Title: Decoherence, Entanglement and the Foundation of Quantum Mechanics

Date: Mar 15, 2007 04:00 PM

URL: http://pirsa.org/07030021

Abstract: Results in decoherence theory and entanglement theory will be considered as tools illuminating the foundation of quantum mechanics and the possible relationship of quantum information to it.

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Decoherence, Entanglement, and the Foundation of Quantum Mechanics:

Toward the quantum-classical transition

Gregg Jaeger

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$$|\Psi_{SA}\rangle|\varepsilon_0\rangle = \left(\sum_j \alpha_j |s_j\rangle|A_j\rangle\right)|\varepsilon_0\rangle \rightarrow \sum_j \alpha_j |s_j\rangle|A_j\rangle|\varepsilon_j\rangle$$

e.g., GHZ state

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    a generalized Schmidt decomposition for three-systems
    when it exists (Peres) shows one the preferred basis

The Decoherence Approach

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Zurek: *trace out environment* (observations as local)

$\rightarrow$ "diagonal" reduced state (*well*, nearly so)

Bub: *three-subsystem method still viable in less than ideal cases*

- the ``tridecompositional theorem''

component states need merely be linearly independent

this also generalizes to N-partite-decompositions

- important in good model environment; for example, a phonon bath
- it isn't full decoherence *per se* that is required,

rather obtaining *the preferred basis* is

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Environmental Monitoring and Disentanglement

"Pointer basis of quantum apparatus:
Into what mixture does the wave packet collapse?" (Zurek's story: PRD, 1516 `81)

$$|\Psi_{SA}\rangle|\varepsilon_0\rangle = \left(\sum_j \alpha_j |s_j\rangle|A_j\rangle\right)|\varepsilon_0\rangle \rightarrow \sum_j \alpha_j |s_j\rangle|A_j\rangle|\varepsilon_j\rangle$$

Measure z-spin in an (atom) environment:

"Because of the correlations with the environment, knowing the state of the apparatus in the {| A_{X+} >, | A_{X-} >} basis does not suffice any more to determine the state of the system.

Part of the information about the state of the spin has been "transferred" from the apparatus to the environment. And both the environment *and* the apparatus are correlated with |s_{Z+}> or | s_{Z-}> states of the spin.

We can therefore conclude that when the environment atom is present and interacts with the apparatus via [the appropriate H] …, then *the apparatus-spin system will retain perfect correlation in only one product basis* {| s_{Z+} > | A_{Z+} >, | s_{Z-} >| A_{Z-} >} *of the direct-product space.*

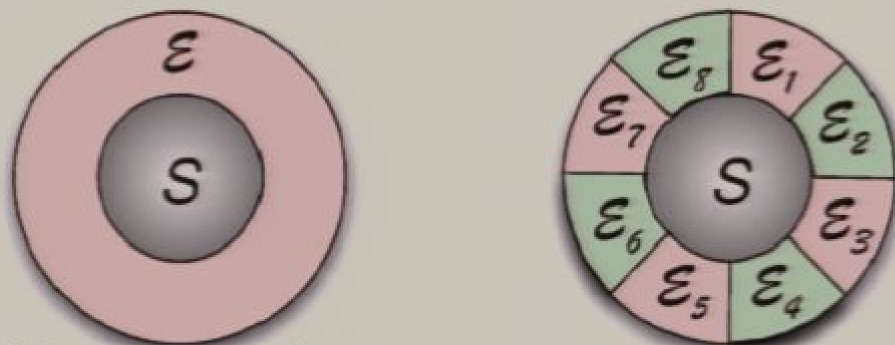
Hence, {| A_{Z+} >, | A_{Z-} >} is the pointer basis of the apparatus, which will eventually appear on the diagonal of the density matrix obtained by tracing out "environmental degrees of freedom," *i.e.* the state of the environment atom.

Measurements made by the apparatus on a spin eigenfunction along any other direction are to some degree obliterated by the interaction with the environment. In particular, no information about the orientation of the spin in the direction of the x-axis can be derived from the state of the apparatus alone."

Doesn't solve the measurement problem, but is presumably good physics that can tell us about Nature.

