Title: Multi-level, multi-party singlets as ground states and their role in entanglement distribution

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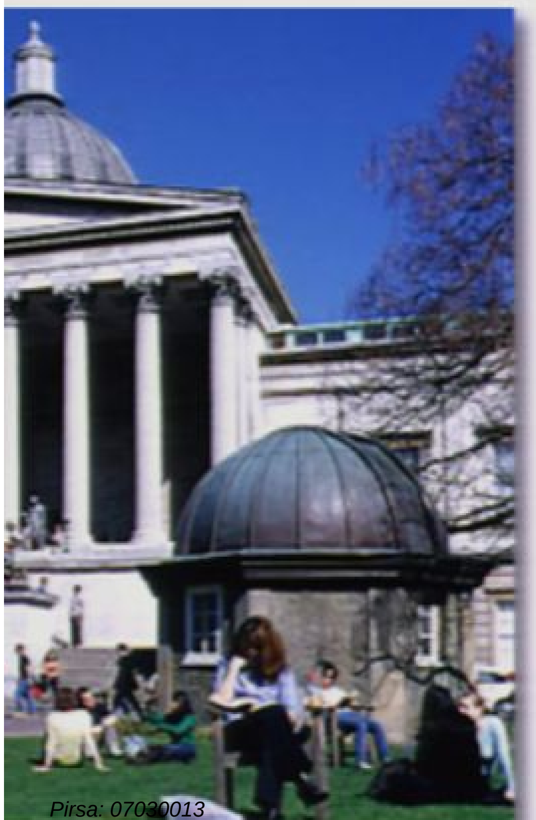
Abstract: We show that singlets composed of multiple multi-level quantum systems can naturally arise as the ground state of a physically-motivated Hamiltonian. The Hamiltonian needs to be one which simply exchanges the states of nearest neighbours in any graph of interacting d-level quantum systems (qudits) as long as the graph also has d sites. We point out that local measurements on some of these qudits, with the freedom of choosing a distinct measurement basis at each qudit randomly from an infinite set of bases, project the remainder onto a singlet state. One implication of this is that the entanglement in these states is very robust (persistent), while an application is in establishing an arbitrary amount of entanglement between well-separated parties (for subsequent use as a communication
resource) by local measurements on an appropriate graph. Based on quant-ph/0602139.

Multi-level, multi-party singlets as ground states and their role in entanglement distribution



Christopher Hadley
University College London
www.tampa.phys.ucl.ac.uk/quinfo/

Perimeter Institute, 7 March 2007

Based on C. Hadley and S. Bose, *quant-ph/0602139* (2006)

Overview of talk

Overview of talk

- Introduction to qudit singlets: setting the scene and formal definition
- Summary of previous results
- Proof that qudit singlets are the ground state of a 2-local Hamiltonian for any graph
- Proof that you can take arbitrary measurements and always obtain a smaller singlet
- Proof that these state have the highest possible persistency of entanglement
- Potential uses of such states / directions for future research
- Preparation of qudit singlets in an optical lattice
- Summary

Introduction to qudit singlets: setting the scene

Secret sharing problem

- Suppose A_1 wants to have a secret action taken on her behalf at a distant location
- She has agents $A_2, \ldots, A_N$ to do it for her
- A_1 knows that some are dishonest, but does not know which

A. Cabello, PRL **89**, 100402 (2002) & J. Mod. Opt. **50**, 1049 (2003)

Introduction to qudit singlets: setting the scene

- She cannot send a secure message to all, because the dishonest parties will sabotage it
- Assume that if they carry it out together, honest ones will prevent dishonest ones damaging task
- So: A_1 needs to convey a cryptographic key to $A_2, \ldots, A_N$, such that they can only read it if they all collaborate

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Introduction to qudit singlets: setting the scene

Problem can be solved if each of the N parties has a sequence of numbers that:

i. Is truly random
ii. Possible numbers are integers from 0, ..., $N-1$
iii. If number i is a position j in k's sequence, it does not appear at position j in another's sequence
iv. Each party knows only his/her sequence

Properties are impossible classically, since eavesdroppers could listen in

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Introduction to qudit singlets: setting the scene

So we need a method to generate such sequences

A_1's sequence defined as the key

Only way to reveal it is to make remaining parties share their respective sequences

Key then composed by the missing results

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Introduction to qudit singlets: setting the scene

A_1	A_2	A_3	A_4
4	2	1	3
3	1	2	4
2	4	3	1
1	3	4	2

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Introduction to qudit singlets: setting the scene

If a dishonest party D declares an incorrect result, there is a probability $1/(r-1)$ where r = number of honest parties, that other honest party H has already obtained that result. Then H would stop the process, so A_1's key would remain safe

Order in which agents declare respective results must change from round to round to avoid D always being last.

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Introduction to qudit singlets: setting the scene

There are two other examples, which reduce to the same problem:

N-strangers problem and *liar detection problem*

So these states are useful

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Definition of qudit singlets

- "Qudit singlets" are N-party, d-level states, with the property $U^{\otimes N}\left|S_N^{(d)}\right\rangle = \left|S_N^{(d)}\right\rangle$

- Each party is a spin $(d-1)/2$; total spin is zero
- For $d = N$, they take the form

$$\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle = \frac{1}{\sqrt{N!}}\sum_{\{n_l\}} \epsilon_{n_1,\ldots,n_N}\left|\alpha_{n_1},\ldots,\alpha_{n_N}\right\rangle$$

- We call this an N-singlet

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Definition of qudit singlets

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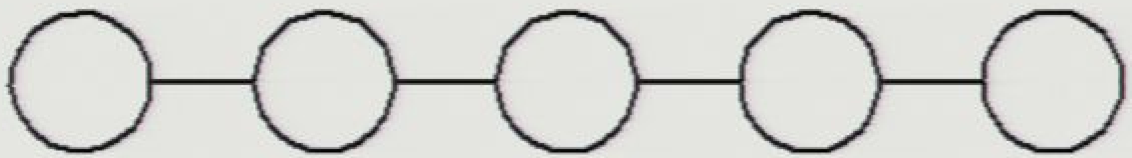
- Sum is taken over all combinations
- State is antisymmetric under all permutation operators P_{ij} where i, j run between 1 and N
- They are essentially *pure* multipartite Werner states[1]

[1]R. F. Werner, PRA **40**, 4277 (1989)

Properties of qudit singlets



- If all parties measure in any basis (must be the same for each), a smaller qudit singlet is established at each stage between the other parties

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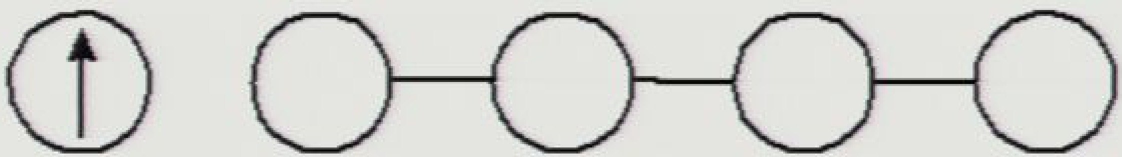

