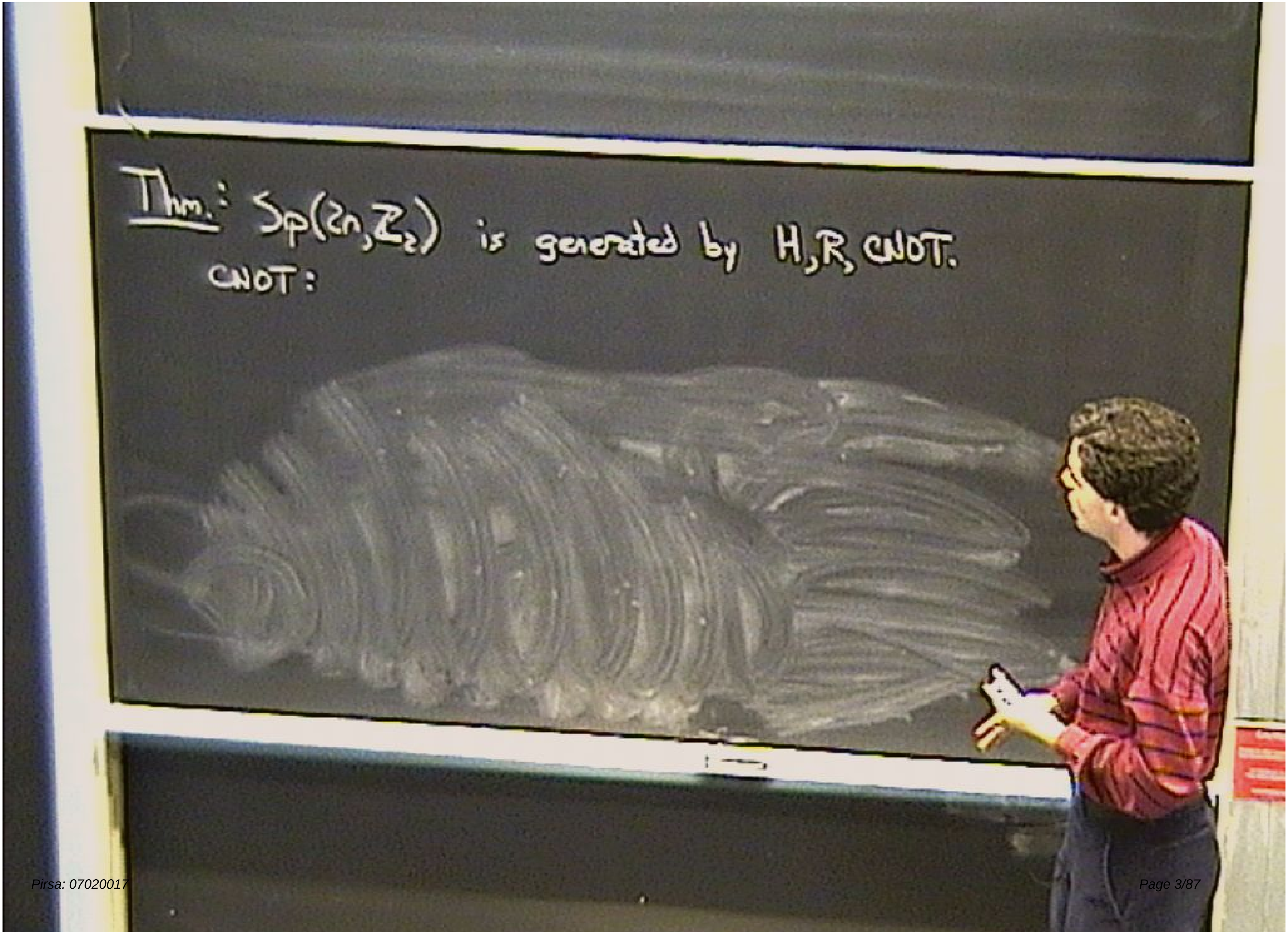
Title: Quantum Error Correction 5A

Date: Feb 06, 2007 03:30 PM

URL: http://pirsa.org/07020017

Abstract: Generators of symplectic group, quantum Gilbert-Varshamov bound, quantum Hamming bound, quantum Singleton bound

Thm.: $Sp(2n, \mathbb{Z}_2)$ is generated by H, R, CNOT.

