Title: Quantum Error Correction 8B

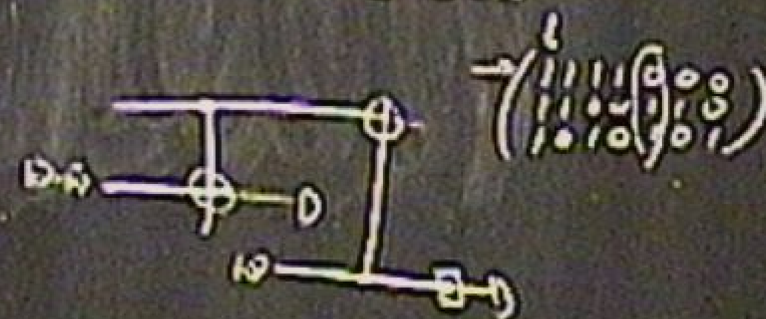
Date: Feb 27, 2007 05:00 PM

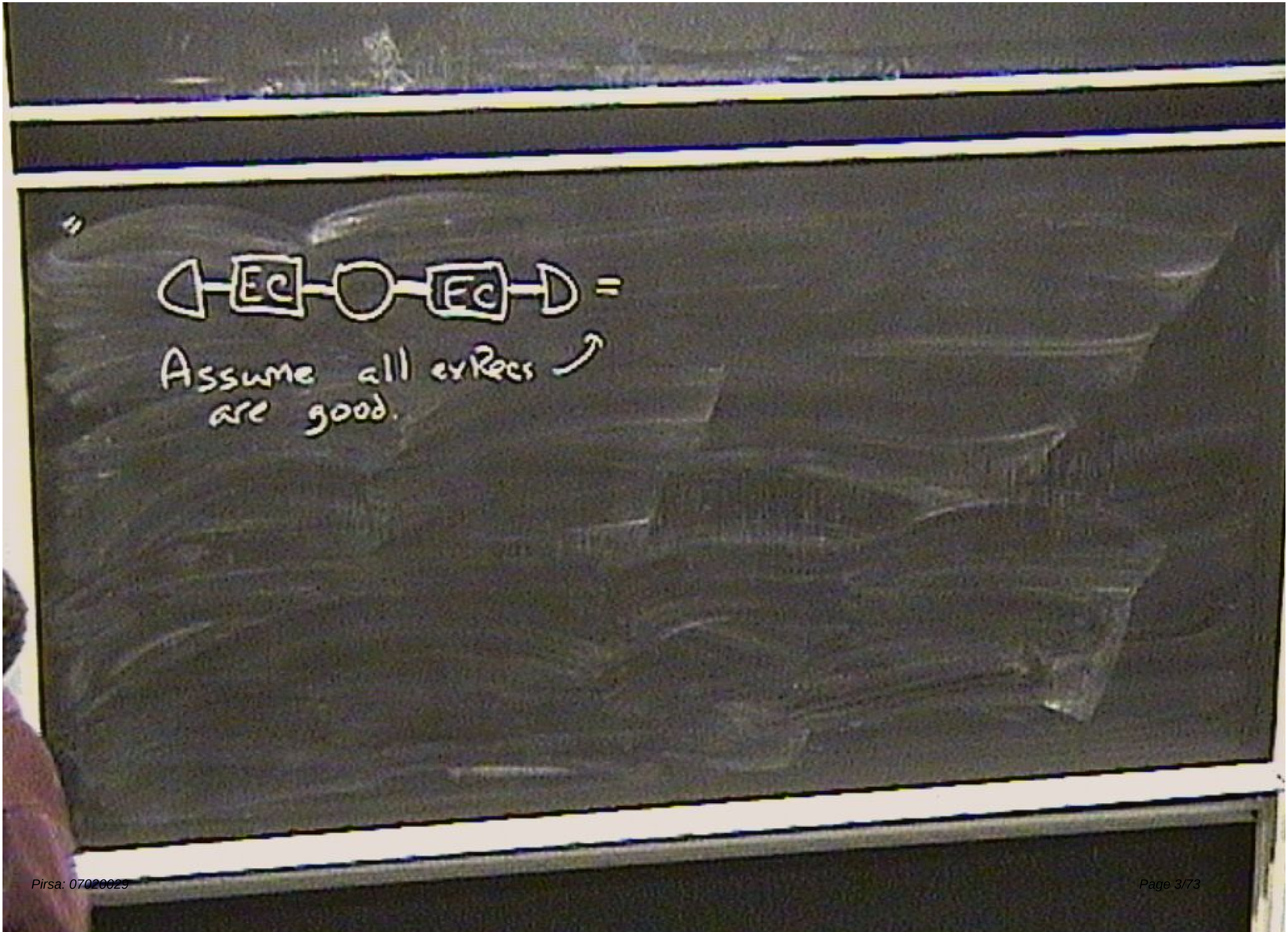
URL: http://pirsa.org/07020029

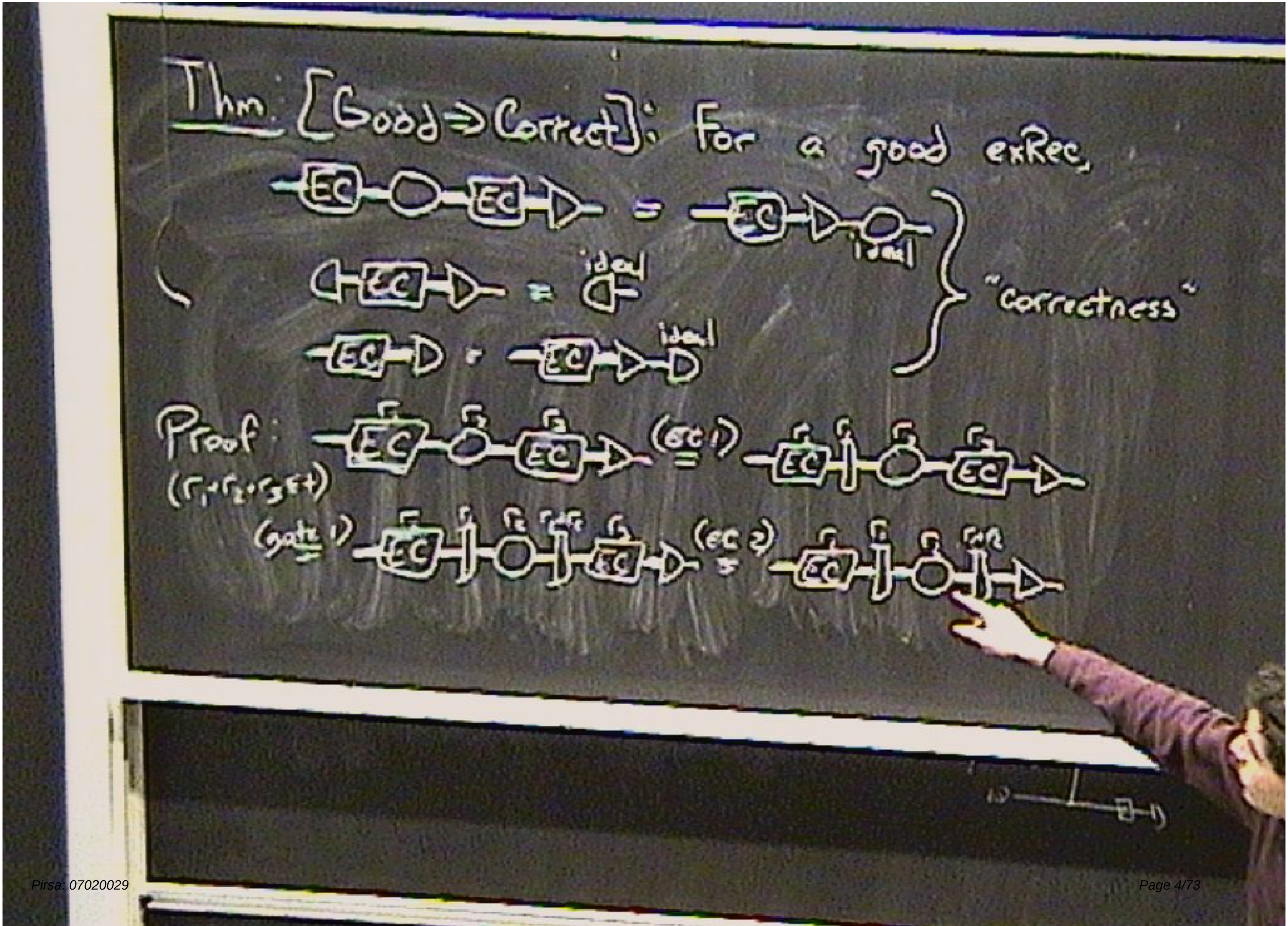
Abstract: Equivalence of fault-tolerant circuit to less noisy unencoded circuits, threshold theorem, calculation of the threshold.

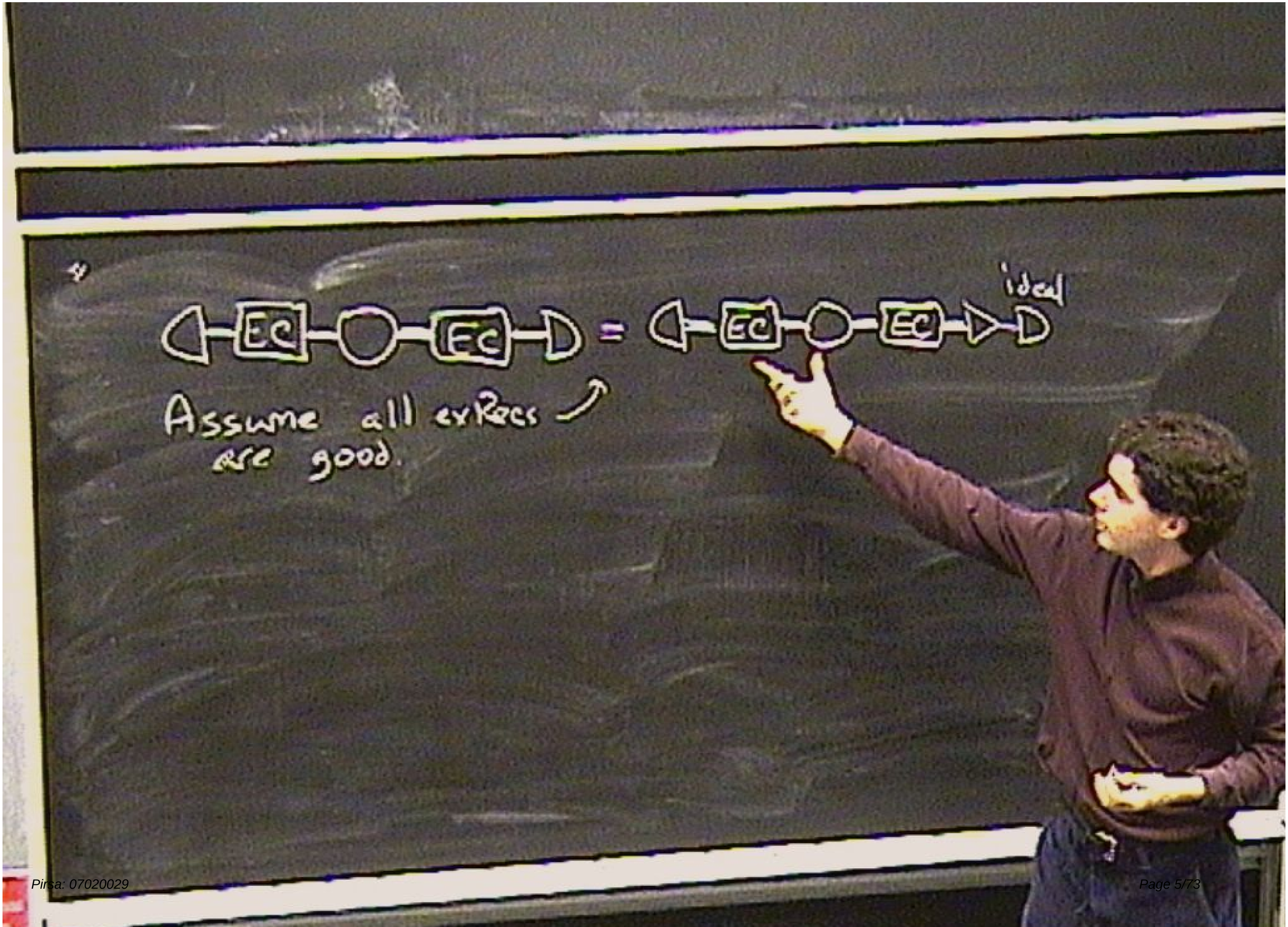

Corollary: For a truncated exRec, good ⇒ correct.

Proof: Insert EC steps w/ no errors to replace truncated ones.




EC
EC
=
Assume all exRecs
are good.


Thm. [Good ⇒ Correct]: For a good exRec,
ideal
ideal
ideal
"correctness"
Proof:
(r1+r2+r3 ≤ t)
(EC 1)
(gate 1)
(EC 2)


EC
EC
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EC
ideal
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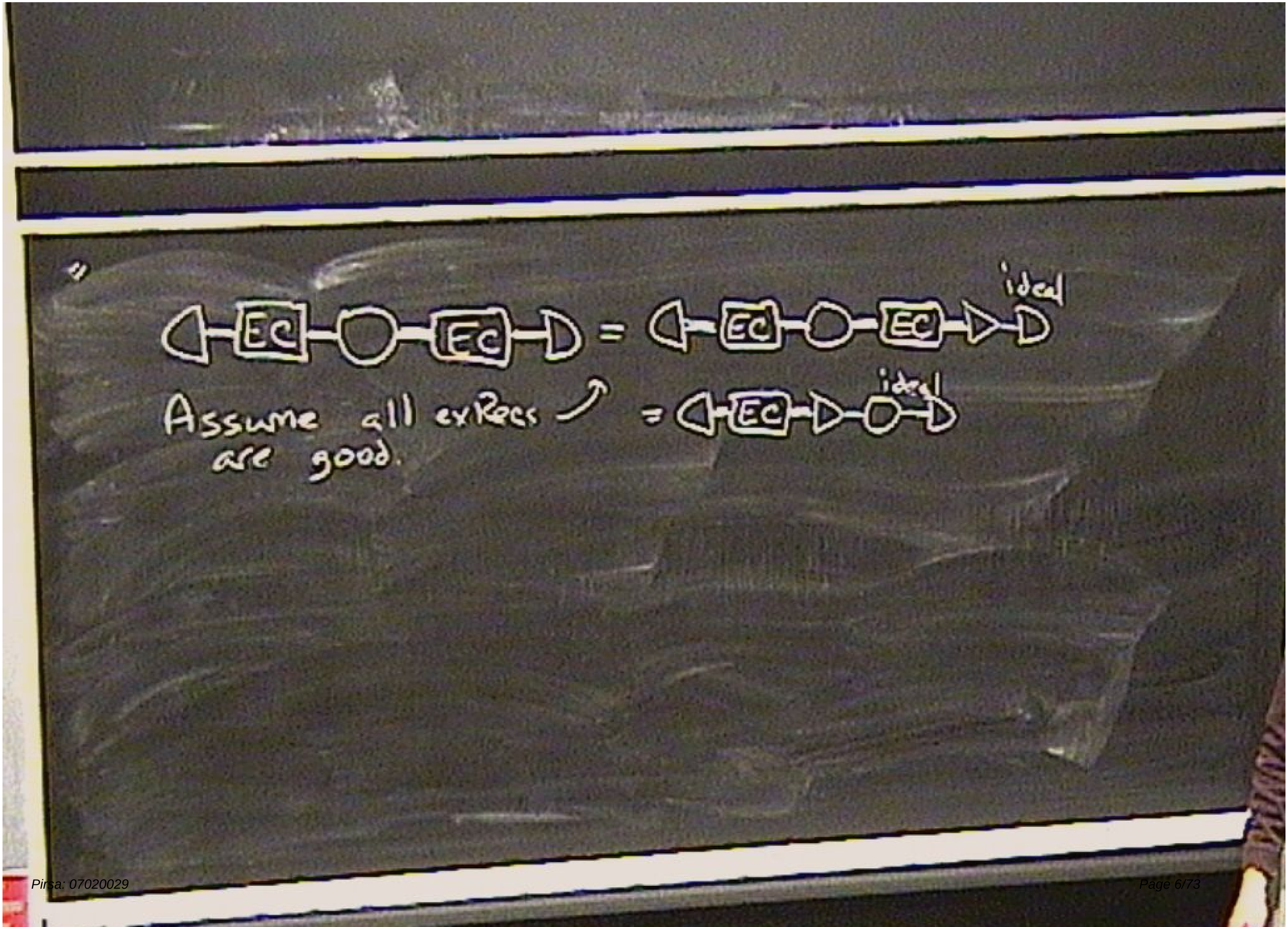
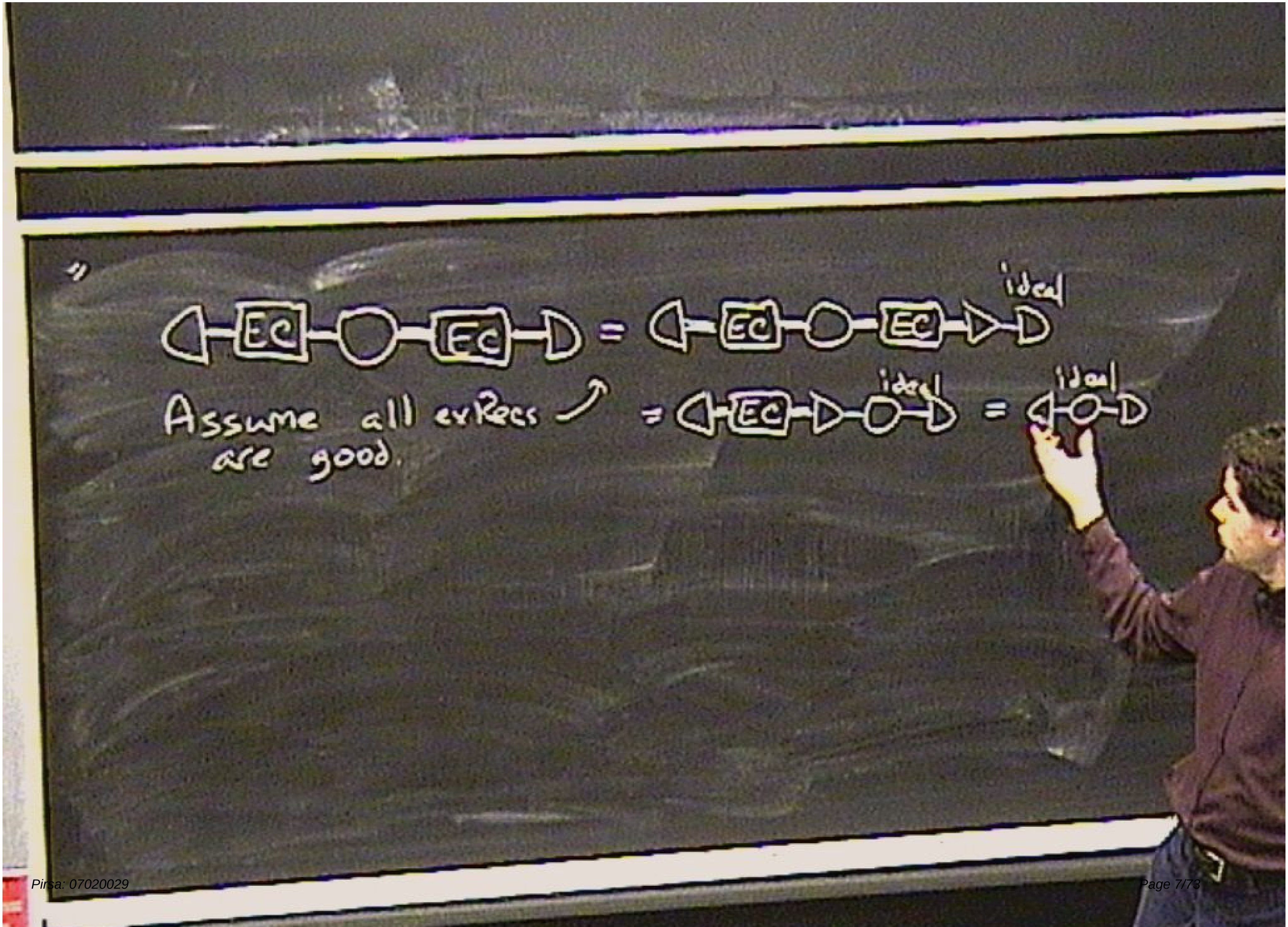

ideal
EC
EC
=
EC
EC
Assume all exRecs
are good.
= ideal
EC


EC
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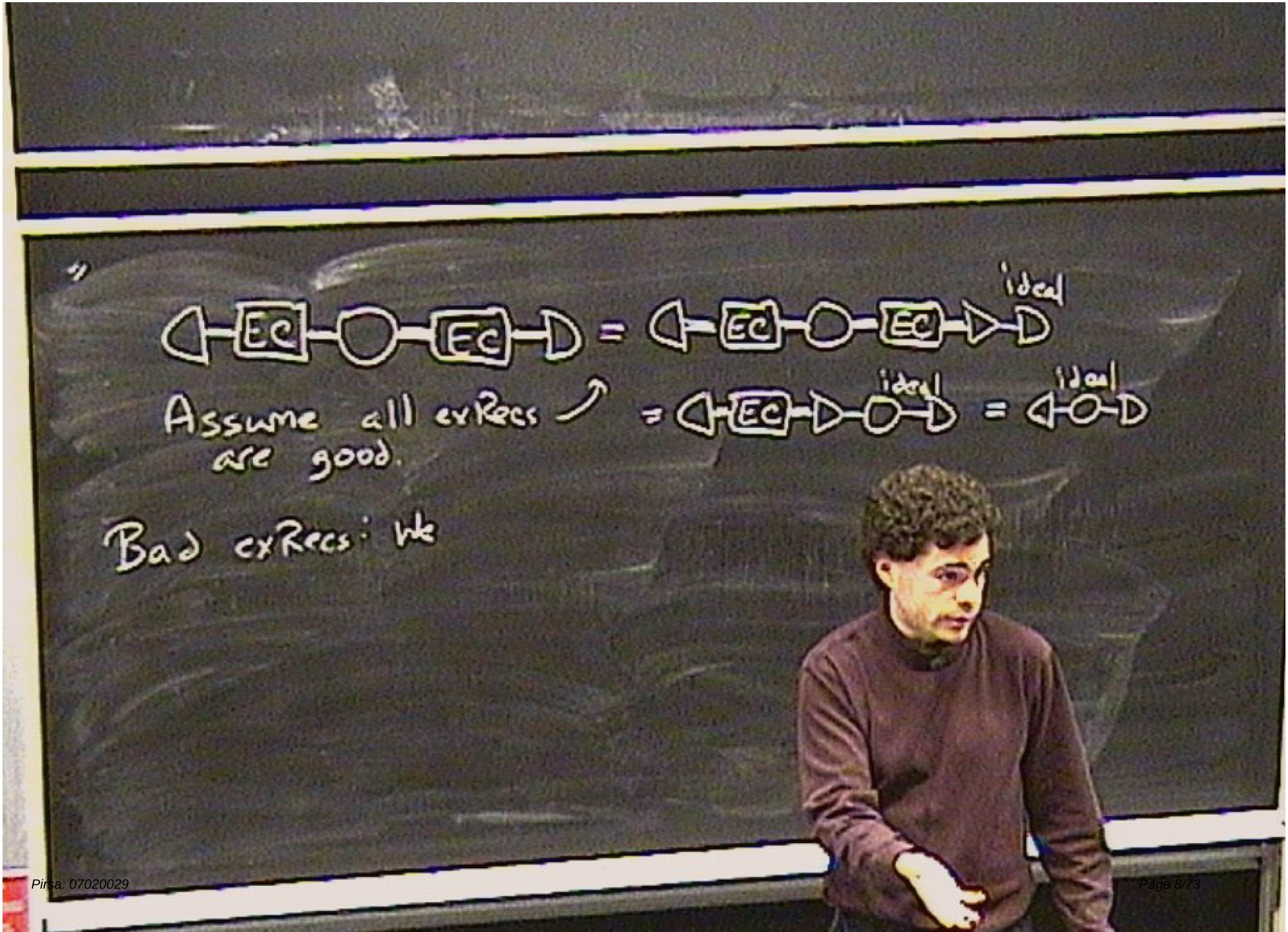
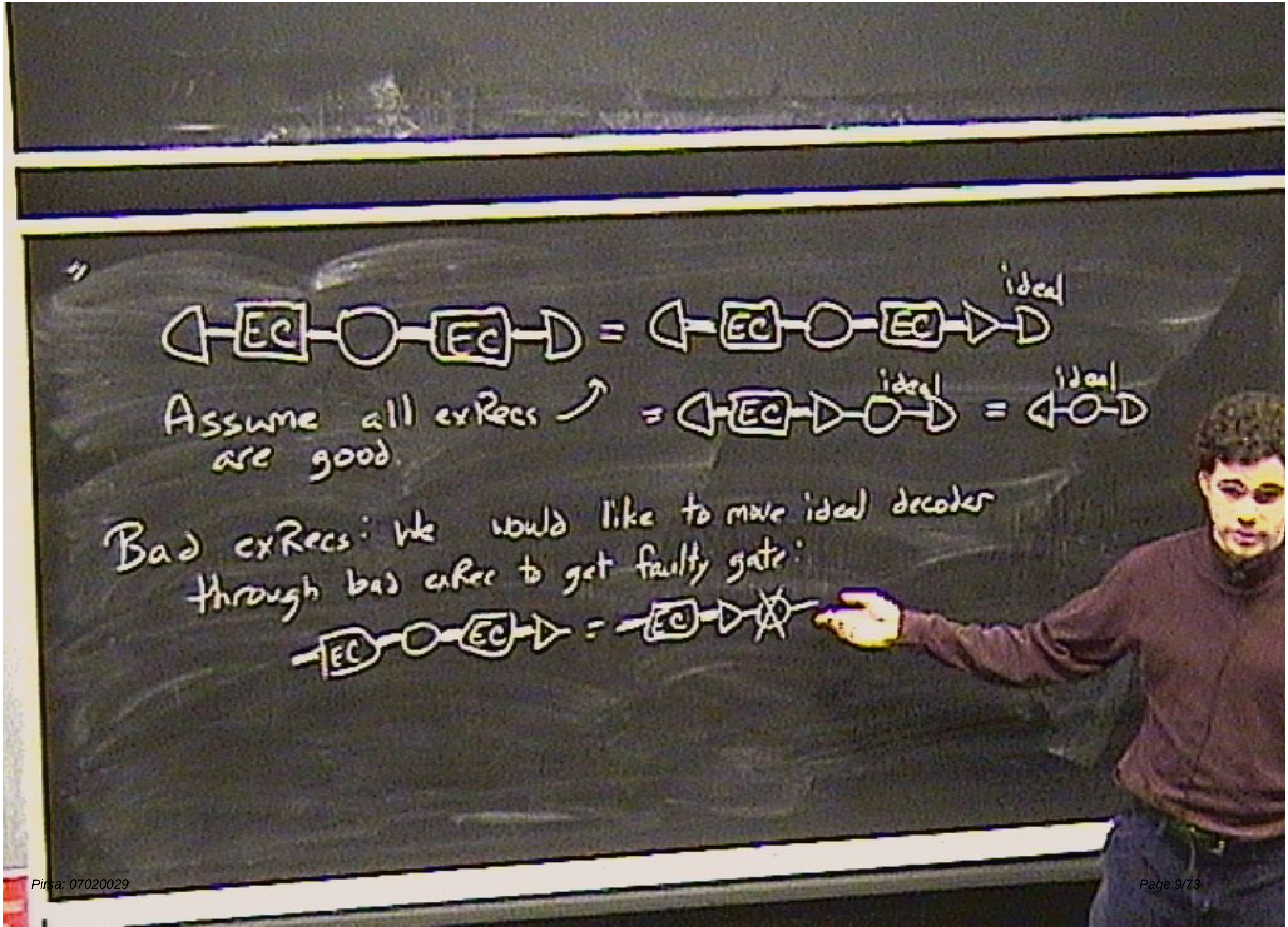

