

Title: The Monster and its moonshine functions

Date: Mar 13, 2007 02:00 PM

URL: <http://pirsa.org/07030014>

Abstract: Abstract: This group of astronomical order is slowly yielding its secrets. It is the symmetry group of a rational conformal field theory. In this introductory talk, I will discuss the functions that constitute monstrous moonshine and explain the importance of the monster group and its connections with better established parts of mathematics

1861  
1873

1861

1873

1898



1861

1873

1898.

JANKO

1861

1873

1898

JANKO  
1964  $\mathcal{T}_1 \subseteq G_2^{(1)}$

1861

1873

1898.

26

JANKO

{ 1964

1974

$\overline{J}_1 \subseteq G_2^{(1)}$

1861

1873

1898

20

JANKO  
1964       $J_1 \subseteq G_2^{(II)}$   
26  
SPORADIC  
1974

1861

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1898

20 ARE INVOLVED IN M.

26

SPORADIC

JANKO

{ 1964

$\bar{J}_1 \subseteq G_2^{(II)}$

1974

1861

1873

1898

20 ARE INVOLVED IN

26

SPORADIC

JANKO

{ 1964

1974

$\overline{J}_1 \subseteq G_2^{(1)}$

M

## Kepler Platonic



## Scottish Late Neolithic





## Scottish Late Neolithic



(Toy) Example: Symmetric group  $S_3$  order 3!

shapes:  $1^3 \quad 2,1 \quad 3$

$$\rightarrow R: \begin{matrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{matrix} \quad r = \text{rank } R = 2$$

$$P_g(q) = \det(1 - R(g)q)$$

$$f_{\langle g \rangle} = 1/P_g(q)$$

$$f_{(1^3)} = \frac{1}{(1-q)^2}, \quad f_{(2,1)} = \frac{1}{1-q^2}, \quad f_{(3)} = \frac{1-q}{1-q^3}$$

$$\left. \begin{array}{cccc} q^0 & 1 & 1 & 1 \\ q^1 & 2 & 0 & -1 \\ q^2 & 3 & 1 & 0 \\ q^3 & 4 & 0 & 1 \\ q^4 & 5 & 1 & -1 \\ q^5 & 6 & 0 & 0 \\ q^6 & 7 & 1 & 1 \\ q^7 & 8 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots \end{array} \right\} = B_0$$

$$B_k = B_0 + k[6, 0, 0]$$

Less naïve:

Replace  $P_g(q)$  by

$$\prod_{k \geq 1} P_g(q^k)$$

to get:

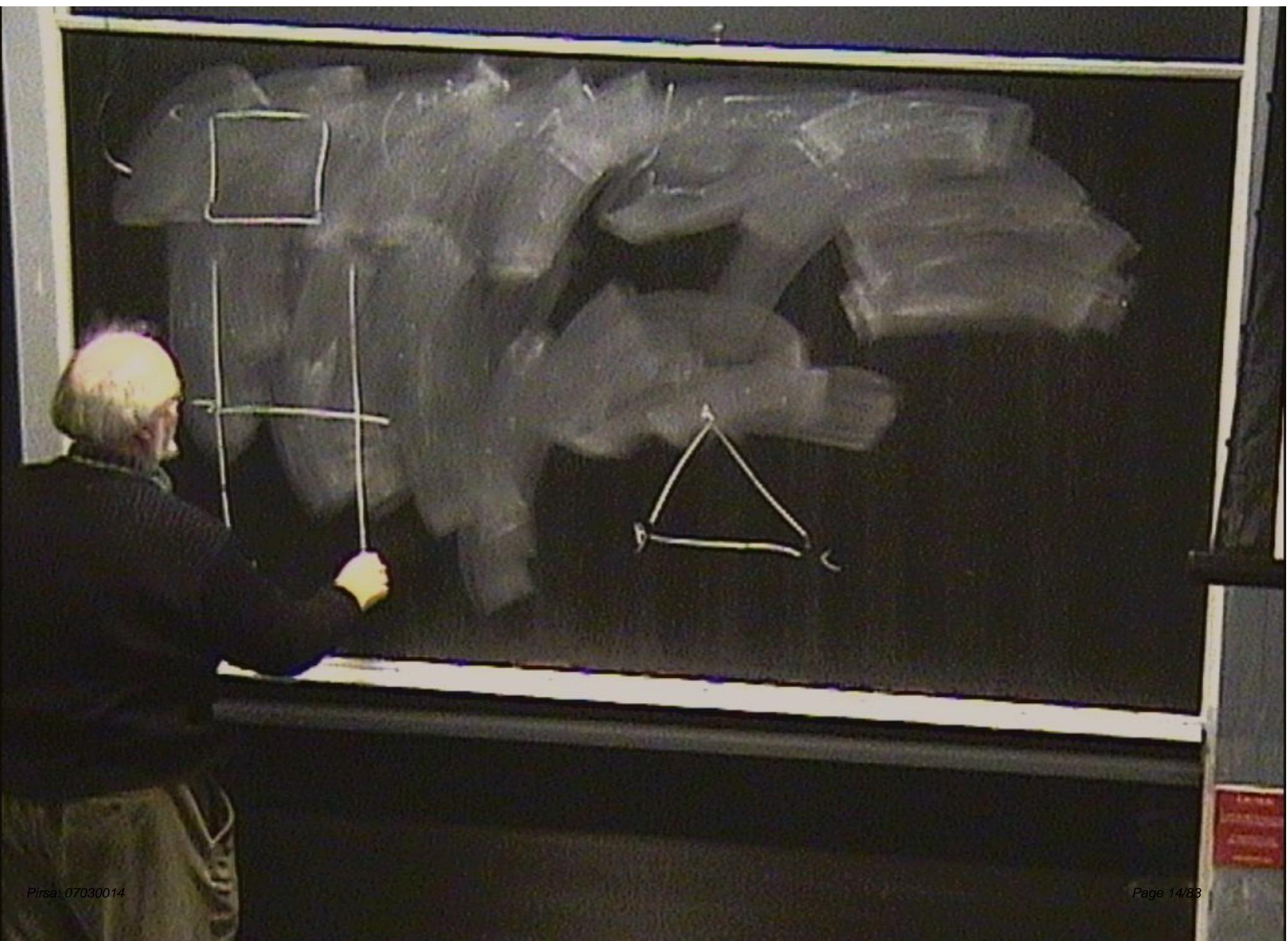
$$f_{(1^3)} = \prod_{k \geq 1} \frac{1}{(1 - q^k)^2}, \quad f_{(2,1)} = \prod_{k \geq 1} \frac{1}{1 - q^{2k}}, \quad f_{(3)} = \prod_{k \geq 1} \frac{1 - q^k}{1 - q^{3k}}.$$