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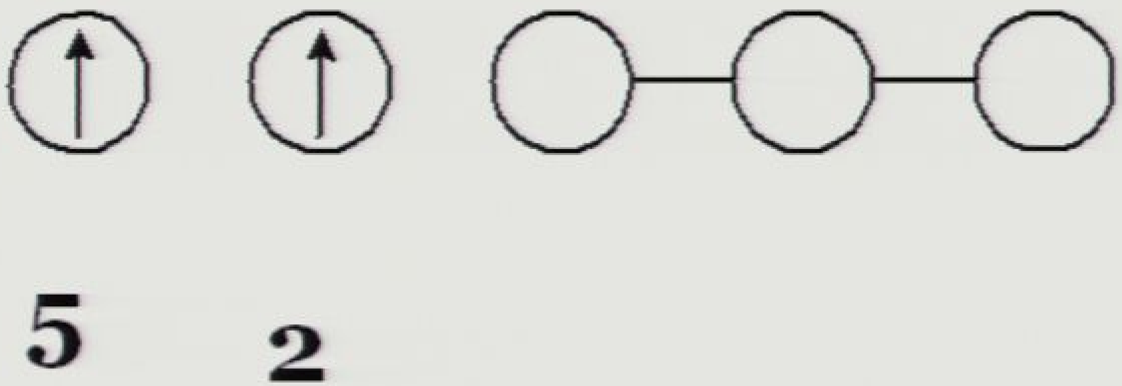

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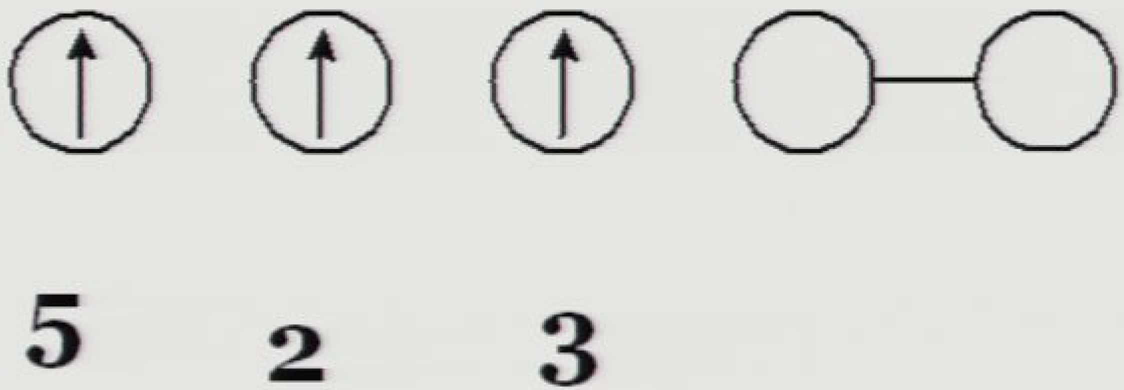

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Properties of qudit singlets

- This is how the sequences are distributed to all the parties
- If the parties share a large number of copies of an N-singlet, they can solve the problem

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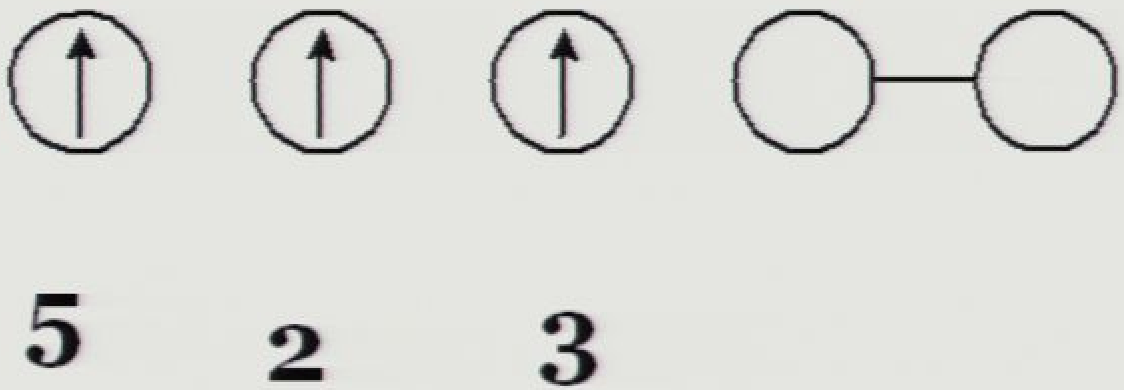

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Secret sharing problem solved

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Previous results: summary

- If the same arbitrary rotation is applied to each qudit, the state is the same (up to a phase)
- If all parties measure in any basis (must be the same for each), a smaller qudit singlet is established at each stage between the other parties

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Previous results: for general *d*, *N*

- Qudit singlets are essentially *pure* multipartite Werner states[1] for special case $d = N$
- They do not exist[2] for $N < d$
- May be used as basis of decoherence-free subspace[2, 3, 4]
- Conjectured that for $N = md$ qudits, there are m orthogonal such states[2]
- May be used for *multi-party remote state preparation*[5]

[1] R. F. Werner, PRA **40**, 4277 (1989)
[2] P. Kok *et al.*, quant-ph/02011038 (2002)
[3] A. Cabello, PRL **89**, 100402 (2002) & J. Mod. Opt. **50**, 1049 (2003)
[4] A. Cabello, PRA **75** (2007) 020301(R)
[5] P. Agrawal *et al.*, quant-ph/0304006 (2003)

Motivation

- Given this multitude of applications, we would like to be able to make these states
- Little progress has been made to date, and only then in the case of entangled photons[1,2]
- Here we consider a condensed matter Hamiltonian, which also may be implemented in optical lattices, for the special case $d = N$

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[2] N. Gisin, conference proceedings, Quantum information: quantum entanglement (Sant Feliu de Guíxols, Spain, 2002)

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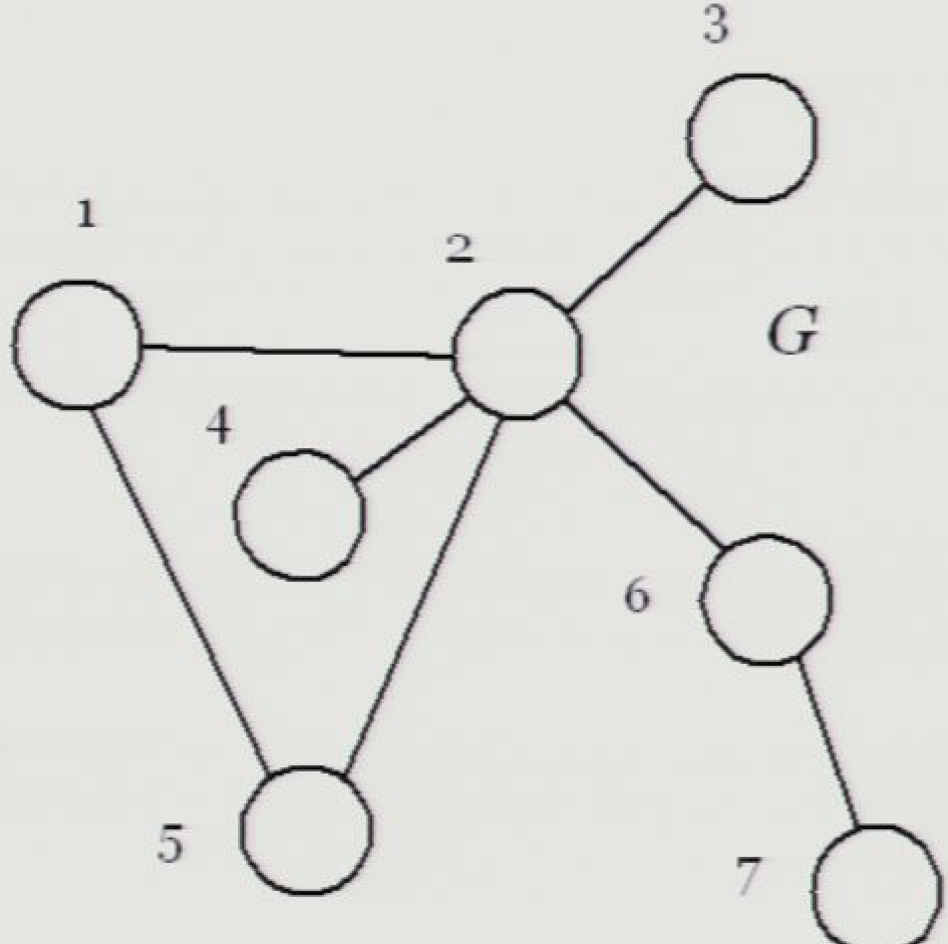
Our three main results

