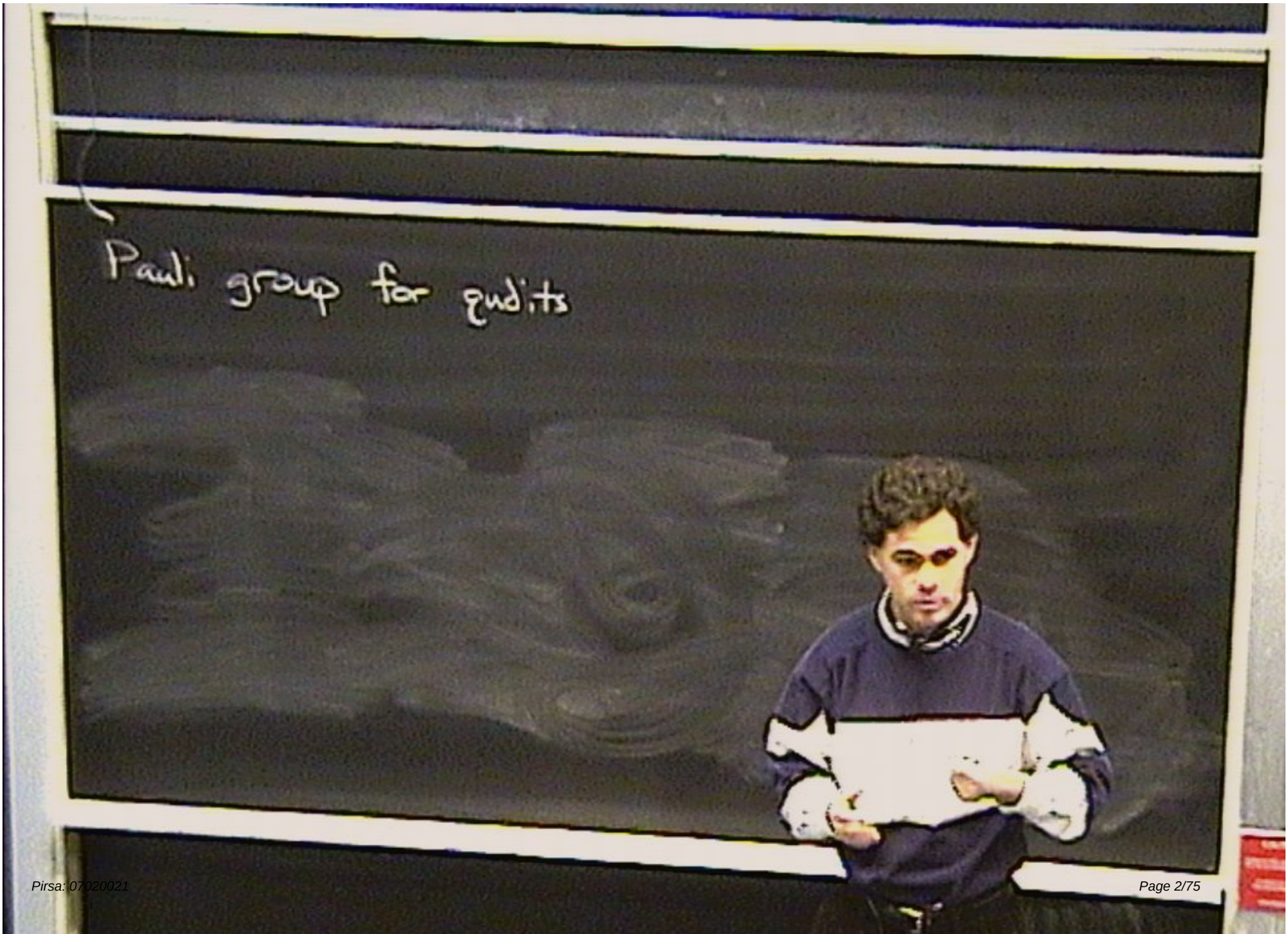
Title: Quantum Error Correction 6A

Date: Feb 13, 2007 03:30 PM

URL: http://pirsa.org/07020021

Abstract: Examples of qudit stabilizer codes, polynomial codes, Clifford group for qudits, introduction to fault-tolerance, definition of transversal gates, definition of fault-tolerant gates


Pauli group for qudits

Pauli group for qudits (dimension p prime):

$$\mathcal{P}_n = \{\omega^a X^b Z^c\} \quad \omega = e^{2\pi i/p}, \; X|j\rangle = |j+1\rangle, \; Z = \omega^j |j\rangle$$

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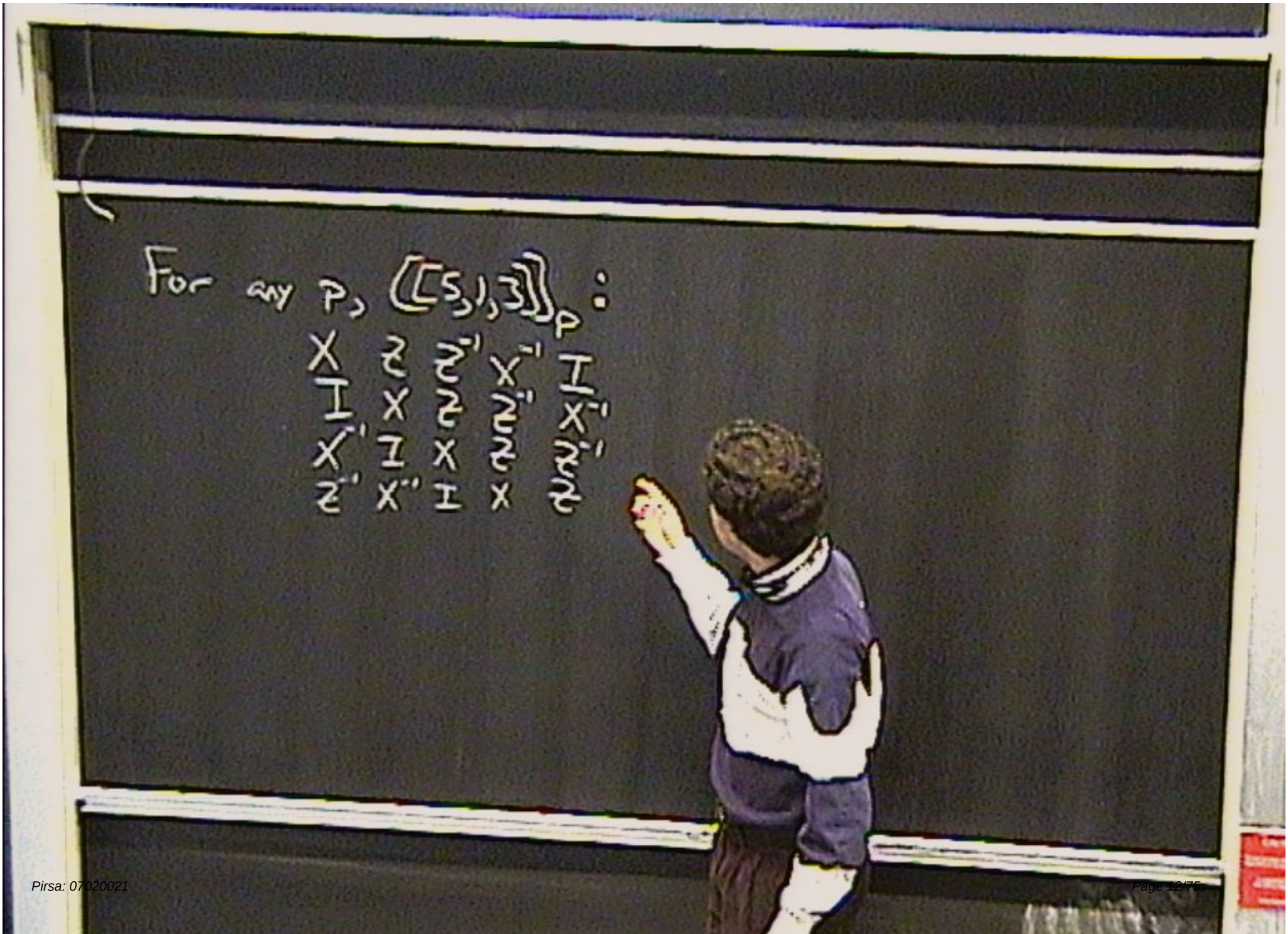
For any p, ([S,1],3]]p :
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$$
\begin{array}{ccccc}
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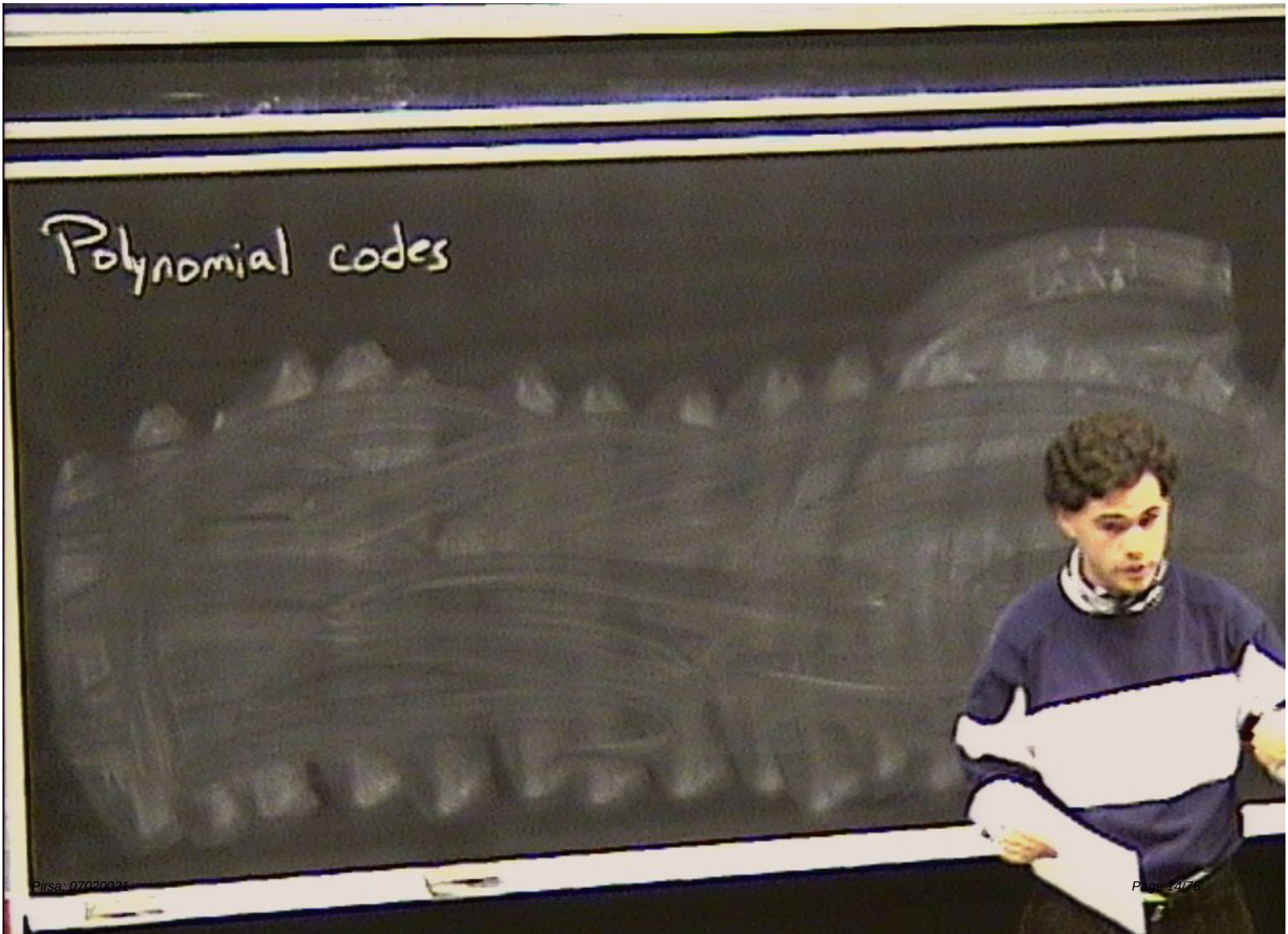
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$$|a\rangle \longmapsto |$$