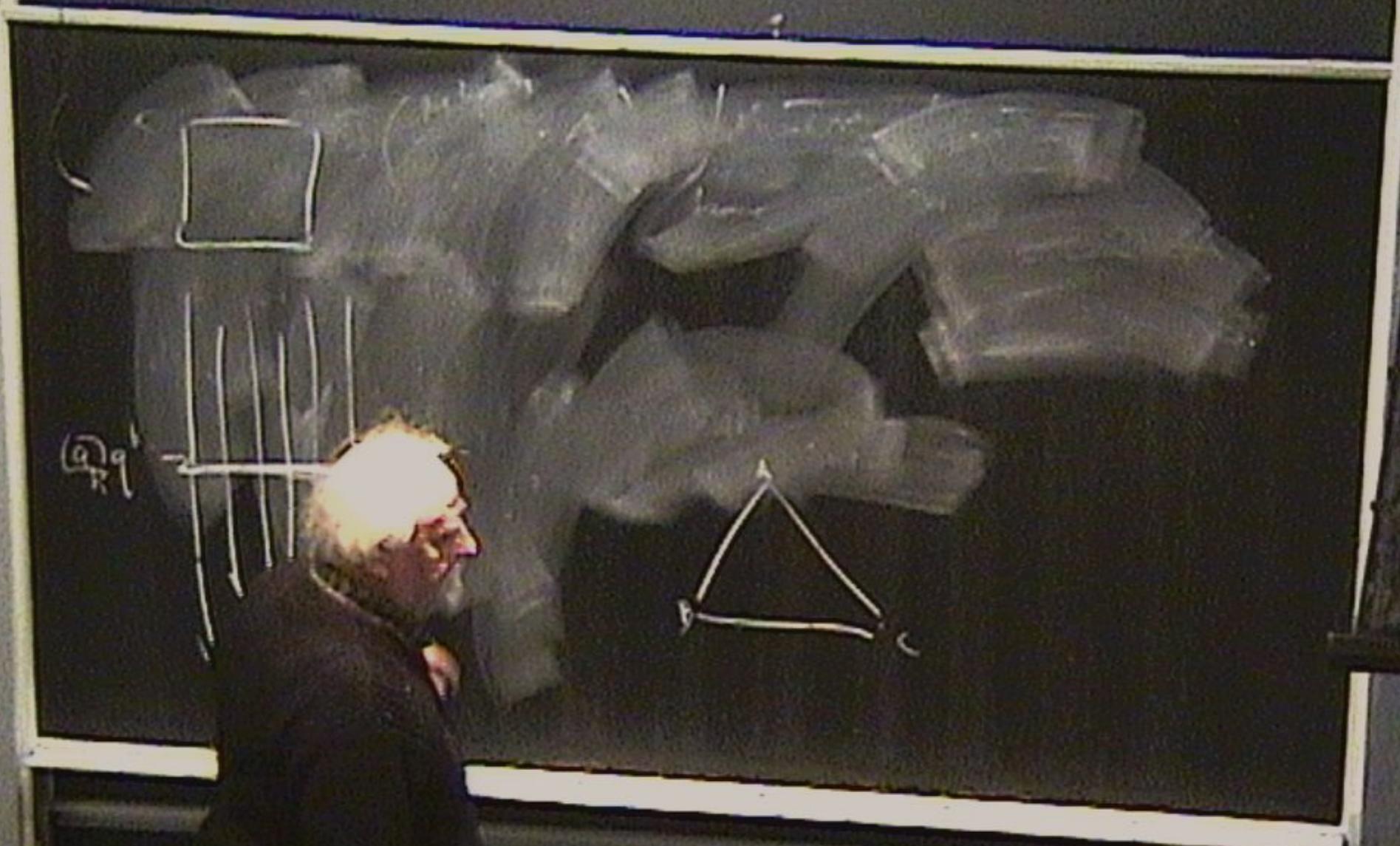
Now replace  $f_{(\cdot)}$  by  $q^{-\frac{r}{24}} f_{(\cdot)}$  to get