- N-singlets are the **ground state** of a 2-local Hamiltonian
- One can use **any measurement basis** to establish smaller qudit singlets at successive measurements: it follows that these states are the **most persistent possible**
- Discuss a potential realisation of this state in an **optical lattice**

C. Hadley & S. Bose, quant-ph/0602139 (2006)

Result 1: Qudit singlets are the ground state of permutation Hamiltonians

- Let G be a graph, $E(G)$ its set of edges, and $V(G)$ its set of vertices.
- Let there be a qudit at each vertex (levels 1, ..., N)
- Let connected vertices i, j interact through a permutation operator P_{ij}
- This operator permutes all states at sites i and j, and is an element of SU(d)



Result 1: Qudit singlets are the ground state of permutation Hamiltonians

- Then the Hamiltonian for this system is: $$H = \sum_{i,j \in E(G)} P_{ij}$$
- This is the $SU(d)$ generalisation of the Heisenberg interaction
- We can show that the ground state of this a qudit singlet, *independently* of the choice of graph G!

Permutation Hamiltonians: physical realisation

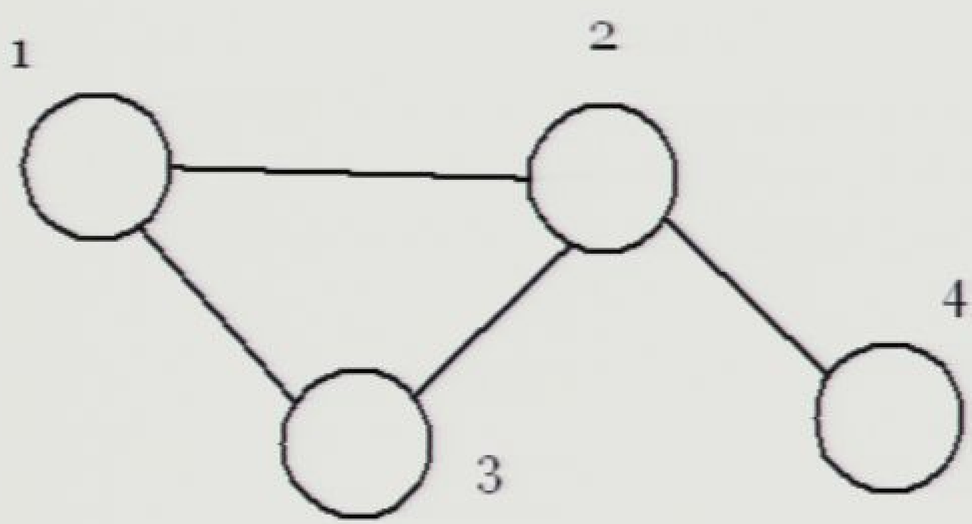
- Permutation operator: $P_{ij}\,|\psi\rangle_i\,|\phi\rangle_j = |\phi\rangle_i\,|\psi\rangle_j$

- This can be written:
  $$P_{ij} = \sum_{\alpha,\beta=1}^{d} S_\alpha^\beta(i) S_\beta^\alpha(j)$$
  where $\{S_\beta^\alpha(n)\}$ are the generators of SU(d) at nth vertex

Permutation Hamiltonians: Lemma 1

$$H = \sum_{i,j \in E(G)} P_{ij}$$

- *Lowest energy state of a permutation Hamiltonian has energy equal to that of an eigenstate of all P_{ij} terms included in the Hamiltonian, all with eigenvalue – 1*



Example: in this case, the lowest energy state will be an eigenstate of P_{12}, P_{13}, P_{23}, P_{24}

Permutation Hamiltonians: Lemma 1

$$H = \sum_{i,j \in E(G)} P_{ij}$$

- *Proof*

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- To minimise the energy: $\min_{|\psi\rangle \in (\mathcal{C}^d)^{\otimes N}} \langle \psi | H | \psi \rangle \geq \sum_{i,j \in E(G)} \min_{|\psi\rangle \in (\mathcal{C}^d)^{\otimes N}} \langle \psi | P_{ij} | \psi \rangle$

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- Minimum of each term is -1, so $\min_{|\psi\rangle \in (\mathcal{C}^d)^{\otimes N}} \langle \psi | H | \psi \rangle \geq -N_c$

where N_c is number of terms in Hamiltonian.

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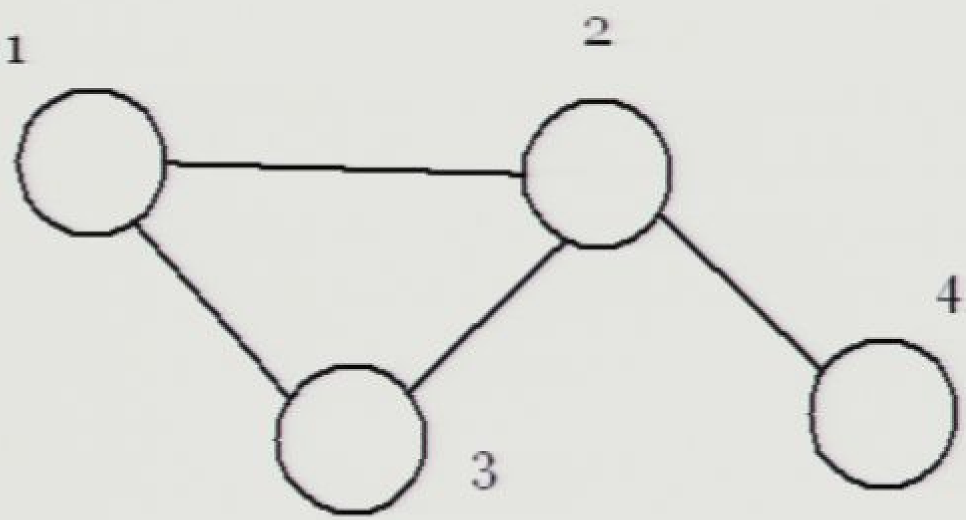
  where N_c is number of terms in Hamiltonian.
- Equality exists for a state that is individually an eigenstate of all terms in Hamiltonian, and if this exists it is a ground state

Permutation Hamiltonians: Lemma 2

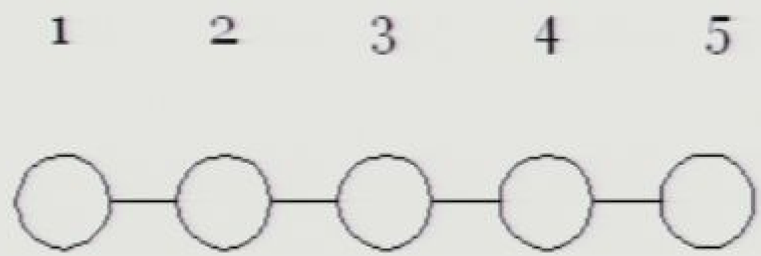
$$H = \sum_{i,j \in E(G)} P_{ij}$$

- *If a state is an eigenstate of all permutation operators in the Hamiltonian, it is an eigenstate of all possible permutation operators*



Example: in this case, *if* the state is be an eigenstate of P_{12}, P_{13}, P_{23}, P_{24}, it will also be an eigenstate of **all** P_{ij}; e.g. P_{34}, P_{14}, ...

