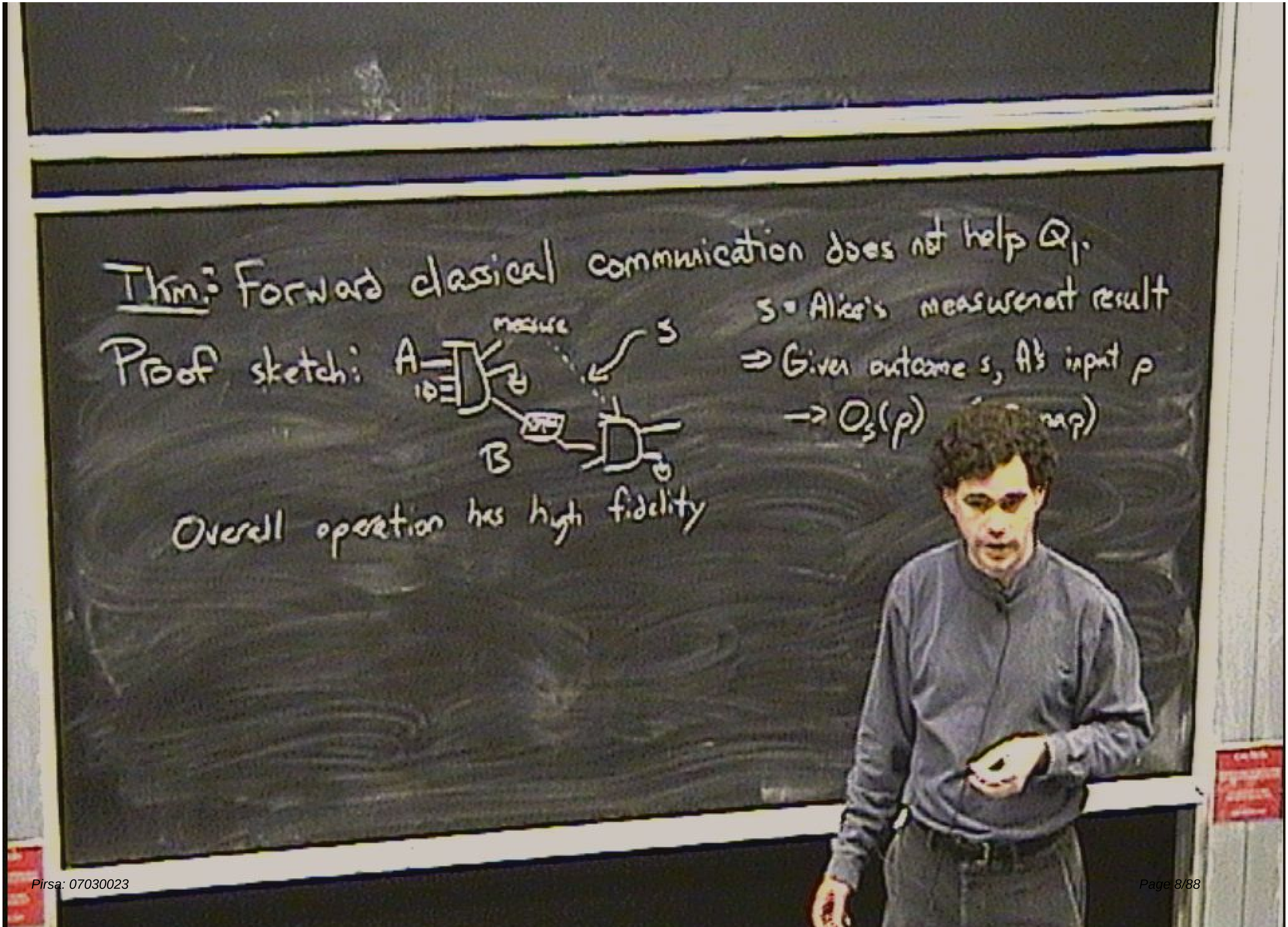
s
B
⇒ Given outcome s, A's input ρ
→ $O_s(\rho)$ (CP map)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Proof sketch:
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⇒ Given outcome s, A's input ρ
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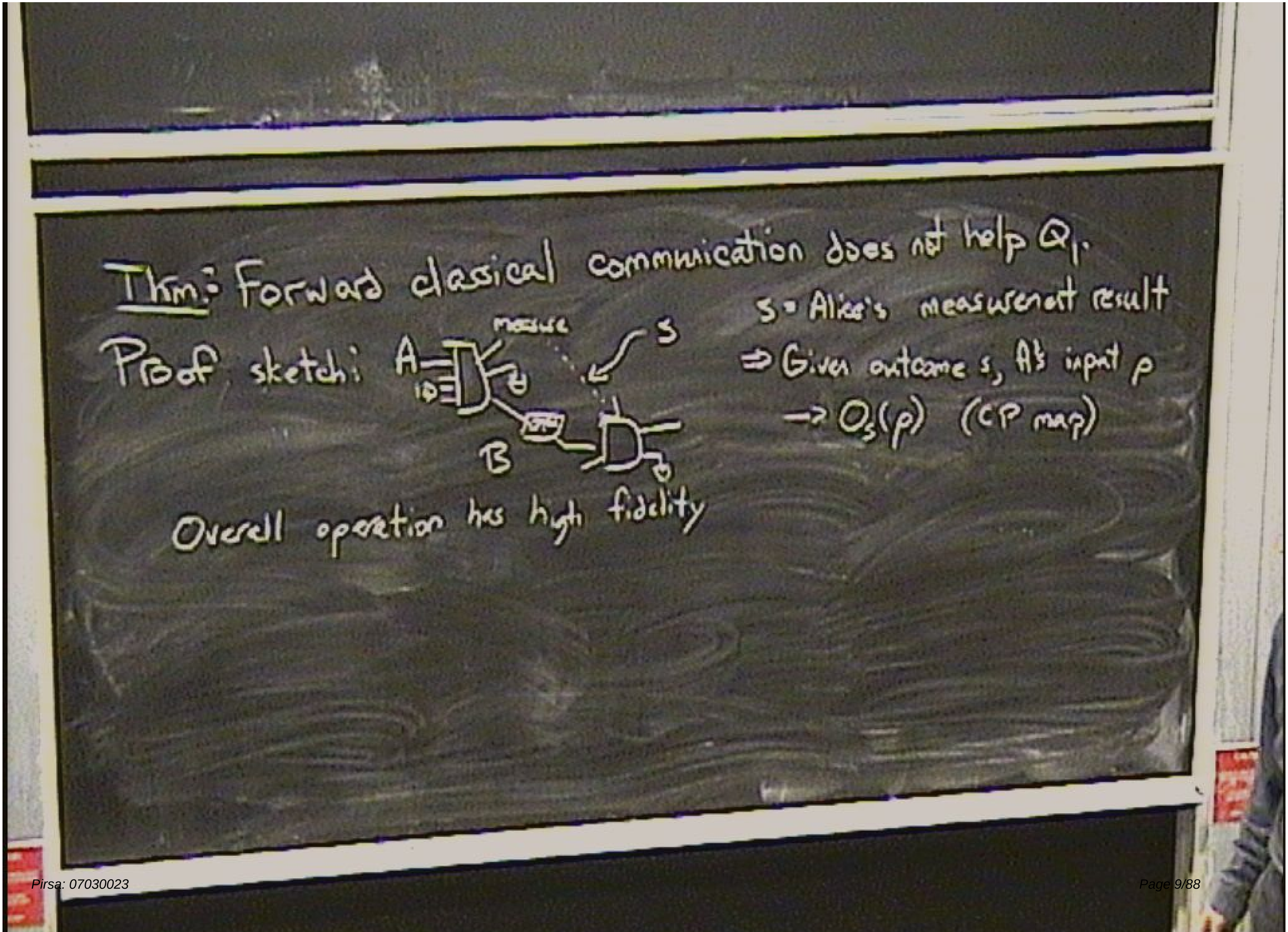
Overall operation has high fidelity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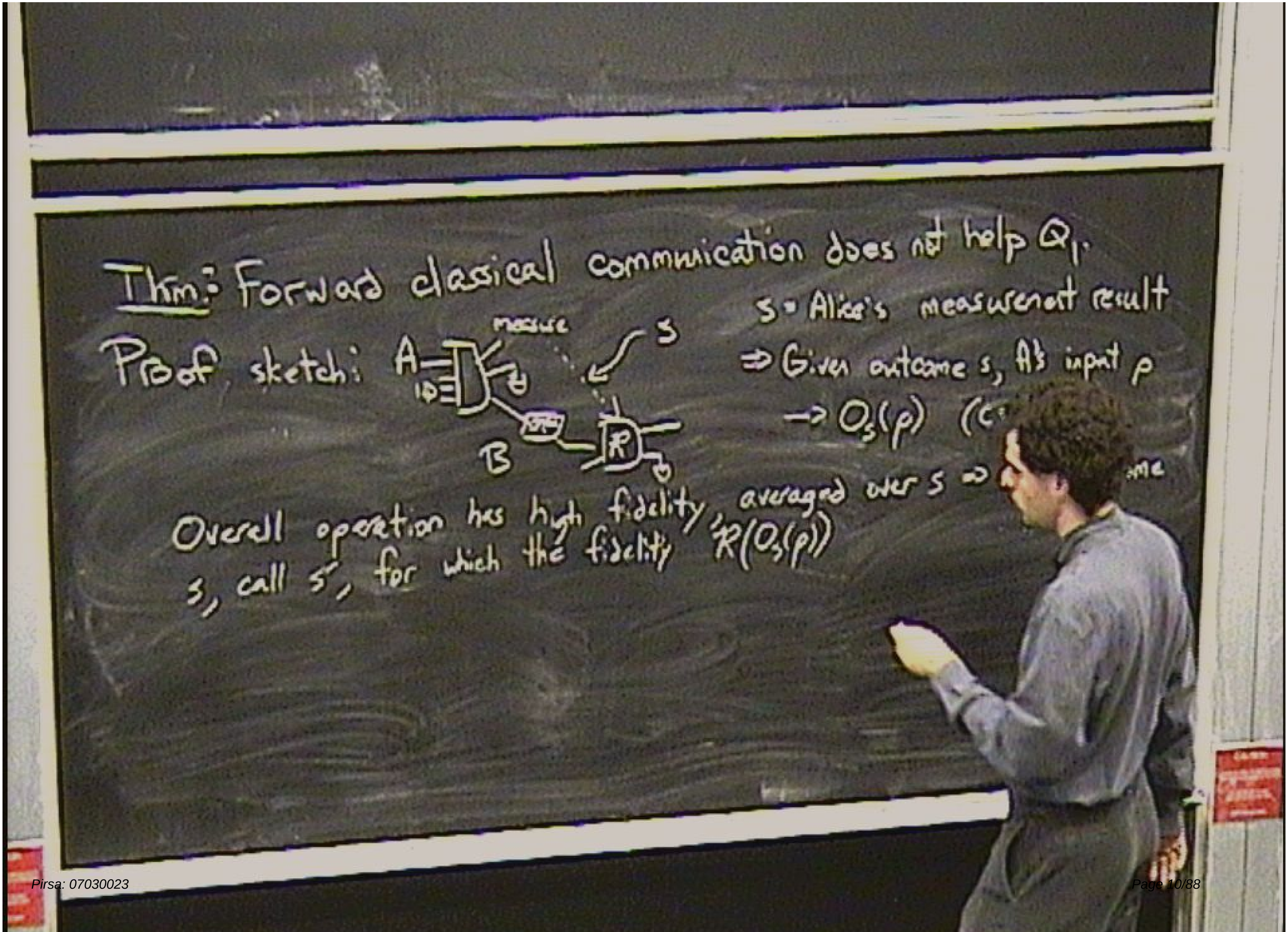
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Proof sketch:
measure
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s = Alice's measurement result
⇒ Given outcome s, A's input ρ
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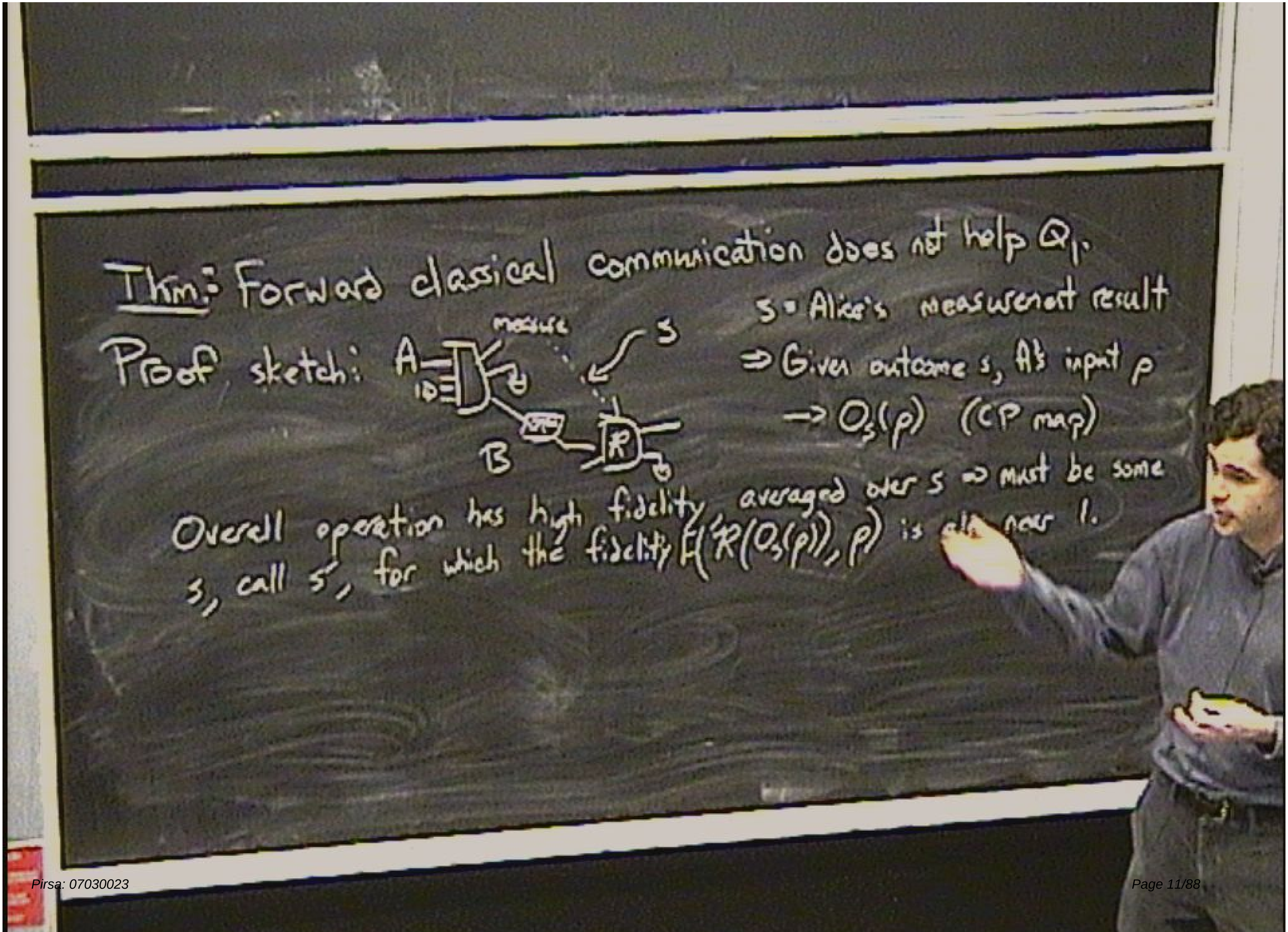
averaged over s ⇒
Overall operation has high fidelity, averaged over s
s, call s', for which the fidelity $R(O_s(\rho))$


Thm: Forward classical communication does not help Q1.
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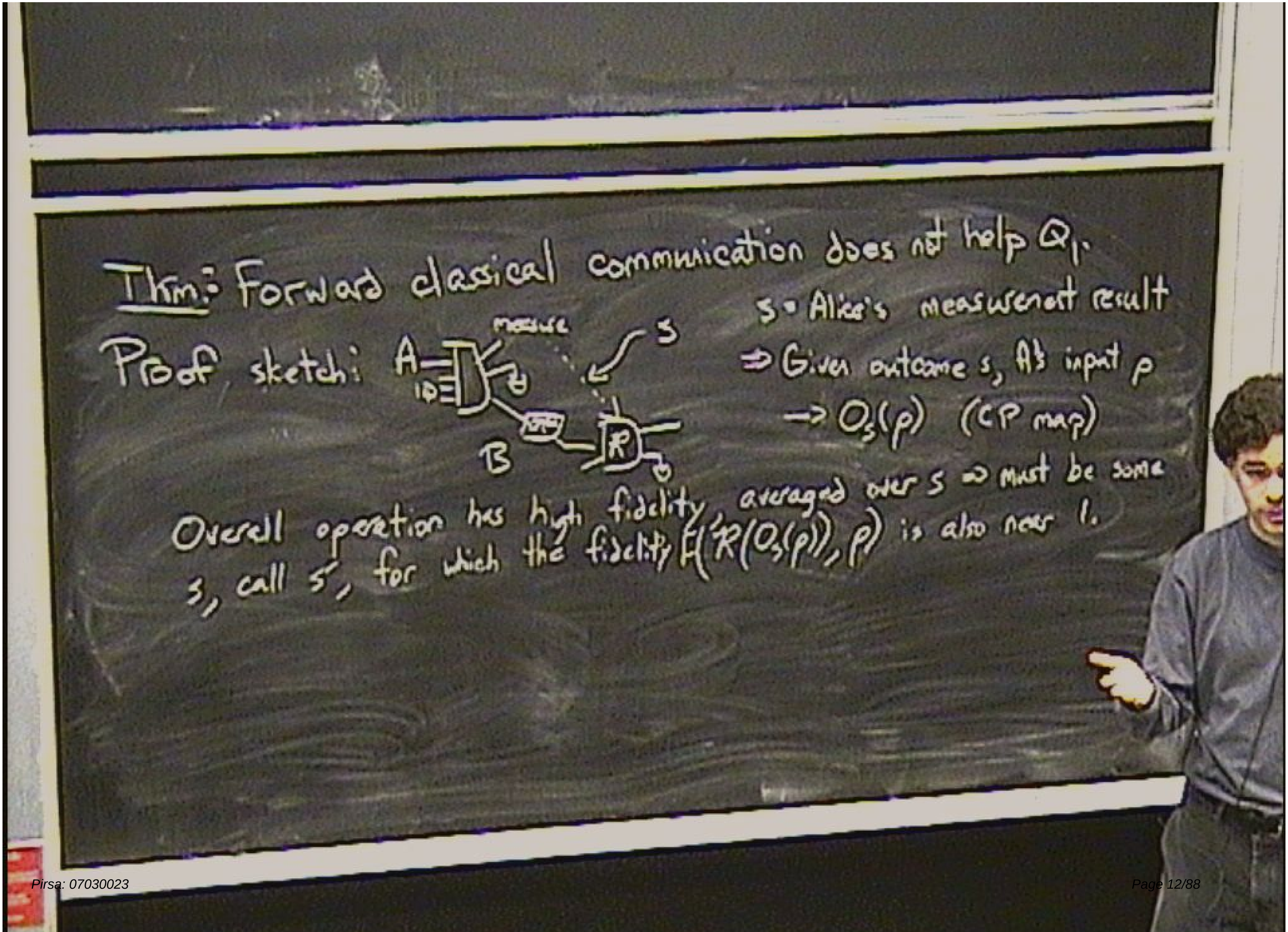
Proof sketch:
A
B
measure
s
⇒ Given outcome s, A's input ρ
→ Os(ρ) (CP map)
Overall operation has high fidelity averaged over s ⇒ must be some
s, call s1, for which the fidelity F(R(Os(ρ)), ρ) is also near 1.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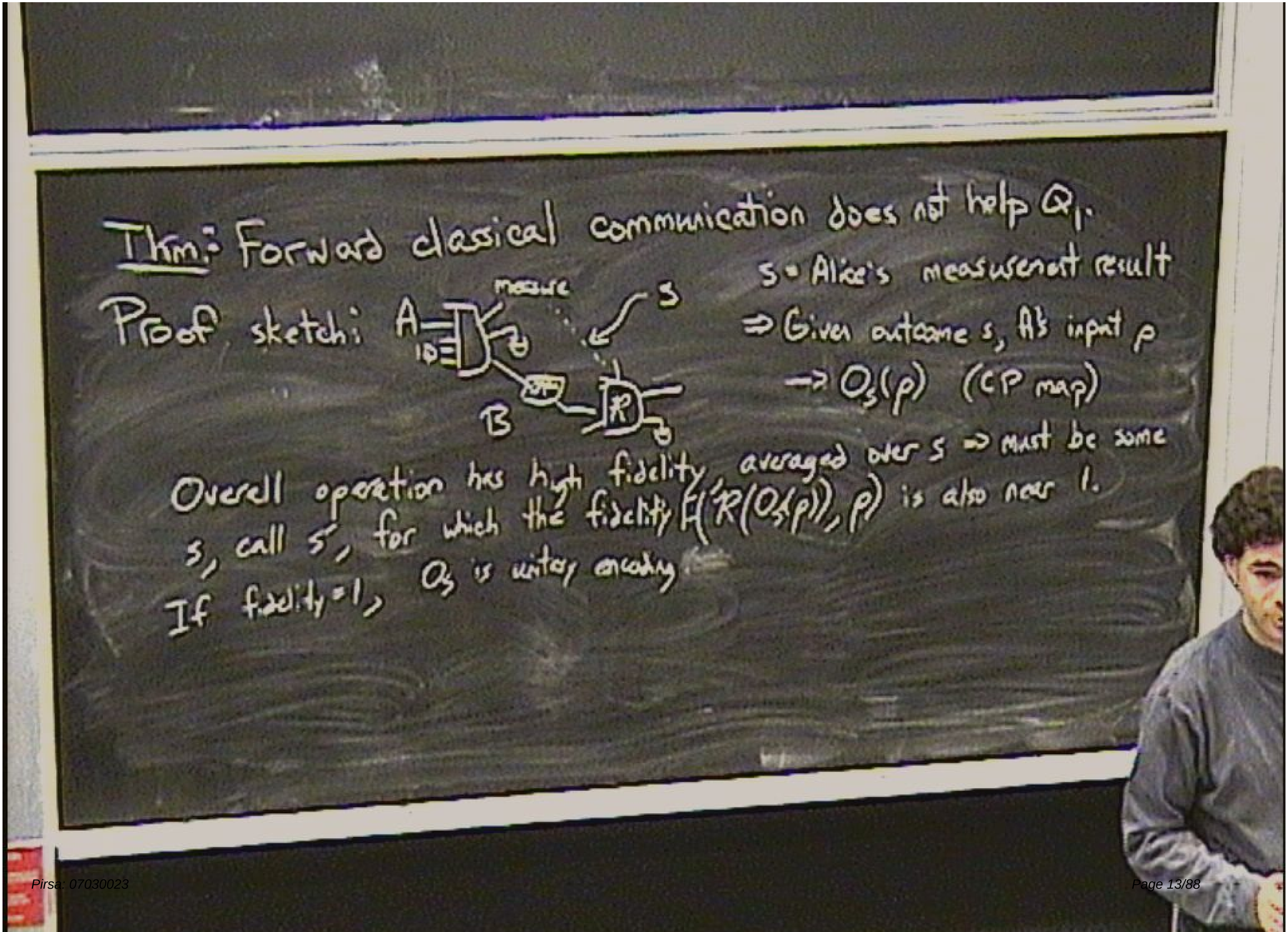
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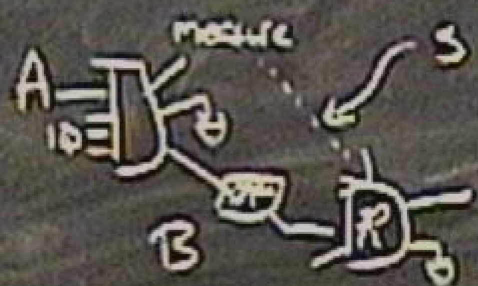
If fidelity = 1, Os is unitary encoding

Thm: Forward classical communication does not help Q_1.

S = Alice's measurement result

Proof sketch:

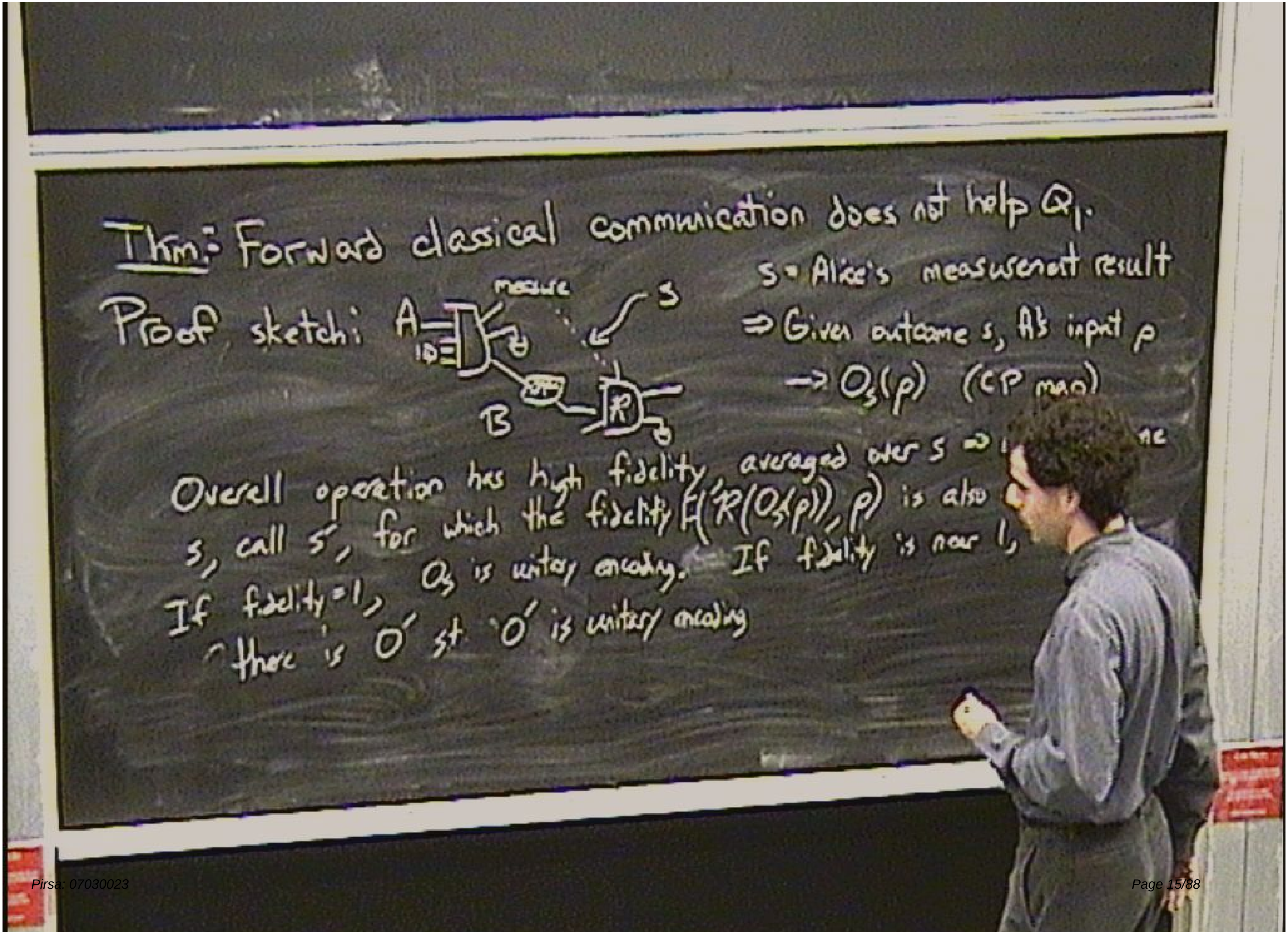


$\Rightarrow$ Given outcome s, A's input ρ

$\rightarrow O_s(\rho)$ (CP map)

Overall operation has high fidelity averaged over s $\Rightarrow$ must be some s, call s', for which the fidelity $F(\mathcal{R}(O_s(\rho)), \rho)$ is also near 1.