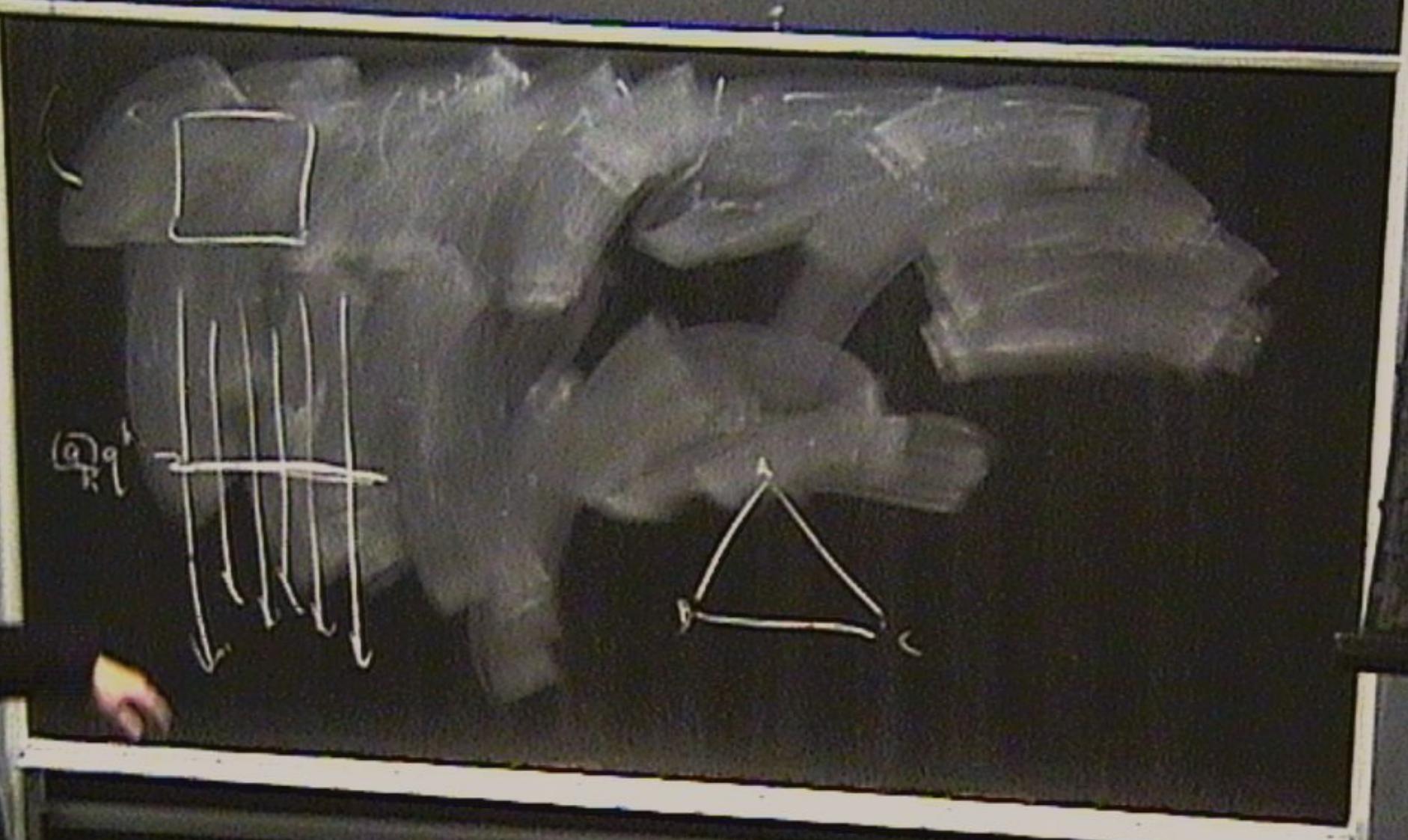
$$f_{(1^3)} = \eta(q)^{-2}, \quad f_{(2,1)} = \eta(q^2)^{-1}, \quad f_{(3)} = \frac{\eta(q)}{\eta(q^3)}$$

with Dedekind's  $\eta$  function:

$$\eta(q) = q^{\frac{1}{24}} \prod_{k \geq 1} (1 - q^k).$$







$|M| =$

8080174247945...617107570057543680000000000

$= 2^{46}3^{15}5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

a product of the (15) supersingular primes.

The sporadic finite simple groups:

The classical ones:  $M_{11}$ ,  $M_{12}$ ,  $M_{22}$ ,  $M_{23}$  and  $M_{24}$  were discovered in 1861 by Emile Mathieu. They caused him to write a paper on the small ones, in which he announced the large ones. In 1873 his second paper describes the large ones.

In 1898 G.A. Miller attempted to show that  $M_{24}$  did not exist! This he retracted (in French!) in 1900.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ nc & d \end{pmatrix} \mid ad - Ncb = 1 \right\}$$



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$\xrightarrow{\text{det}} \mathfrak{X}$   
stabilisier

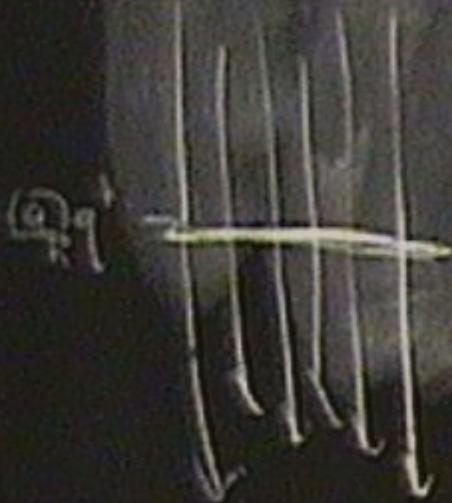


$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - Ncb = 1 \right\}$$

$\xrightarrow{\text{det } \neq 1}$   
stabilis

$$\Gamma_{0(p)}^+ \approx \left\{ \Gamma_0(p) \cup \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

$\xrightarrow{\text{genus zero}}$



$|M| =$

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$= 2^{46}3^{15}5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

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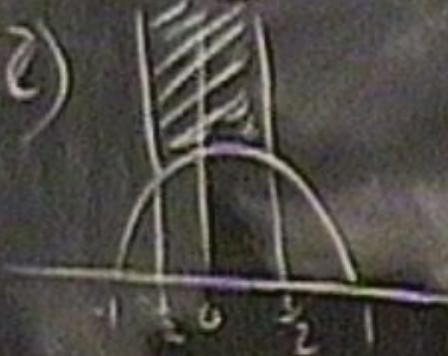
$$j(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi iz}, \quad |q| > 0.$$

$$j(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi iz}, \quad q = z > 0.$$



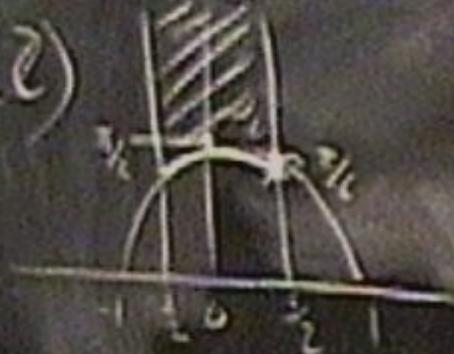
$$j(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad q, z > 0.$$

$$\mathbb{G}_m \cong \mathrm{PSL}_2(\mathbb{C})$$



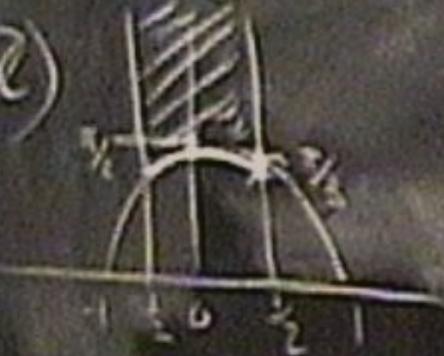
$$j(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad \Im z > 0.$$