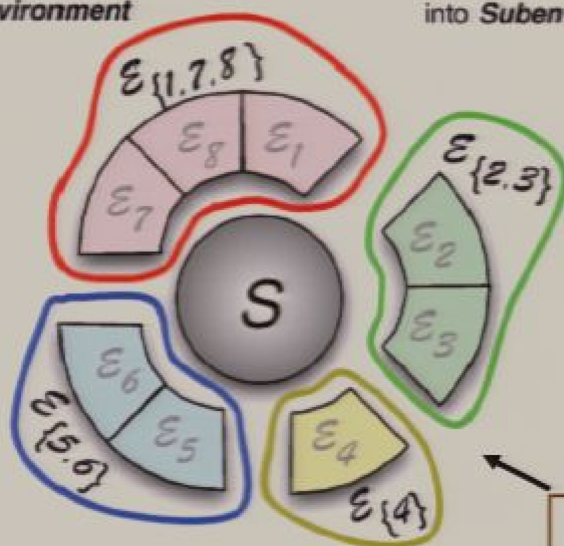
Elements of the "Redundancy Paradigm"

BLUME-KOHOUT AND ZUREK PHYSICAL REVIEW A 73, 062310 (2006)



(a) Decoherence Paradigm: ***Universe*** is divided into ***System*** & ***Environment***

(b) Redundancy Paradigm: ***Environment*** is divided into ***Subenvironments***



(c) ***Subenvironments*** are combined into ***Fragments*** that each have nearly-complete information.

Generalized GHZ states are the "perfectly redundant" ones

Explore this situation in detail
What matters here?

Decoherence and Disentanglement: Size

- **N-partite GHZ state dephasing rate is linear in number of qubits N**

$$|GHZ\rangle \equiv 1/\sqrt{2}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$$

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But* cluster states *have an entanglement lifetime is independent of N

Question: are the latter relevant to measurement?

Likely not, but they stretch the boundary.

Temperature

Entanglement *e.g.*, in spin pairs (AB)

Quantum correlations can exist only for finite temperatures:

simple argument: for infinite T, ρ_{AB} is $\mathbb{I}/16$, which is surrounded by a neighborhood of separable states; *hence*, entanglement is zero after some finite value, *thermally bounding* the set of entangled 2-qubit states.

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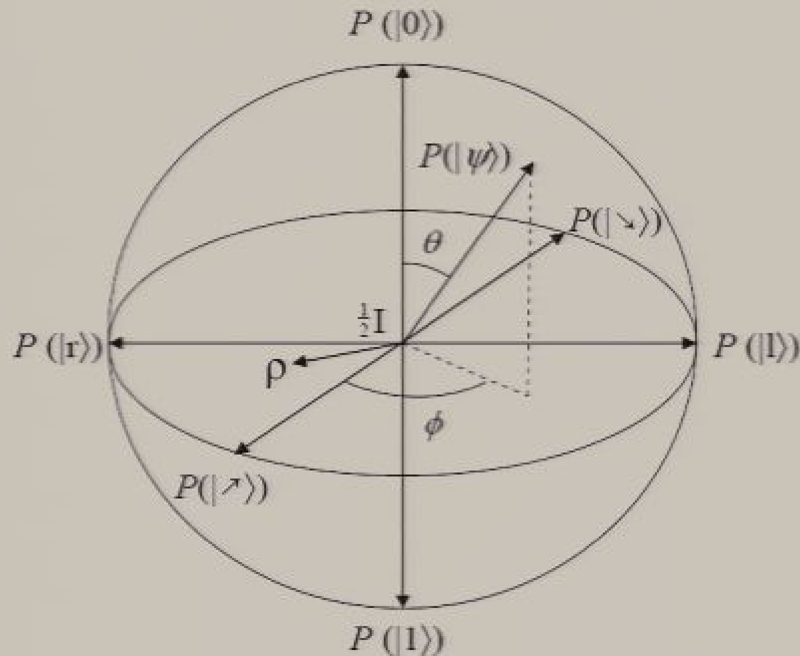
We recently started out more simply, will develop a more complete picture, carefully beginning with environmental phase noise acting on composite systems (cf. Eberly)