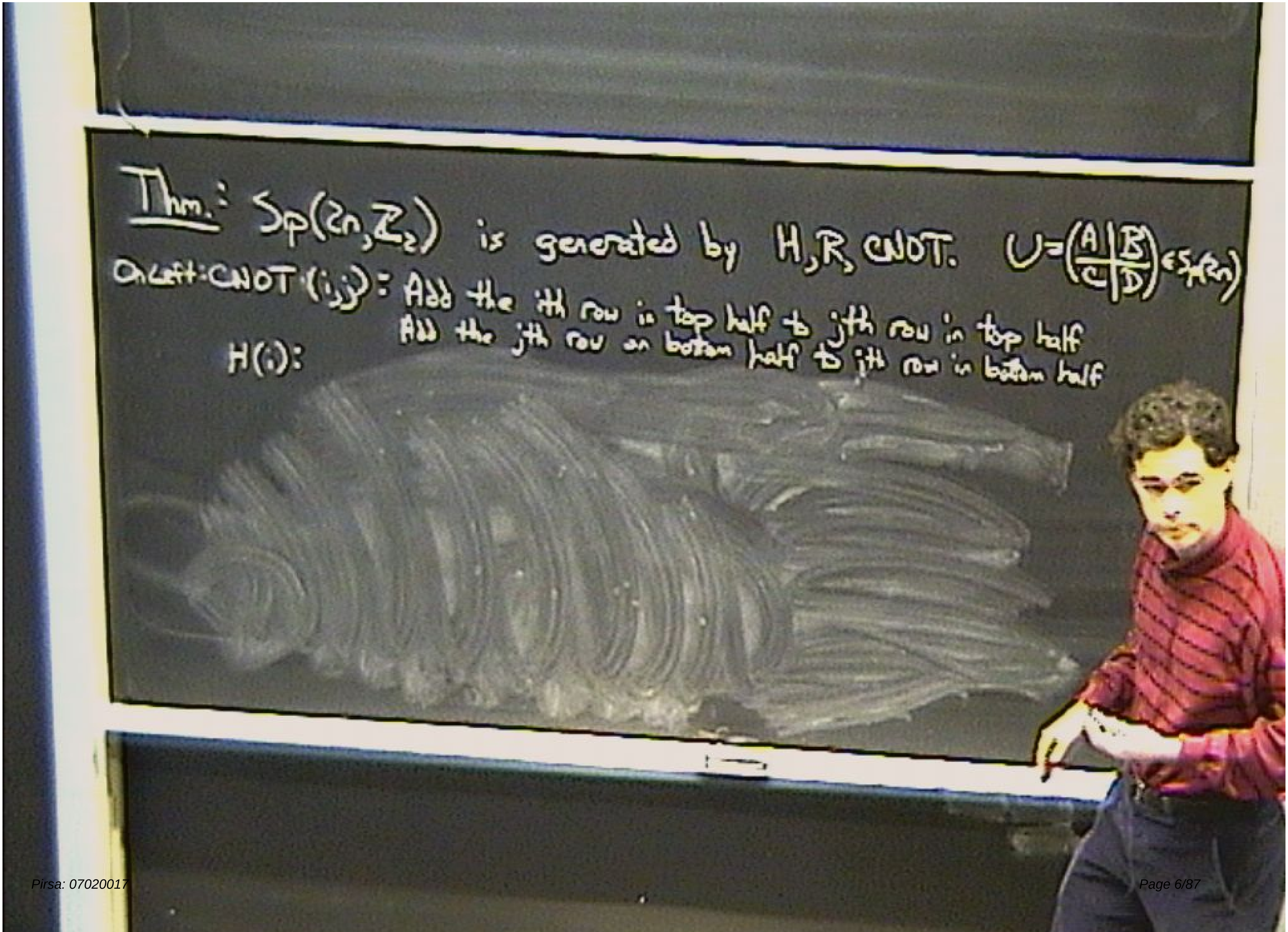

Thm.: $Sp(2n, \mathbb{Z}_2)$ is generated by H, R, CNOT.
CNOT:

Thm.: $Sp(2n,\mathbb{Z}_2)$ is generated by $H, R,$ CNOT. $U=\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)\in Sp(2n)$

CNOT:

Thm.: $Sp(2n, \mathbb{Z}_2)$ is generated by $H, R,$ CNOT. $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in Sp(2n)$

On Left: CNOT(i,j) = Add the ith row in top half to jth row in top half
Add the jth row in bottom half to ith row in bottom half


Thm.: Sp(2n, Z2) is generated by H, R, CNOT. U = (A|B / C|D) ∈ Sp(2n)
On Left: CNOT(i,j): Add the ith row in top half to jth row in top half
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H(i):

Thm.: $Sp(2n,\mathbb{Z}_2)$ is generated by H, R, CNOT. $U=\left(\begin{array}{c|c}A & B\\ \hline C & D\end{array}\right)\in Sp(2n)$

On Left: CNOT(i,j): Add the ith row in top half to jth row in top half

Add the jth row in bottom half to ith row in bottom half

$H(i)$: Switches ith row from top half with ith row in bottom half

$R(i)$:

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H(i): Switches ith row from top half with ith row in bottom half

R(i): Adds ith row from top half to ith row in bottom half

= SWAP(i,j): Swaps ith & jth rows in both top & bottom half

Controlled-Z(i,j):

Thm.: $Sp(2n,\mathbb{Z}_2)$ is generated by H, R, CNOT. $U=\left(\begin{array}{c|c}A&B\\\hline C&D\end{array}\right)\in Sp(2n)$

On Left: CNOT(i,j): Add the ith row in top half to jth row in top half
Add the jth row in bottom half to ith row in bottom half

H(i): Switches ith row from top half with ith row in bottom half

R(i): Adds ith row from top half to ith row in bottom half

= SWAP(i,j): Swaps ith & jth rows in both top & bottom half

Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

Thm.: $Sp(2n, \mathbb{Z}_2)$ is generated by H, R, CNOT. $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in Sp(2n)$

On Left: CNOT(i,j): Add the ith row in top half to jth row in top half
Add the jth row in bottom half to ith row in bottom half

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R(i): Adds ith row from top half to ith row in bottom half

= SWAP(i,j): Swaps ith & jth rows in both top & bottom half

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
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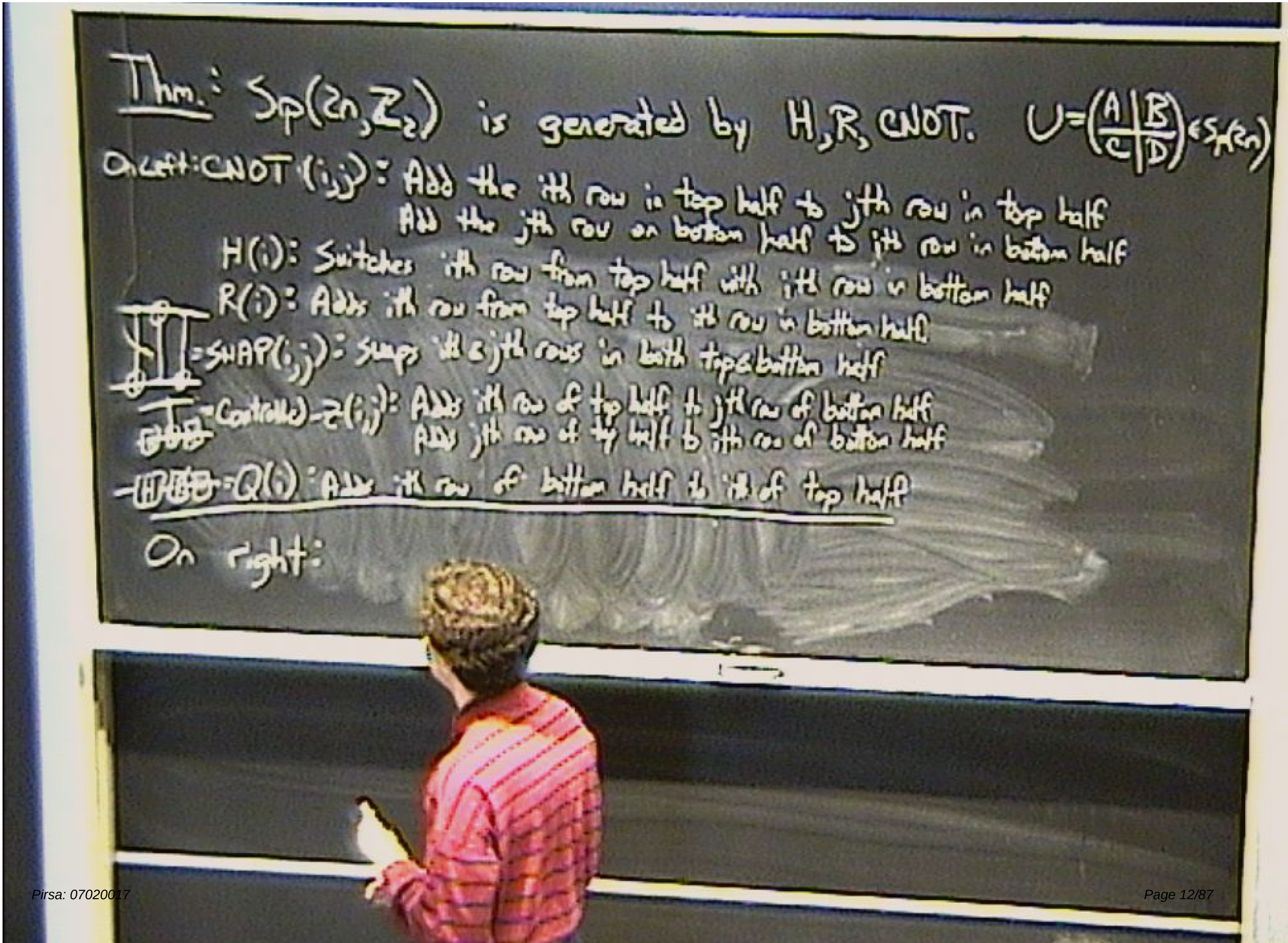
$-H-R-H- = Q(i)$