If fidelity = 1, O_s is unitary encoding. If fidelity is near 1, O_s is close to a un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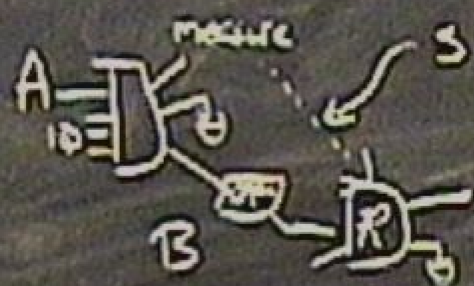

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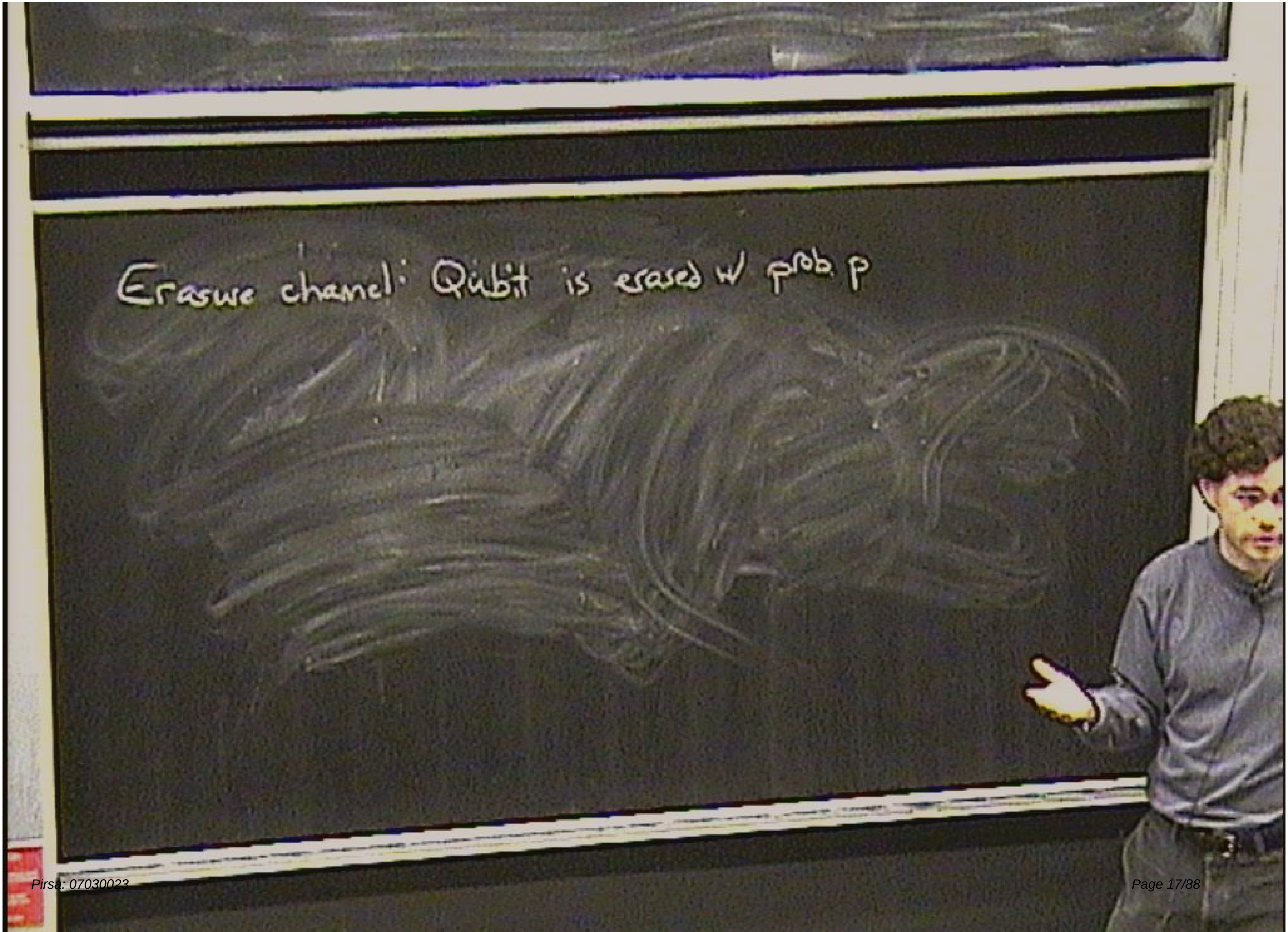
⇒ Given outcome s, A's input ρ

$\rightarrow O_s(\rho)$ (CP map)

Overall operation has high fidelity, averaged over s ⇒ must be some s, call s', for which the fidelity $F(\mathcal{R}(O_{s'}(\rho)), \rho)$ is also near 1.

If fidelity = 1, O_s is unitary encoding. If fidelity is near 1, there is O' s.t. O' is unitary encoding, $F(\mathcal{R}(O'(\rho)), \rho)$ is close to 1.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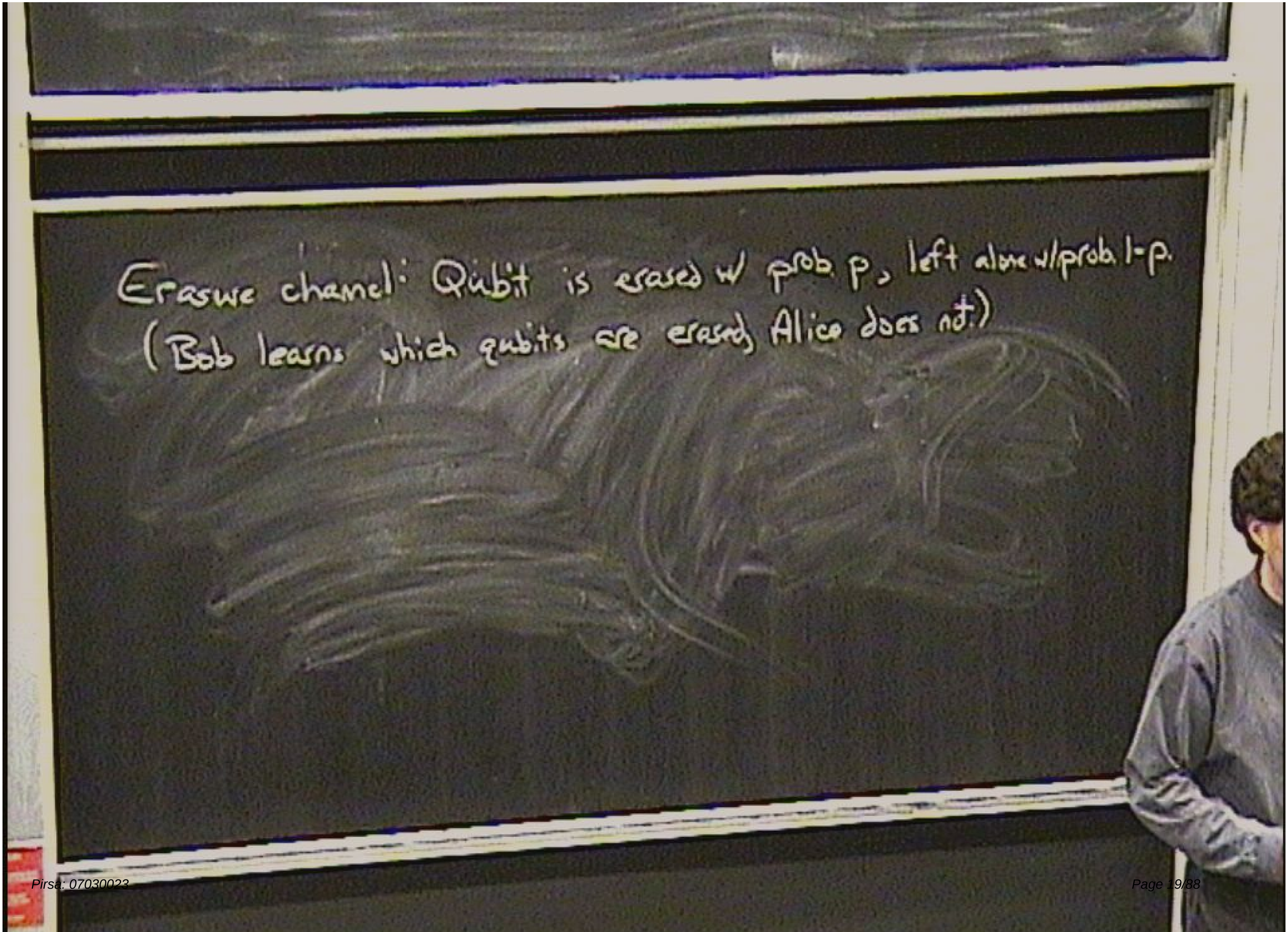
Erasure channel: Qubit is erased w/ prob. p, left alone w/ prob 1-p.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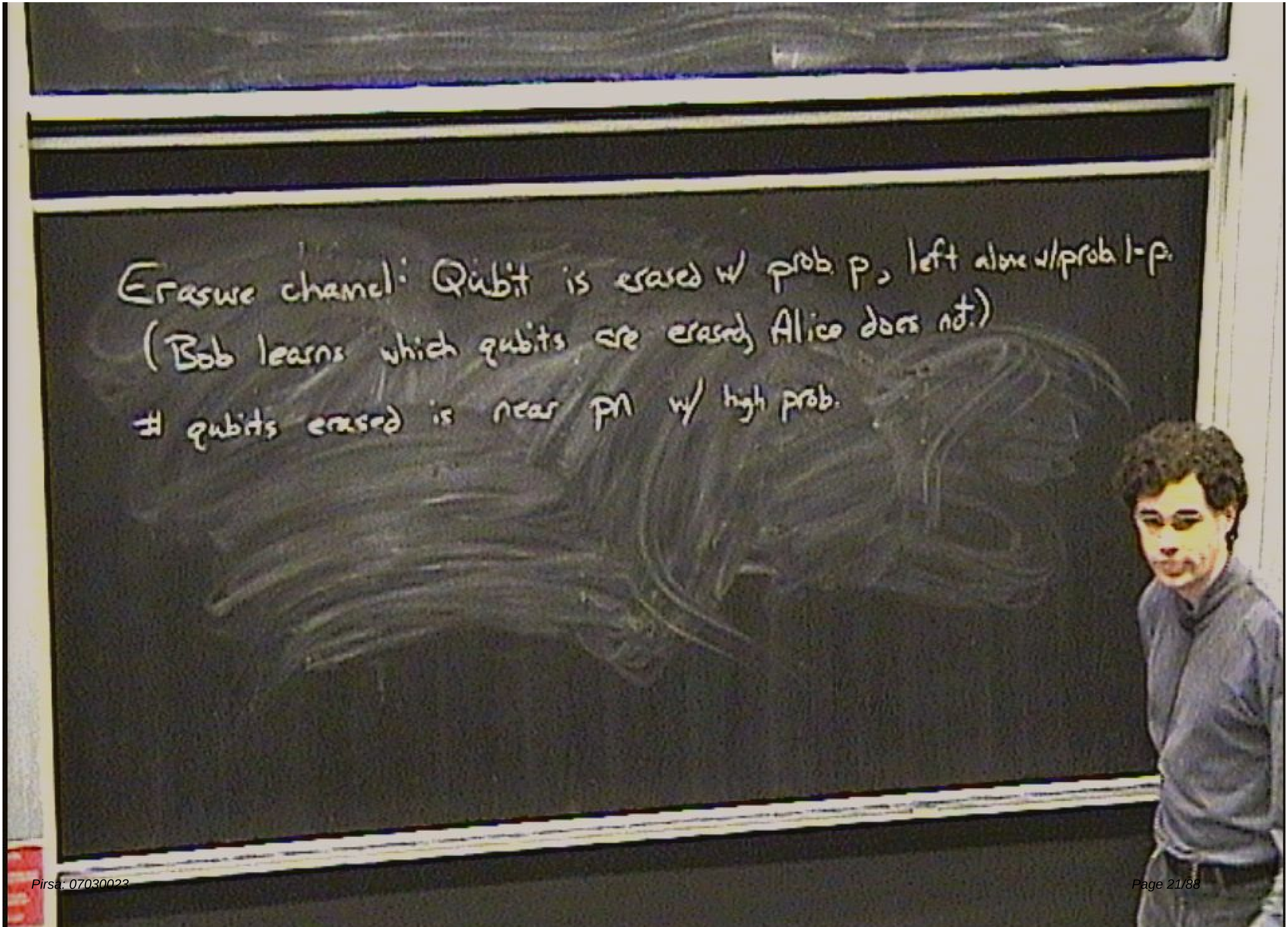
qubits erased is near pn w/ high prob.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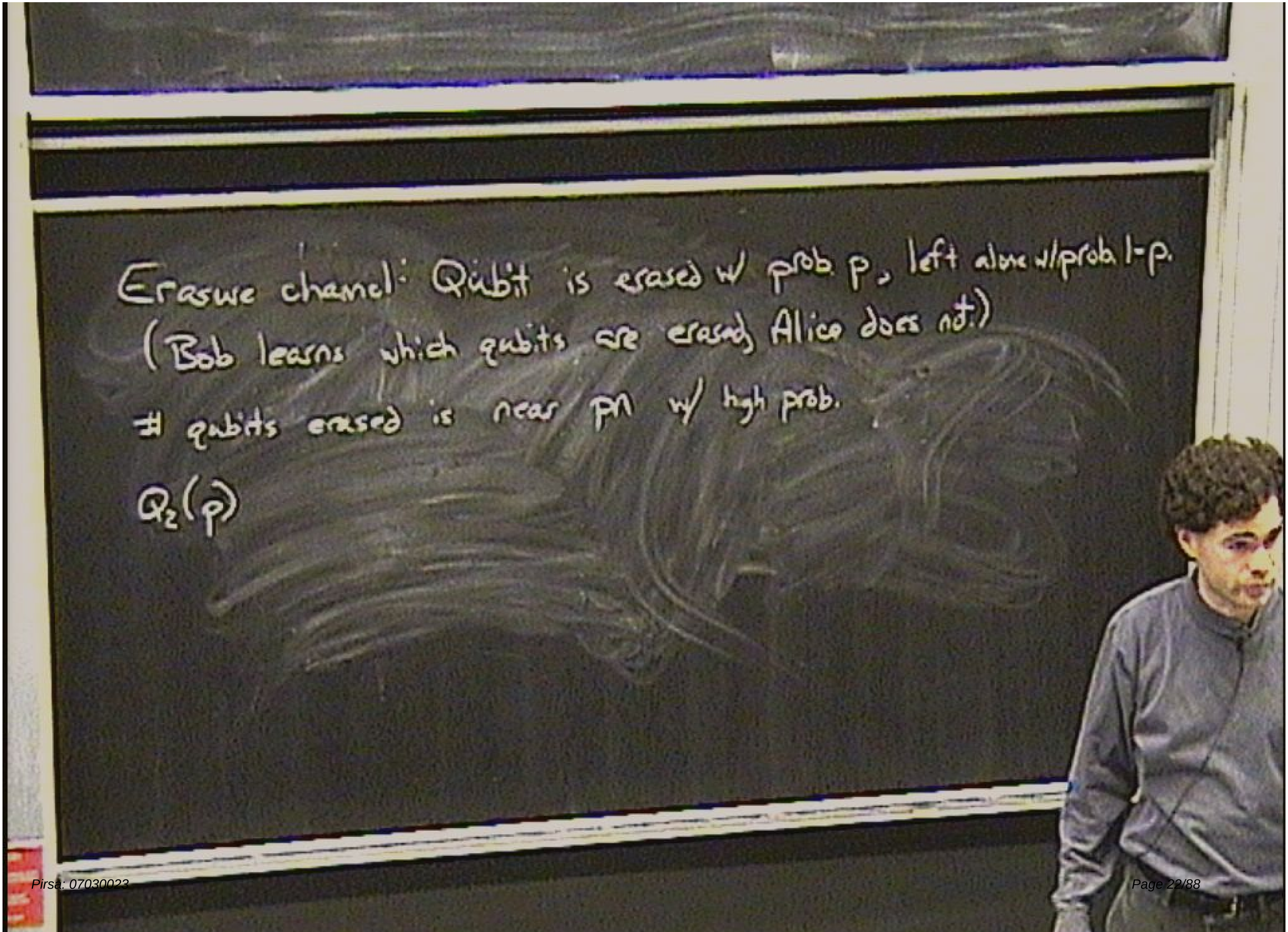
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$Q_2(p)$

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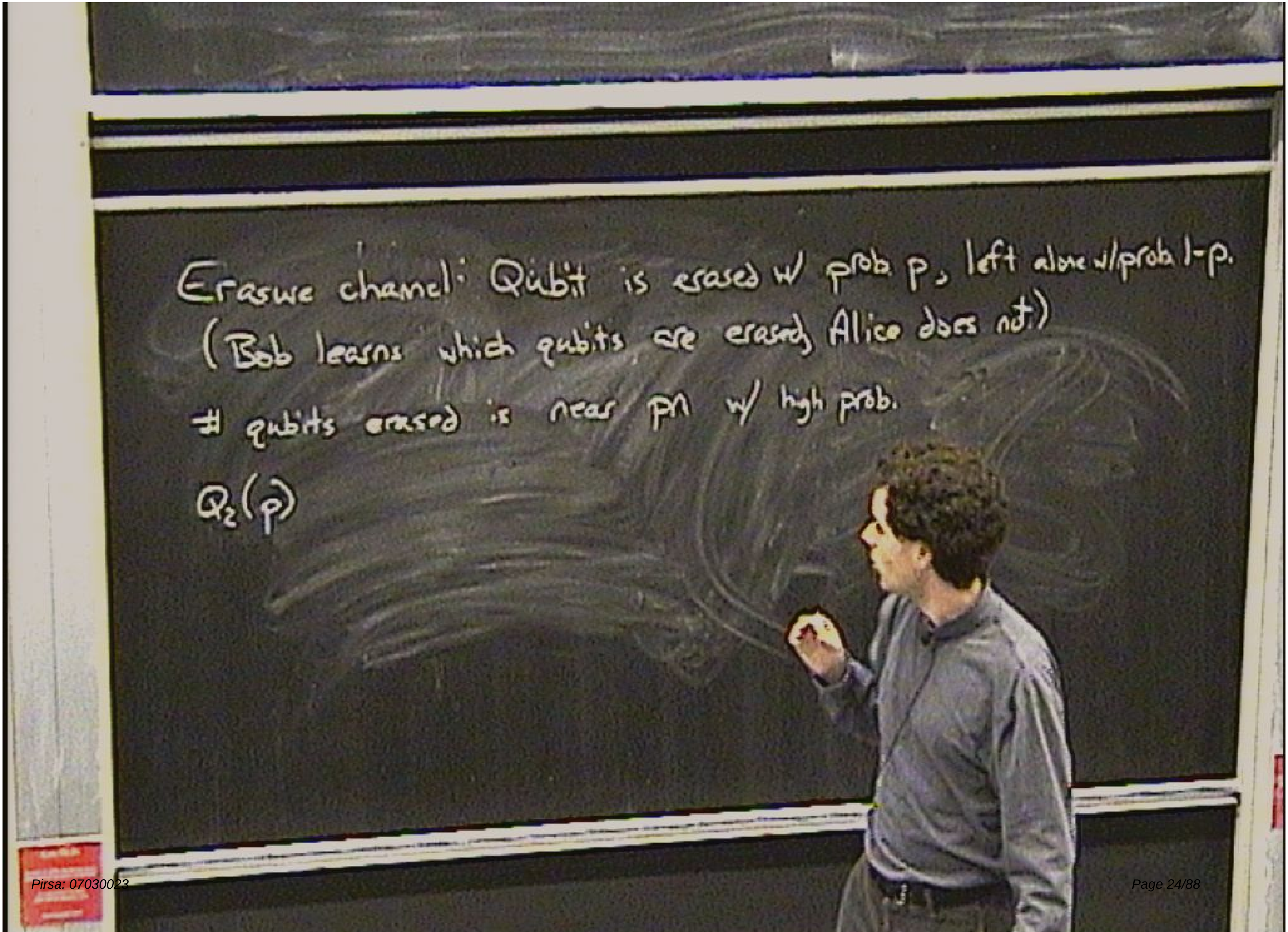
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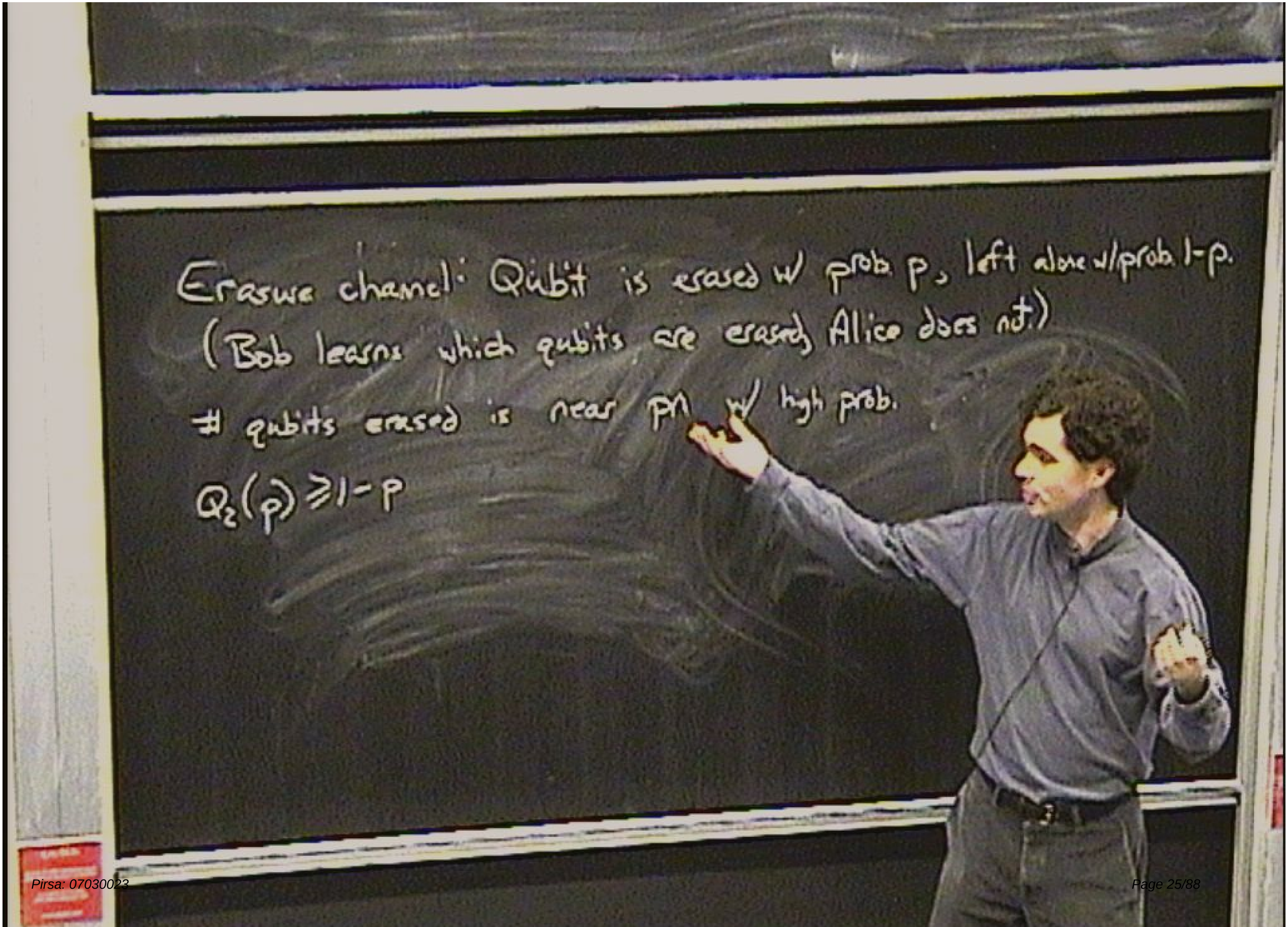
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Any two errors E, F what is the probability E & F have same syndrome?

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Random stabilizer code w/ n generators.

Any two distinct errors $E \neq F$ what is the probability E & F have same syndrome? $= \text{Prob}(E^\dagger F \in N(S))$

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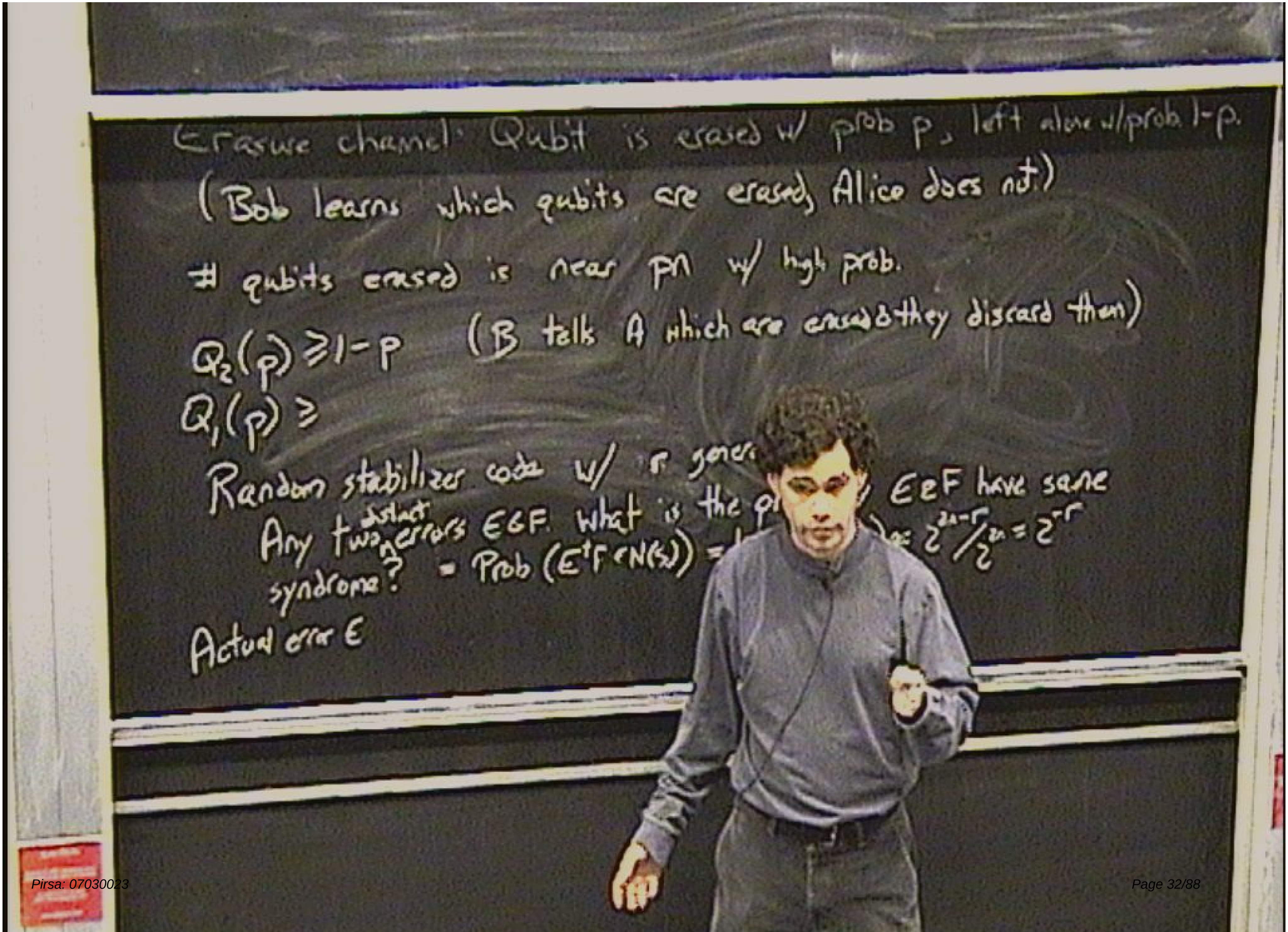

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Q2(p) ≥ 1-p (B tells A which are erased & they discard them)
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Random stabilizer code w/ r gener
E & F have same
Any two distinct errors E&F. what is the
syndrome? = Prob (E†F ∈ N(S)) =
2^{2n-r}/2^{2n} = 2^{-r}
Actual error E

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Random stabilizer code w/ r generators.

Any two distinct errors E, F: what is the probability E & F have same syndrome? $= \mathrm{Prob}(E^\dagger F \in N(S)) = |N(S)| / (|\mathcal{P}_n| - 1) \approx 2^{2n-r} / 2^{2n} = 2^{-r}$

Actual error E – What is prob that there is another F w/ same syndrome?

Erasure channel: Qubit is erased w/ prob p, left alone w/ prob $1-p$.
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Random stabilizer code w/ r generators.