EC
EC
=
EC
EC
ideal
Assume all exRecs are good.
=
EC
ideal
=
ideal
Bad exRecs: We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ideal
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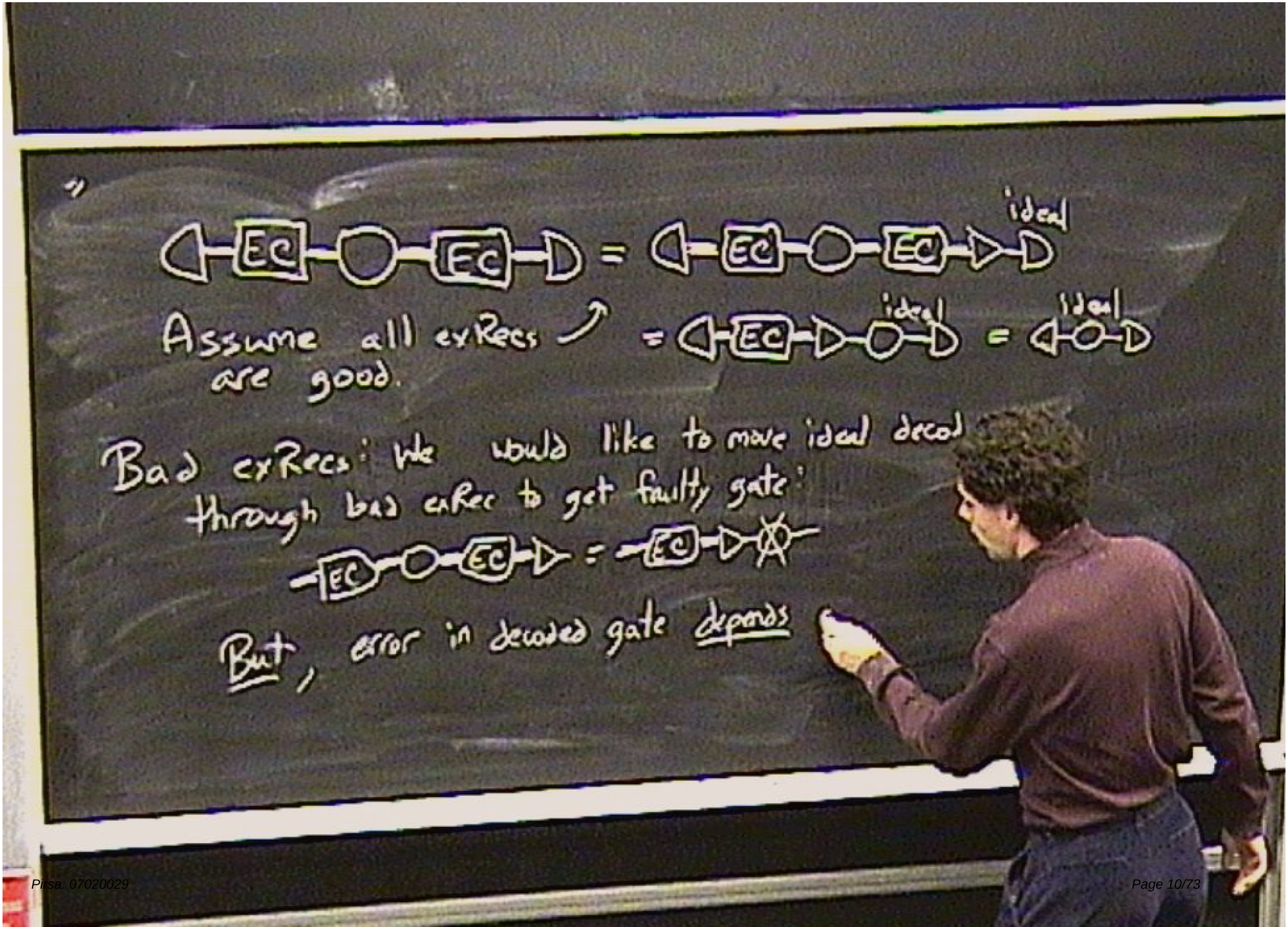
Bad exRecs: We would like to move ideal decoder through bad exRec to get faulty gate:
EC
EC
EC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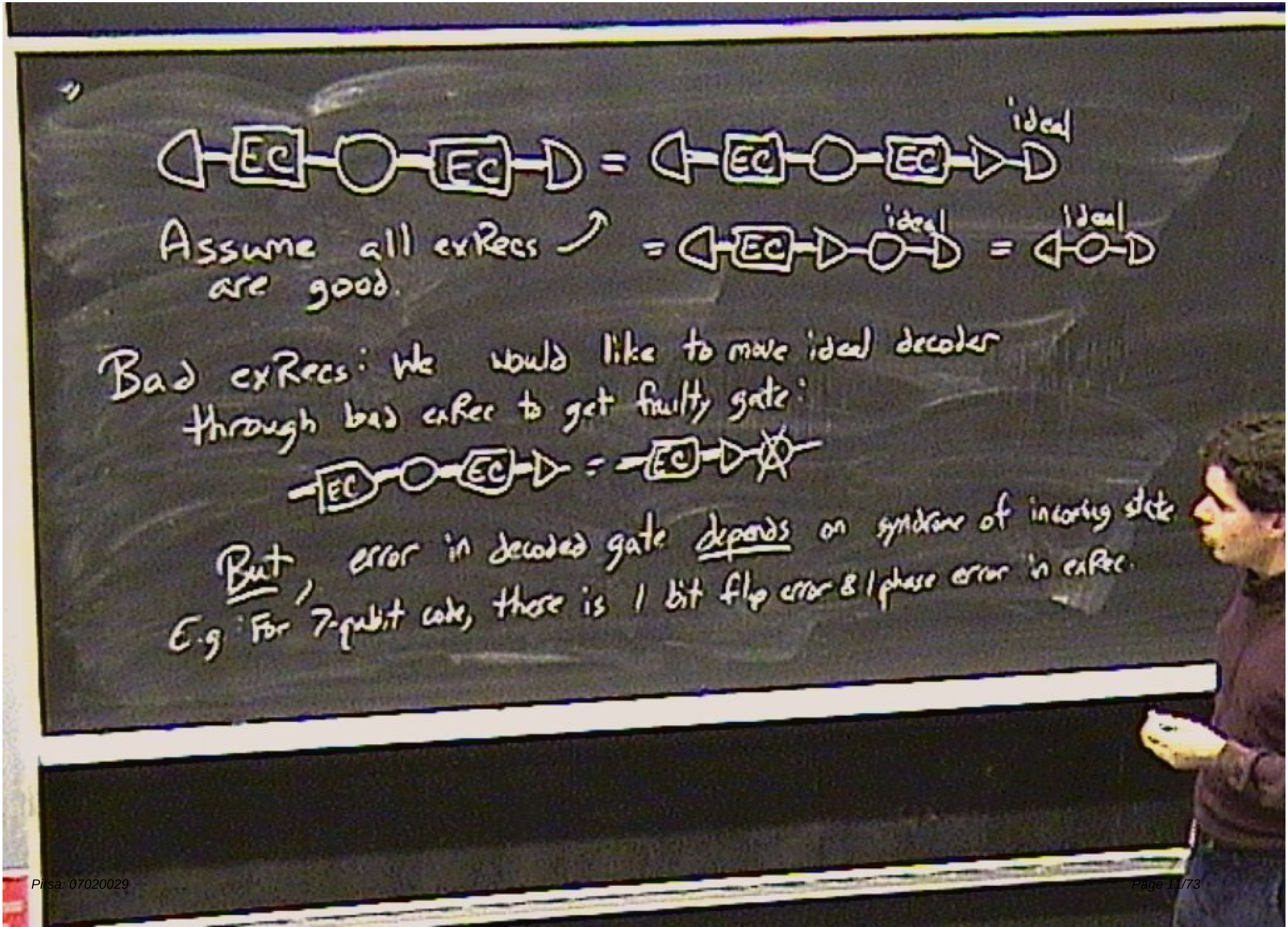
But, error in decoded gate depends


ideal
EC
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=
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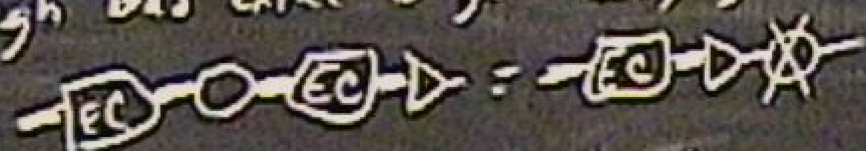
Assume all exRecs are good.
ideal
ideal
=
EC
=
Bad exRecs: We would like to move ideal decoder through bad exRec to get faulty gate:
EC
EC
=
EC
But error in decoded gate depends on syndrome of incoming state
E.g. For 7-qubit code, there is 1 bit flip error & 1 phase error in exRec.


Assume all exRecs ↗ are good.


Bad exRecs: We would like to move ideal decoder through bad exRec to get faulty gate:




<u>But</u>, error in decoded gate <u>depends</u> on syndrome of incoming state

E.g.: For 7-qubit code, there is 1 bit flip error & 1 phase error in exRec.
If incoming syndrome = 0, no error; if syndrome is a bit flip (phase), gate has bit flip error (" phase error)

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If incoming syndrome = 0, no error; if syndrome is a bit flip (phase), gate has bit flip error / phase error

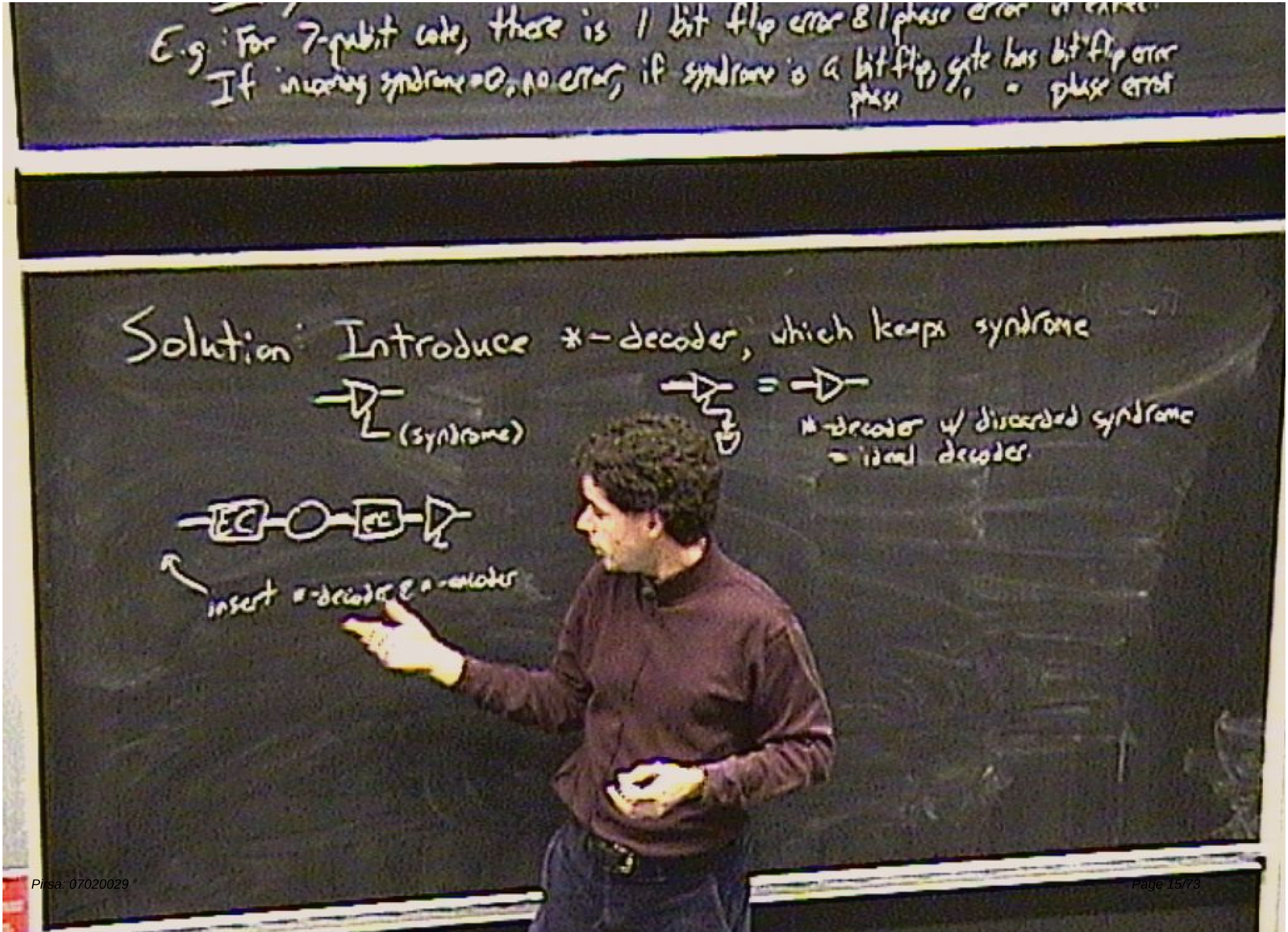
Solution:

E.g: For 7-qubit code, there is 1 bit flip error & 1 phase error in extract...

If incoming syndrome = 0, no error; if syndrome is a bit flip, gate has bit flip error
phase " " phase error