Thm.: Sp(2n, Z2) is generated by H, R, CNOT. U=(A|B / C|D) ∈ Sp(2n)
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R(i): Adds ith row from top half to ith row in bottom half
=SWAP(i,j): Swaps ith & jth rows in both top & bottom half
=Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half
=Q(i): Adds ith row of bottom half to ith of top half
On right:

<u>Thm.</u>: $Sp(2n,\mathbb{Z}_2)$ is generated by H, R, CNOT. $U=\left(\begin{array}{c|c}A&B\\\hline C&D\end{array}\right)\in Sp(2n)$

On Left: CNOT(i,j): Add the ith row in top half to jth row in top half
Add the jth row on bottom half to ith row in bottom half
$H(i)$: Switches ith row from top half with ith row in bottom half
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= SWAP(i,j): Swaps ith & jth rows in both top & bottom half
= Controlled-$Z(i,j)$: Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half
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On right: row → column, top → left, bottom → right, opposite direction

Thm.: $Sp(2n, \mathbb{Z}_2)$ is generated by H, R, CNOT. $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in Sp(2n)$

On Left: CNOT(i,j): Add the ith row in top half to jth row in top half
Add the jth row on bottom half to ith row in bottom half

$H(i)$: Switches ith row from top half with ith row in bottom half

$R(i)$: Adds ith row from top half to ith row in bottom half

= SWAP(i,j): Swaps ith & jth rows in both top & bottom half

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
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= $Q(i)$: Adds ith row of bottom half to ith of top half

On right: row $\rightarrow$ column, top $\rightarrow$ right, bottom $\rightarrow$ left

= SWAP(i,j): swaps [illegible] in both top&bottom half

= Controlled-$Z(i,j)$: Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

= $Q(i)$: Adds ith row of bottom half to ith of top half

On right: row $\to$ column, top $\to$ right, bottom $\to$ left

1. Put 1 in upper left corner of A

= SWAP(i,j): Swaps ith & jth rows in each top & bottom half

= Controlled-$Z(i,j)$: Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

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On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using

= SWAP(i,j): swaps i,j entries in both top & bottom half

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Adds jth row of top half to ith row of bottom half

= Q(i): Adds ith row of bottom half to ith of top half

On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT(1,i) to make rest of 1st column of A = 0

= SWAP(i,j): Swaps ith,jth rows in both top & bottom half

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
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= Q(i): Adds ith row of bottom half to ith of top half

On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT(1,i) to make rest of 1st column of A = 0

$$A = \begin{pmatrix} 1 & ? \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}$$

= SWAP(i,j): Swaps ... entries in each top/bottom half

Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
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On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT(1,i) to make rest of 1st column of A = 0
3.

$$A = \underbrace{\begin{pmatrix} 1 & ? \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}}_{n} \Big\}\, n$$

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On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT(1,i) to make rest of 1st column of A = 0
3. Use R(1) & Controlled-Z(1,i) to make 1st column of C = 0

$$A = \underbrace{\left(\begin{array}{c|c} 1 & \\ 0 & ? \\ \vdots & \\ 0 & \end{array} \right)}_{n} \Big\}{n}$$

= SWAP(i,j): Swaps ith, jth rows in each top & bottom half

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
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= Q(i): Adds ith row of bottom half to ith of top half

On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT(1,i) to make rest of 1st column of A = 0
3.
4. Use R(1) & Controlled-Z(1,i) to make 1st column of C = 0

$A = \underbrace{\left(\begin{array}{c|c} 1 & ? \\ 0 & \vdots \\ \vdots & \\ 0 & \end{array}\right)}_{n} \Big\} n$

= SWAP(i,j): swaps ith, jth rows in both top & bottom half

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On right: row $\to$ column, top $\to$ right, bottom $\to$ left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT$(1,i)$ to make rest of 1st column of A $= 0$
3. Use CNOT on right to eliminate 1st row of A
4. Use $R(1)$ & Controlled-$Z(1,i)$ to make 1st column of $C = 0$

$$A = \underbrace{\left(\begin{array}{c|c} 1 & ? \\ 0 & \\ \vdots & \vdots \\ 0 & \end{array}\right)}_{n} \Big\}n$$