Qubit Pure Dephasing Dynamics

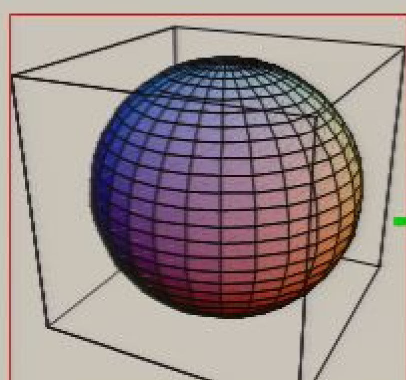


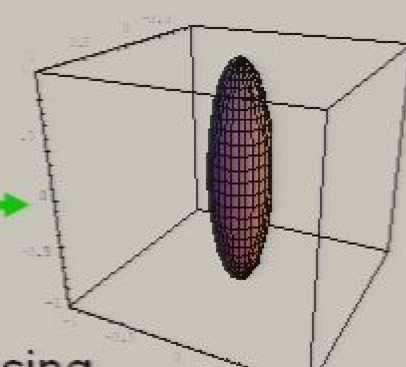
Operator-sum description:

$$\rho(t) = \sum_{\mu=1}^{N} K_{\mu}\,\rho(0)\,K_{\mu}^{\dagger}$$

Local phase damping

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{1-\gamma^2} \end{pmatrix}$$





Local dephasing

Decoherence vs. Disentanglement: Qubit Pairs

Entanglement is bounded by coherence-in-a-preferred-basis under *local* dephasing noises

- When decoherence occurs entanglement decays equally fast or faster than coherence (either local or global):

  e.g., for "fragile states" $\left|\phi_1^F\right\rangle = a\left|1\right\rangle_{AB} + b\left|2\right\rangle_{AB} + d\left|4\right\rangle_{AB}$

$|1\rangle_{AB} = |++\rangle_{AB}, |2\rangle_{AB} = |+-\rangle_{AB},$
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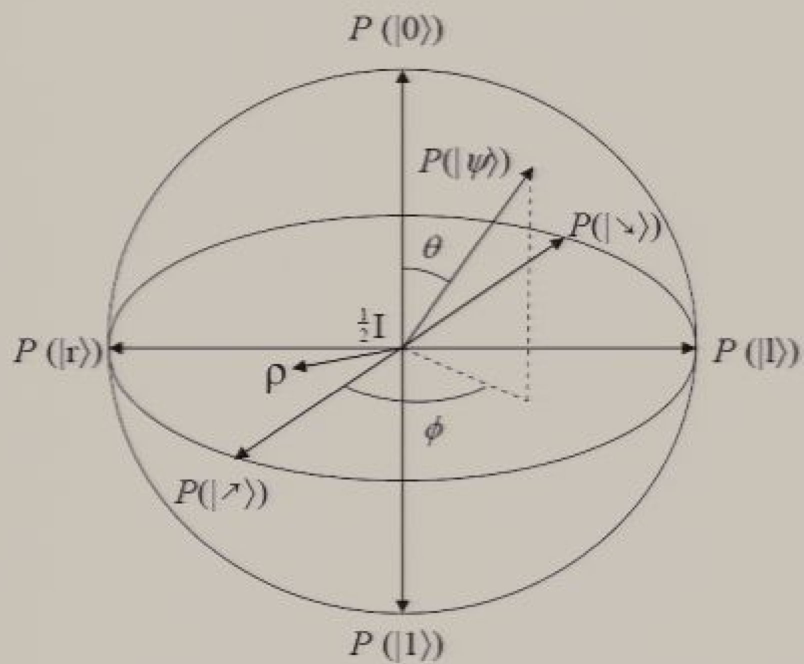
$$\rho(t) = \tilde{\mathcal{E}}(\rho(0)) = \sum_{i,j=1}^{2} D_k^\dagger E_j^\dagger F_i^\dagger \rho(0) F_i E_j D_k$$

D_k being the identity
In this case

Qubit Pure Dephasing Dynamics



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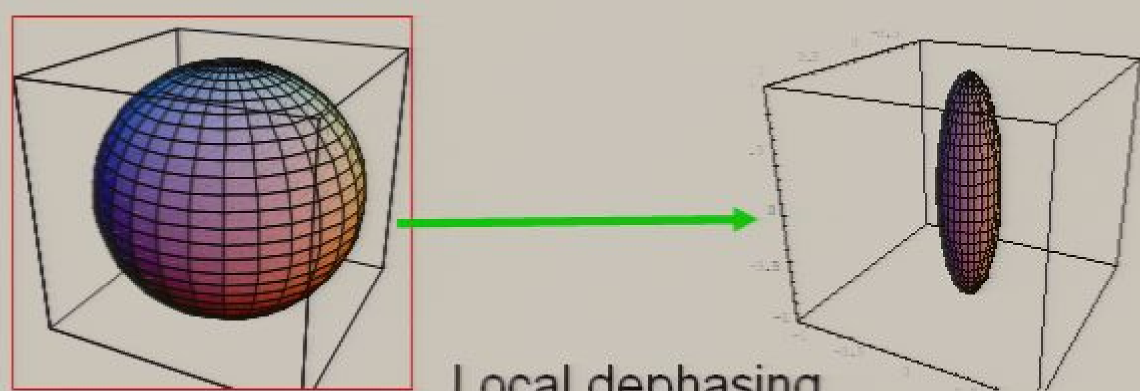