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$$|a\rangle \longmapsto |a, a, a\rangle + |a+1, a+2, a+3\rangle + \ldots + |a+c_1, a+2c_1, a+3c_1\rangle$$

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$$f = a + c_1 x + c_2 x^2 \quad + |a+1, a+4, a+4\rangle + |a+2, a+1, a+2\rangle + \ldots$$

$$+ \ldots |a+c_1+c_2, a+2c_1+4c_2, a+3c_1+4c_2\rangle + \ldots$$

$\alpha_0 = 1$, $\alpha_1 = 2$, $\alpha_2 = 3$

CSS code: $C_1 = \{(f(\alpha_0), \ldots, f(\alpha_{n-1})) \mid f \text{ degree} < g\}$

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$\Rightarrow n-(g+1)$ erasures can be corrected

$d_1 = n-g$

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$(C_1^\perp \subset C_2)$ $d_Q = \min(d_1, d_2) = \min(n-g, g+2-k)$

Choose g so that $n-g = g+2-k \Rightarrow 2(g+1)$

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$(C_1^\perp \subset C_2)$ $d_Q = \min(d_1, d_2) = \min(n-g, g+2-k) = \frac{n-k}{2}+1$ (Singleton bound $n-k \ge 2(d-1)$)

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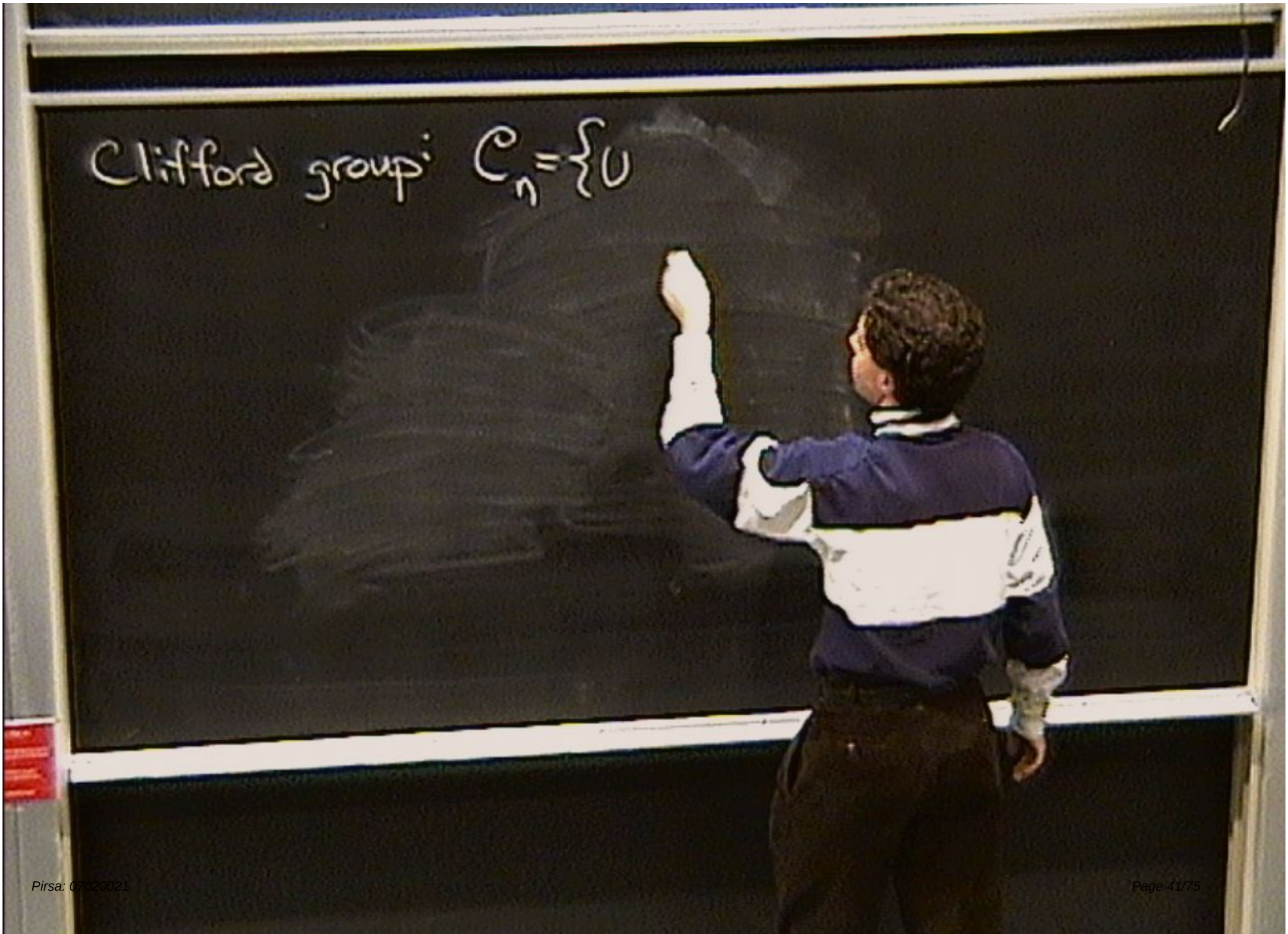
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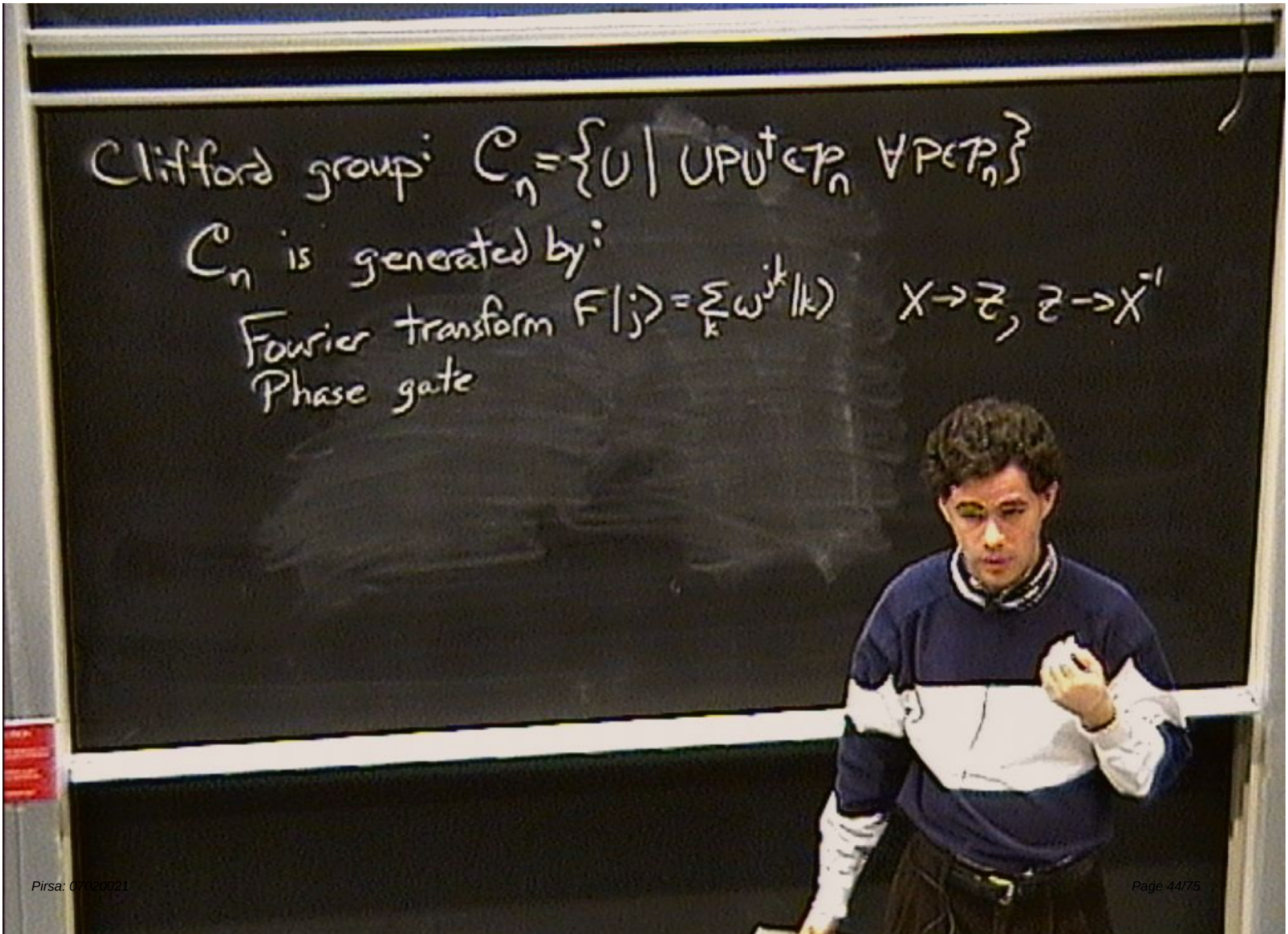
$\mathcal{C}_n$ is generated by:

Fourier transform $F|j\rangle = \sum_k \omega^{jk} |k\rangle$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$X \to Z$, $Z \to X^{-1}$
Phase gate

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Phase gate $R|j\rangle = \omega^{j(j-1)/2}|j\rangle$ $\quad X \to XZ,\ Z \to Z$

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SUM: $\mathrm{SUM}|j\rangle|k\rangle = |j\rangle|k-j \pmod p\rangle$

$X \otimes I \to X \otimes X$

$I \otimes X \to I \otimes X$

$Z \otimes I \to Z \otimes I$

$I \otimes Z \to$

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$Z \otimes I \to Z \otimes I$

$I \otimes Z \to Z^{-1} \otimes Z$

Scalar multiplication:

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$\mathcal{C}_n$ is generated by:

Fourier transform: $F|j\rangle = \sum_k \omega^{jk}|k\rangle$ $\quad X \to Z,\ Z \to X^{-1}$

Phase gate: $R|j\rangle = \omega^{j(j-1)/2}|j\rangle$ $\quad X \to XZ,\ Z \to Z$

SUM: $\mathrm{SUM}|j\rangle|k\rangle = |j\rangle|k-j \pmod p\rangle$

$X \otimes I \to X \otimes X$

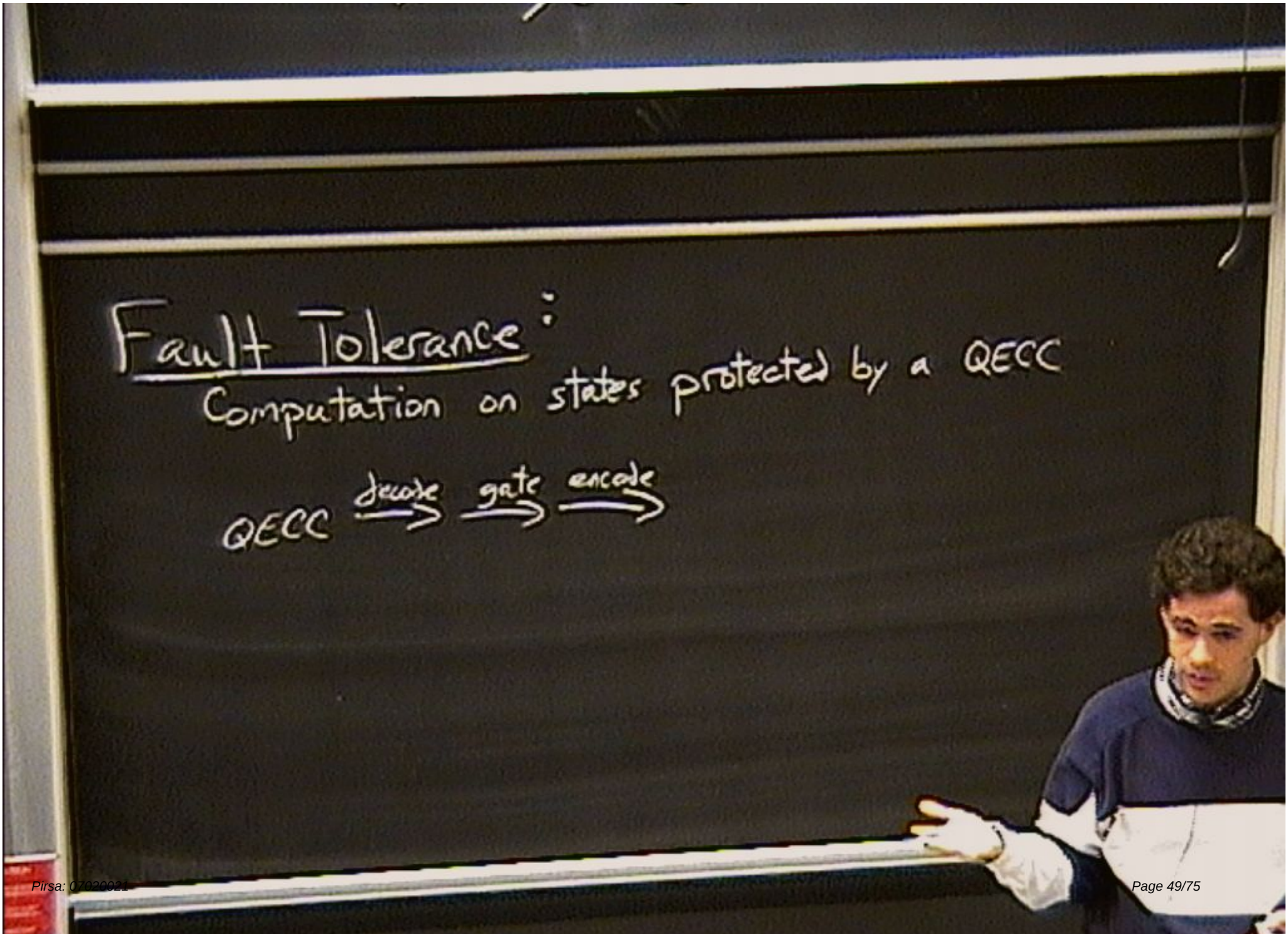
$I \otimes X \to I \otimes X$

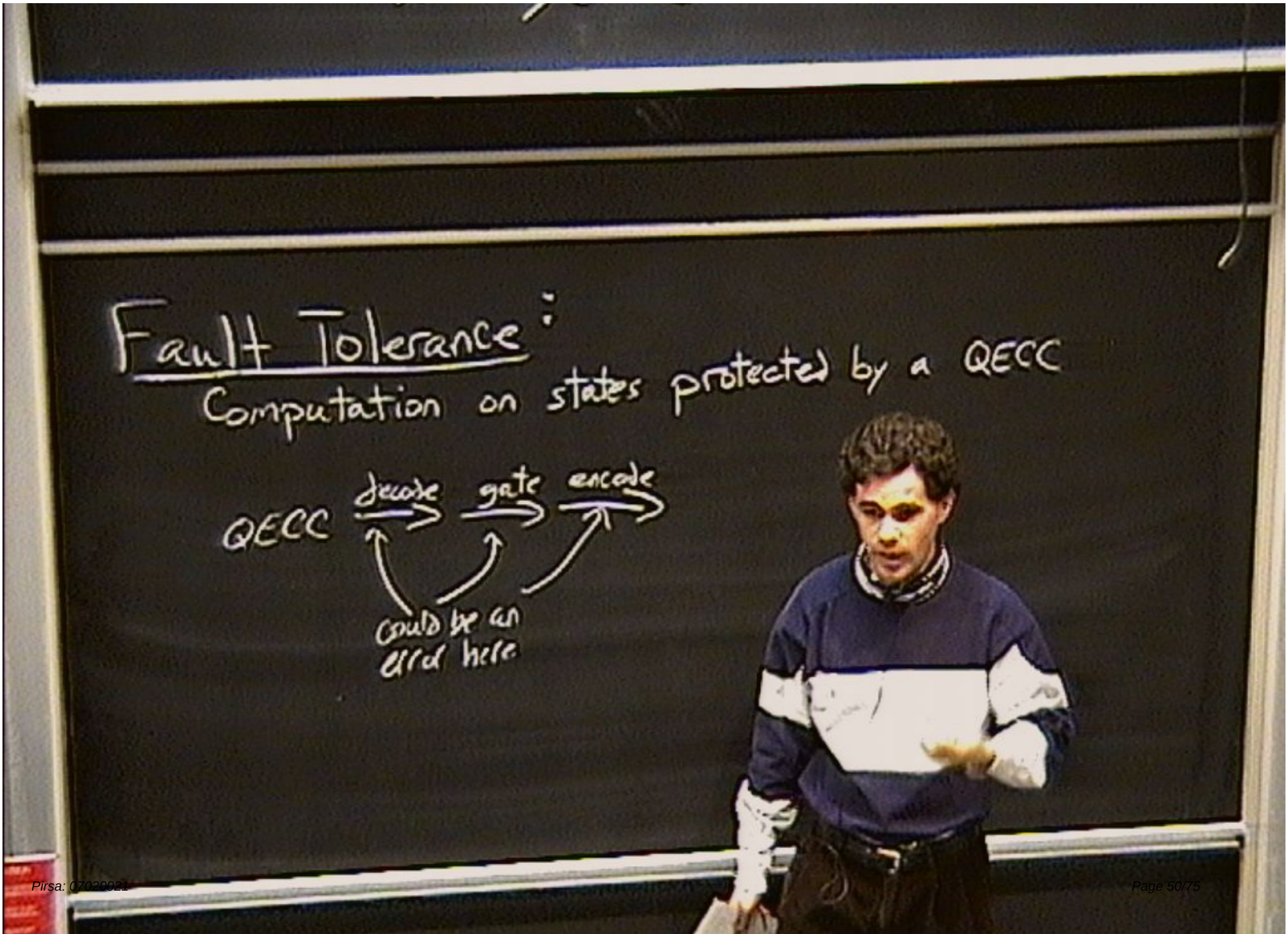
$Z \otimes I \to Z \otimes I$

$I \otimes Z \to Z^{-1} \otimes Z$

Scalar multiplication: $S_c|j\rangle = |cj\rangle$ $(c \neq 0)$

$X \to X^c,\ Z \to Z^{1/c}$


Fault Tolerance:
Computation on states protected by a QECC
QECC $\xrightarrow{\text{decode}}$ $\xrightarrow{\text{gate}}$ $\xrightarrow{\text{encode}}$