Solution: Introduce *-decoder, which keeps syndrome


*-decoder w/ discarded syndrome
= ideal decoder.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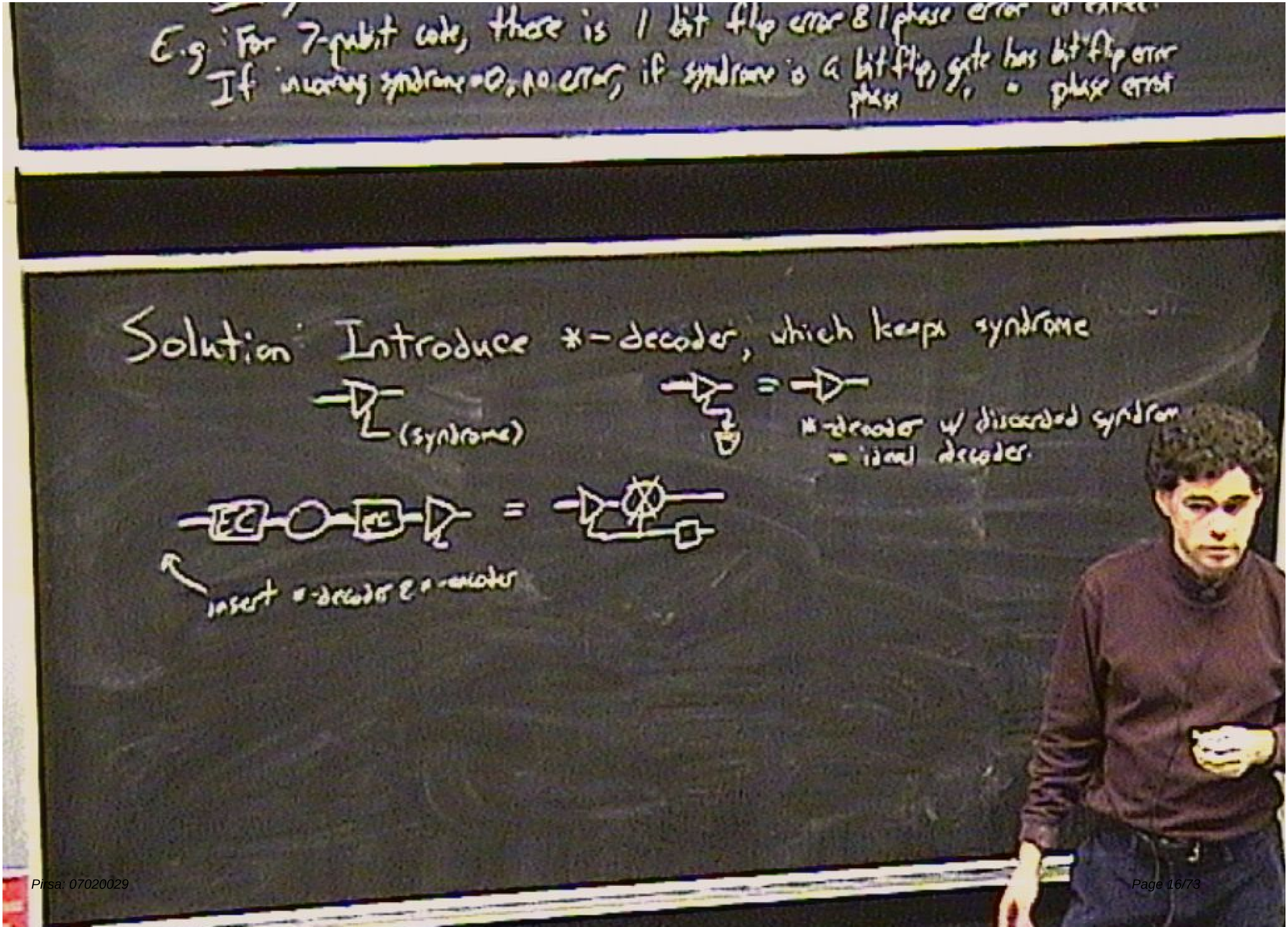
Solution: Introduce *-decoder, which keeps syndrome
(syndrome)
*-decoder w/ discarded syndrome
= ideal decoder.
EC
EC
insert *-decoder & *-encoder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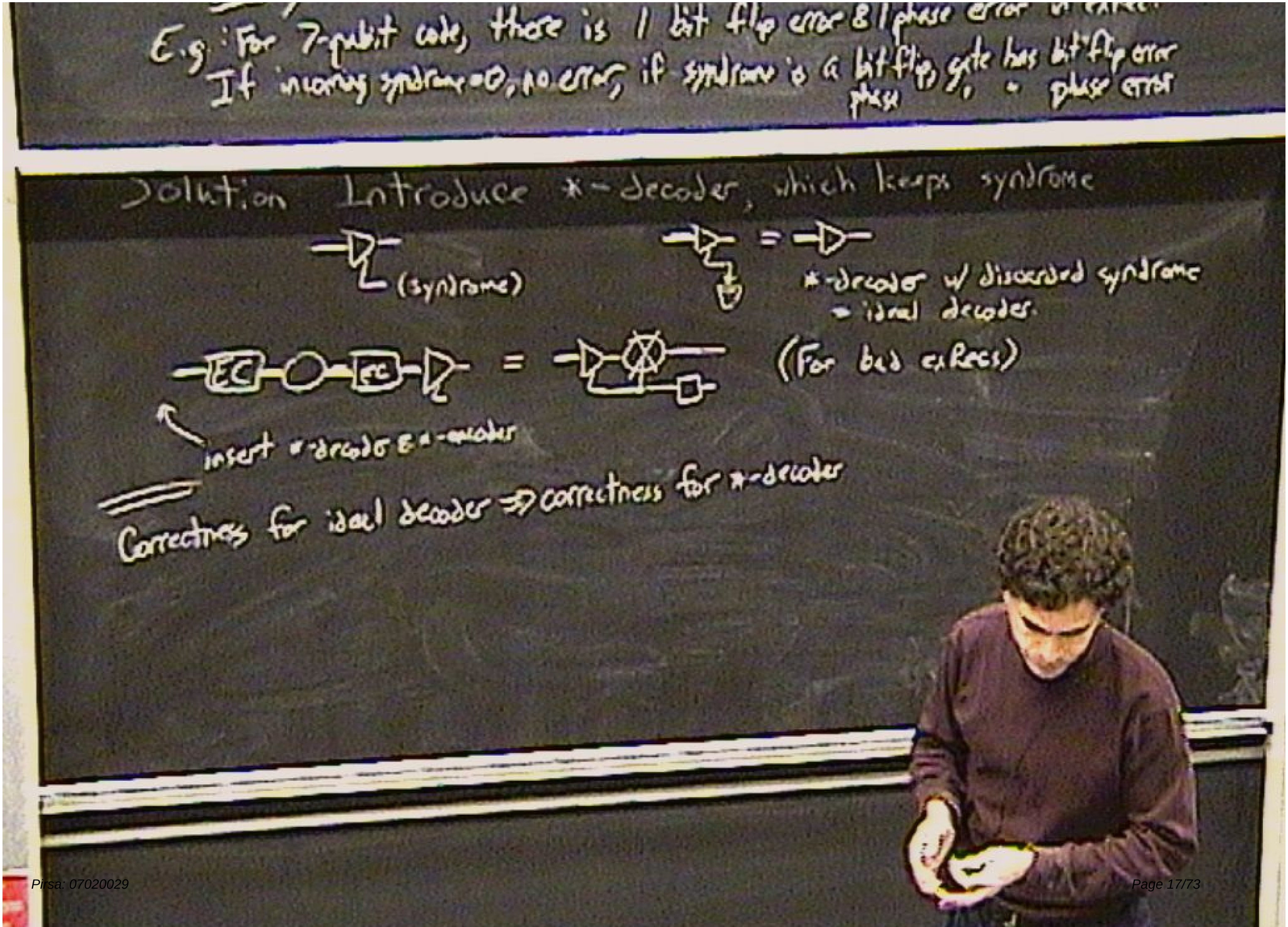
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Solution Introduce *-decoder, which keeps syndrome
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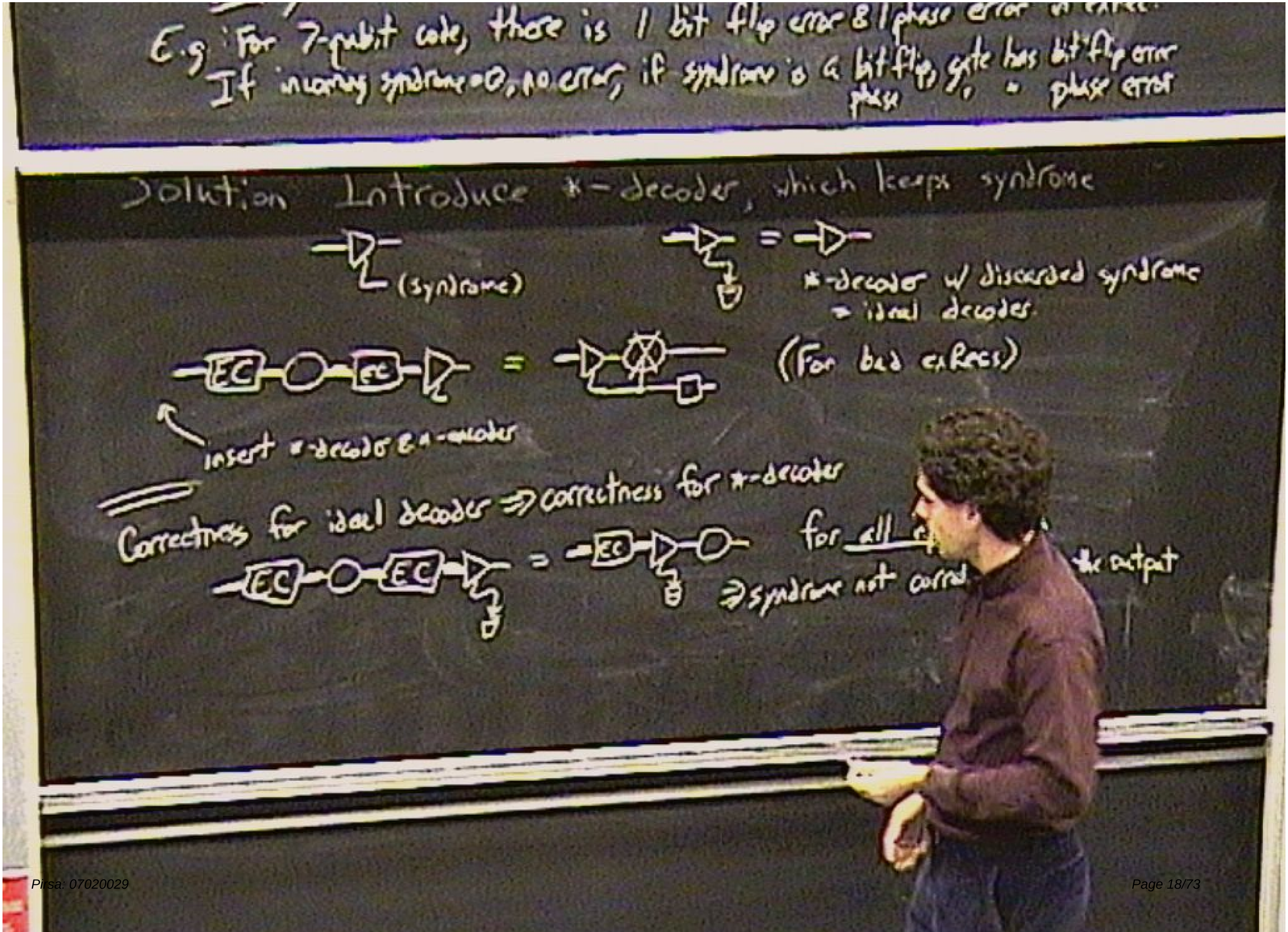
= ideal decoder.
EC
EC
(For bad exRecs)
insert *-decoder & *-encoder
Correctness for ideal decoder ⇒ correctness for *-decoder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phase phase error
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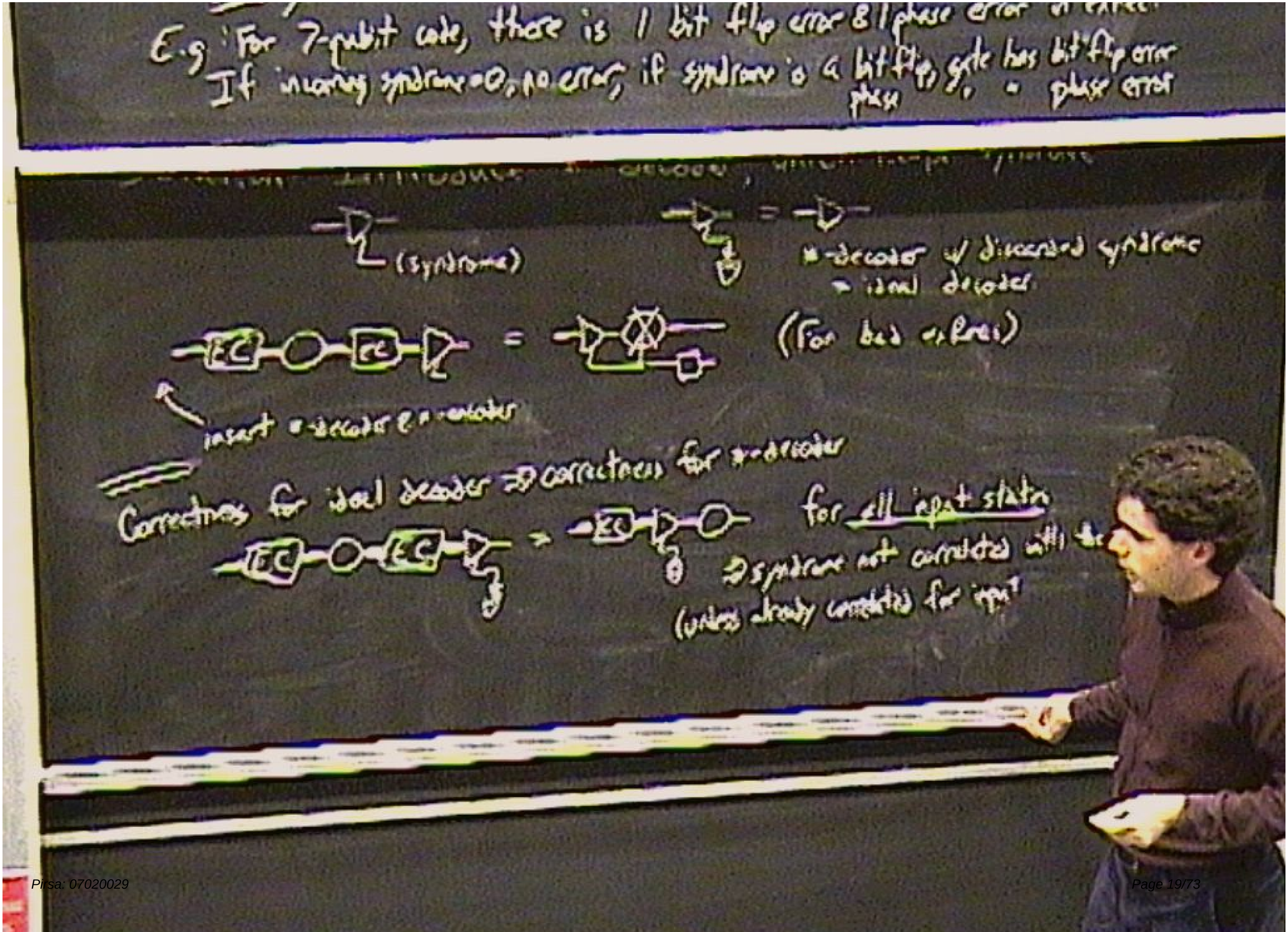
(syndrome)
*-decoder w/ discarded syndrome = ideal decoder.
(For bad exRecs)
insert *-decoder & *-encoder
Correctness for ideal decoder ⇒ correctness for *-decoder
Correctness
for all
the output
⇒ syndrome not correct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decoder w/ discarded syndrome
= ideal decoder
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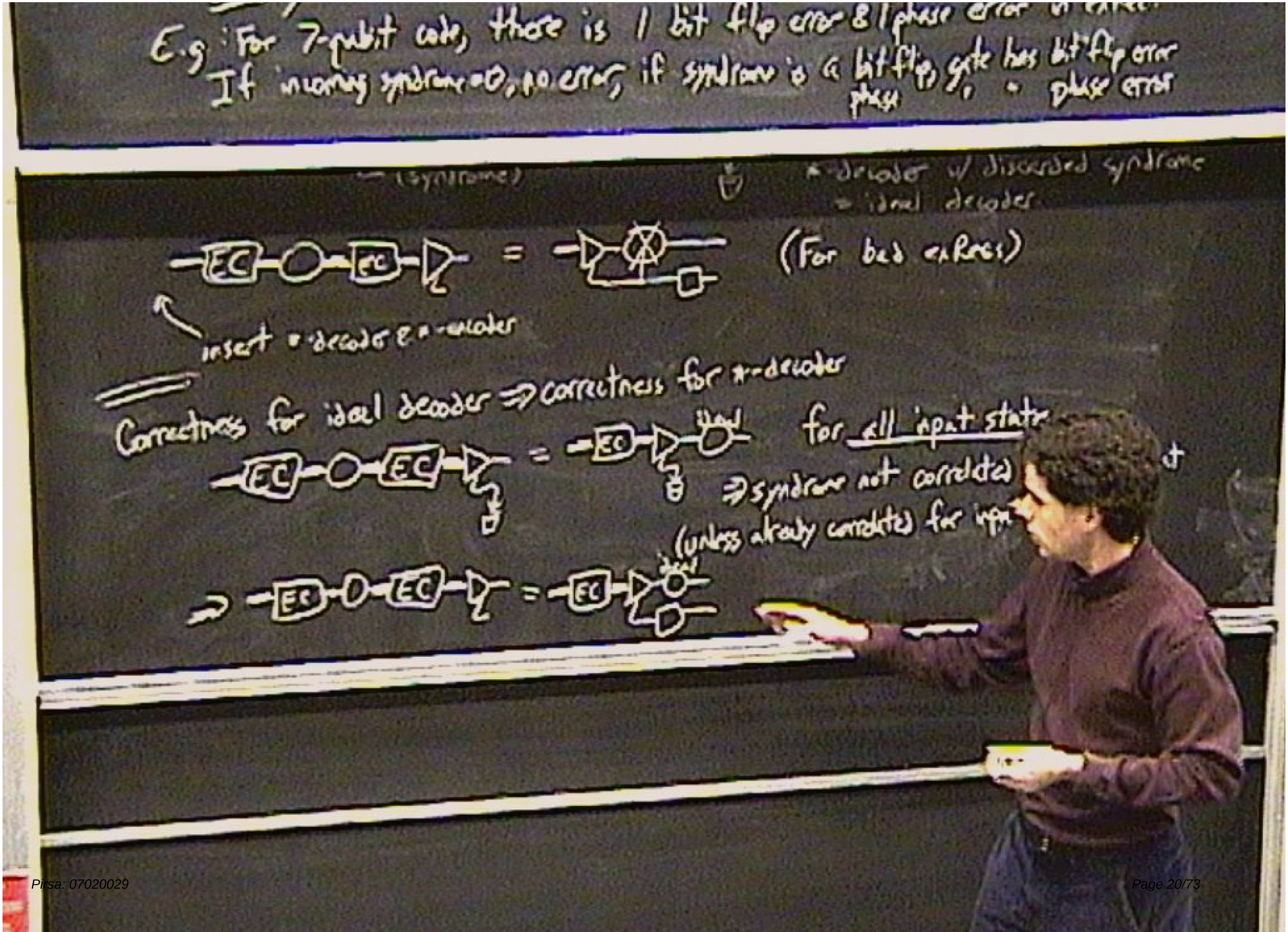
Correctness for ideal decoder ⇒ correctness for decoder
Correctness for ideal decoder
for all input states
syndrome not correlated with the
(unless already correlated for input)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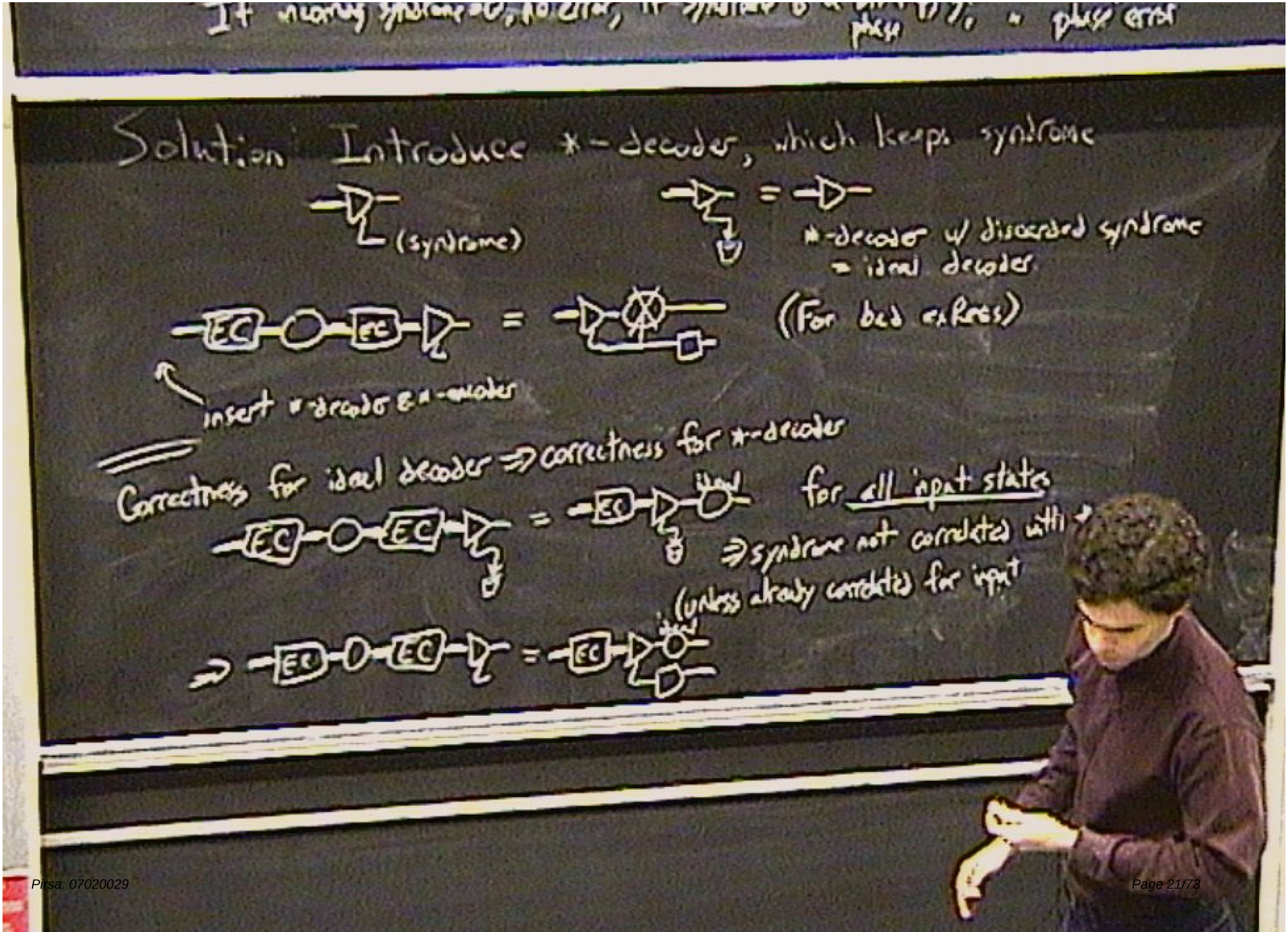
Correctness for ideal decoder
ideal
for all input state
⇒ syndrome not correlated
(unless already correlated for input
ideal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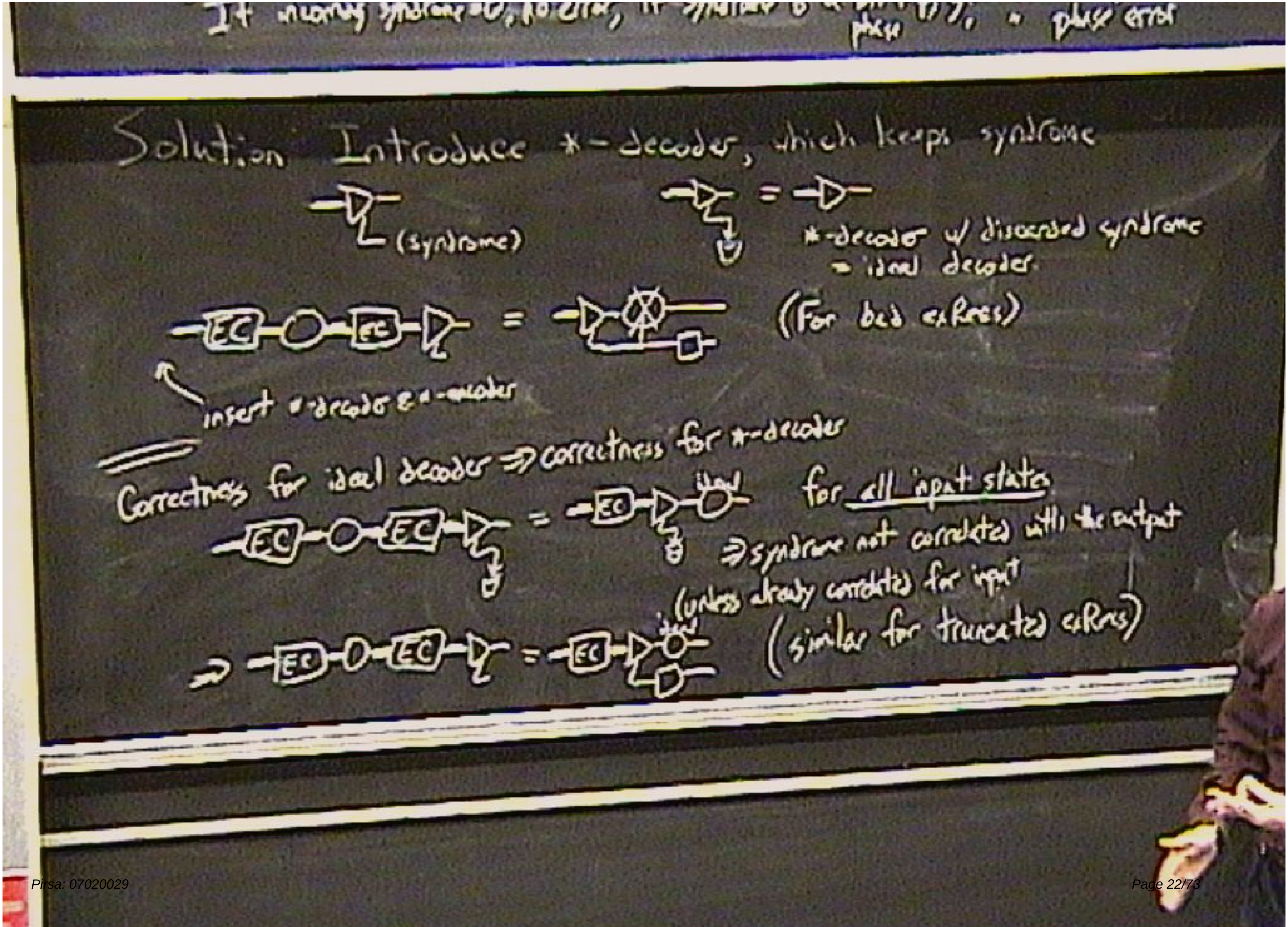
phase error
Solution: Introduce *-decoder, which keeps syndrome
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EC
EC
((For bad exRecs)
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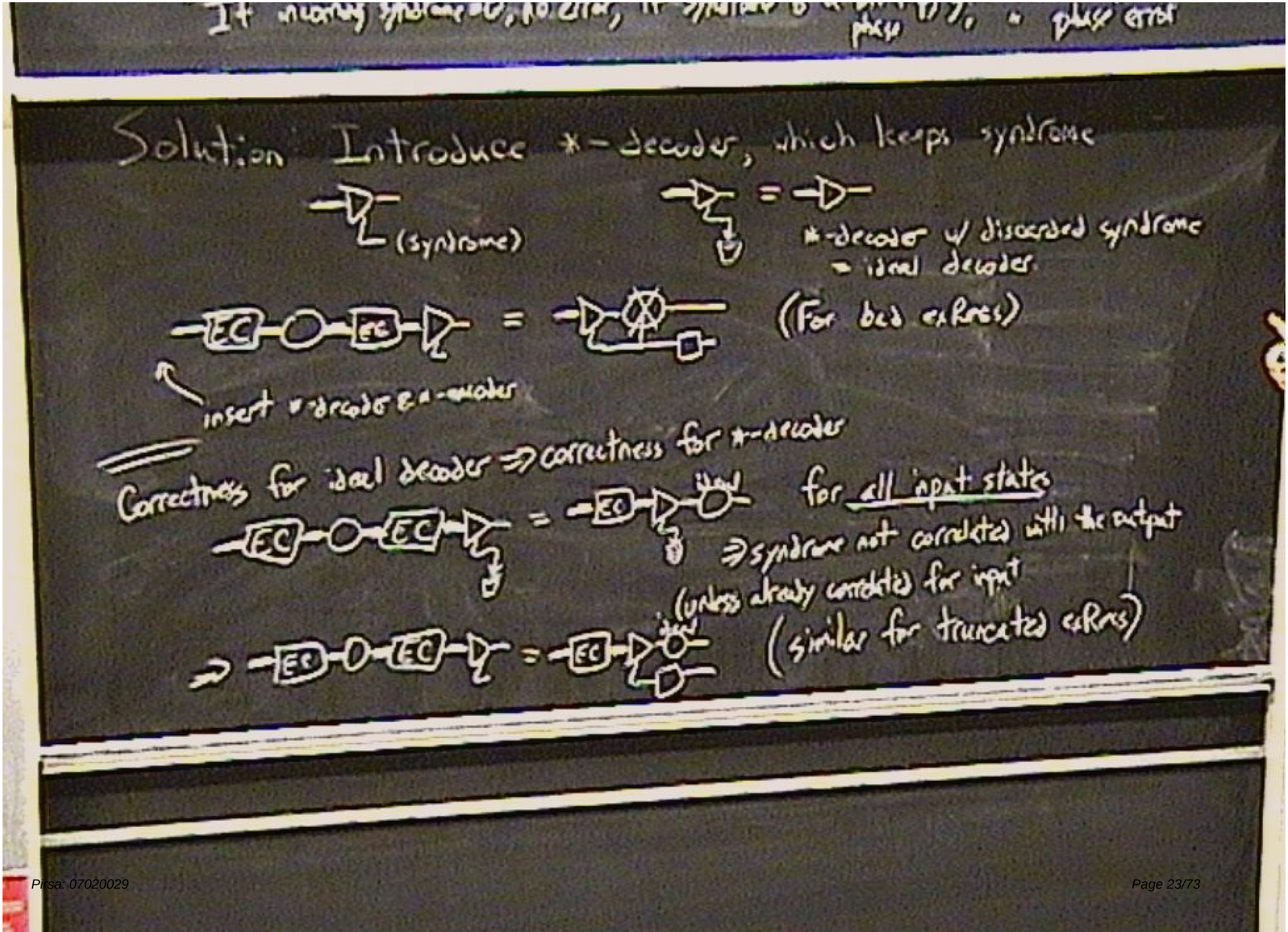
Correctness for all input states
EC
EC
EC
ideal
⇒ syndrome not correlated with the output
(unless already correlated for input
(similar for truncated exRecs)
⇒
EC
EC
EC
ideal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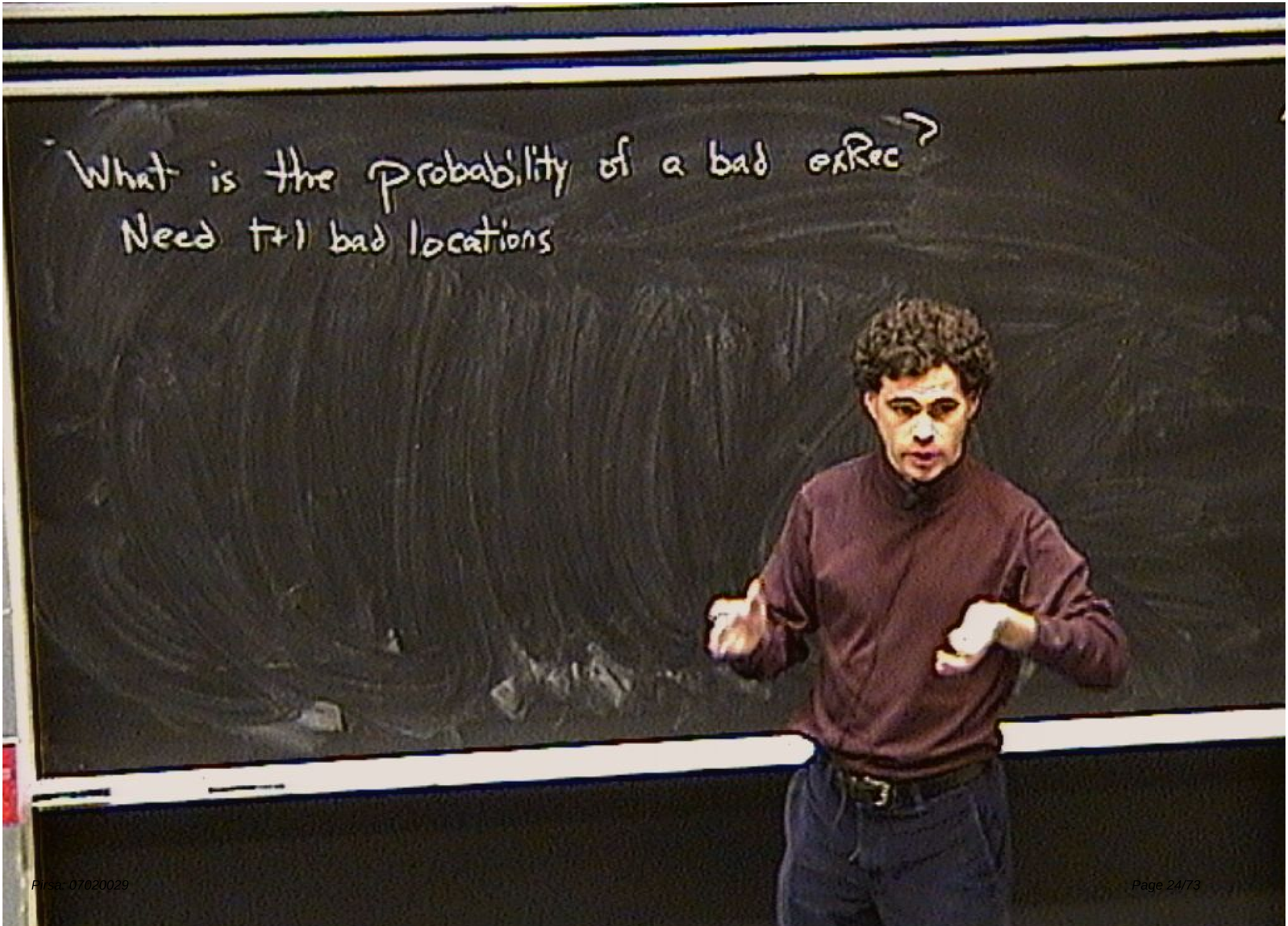
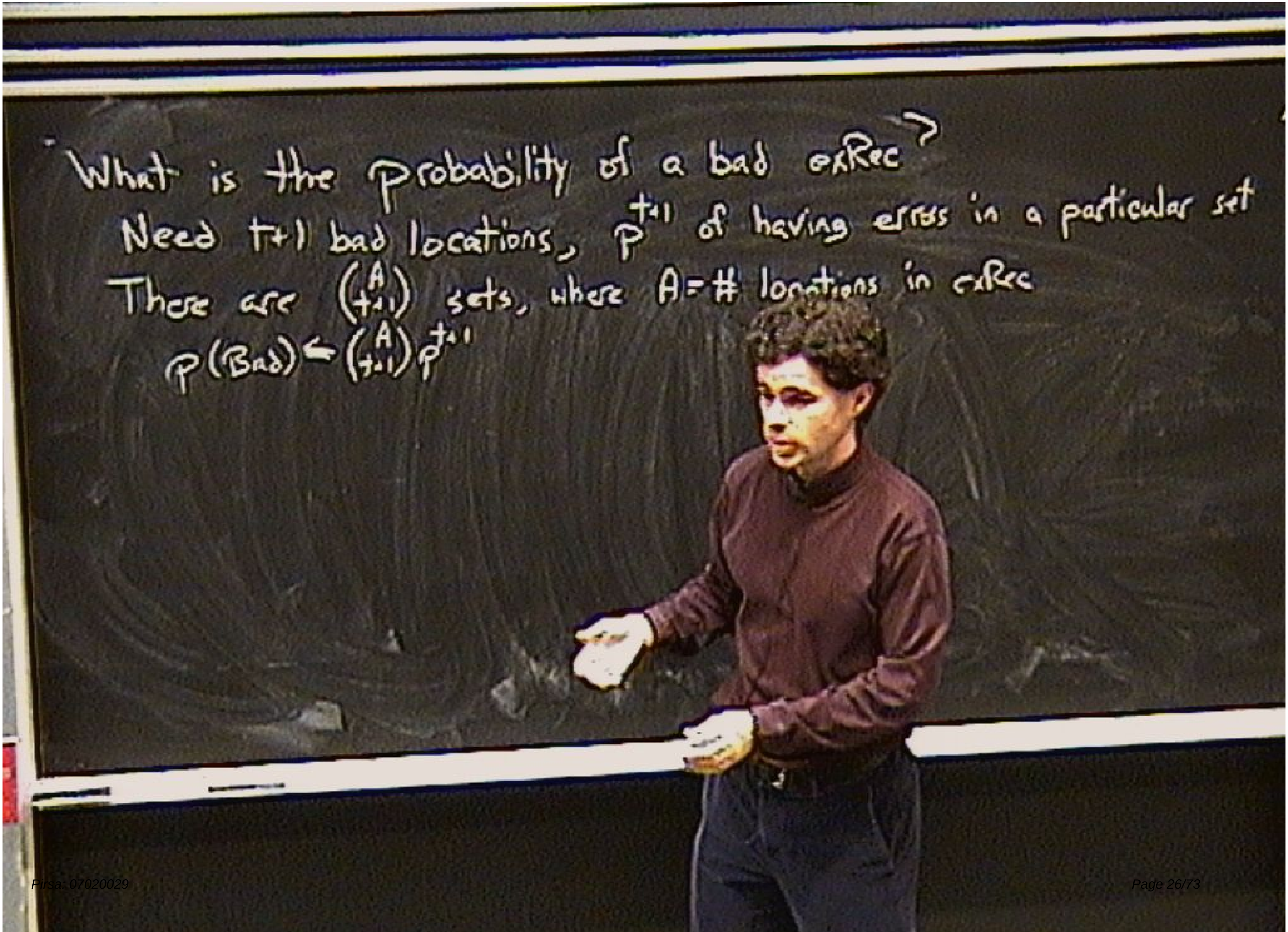
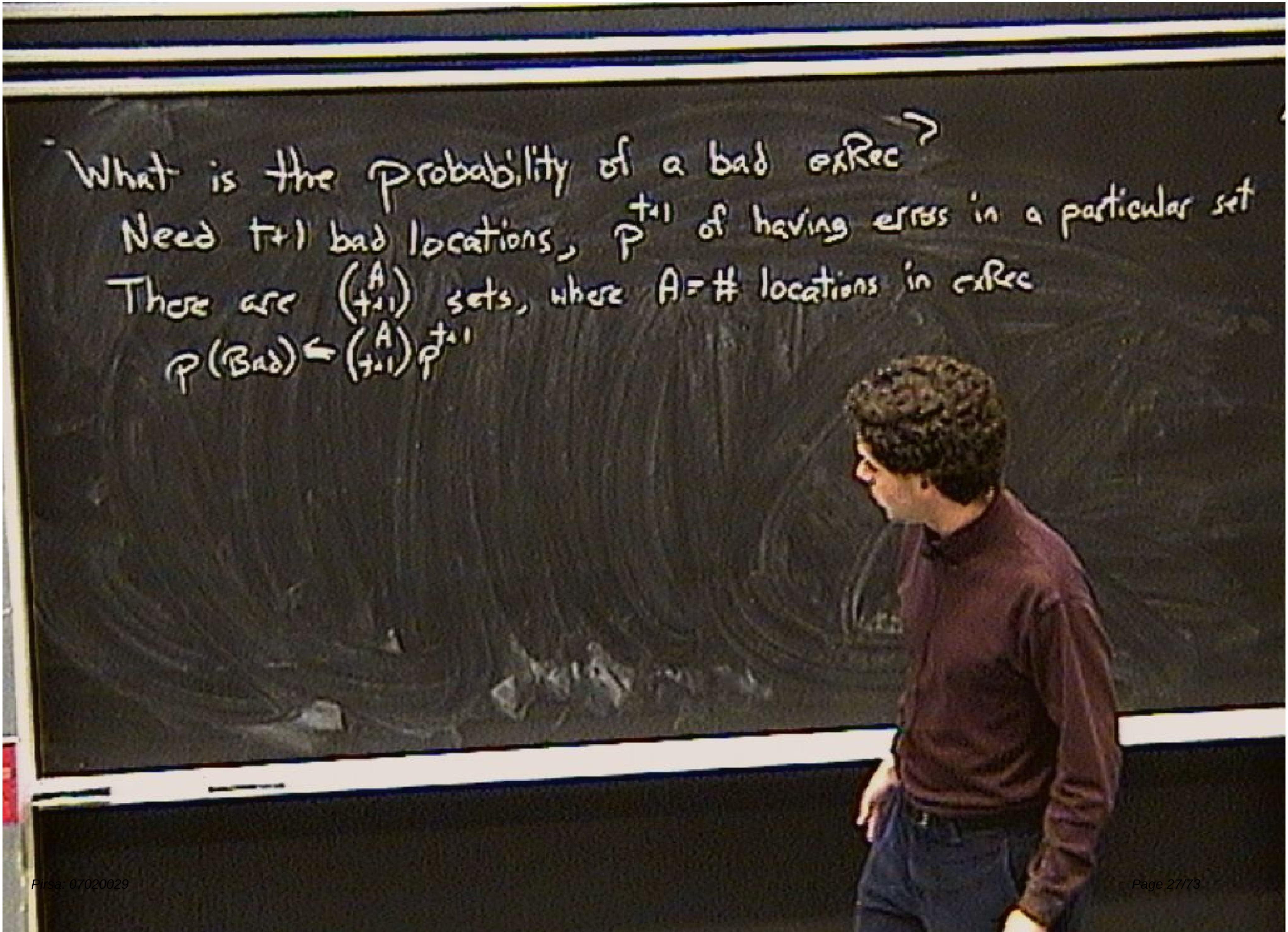