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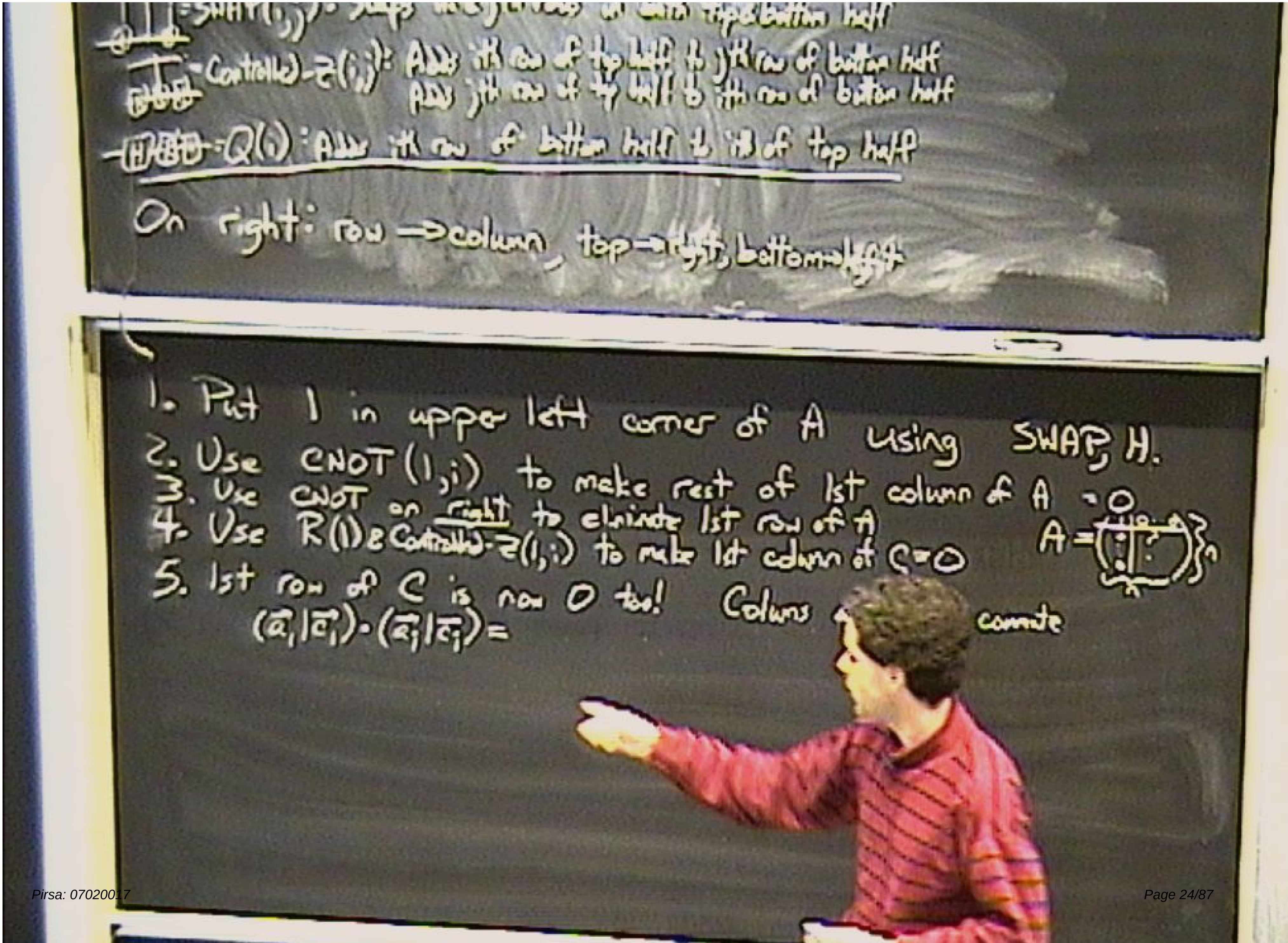
2. Use CNOT(1,i) to make rest of 1st column of A = 0
3. Use CNOT on right to eliminate 1st row of A
4. Use R(1) & Controlled-Z(1,i) to make 1st column of C = 0
5. 1st row of C is now 0 too!

$A = \underbrace{\begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & \\ 0 & \end{pmatrix}}_{n} \Big\} n$



Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
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commute

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On right: row $\Rightarrow$ column, top $\Rightarrow$ right, bottom $\Rightarrow$ left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT$(1,i)$ to make rest of 1st column of $A = 0$
3. Use CNOT on right to eliminate 1st row of A
4. Use $R(1)$ & Controlled-$Z(1,i)$ to make 1st column of $C = 0$
5. 1st row of C is now 0 too! Columns on left commute

$$(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_j|\vec{c}_j) = c_{1j} = 0$$

$$A = \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & ? \end{array}\right)}_{n}\Big\}n \qquad C = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & ? \end{array}\right)$$

= SWAP(i,j): swaps ... in each top/bottom half

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
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On right: row → column, top → right, bottom → left

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2. Use CNOT(1,i) to make rest of 1st column of A = 0
3. Use CNOT on right to eliminate 1st row of A
4. Use R(1) & Controlled-Z(1,i) to make 1st column of C = 0
5. 1st row of C is now 0 too! Columns on left commute

$(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_j|\vec{c}_j) = c_{1j} = 0$

$A = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & ? \end{array}\right)\}n$

$C = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & ? \end{array}\right)\}n$

SWAP(i,j): swaps ith & jth rows in both top & bottom half

Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
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Q(i): Adds ith row of bottom half to ith of top half

On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT(1,i) to make rest of 1st column of A = 0
3. Use CNOT on right to eliminate 1st row of A
4. Use R(1) & Controlled-Z(1,i) to make 1st column of C = 0
5. 1st row of C is now 0 too! Columns on left commute

$$(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_i|\vec{c}_i) = c_{1i} = 0$$

6. Repeat 1-5 with 2nd row & column, etc. to get $A = I$, $C = 0$

$A = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & ? \end{pmatrix}}_{n}\Big\}n$

$C = \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & ? \end{pmatrix}}_{n}\Big\}n$

= SWAP(i,j): Swaps ith & jth rows in both top & bottom half

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4. Use R(1) & Controlled-Z(1,i) to make 1st column of C = 0
5. 1st row of C is now 0 too! Columns on left commute

$$(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_j|\vec{c}_j) = c_{1j} = 0$$

$$A = \underbrace{\left(\begin{array}{c|c} 1 & 0 \\ \hline \vdots & \vdots \end{array}\right)}_{n}\Big\}n \qquad C = \underbrace{\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & ? \\ \vdots & \end{array}\right)}_{n}\Big\}n$$