Fault Tolerance:
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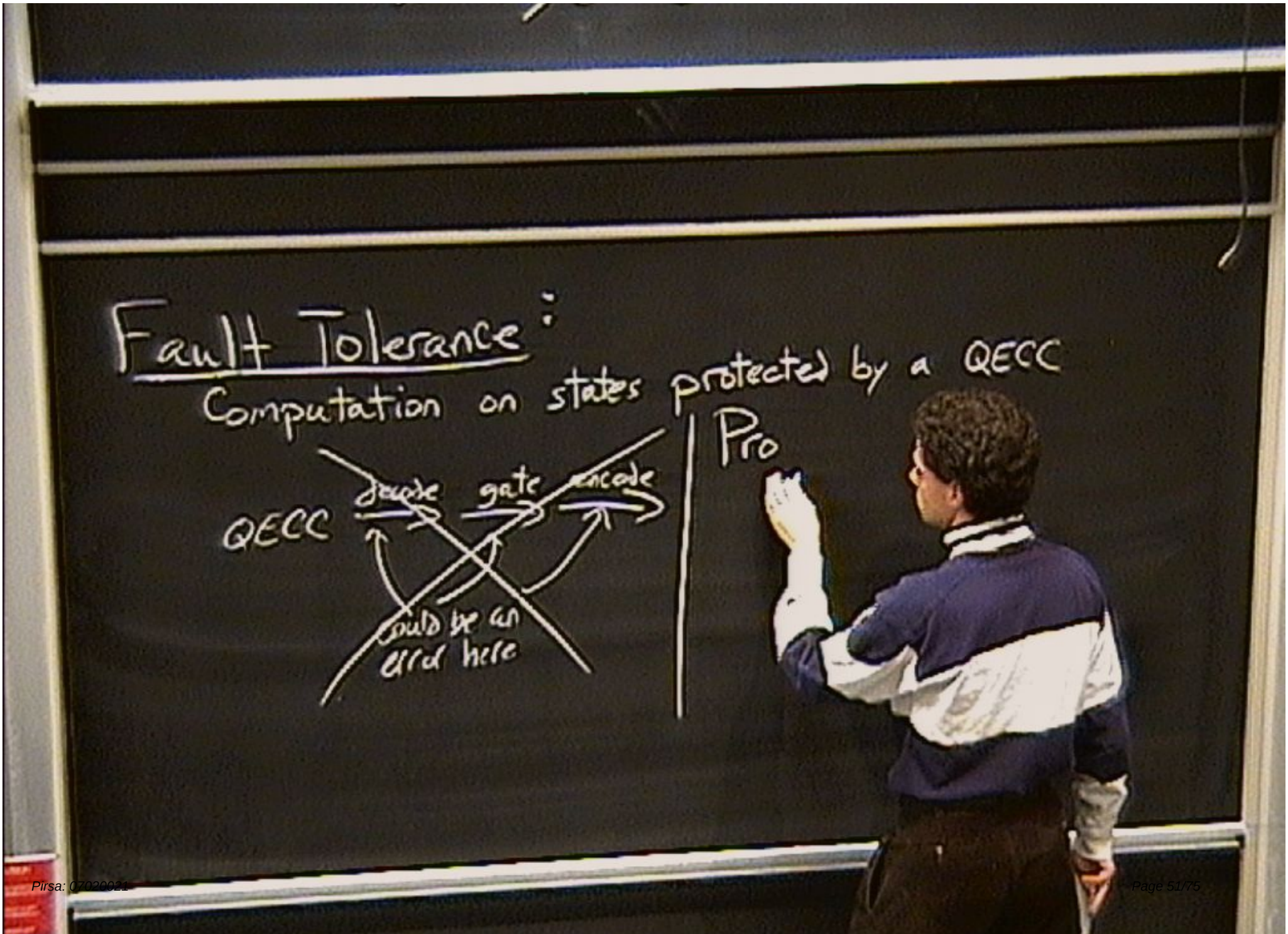
QECC
decode
gate
encode
could be an error here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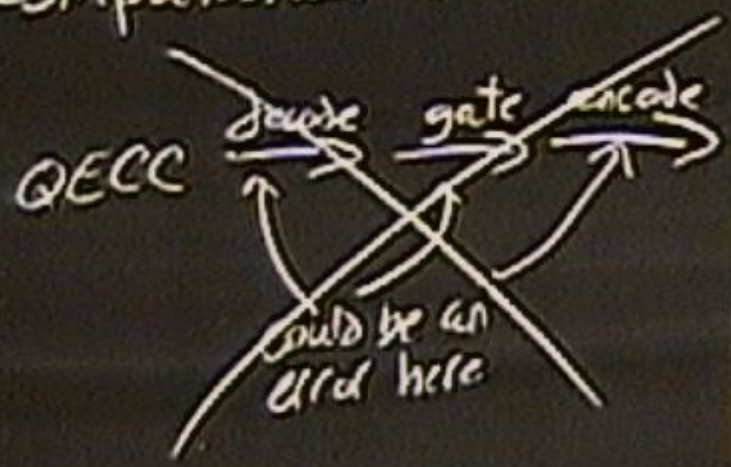
Computation on states protected by a QECC
QECC
decode
gate
encode
could be an error here
Pro

Fault Tolerance:

Computation on states protected by a QECC



Propagation of errors:

$X_1|\bar{\psi}\rangle$

Fault Tolerance:

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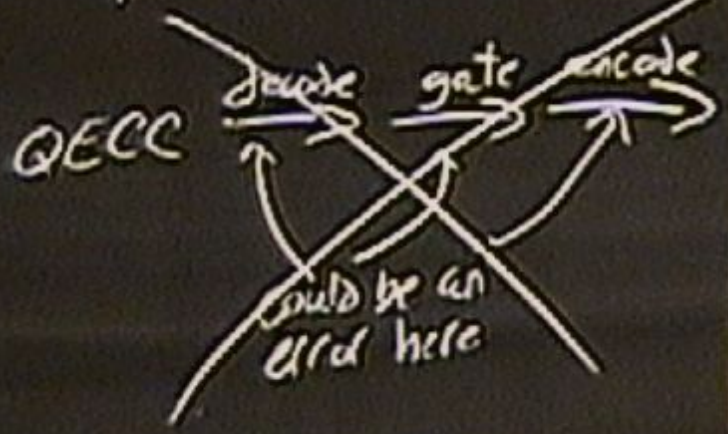

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X propagates forward in CNOT

Z propagates backwards

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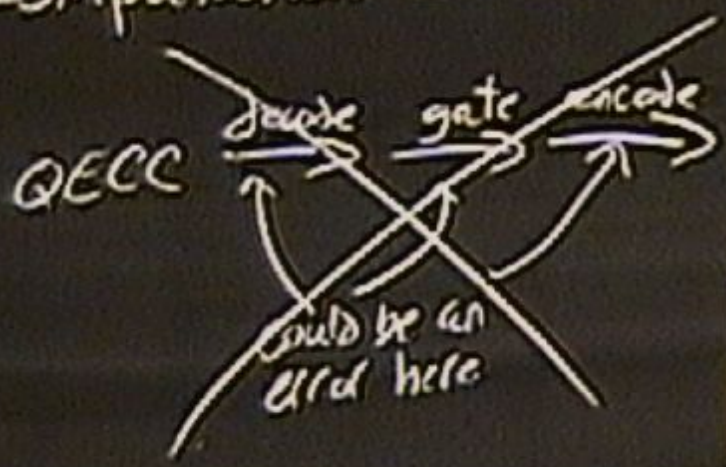