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•*Yu/Eberly Phys Rev B, 165322 `03.*

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$$\tau_e = \frac{\tau_A}{2} = \frac{\tau_B}{2} = \frac{\tau}{2}.$$

“This relation reminds us of the well-known relation between the phase coherence relaxation rate T_2 and the diagonal element decay rate T_1 in open quantum systems.[37]”

Current Work in Dephasing and Disentanglement in qudit systems

Entanglement appears generally bounded by coherence-in-preferred-basis (details to follow):

- *Pairs within tri-qubits*: Ann/Jaeger *Phys Rev B 75 115307 `07.*

Under all noise combinations:

multi-local: *subglobal+local:*

W: $\tau^{W}_{\text{dis},\mathcal{D}^A\mathcal{D}^B\mathcal{D}^C} < \tau^{W}_{\text{3-dec},\mathcal{D}^A\mathcal{D}^B\mathcal{D}^C}$ $\tau^{W}_{\text{dis},\mathcal{D}\mathcal{E}} < \tau^{W}_{\text{3-dec},\mathcal{D}\mathcal{E}},$

$\tau^{W}_{\text{dis},\mathcal{D}^A\mathcal{D}^B\mathcal{D}^C} < \tau^{W}_{\text{2-dec},\mathcal{D}^A\mathcal{D}^B\mathcal{D}^C},$ $\tau^{W}_{\text{dis},\mathcal{D}\mathcal{E}} < \tau^{W}_{\text{2-dec},\mathcal{D}\mathcal{E}},$

$$|W^g\rangle = \bar{a}_1|001\rangle + \bar{a}_2|010\rangle + \bar{a}_4|100\rangle$$

GHZ:

$$\rho^{\text{GHZ}}_{\text{AB}}(t) = \begin{pmatrix} |\bar{a}_0|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\bar{a}_7|^2 \end{pmatrix}$$

$$\rho^{\text{GHZ}}_{\text{AC}}(t) = \begin{pmatrix} |\bar{a}_0|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\bar{a}_7|^2 \end{pmatrix}$$

$$\rho^{\text{GHZ}}_{\text{BC}}(t) = \begin{pmatrix} |\bar{a}_0|^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & |\bar{a}_7|^2 \end{pmatrix}$$

$$C^2_{\text{AB}} = 0$$
$$C^2_{\text{AC}} = 0$$
$$C^2_{\text{BC}} = 0$$

- ***qutrit pairs***: Jaeger/Ann *(J Mod Opt, submitted)*

Similar results are obtained.

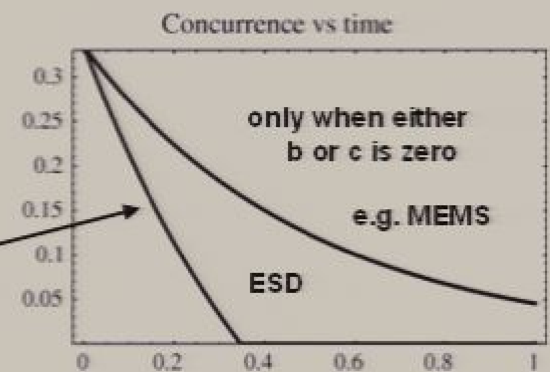
An Intriguing Effect for Mixed States: entanglement ``sudden death''

Global dephasing noise can kill entanglement of qubit pairs in *finite time* for *initially mixed states* (emergence of classicality)

Consider the following "standard" class of states that

$$\rho^{AB} = \begin{pmatrix} a & 0 & 0 & w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ w^* & 0 & 0 & d \end{pmatrix}.$$

includes the Werner states (take z=0; $|\Phi^+\rangle$-type)



Global noise case:

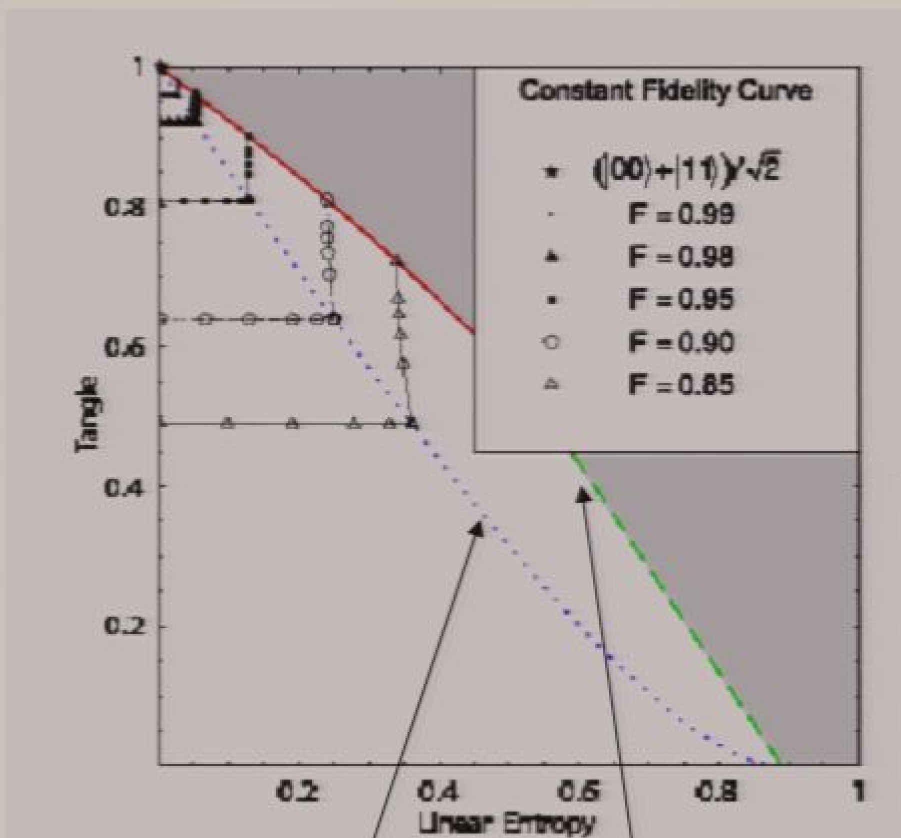
$$\rho^{AB}(t) = \mathcal{E}_D(\rho^{AB}(0)) = \begin{pmatrix} a & 0 & 0 & \gamma^4 w \\ 0 & b & z & 0 \\ 0 & z^* & c & 0 \\ \gamma^4 w^* & 0 & 0 & d \end{pmatrix},$$

$$C(\rho) = 2\max\{0, |z| - \sqrt{ad}, |w| - \sqrt{bc}\}.$$

Yu/Eberly Opt. Comm. 264, 393, `06.

E.g., MEMS (Kwiat group)

quant-ph/0407172 v2 22 Oct 2004



Werner states with $|\Phi^+\rangle$

MEMS

$$\rho_{MEMS\ I} = \begin{pmatrix} \frac{r}{2} & 0 & 0 & \frac{r}{2} \\ 0 & 1-r & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r}{2} & 0 & 0 & \frac{r}{2} \end{pmatrix}, \quad \frac{2}{3} \le r \le 1,$$

$$\rho_{MEMS\ II} = \begin{pmatrix} \frac{1}{3} & 0 & 0 & \frac{r}{2} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{r}{2} & 0 & 0 & \frac{1}{3} \end{pmatrix}, \quad 0 \le r \le \frac{2}{3};$$

the parameter r is the concurrence of the MEMS.

FIG. 1: Constant fidelity curves for the maximally entangled state $(|00\rangle + |11\rangle)/\sqrt{2}$ (star, upper left corner). Also shown are the Werner state curve (dotted line) and, bounding the gray region of nonphysical entropy-tangle combinations, the MEMS curve, which is solid for $\rho_{MEMS\ I}$ and dashed for $\rho_{MEMS\ II}$. The (horizontal) constant fidelity curves below the Werner state curve are swept out by comparing the starting state with states of the form $\rho_1(\epsilon, \theta)$, equation (20), while the (nearly vertical) curves above the Werner state line are generated by varying the parameters of $\rho_2(\epsilon, r)$ given by equation (22). For comparison, the pure product state $|00\rangle$ (lower left corner) has fidelity of 0.5 with this target.