Any two distinct n errors $E \neq F$. What is the probability $E \neq F$ have same syndrome? $= \text{Prob}(E^\dagger F \in N(S)) = |N(S)| / (|\mathcal{P}_n| - 1) \approx 2^{2n-r} / 2^{2n} = 2^{-r}$

Actual error E — What is prob that there is another F w/ same syndrome? $4^{pn} / 2^r < 1$ if $r > 2pn$

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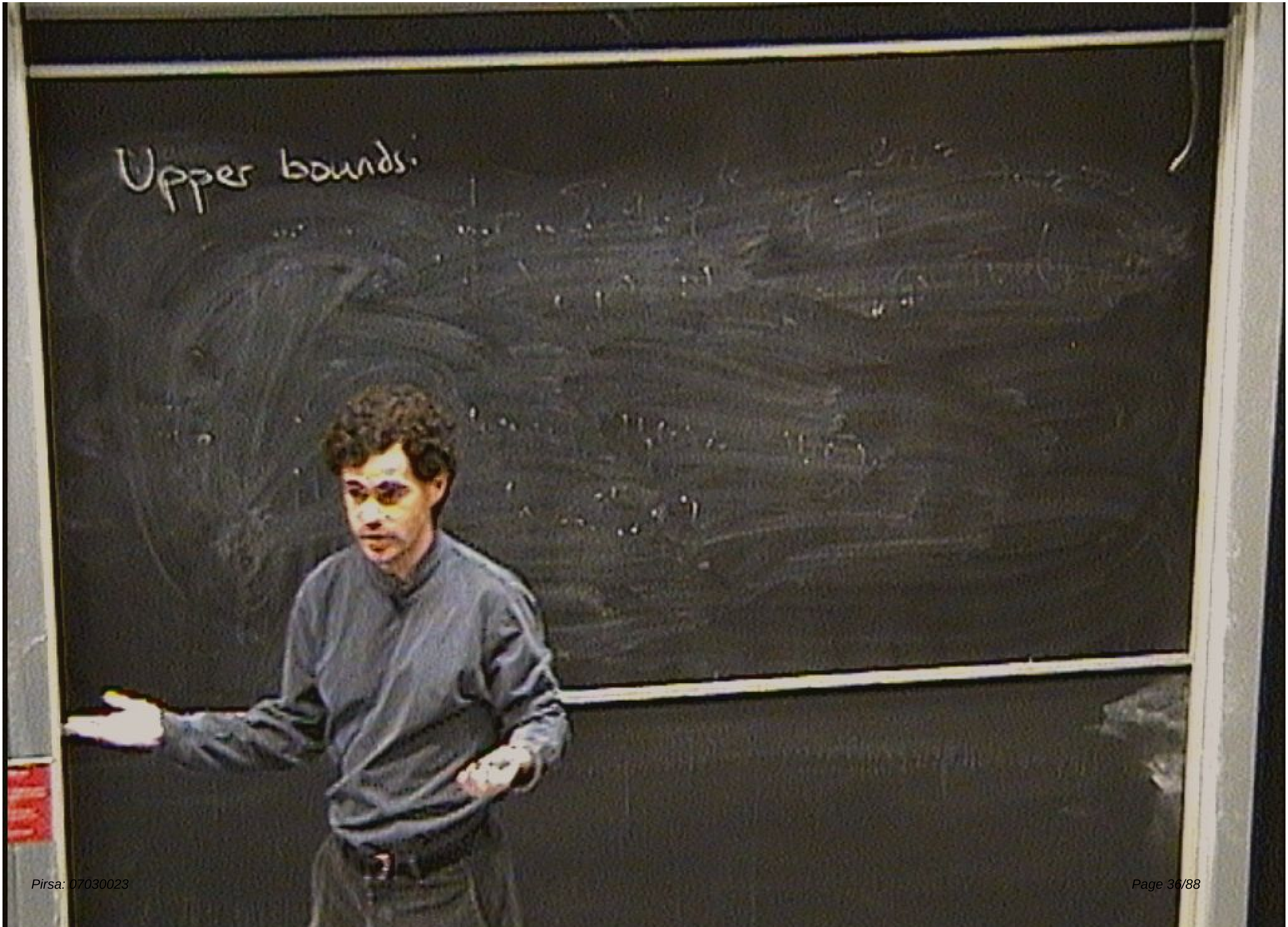
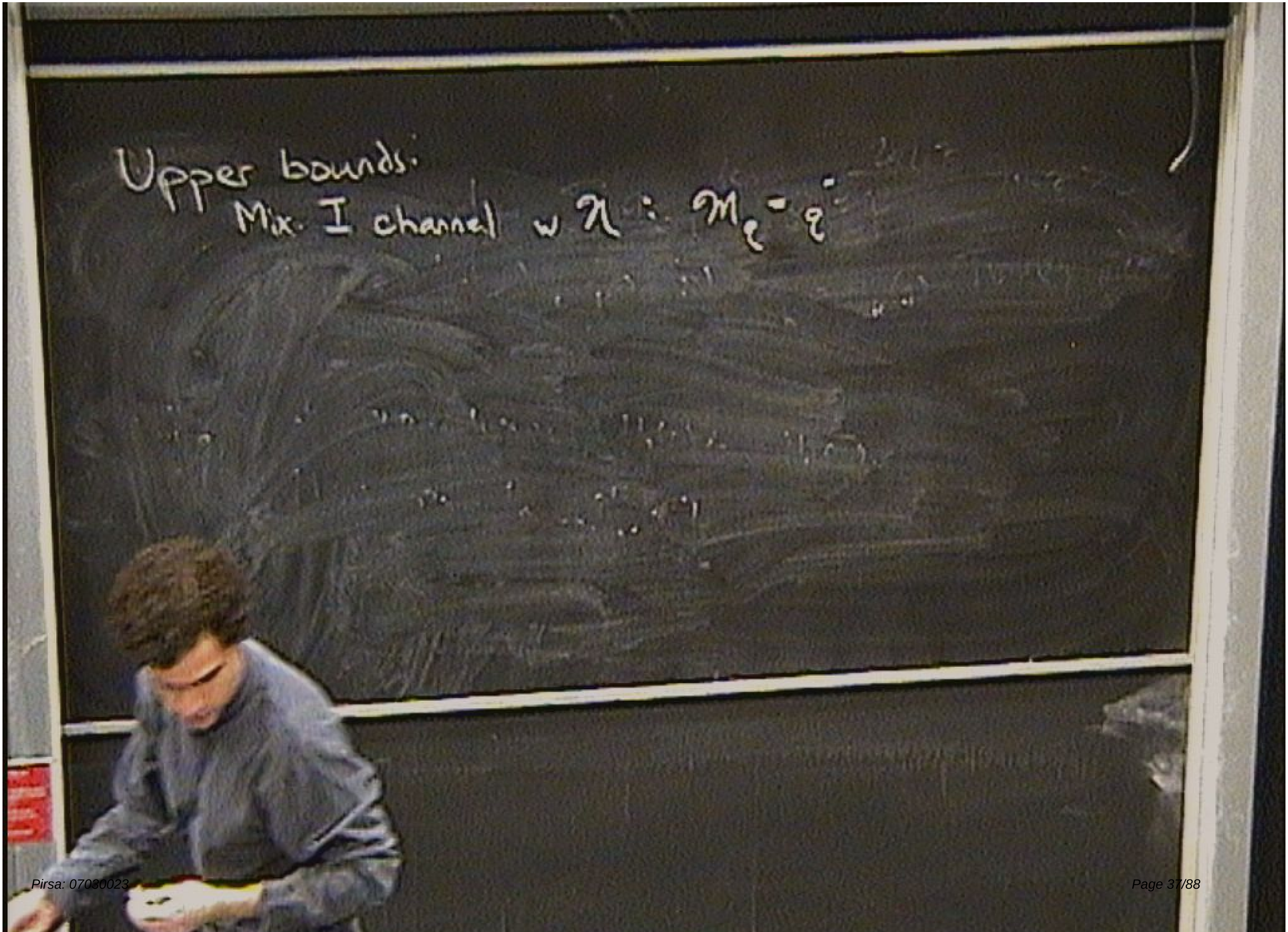
Random stabilizer code w/ r generators.

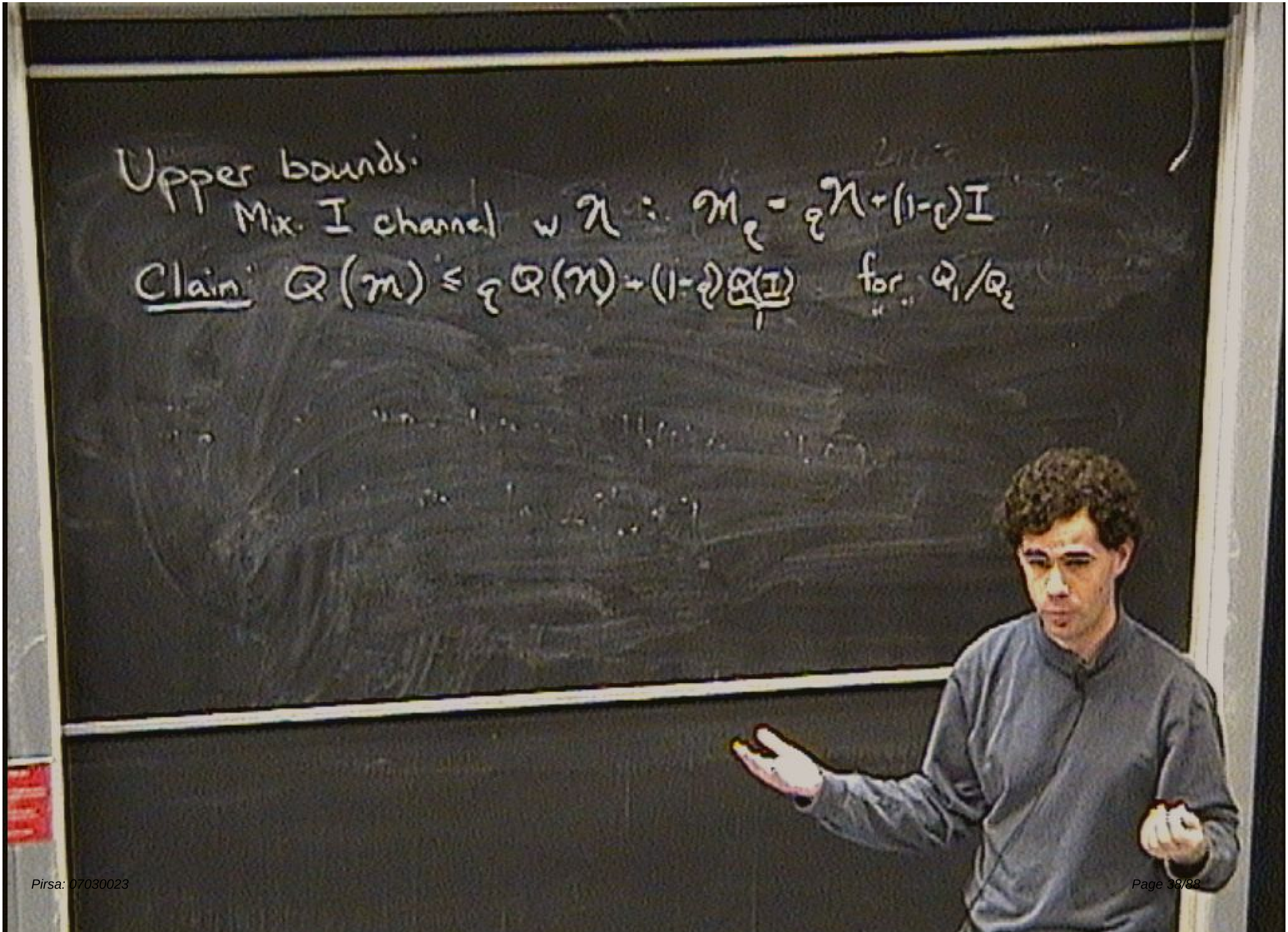
Any two distinct errors E & F what is the probability E & F have same syndrome? $= \text{Prob}(E^{\dagger}F \in N(S)) = |N(S)|/(|P_n|-1) \approx 2^{2n-r}/2^{2n} = 2^{-r}$

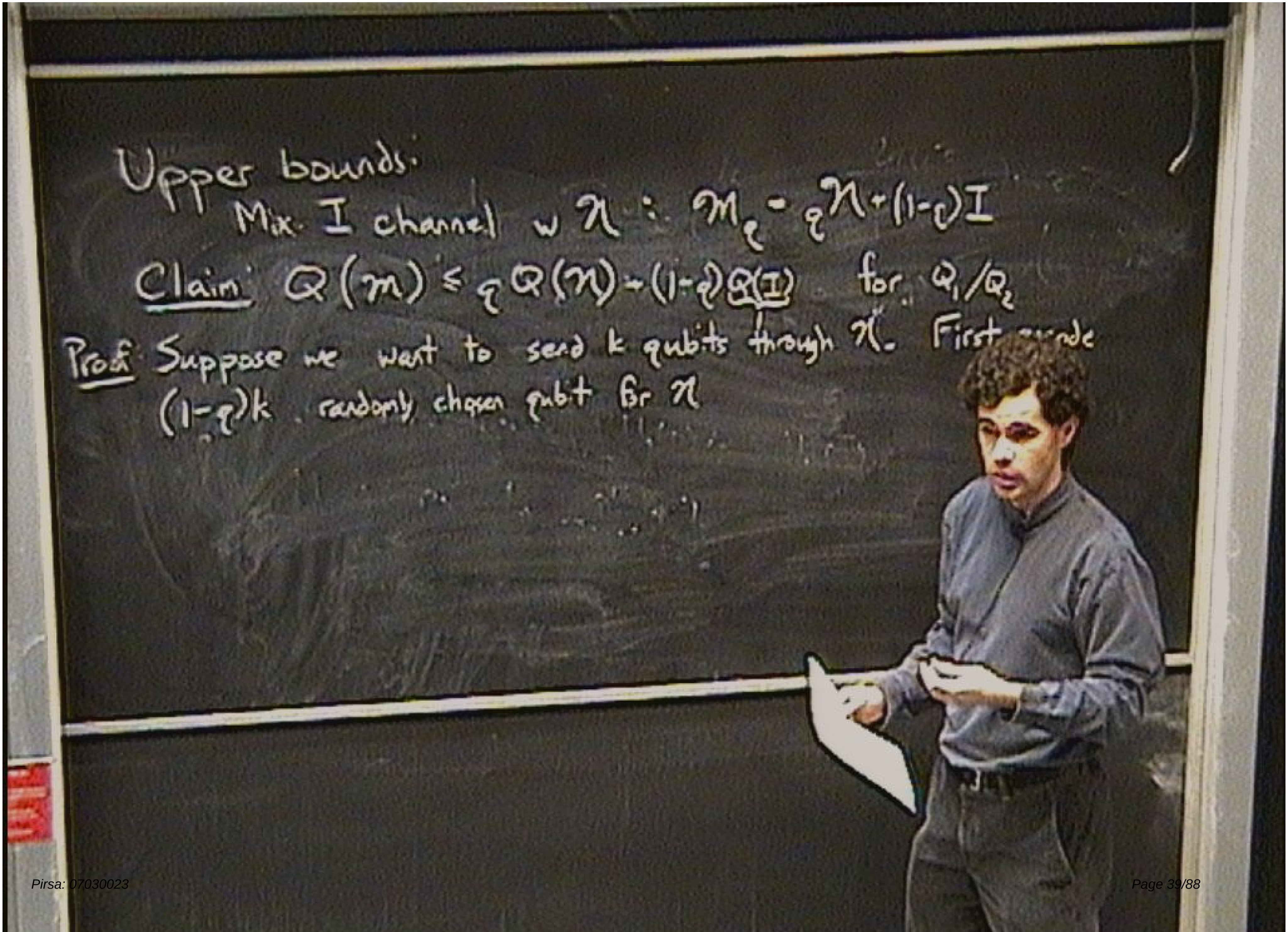
Actual error E – what is prob that there is another F w/ same syndrome? $4^{pn}/2^{r} < 1$ if $r > 2pn$

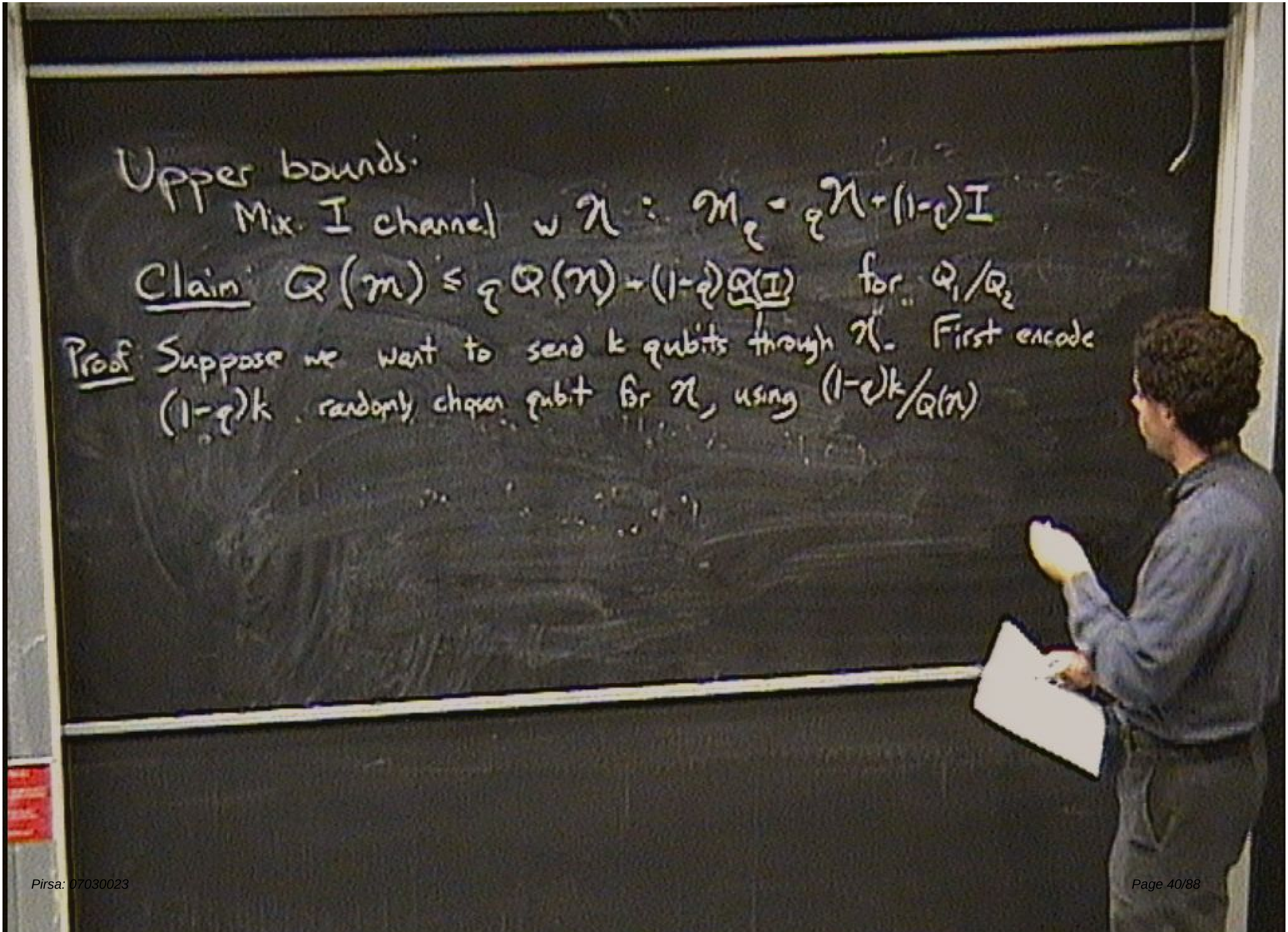
Choose $r \approx 2pn \Rightarrow k = n-2pn \Rightarrow \frac{k}{n} = 1-2p$


Upper bounds:


Upper bounds:
Mix. I channel w


Upper bounds:
Mix. I channel w N : M_p = pN + (1-p)I
Claim: Q(M) ≤ pQ(N) + (1-p)Q(I) for Q_1/Q_2


Upper bounds:
Mix I channel w N : M_q = qN + (1-q)I
Claim: Q(M) ≤ q Q(N) + (1-q) Q(I) for Q_1/Q_2
Proof Suppose we want to send k qubits through N. First
(1-q)k randomly chosen qubit for N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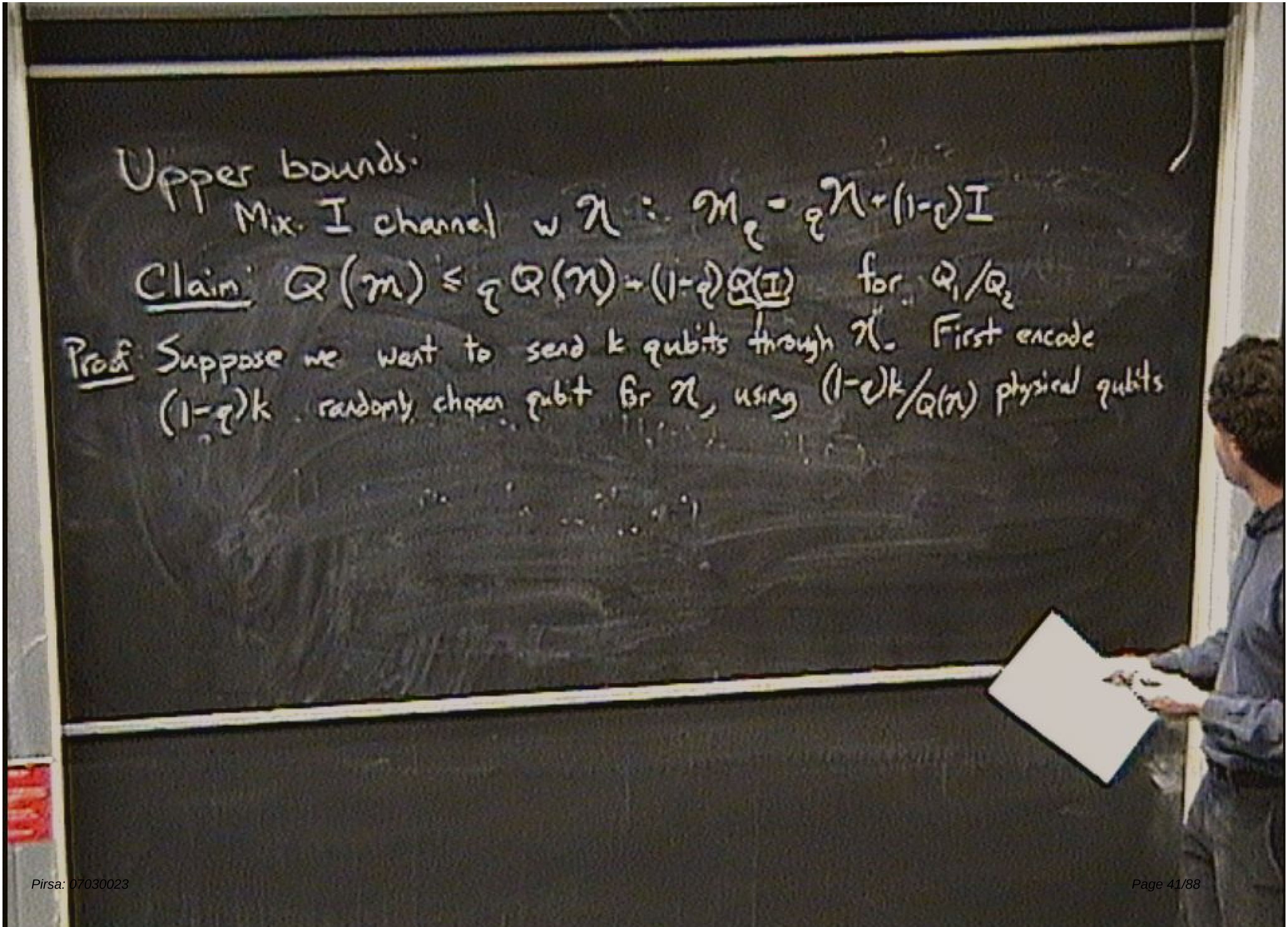
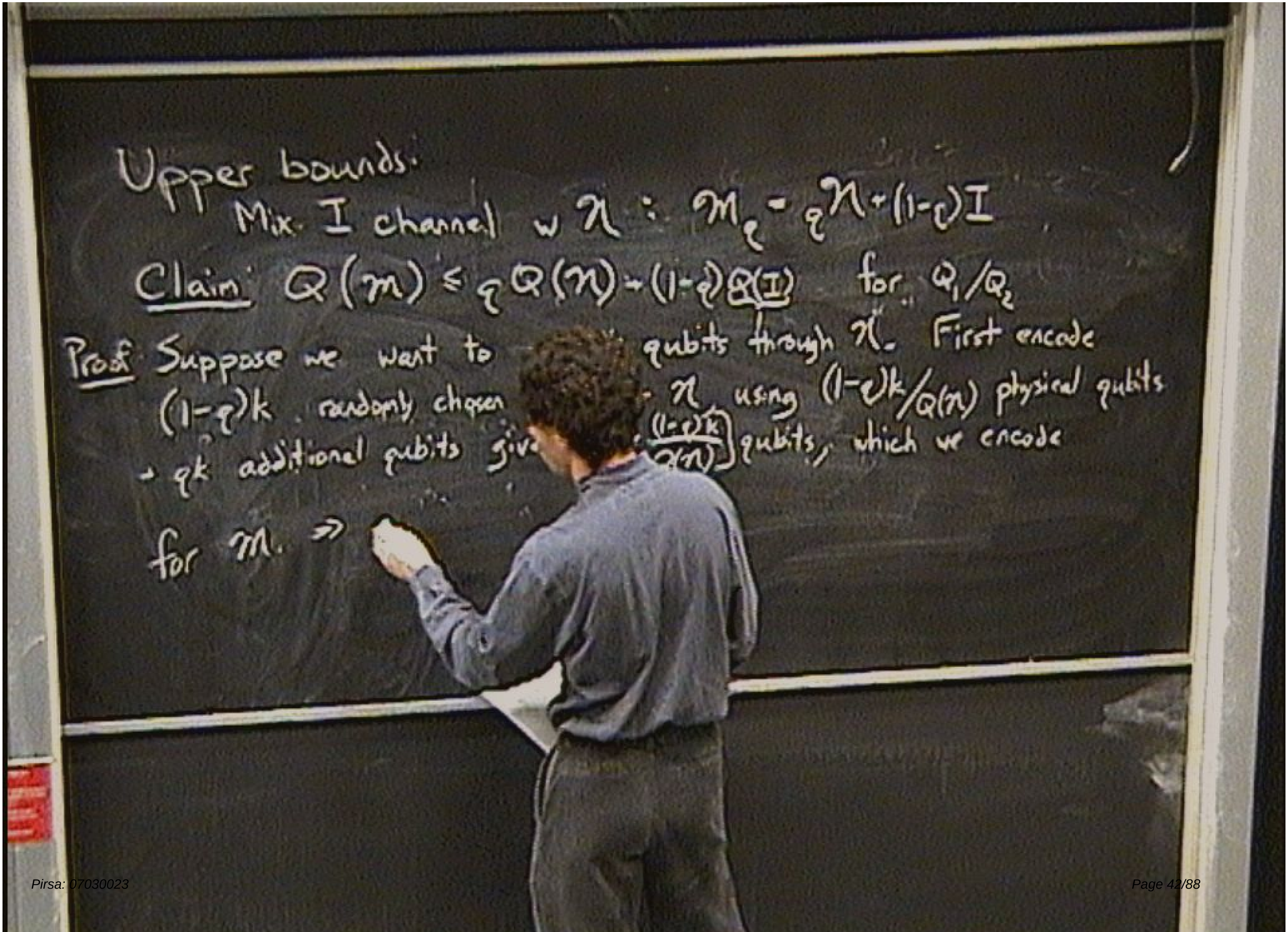
Claim: Q(M) ≤ q Q(N) + (1-q) Q(I) for Q_1/Q_2
Proof: Suppose we want to send k qubits through N. First encode
(1-q)k randomly chosen qubit for N, using (1-q)k/Q(N)