$\text{PSL}_2(\mathbb{C})$



$$\varphi(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad \operatorname{Im} z > 0.$$

$$G_0 \cong PSL_2(\mathbb{C})$$



$$2 \cdot \frac{w_1}{w_2}$$



$$\varphi(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad \operatorname{Im} z > 0.$$

$$\Gamma \subseteq PSL_2(\mathbb{C})$$

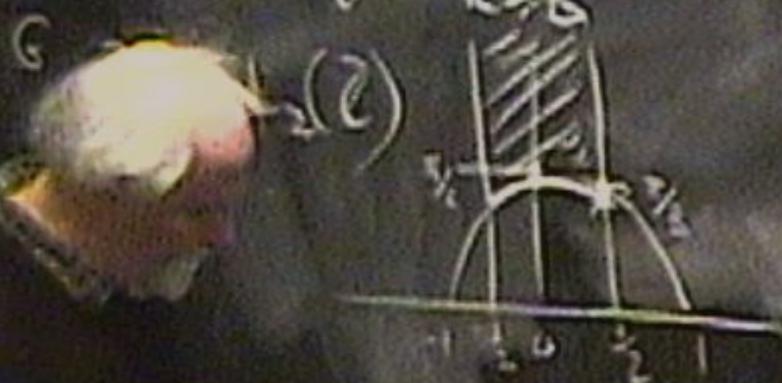


$$2 \cdot \frac{w_1}{w_2}$$

$$\sqrt{\frac{w_1}{w_2}}$$

$T_n$  Hedge operte.

$$j(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad \operatorname{Im} z > 0.$$



$$2 \cdot \frac{w_1}{w_2} \quad \begin{array}{c} \swarrow \\ w_1 \\ \searrow \\ w_2 \end{array}$$

$T_n$  Hodge operator  $\{L\} \rightarrow \{\{L_i\}\}$

$$[L:L_i] = n$$

$$\gamma(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad \Im z > 0.$$



$$2 \cdot \frac{w_1}{w_2}$$



T<sub>n</sub>

Hedie opente.

{L} → {{L; 3}}

[L:L;] = n

$$\phi(q) = L + \sum_{k=0}^{\infty} q^k L^k, \quad q = e^{2\pi i z}, \quad q, z > 0.$$

$$G_i \cong P$$

(1)



$$n=2 \quad \alpha = \frac{\pi - 1}{2} = \frac{\pi}{2}$$

$$\omega = \frac{\omega_0}{\omega_0}$$



$T_n$

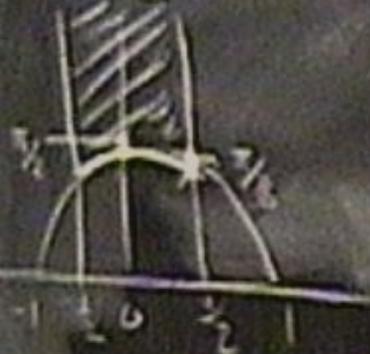
Hedie opata.

$\{L\} \rightarrow \{\{L_i\}\}$

$$[L:L_i] = n$$

$$\psi(q) = -\sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad \Im z > 0.$$

$$n=2 \quad z = \frac{\pi i}{2} \quad \bar{z} = \frac{\pi i}{2}$$



$$z = \frac{w_1}{w_2}$$

$$\begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

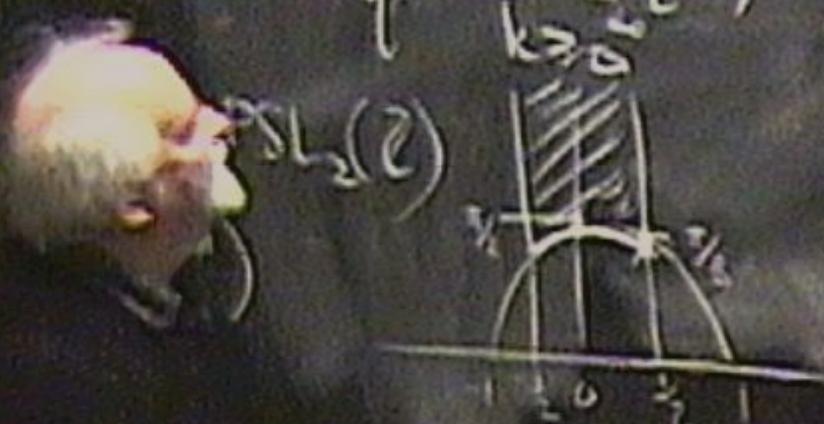
$T_n$

Hedde opera-

$\{L\} \rightarrow \{\{L_i\}\}$

$$[L:L_i] = n$$

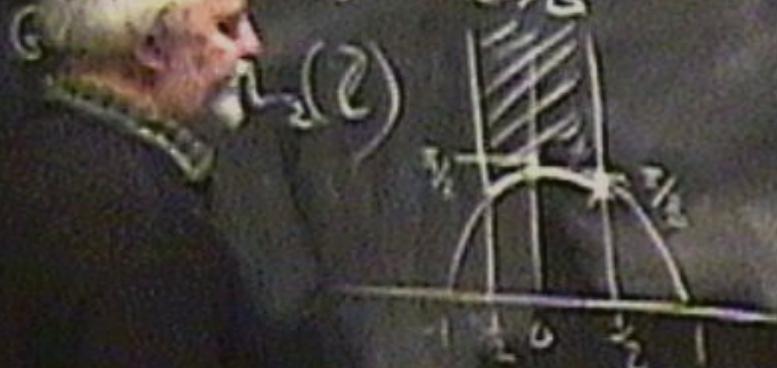
$$J(q) = \frac{1}{q} + \sum_{k>0} c_k q^{kR}, \quad q = e^{\frac{2\pi i z}{n}}, \quad n \geq 2.$$