What is the probability of a bad exRec?
Need t+1 bad locations

What is the probability of a bad exRec?

Need t+1 bad locations, $p^{t+1}$ of having errors in a particular set


What is the probability of a bad exRec?
Need t+1 bad locations, $p^{t+1}$ of having errors in a particular set
There are $\binom{A}{t+1}$ sets, where A = # locations in exRec
$p(\text{Bad}) \leq \binom{A}{t+1} p^{t+1}$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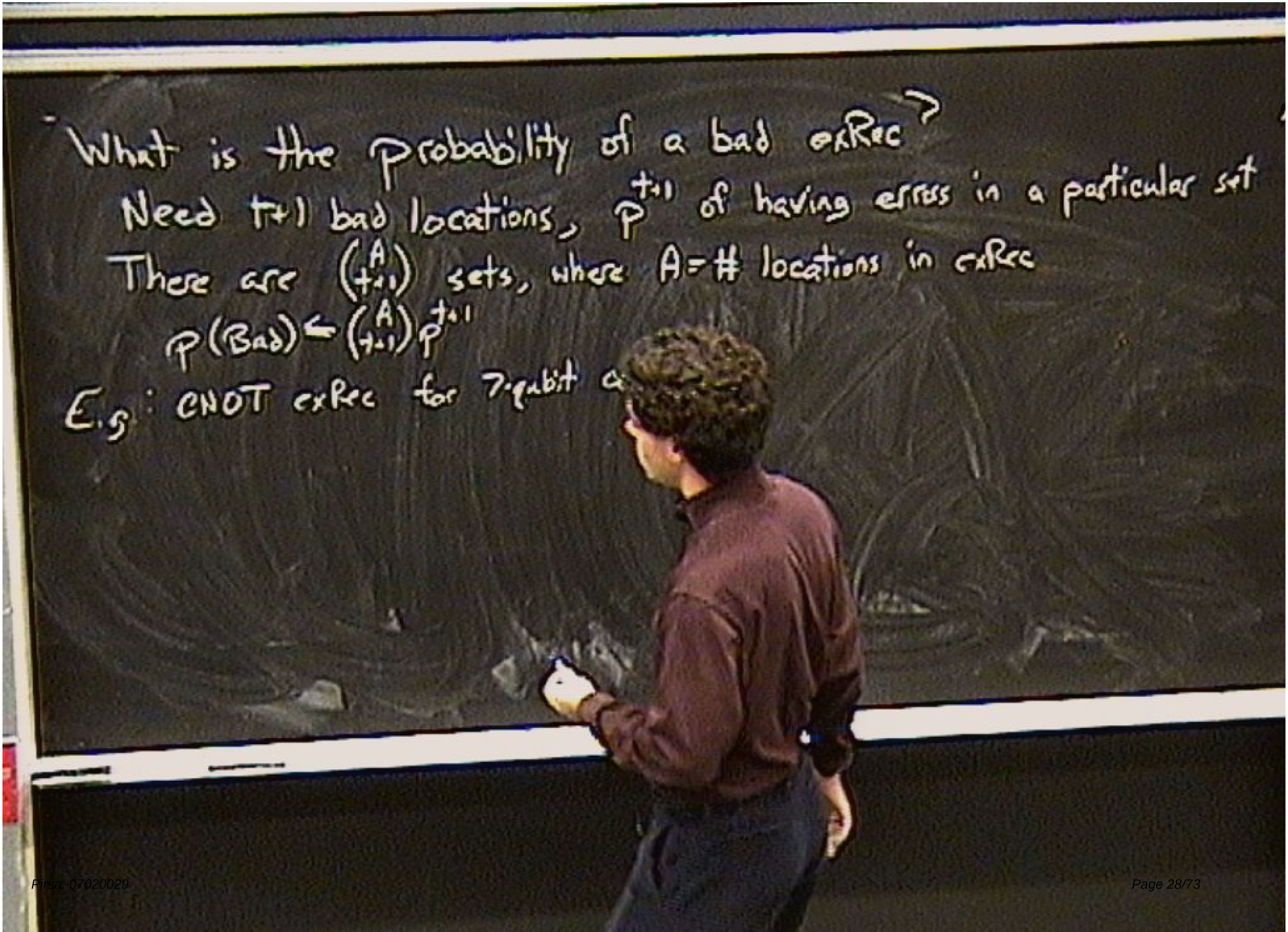
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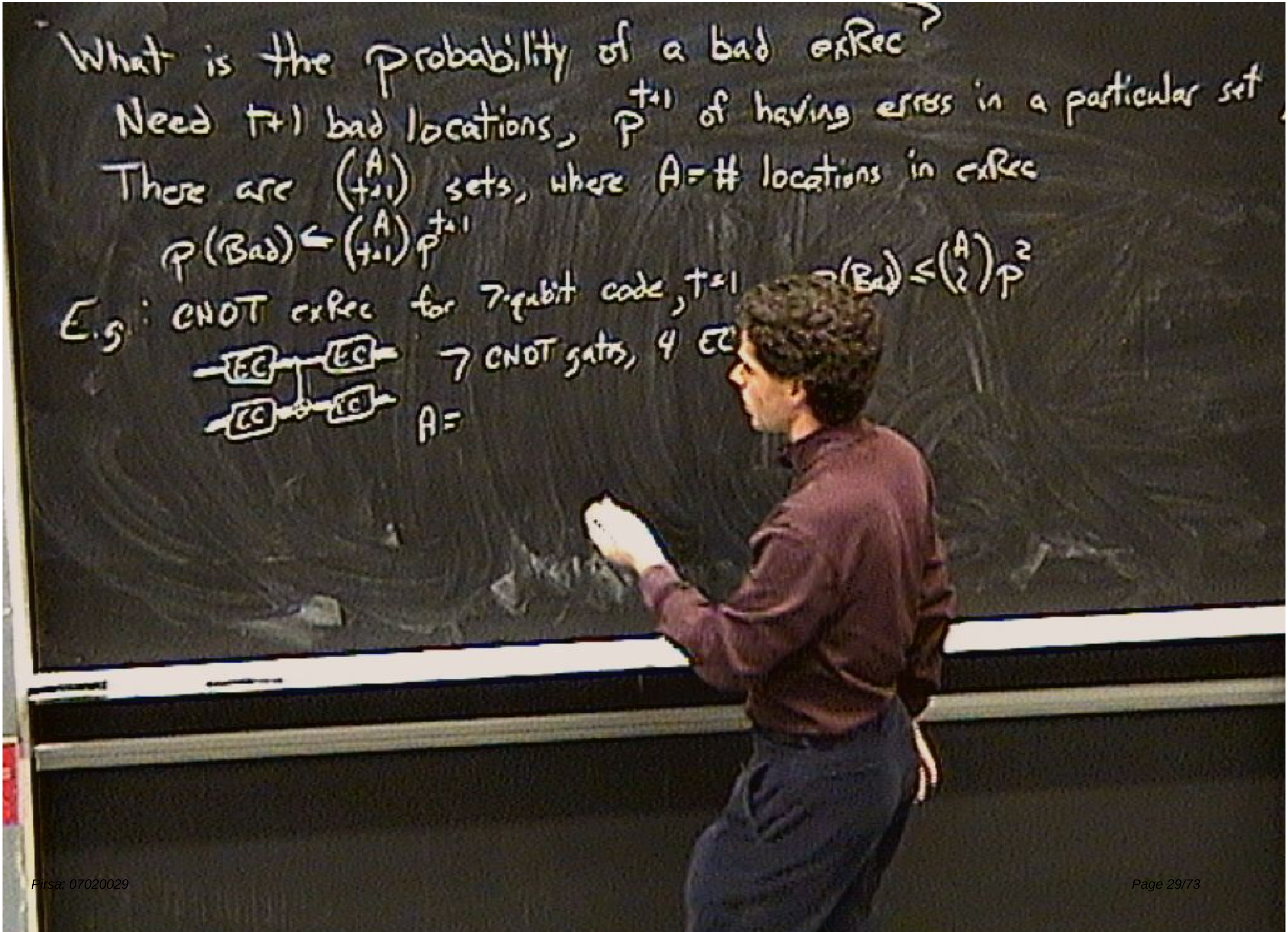
E.g.: CNOT exRec for 7-qubit c


What is the probability of a bad exRec?
Need t+1 bad locations, p^{t+1} of having errors in a particular set
There are (A choose t+1) sets, where A = # locations in exRec
p(Bad) ≤ (A choose t+1) p^{t+1}
E.g: CNOT exRec for 7-qubit code, t=1
p(Bad) ≤ (A choose 2) p^2
EC EC EC EC
7 CNOT gates, 4 EC
A =

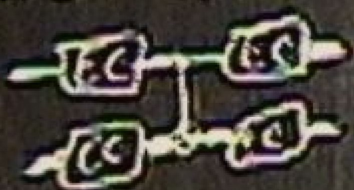
What is the probability of ... c ?

Need t+1 bad locations, p ... of having errors in a particular set

There are $\binom{A}{t+1}$ sets, where A = # locations in exRec

$p(\text{Bad}) \leq \binom{A}{t+1} p^{t+1}$

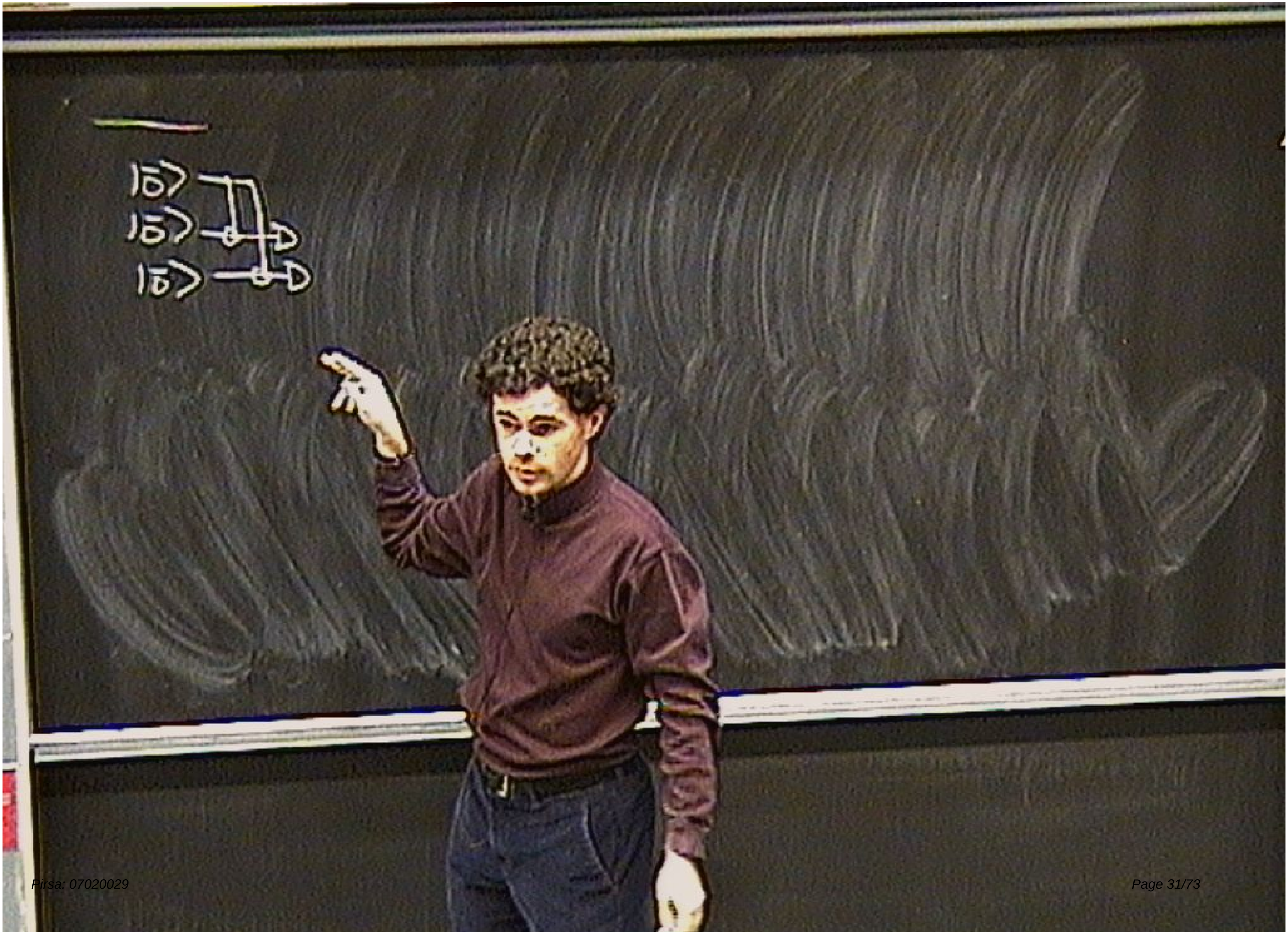
E.g.: CNOT exRec for 7-qubit code, t=1 $\quad p(\text{Bad}) \leq \binom{A}{2} p^2$