Upper bounds:
Mix. I channel w $\mathcal{N}$: $\mathcal{M}_q = q\mathcal{N} + (1-q)I$
Claim: $Q(\mathcal{M}) \leq q\,Q(\mathcal{N}) + (1-q)\,Q(I)$ for Q_1/Q_2
Proof: Suppose we want to send k qubits through $\mathcal{N}$. First encode
$(1-q)k$ randomly chosen qubit for $\mathcal{N}$, using $(1-q)k/Q(\mathcal{N})$ physical qubits


Upper bounds:
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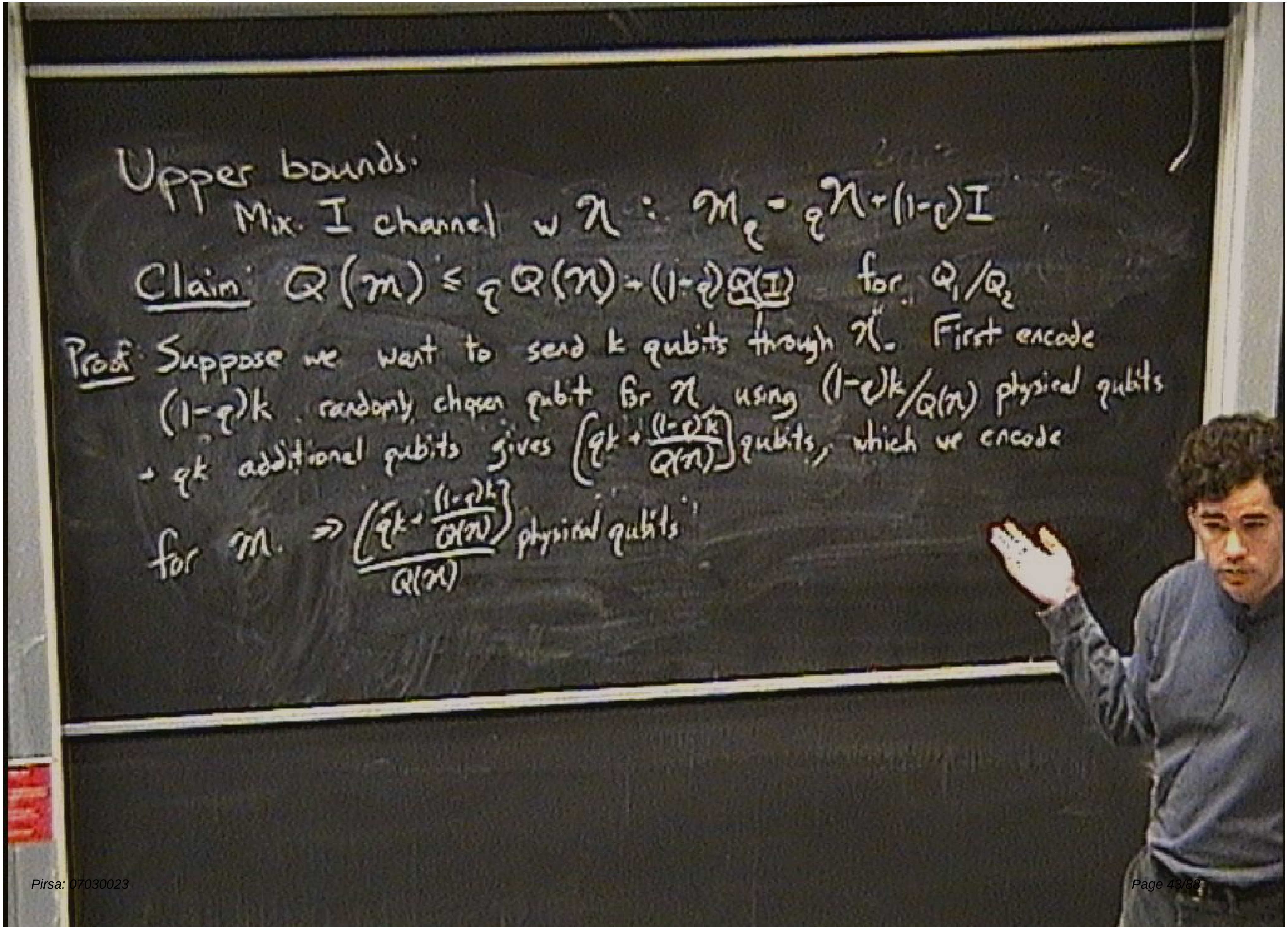
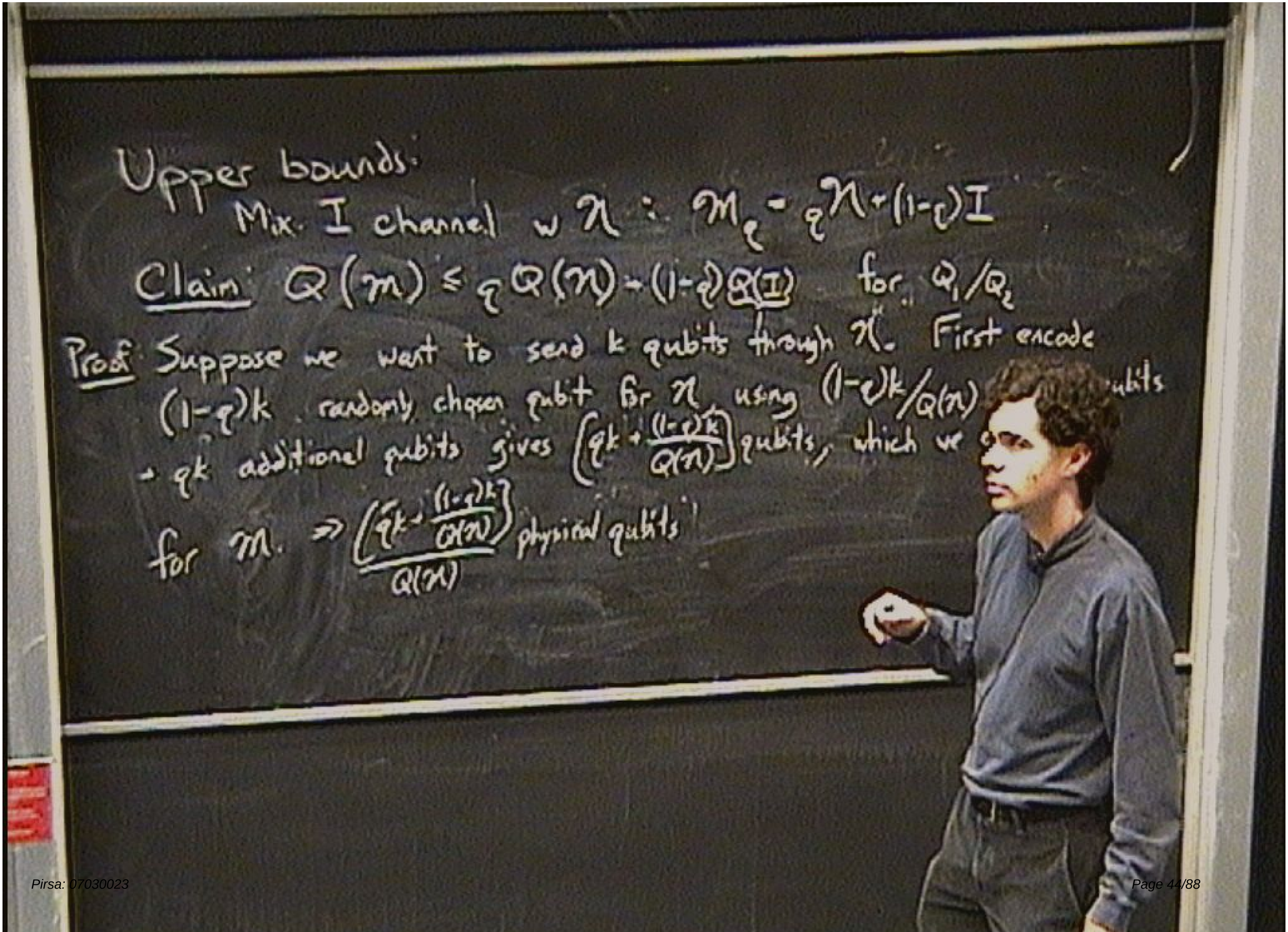
Claim: Q(M) ≤ qQ(N) + (1-q)Q(I) for Q_1/Q_2
Proof Suppose we want to qubits through N. First encode
(1-q)k randomly chosen N using (1-q)k/Q(N) physical qubits
+ qk additional qubits giv (1-q)k/Q(N) qubits, which we encode
for M. ⇒


Upper bounds:
Mix. I channel w N : M_q = qN + (1-q)I
Claim: Q(M) ≤ q Q(N) + (1-q) Q(I) for Q_1/Q_2
Proof: Suppose we want to send k qubits through N. First encode
(1-q)k randomly chosen qubit for N using (1-q)k/Q(N) physical qubits
+ qk additional qubits gives [qk + (1-q)k/Q(N)] qubits, which we encode
for M. ⇒ (qk + (1-q)k/Q(N))/Q(N) physical qubits


Upper bounds:
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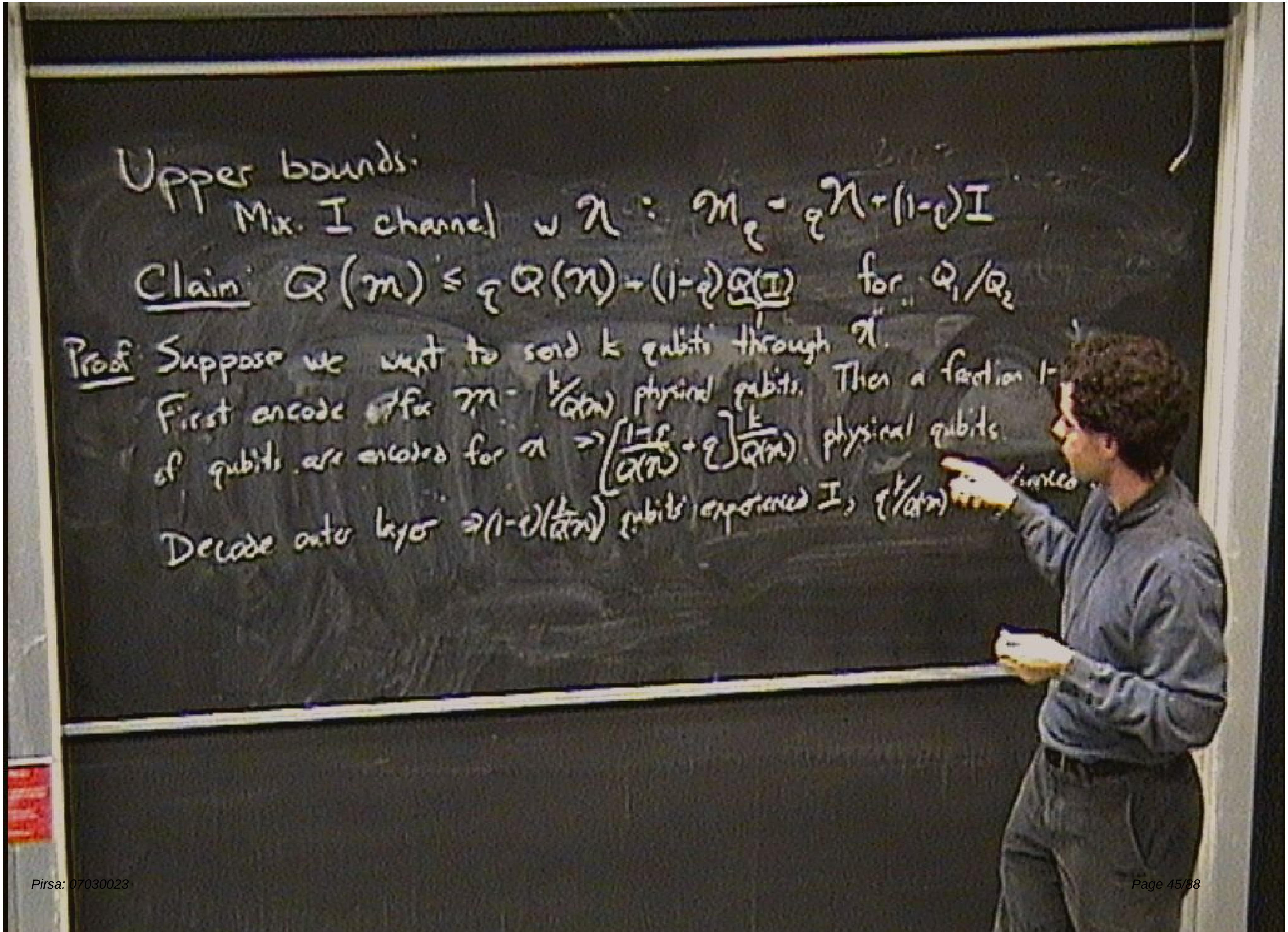
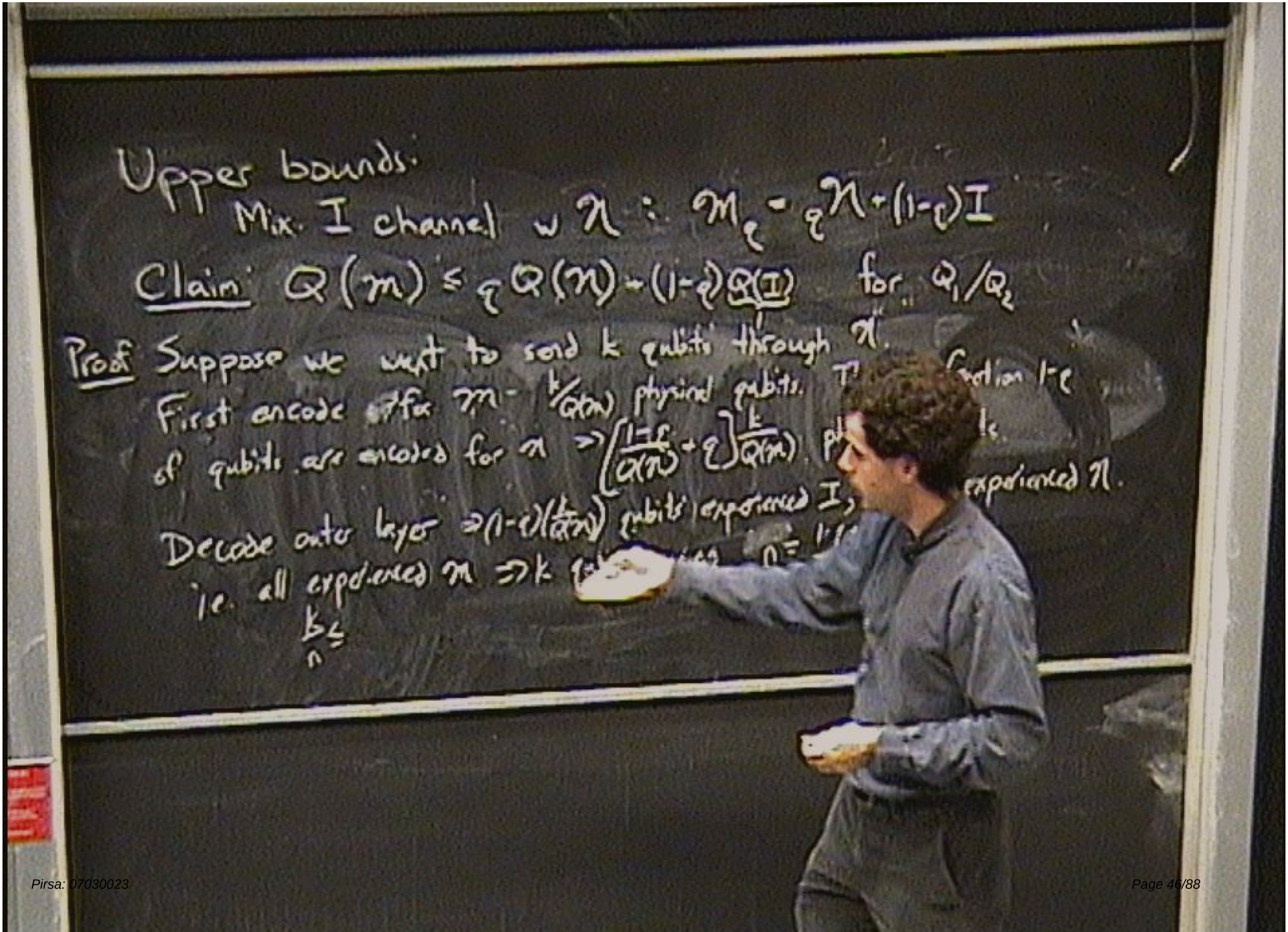
Claim: $Q(\mathcal{M}) \leq q Q(\mathcal{N}) + (1-q) Q(I)$ for Q_1/Q_2
Proof: Suppose we want to send k qubits through $\mathcal{N}$. First encode
$(1-q)k$ randomly chosen qubit for $\mathcal{N}$ using $(1-q)k/Q(\mathcal{N})$
+ qk additional qubits gives $\left(qk + \frac{(1-q)k}{Q(\mathcal{N})}\right)$ qubits, which we
for $\mathcal{M}$. $\Rightarrow \frac{\left(qk + \frac{(1-q)k}{Q(\mathcal{N})}\right)}{Q(\mathcal{N})}$ physical qubits


Upper bounds:
Mix. I channel w N : M_q = qN + (1-q)I
Claim: Q(M) ≤ q Q(N) + (1-q) Q(I) for Q1/Q2
Proof: Suppose we want to send k qubits through N.
First encode for M - k/Q(M) physical qubits. Then a fraction 1-
of qubits are encoded for N ⇒ [(1-q)/Q(N) + q] k/Q(M) physical qubits
Decode outer layer ⇒ (1-q)(k/Q(M)) qubits experienced I, q k/Q(M)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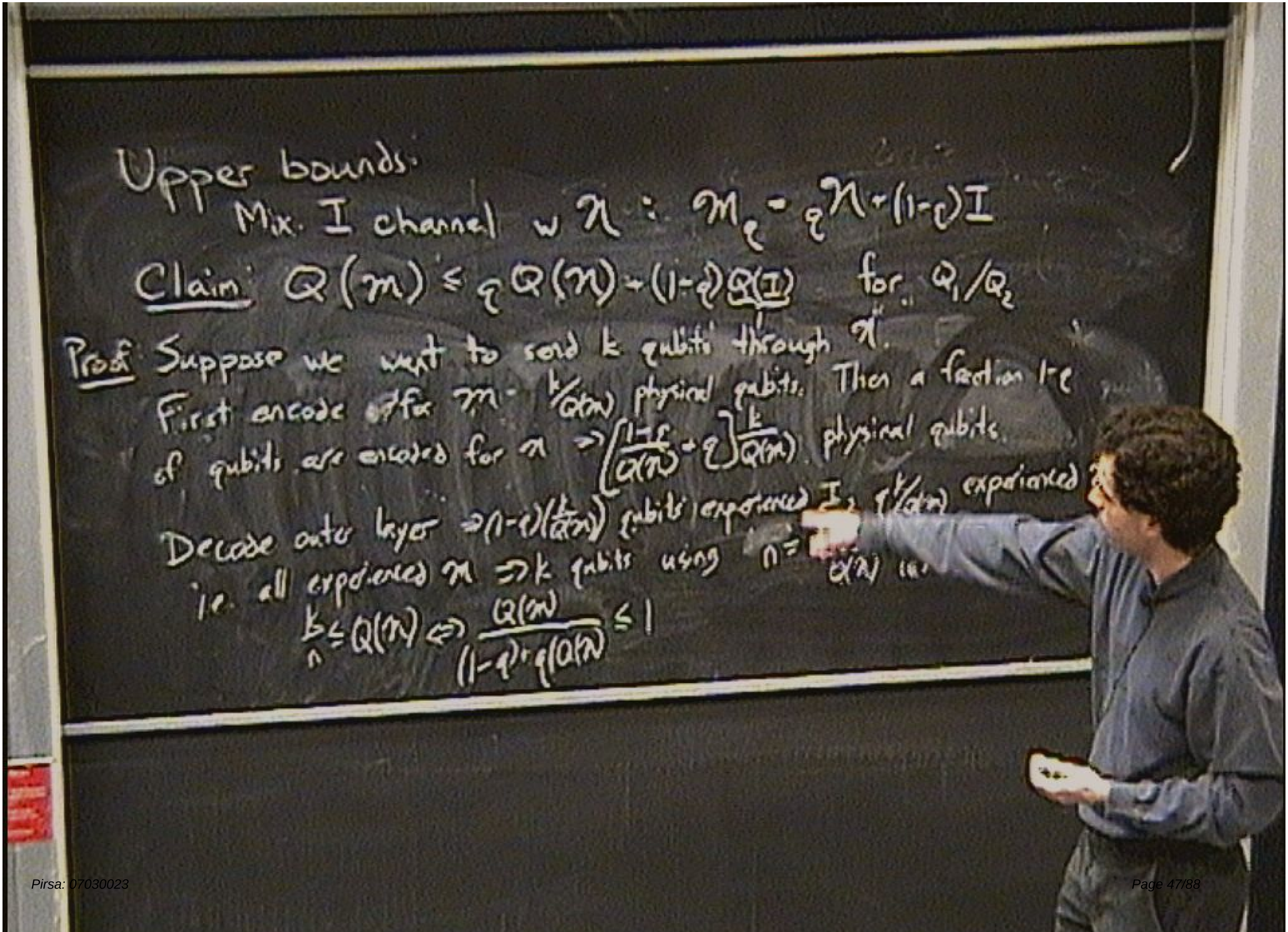
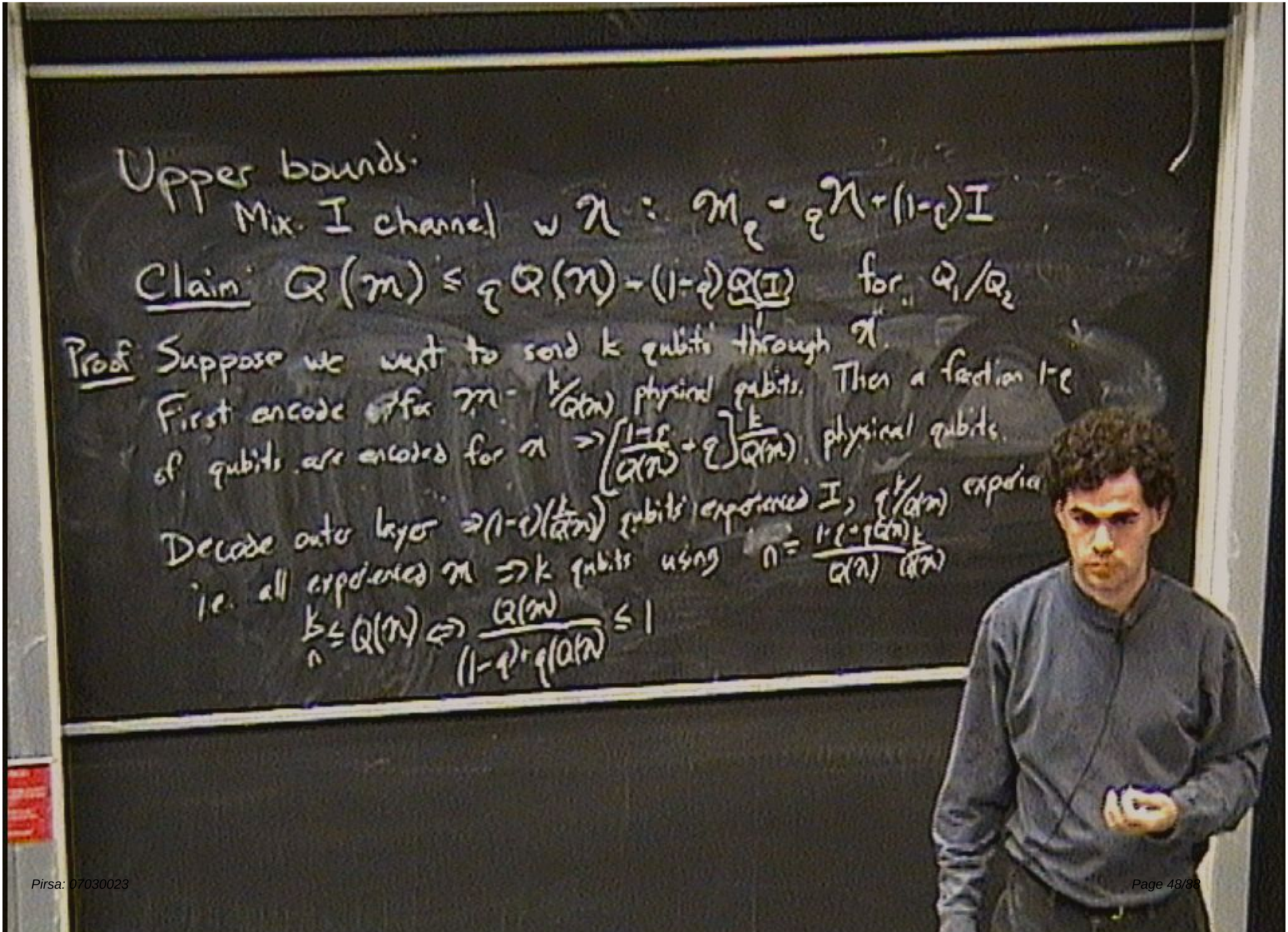
qubits experienced I
i.e. all experienced M ⇒ k


Upper bounds:
Mix. I channel w N : M_q = qN + (1-q)I
Claim: Q(M) ≤ qQ(N) + (1-q)Q(I) for Q_1/Q_2
Proof Suppose we want to send k qubits through N.
First encode for M: k/Q(M) physical qubits. Then a fraction 1-q
of qubits are encoded for N ⇒ [(1-q)/Q(N) + q] k/Q(M) physical qubits.
Decode outer layer ⇒ (1-q)(k/Q(M)) qubits experienced I, q k/Q(M) experienced
i.e. all experienced M ⇒ k qubits using
k/n ≤ Q(N) ⟺ Q(N) / ((1-q) + qQ(N)) ≤ 1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Decode outer layer ⇒ (1-q)(k/Q(M)) qubits experienced I, q k/Q(M) experienced
i.e. all experienced M ⇒ k qubits using n = (1-q+qQ(M))/Q(N) · k/Q(M)
k/n ≤ Q(N) ⇔ Q(N) / ((1-q) + qQ(M)) ≤ 1

Upper bounds:

Mix. I channel w $\mathcal{N}$: $\mathcal{M}_q = q\mathcal{N} + (1-q)I$

<u>Claim</u>: $Q(\mathcal{M}) \leq q\,Q(\mathcal{N}) + (1-q)\,Q(I)$ for Q_1/Q_2

<u>Proof</u>: Suppose we want to send k qubits through $\mathcal{N}$.

First encode for $\mathcal{M}$ - $k/Q(\mathcal{M})$ physical qubits. Then a fraction $1-q$ of qubits are encoded for $\mathcal{N}$ $\Rightarrow \left(\frac{1-q}{Q(\mathcal{N})} + q\right)\frac{k}{Q(\mathcal{M})}$ physical qubits.