Permutation Hamiltonians: Lemma 2



- *Proof:*
- Consider a chain:
- Any permutation may be written as a product of nearest-neighbour operators, e.g.:

$P_{14} = P_{12} P_{23} P_{34} P_{23} P_{12}$

	1	2	3	4
	A	B	C	D
P_{12}:	B	A	C	D
P_{23}:	B	C	A	D
P_{34}:	B	C	D	A
P_{23}:	B	D	C	A
P_{12}:	D	B	C	A

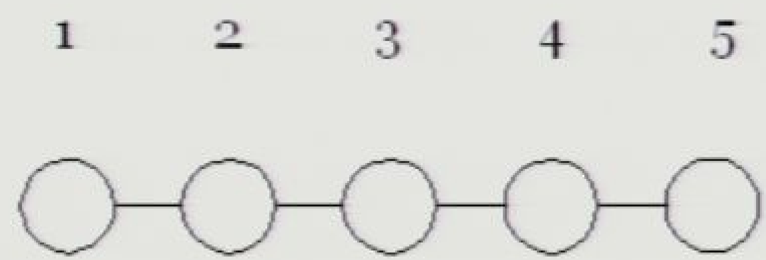
Permutation Hamiltonians: Lemma 2

- So, an eigenstate of all nearest-neighbour permutations must be an eigenstate of *all* permutations

- Always an odd number of permutations

- Can be readily generalised to *any* connected graph

Permutation Hamiltonians: Lemma 2



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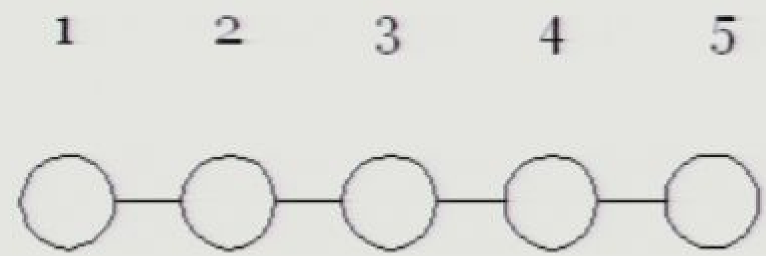

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P_{12}:	D	B	C	A

Permutation Hamiltonians: Proof of ground state

- *Theorem: The ground state of a SU(N) permutation Hamiltonian on a lattice of N sites is an N-singlet*
- From above, we know a state completely antisymmetric under all permutations is a valid ground state
- A qudit singlet $d = N$ satisfies this by definition
- Uniqueness can be proven by contradiction

Result 2: General measurements and establishing smaller qudit singlets

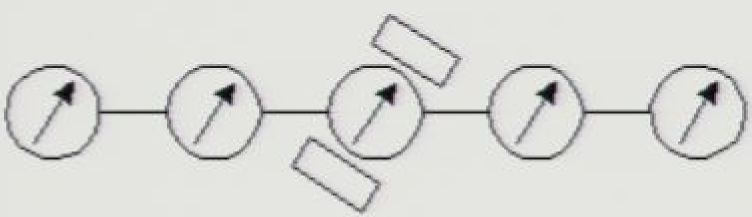
- Recall that if all parties measure in the same basis, a smaller qudit singlet is established at each stage between the other parties

- Can we measure in *any* basis?

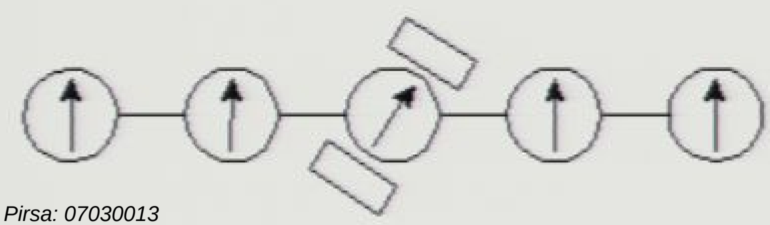
Result 2: General measurements and establishing smaller qudit singlets

What if we want to measure in some new basis, as shown?

Perform some U on all ➔ still qudit singlet



Measure



Perform $U^{\dagger}$ on others ➔ should have same effect?

Result 2: General measurements and establishing smaller qudit singlets

- We find that this is true
- First, we prove the property

$$U \otimes \mathcal{I}^{\otimes N-1} |S_N^{(N)}\rangle = \mathcal{I} \otimes U^{\dagger \otimes N-1} |S_N^{(N)}\rangle$$

i.e. performing U on one qudit is equivalent to performing the adjoint operation on all other qudits

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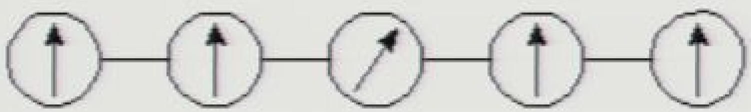

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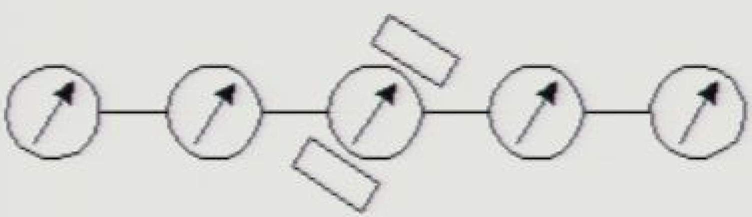

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Result 2: General measurements and establishing smaller qudit singlets

- For a 2-singlet, we can prove this

$$\begin{aligned}(U\otimes\mathcal{I})\left|S_2^{(2)}\right\rangle &= (U\otimes U^{\dagger}U)\left|S_2^{(2)}\right\rangle \\ &= (\mathcal{I}\otimes U^{\dagger})(U\otimes U)\left|S_2^{(2)}\right\rangle \\ &= (\mathcal{I}\otimes U^{\dagger})\left|S_2^{(2)}\right\rangle\end{aligned}$$

- Making use of the invariance property
- This can be generalised to:

$$U\otimes\mathcal{I}^{\otimes N-1}|S_N^{(N)}\rangle = \mathcal{I}\otimes U^{\dagger\otimes N-1}|S_N^{(N)}\rangle$$

Result 2: General measurements and establishing smaller qudit singlets

- *Claim: When we measure an N-singlet written in a basis* $\{|\alpha_i>\}$ *at one qudit using an arbitrary basis* $\{|\beta_i><\beta_i|\}$*, we get a product of* $|\beta_i>$ *at the measured site and an* $(N-1)$*-singlet* $\left|S_{N-1}^{(N-1)}(\beta;\beta_l)\right\rangle$ *in basis* $\{|\beta_i\rangle\}^{\otimes N-1}$ *at the other qudits*