6. Repeat 1-5 with 2nd row & column, etc. to get $A = I$, $C = 0$
7.

= SWAP(i,j): swaps ith & jth rows in both top & bottom half

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

= Q(i): Adds ith row of bottom half to ith of top half

On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT(1,i) to make rest of 1st column of A = 0
3. Use CNOT on right to eliminate 1st row of A
4. Use R(1) & Controlled-Z(1,i) to make 1st column of C = 0
5. 1st row of C is now 0 too! Columns on left commute

$(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_j|\vec{c}_j) = c_{1j} = 0$

6. Repeat 1-5 with 2nd row & column, etc. to get $A = I$, $C = 0$
7. Use Q & Controlled-Z on R

Thm.: $Sp(2n,\mathbb{Z}_2)$ is generated by H, R, CNOT. $U = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \in Sp(2n)$

On Left: CNOT(i,j): Add the ith row in top half to jth row in top half
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H(i): Switches ith row from top half with ith row in bottom half

R(i): Adds ith row from top half to ith row in bottom half

= SWAP(i,j): Swaps ith & jth rows in both top & bottom half

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right)$$

= Q(i): Adds ith row of bottom half to ith of top half

On right: row → column, top → right, bottom → left

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

= $Q(i)$: Adds ith row of bottom half to ith of top half

On right: row $\to$ column, top $\to$ right, bottom $\to$ left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT$(1,i)$ to make rest of 1st column of A = 0
3. Use CNOT on right to eliminate 1st row of A
4. Use $R(1)$ & Controlled-Z$(1,i)$ to make 1st column of $C = 0$
5. 1st row of C is now 0 too! Columns on left commute
   $(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_j|\vec{c}_j) = c_{1j} = 0$
6. Repeat 1–5 with 2nd row & column, etc. to get $A = I$, $C = 0$
7. Use $Q^{(new)}$ & (Controlled-Z)$^{(new)}$ on R

Controlled-$Z(i,j)$: Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

$-H-$ $= Q(i)$: Adds ith row of bottom half to ith of top half

On right: row $\rightarrow$ column, top $\rightarrow$ right, bottom $\rightarrow$ left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT$(1,i)$ to make rest of 1st column of A = 0
3. Use CNOT on right to eliminate 1st row of A
4. Use $R(1)$ & Controlled-$Z(1,i)$ to make 1st column of $C = 0$
5. 1st row of C is now 0 too! Columns on ... te
   $(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_i|\vec{c}_i) = c_{1i} = 0$
6. Repeat 1-5 with 2nd row & column, etc. to get $A = I$
7. Use $Q^{(new)}$ & (Controlled-$Z)^{(new)}$ on R make $B = 0$
8. This forces $D = I$: $(\vec{a}_i|\vec{c}_i)\cdot(\vec{b}_j|\vec{d}_j) =$

$A = \left(\begin{array}{c|c} 1 & 0 \\ 0 & ? \end{array}\right)\Big\}n$

$C = \left(\begin{array}{c|c} 0 & 0 \\ 0 & ? \end{array}\right)\Big\}n$

Controlled-$Z(i,j)$: Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

$H = Q(i)$: Adds ith row of bottom half to ith of top half

On right: row $\rightarrow$ column, top $\rightarrow$ right, bottom $\rightarrow$ left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT$(1,i)$ to make rest of 1st column of $A = 0$
3. Use CNOT on right to eliminate 1st row of A
4. Use $R(1)$ & Controlled-$Z(1,i)$ to make 1st column of $C = 0$
5. 1st row of C is now 0 too! Columns on left commute

   $(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_i|\vec{c}_i) = c_{1i} = 0$
6. Repeat 1-5 with 2nd row & column, etc. to get $A = I$, $C = 0$
7. Use $Q^{(row)}$ & (Controlled-$Z)^{(row)}$ on R make $B = 0$
8. This forces $D = I$: $(\vec{a}_i|\vec{c}_i)\cdot(\vec{b}_j|\vec{d}_j) = \delta_{ij} = d_{ij}$ $\Rightarrow U = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right)$

$A = \underbrace{\left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline 0 & \\ \vdots & ? \\ 0 & \end{array}\right)}_{n} \Big\} n$

$C = \underbrace{\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & ? \end{array}\right)}_{n-1} \Big\} n-1$

On Left: CNOT(i,j): Add the ith row in top half to jth row in top half
Add the jth row in bottom half to ith row in bottom half