Decode outer layer $\Rightarrow (1-q)\left(\frac{k}{Q(\mathcal{M})}\right)$ qubits experienced I, $q\,k/Q(\mathcal{M})$ experienced $\mathcal{N}$.

i.e. all experienced $\mathcal{M}$ $\Rightarrow k$ qubits using $n = \frac{1-(1-q)Q(\mathcal{M})}{Q(\mathcal{N})}\frac{k}{Q(\mathcal{M})}$

$$\frac{k}{n} \leq Q(\mathcal{N}) \Leftrightarrow \frac{Q(\mathcal{M})}{(1-q) + q\,Q(\mathcal{N})} \leq 1$$

Upper bounds:

Mix. I channel w $\mathcal{N}$: $\mathcal{M}_q = q\mathcal{N} + (1-q)I$

<u>Claim</u>: $Q(\mathcal{M}) \leq q\,Q(\mathcal{N}) + (1-q)\,Q(I)$ for Q_1/Q_2 (if $Q(\mathcal{M})>0$)

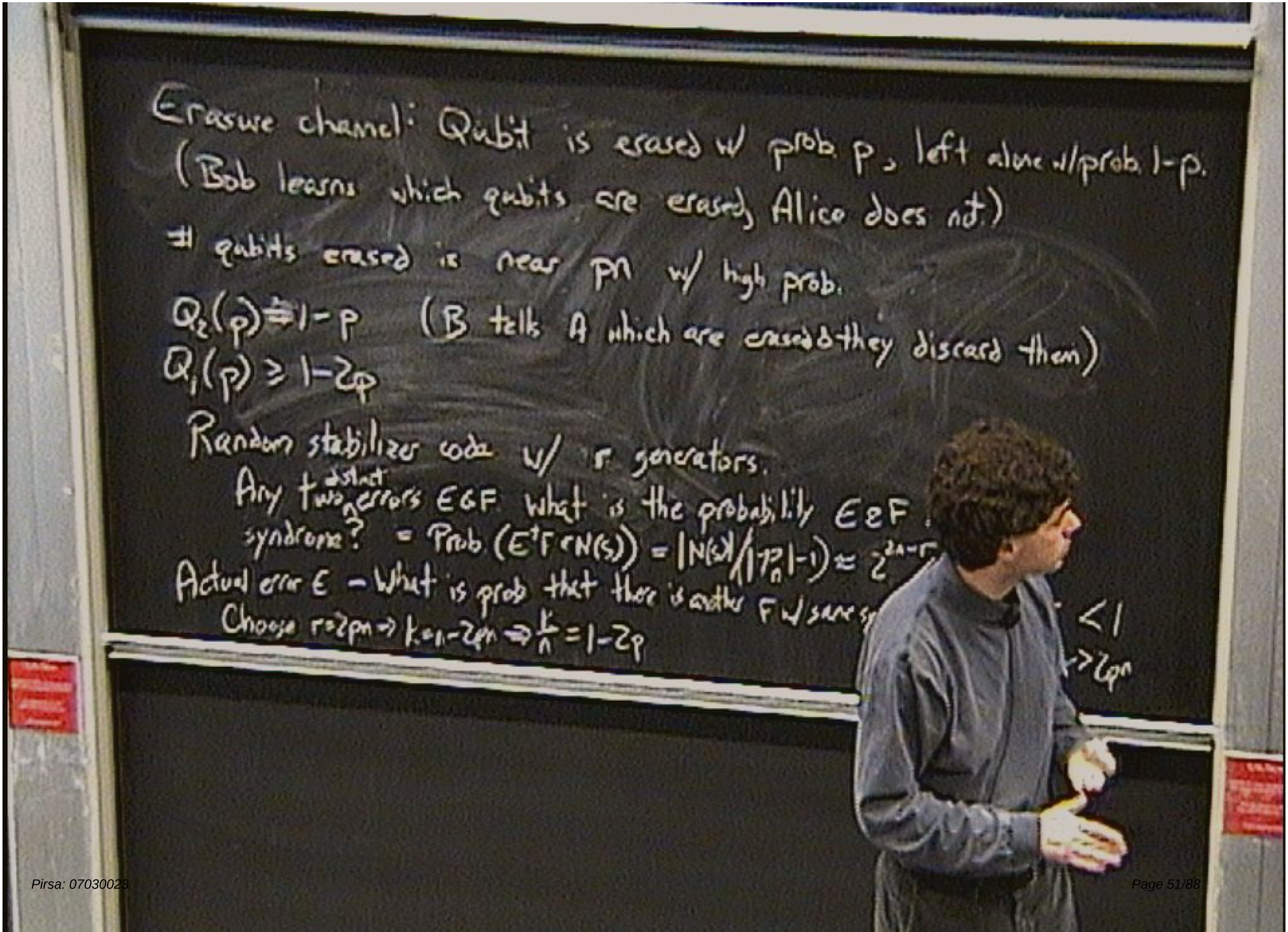
<u>Proof</u> Suppose we want to send k qubits through $\mathcal{N}$.

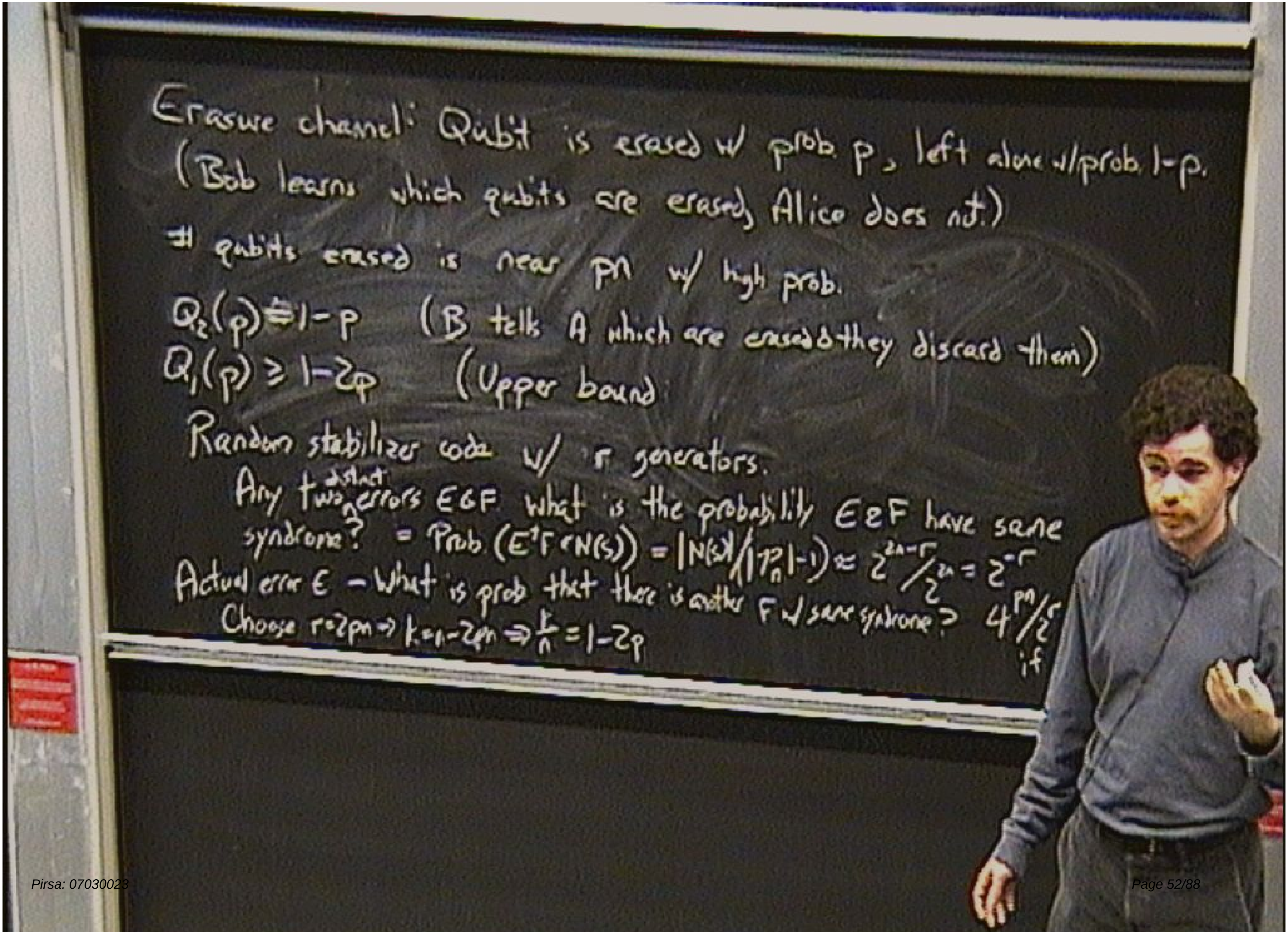
First encode for $\mathcal{M}$ - $k/Q(\mathcal{M})$ physical qubits. Then a fraction $1-q$ of qubits are encoded for $\mathcal{N}$ $\Rightarrow \left[\frac{1-q}{Q(\mathcal{N})} + q\right]\frac{k}{Q(\mathcal{M})}$ physical qubits.

Decode outer layer $\Rightarrow (1-q)\left(\frac{k}{Q(\mathcal{M})}\right)$ qubits experienced I, $q\,k/Q(\mathcal{M})$ experienced $\mathcal{N}$.

i.e. all experienced $\mathcal{N}$ $\Rightarrow k$ qubits using $n = \frac{1-q+q\,Q(\mathcal{N})}{Q(\mathcal{N})}\frac{k}{Q(\mathcal{M})}$

$$\frac{k}{n} \leq Q(\mathcal{N}) \Leftrightarrow \frac{Q(\mathcal{M})}{(1-q)+q\,Q(\mathcal{N})} \leq 1$$


Erasure channel: Qubit is erased w/ prob. p, left alone w/prob. 1-p.
(Bob learns which qubits are erased, Alice does not.)
# qubits erased is near pn w/ high prob.
Q2(p) = 1-p (B tells A which are erased & they discard them)
Q1(p) ≥ 1-2p
Random stabilizer code w/ r generators.
Any two distinct errors E&F what is the probability E&F
syndrome? = Prob.(E†F ∈ N(S)) = |N(S)|/(|Pn|-1) ≈
Actual error E - What is prob that there is another F w/ same
Choose r≥2pn ⇒ k=n-2pn ⇒ k/n = 1-2p
< 1


Erasure channel: Qubit is erased w/ prob. p, left alone w/prob. 1-p.
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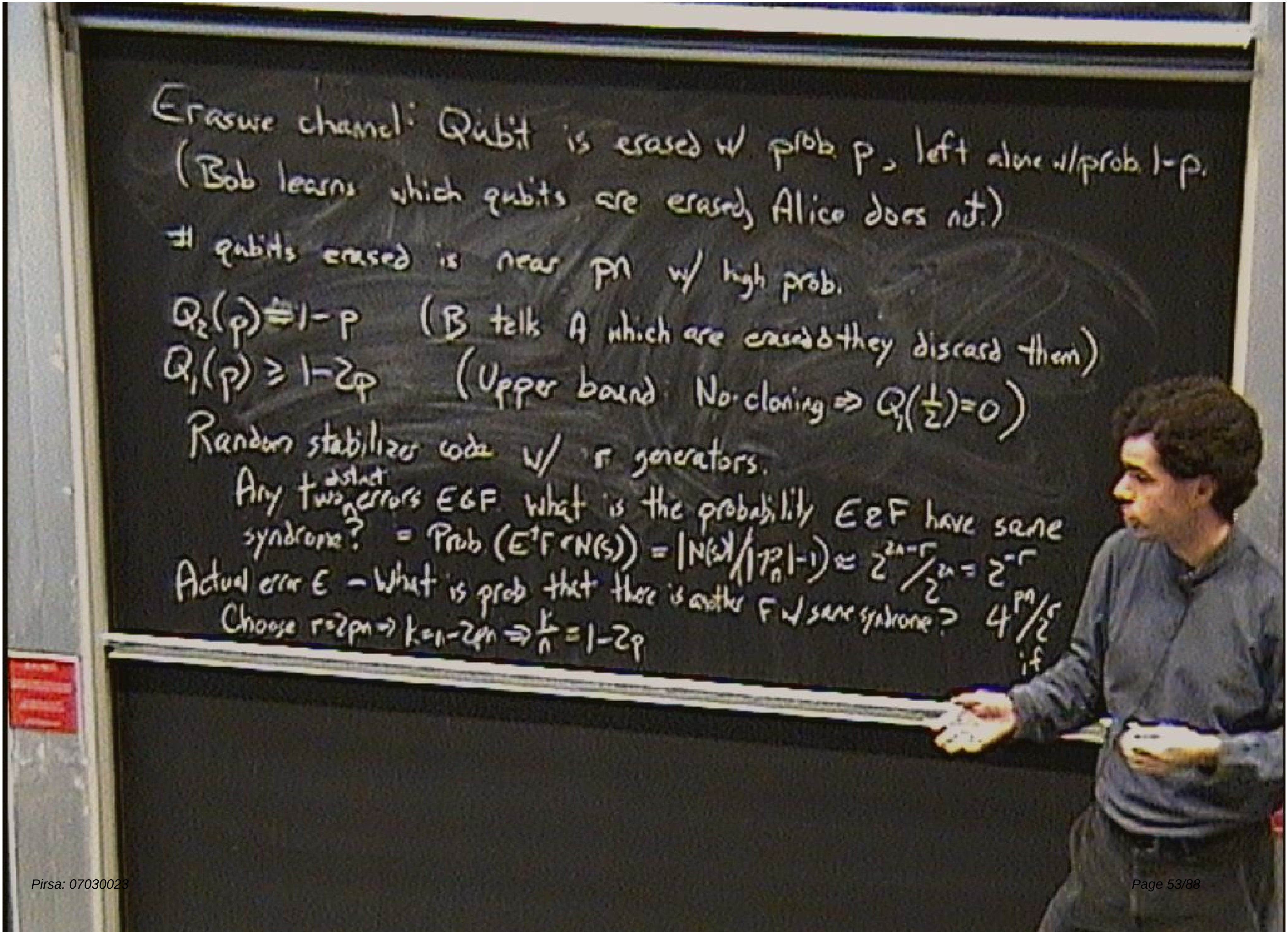
Q2(p) = 1-p (B tells A which are erased & they discard them)
Q1(p) ≥ 1-2p (Upper bound
Random stabilizer code w/ r generators.
Any two distinct errors E≠F. what is the probability E&F have same
syndrome? = Prob. (E†F ∈ N(S)) = |N(S)|/(|Pn|-1) ≈ 2^2n-r / 2^2n = 2^-r
Actual error E – What is prob that there is another F w/ same syndrome? 4^pn / 2^r
Choose r=2pn ⇒ k=n-2pn ⇒ k/n = 1-2p
if


Erasure channel: Qubit is erased w/ prob. p, left alone w/ prob. 1-p.
(Bob learns which qubits are erased, Alice does not.)
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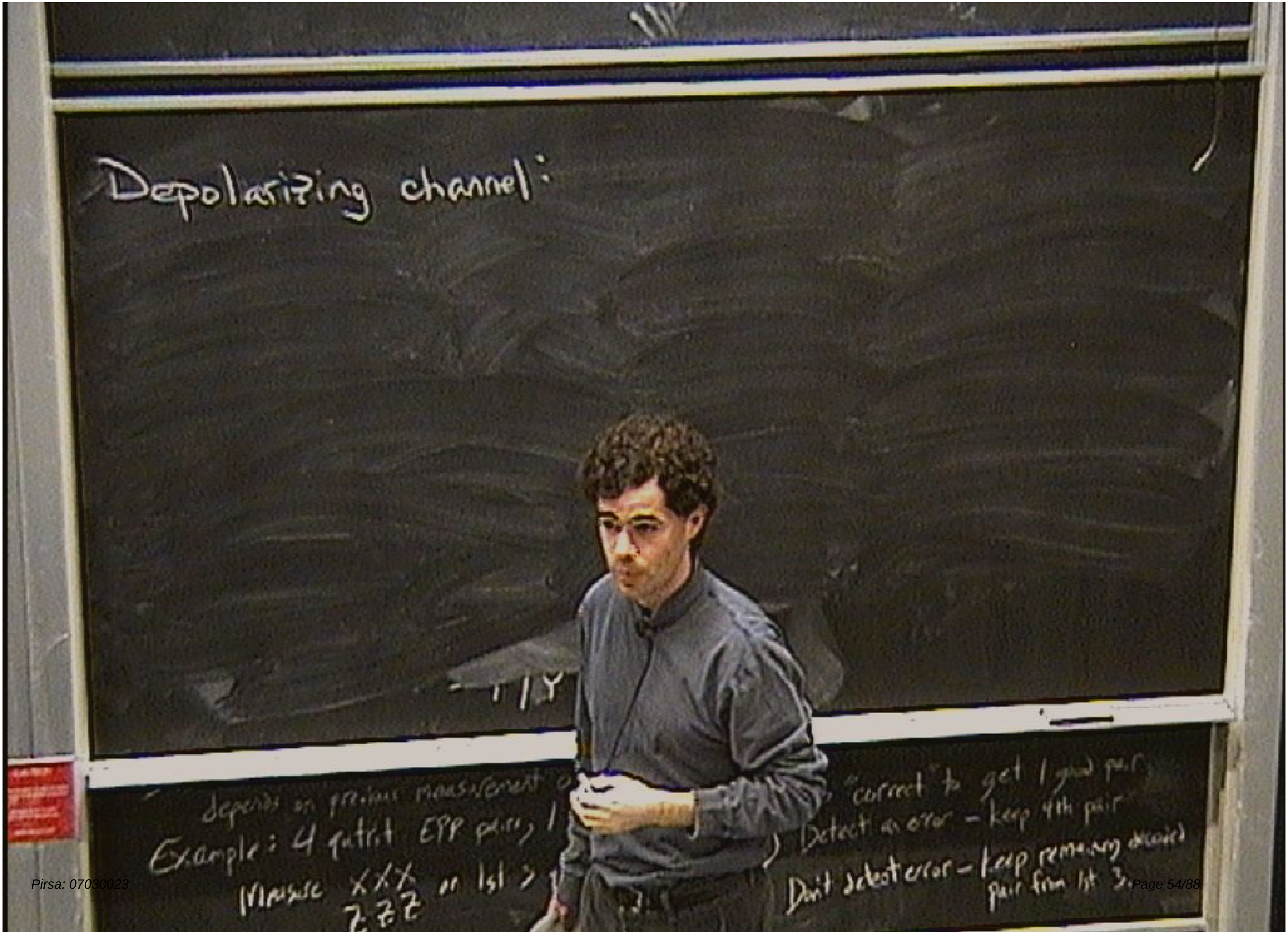
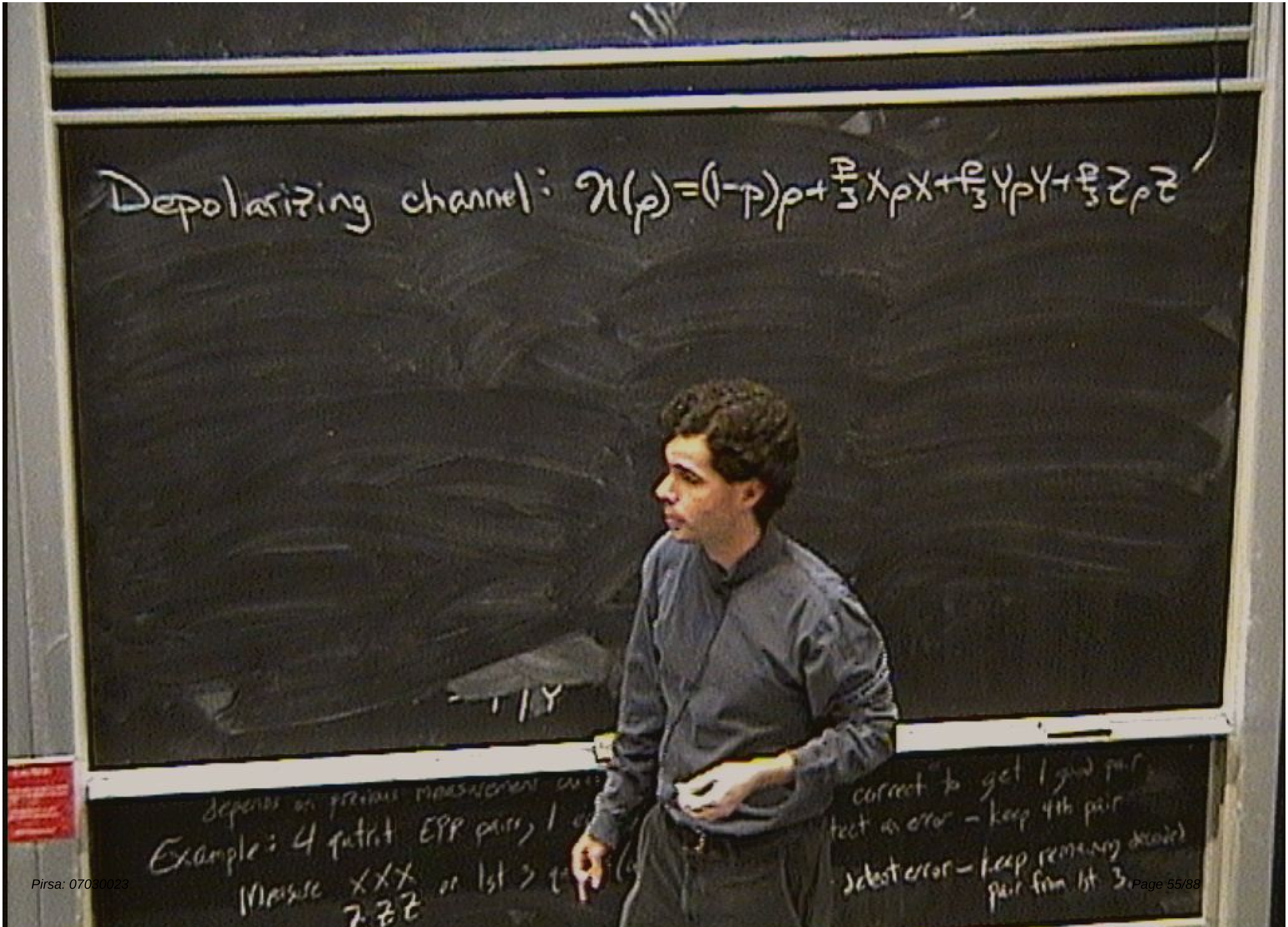
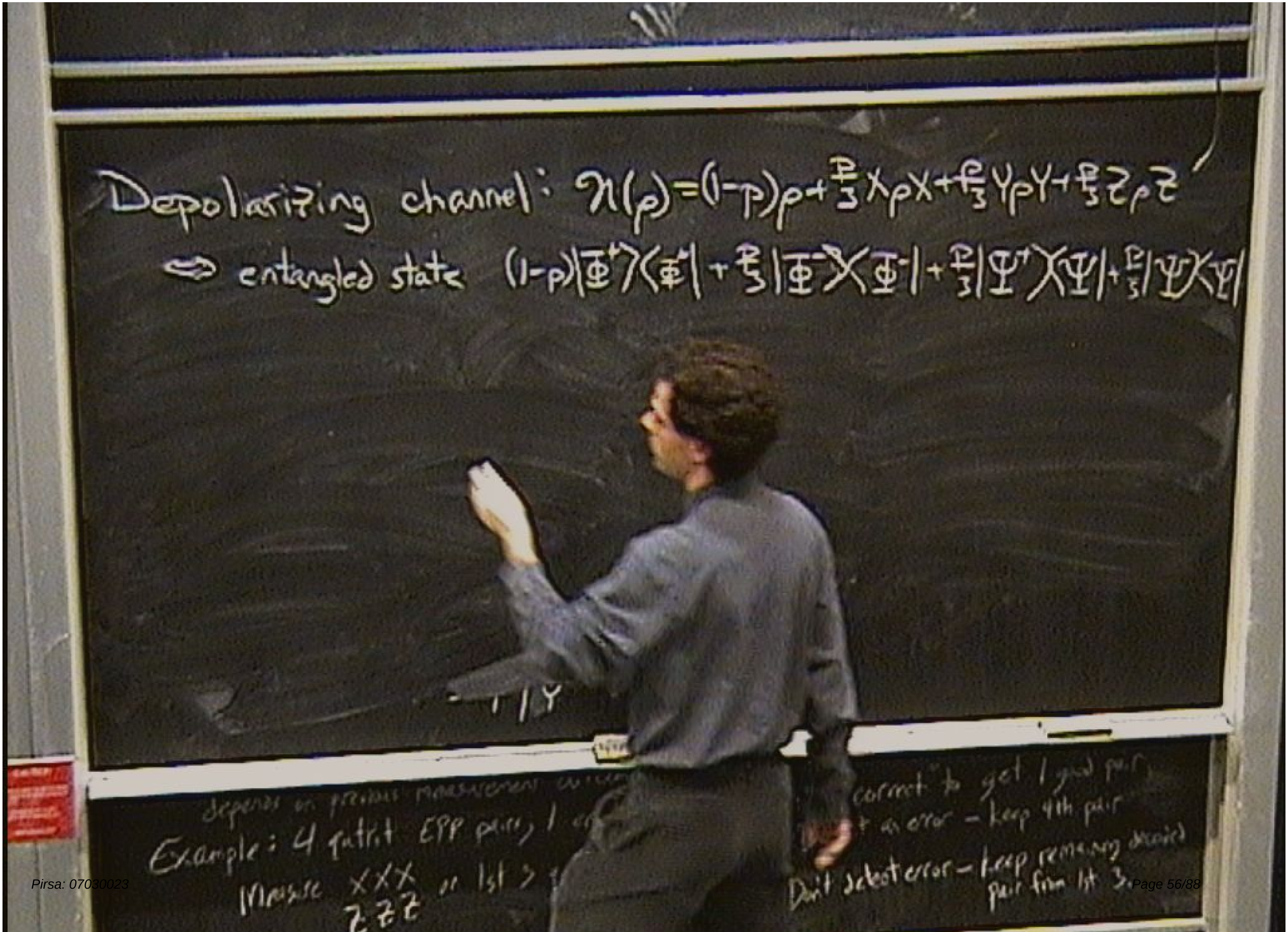
$Q_2(p) = 1-p$ (B tells A which are erased & they discard them)
$Q_1(p) \geq 1-2p$ (Upper bound: No-cloning $\Rightarrow Q_1(\frac{1}{2})=0$)
Random stabilizer code w/ r generators.
Any two distinct errors E & F. What is the probability E & F have same syndrome? = Prob $(E^\dagger F \in N(S)) = |N(S)|/(|P_n|-1) \approx 2^{2n-r}/2^{2n} = 2^{-r}$
Actual error E - What is prob that there is another F w/ same syndrome? $4^{pn}/2^r$
if
Choose $r=2pn \Rightarrow k=n-2pn \Rightarrow \frac{k}{n}=1-2p$


Depolarizing channel:
Example: 4 qutrit EPP pairs,
correct to get 1 good pair
Detect an error – keep 4th pair
Don't detect error – keep remaining decoded pair from 1st 3.