7 CNOT gates, 4 EC steps

A = 7 + 4B, B = # locations in EC

Sometimes we are interested in exRecs which are missing ... or more trailing EC steps. These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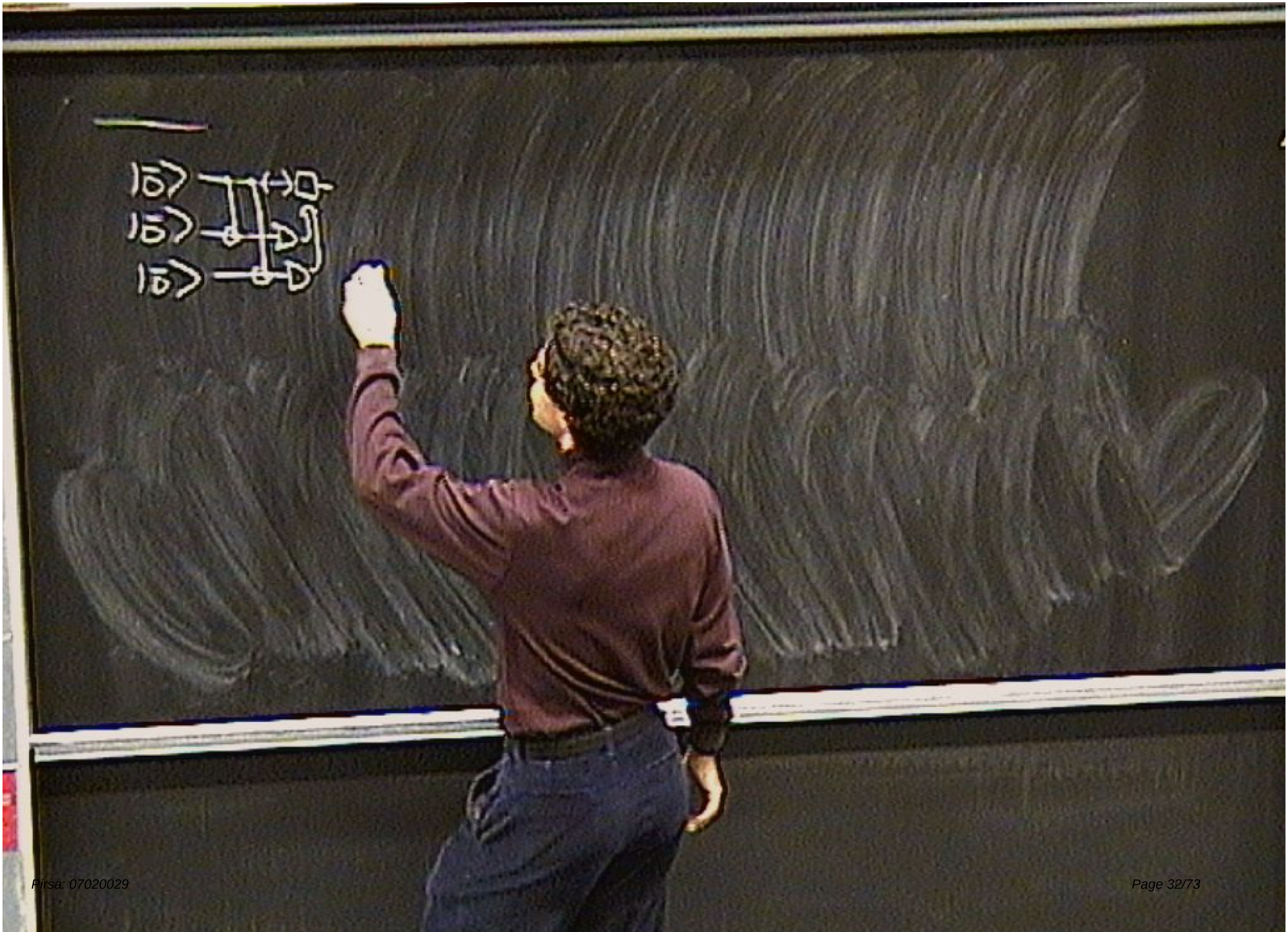
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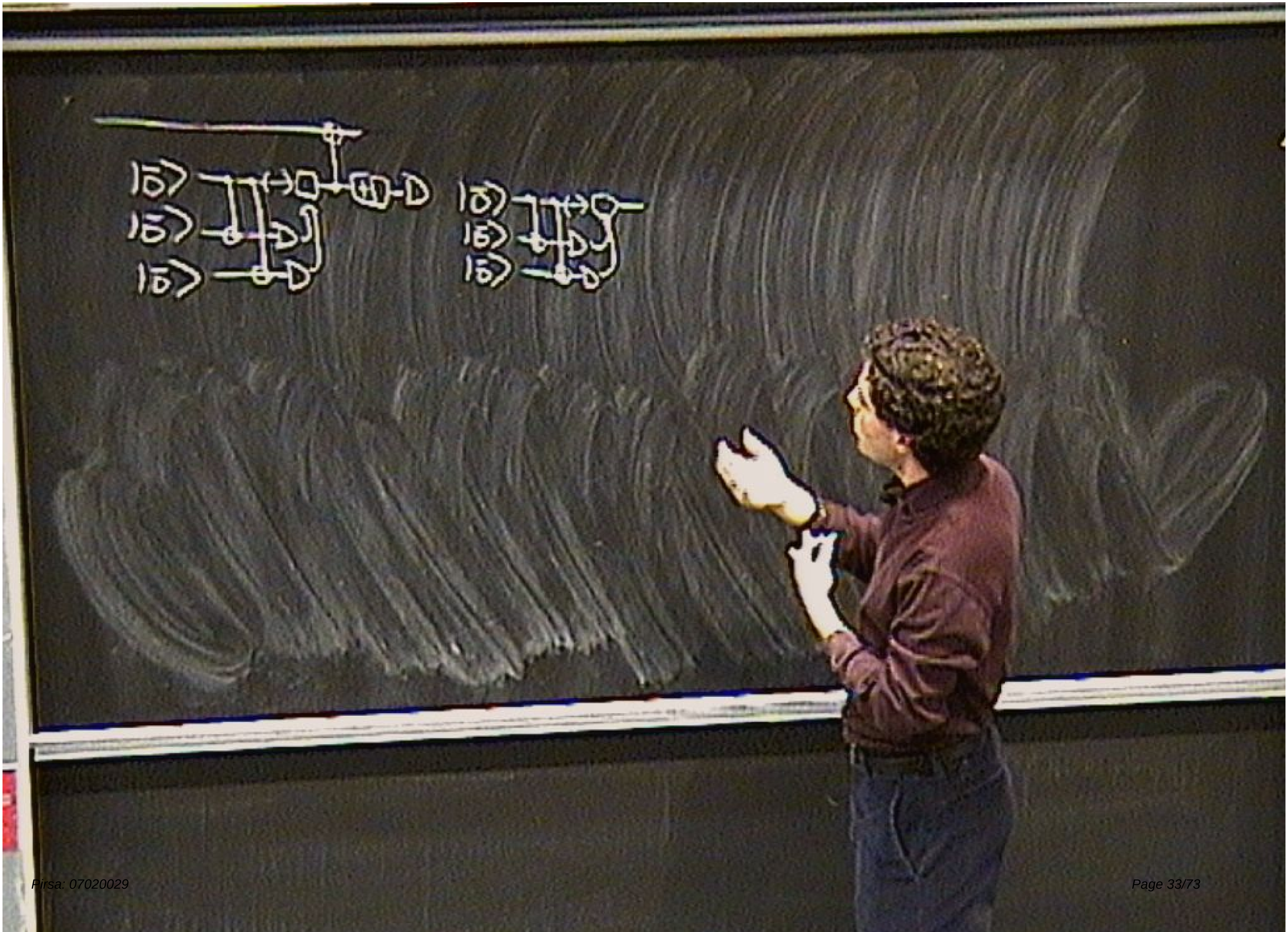
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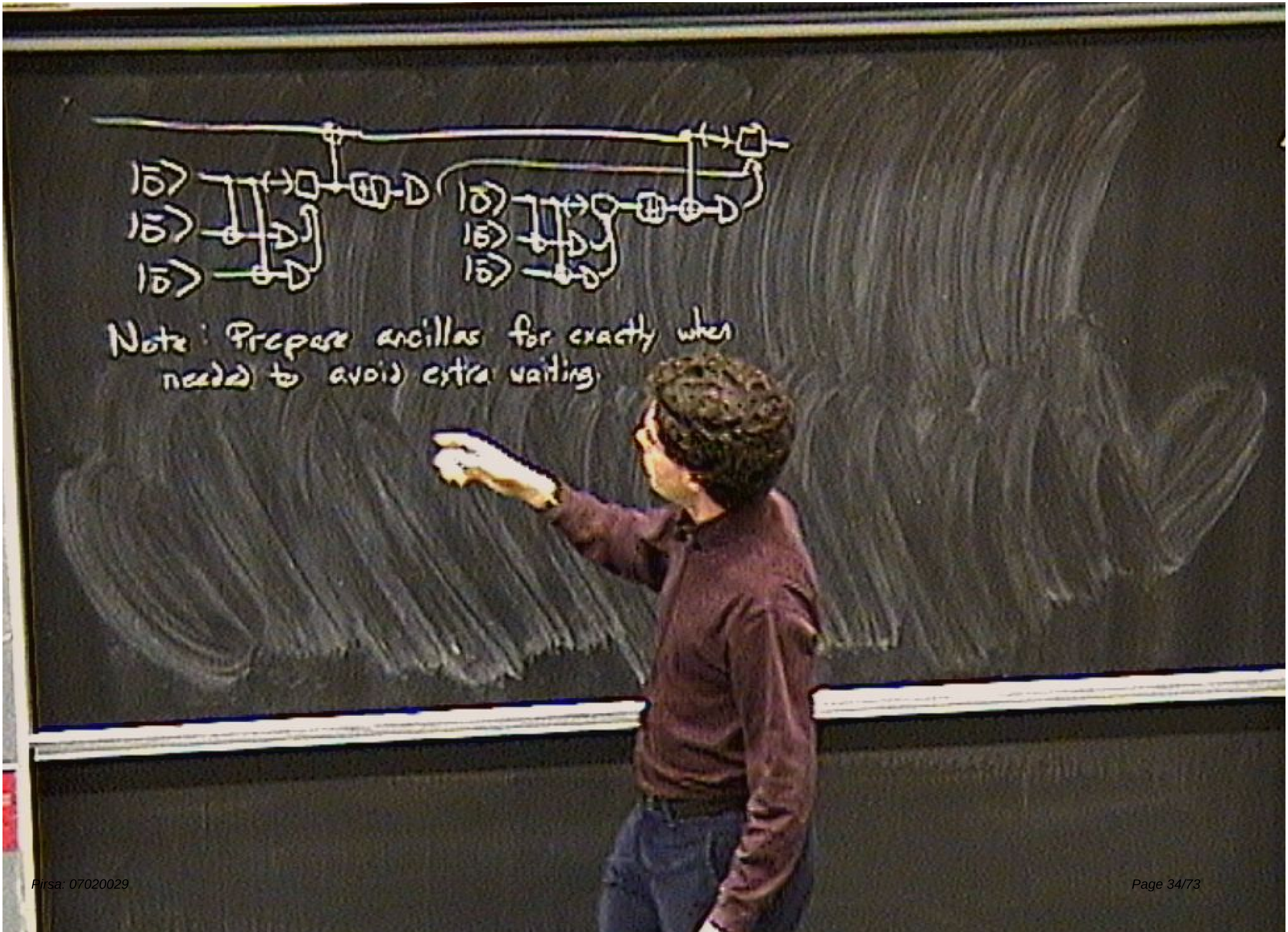
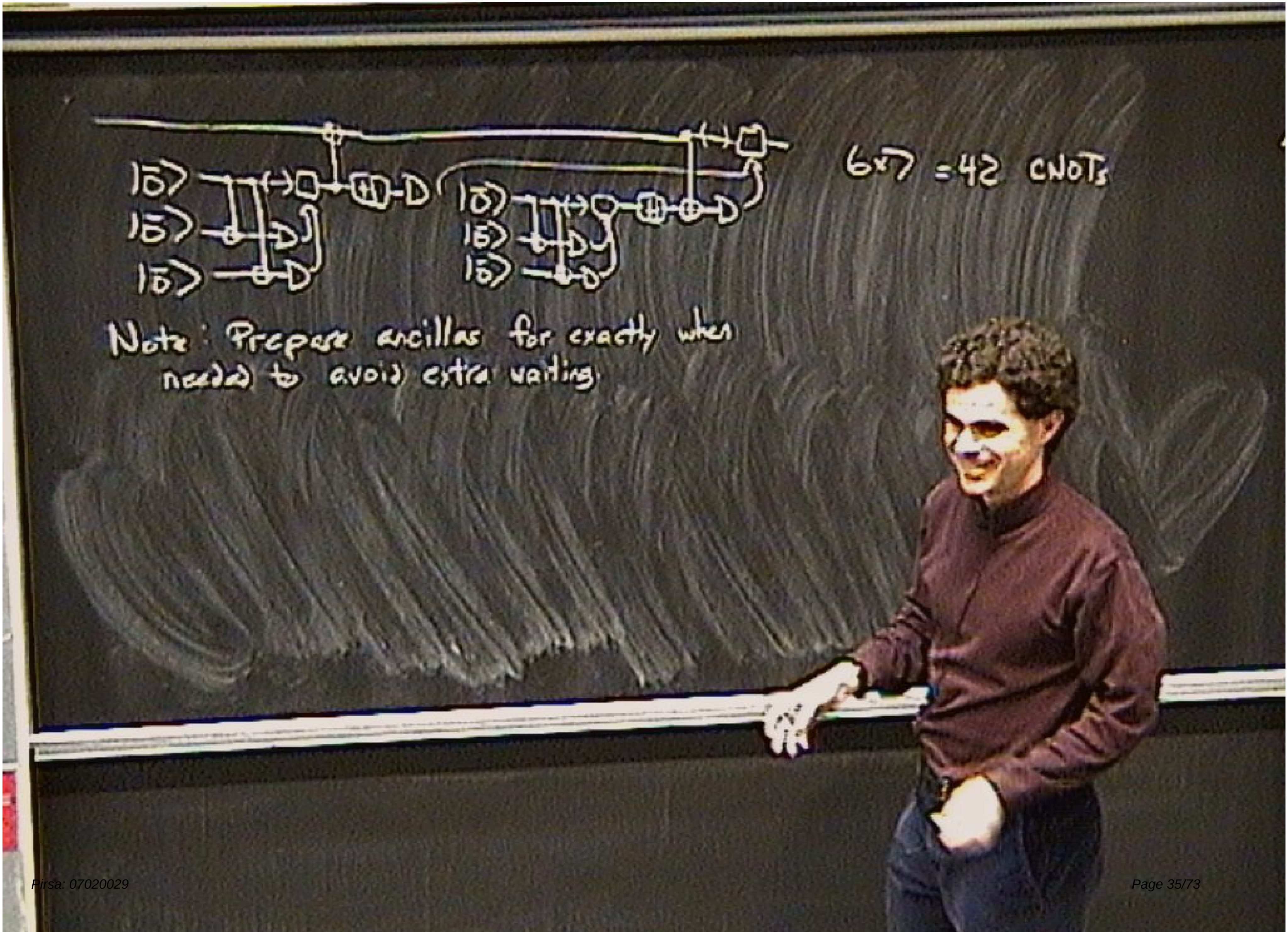

Note: Prepare ancillas for exactly when
needed to avoid extra waiting


|0⟩
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6×7 = 42 CNOTs
Note: Prepare ancillas for exactly when
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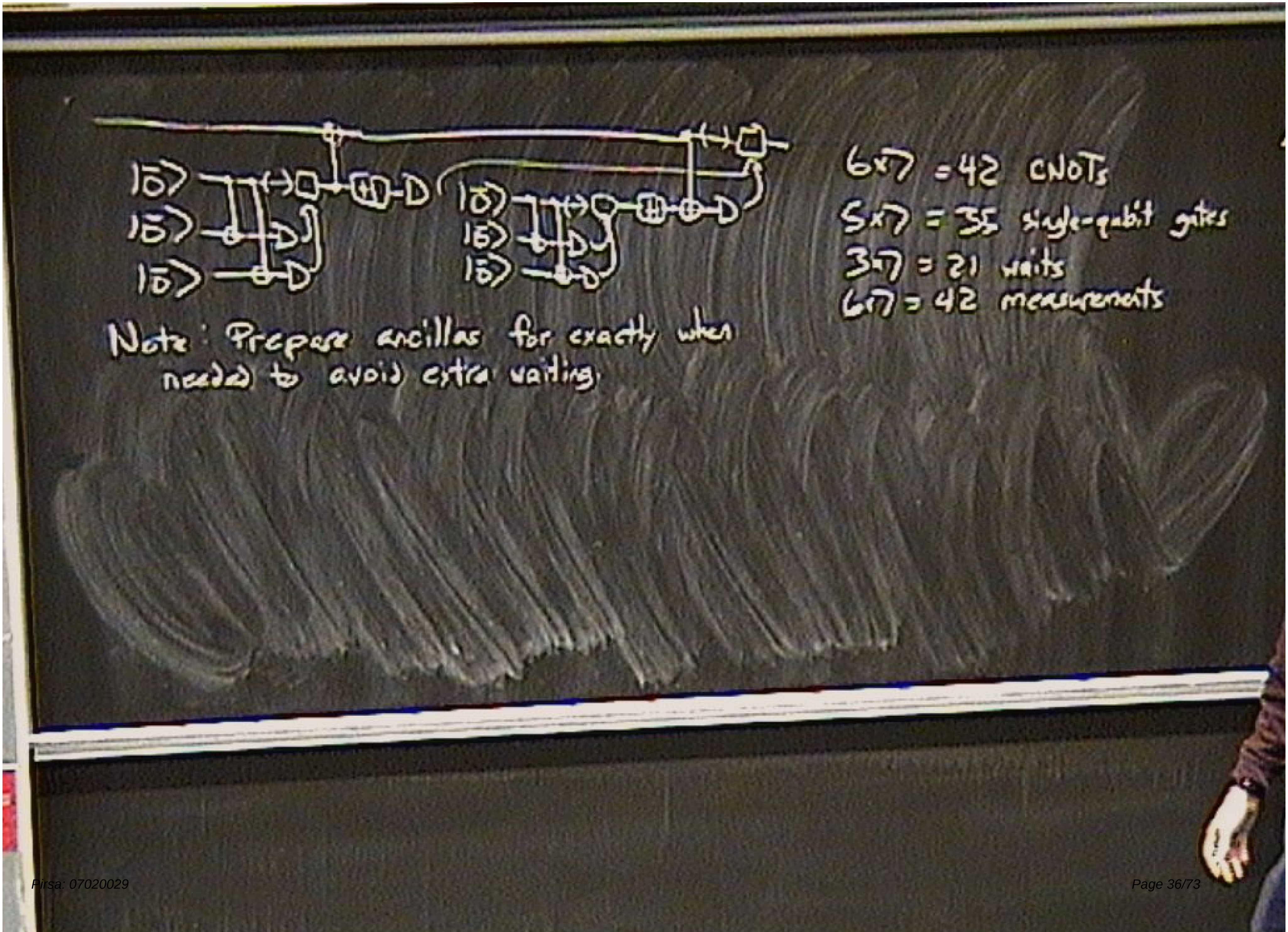
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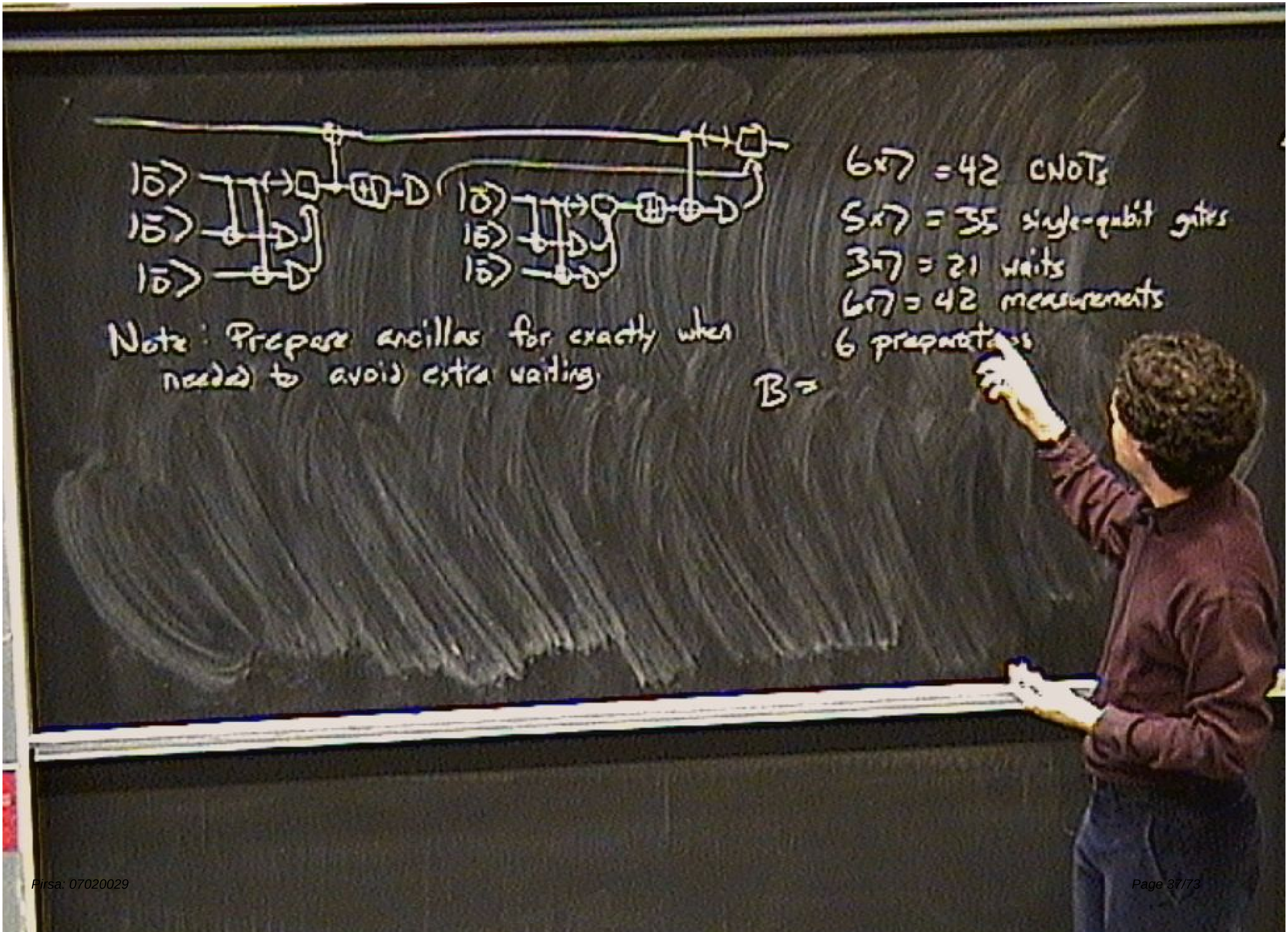
6×7 = 42 CNOTs
5×7 = 35 single-qubit gates
3×7 = 21 waits
6×7 = 42 measurements
Note: Prepare ancillas for exactly when needed to avoid extra waiting.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6 preparat
Note: Prepare ancillas for exactly when
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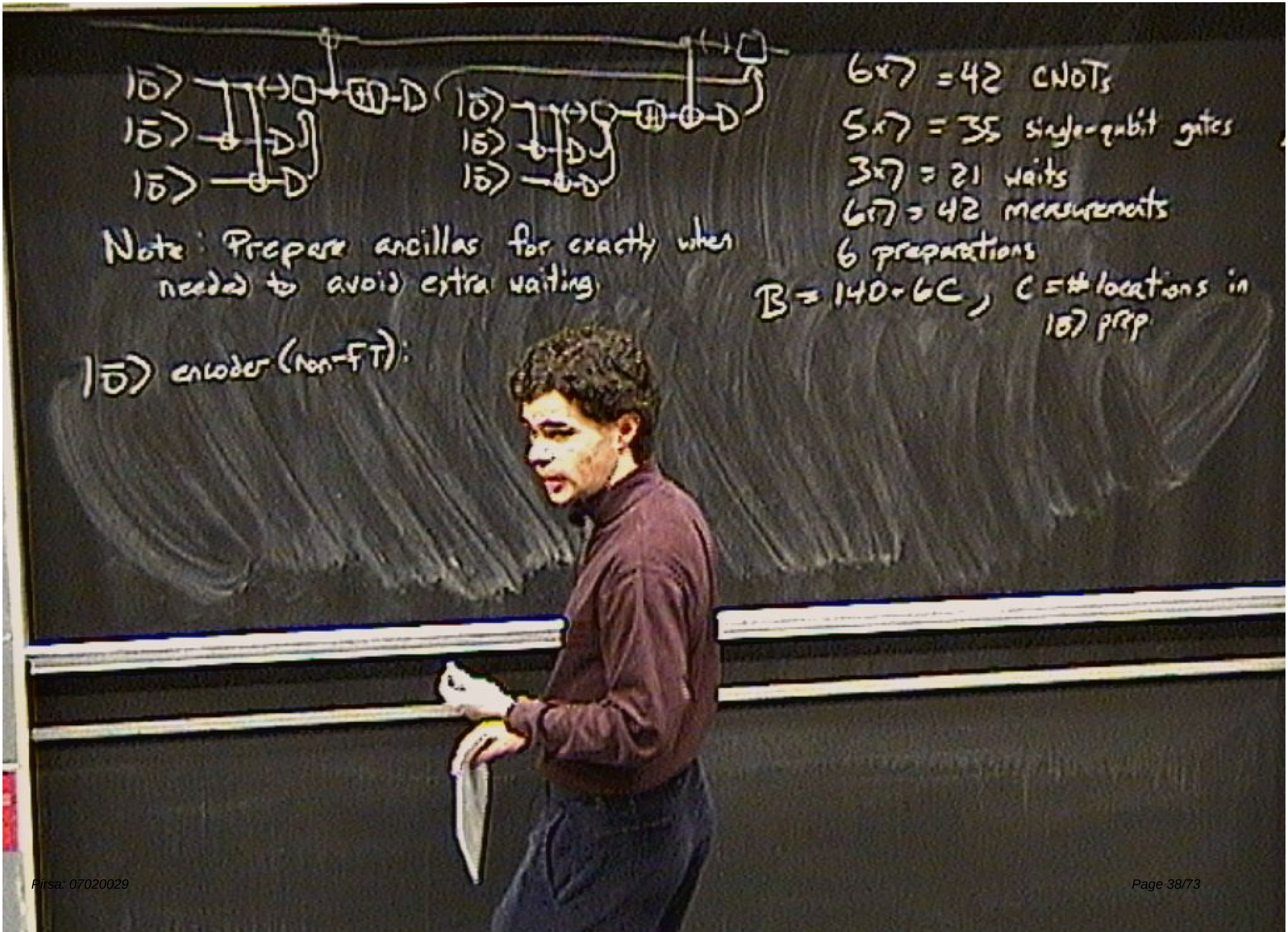
B =