$$\rho_2(\epsilon, r) \equiv (1-\epsilon)\rho_{MEMS}(r) + \frac{\epsilon}{4}\mathbf{1}_4$$

Main Foundational Motivation

E.g., the environmental conditions giving rise to entanglement sudden death are *pervasive in Nature*.
(mixed states under phase noise, amplitude damping)

A more nuanced view of decoherence may qualitatively improve our understanding of the onset of classicality
(say, as entanglement loss)

In particular, investigating the behavior of the various interesting classes of state under various environmental conditions *provides a better understanding of the flexibility/permeability of the quantum-classical boundary*

Decoherence and Disentanglement in Multi-Qubit Systems

Model: pure dephasing noise

acting individually or collectively at

1- and/or 2- or 3-qubit levels

$$\rho(0)=\begin{pmatrix}\rho_{11} & \rho_{12} & \cdots & \rho_{1n}\\ \rho_{21} & \rho_{22} & \cdots & \rho_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ \rho_{n1} & \rho_{n2} & \cdots & \rho_{nn}\end{pmatrix}$$

Decoherence : decay of off-diagonal elements

Basis-dependent

Disentanglement (bipartite) : loss of concurrence

Basis-independent

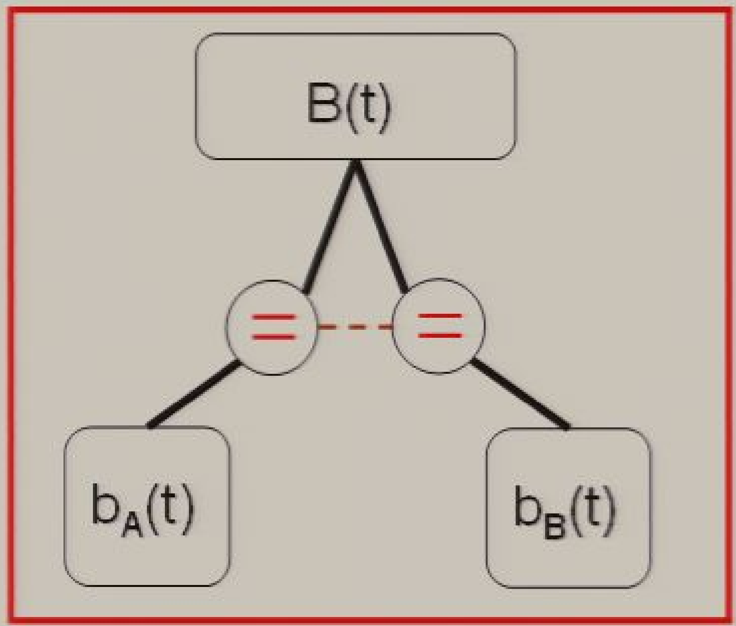
$$C(\rho)=\max\left\{0,\sqrt{\lambda_1}-\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4}\right\}$$

$$\lambda_i = \text{eigenvalues}[\rho\tilde{\rho}] \quad \tilde{\rho}=\sigma_y^{\otimes 2}\rho^*\sigma_y^{\otimes 2}$$

Compare timescales

$$\tau_{dis} \text{ vs } \tau_{dec}$$

2-Qubit System (3 noise fields)

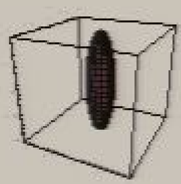


Open-system Hamiltonian:

$$H(t) = -\frac{1}{2}\mu\left[B(t)(\sigma_z^A + \sigma_z^B) + b_A(t)\sigma_z^A + b_B(t)\sigma_z^B\right] \qquad \sigma_z^{AB} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Markovian noise

"Preferred basis"



$$\langle B(t)\rangle = 0 \qquad \langle B(t)B(t')\rangle = \frac{\Gamma}{\mu^2}\delta(t-t')$$

$$\langle b_i(t)\rangle = 0 \qquad \langle b_i(t)b_i(t')\rangle = \frac{\Gamma_i}{\mu^2}\delta(t-t')$$