Result 2: General measurements and establishing smaller qudit singlets

- *Proof:*
- Introduce notation: $\left|S_{N-1}^{(N-1)}(\beta;\beta_l)\right\rangle$ is a singlet written in basis $\{|\beta_i\rangle\}^{\otimes N-1}$ with state $|\beta_l\rangle$ missing
- Perform measurement $|\beta_i\rangle\langle\beta_i|\}$
- Since $|\beta_i\rangle\langle\beta_i| = U|\alpha_i\rangle\langle\alpha_i|U^\dagger$ we have

$$
\begin{aligned}
&|\beta_i\rangle\langle\beta_i|\otimes\mathcal{I}^{\otimes N-1}\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle \\
&\quad= U|\alpha_i\rangle\langle\alpha_i|U^\dagger\otimes\mathcal{I}^{\otimes N-1}\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle \\
&\quad= U|\alpha_i\rangle\langle\alpha_i|\mathcal{I}\otimes U^{\otimes N-1}\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle.
\end{aligned}
$$

Result 2: General measurements and establishing smaller qudit singlets

- To proceed, we write the N-singlet in the form

$$\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(-)^{i+1}\left|\alpha_i\right\rangle_1\left|S_{N-1}^{(N-1)}(\boldsymbol{\alpha};\alpha_i)\right\rangle_{2,\dots,N}$$

and thus $I\otimes U^{\otimes N-1}\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle_{1,\dots N} =$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(-)^{i+1}\left|\alpha_i\right\rangle_1\otimes U^{\otimes N-1}\left|S_{N-1}^{(N-1)}(\boldsymbol{\alpha};\alpha_i)\right\rangle_{2,\dots,N}$$

- This is a qudit singlet in the new basis

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- Since $|\beta_i\rangle\langle\beta_i| = U|\alpha_i\rangle\langle\alpha_i|U^\dagger$ we have

$$\begin{aligned}
&|\beta_i\rangle\langle\beta_i| \otimes \mathcal{I}^{\otimes N-1}\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle \\
&= U|\alpha_i\rangle\langle\alpha_i|U^\dagger \otimes \mathcal{I}^{\otimes N-1}\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle \\
&= U|\alpha_i\rangle\langle\alpha_i|\mathcal{I} \otimes U^{\otimes N-1}\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle.
\end{aligned}$$

Result 2: General measurements and establishing smaller qudit singlets

- To proceed, we write the N-singlet in the form

$$\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(-)^{i+1}\left|\alpha_i\right\rangle_1\left|S_{N-1}^{(N-1)}(\boldsymbol{\alpha};\alpha_i)\right\rangle_{2,\ldots,N}$$

and thus $I\otimes U^{\otimes N-1}\left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle_{1,\ldots N} =$

$$\frac{1}{\sqrt{N}}\sum_{i=1}^{N}(-)^{i+1}\left|\alpha_i\right\rangle_1\otimes U^{\otimes N-1}\left|S_{N-1}^{(N-1)}(\boldsymbol{\alpha};\alpha_i)\right\rangle_{2,\ldots,N}$$

- This is a qudit singlet in the new basis

Result 2: General measurements and establishing smaller qudit singlets

So the outcome is $|\beta_i\rangle\langle\beta_i| \otimes \mathcal{I}^{\otimes N-1} \left|S_N^{(N)}(\boldsymbol{\alpha})\right\rangle_{1,\ldots,N} / ||\ldots||$

$$= |\beta_i\rangle_1 \otimes \left|S_{N-1}^{(N-1)}(\boldsymbol{\beta}; \beta_i)\right\rangle_{2,\ldots,N}.$$

...and we have proved the claim

The significance of this is seen if we iterate ...

Result 2: General measurements and establishing smaller qudit singlets

Claim: If M parties perform successive measurements in arbitrary bases $B_m = \{|\alpha_i^{(m)}\rangle\}_{i\neq 1,\ldots,m-1} = \{\prod_{l=1}^{m} U^{(l)}|\alpha_i^{(0)}\rangle\}$

the (N – M) remaining parties share an (N – M)-singlet in the basis

$$B_M^{\otimes N-M}$$

Restriction: each basis transformation operates on a space one dimension smaller than previous (lifted later)

Result 2: General measurements and establishing smaller qudit singlets

Proof:

Consider effect of measuring in basis $\{|\alpha_i^{(2)}\rangle\}$ on outcome of previous measurement. End result is

$$\left|\alpha_i^{(1)}\right\rangle_1 \left|\alpha_j^{(2)}\right\rangle_2 \left|S_{N-2}^{(N-2)}(\boldsymbol{\alpha}^{(2)};\alpha_i^{(2)},\alpha_j^{(2)})\right\rangle_{3,\dots,N}.$$

In general for M measurements

$$\left|\alpha_{n_1}^{(1)}\right\rangle_1 \cdots \left|\alpha_{n_M}^{(M)}\right\rangle_M \left|S_{N-M}^{(N-M)}(\boldsymbol{\alpha}_{n_M}^{(M)};\boldsymbol{n}^{(M)})\right\rangle_{M+1,\dots,N}$$

Elements of vector $\boldsymbol{n}$ are the levels excluded

Result 2: General measurements and establishing smaller qudit singlets

Restriction arises because the property

$$U^{\otimes N}\left|S_N^{(d)}\right\rangle = \left|S_N^{(d)}\right\rangle$$

only holds when U operates on space inhabited by the singlet

So: at lth measurement, we restrict basis transformation $U^{(l)}$ to operate only on $(N-l)$ levels

But ...

Result 2: General measurements and establishing smaller qudit singlets

- It is well known that a *d*-level unitary can be written in terms of two-level unitaries:

$$U_d = V_1 \ldots V_k$$

- So:

$$U_d \otimes U_d = (V_1 \otimes V_1) \ldots (V_k \otimes V_k)$$

- We can now use $d \times d$ unitaries that the V_i act *either* within the singlet subspace *or* its complement

Result 2: General measurements and establishing smaller qudit singlets

- But what happens if we really take measurements in *any* basis?
- Consider effect of arbitrary unitary on 2-singlet of levels *j*, *m*:

$$(V_1 \otimes V_1) \ldots (V_i \otimes V_i)(|jm\rangle - |mj\rangle)$$
$$= (V_1 \otimes V_1) \ldots (V_{i-1} \otimes V_{i-1})((V_i |j\rangle) |m\rangle - |m\rangle (V_i |j\rangle))$$

- Each of the factors on the l.h.s. take the singlet into a different space, but its entanglement properties are unaffected
- So we end up with a singlet!

Result 2: General measurements and establishing smaller qudit singlets

- So we conclude that:

  when N parties share a qudit singlet, and M of them perform local measurements, the remaining $(N - M)$ of them share a smaller qudit singlet, *regardless of the measurement choices or outcomes*!