H(i): Switches ith row from top half with ith row in bottom half

R(i): Adds ith row from top half to ith row in bottom half

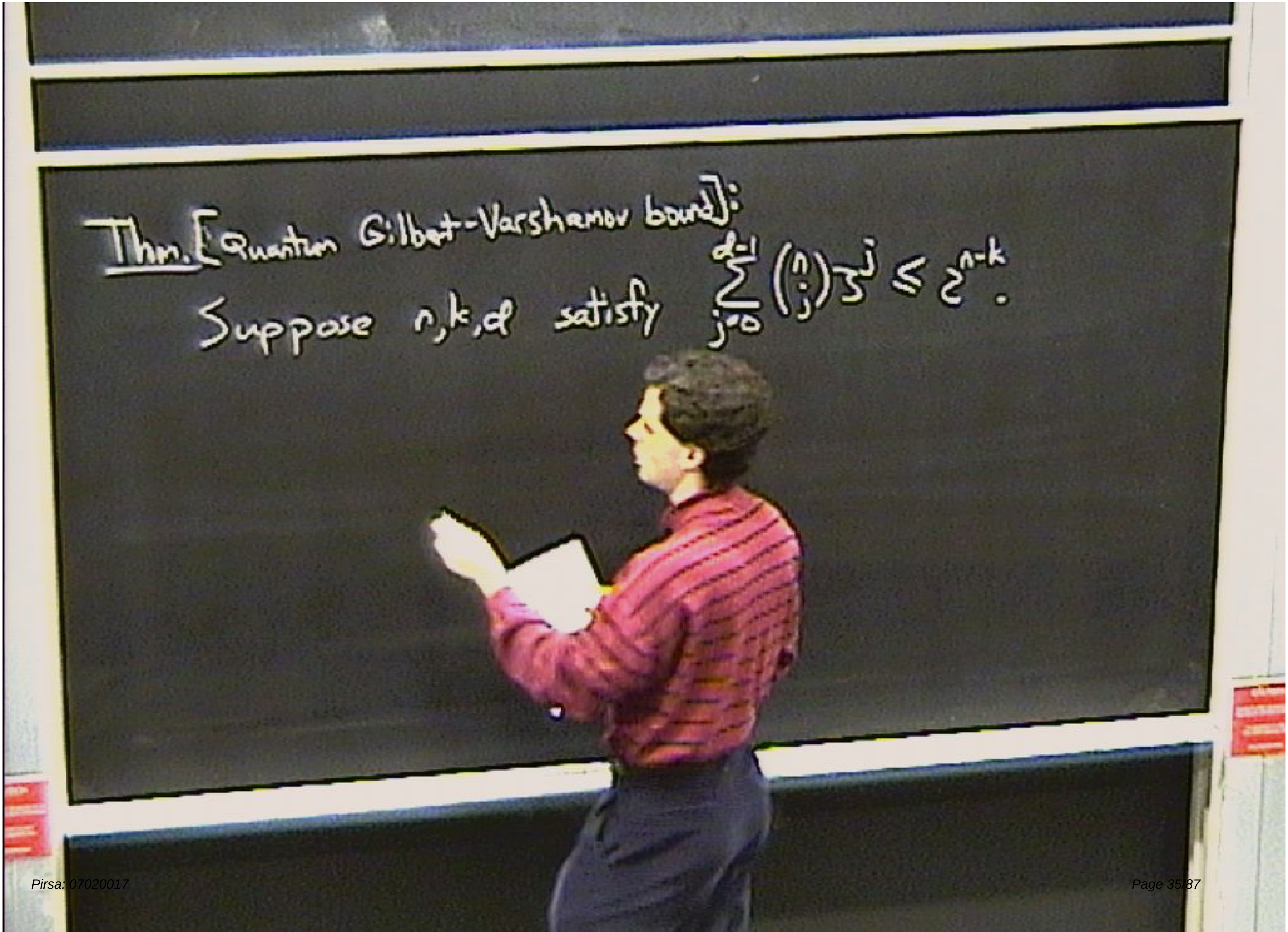
= SWAP(i,j): Swaps the i,jth rows in both top & bottom half

= Controlled-Z(i,j): Adds ith row of top half to jth row of bottom half
Adds jth row of top half to ith row of bottom half

= Q(i): Adds ith row of bottom half to ith of top half

On right: row → column, top → right, bottom → left

1. Put 1 in upper left corner of A using SWAP, H.
2. Use CNOT$(1,i)$ to make rest of 1st column of A = 0
3. Use CNOT on right to eliminate 1st row of A
4. Use R(1) & Controlled-Z$(1,i)$ to make 1st column of $C=0$
5. 1st row of C is now 0 too! Columns on left commute
$(\vec{a}_1|\vec{c}_1)\cdot(\vec{a}_j|\vec{c}_j) = c_{1j} = 0$
6. Repeat 1-5 with 2nd row/column, etc. to get $A=I$, $C=0$
7. Use Q (row) & Controlled-Z (pair) on R make $B=0$
8. This forces $D=I$: $(\vec{a}_i|\vec{c}_i)(\vec{b}_j|\vec{d}_j) = \delta_{ij} = d_{ij}$ $\Rightarrow U = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right)$


Thm. [Quantum Gilbert-Varshamov bound]:
Suppose n,k,d satisfy $\sum_{j=0}^{d-1} \binom{n}{j} 3^j \leq 2^{n-k}$.

Thm. [Quantum Gilbert-Varshamov bound]:

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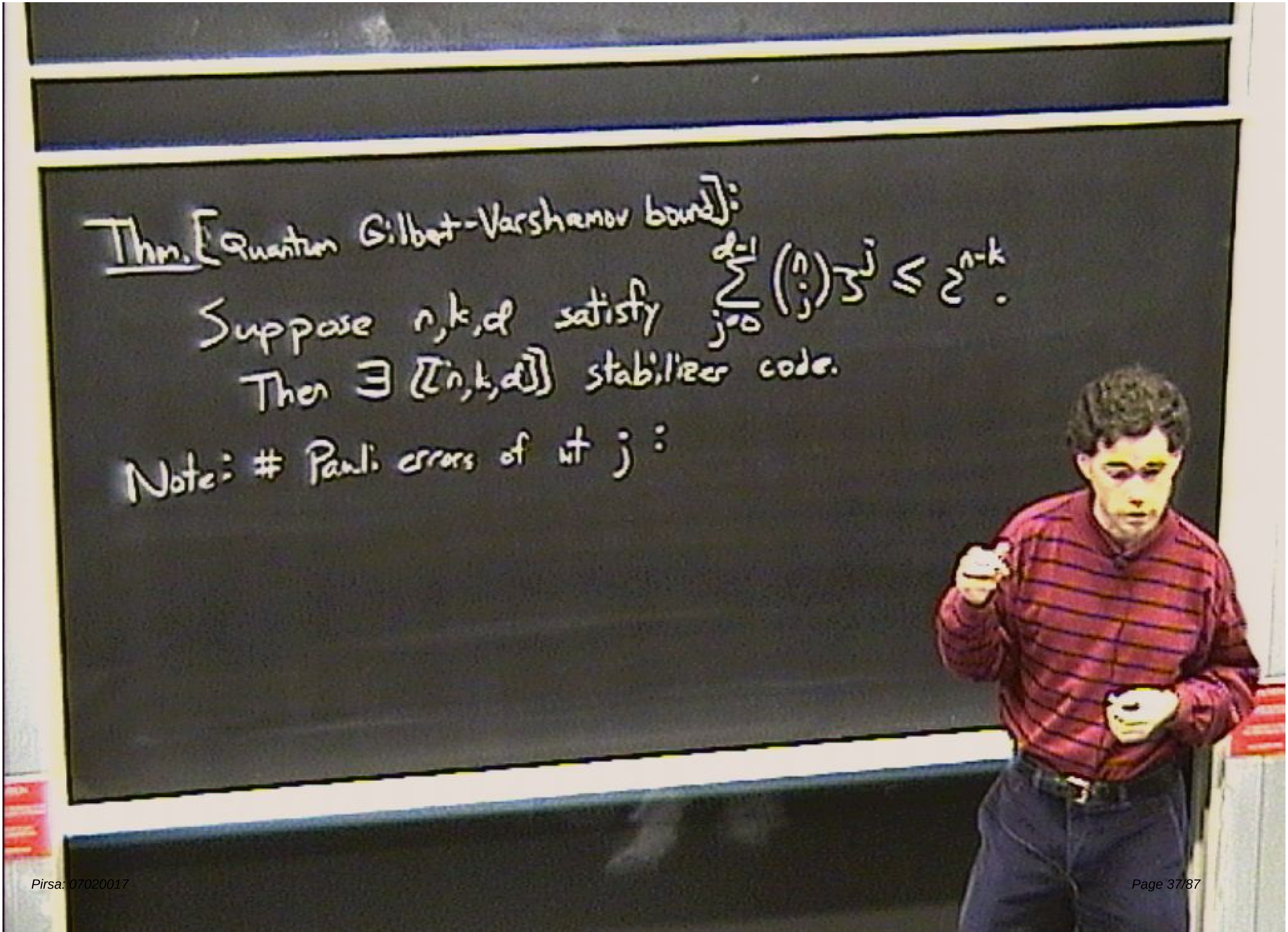
Then $\exists$ $[[n,k,d]]$ stabilizer code.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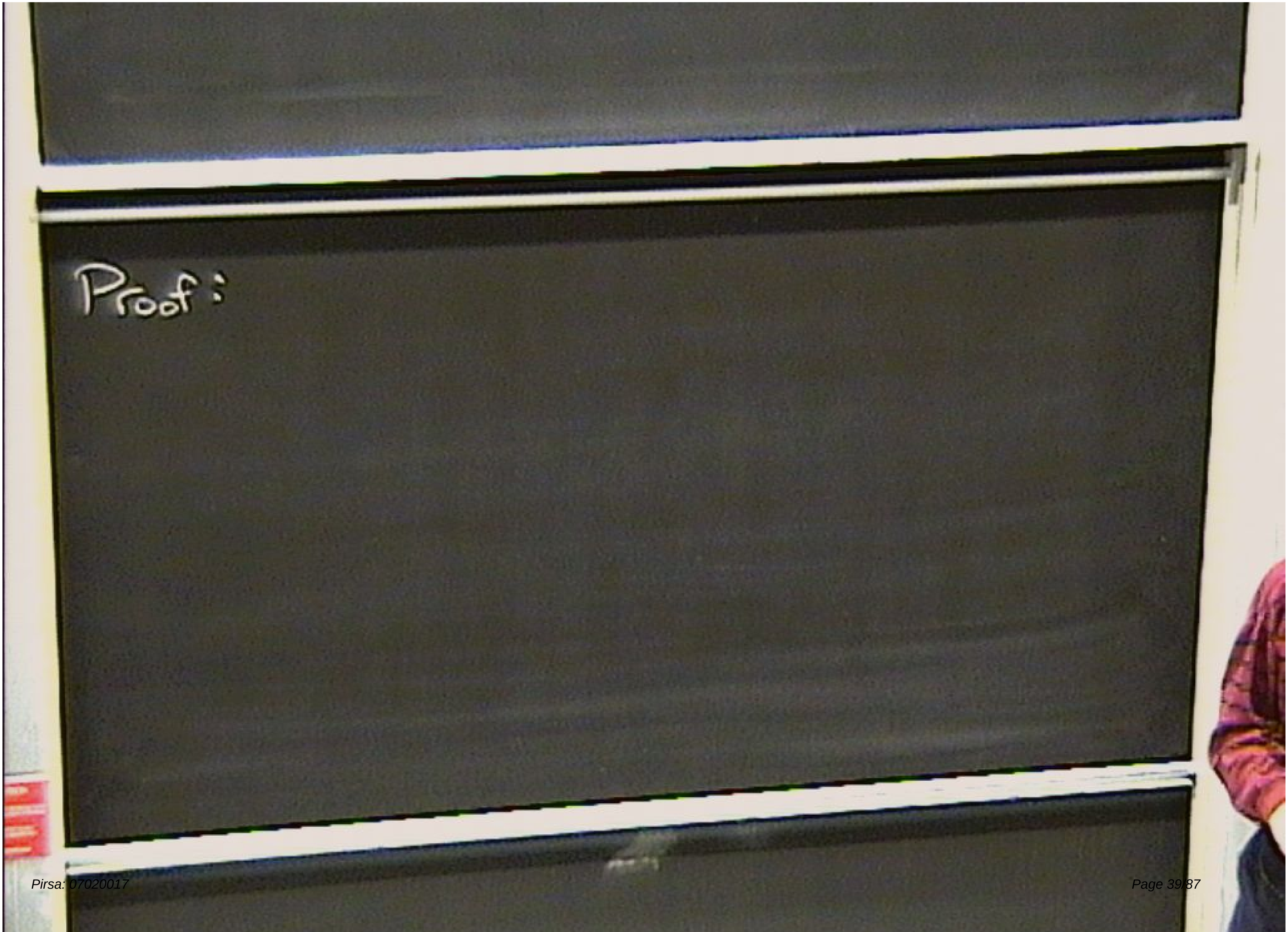