6x7 = 42 CNOTs
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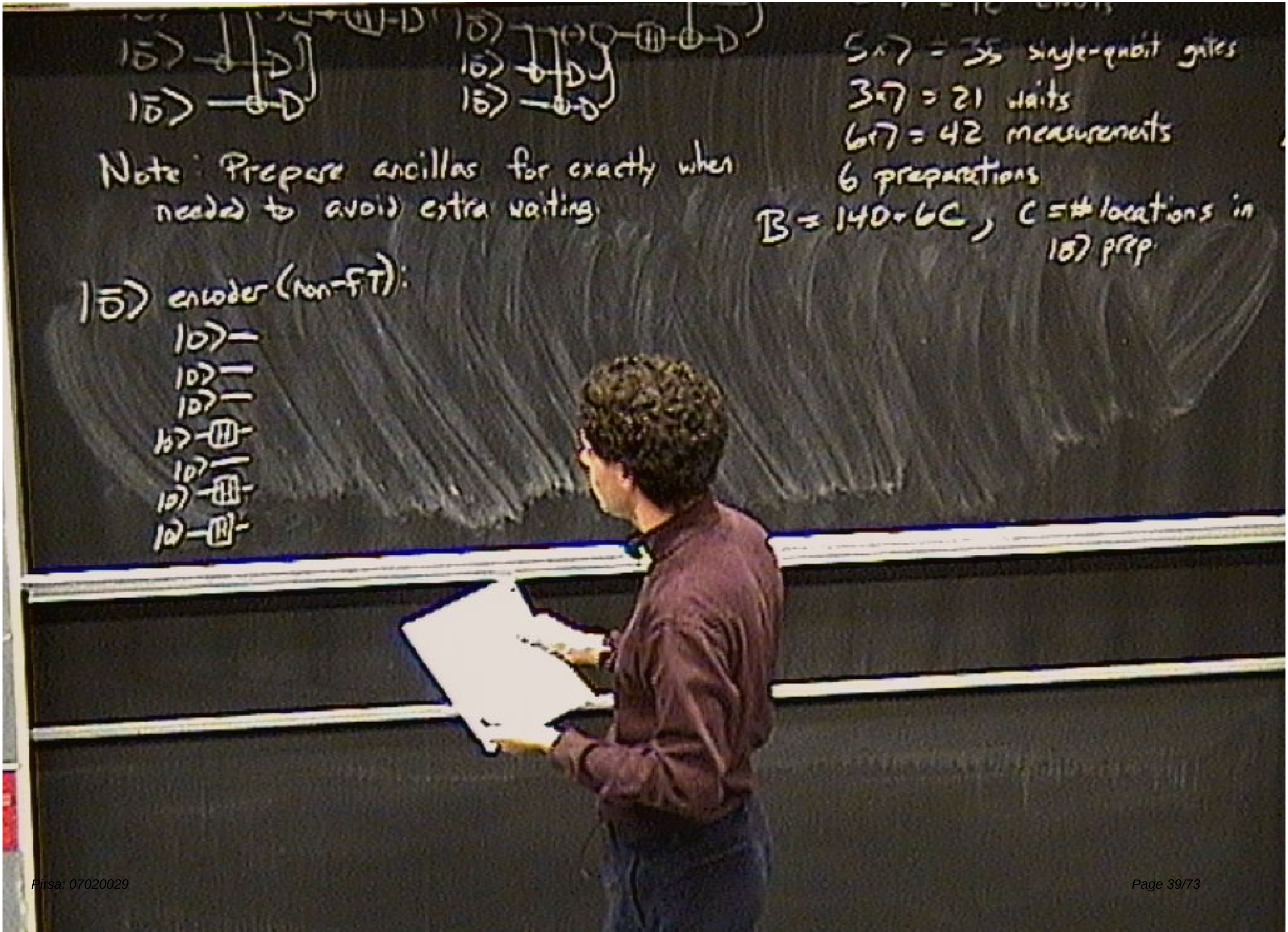
6x7 = 42 measurements
6 preparations
B = 140 + 6C, C = # locations in |0> prep
Note: Prepare ancillas for exactly when needed to avoid extra waiting.
|0> encoder (non-FT):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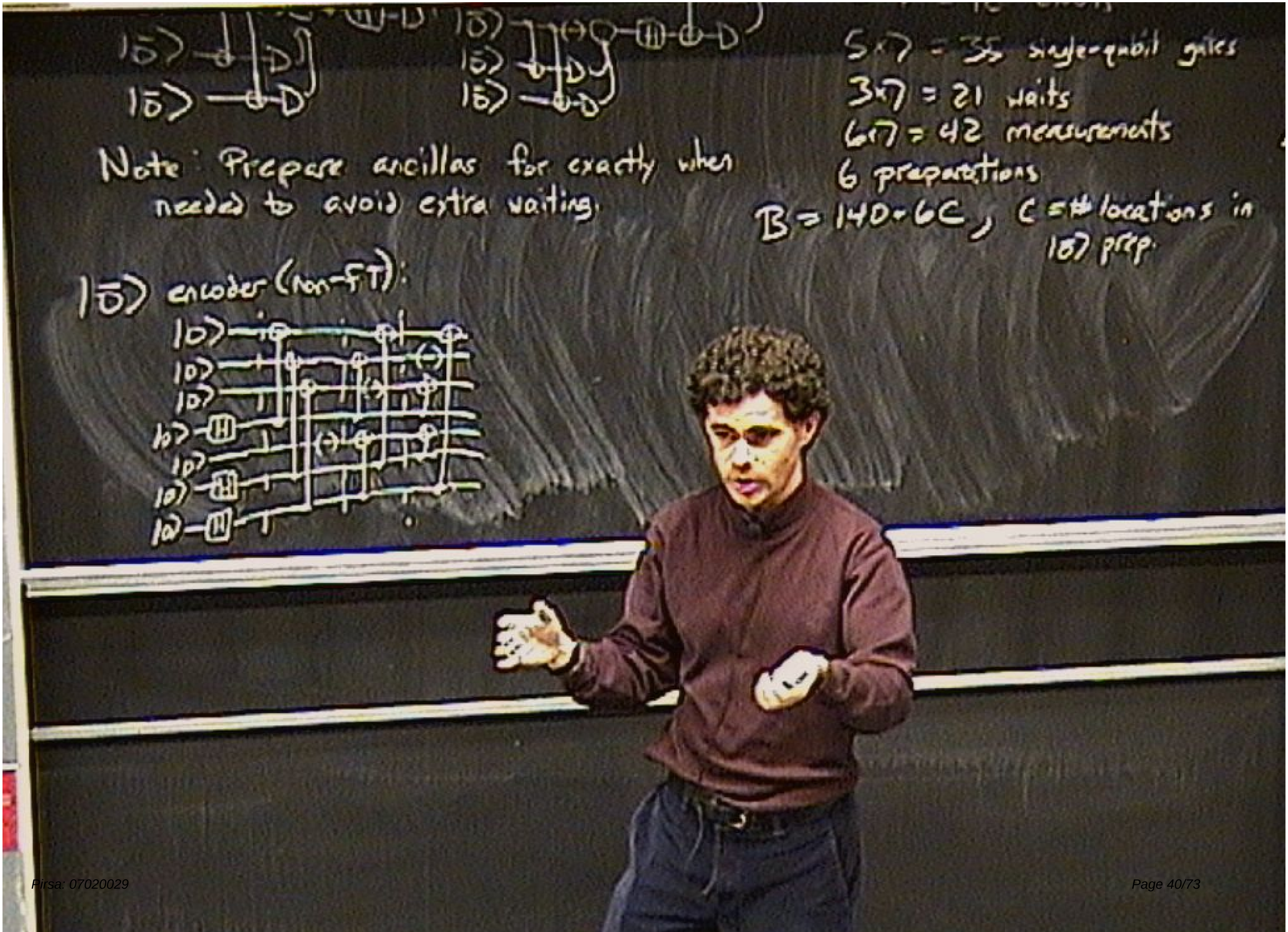

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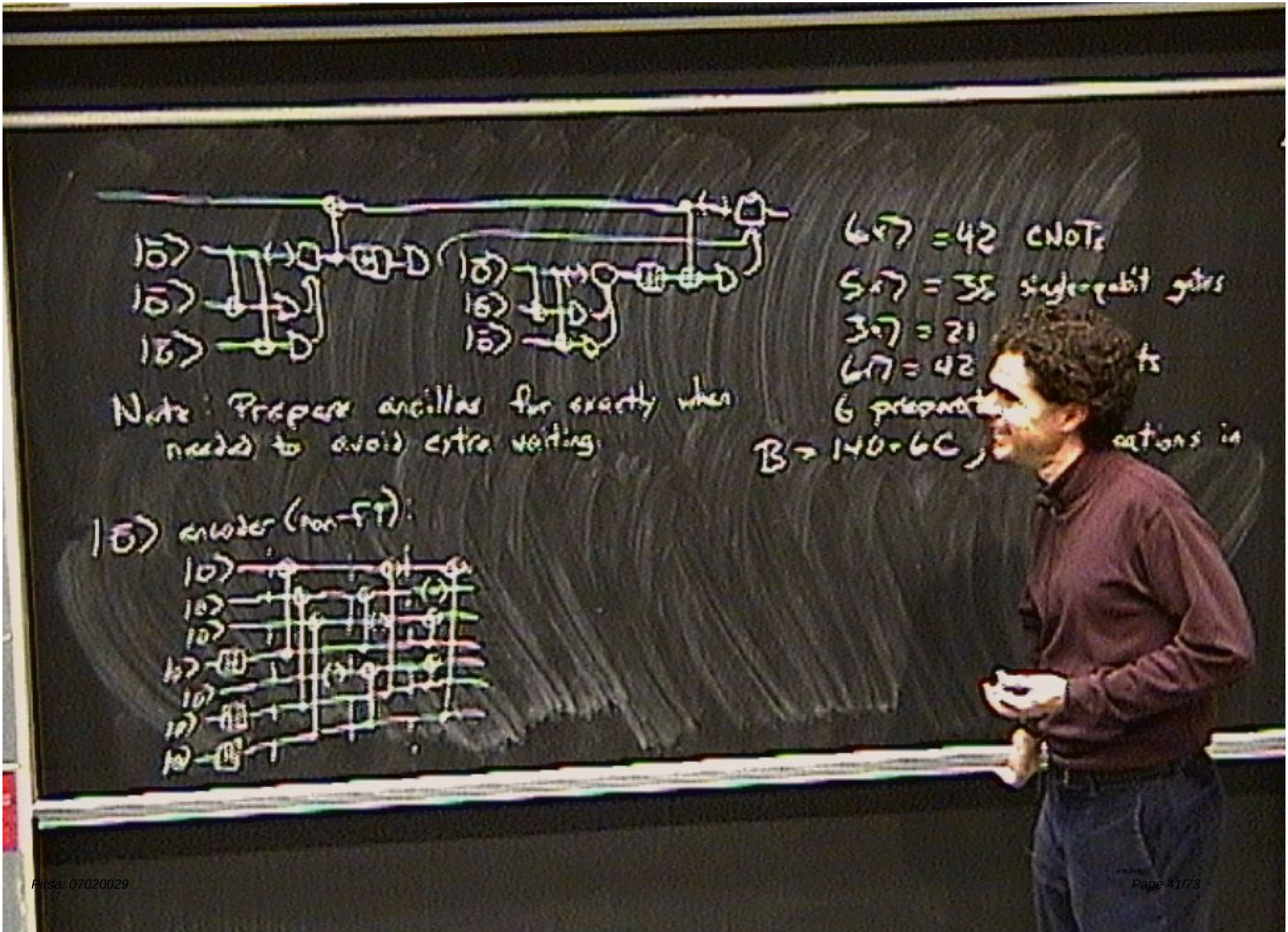
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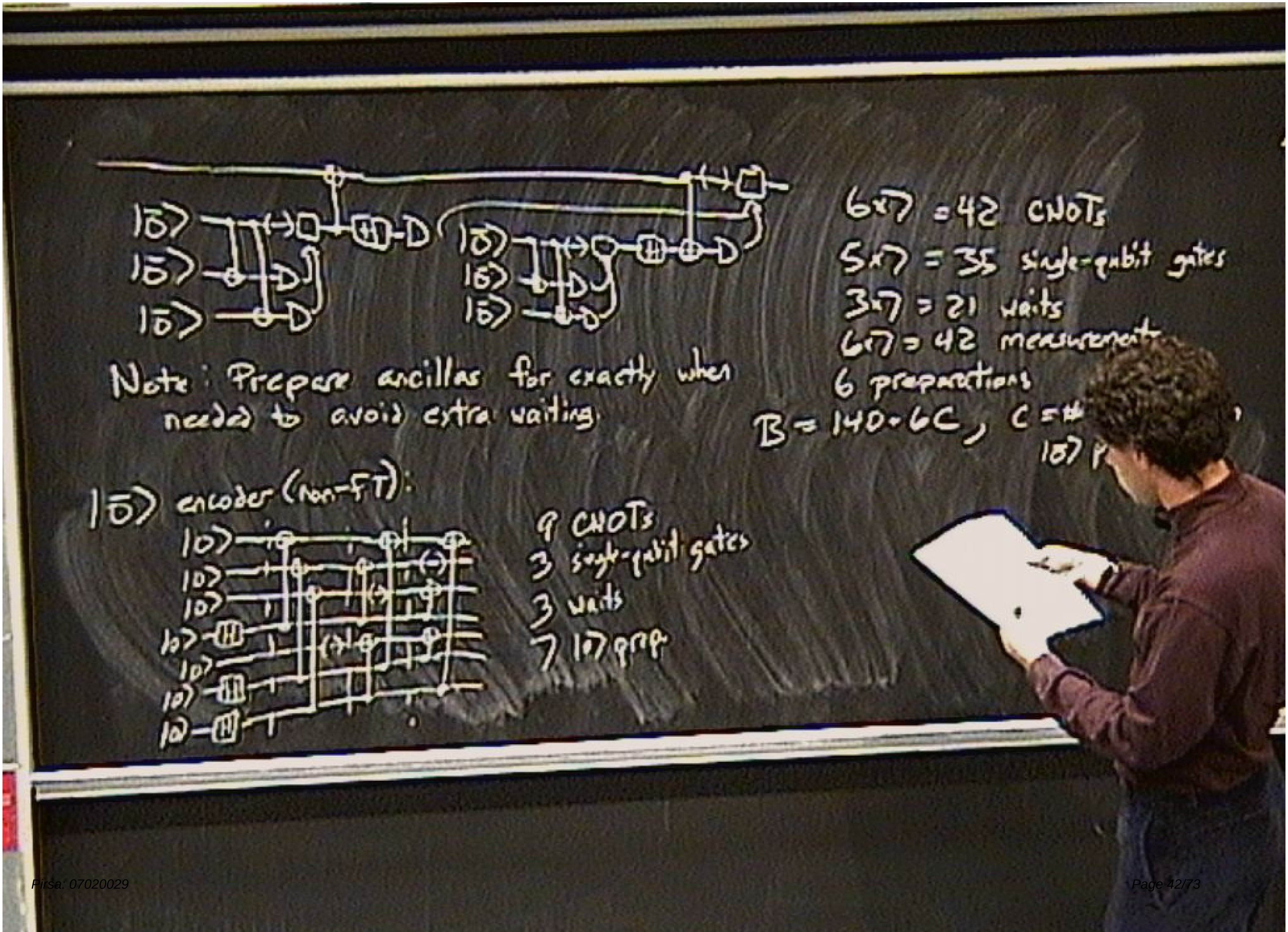
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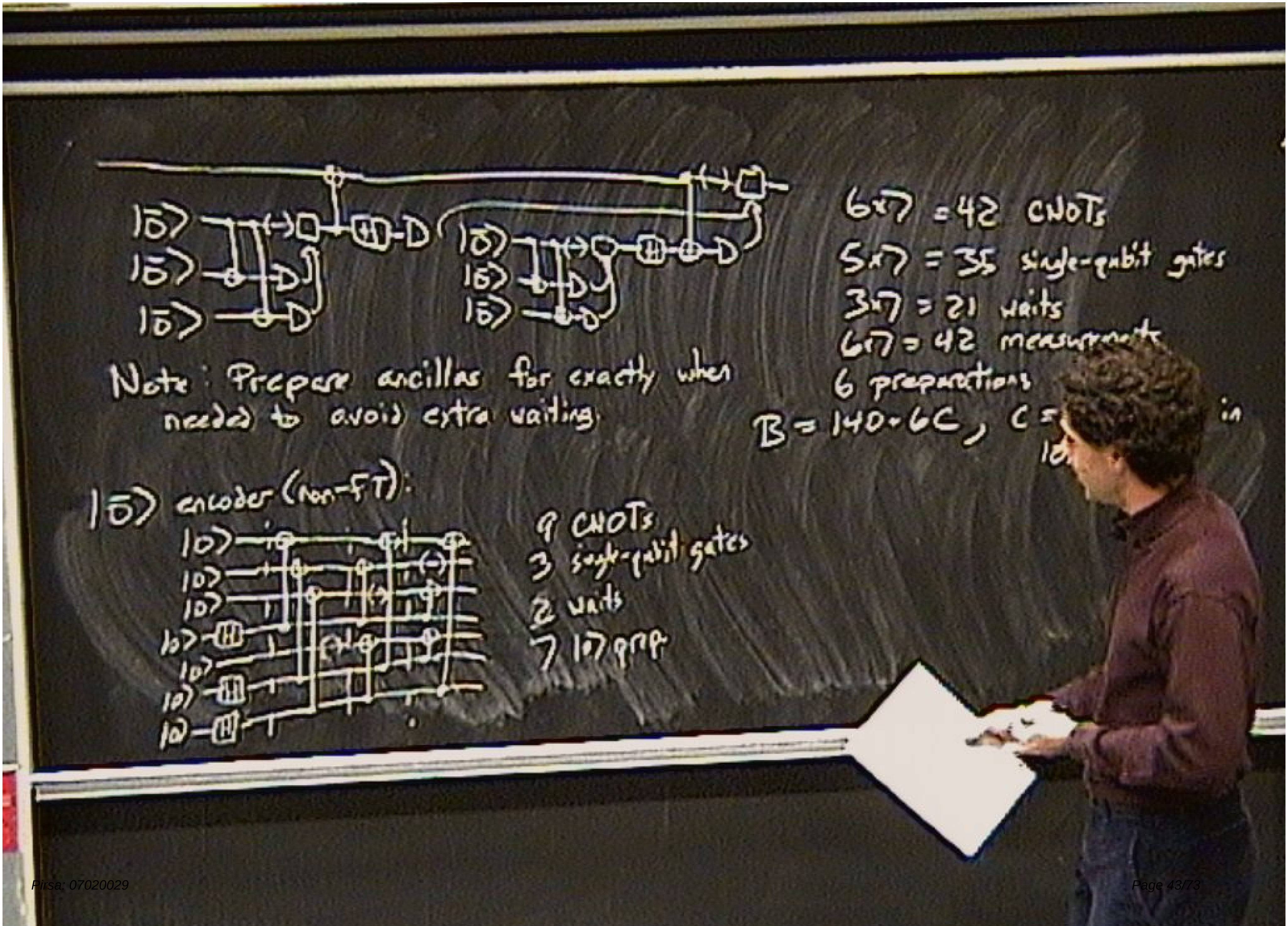
|0̄⟩ encoder (non-FT):
9 CNOTs
3 single-qubit gates
3 waits
7 |0⟩ prep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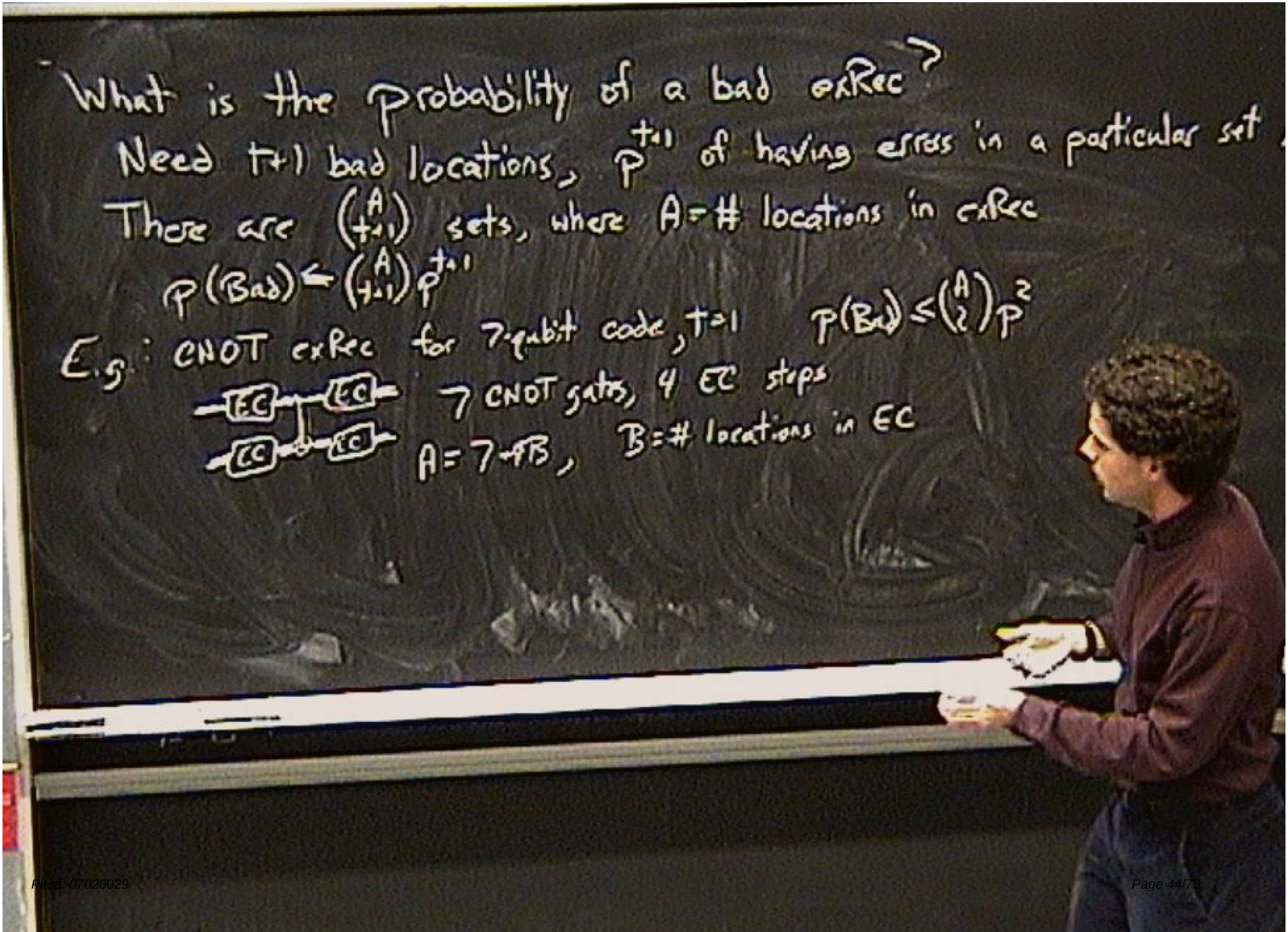
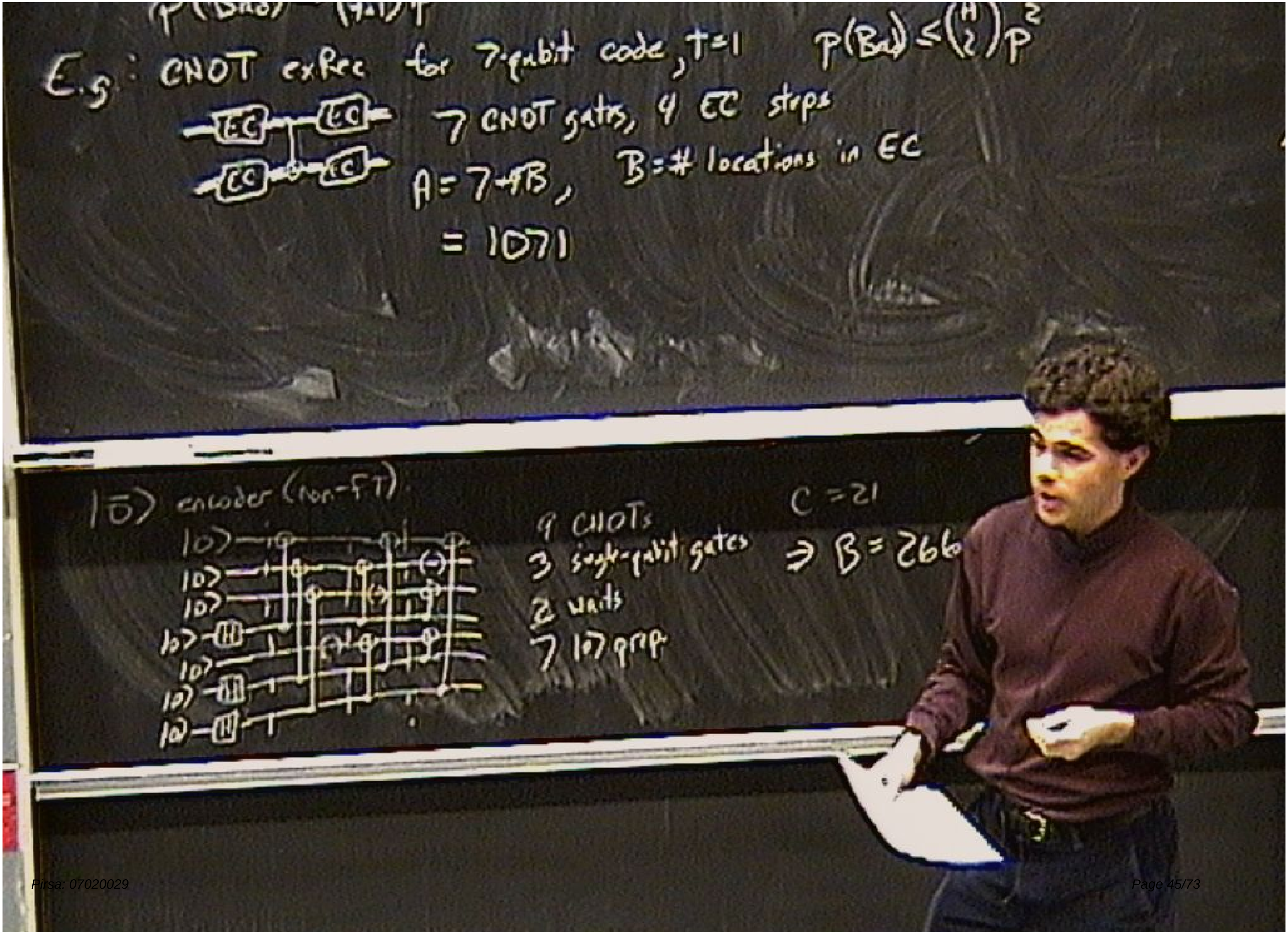
|0̄⟩ encoder (non-FT):
9 CNOTs
3 single-qubit gates
2 waits
7 |0⟩ prep


What is the probability of a bad exRec?
Need t+1 bad locations, p^{t+1} of having errors in a particular set
There are (A choose t+1) sets, where A = # locations in exRec
p(Bad) ≤ (A choose t+1) p^{t+1}
E.g.: CNOT exRec for 7-qubit code, t=1
p(Bad) ≤ (A choose 2) p^2
EC
EC
EC
EC
7 CNOT gates, 4 EC steps
A = 7+4B, B = # locations in EC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EC
7 CNOT gates, 4 EC steps
A = 7+4B, B = # locations in EC
= 1071
|0̄⟩ encoder (non-FT)
|0⟩
H
9 CNOTs
3 single-qubit gates
2 waits
7 |0⟩ prep
C = 21
⇒ B = 266

What is the probability of a bad exRec?

Need $t+1$ bad locations, $p^{t+1}$ of having errors in a particular set

There are $\binom{A}{t+1}$ sets, where $A = \#$ locations in exRec

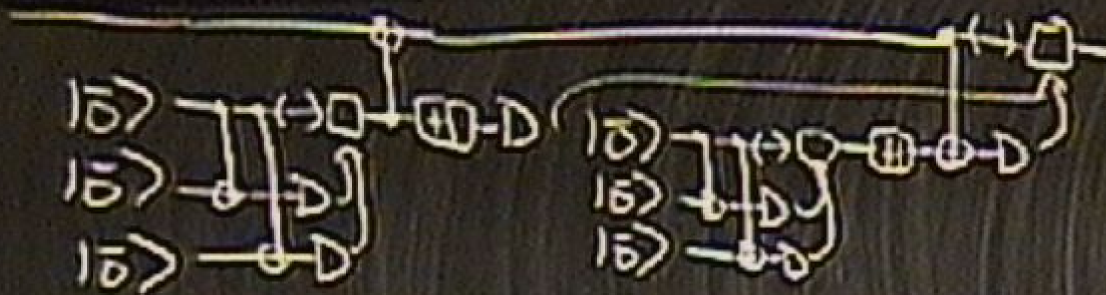
$p(\text{Bad}) \leq \binom{A}{t+1} p^{t+1}$

E.g.: CNOT exRec for 7-qubit code, $t=1$ $\quad p(\text{Bad}) \leq \binom{A}{2} p^2 =$


7 CNOT gates, 4 EC steps

$A = 7 + 4B$, $\quad B = \#$ locations in EC

$= 1071$




$6\times7 = 42$ CNOTs

$5\times7 = 35$ single-qubit gates

$3\times7 = 21$ waits

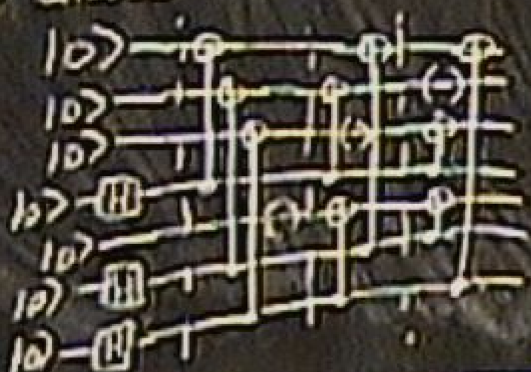
$6\times7 = 42$ measurements

6 preparations

$B = 140 + 6C$, $C$ = # locations in $|0\rangle$ prep

Note: Prepare ancillas for exactly when needed to avoid extra waiting.