$$n=2 \quad z = \frac{\pi - i}{2} \approx \frac{\pi}{2}$$

$\omega \cdot \frac{w_1}{w_2}$        $\omega$        $T_n$  Hecke operator.  $\{L\} \rightarrow \{\{L_i\}\}$   
 $[L:L_i] = n$

$$j(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad \Im z > 0.$$



$$n=2 \quad \alpha = \frac{\pi-1}{2} = \frac{\pi}{2}$$

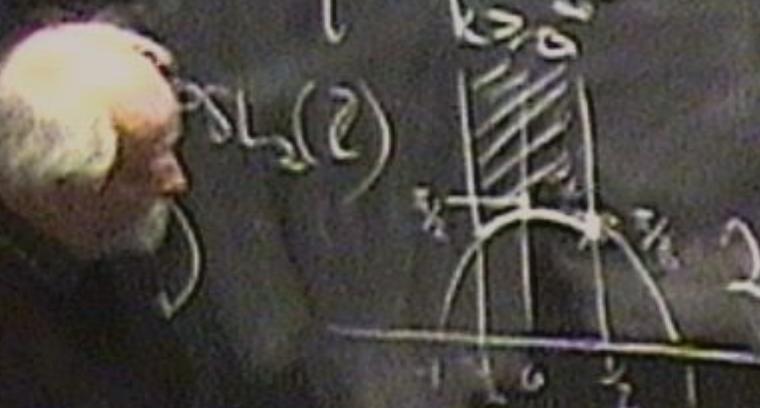
$$j(z) \rightarrow j(\frac{z+1}{2}) + j(\frac{z-1}{2}) + j(\frac{z-i}{2})$$

$$2 \cdot \frac{w_1}{w_2} \quad \begin{array}{c} \diagup \\ w_1 \\ \diagdown \\ w_2 \end{array}$$

$T_n$  Hecke operte.  $\{L\} \rightarrow \{\{L_i\}\}$

$$[L : L_i] = n$$

$$j(q) = \frac{1}{q} + \sum_{k>0} c_k q^{kz}, \quad q = e^{\frac{2\pi iz}{n}}, \quad n > 0.$$



$$n=2 \quad z = \frac{z+1}{2} \quad \bar{z} = \frac{z-1}{2}$$

$$2 j(z) \rightarrow j\left(\frac{z+1}{2}\right) + j\left(\frac{z-1}{2}\right) + j\left(\frac{z-i}{2}\right)$$

$$j(z) = \frac{1}{z} + \sum_{k>0} c_k z^k, \quad k=0, \quad c_k > 0.$$

$\mathcal{G} \cong PSL_2(\mathbb{C})$

$\langle \rangle$



$$n=2 \quad \omega = \frac{z+1}{2} \quad \bar{z}_2$$

$$\begin{aligned} 2 j(z) &\rightarrow j\left(\frac{z}{2}\right) + j\left(\bar{z}_2\right) + j\left(\frac{z+1}{2}\right) \\ &= j^2(z) - 2 c_1 \end{aligned}$$



$$\text{PROOF: } \frac{\partial}{\partial z} \left( \frac{f(z)}{z^2 - 1} \right) = \frac{(z^2 - 1)f'(z) - 2zf(z)}{(z^2 - 1)^2} \rightarrow \frac{1}{z^2 - 1} f'(z) + \frac{2z}{z^2 - 1} f(z)$$
$$= j'(z) - 2c_1$$

FABER

$$c_1 = 196884$$

$$\text{Diagram: } \begin{array}{c} \text{O} \\ \text{---} \\ \text{A} \end{array} \quad \begin{array}{c} \text{O} \\ \text{---} \\ \text{B} \end{array} \quad \begin{array}{c} \text{O} \\ \text{---} \\ \text{C} \end{array} \quad \begin{array}{c} \text{O} \\ \text{---} \\ \text{D} \end{array}$$
$$f(z) = \frac{1}{z-1} + \frac{1}{z-2} + \frac{1}{z-3}$$
$$= \frac{z^2 - 2z}{z^2 - 6z + 11}$$

FABER

$$C_1 = 196884$$

$$= 196883 + \epsilon$$

(b)

$$\frac{1}{(1-z)(1-\bar{z})} = \frac{1}{1-z} + \frac{1}{1-\bar{z}} = j(z) + \bar{j}(\bar{z})$$
$$= j^2(z) - 2c_1$$

FABER

$c_1 = 196884$

$= 196883 + \epsilon$

MOONSHINE

Kronecker: Die Zahlen hat der liebe Gott gemacht. Alles andere ist Menschenwerk.

1,2,3,... are the degrees of irreducible representations of  $SU_2(\mathbb{C})$ .

The Oxford English Dictionary tells us:

**MONSTROUS:** of unnaturally or extraordinarily huge dimensions; greatly to be marvelled at; astounding.

**MOONSHINE:** something unsubstantial, of dubious quality.

$$z \cdot \frac{w_1}{w_2}$$

 $T_n$ 

Hedie opnb.

$$\{L\} \rightarrow \{\{L_i\}\}$$

$$[L:L_i] = n$$

$$j(z) = \frac{1}{q} - \frac{1}{q} \sum_{k=0}^{\infty} q^k L_k, \quad q = e^{2\pi i z}, \quad q \approx 0.$$

$$n=2 \quad z = \frac{z-1}{2} \quad \bar{z} = \frac{z+1}{2}$$

$$2 j(z) \rightarrow j\left(\frac{z-1}{2}\right) + j\left(\bar{z}\right) + j\left(\frac{z+1}{2}\right)$$

$$= j(z) - 2c_1$$

FABER

$$z \cdot \frac{w_1}{w_2}$$



$$T = \text{diag}(H_L(\omega))$$

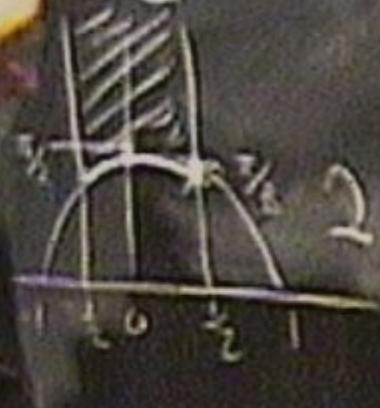
Hedie opera.