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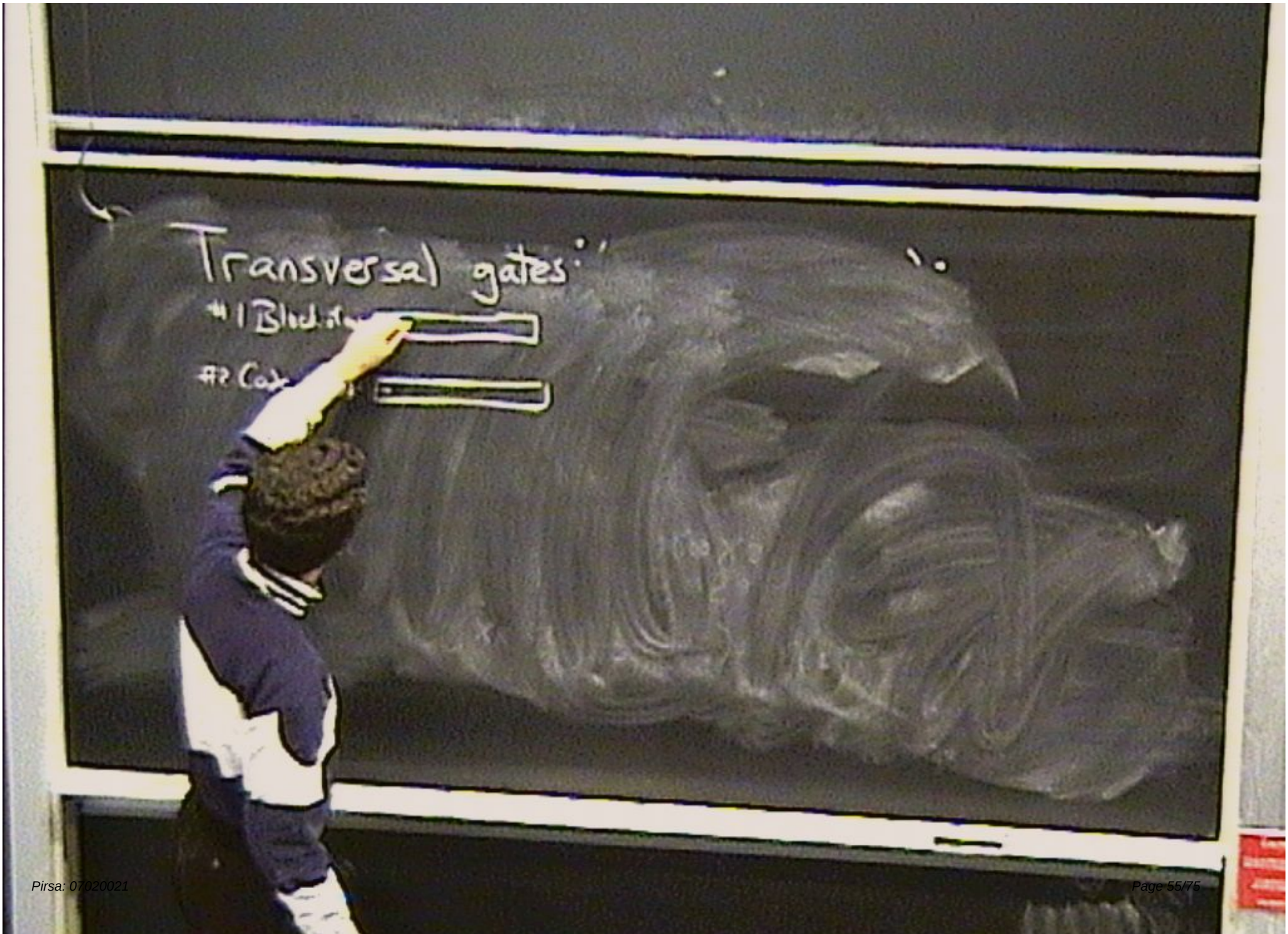
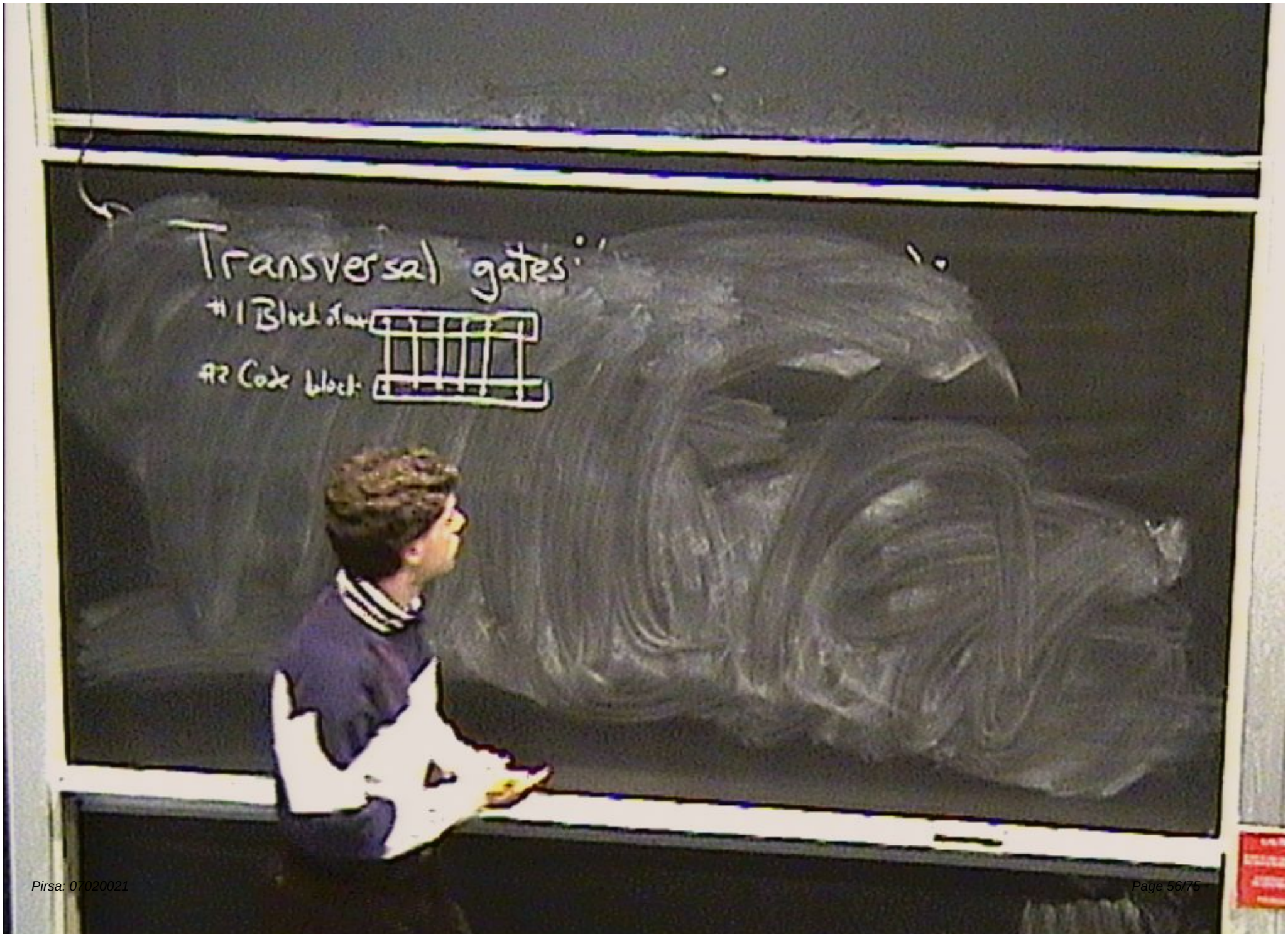

Transversal gates:

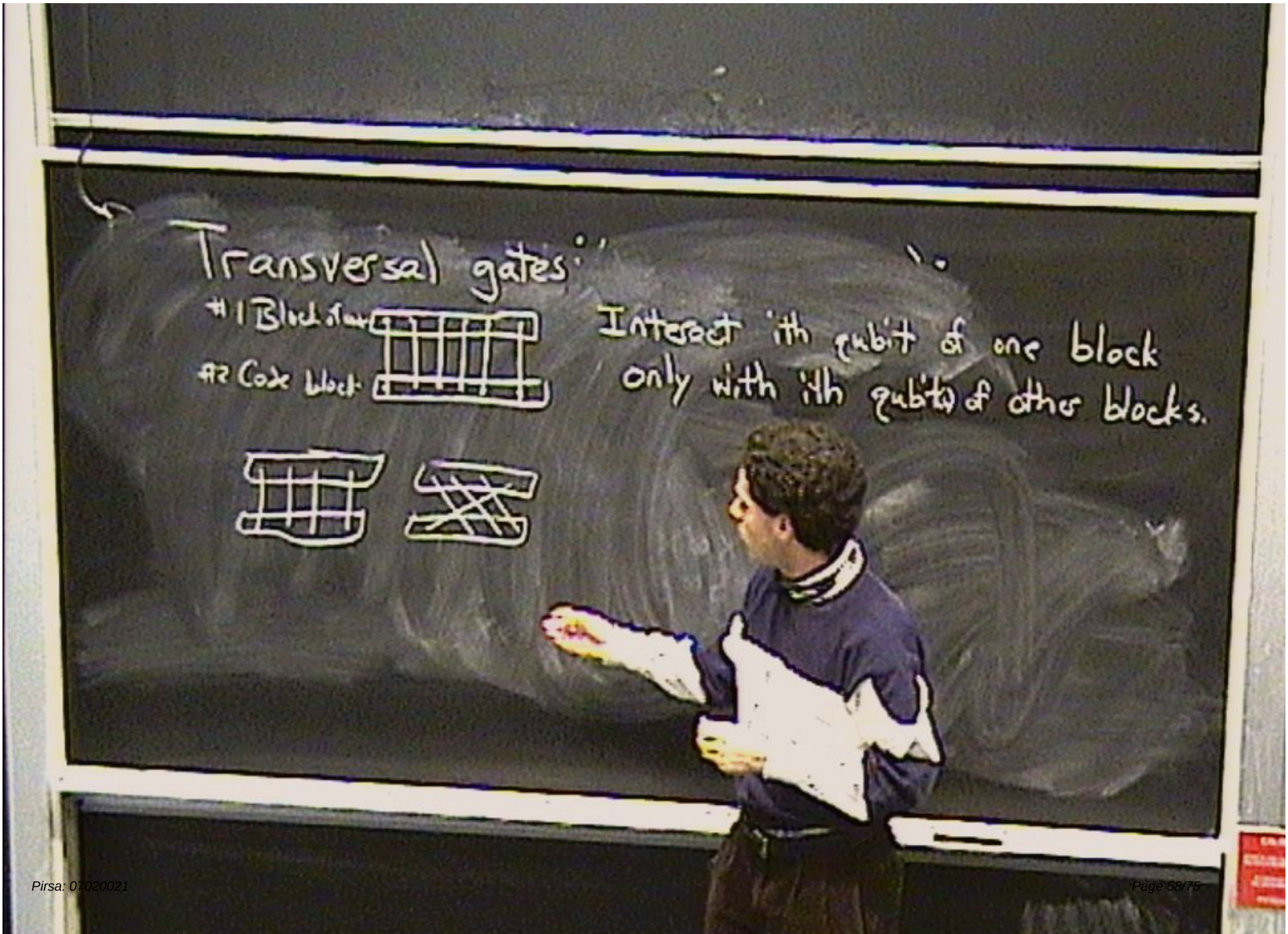

Transversal gates:
#1 Block of one
#2 Code block

Transversal gates:

#1 Block of code

#2 Code block

Interact ith qubit of one block only with ith qubit(s) of other blocks.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Interact ith qubit of one block
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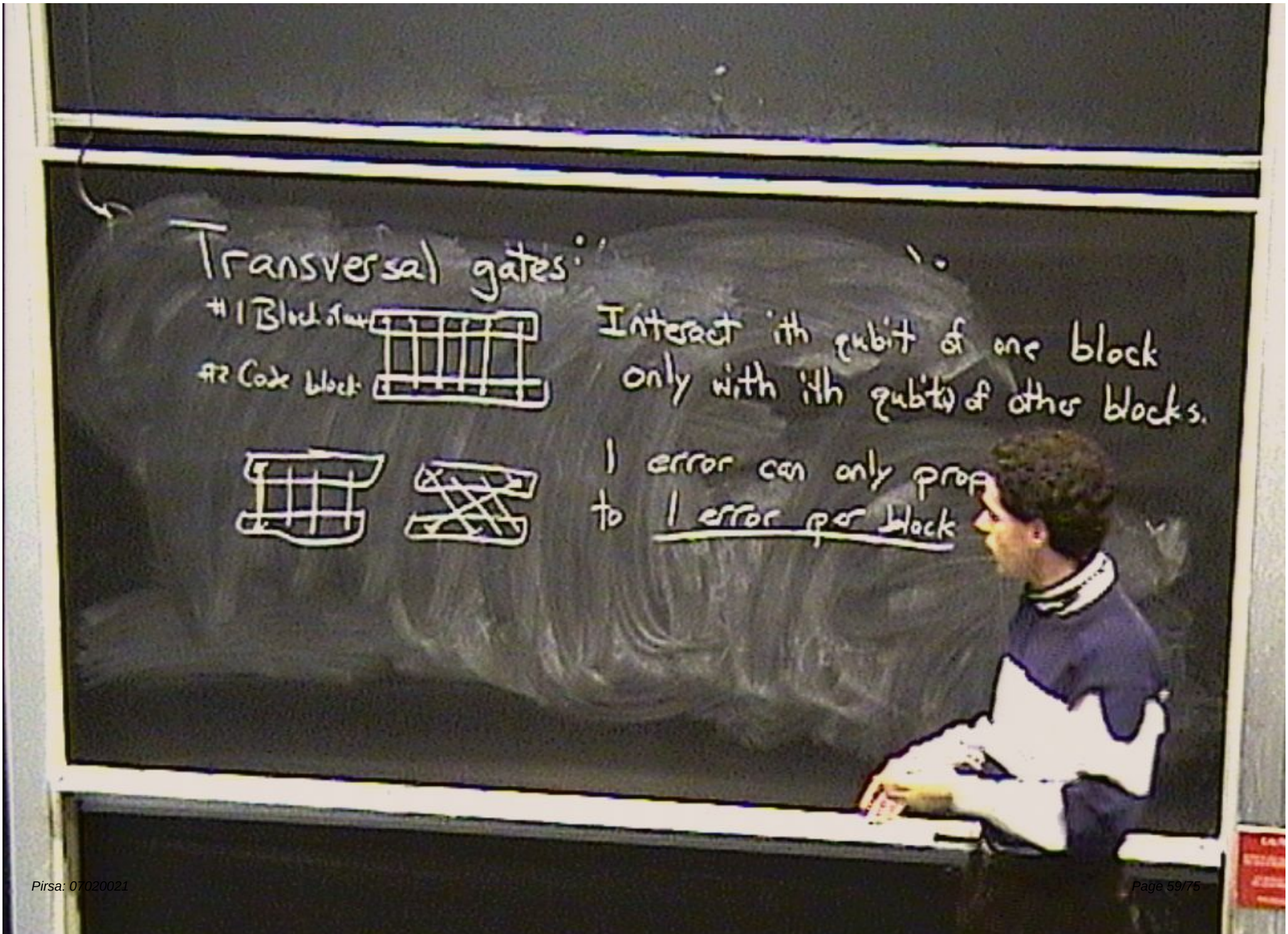
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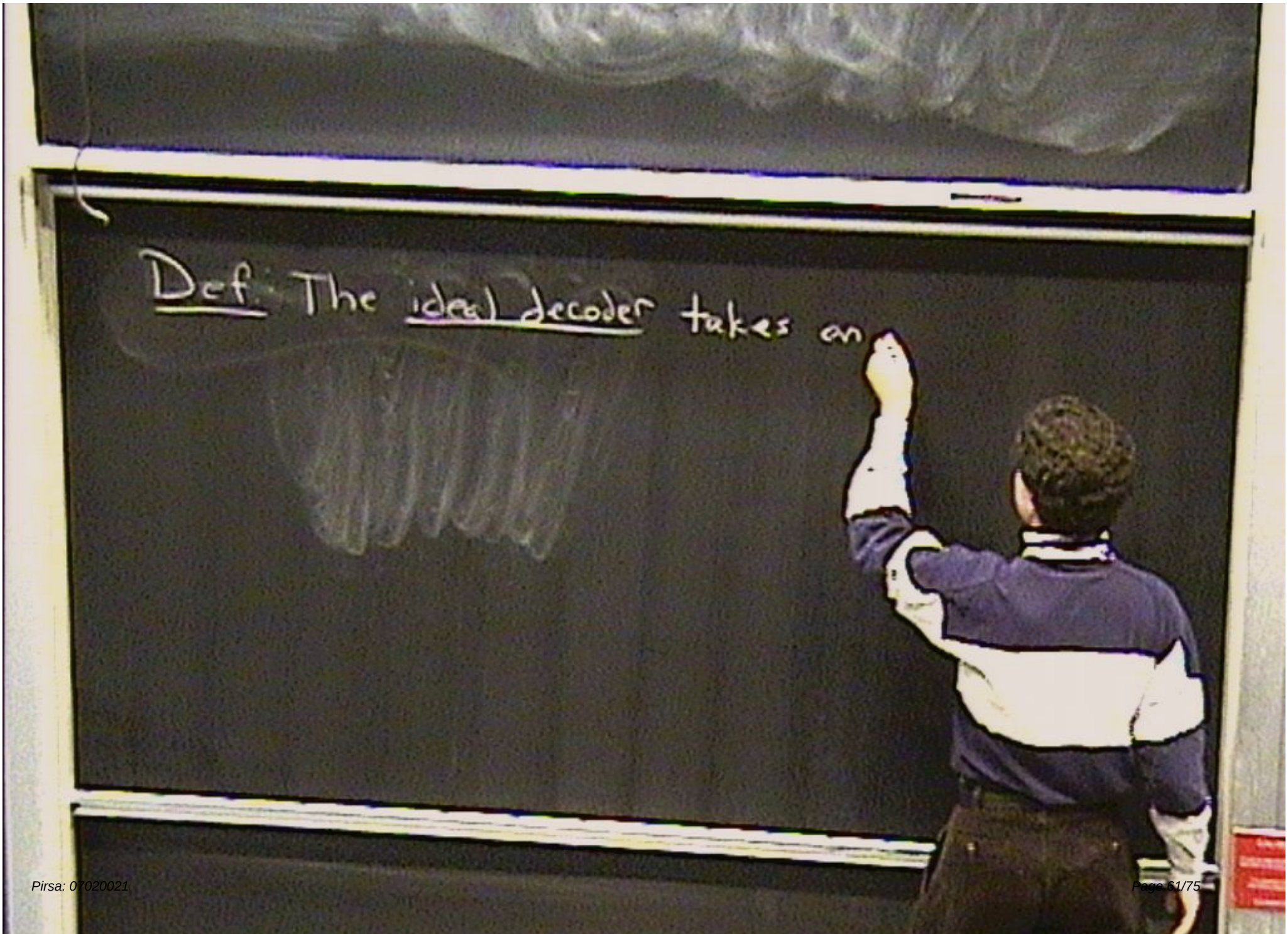
(even in a sequence of transversal gates)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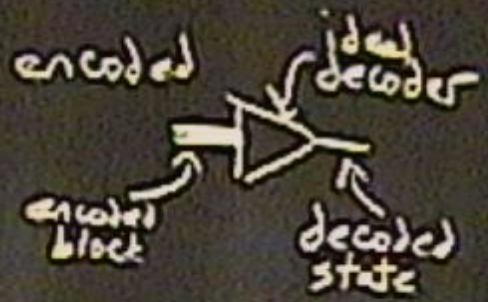

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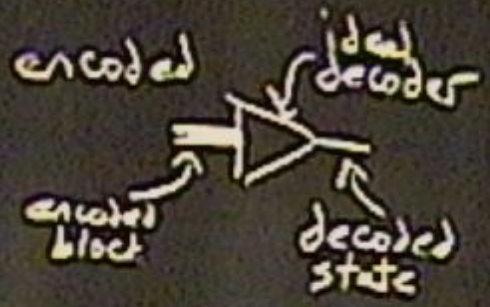

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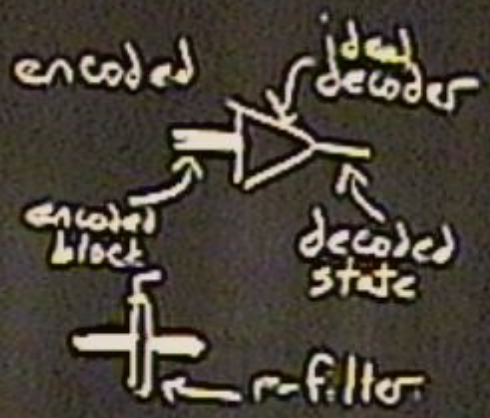