$|\bar{0}\rangle$ encoder (non-FT):




9 CNOTs

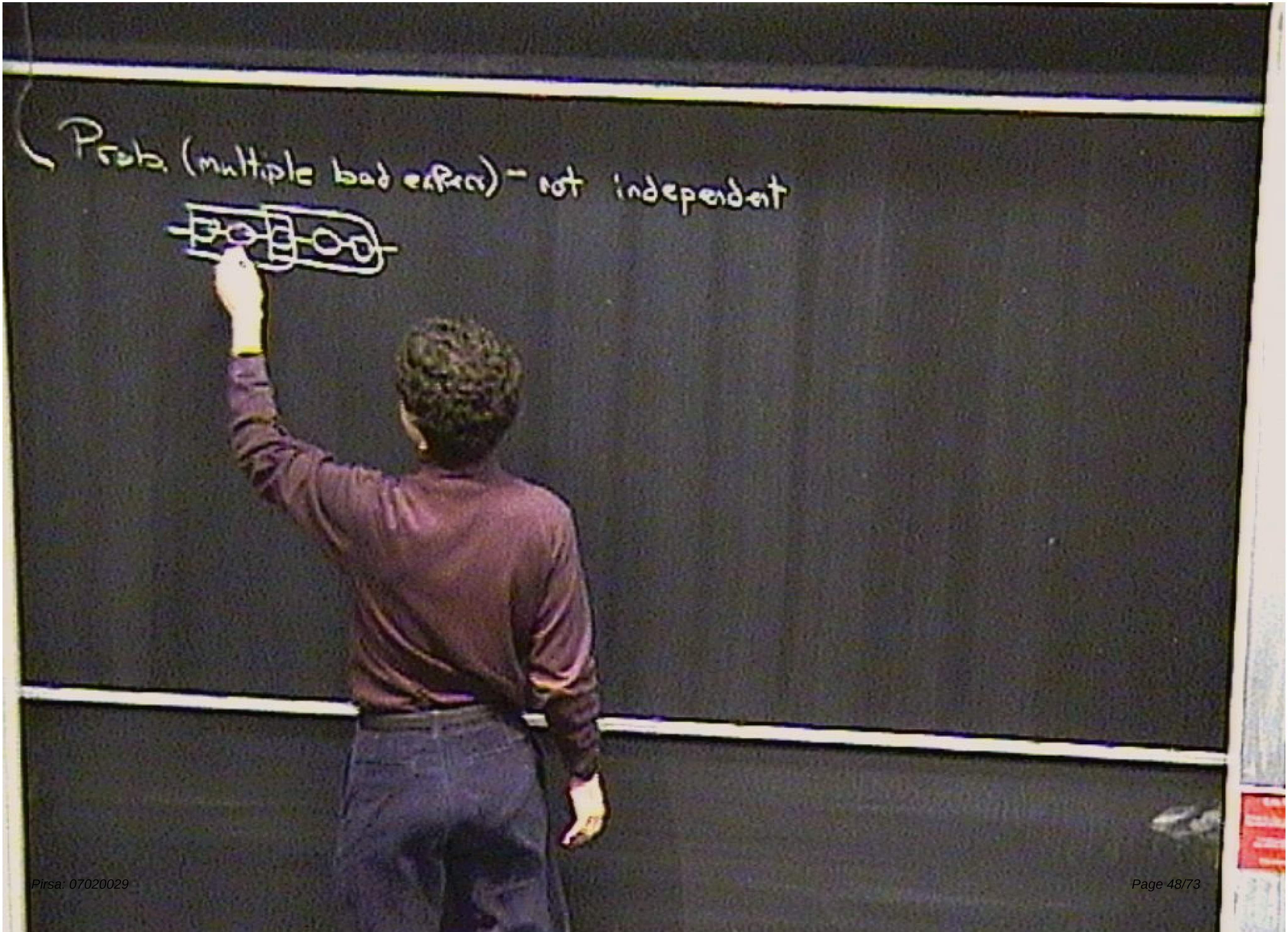
3 single-qubit gates

2 waits

7 $|0\rangle$ prep

$C = 21$

$\Rightarrow B = 266$


Prob. (multiple bad effects) – not independent

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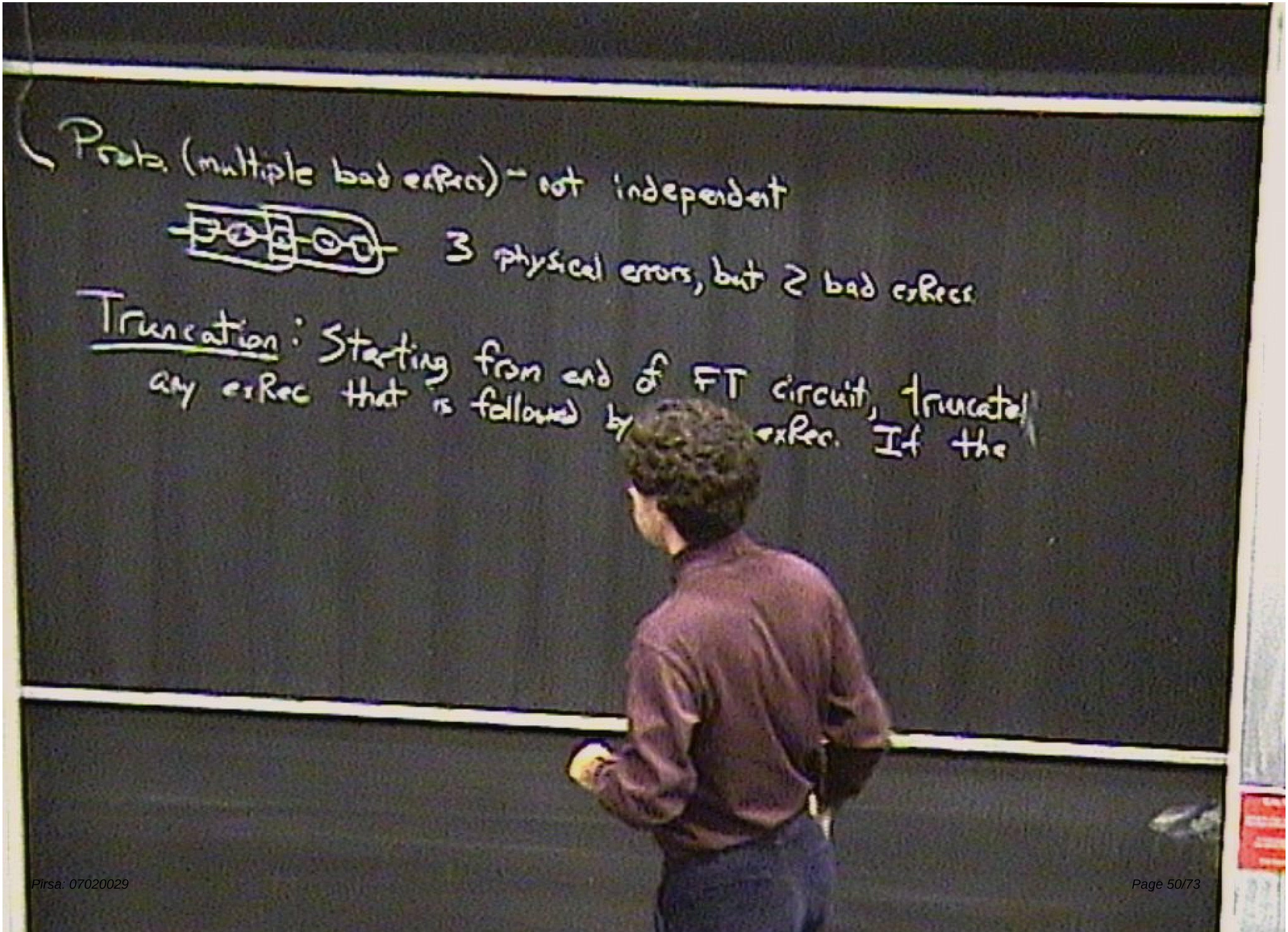
3 physical errors, but 2 bad effects


Prob. (multiple bad exRecs) ~ not independent
3 physical errors, but 2 bad exRecs
Truncation: Starting from end of FT circuit, truncate
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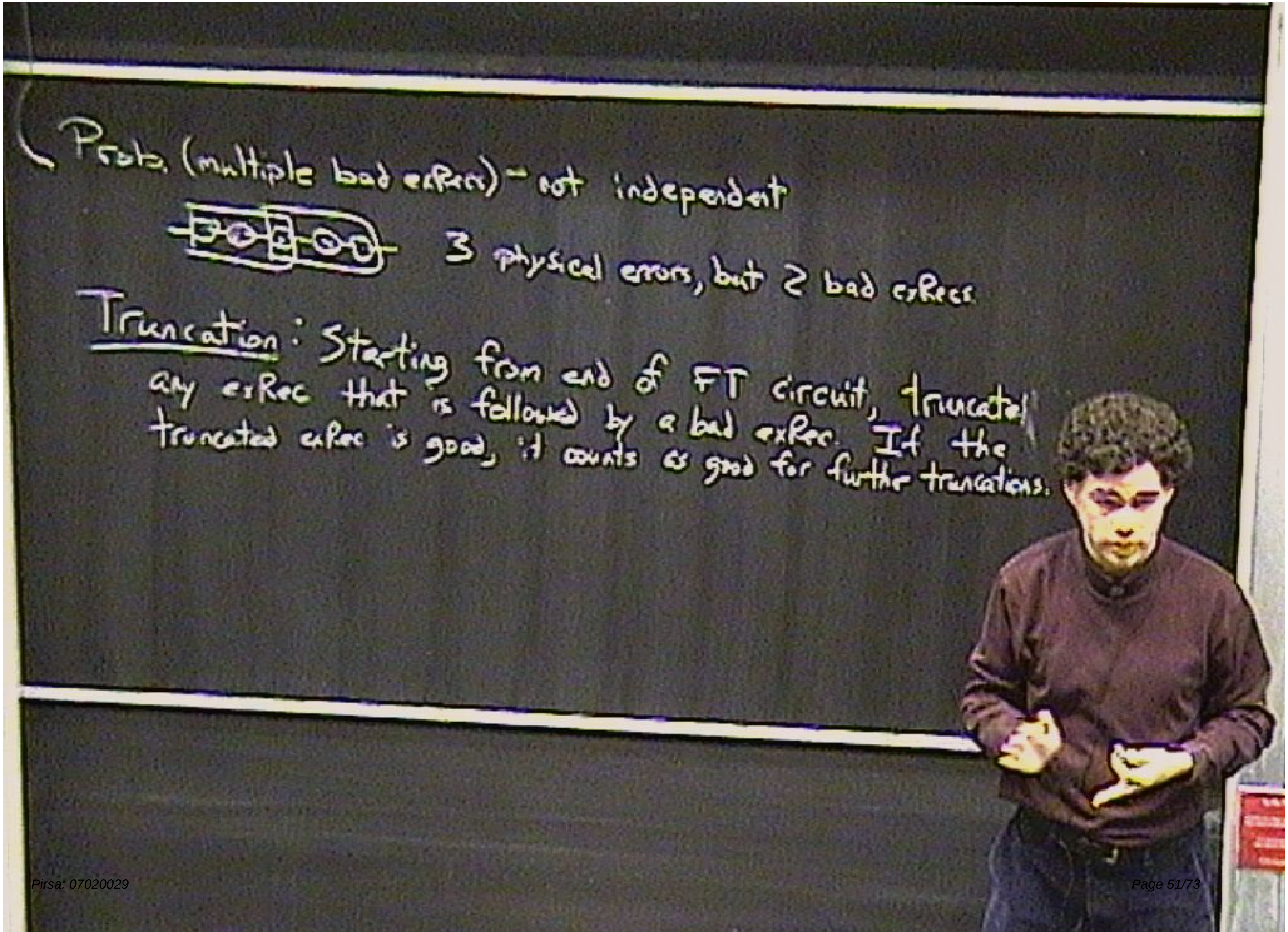
Prob. (multiple bad exRecs) — not independent
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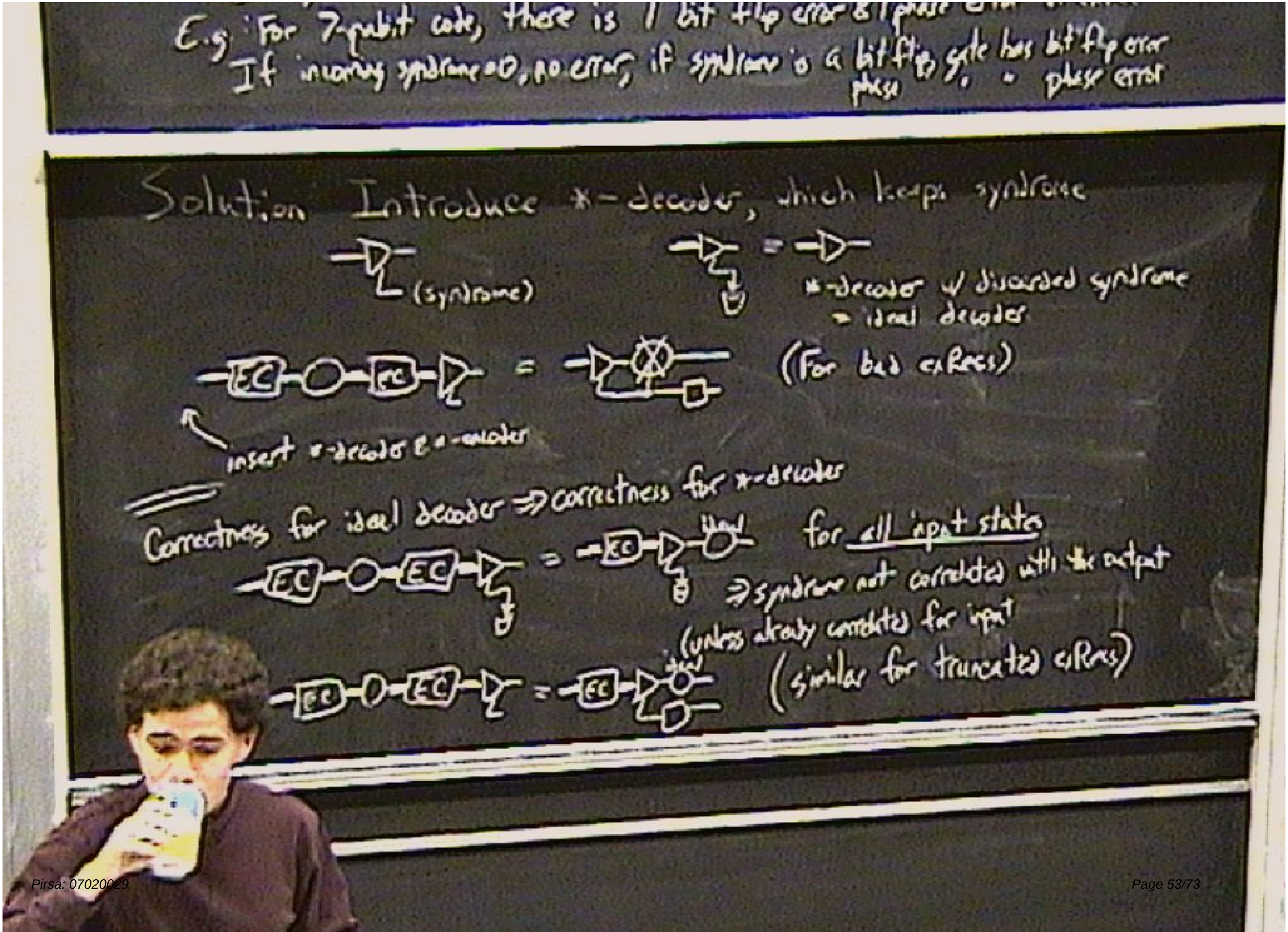
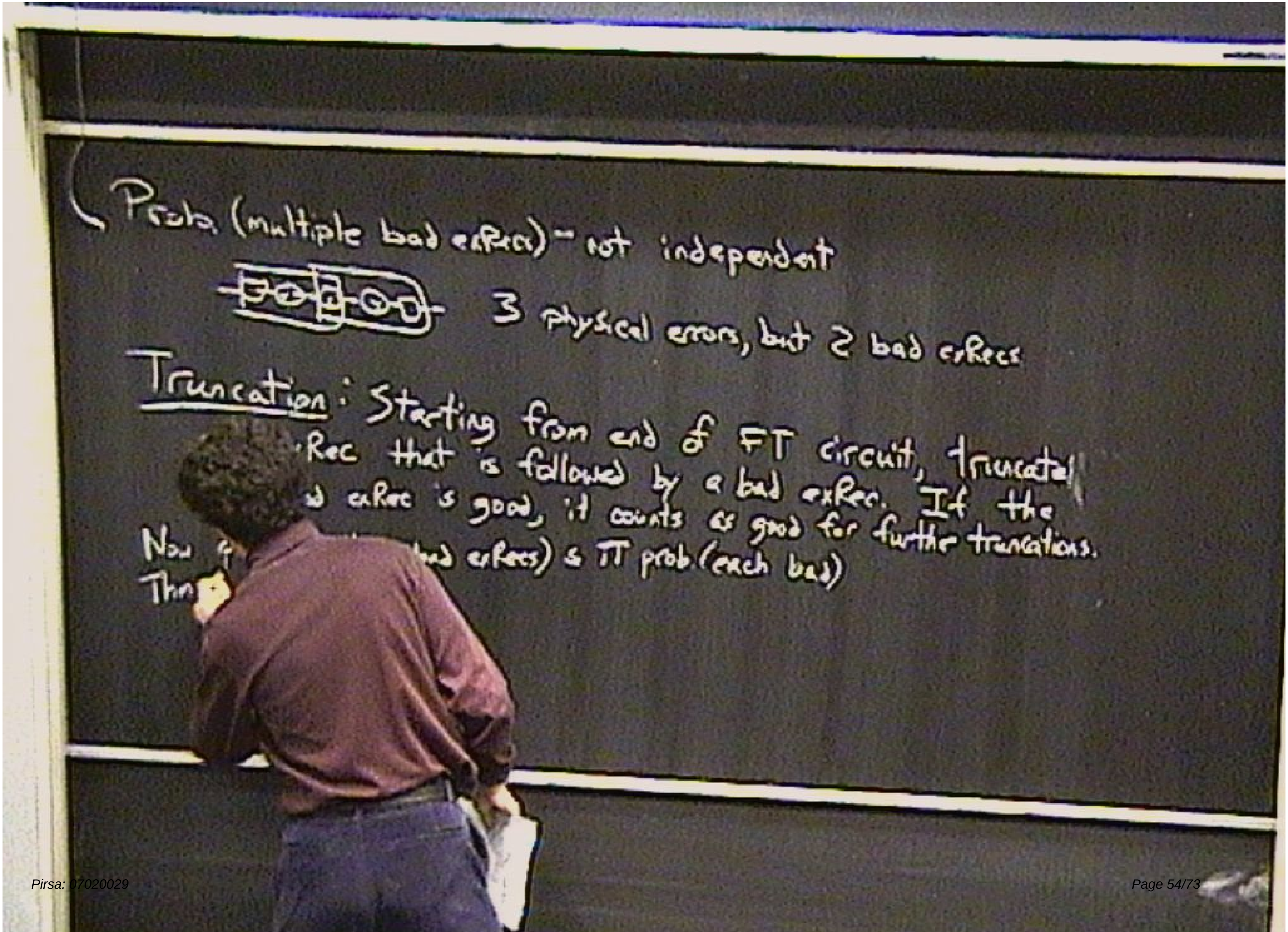

E.g. For 7-qubit code, there is 1 bit flip error
If incoming syndrome = 0, no error; if syndrome is a bit flip, gate has bit flip error
phase phase error
Solution Introduce *-decoder, which keeps syndrome
(syndrome)
*-decoder w/ discarded syndrome
= ideal decoder
EC
EC
(For bad exRecs)
insert *-decoder & *-encoder
Correctness for ideal decoder ⇒ correctness for *-decoder
ideal
for all input states
⇒ syndrome not correlated with the output
(unless already correlated for input)
(similar for truncated exRecs)


Prob (multiple bad exRecs) — not independent
3 physical errors, but 2 bad exRecs
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Now
bad exRecs) ≤ ∏ prob (each bad)
Thm

Prob. (multiple bad exRecs) – not independent

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Truncation: Starting from end of FT circuit, truncate any exRec that is followed by a bad exRec. If the truncated exRec is good, it counts as good for further truncations.

Now prob (multiple bad exRecs) $\leq \prod$ prob (each bad)

Thm (Level reduction): A FT circuit can be replaced by a [illegible] unencoded circuit, with errors bounded by prob $p' \leq \binom{A}{t+1} p^{t+1}$

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**Thm (Level reduction)**: A FT circuit can be replaced by a noisy unencoded circuit, with errors bounded by prob $p' \le \binom{A}{t+1} p^{t+1}$, where $A$ is the number of locations in the largest exRec.