New property of qudit singlets: *highest possible persistency of entanglement*

- We can now make some new claims about qudit singlets
- *Persistency of entanglement* defined[1] as minimum number of local von Neumann measurements needed to completely disentangle the state

[1]H. J. Briegel & R. Raussendorf, PRL **86**, 910 (2001)

New property of qudit singlets: *highest possible persistency of entanglement*

- Can be considered a measure of multi-partite entanglement
- For cluster states[1], it is $\sim N/2$
- Here, it is $(N-1)$, the *highest possible value for an N-partite state*

[1]H. J. Briegel & R. Raussendorf, PRL **86**, 910 (2001)

Entanglement distribution

- For most quantum computing tasks, we need entanglement to be shared by well separated parties
- This is difficult to achieve by distributing photons
- One approach: localisable entanglement

[1]F. Verstraete *et al.*, PRL **92**, 027901 (2004); *ibid.*, 087201; B.-Q. Jin & V. E. Korepin, PRA **69**, 062314 (2004); M. Popp *et al.*, PRA **71**, 042306 (2005)

Localisable entanglement

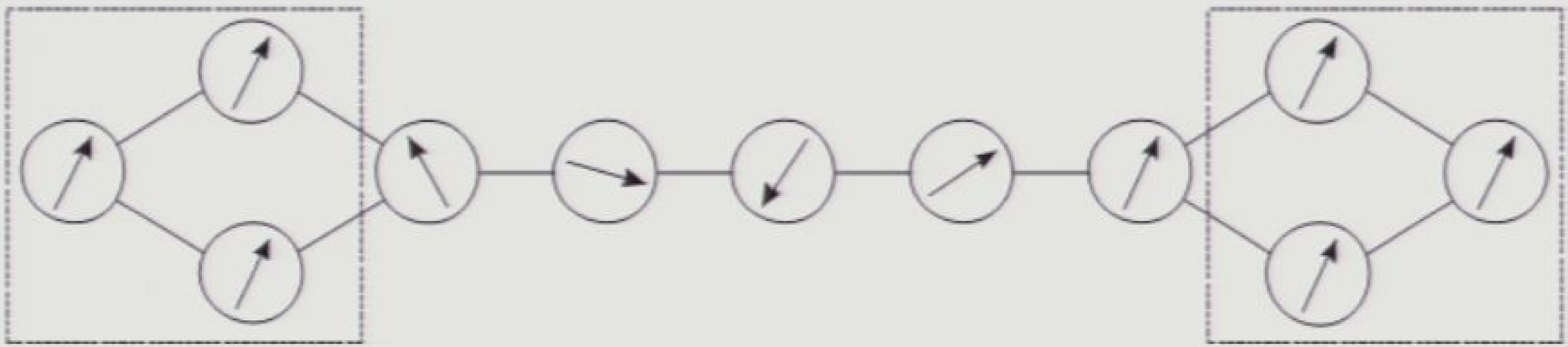
- *Localisable entanglement (LE)* = maximum amount of entanglement one can concentrate between two parts of system, by local measurements on others
- Normally have to optimise over measurement basis
- Here, it is *basis independent*

Consequences for localisable entanglement

- We introduce *two subsystem LE*:
- Normally LE relates to two qudits
- Now suppose Alice and Bob have control of small parts of the graph, each of n qudits
- They can establish $\log_2 {}^{2n}C_n$ ebits between them

[1]F. Verstraete *et al.*, PRL **92**, 027901 (2004); *ibid.*, 087201; B.-Q. Jin & V. E. Korepin, PRA **69**, 062314 (2004); M. Popp *et al.*, PRA **71**, 042306 (2005)

Consequences for localisable entanglement



Alice

Bob

- Alice and Bob have access to the boxes
- By performing arbitrary measurements on the other qudits, they can establish a 6-singlet, with $\log_2 {}^6C_3$ ebits shared between them
- This is very relevant physically, as this can now be used for short distance communications or networking distinct quantum registers

Result 3: physical realisation

- These have fermionic and bosonic representations:

$$S_{\alpha}^{\beta}(n) = c_{\beta,n}^{\dagger} c^{\alpha,n}$$

- Potentially there could be many implementations, including optical traps, quantum dots, spin tubes
- Indeed, this is a natural interaction arising when qudits simply hop along a lattice with site occupancy of at most one qudit

Hubbard model

- The permutation Hamiltonian may be obtained from the Hubbard model in a certain limit[1]

- The standard, two-level, one-band Hubbard model Hamiltonian is:

$$H = -t \sum_{\langle ij \rangle} \sum_{\sigma=\uparrow,\downarrow} \left(c^\dagger_{\sigma i} c_{\sigma j} + c_{\sigma i} c^\dagger_{\sigma j} \right) + U \sum_{i=1}^{N} n_{\uparrow i} n_{\downarrow i}$$

[1]E. Fradkin, Field theories of condensed matter systems (Addison Wesley, 1991)

Hubbard model

- Consider this in the half-filled regime (there are no unoccupied sites, and no sites with multiple occupancy)

- Consider strong-coupling limit: $U >> t$ (number fluctuations eliminated, energy cost of leaving half-filled subspace is very large)

[1] E. Fradkin, Field theories of condensed matter systems (Addison Wesley, 1991)

Hubbard model

- Treating the hopping term as a perturbation, we obtain an effective Hamiltonian $H' = H_0{}^2/U$

$$\frac{t^2}{U}\sum_{\alpha,\beta=\uparrow,\downarrow}\sum_{\langle ij\rangle}\left(c^\dagger_{\alpha i}c_{\alpha j}+c_{\alpha i}c^\dagger_{\alpha j}\right)\left(c^\dagger_{\beta i}c_{\beta j}+c_{\beta i}c^\dagger_{\beta j}\right)$$

- Expand and only keep terms which confine state to single-occupancy subspace

(since U is large these state are suppressed)

Hubbard model

$$H_0' = \frac{t^2}{U} \sum_{\alpha=\uparrow,\downarrow} \left(c^\dagger_{\alpha i} c_{\beta i} c^\dagger_{\beta j} c_{\alpha j} + c^\dagger_{\beta i} c_{\alpha i} c^\dagger_{\alpha j} c_{\beta j} \right)$$

$$= \frac{2t^2}{U} \sum_{\alpha=\uparrow,\downarrow} c^\dagger_{\alpha i} c_{\beta i} c^\dagger_{\beta j} c_{\alpha j}.$$

- This is the two-level permutation Hamiltonian *i.e.* the Heisenberg Hamiltonian with $J = 2t^2/U$

$$H_0' = J \sum_{\langle ij \rangle} \mathbf{s}_i \cdot \mathbf{s}_j$$

Hubbard model

- Now we add more levels:

$$H = -t \sum_{\langle ij \rangle} \sum_{\sigma=1}^{d} \left(c^{\dagger}_{\sigma i} c_{\sigma j} + c_{\sigma i} c^{\dagger}_{\sigma j} \right) + U \sum_{i} \sum_{\sigma \neq \sigma'} n_{\sigma i} n_{\sigma' i}.$$

- We repeat the derivation, but in the $1/d$-filling regime

Hubbard model

- We now obtain the effective Hamiltonian:

$$H_0' = \frac{2t^2}{U} \sum_{\alpha=1}^{d} c_{\alpha i}^\dagger c_{\beta i} c_{\beta j}^\dagger c_{\alpha j}$$

- which is equivalent to the permutation Hamiltonian required:

$$H = \sum_{i,j \in E(G)} P_{ij}$$

Hubbard model

- This derivation is also equivalent to a generalised Schrieffer–Wolff transformation[1]

[1] L.-M. Duan, E. Demler and M. D. Lukin, PRL **90**, 100401 (2003)