Depolarizing channel: $\mathcal{N}(\rho) = (1-p)\rho + \frac{p}{3} X\rho X + \frac{p}{3} Y\rho Y + \frac{p}{3} Z\rho Z$
correct to get 1 good pair
Example: 4 qutrit EPR pairs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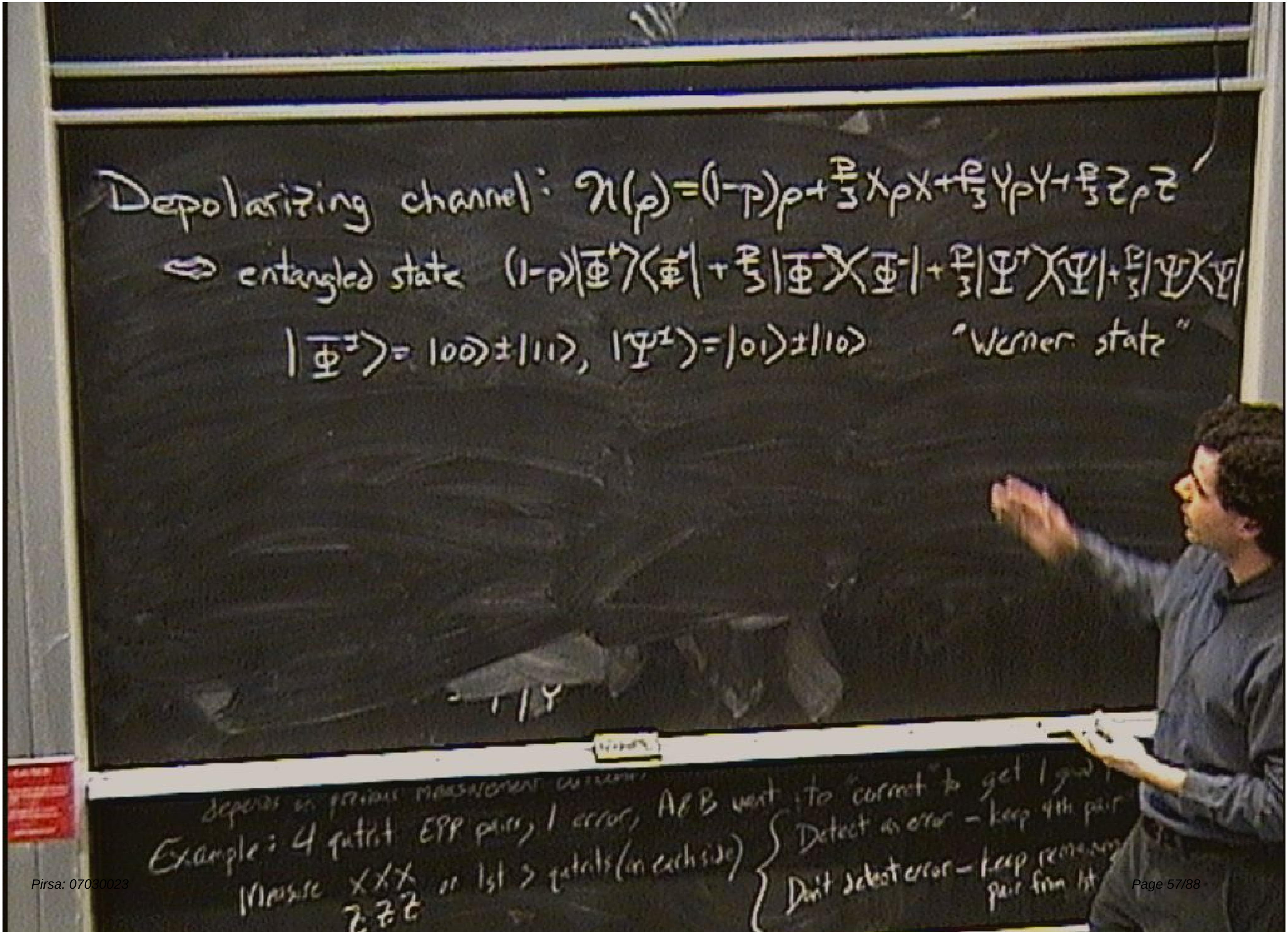
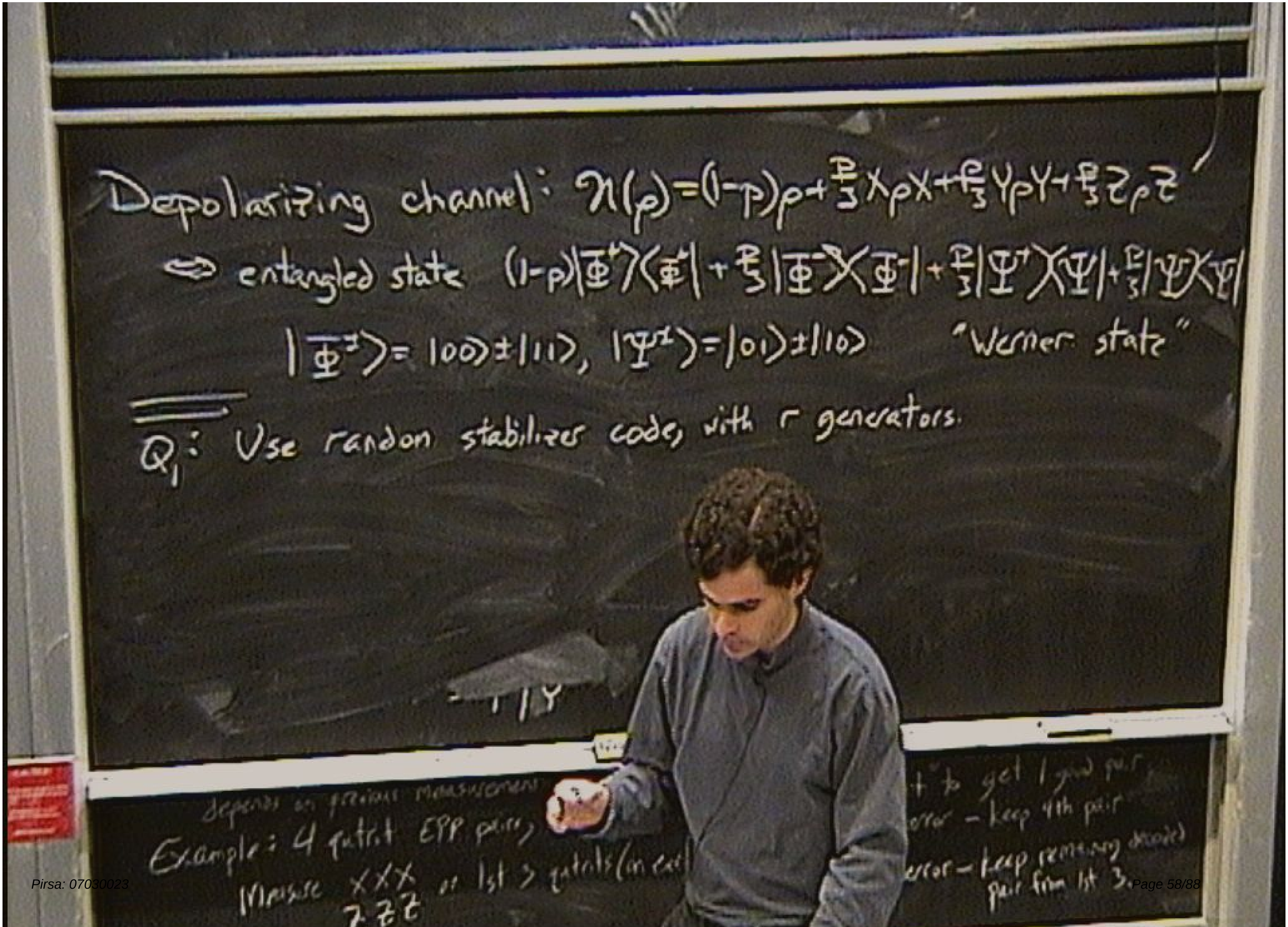
⟺ entangled state $(1-p)|\Phi^+\rangle\langle\Phi^+| + \frac{p}{3}|\Phi^-\rangle\langle\Phi^-| + \frac{p}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{p}{3}|\Psi^-\rangle\langle\Psi^-|$
correct to get 1 good pair
an error – keep 4th pair
Example: 4 qutrit EPP pairs, 1
Don't detect error – keep remaining desired pair from 1st 3.
Measure XXX on 1st 3
ZZZ


Depolarizing channel: N(ρ) = (1−p)ρ + p/3 XρX + p/3 YρY + p/3 ZρZ
⟺ entangled state (1−p)|Φ+⟩⟨Φ+| + p/3|Φ−⟩⟨Φ−| + p/3|Ψ+⟩⟨Ψ+| + p/3|Ψ−⟩⟨Ψ−|
|Φ±⟩ = |00⟩ ± |11⟩, |Ψ±⟩ = |01⟩ ± |10⟩
"Werner state"
Example: 4 qutrit EPR pairs, 1 error, A & B want to "correct" to get 1 good
Measure XXX on 1st 3 qutrits (on each side)
ZZZ
Detect an error – keep 4th pair
Don't detect error – keep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$|\Phi^\pm\rangle = |00\rangle \pm |11\rangle$, $|\Psi^\pm\rangle = |01\rangle \pm |10\rangle$
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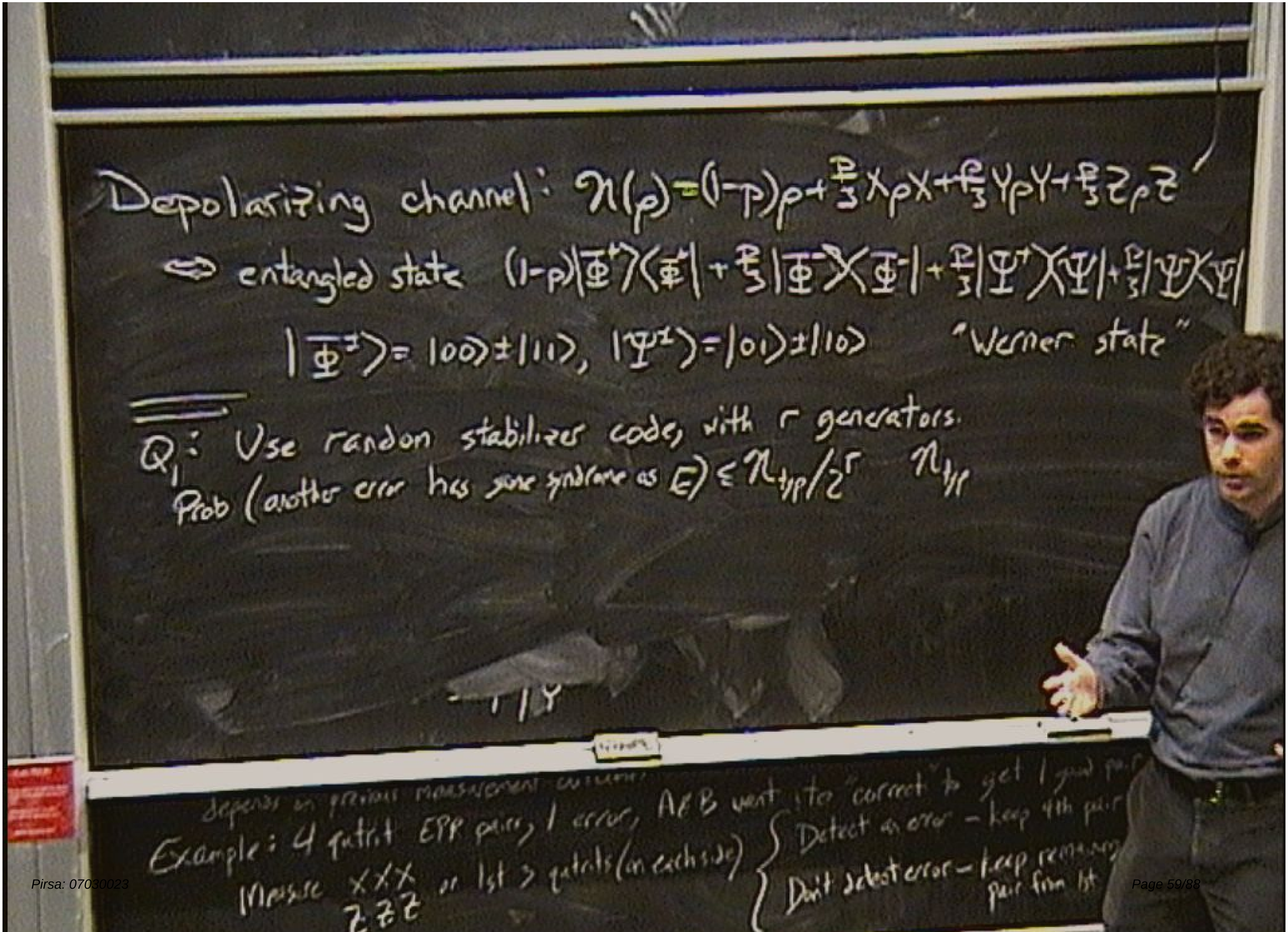
Q_1: Use random stabilizer codes with r generators.
Example: 4 qutrit EPR pairs
to get 1 good pair
error – keep 4th pair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Q1: Use random stabilizer codes, with r generators.
Prob (another error has same syndrome as E) ≤ N_typ / 2^r
Example: 4 qutrit EPR pairs, 1 error, A & B want to "correct" to get 1 good pair
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depends on previous measurement outcomes

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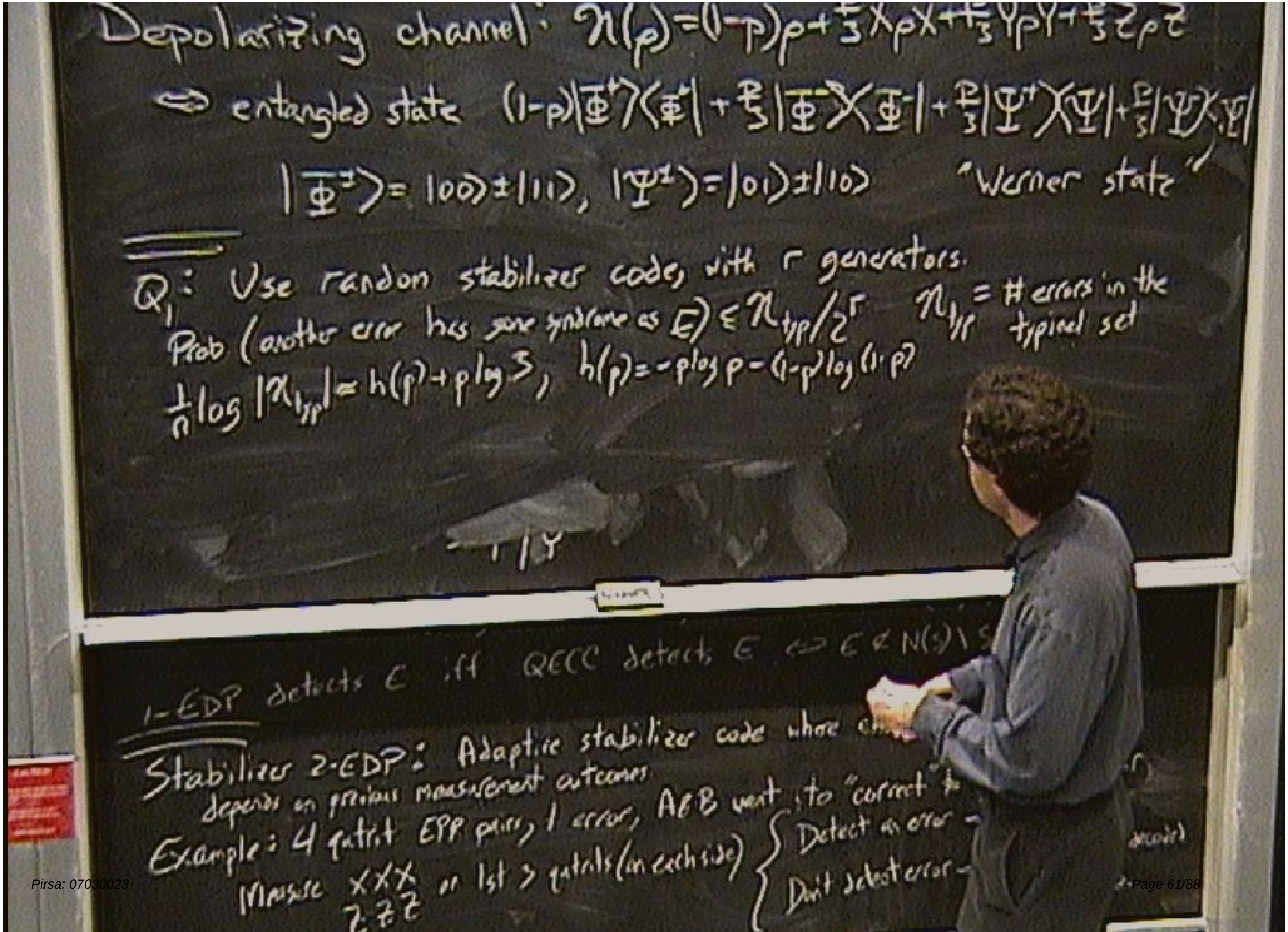
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1/n log |N_typ| = h(p) + p log 3, h(p) = −p log p − (1−p) log (1−p)
1-EDP detects E iff QECC detects E ⇔ E ∉ N(S) \ S
Stabilizer 2-EDP: Adaptive stabilizer code where
depends on previous measurement outcomes
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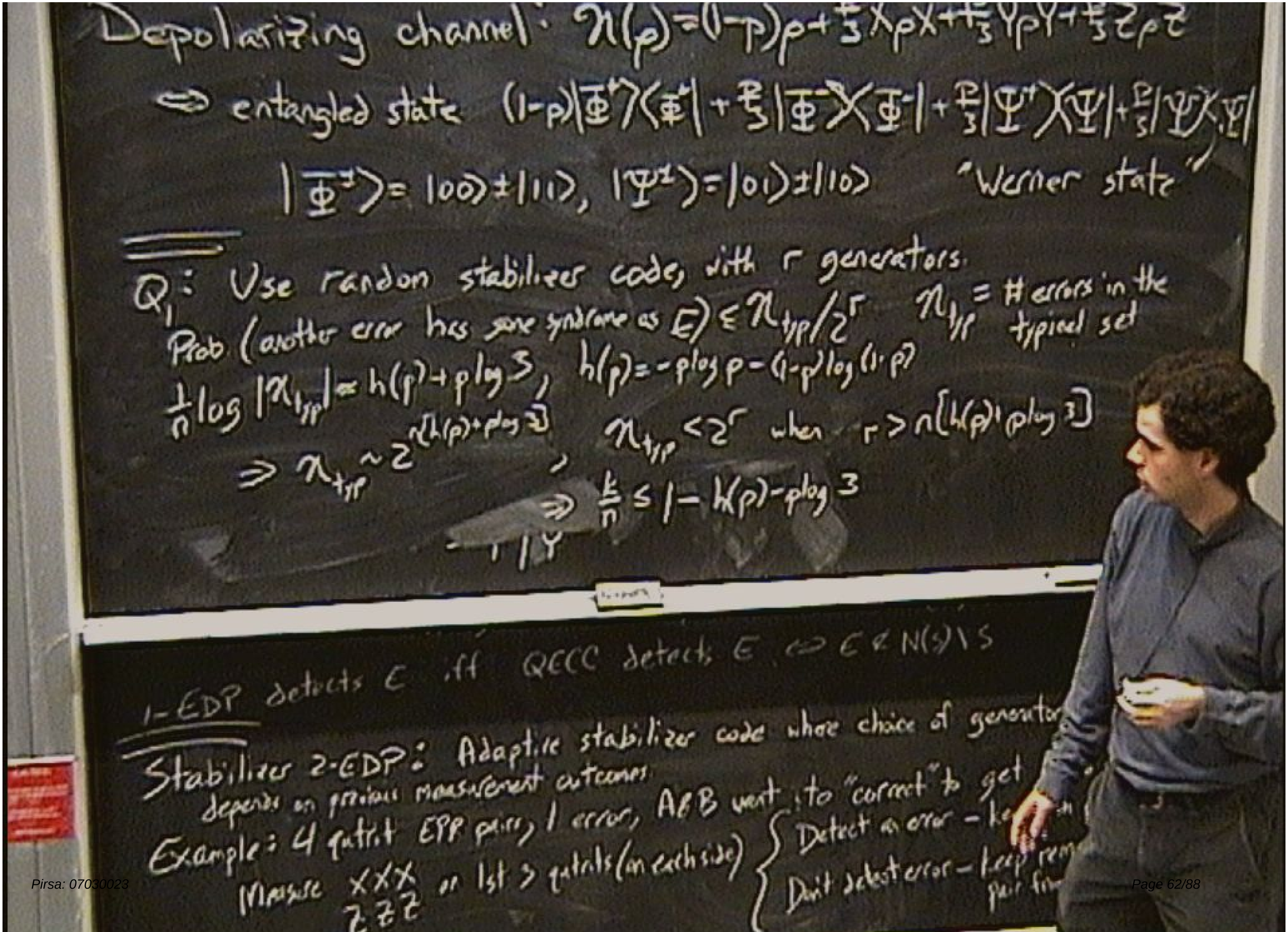
Detect an error
Don't detect error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⇒ N_typ ~ 2^{n[h(p)+p log 3]}
N_typ < 2^r when r > n[h(p)+p log 3]
⇒ k/n ≤ 1 - h(p) - p log 3
1-EDP detects E iff QECC detects E ⟺ E ∉ N(S)\S
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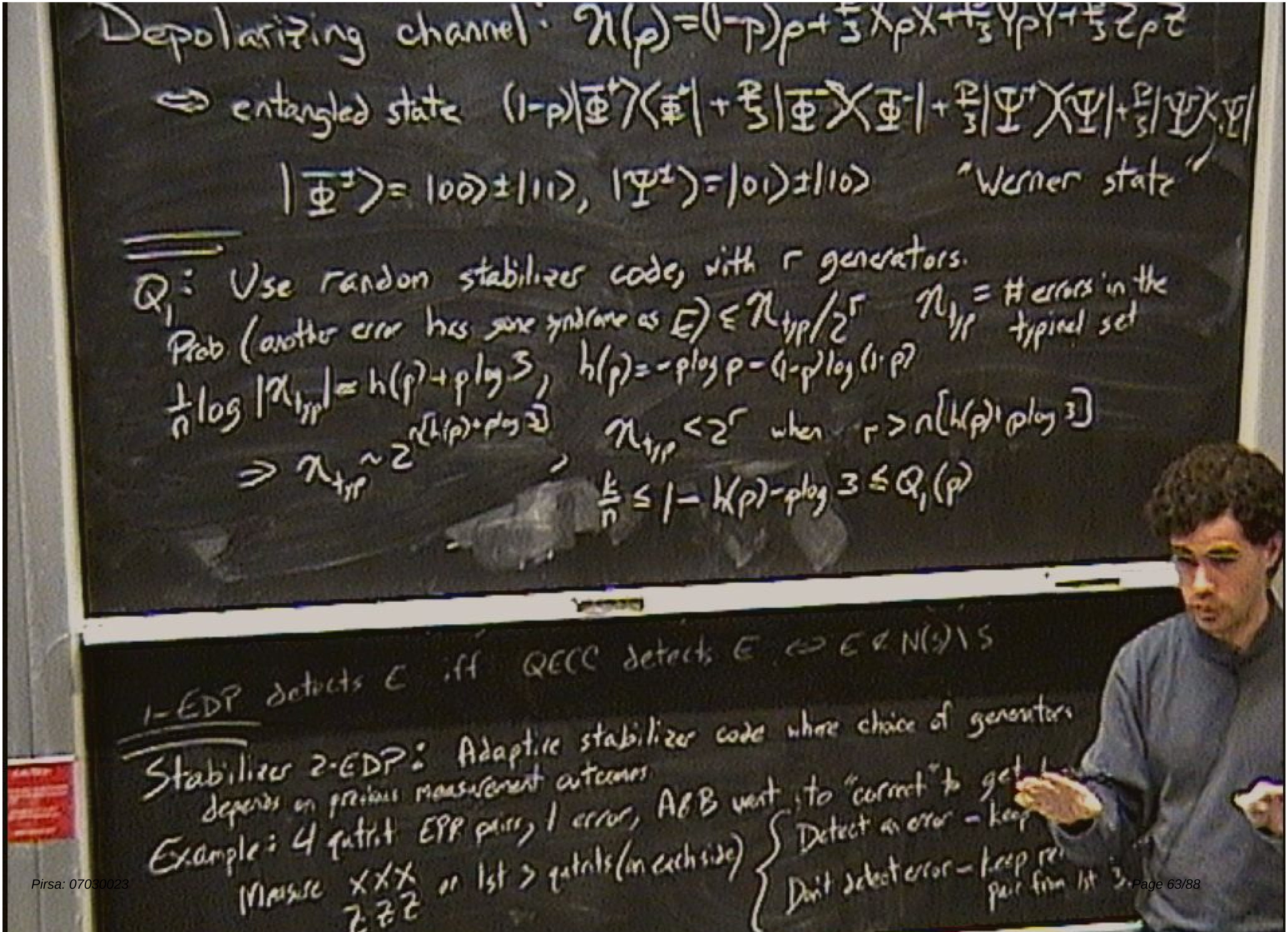
Measure XXX, ZZZ on 1st 3 qutrits (on each side)
Detect an error
Don't detect error - keep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⇒ N_typ ~ 2^{n[h(p)+p log 3]} N_typ < 2^r when r > n[h(p) + p log 3]
k/n ≤ 1 − h(p) − p log 3 ≤ Q1(p)
1-EDP detects E iff QECC detects E ⟺ E ∉ N(S)\S
Stabilizer 2-EDP: Adaptive stabilizer code where choice of generators depends on previous measurement outcomes
Example: 4 qutrit EPR pairs, 1 error, A&B want to "correct" to get
Measure XXX, ZZZ on 1st 3 qutrits (on each side)
Detect an error – keep
Don't detect error – keep pair from 1st 3

Depolarizing channel: $\mathcal{N}$ [illegible] $\frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z$

$\Leftrightarrow$ entangled state $(1-p)|\Phi^+\rangle\langle\Phi^+| + \frac{p}{3}|\Phi^-\rangle\langle\Phi^-| + \frac{p}{3}|\Psi^+\rangle\langle\Psi^+| + \frac{p}{3}|\Psi^-\rangle\langle\Psi^-|$

$|\Phi^\pm\rangle = |00\rangle \pm |11\rangle$, $|\Psi^\pm\rangle = |01\rangle \pm |10\rangle$ "Werner state"

Q_1: Use random stabilizer codes with r generators.

Prob (another error has same syndrome as E) $\le \mathcal{N}_{typ}/2^r$ $\quad \mathcal{N}_{typ}$ = # errors in the typical set

$\frac{1}{n}\log|\mathcal{N}_{typ}| \approx h(p) + p\log 3$, $\quad h(p) = -p\log p - (1-p)\log(1-p)$

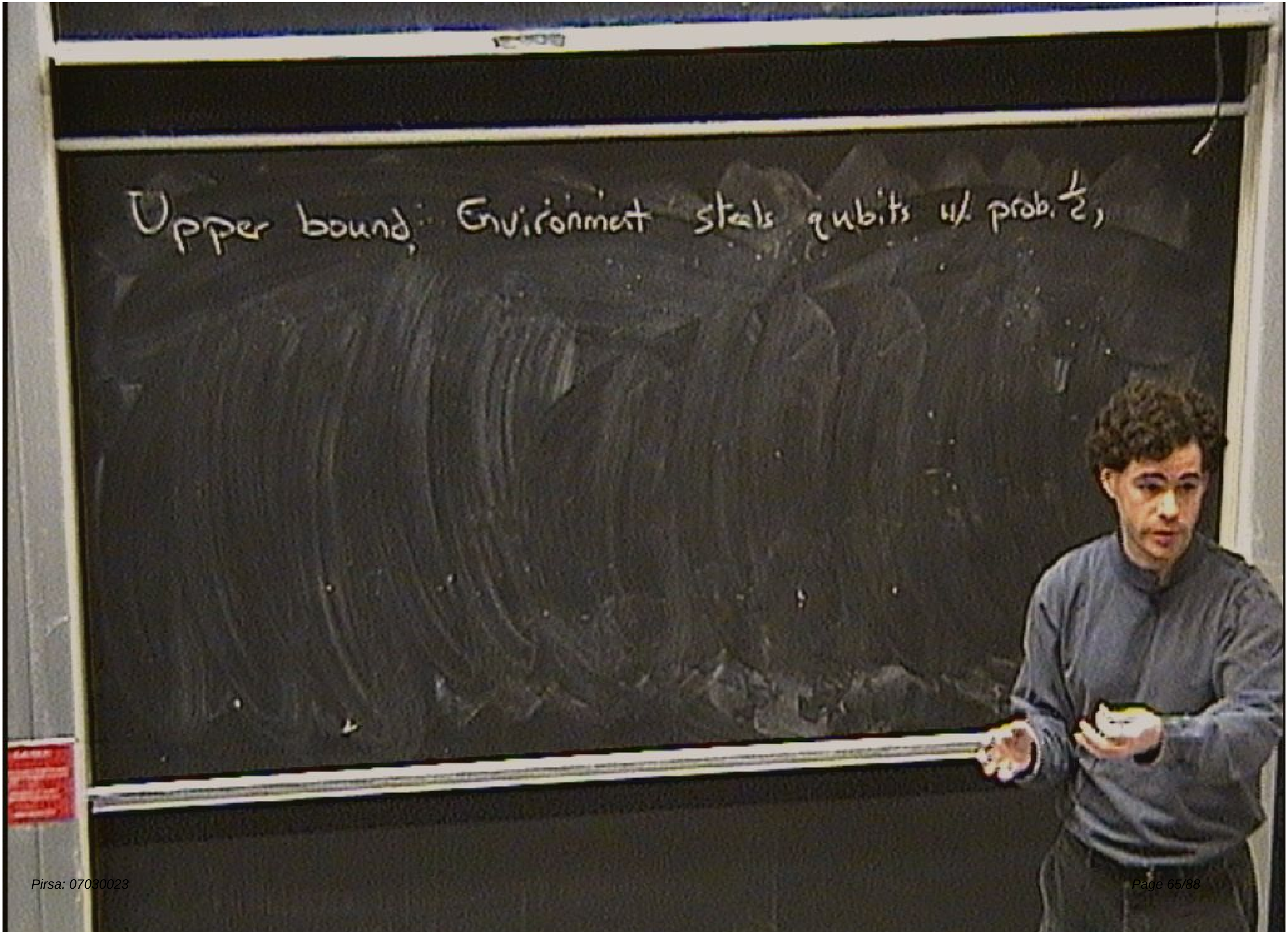
$\Rightarrow \mathcal{N}_{typ} \sim 2^{n[h(p)+p\log 3]}$, $\quad \mathcal{N}_{typ} < 2^r$ when $r > n[h(p)+p\log 3]$