Prob (multiple bad exRecs) – not independent

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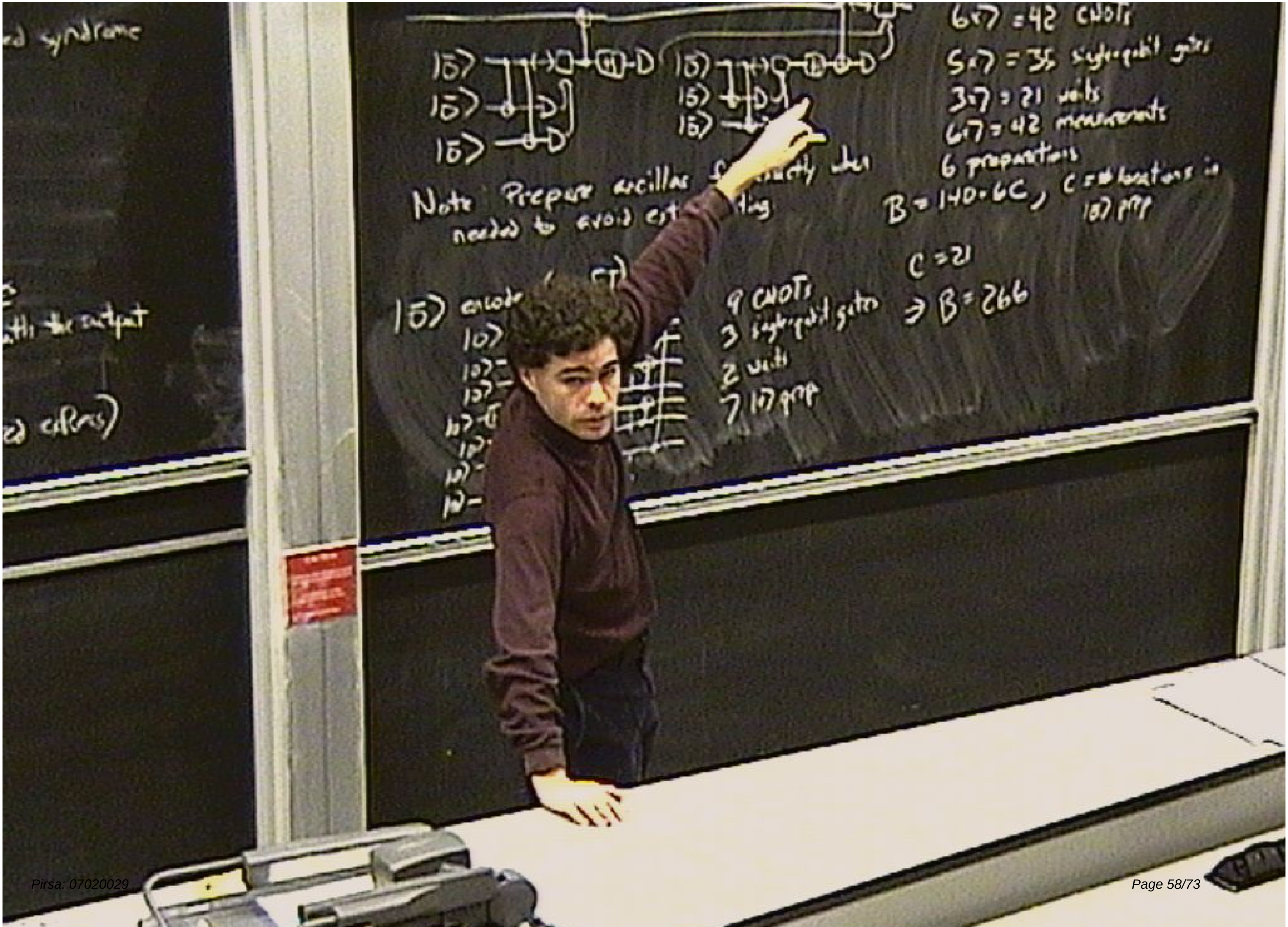
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⇒ B = 266

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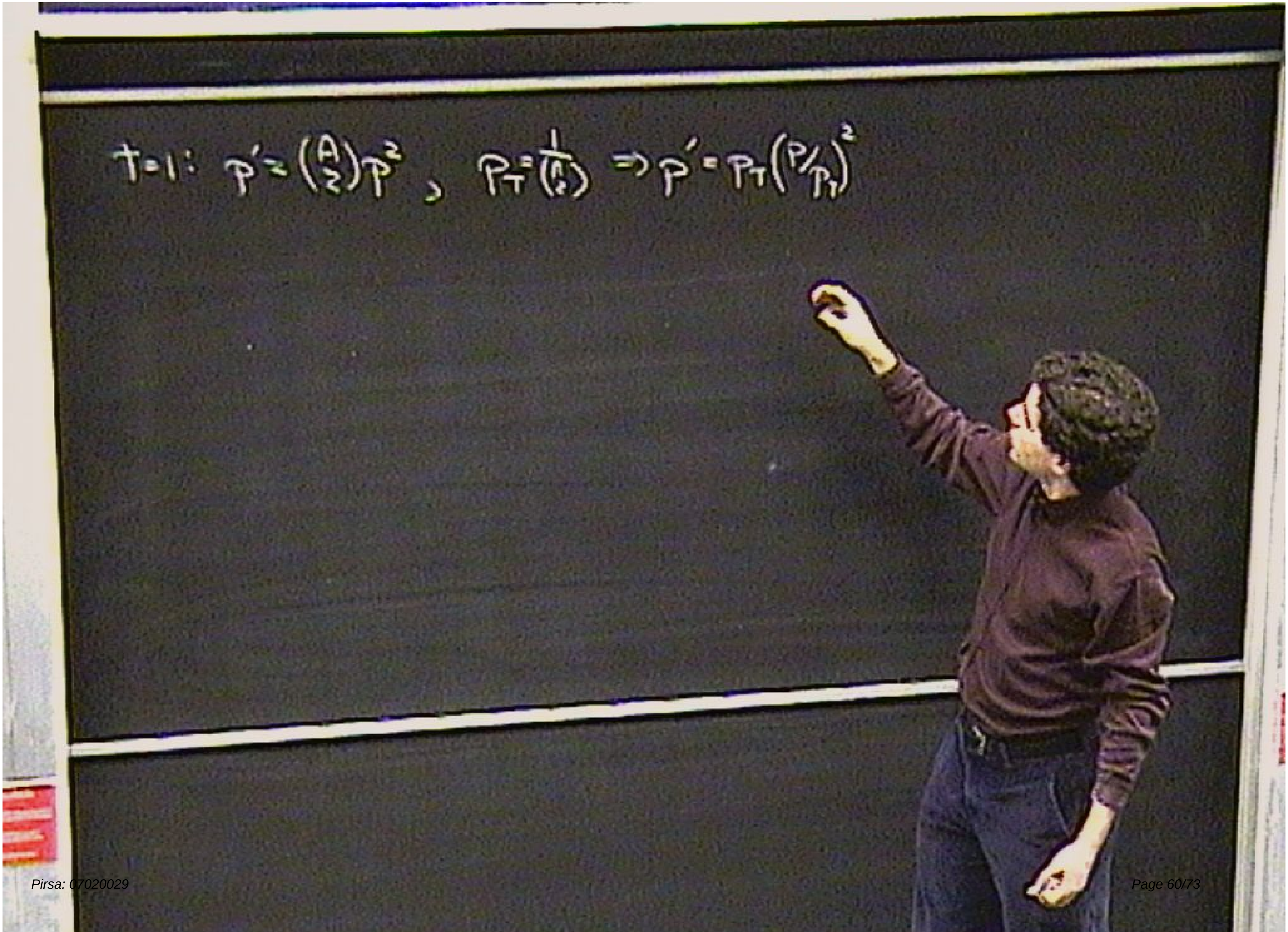
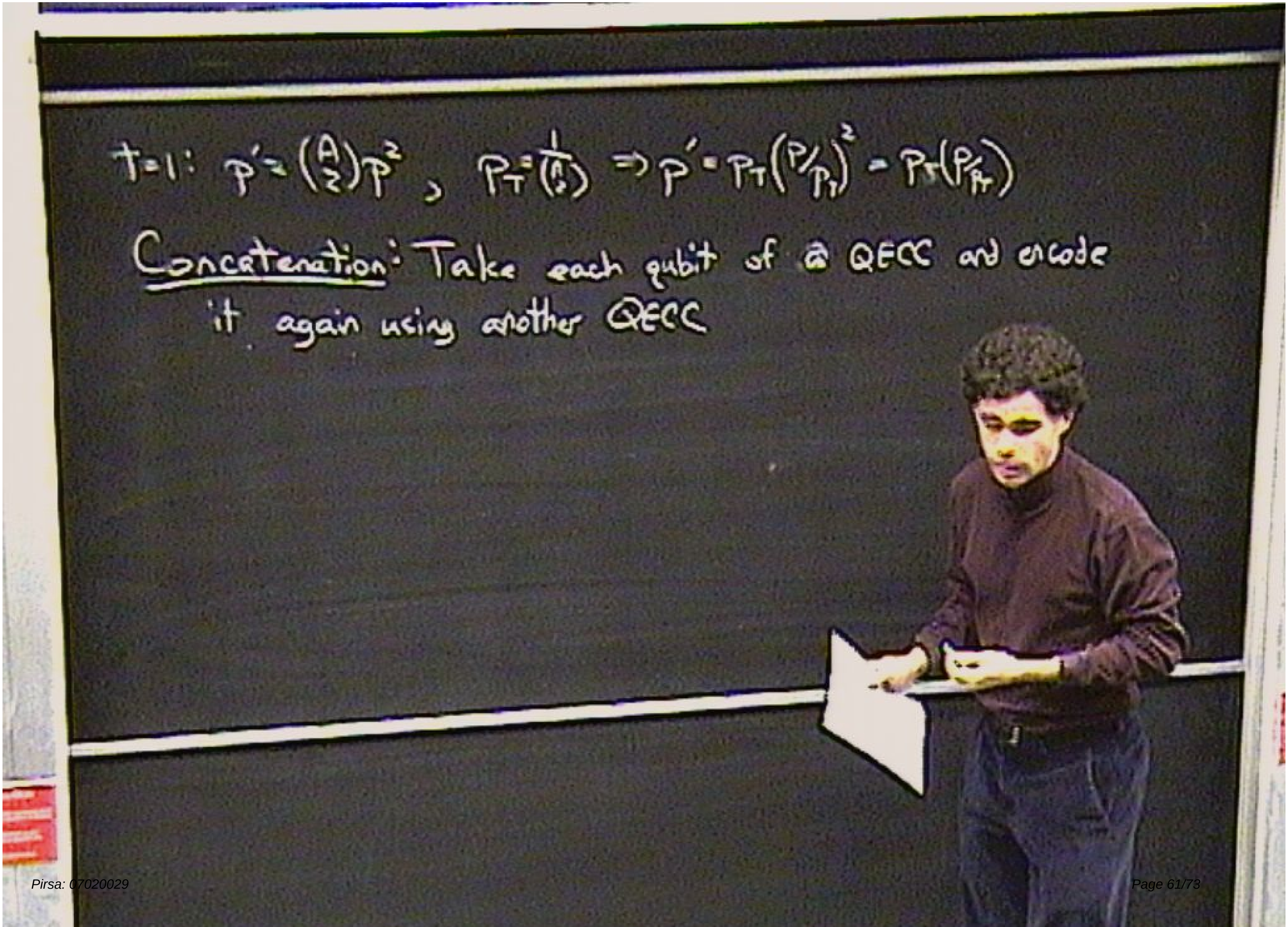

$t=1: \; p' = \left(\frac{A}{2}\right) p^2 \, , \quad p_T = \frac{1}{\left(\frac{A}{2}\right)} \Rightarrow p' = p_T \left(p/p_T\right)^2$


t=1: p' = (A 2) p^2 , p_T = 1/(A 2) ⇒ p' = p_T(p/p_T)^2 = p_T(p/p_T)
Concatenation: Take each qubit of a QECC and encode it again using another QECC

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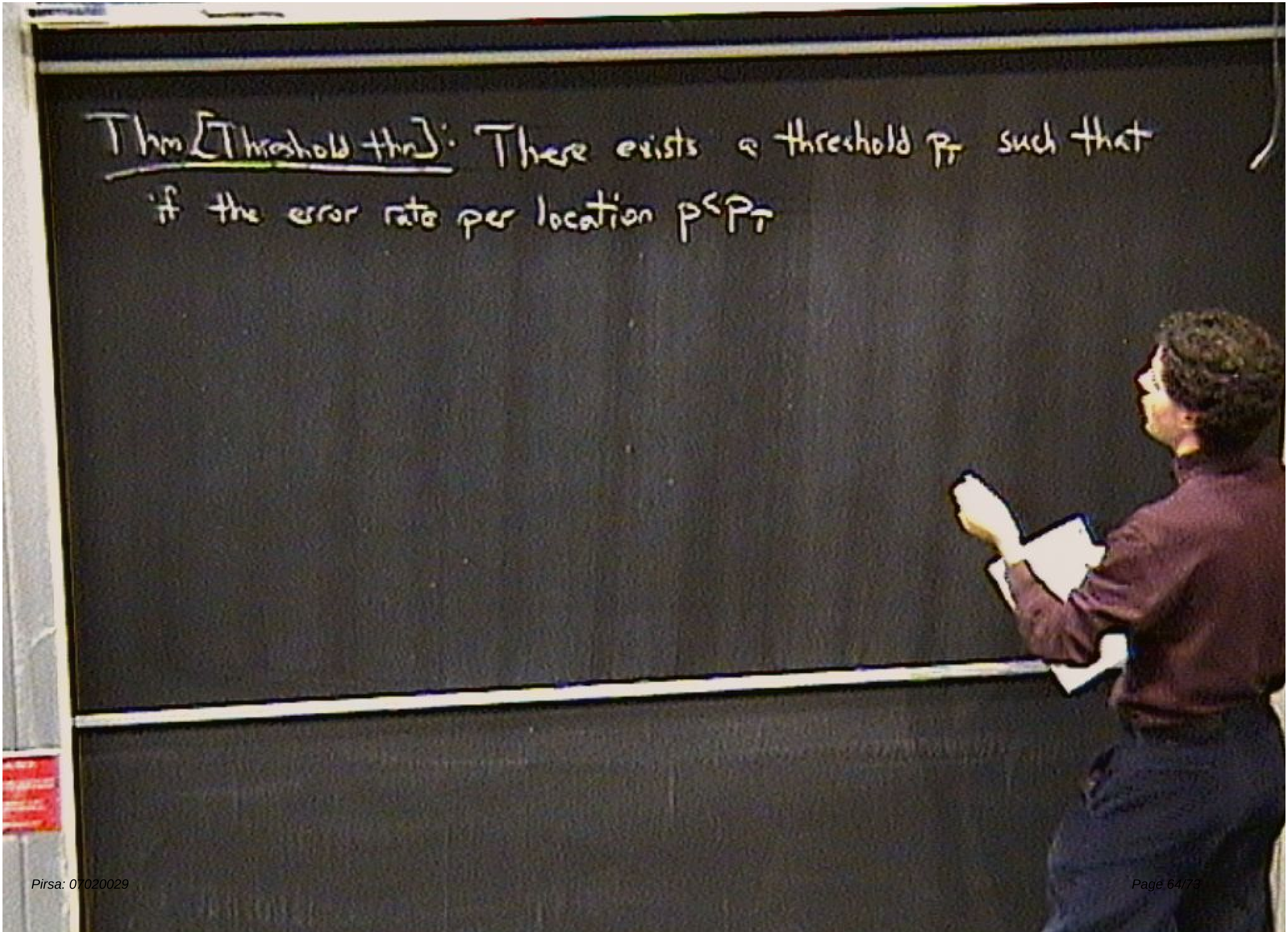
<u>Concatenation</u>: Take each qubit of a QECC and encode it again using another QECC. Take a FT circuit, treat it as a circuit of physical gates – make it FT again.

Each time we do this, it adds one <u>level</u> of concatenation.

Level 0 = physical qubits, top level = logical qubits.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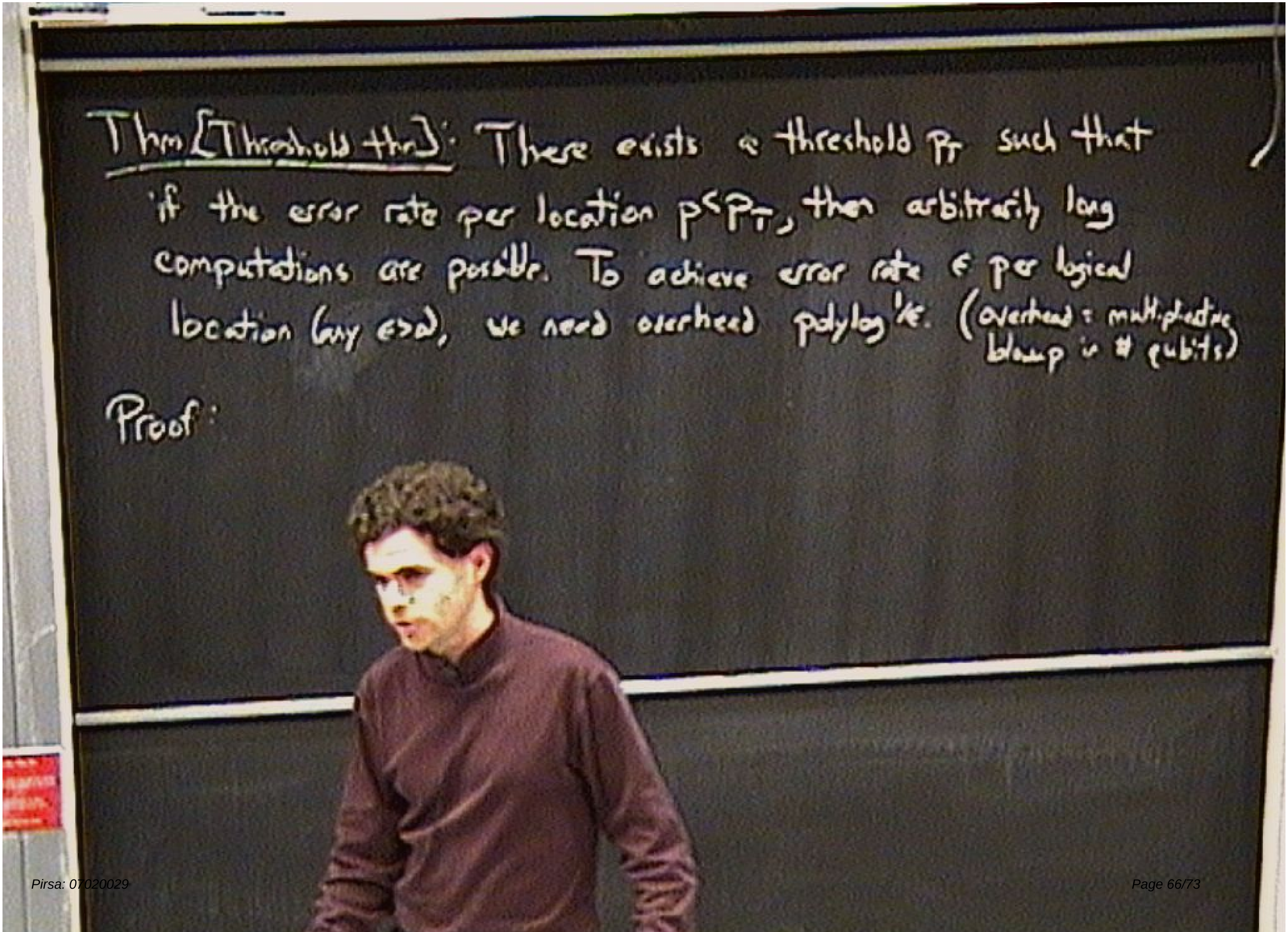
Thm [Threshold thm]: There exists a threshold p_T such that
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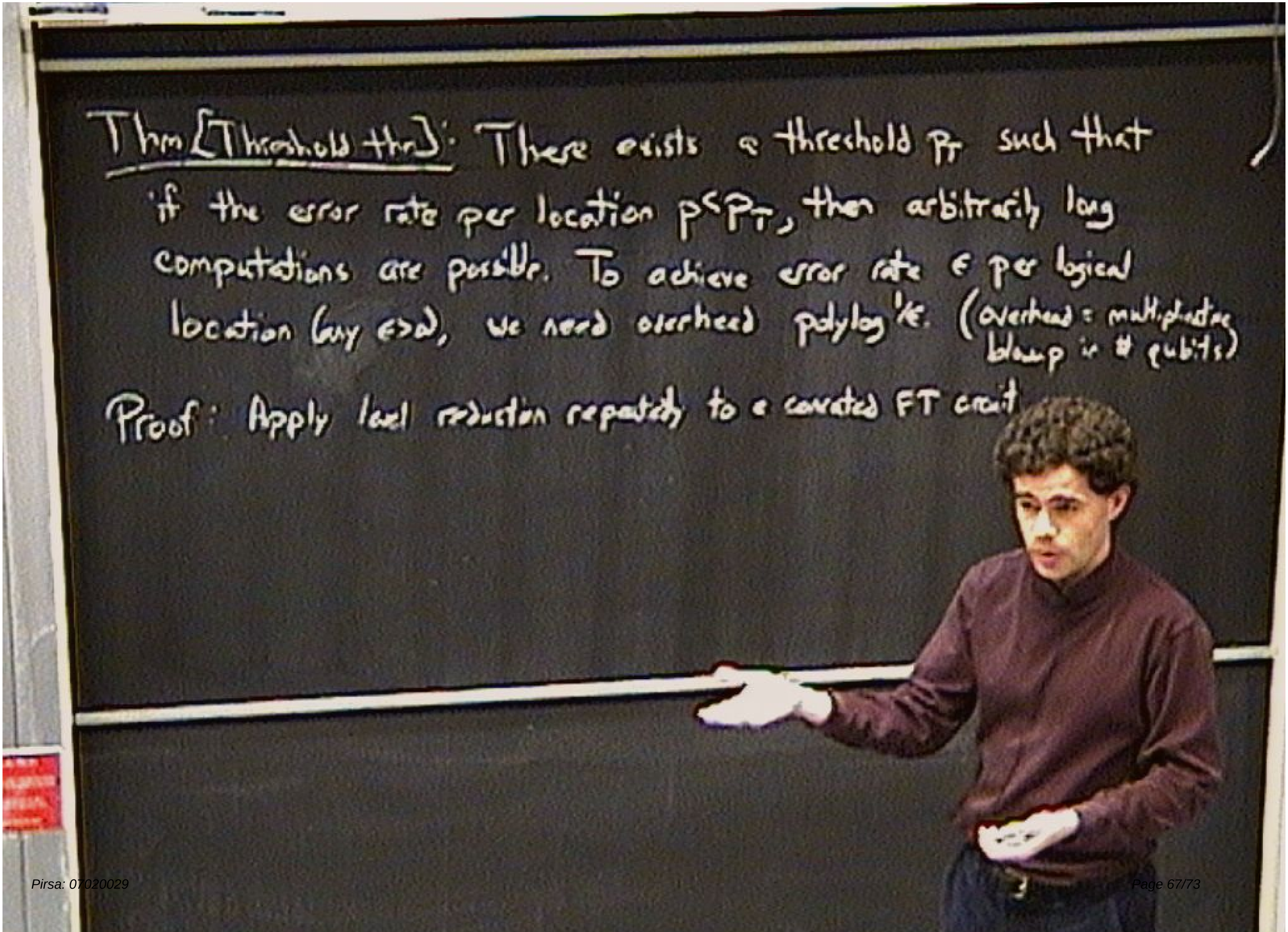
Proof: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Proof: Apply level reduction repeatedly to a concatenated FT circuit

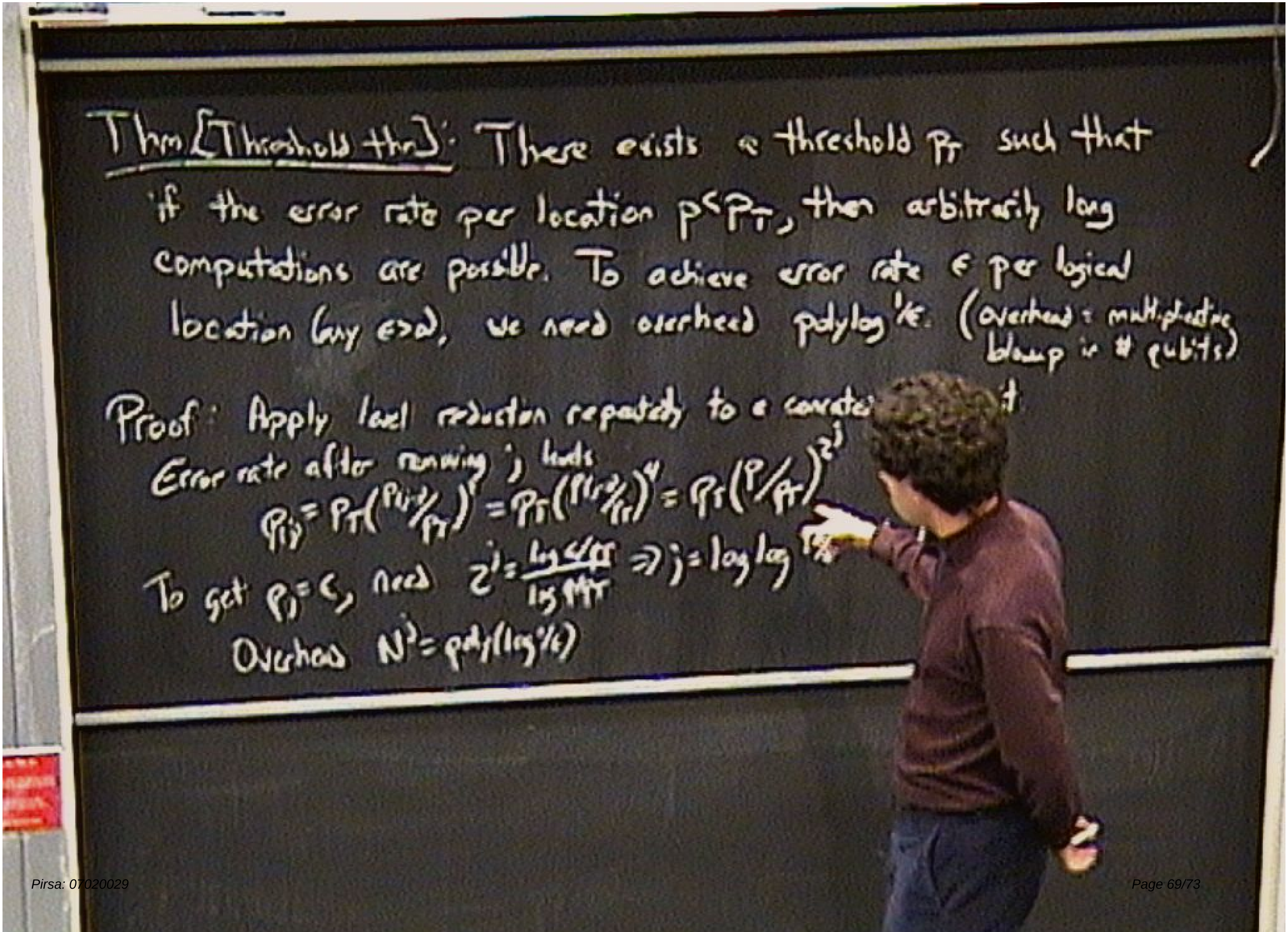
**Thm [Threshold thm]:** There exists a threshold $p_T$ such that if the error rate per location $p < p_T$, then arbitrarily long computations are possible. To achieve error rate $\epsilon$ per logical location (any $\epsilon > 0$), we need overhead $\mathrm{polylog}\, 1/\epsilon$. (overhead = multiplicative blowup in # qubits)

**Proof:** Apply local reduction repeatedly to a concatenated FT circuit.

Error rate after running $j$ levels

$$p_{(j)} = p_T \left(\frac{p_{(j-1)}}{p_T}\right)^2 = p_T \left(\frac{p_{(j-2)}}{p_T}\right)^4 = p_T \left(\frac{p}{p_T}\right)^{2^j}$$

To get $p_{(j)} = \epsilon$, need $2^j = \frac{\log \epsilon/p_T}{\log p/p_T} \Rightarrow j = \log\log \frac{p_T}{\epsilon} - \log\log \frac{p_T}{p}$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What is the probability of a bad exRec?

Need $t+1$ bad locations, $p^{t+1}$ of having errors in a particular set

There are $\binom{A}{t+1}$ sets, where $A = \#$ locations in exRec

$p(\text{Bad}) \leq \binom{A}{t+1} p^{t+1}$

E.g.: CNOT exRec for 7-qubit code, $t=1$ $\quad p(\text{Bad}) \leq \binom{A}{2} p^2 = (572{,}985)\, p^2$


7 CNOT gates, 4 EC steps

$A = 7 + 4B$, $\quad B = \#$ locations in EC $\Rightarrow p_T \geq 1.7 \times 10^{-6}$

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(Better thresholds:

For 7-qubit code, $p_T \gtrsim 2.7\times10^{-5}$

Best proofs threshold $\sim 10^{-3}$

Best simulations: threshold 5–6%.

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Error rate at level j ...

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