CSS code: $C_1 = \{(f(\alpha_0), \ldots, f(\alpha_{n-1})) \mid f \text{ degree} \leq e\}$

Gate property 1:

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$(r+$

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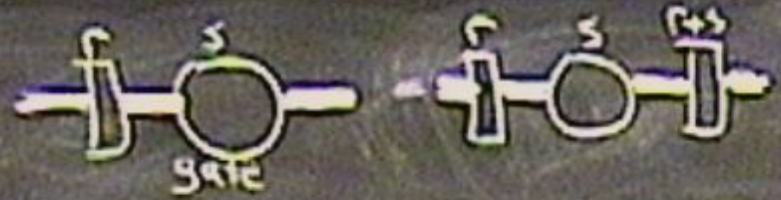
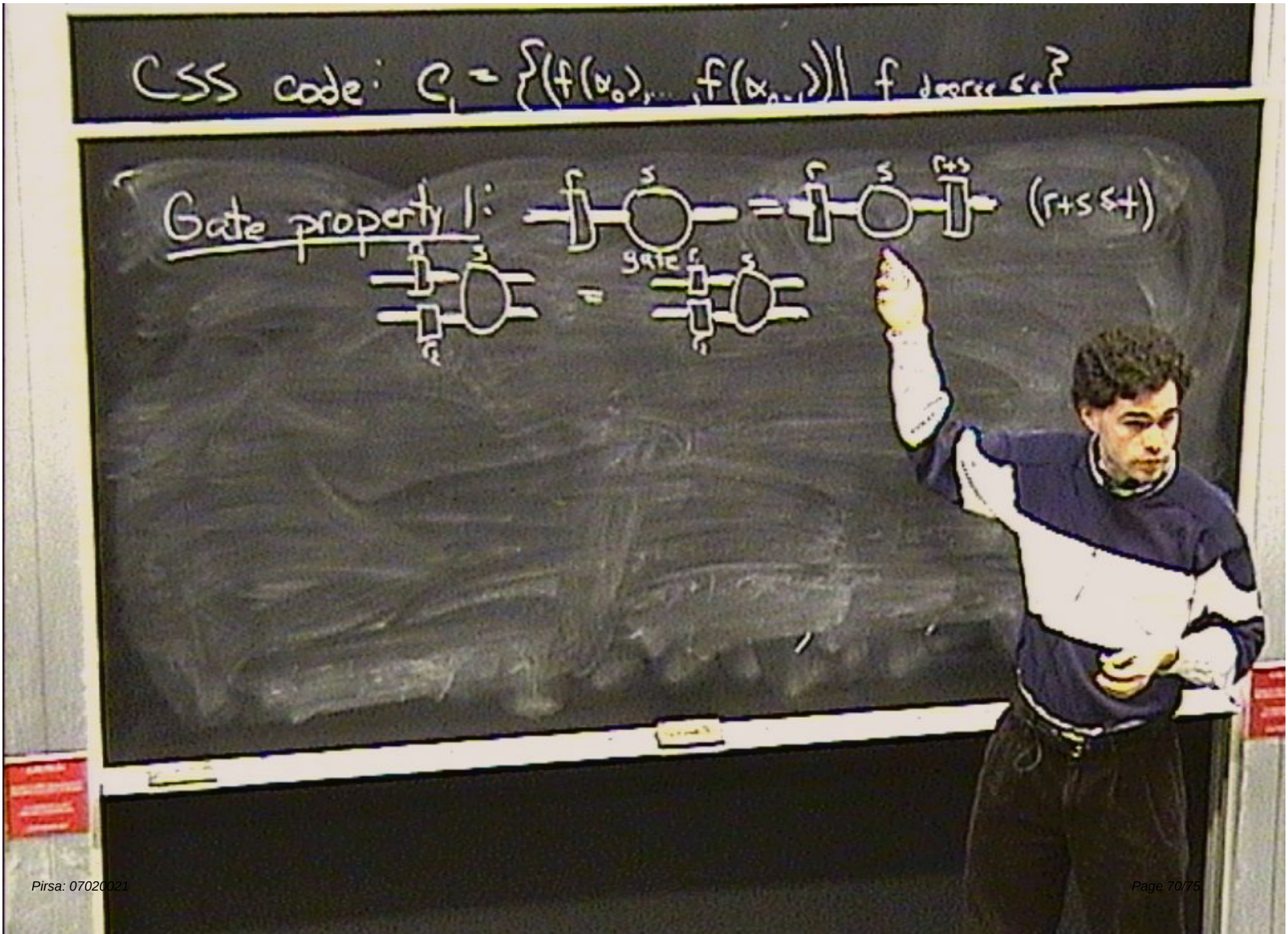
<u>Gate property 1:</u> 

$(r+s \leq t)$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Gate property 1:
(r+s≤t)
gate

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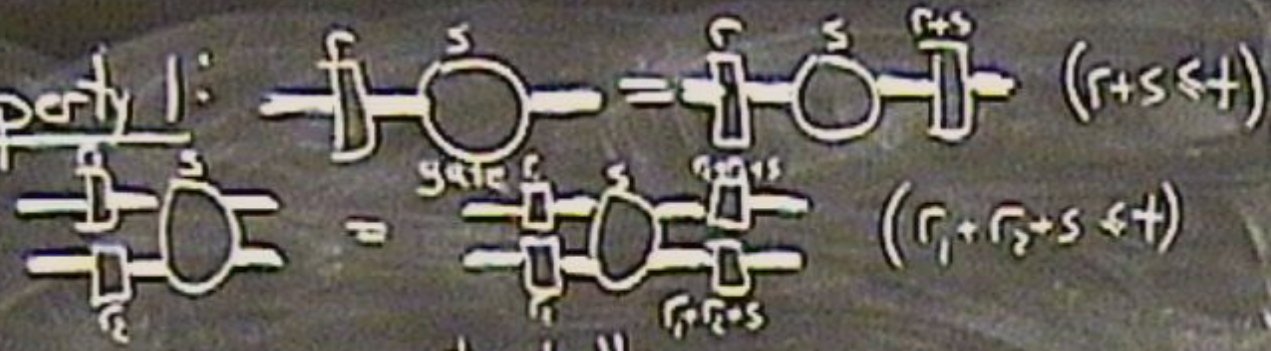

$(r+s \leq t)$

$(r_1 + r_2 + s \leq t)$

Errors do not propagate badly.

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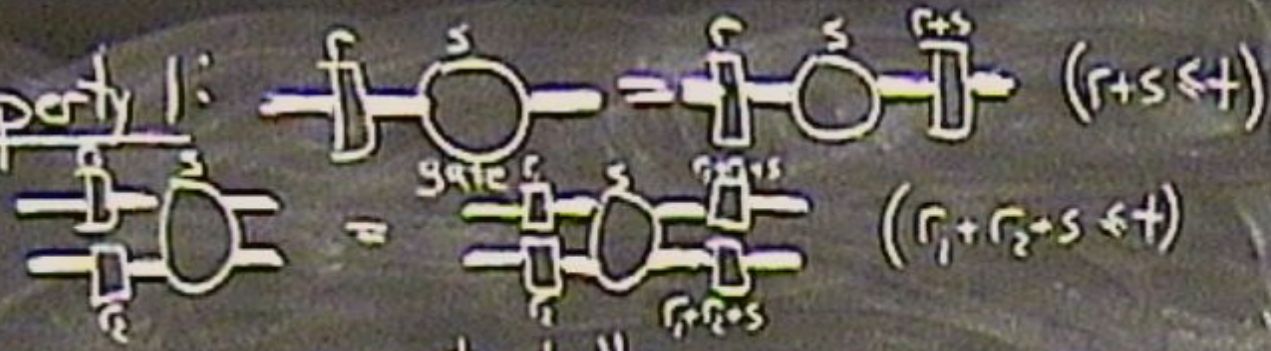