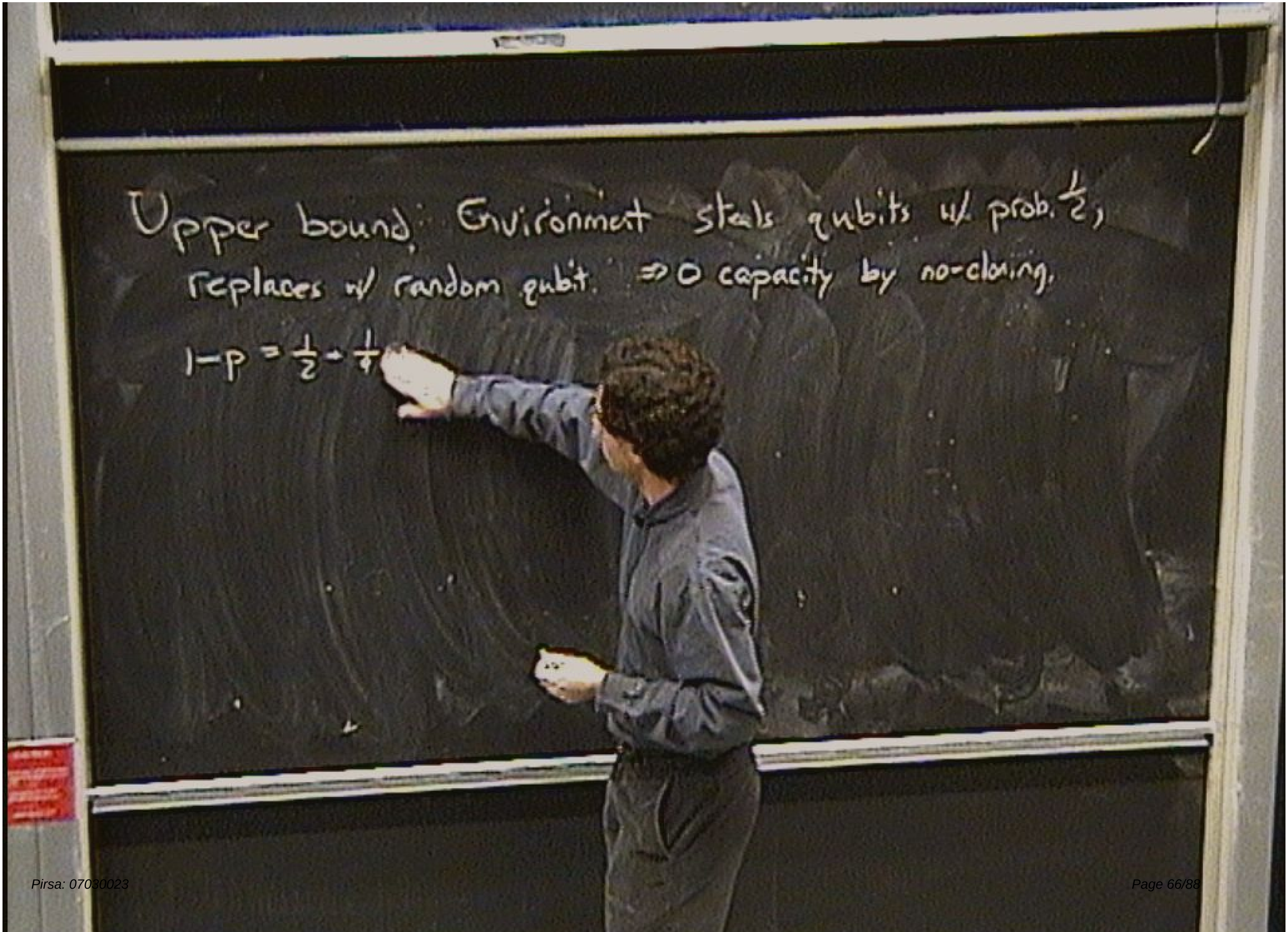
$\frac{k}{n} \le 1 - h(p) - p\log 3 \le Q_1(p)$

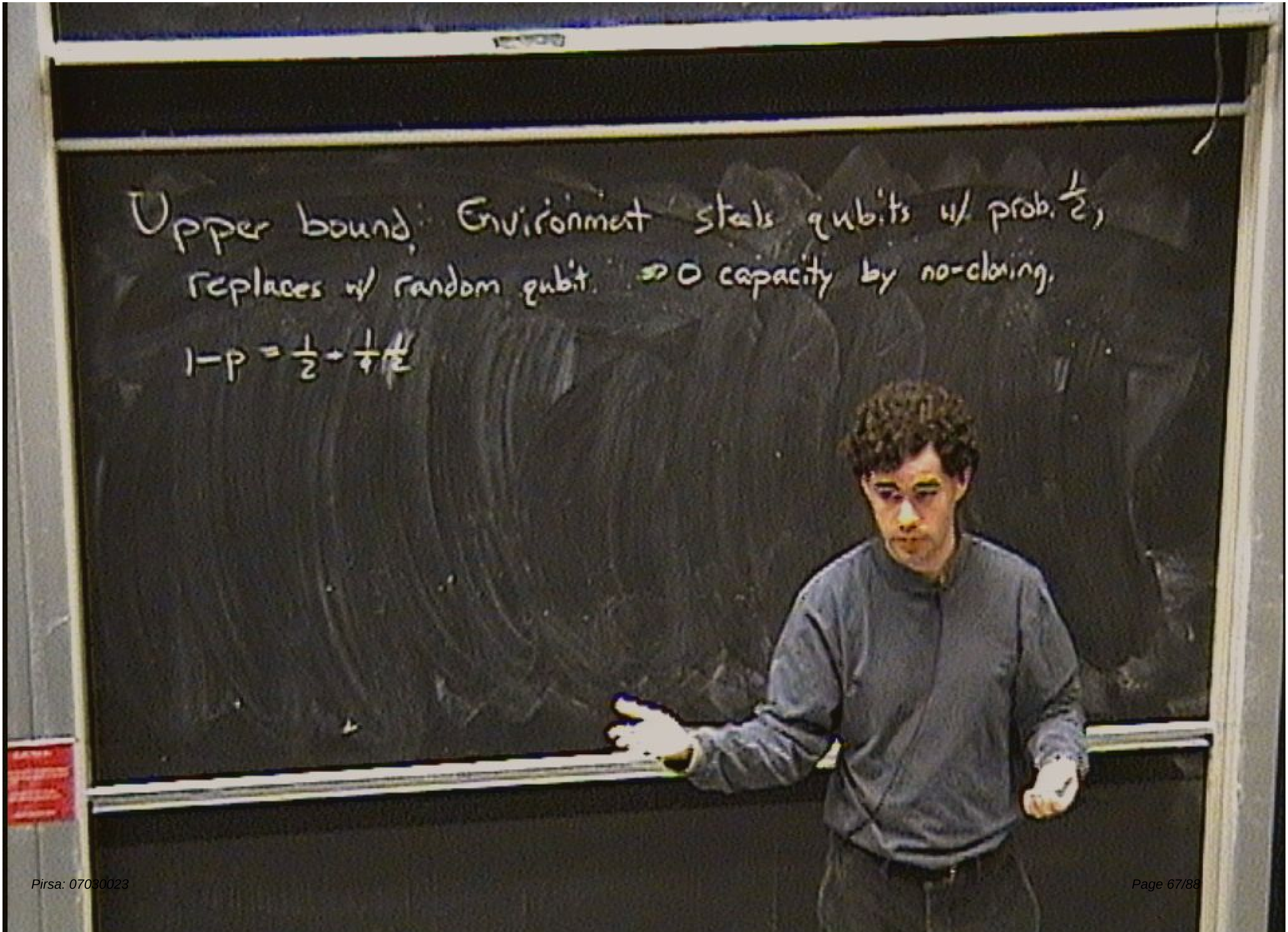
$\rightarrow 0$ at $p = 0.18929$

Upper bounds.

Mix I channel w $\mathcal{N}$: $\mathcal{M}_q = q\mathcal{N} + (1-$


Upper bound: Environment steals qubits w/ prob. ½,


Upper bound: Environment steals qubits w/ prob. 1/2,
replaces w/ random qubit. ⇒ 0 capacity by no-cloning.
1-p = 1/2 + 1/4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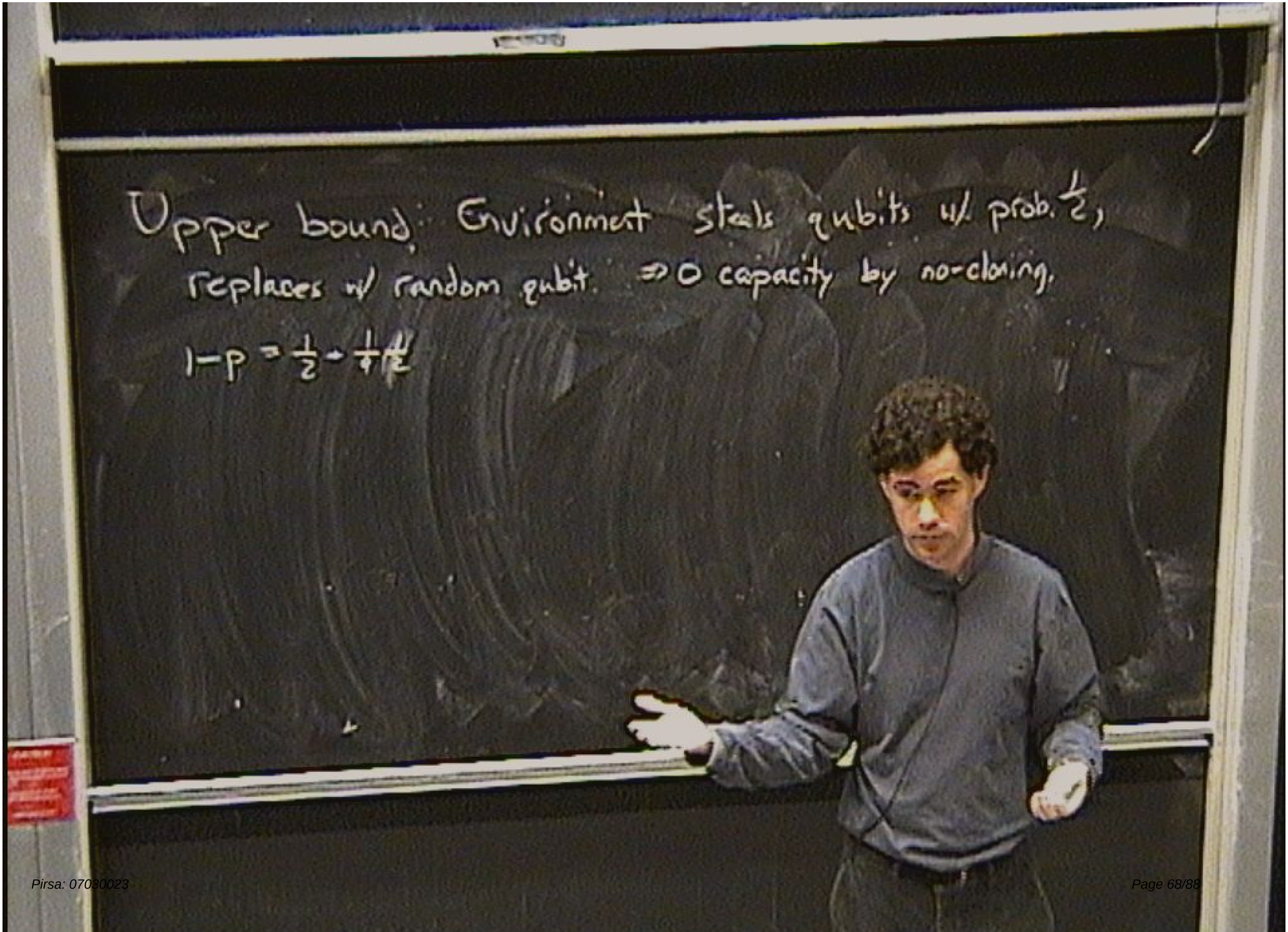
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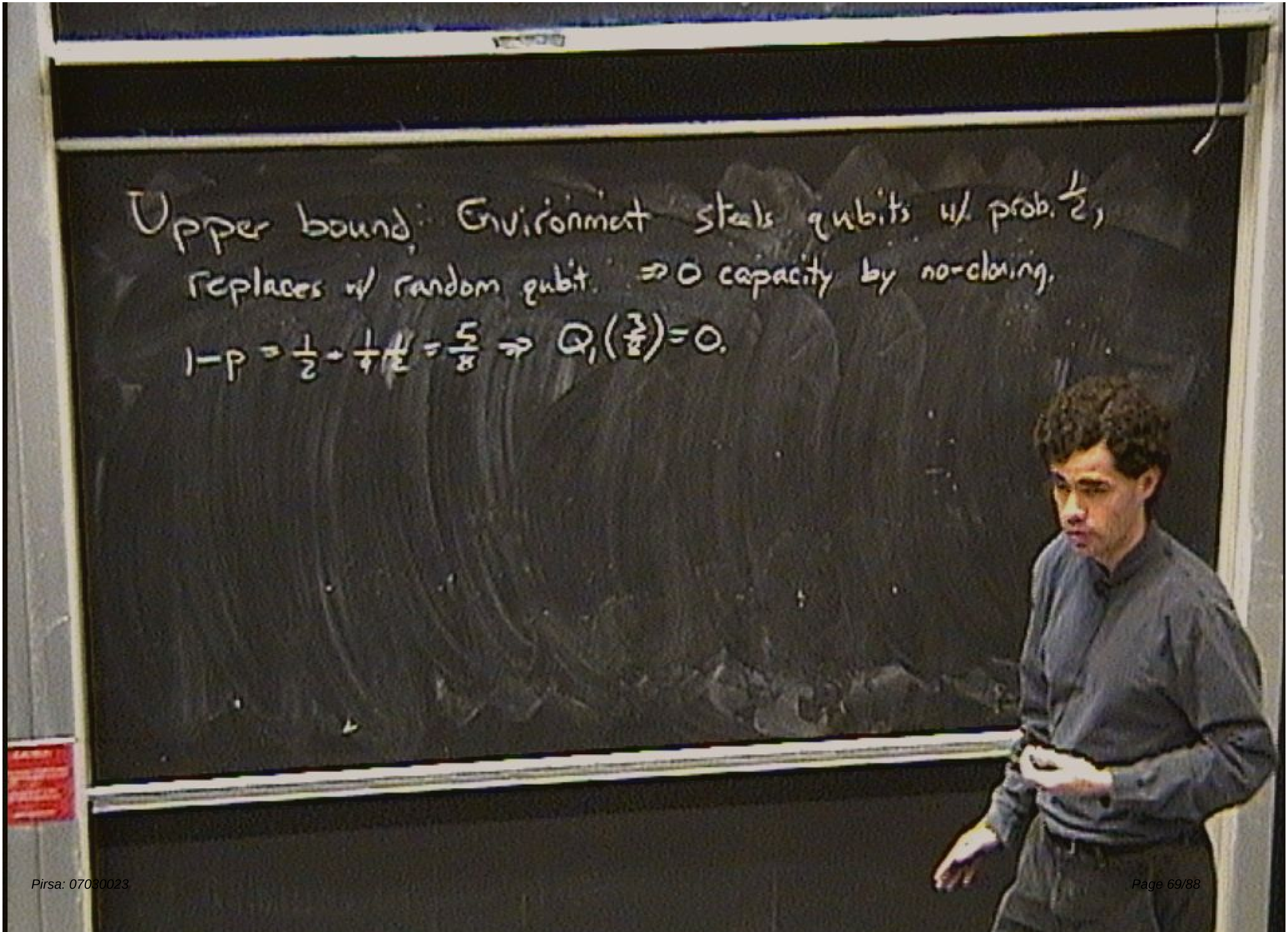
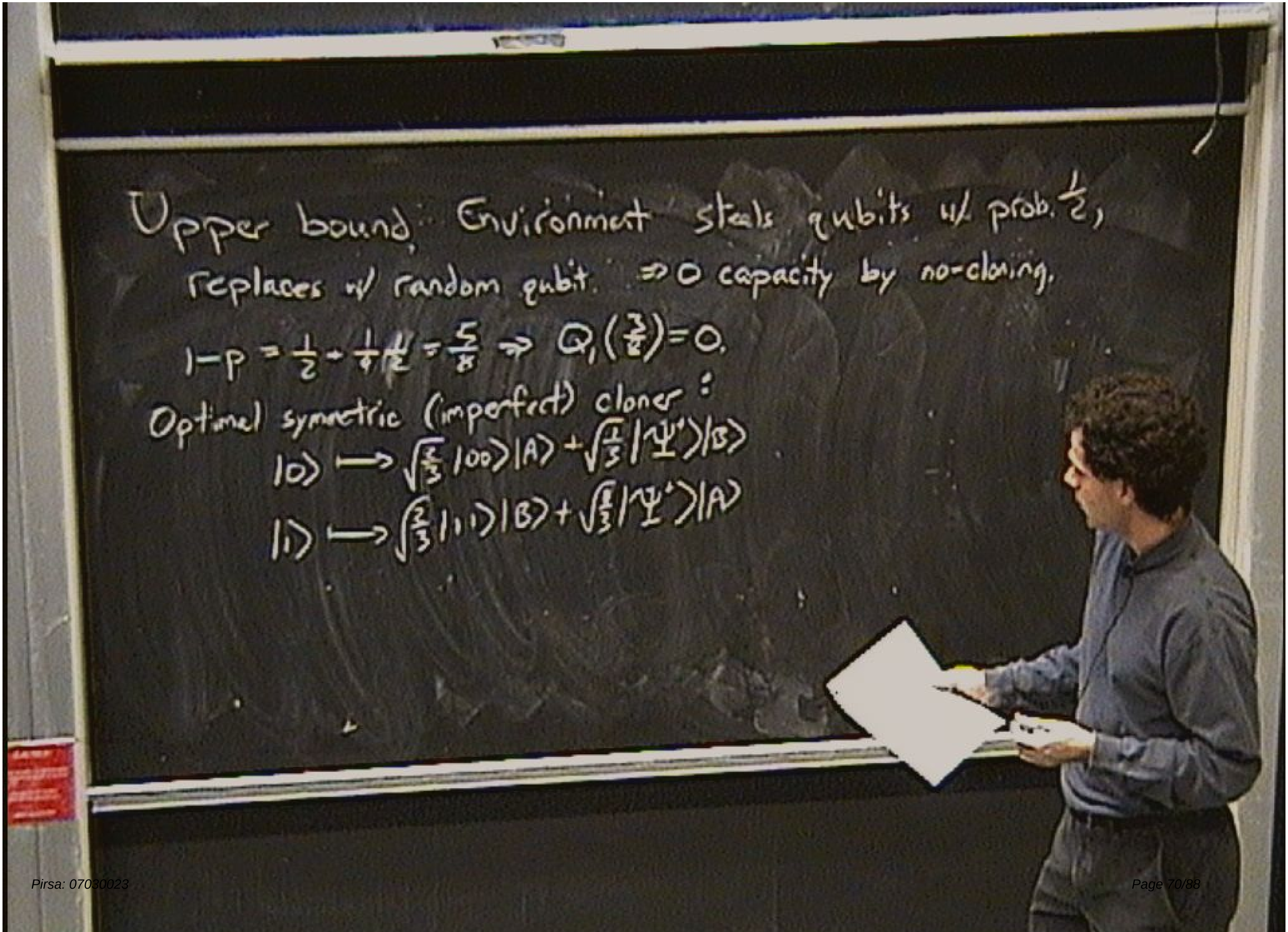

Upper bound: Environment steals qubits w/ prob. 1/2,
replaces w/ random qubit ⇒ 0 capacity by no-cloning.
1−p = 1/2 + 1/4·1/2 = 5/8 ⇒ Q1(3/8) = 0.


Upper bound: Environment steals qubits w/ prob. 1/2,
replaces w/ random qubit. ⇒ 0 capacity by no-cloning.
1−p = 1/2 + 1/4·1/2 = 5/8 ⇒ Q1(5/8)=0.
Optimal symmetric (imperfect) cloner:
|0⟩ ⟶ √(2/3) |00⟩|A⟩ + √(1/3) |Ψ+⟩|B⟩
|1⟩ ⟶ √(2/3) |11⟩|B⟩ + √(1/3) |Ψ+⟩|A⟩

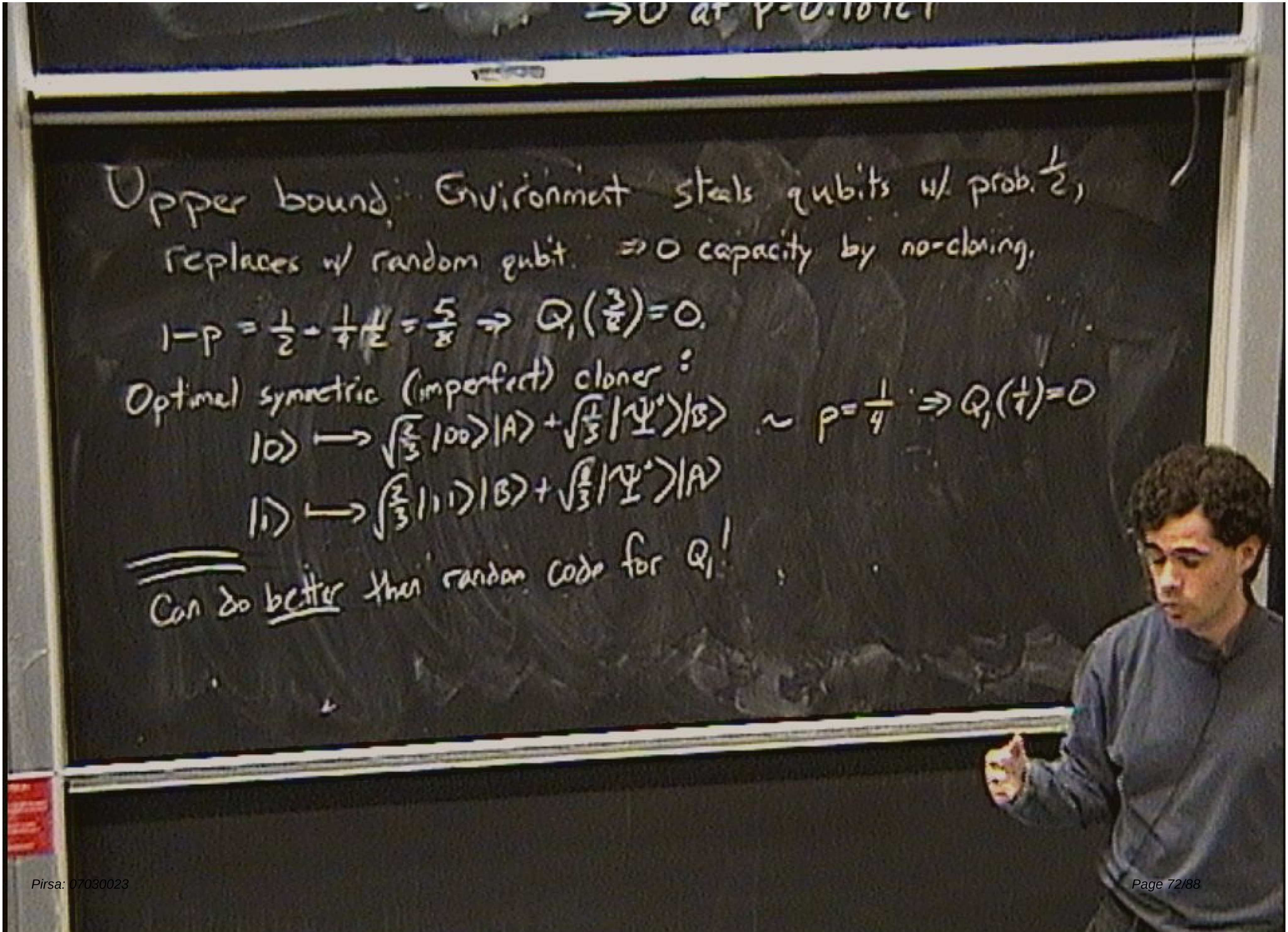
Upper bound: Environment steals qubits w/ prob. $\frac{1}{2}$, replaces w/ random qubit. $\Rightarrow 0$ capacity by no-cloning.

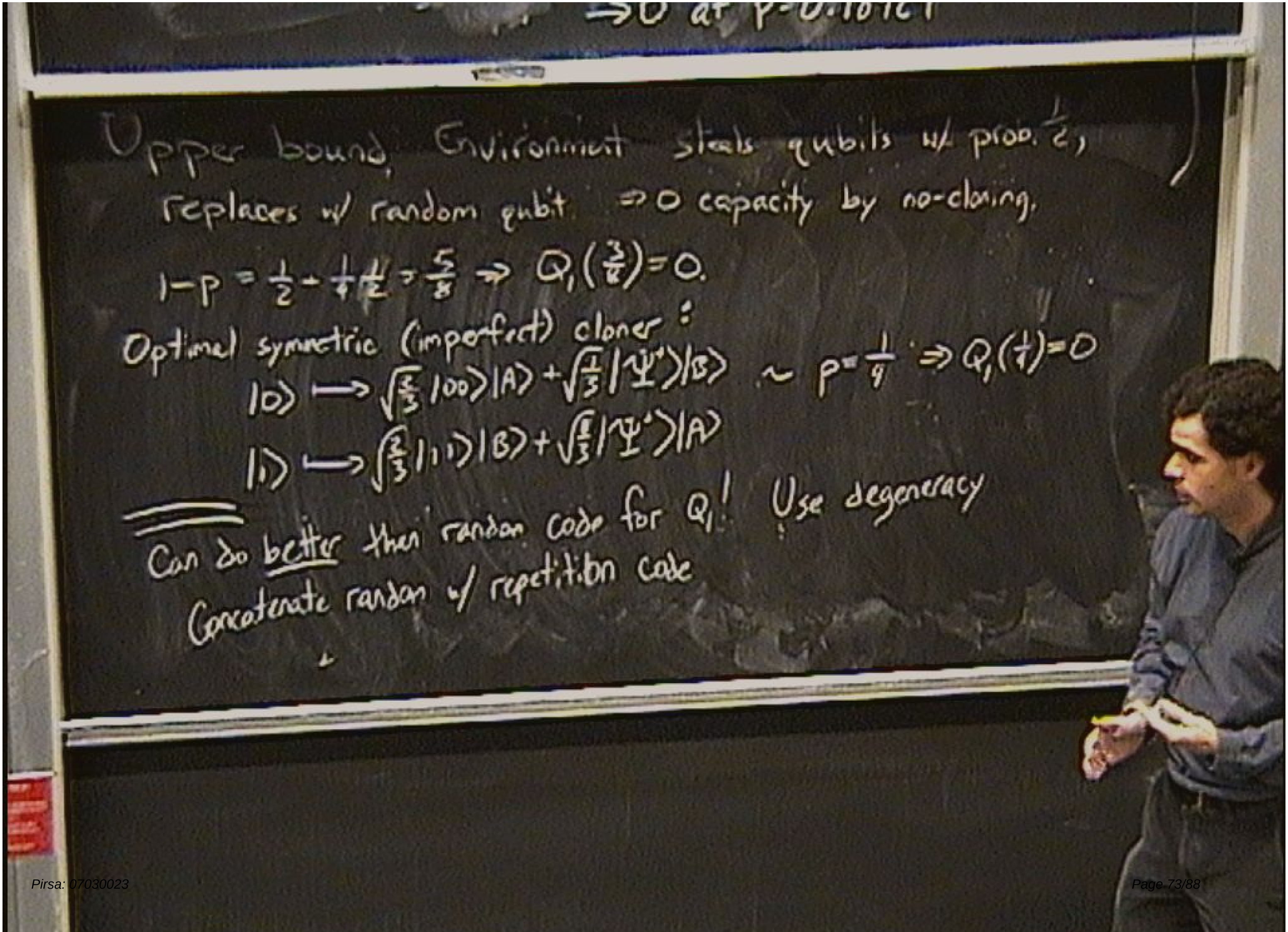
$$1-p = \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{5}{8} \Rightarrow Q_1\left(\frac{3}{8}\right) = 0.$$

Optimal symmetric (imperfect) cloner:

$$|0\rangle \longmapsto \sqrt{\tfrac{2}{3}}\,|00\rangle|A\rangle + \sqrt{\tfrac{1}{3}}\,|\Psi^+\rangle|B\rangle \quad \sim \quad p = \frac{1}{4} \Rightarrow Q_1\left(\frac{1}{4}\right) = 0$$

$$|1\rangle \longmapsto \sqrt{\tfrac{2}{3}}\,|11\rangle|B\rangle + \sqrt{\tfrac{1}{3}}\,|\Psi^+\rangle|A\rangle$$


Upper bound: Environment steals qubits w/ prob. $\frac{1}{2}$,
replaces w/ random qubit. $\Rightarrow$ 0 capacity by no-cloning.
$1-p = \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{5}{8} \Rightarrow Q_1(\frac{3}{8})=0.$
Optimal symmetric (imperfect) cloner:
$|0\rangle \longrightarrow \sqrt{\frac{2}{3}}|00\rangle|A\rangle + \sqrt{\frac{1}{3}}|\Psi^+\rangle|B\rangle \sim p=\frac{1}{4} \Rightarrow Q_1(\frac{1}{4})=0$
$|1\rangle \longrightarrow \sqrt{\frac{2}{3}}|11\rangle|B\rangle + \sqrt{\frac{1}{3}}|\Psi^+\rangle|A\rangle$
Can do better than random code for Q_1!