B, b_i fluctuations, Γ, Γ_i damping rates

Compound System Dynamics

Ensemble average state over noise fields: $\rho(t) = \langle\langle\langle \rho_{st}(t) \rangle\rangle\rangle$

statistical state $\rho_{st}(t) = U(t)\rho(0)U^{\dagger}(t)$

Qubits initially uncorrelated with fields $U(t) = \exp[-i\int_0^t H(t')dt']$

Operator-sum representation of evolution in Markov approximation

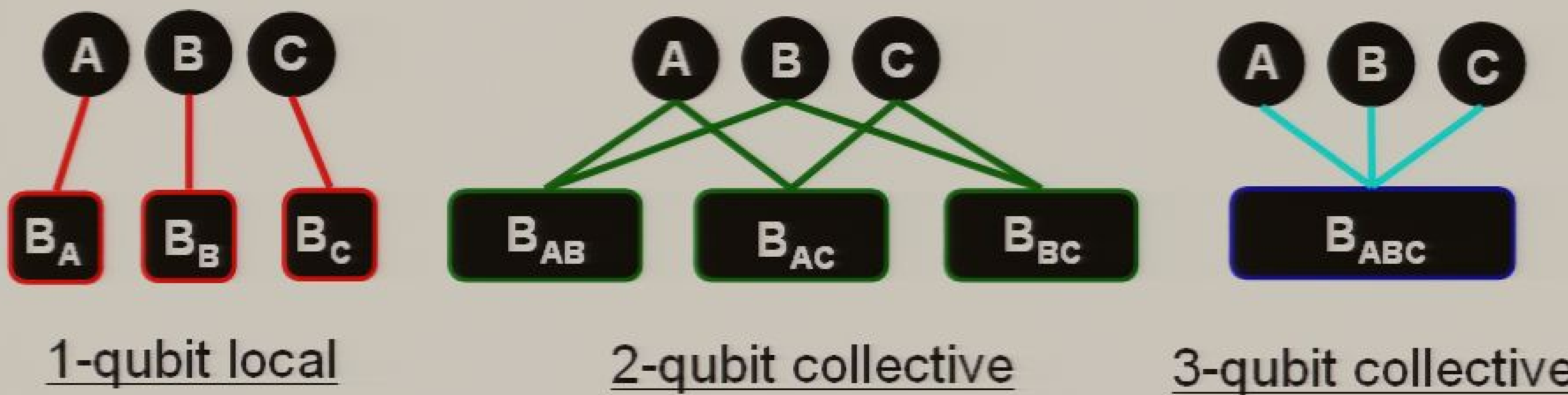
$$\rho(t) = \sum_{i,j=1}^{2} \sum_{k=1}^{3} D_k^{\dagger} E_j^{\dagger} F_i^{\dagger} \rho(0) F_i E_j D_k$$

e.g. local noise at qubit A

$$\gamma_A = e^{-t/2T_A} \qquad E_1 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix} \otimes \mathbf{I}$$

$$\omega_A = \sqrt{1-\gamma_A^2} \qquad E_2 = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix} \otimes \mathbf{I}$$

3-Qubit Phase Noise
(7 noise fields)



$$H(t) = -\frac{1}{2}\mu\left[\begin{array}{l} B_A^{(1)}(t)\sigma_z^A + B_B^{(1)}(t)\sigma_z^B + B_C^{(1)}(t)\sigma_z^C + \\ B_{AB}^{(2)}(t)(\sigma_z^A + \sigma_z^B) + B_{BC}^{(2)}(t)(\sigma_z^B + \sigma_z^C) + B_{AC}^{(2)}(t)(\sigma_z^A + \sigma_z^C) + \\ B_{ABC}^{(3)}(t)(\sigma_z^A + \sigma_z^B + \sigma_z^C) \end{array}\right]$$

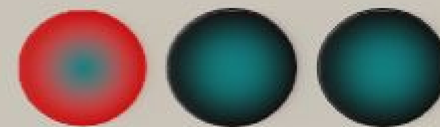
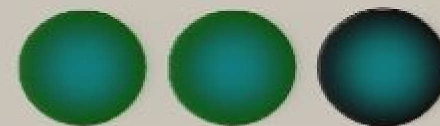
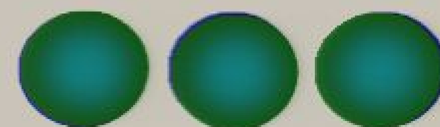
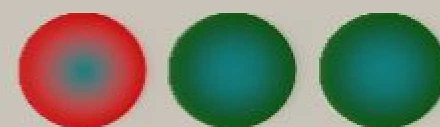
$$\langle B_i(t)\rangle = 0$$