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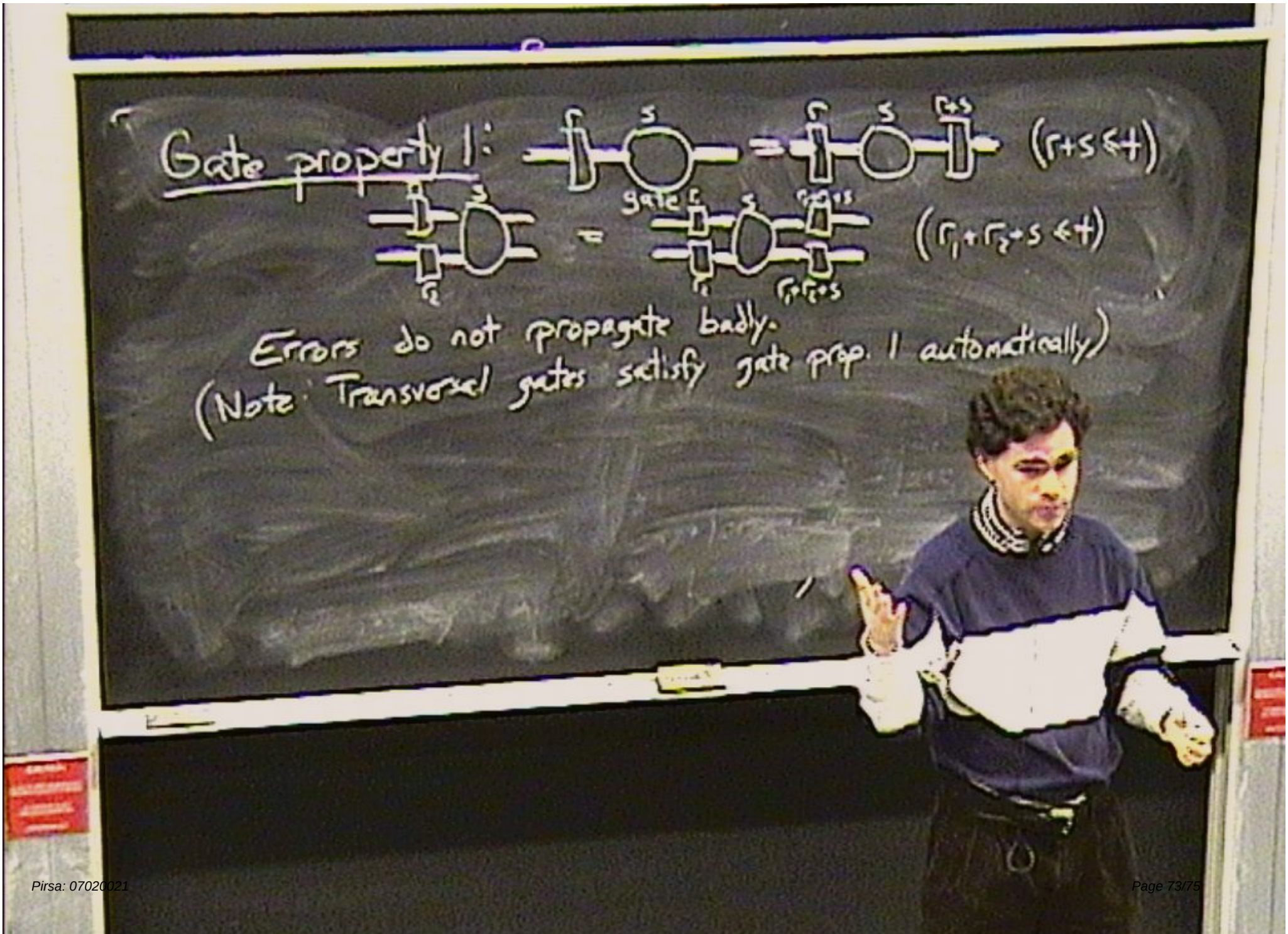
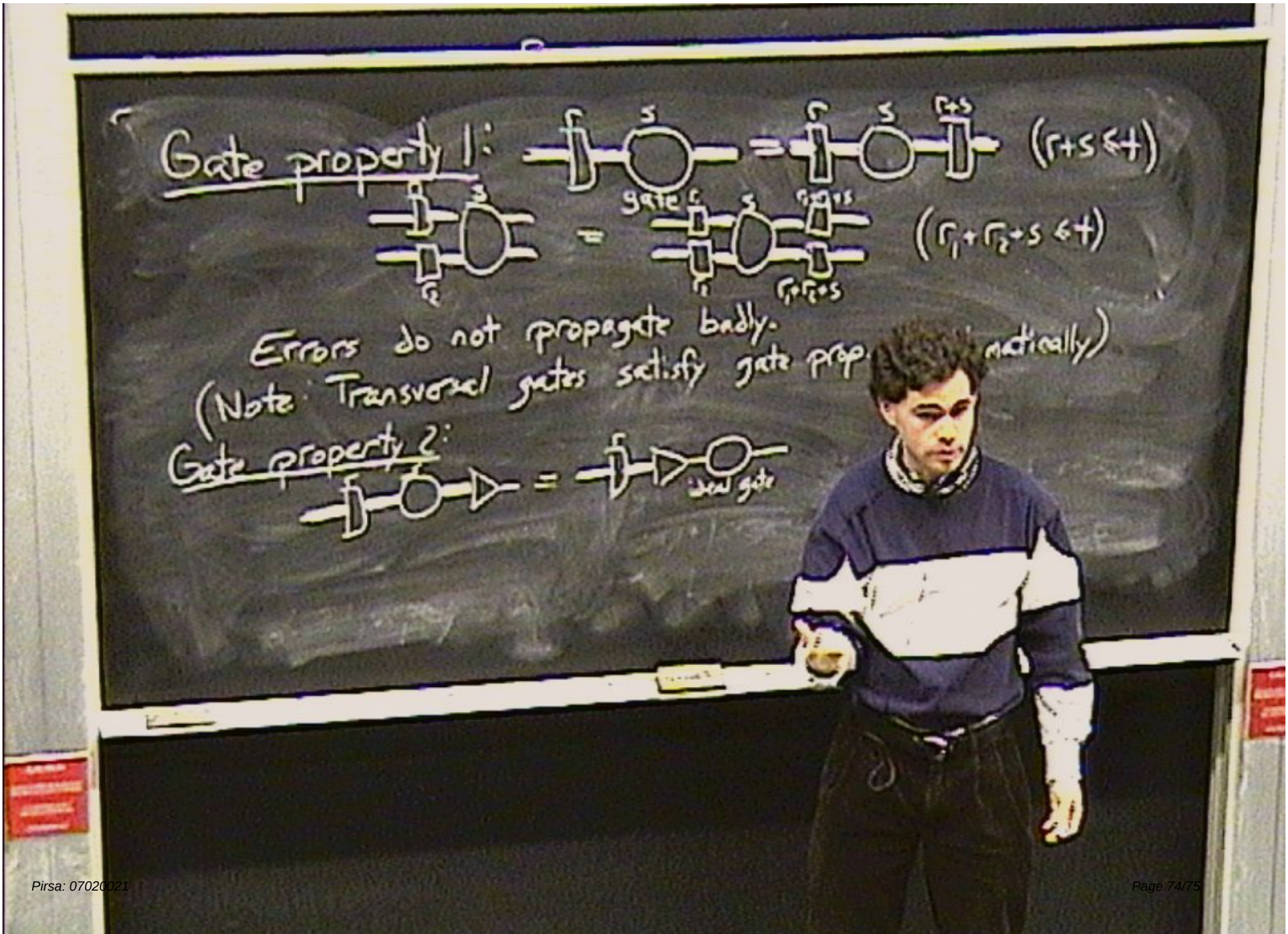

Gate property 1:
(r+s ≤ t)
gate
(r1+r2+s ≤ t)
Errors do not propagate badly.
(Note: Transversal gates satisfy gate prop. 1 automatically)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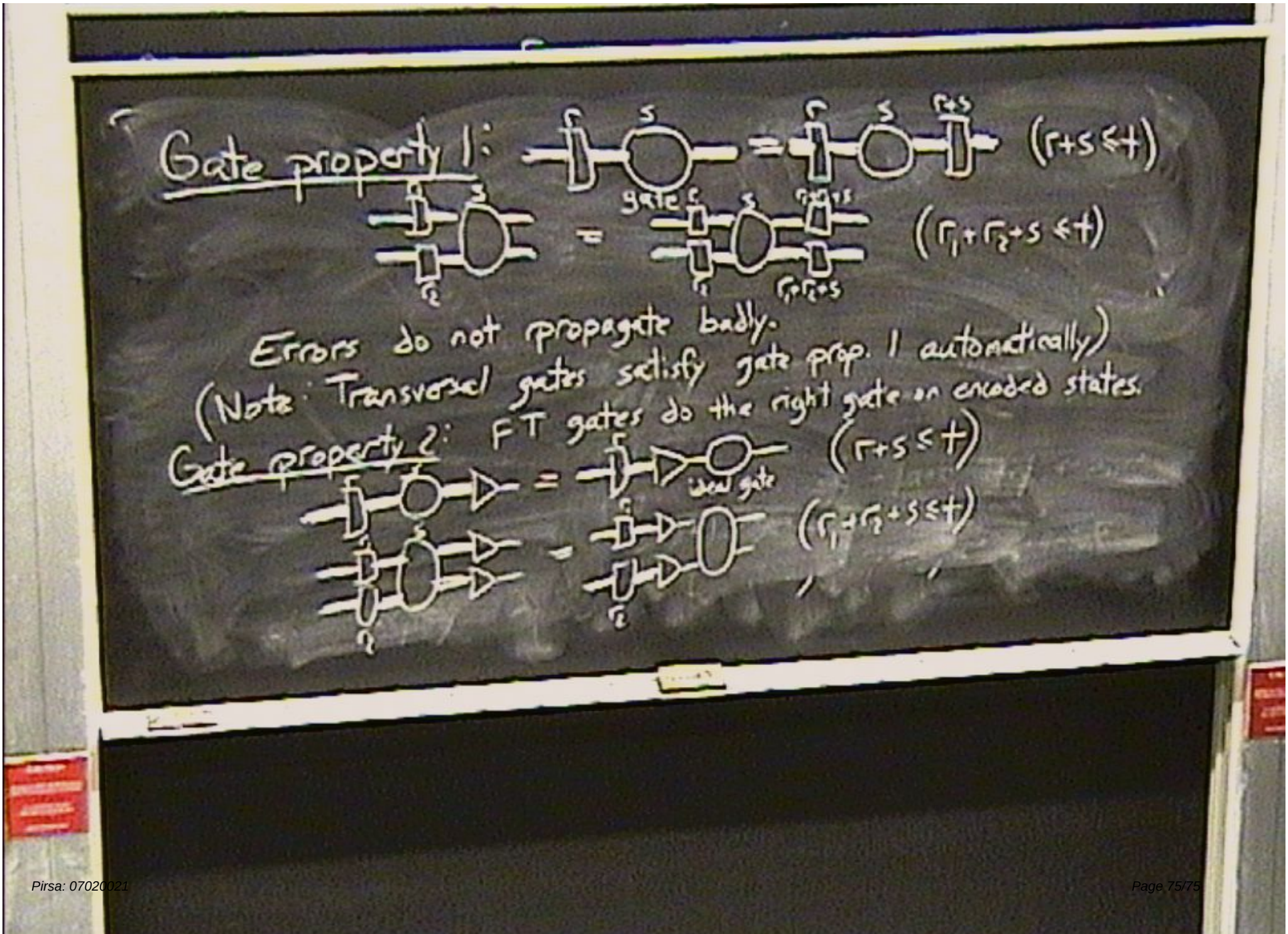
Gate property 2:
seal gate


Gate property 1:
(r+s≤t)
gate
(r1+r2+s≤t)
Errors do not propagate badly.
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Gate property 2: FT gates do the right gate on encoded states.
(r+s≤t)
ideal gate
(r1+r2+s≤t)