Upper bound: Environment steals qubits w/ prob. 1/2,
replaces w/ random qubit ⇒ 0 capacity by no-cloning.
1-p = 1/2 + 1/4 · 1/2 = 5/8 ⇒ Q1(3/8)=0.
Optimal symmetric (imperfect) cloner:
|0> ⟼ √(2/3) |00>|A> + √(1/3) |Ψ+>|B> ~ p=1/4 ⇒ Q1(1/4)=0
|1> ⟼ √(2/3) |11>|B> + √(1/3) |Ψ+>|A>
Can do better than random code for Q1! Use degeneracy
Concatenate random w/ repetition code

$\Rightarrow 0$ at $p = 0.1892$

Upper bound: Environment steals qubits w/ prob. $\frac{1}{2}$, replaces w/ random qubit $\Rightarrow 0$ capacity by no-cloning.

$$1-p = \frac{1}{2} + \frac{1}{4}\cdot\frac{1}{2} = \frac{5}{8} \Rightarrow Q_1\left(\frac{3}{8}\right) = 0.$$

Optimal symmetric (imperfect) cloner:

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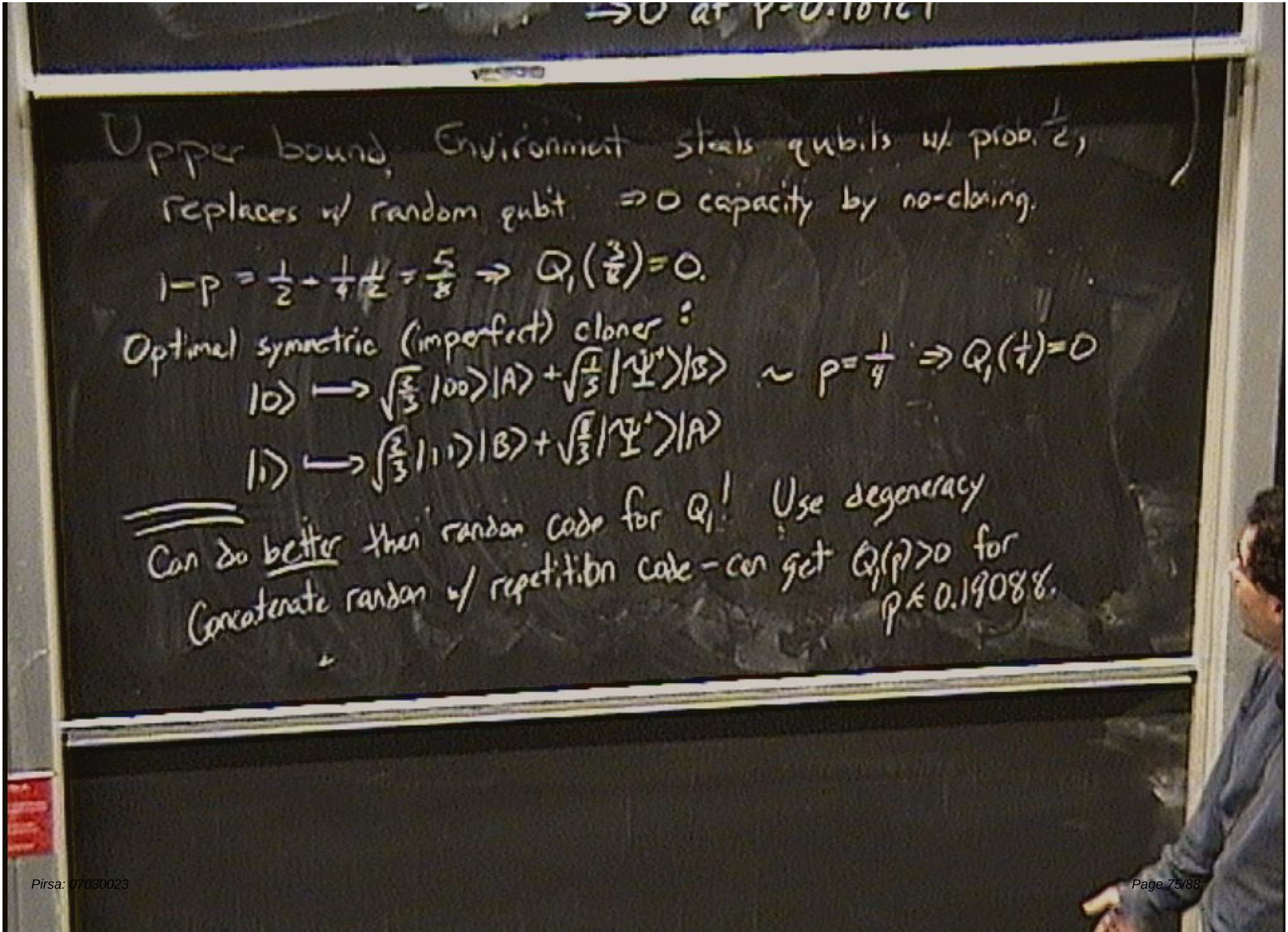
Can do better than random code for Q_1! Use degeneracy

Concatenate random w/ repetition code – can get $Q_1(p) > 0$ for $p \leq 0.1908\%$.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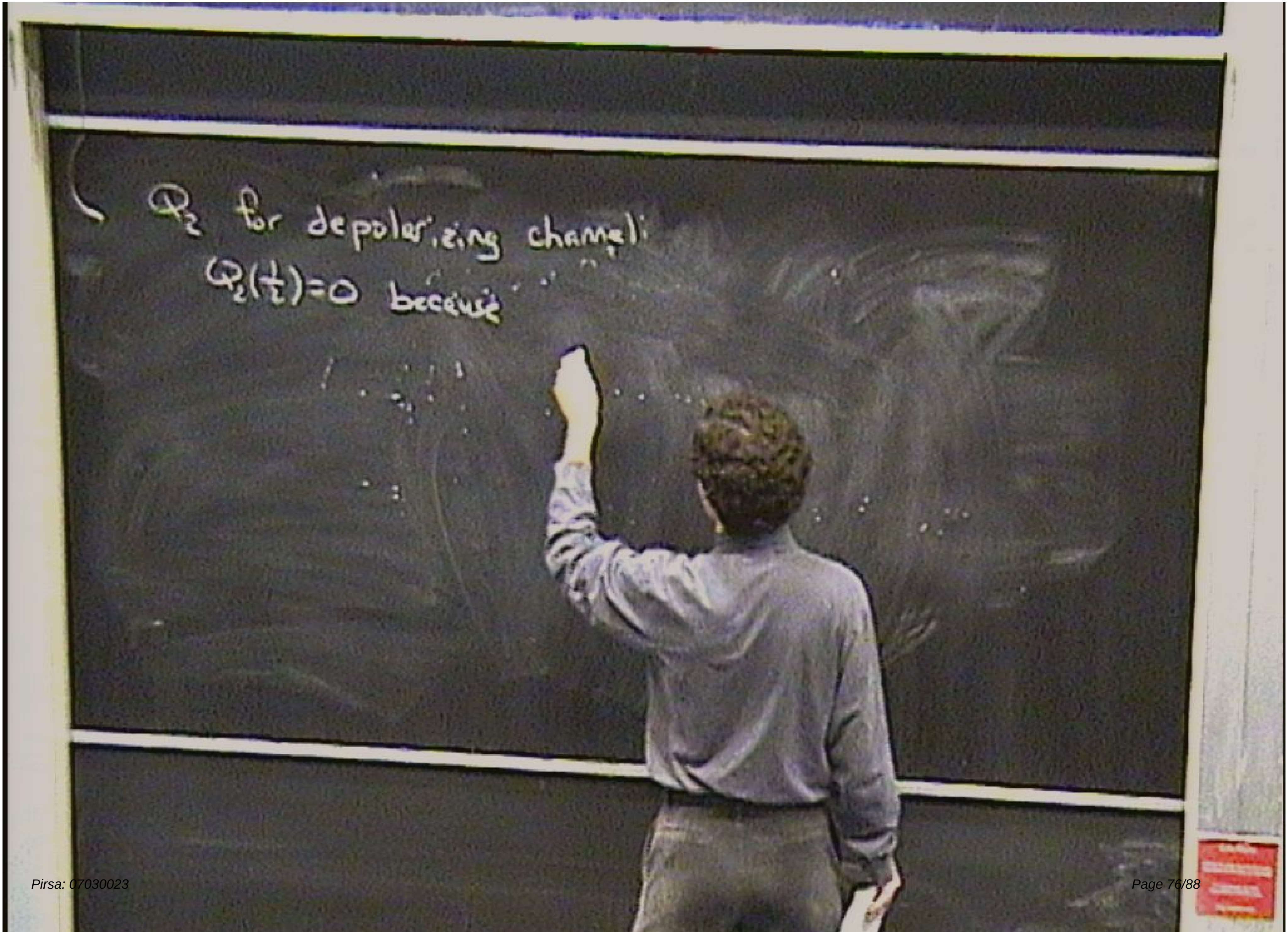
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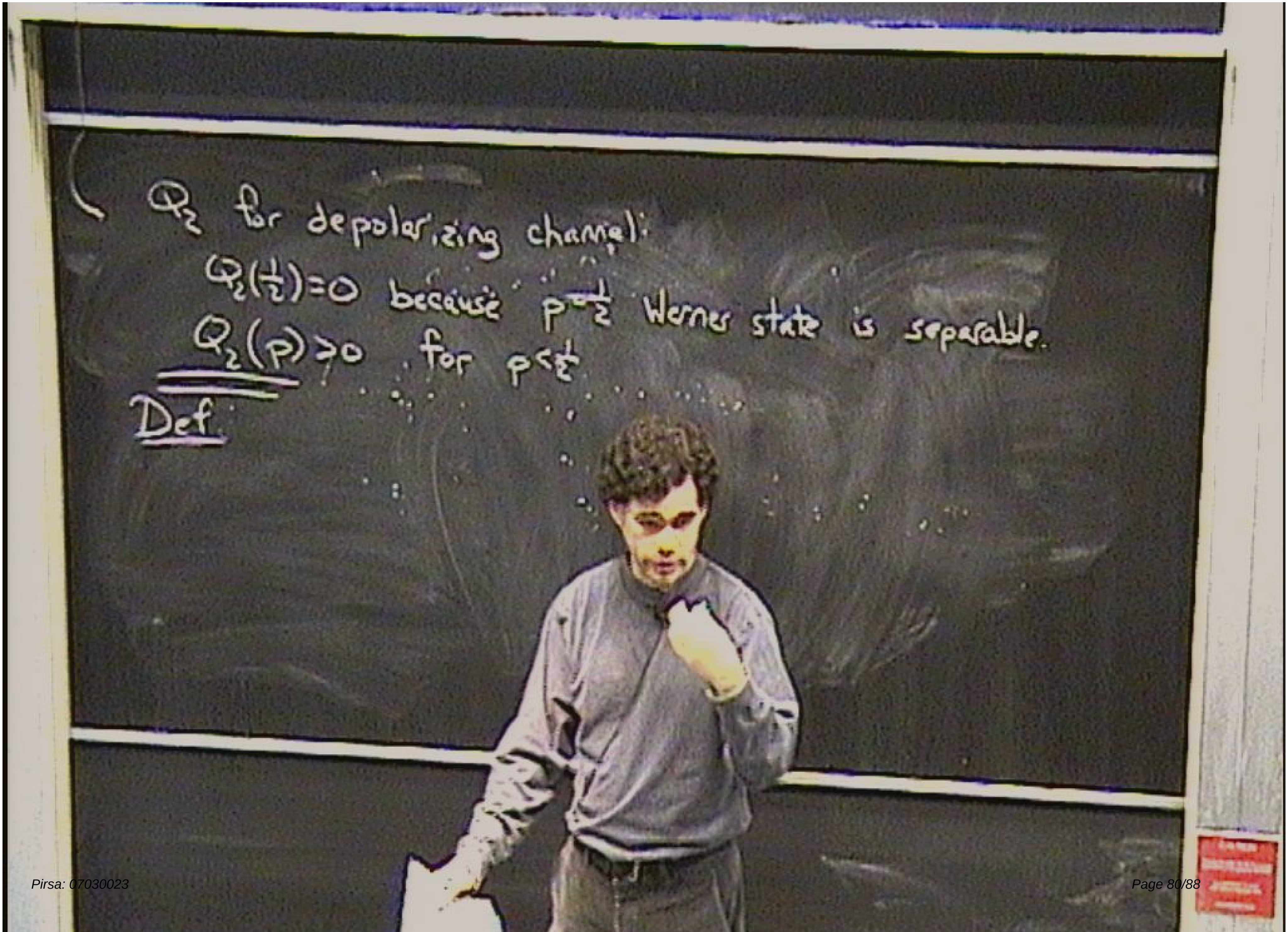
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The coherent information is $I(\rho,\mathcal{N}) = S(\mathcal{N}(\rho)) - S(E)$,

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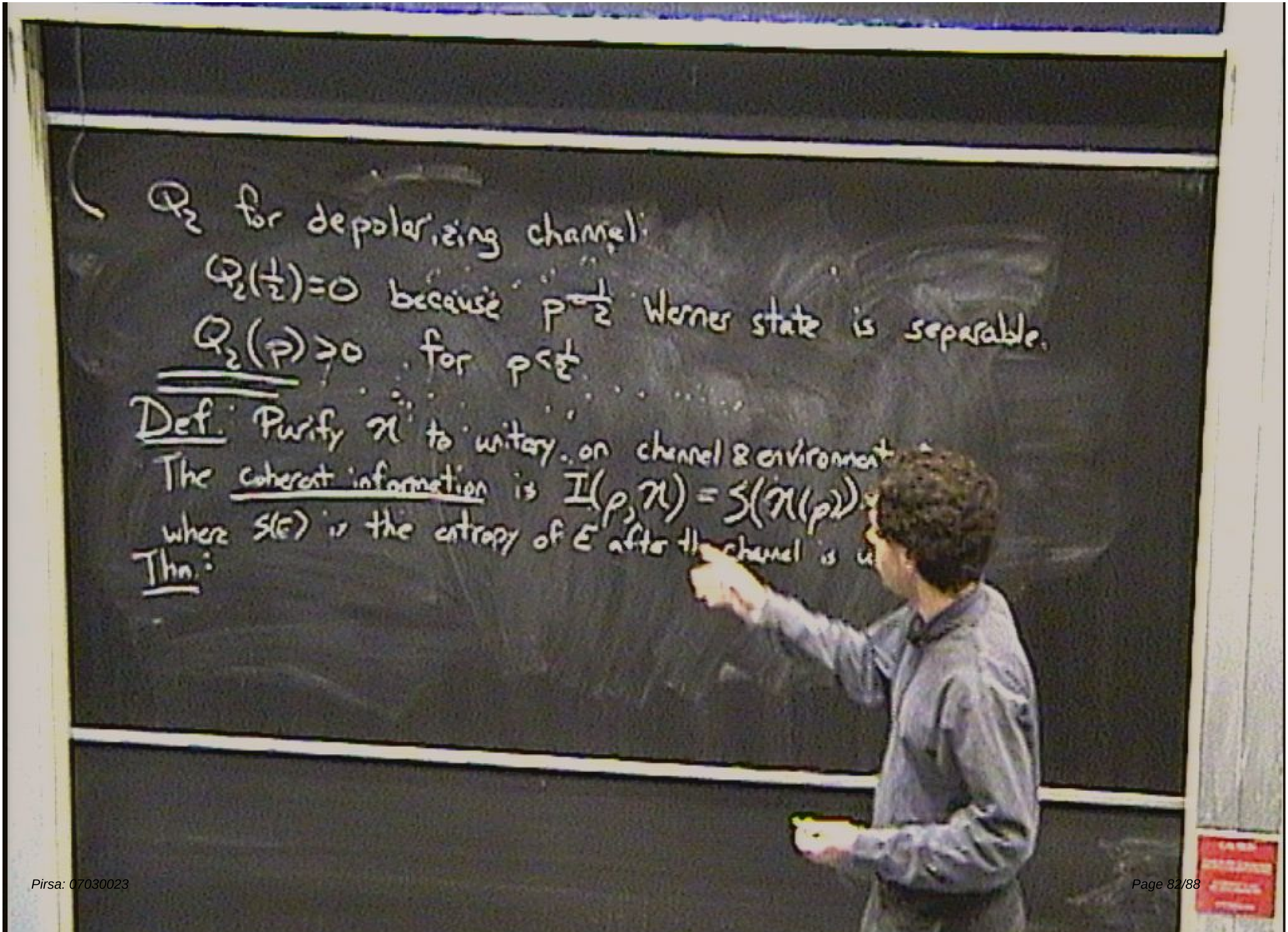
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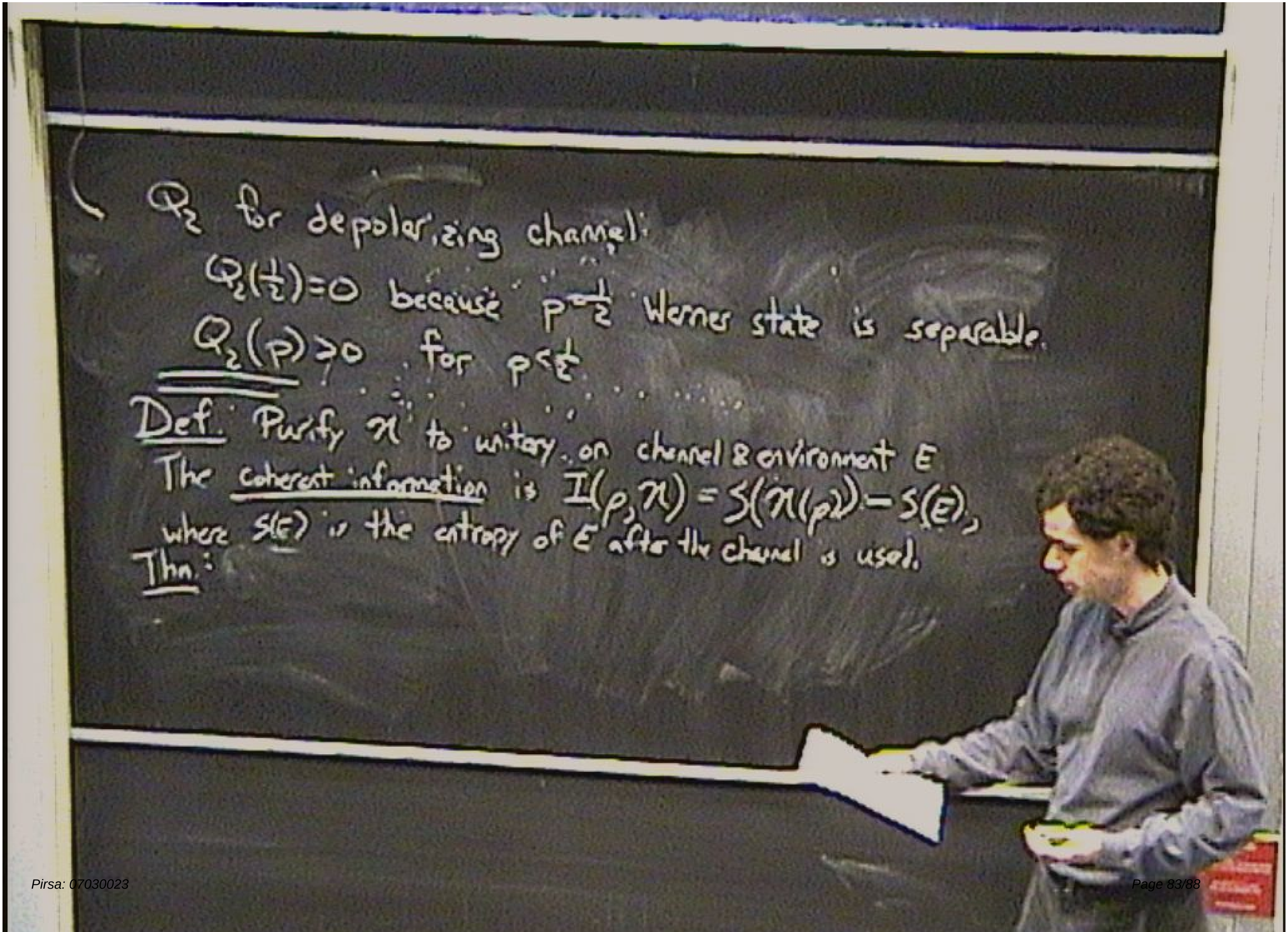
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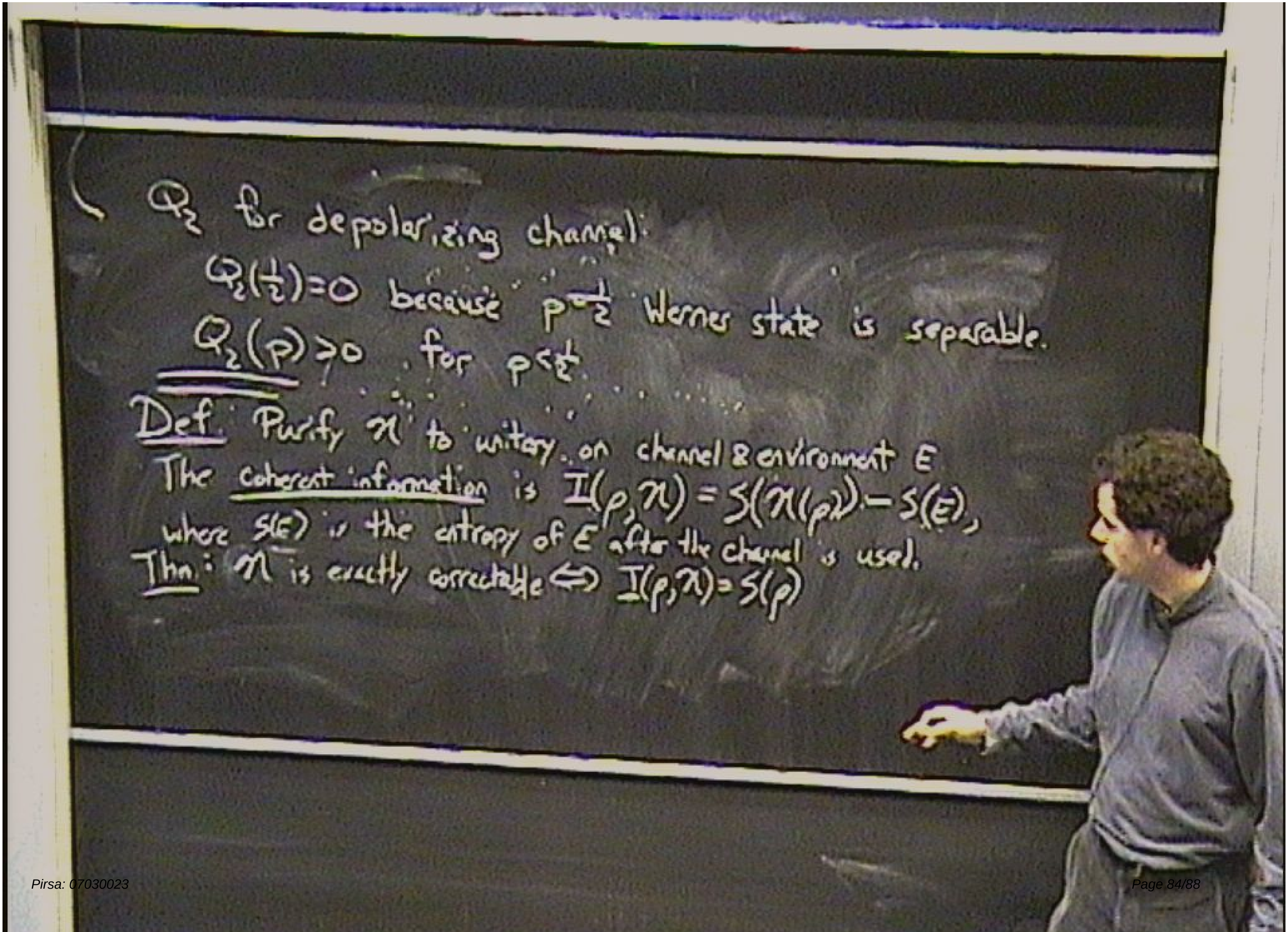

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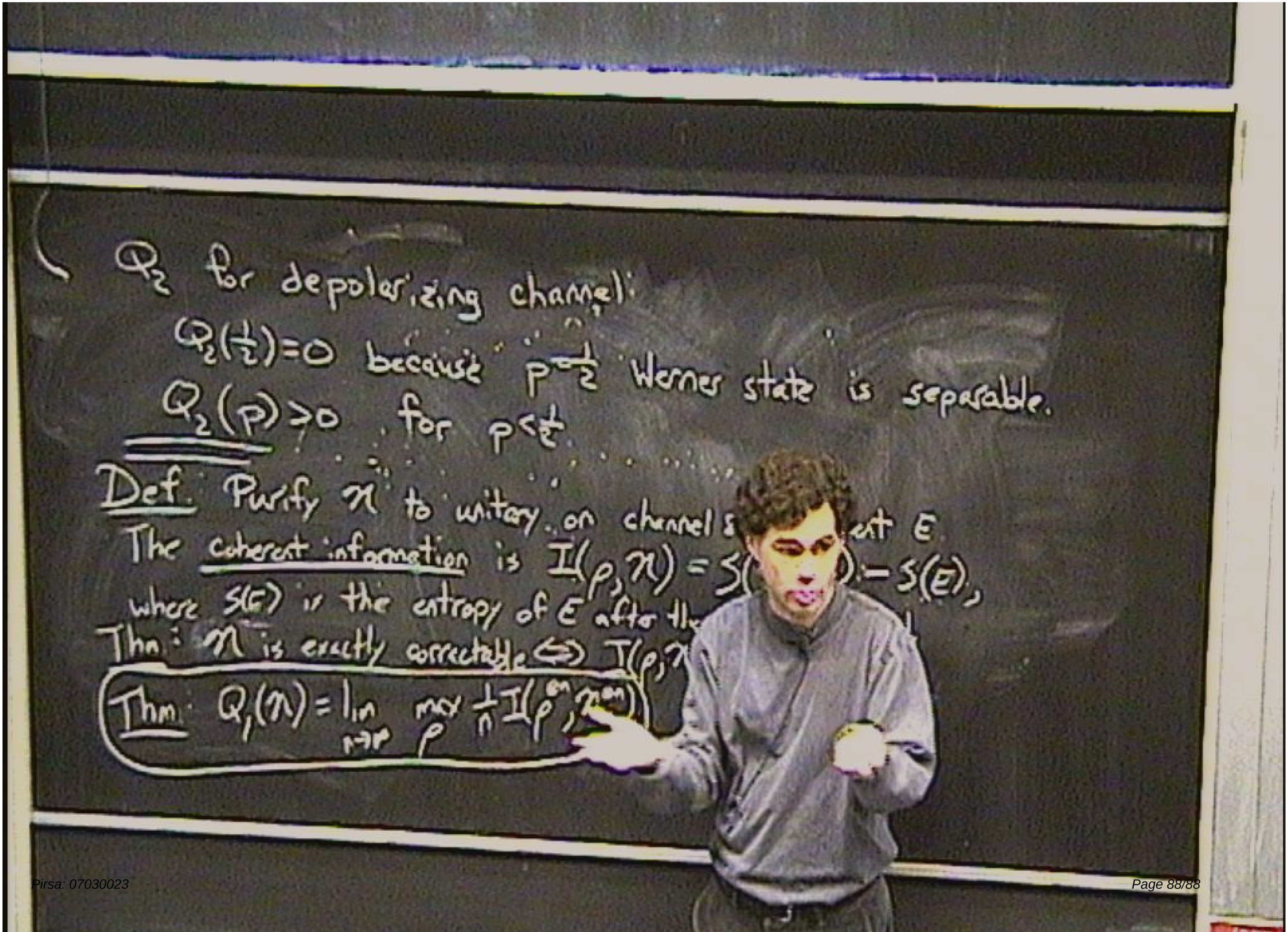
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