$$\langle B_i(t)B_i(t')\rangle = \frac{\Gamma_i}{\mu^2}\delta(t-t')$$

$$i = A(B,C), AB(AC,BC), ABC$$

3-Qubits in Phase-Noise Environments

$$\rho(t) = \sum_{i,j,k=1}^{2} \sum_{l,m,n,p=1}^{3} \left(F_p^{ABC} E_n^{AC} E_m^{BC} E_l^{AB} D_k^{C} D_j^{B} D_i^{A} \right) \rho(0) \left(D_i^{A\dagger} D_j^{B\dagger} D_k^{C\dagger} E_l^{AB\dagger} E_m^{BC\dagger} E_n^{AC\dagger} F_p^{ABC\dagger} \right)$$

- 1-qubit local $\mathcal{D}^{A}$ 
- 2-qubit collective $\mathcal{E}^{AB}$ 
- 3-qubit collective $\mathcal{F}^{ABC}$ 
- 3-qubit multi-local $\mathcal{D}^{A}\mathcal{D}^{B}\mathcal{D}^{C}$ 
- 1-qubit local + 2-qubit collective $\mathcal{D}^{A}\mathcal{E}^{BC}$ 

3-Qubit Dephasing Dynamics

$$\rho(t) = \sum_{i,j,k=1}^{2} \sum_{l,m,n,p=1}^{3} \left(F_p^{ABC} E_n^{AC} E_m^{BC} E_l^{AB} D_k^{C} D_j^{B} D_i^{A} \right) \rho(0) \left(D_i^{A\dagger} D_j^{B\dagger} D_k^{C\dagger} E_l^{AB\dagger} E_m^{BC\dagger} E_n^{AC\dagger} F_p^{ABC\dagger} \right)$$

$$\gamma_i = \exp\left[\frac{-t}{2T_i}\right] \qquad i = \{A, B, C, AB, ABC, \ldots\}$$

$$\omega_i = \sqrt{1-\gamma_i^2} \qquad \omega_{i2} = -\gamma_i^2\sqrt{1-\gamma_i^2}$$

$$\omega_{i1} = \sqrt{1-\gamma_i^2} \qquad \omega_{i3} = \sqrt{(1-\gamma_i^2)(1-\gamma_i^4)}$$

$$D_1^A = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_i \end{pmatrix} \otimes \mathbf{I} \otimes \mathbf{I}$$

$$D_2^A = \begin{pmatrix} 0 & 0 \\ 0 & \omega_i \end{pmatrix} \otimes \mathbf{I} \otimes \mathbf{I}$$

$$E_1^{AB} = \begin{pmatrix} \gamma_i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma_i \end{pmatrix} \otimes \mathbf{I} \quad E_2^{AB} = \begin{pmatrix} \omega_{i1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{i2} \end{pmatrix} \otimes \mathbf{I} \quad E_3^{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_{i3} \end{pmatrix} \otimes \mathbf{I}$$

$$F_1^{ABC} = \begin{pmatrix} \gamma_{i1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \gamma_{i1} \end{pmatrix} \quad F_2^{ABC} = \begin{pmatrix} \omega_{i1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{i2} \end{pmatrix} \quad F_3^{ABC} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{i3} \end{pmatrix}$$

3-Qubits in Phase-Noise Environments

$$\rho(t) = \sum_{i,j,k=1}^{2} \sum_{l,m,n,p=1}^{3} \left(F_p^{ABC} E_n^{AC} E_m^{BC} E_l^{AB} D_k^{C} D_j^{B} D_i^{A}\right) \rho(0) \left(D_i^{A\dagger} D_j^{B\dagger} D_k^{C\dagger} E_l^{AB\dagger} E_m^{BC\dagger} E_n^{AC\dagger} F_p^{ABC\dagger}\right)$$

- 1-qubit local $\mathcal{D}^{\mathrm{A}}$
- 2-qubit collective $\mathcal{E}^{\mathrm{AB}}$
- 3-qubit collective $\mathcal{F}^{\mathrm{ABC}}$
- 3-qubit multi-local $\mathcal{D}^{\mathrm{A}}\mathcal{D}^{\mathrm{B}}\mathcal{D}^{\mathrm{C}}$
- 1-qubit local + 2-qubit collective $\mathcal{D}^{\mathrm{A}}\mathcal{E}^{\mathrm{BC}}$