Hubbard model: candidate systems

- The levels must be degenerate (e.g. hyperfine levels)

- With ^{40}K atoms, one can obtain $2F + 1 = 10$ levels
- With Er atoms, one can obtain $2F + 1 = 22$ levels

[1]M. Köhl *et al.*, PRL **94**, 080403 (2005)

[2]J. J. McClelland and J. L. Hanssen, PRL **96**, 143005 (2006)

Other potential realisations

- Spin ladders and tubes[1]
- Arrays of quantum dots with electrons having both spin and orbital levels[2]

[1]M. T. Batchelor and M. Maslen J. Phys. A **32**, L377 (1999)
[2]Y.-Q. Li *et al.*, PRB **62**, 4866 (2000)

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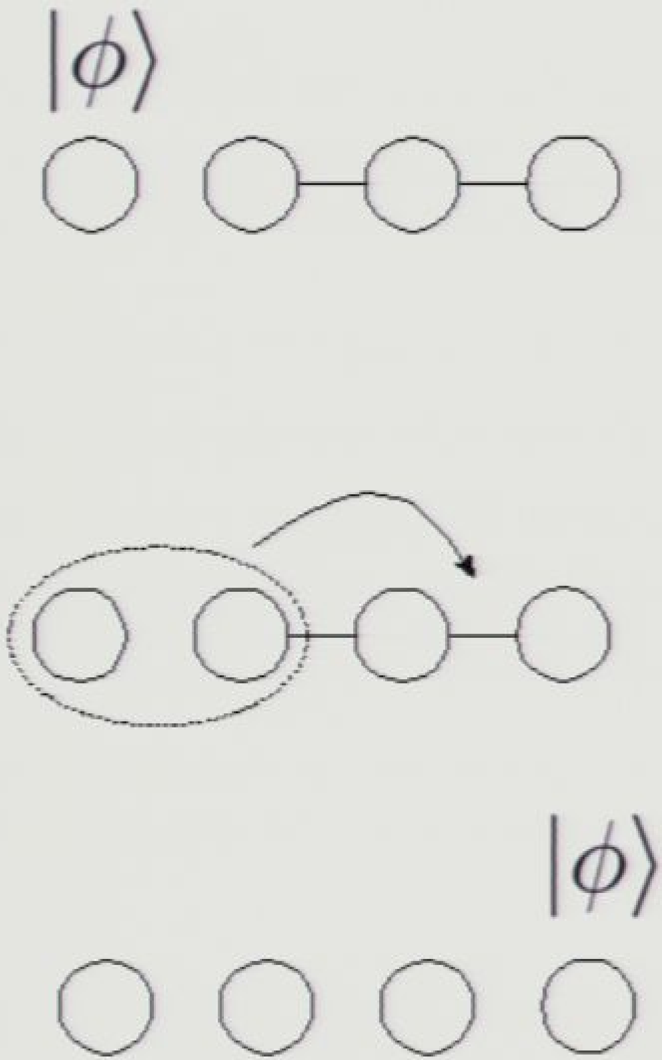
Potential uses of qudit singlets (work in progress and open problems)

- Teleportation, and measurement-based QC
- Making a valence bond solid
- Proving non-locality for arbitrary numbers of observers, measurements and outcomes
- Open problems

Teleportation (work in progress)

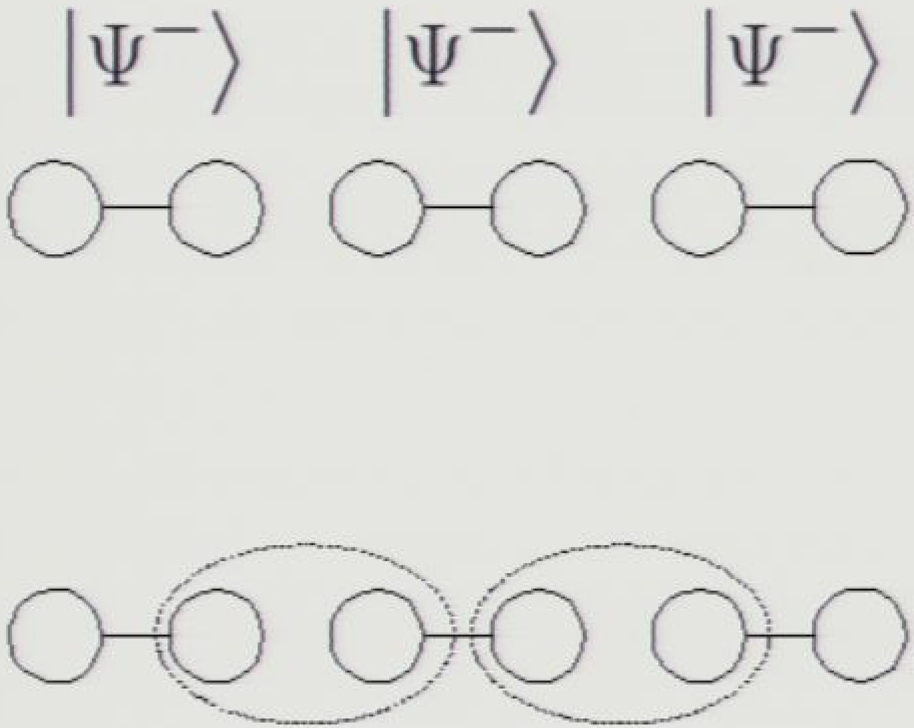


Similar to GHZ and cluster state teleportation

1. Alice, Bob and Charlie share $|S^3_3>$
2. Alice has another qudit in state
3. Alice performs a measurement, so the state is shared by Bob and Charlie
4. Bob measures
5. Charlie reconstructs the state using the classical bits from Alice and Bob

Can this be generalised to measurement-based QC?

Valence bond solid (work in progress)



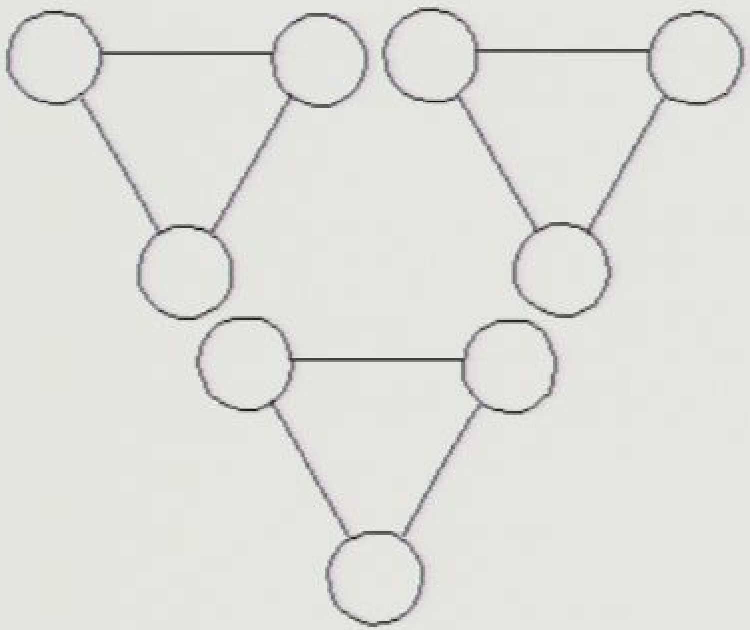
- Bell pairs may be put together to make a spin-1 chain[1,2]
- Between each pair, the qubits are projected to the symmetric spin-1 space
- This makes a spin-1 chain with two spin-1/2's at the ends
- This can be used to implement a qubit cluster state[3]