$$\{L\} \rightarrow \{\{L_i\}\}$$

$$[L:L_i] = n$$

$$j(q) = 1 + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad q_m z > 0.$$

$$G_j \cong \mathbb{P}_n$$



$$n=2 \quad 2 \in \frac{-1}{2} \quad \bar{z}_2$$

$$2 j(z) \rightarrow j\left(\frac{z}{2}\right) + j\left(2\bar{z}\right) + j\left(\frac{z+1}{2}\right)$$

$$= j(z) - 2c_1$$

FABER

1861

1873

1898

20 ARE INVOLVED IN

CONWAY-NORTON

26

SPORADIC

JANKO

1964

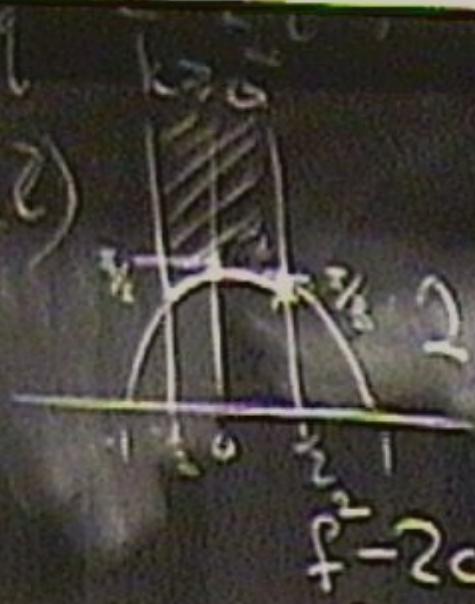
1974

M

$\mathcal{T}_1 \subseteq G_2^{(II)}$

1979 BULL. LOND. MATH.  
SOC.

$$G \cong PSL_2(\mathbb{C})$$
$$\mathcal{J} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right)$$



$$n=2 \quad 2 \approx \frac{z-1}{2} \quad \frac{z}{2}$$
$$j(z) \rightarrow j\left(\frac{z}{2}\right) + j\left(\frac{z-1}{2}\right) + j\left(\frac{z+1}{2}\right)$$

$$= j(z) - 2c_1$$

FABER

1968 83 + 1

MOONSHINE

$$2 \cdot \frac{\omega_1}{\omega_2}$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}$$

$T$  Hodge operator

$$\{L\} \rightarrow \{\{L_i\}\}$$

$\text{Tors}(H_k(\tilde{M}))$

$$[L : L_i] = n$$

$$j(q) = \frac{1}{q} + \sum_{k>0} q^k, \quad q = e^{2\pi i z}, \quad \Im z > 0.$$

$$G_j \cong PSL_2(\mathbb{Z})$$



$$n=2 \quad z = \frac{3\pi i}{2} \quad \frac{\pi i}{2}$$

$$\Rightarrow j\left(\frac{1}{2}\right) + j\left(\frac{2\pi i}{2}\right) + j\left(\frac{\pi i}{2}\right)$$

$$= j(2) - 2c_1$$

FABER

$$2 \cdot \frac{w_i}{\omega_i}$$



T

Hodge operator

$\{L_i \rightarrow \{ \{ L_i \} \}$

$\text{Tors}(H_L(\tilde{\omega}))$

$[L : L_i] = n$

$$j(q) = \frac{1}{q} + \sum_{k>0} c_k q^k, \quad q = e^{2\pi i z}, \quad \Im z > 0.$$

$G_0 \cong PSL_2(\mathbb{Z})$

$\left( \begin{matrix} a & b \\ c & d \end{matrix} \right)$



$$n=2 \quad 2 \in \frac{3\pi}{2} \quad \frac{\pi}{2}$$

$$j(z) \rightarrow j\left(\frac{z}{2}\right) + j\left(2z\right) + j\left(\frac{z+1}{2}\right)$$

$$f = 2c_1$$
$$= j(z) - 2c_1(f)$$

FABER

$M =$

8080174247945...617107570057543680000000000

$= 2^{46} 3^{15} 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$

a product of the (15) supersingular primes.

**Riemann maps:**

$$w(z) = rz + a_0 + \sum_{k \geq 1} a_k/z^k$$

from the exterior of a disk of radius  $r$  to the simply connected complement of a Jordan curve in  $\mathbb{C} \cup \{\infty\}$ . We take  $a_0 = 0$  and  $r = 1$ , and compose with  $z \rightarrow \exp(-2\pi iz) = 1/q$

Our interest is in

$$\begin{aligned} f(z) &= \frac{1}{q} + \sum_{k \geq 1} a_k q^k, \\ q &= e^{2\pi iz}, \quad z \in \mathcal{H}. \end{aligned}$$



Cu{oo}



196884

96883 + E

MOONSHINE

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8080174247945...61710757005754368000000000

=  $2^{46}3^{15}5^97^611^213^3171923293141475971$

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$$f = \frac{1}{q} + \dots$$

$$F_n(f) = \frac{1}{q^n} \in q\mathbb{Q}[q]$$

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$$F_n(f) = \frac{1}{q^n} \in q\mathbb{Q}[[q]]$$

$$F_n(f) = \frac{1}{q^n} + n \sum_{m \geq 1} h_{m,n} q^m$$

$$f = \frac{1}{q} + \dots - h_{m,n} \stackrel{(4)}{=} T_m(f) \Big|_{q^n}$$

$$F_n(f) = \frac{1}{q^n} \in q^n \mathbb{Q}[[q]] \quad h_{n,m}$$

$$F_n(f) = \frac{1}{q^n} + n \sum_{m \geq 1} h_{m,n} q^m$$

$$f = \frac{1}{q} + \dots \quad h_{m,n}(f) = T_m(f) \Big|_q$$

$$F_n(f) = \frac{1}{q^n} \cdot f \in q\mathbb{Q}[[q]] \quad h_{n,m}$$

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$$f = \frac{1}{q} + \dots - h_{m,n} \stackrel{(1)}{=} T_m(f) \Big|_{q^n}$$

$$F_n(f) = \frac{1}{q^n} \in q\mathbb{C}[[q]] \quad h_{n,m}$$

$$F_n(f) = \frac{1}{q^n} + n \sum_{m \geq 1} h_{m,n} q^m$$

$$\forall g \in G, \exists f_{(g)} = \frac{1}{q} + \sum_{k \geq 1} q_k q^k \quad q = e^{2\pi i z} \quad q_m = \text{Trace}(t_k(g))$$