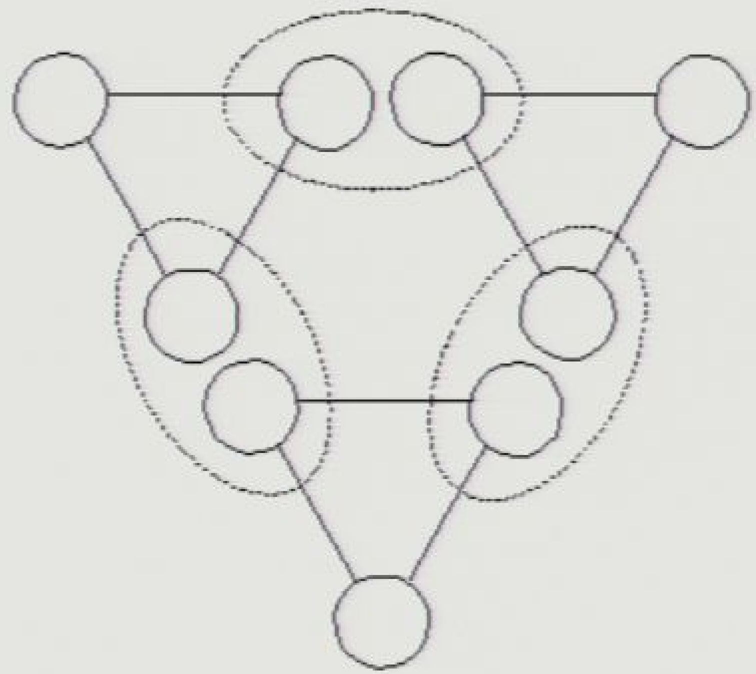
[1] A. Affleck *et al.*, Commun. Math. Phys. **115**, 477 (1988) & PRL **59**, 799 (1987)

[2] H. Fan, V. E. Korepin, V. Roychowdhury, C. Hadley & S. Bose, *quant-ph/0605133* (2006)

[3] F. Verstraete & J. I. Cirac, Phys. Rev. A **70**, 060302(R) (2004)

Valence bond solid (work in progress)





So could we make a VBS from qudit singlets?

May not be physical, but could be interesting ...

cf. H. Fan, V. E. Korepin & V. Roychowdhury, *quant-ph/0511150* (2005)

Other questions (work in progress)

Other questions:

- Do there exist multi-*qubit* states with as much persistency?
- What do these states look like when $d \neq N$?
- For which d, N do they exist?

  (Derive criteria from $\mathrm{Tr}\,\rho^2 \leq 1$, as in bipartite case)
- Prove conjecture that for $N = md$ qudits, there are m orthogonal such states[1]

[1]P. Kok *et al.*, quant-ph/02011038 (2002)

UCL

Summary

- Introduced qudit singlets
- Summarised previous results
- Shown that qudit singlets are the ground state of a 2-local Hamiltonian for *any graph*
- Shown that you can take *arbitrary measurements* and still get a singlet
- Shown that they have the *highest possible persistency of entanglement*
- Reviewed potential uses of states: work in progress and open problems
- Discussed potential implementation of these states

Thank you for listening!

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Further details online
quant-ph/0602139
www.tampa.phys.ucl.ac.uk/quinfo

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highest possible persistency of entanglement

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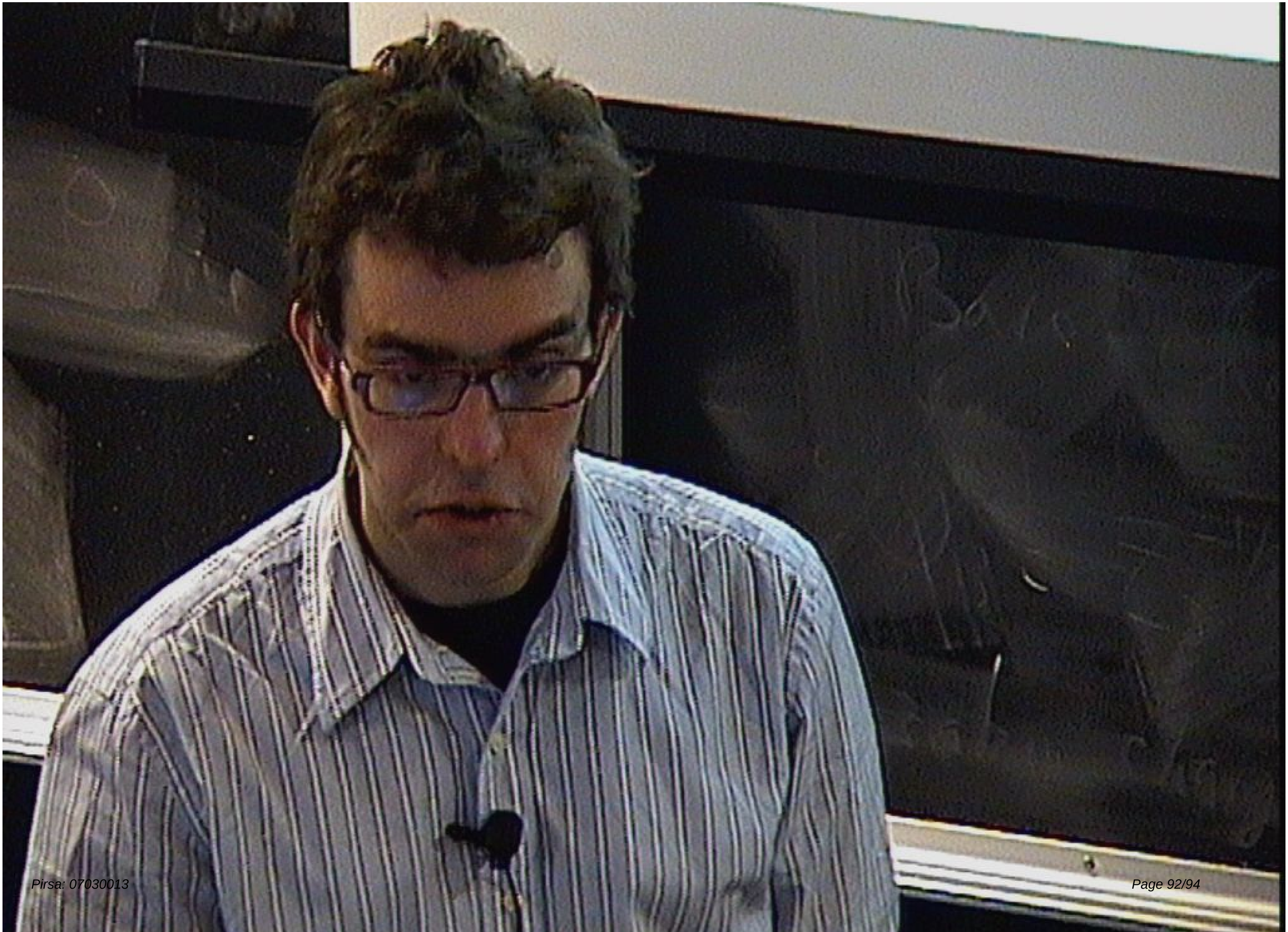

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