\$a\_k = a\_k(g) = \text{Trace}(t\_k(g))\$

$$f = \frac{1}{q} + \dots \quad h_{m,n} = T_m(f)|_n$$

$$F_n(f) = \frac{1}{q^n} \in q\mathbb{Q}[[q]] \quad h_{n,m}$$

GELONSKY 1939

$$F_n(f) = \frac{1}{q^n} + n \sum_{m=1}^n h_{m,n} q^m$$

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$$\begin{aligned} f(z) &= \frac{1}{q} + \sum_{k \geq 1} a_k q^k, \\ q &= e^{2\pi iz}, \quad z \in \mathcal{H}. \end{aligned}$$

Demuth's painting

$$f = \frac{1}{q} + \dots - h_{m,n} \left| \begin{array}{l} T_m(q) \\ q^m \end{array} \right.$$

Grunsky 1939

denoting symmetric  
pure sum

$$\sum_{i_1, i_2, \dots, i_L}$$

$$f = \frac{1}{q} + \dots + h_m$$

$$F(f) = \frac{1}{q} e_q \Phi([1])$$

**Faber polynomials:** For  $f$  as above, the Faber polynomial of degree  $n$  in  $f$  is defined (as are the **Grunsky coefficients**,  $h_{m,n}$ ) by:

$$F_{n,f}(f) = \frac{1}{q^n} + n \sum_{m \geq 1} h_{m,n} q^m$$

They are generated by

$$\frac{qf'(q)}{f(p) - f(q)} = \sum_{n=0}^{\infty} F_n(f(p)) q^n$$

with  $F_0(f) = 1$ ,  $F_1(f) = f$ ,  
 $F_2(f) = f^2 - 2a_1$ ,  $F_3(f) = f^3 - 3a_1f - 3a_2$ ,  
 $F_4(f) = f^4 - 4a_1f^2 - 4a_2f + 2a_1^2 - 4a_3$ ,  
& more generally:

$$F_n(f) = \det(fI - A_n),$$

where

$$A_n = \begin{pmatrix} a_0 & 1 & & & & \\ 2a_1 & a_0 & 1 & & & \\ \vdots & \vdots & \vdots & & & \\ (n-2)a_{n-3} & a_{n-4} & a_{n-5} & \dots & 1 & \\ (n-1)a_{n-2} & a_{n-3} & a_{n-4} & \dots & a_0 & 1 \\ na_{n-1} & a_{n-2} & a_{n-3} & \dots & a_1 & a_0 \end{pmatrix}$$

**Faber polynomials:** For  $f$  as above, the Faber polynomial of degree  $n$  in  $f$  is defined (as are the **Grunsky coefficients**,  $h_{m,n}$ ) by:

$$F_{n,f}(f) = \frac{1}{q^n} + n \sum_{m \geq 1} h_{m,n} q^m$$

They are generated by

$$\frac{qf'(q)}{f(p) - f(q)} = \sum_{n=0}^{\infty} F_n(f(p)) q^n$$

with  $F_0(f) = 1$ ,  $F_1(f) = f$ ,  
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The Faber polynomial,  $F_{n,f}(f)$  is the action of a generalized **Hecke operator**:

For the elliptic modular function,  $j$ , we have:

$$\forall n \in \mathbb{N}$$

$$nT_n(j) = \sum_{\substack{ad=n \\ 0 \leq b < d}} j\left(\frac{az+b}{d}\right) = F_{n,j}(j),$$

This property we generalize to replicable functions by defining replicable functions as those functions  $f = f^{(1)}$ , such that there are functions  $f^{(a)}$  of the same form satisfying

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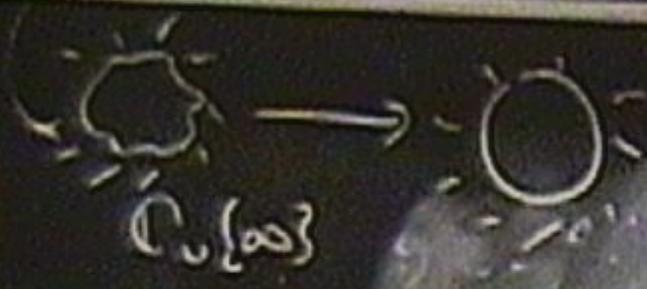
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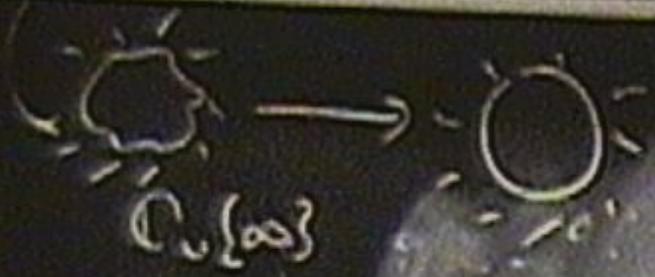
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$f \rightarrow f^{(a)}$   
replication program



$$f \rightarrow f^{(a)}$$

replication probability

$$f_{g^i}^{(a)} = \frac{1}{Z} \langle g^i \rangle$$

Replicability encapsulates  $F_{n,f}(f) - \sum'$  is a series [ $= f^{(n)}(q^n)$ ] in  $q^n$ . For prime  $p$ ,

$$T_p = \frac{1}{p}V_p + U_p$$

$$V_p : f(q) \longrightarrow f(q^p).$$

We define a generalized Hecke operator (cf. Bott: **Cannibalistic class**)

$$\widehat{T}_p = \frac{1}{p}\widehat{V}_p + U_p$$

$$\widehat{V}_p = \Psi^p \circ V_p, \quad \Psi^p = \text{"Adams operator"}$$

$$\widehat{V}_p : f(q) \longrightarrow f^{(p)}(q^p)$$

$$a_n^{(p)} = ph_{pn,p} - pa_{p^2 n}$$

On  $M$

$$f \longrightarrow f^{(k)}$$

$$\downarrow \qquad \downarrow$$

$$f_g \longrightarrow f_{g^k}$$

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$f$  is a cyclic after  $f_0$  &  $b^{\alpha}$

1.  $f = \frac{1}{q}, \frac{1}{q+1}, \frac{1}{q-1}$

$\exp i \sin \cos$

$f$  is a cyclic after  $f_0$ ,  $f \sim$

1.  $f = \frac{1}{q}, \frac{1}{q} \cdot q, \frac{1}{1-q}$

$\Rightarrow f \sim \sin \text{cosec}$

$\alpha^2$

$f$  is a complex reflection

$$1. \quad f = \frac{1}{q}, \frac{1}{q+q}, \frac{1}{q-q}$$

$\exp^{-i\pi} \sin \text{cusp}$

$\alpha_2$

$$\Gamma(N) \subset G_f \subset \text{Normalizer}_{PSL_2(\mathbb{R})}(\Gamma_0(N))$$

$f$  is a regular cokernel for  $f \circ \gamma$ .

1.  $f = \frac{1}{q}, \frac{1}{q} \cdot q, \frac{1}{q} - q$

$\exp: \sin \cos$

$\alpha^2$

$$\Gamma_0(N) \subseteq G_f \subset \text{Normalizer}_{\text{PSL}_2(\mathbb{R})}(\Gamma_0(N))$$

Lemma:  $\geq 2000$ .

$$h_{m,n}(f) = \widehat{T}_n(f)|_{q^m}.$$

Norton's remarkable result is that when the Grunsky coefficients of  $f$  satisfy:

$$h_{m,n} = h_{lcm(m,n), gcd(m,n)}$$

then  $h_{m,n}$  is polynomially dependent on  $\{a_k\}, k \in B$ , the Norton basis,

$$B = \{1, 2, 3, 4, 5, 7, 8, 9, 11, 17, 19, 23\}, |B| = 12.$$

This compatibility is Norton's definition of replicability of  $f$ .

### The Grunsky coefficients:

There are identities,  $h_{m,1} = a_m$ ,  $h_{r,s} = h_{s,r}$ . Norton's basis theorem follows a descent argument on the index. The rows are in decreasing index.

The coefficients  $\hat{h}_{r,s} = (r+s)h_{r,s}$  satisfy

$$\hat{h}_{r,s} = (r+s)a_{r+s-1} + \sum_{m=1}^{r-1} \sum_{n=1}^{s-1} a_{m+n-1} \hat{h}_{r-m, s-n}.$$

...

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1. Dispersionless K-P like "Mott-like" reduction

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1. Disparielen K-P wie "höher reduzieren"
2. Parabolic bundles & symplectic reduction.
3. Algebraic Topology

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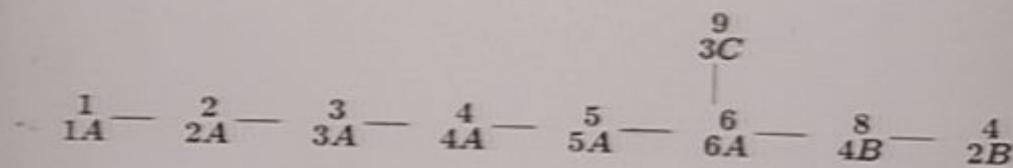
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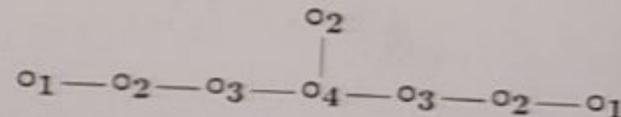
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$E_8$

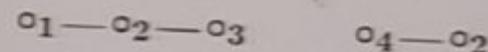


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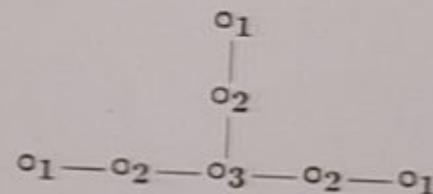
$E_7$



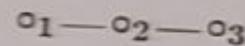
$F_4$



$E_6$



$G_2$



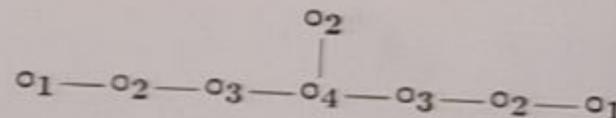
- {
- 1. Dispersions K-P wie "höher reduzieren"
  - 2. Parabolic bundles & symplectic reduction.
  - 3. Algebraic Topology



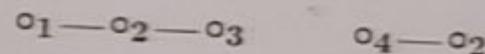
- { 1. Dispersible K-P mit "Knoten reduzieren"  
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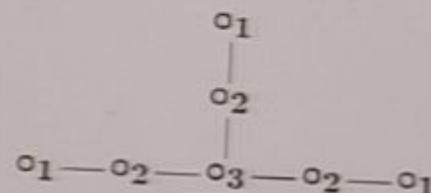
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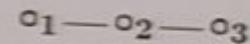
$F_4$



$E_6$



$G_2$



$M =$

8080174247945...61710757005754368000000000

$= 2^{46} 3^{15} 5^9 7^6 11^2 13^3 17 19 23 29 31 41 47 59 71$

a product of the (15) supersingular primes.

**Riemann maps:**

$$w(z) = rz + a_0 + \sum_{k \geq 1} a_k/z^k$$

from the exterior of a disk of radius  $r$  to the simply connected complement of a Jordan curve in  $\mathbb{C} \cup \{\infty\}$ . We take  $a_0 = 0$  and  $r = 1$ , and compose with  $z \rightarrow \exp(-2\pi iz) = 1/q$

Our interest is in

$$\begin{aligned} f(z) &= \frac{1}{q} + \sum_{k \geq 1} a_k q^k, \\ q &= e^{2\pi iz}, \quad z \in \mathcal{H}. \end{aligned}$$