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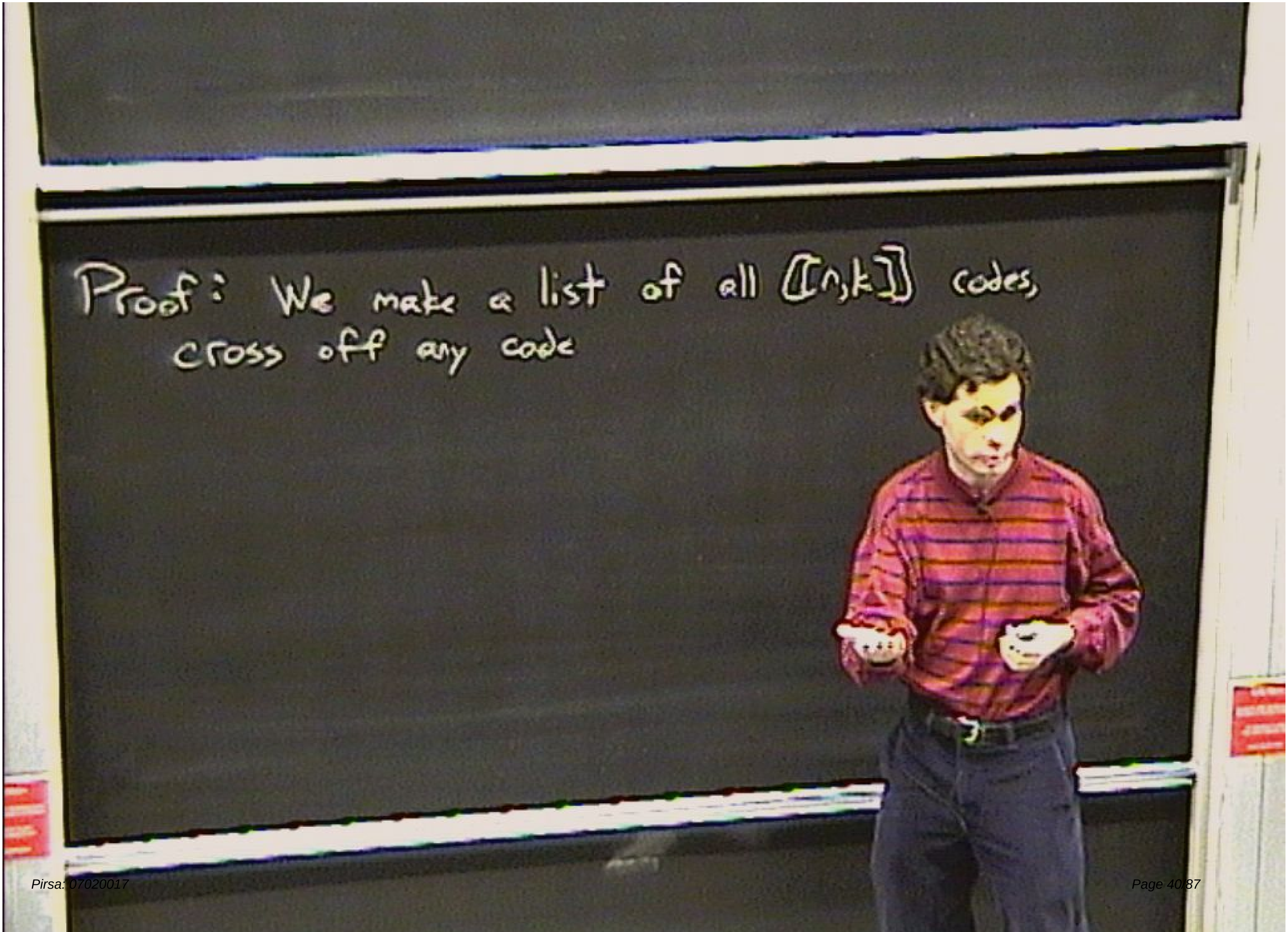
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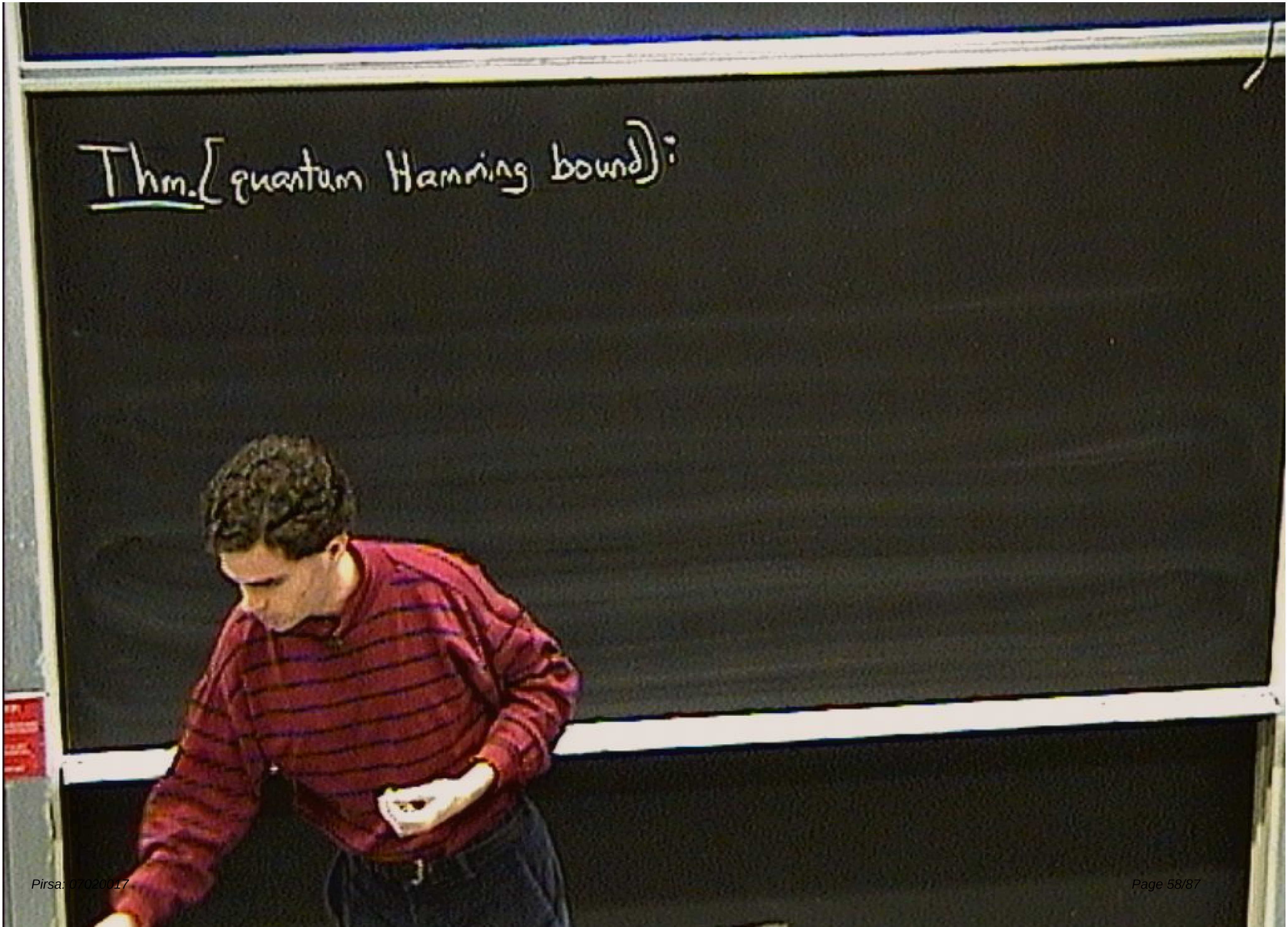
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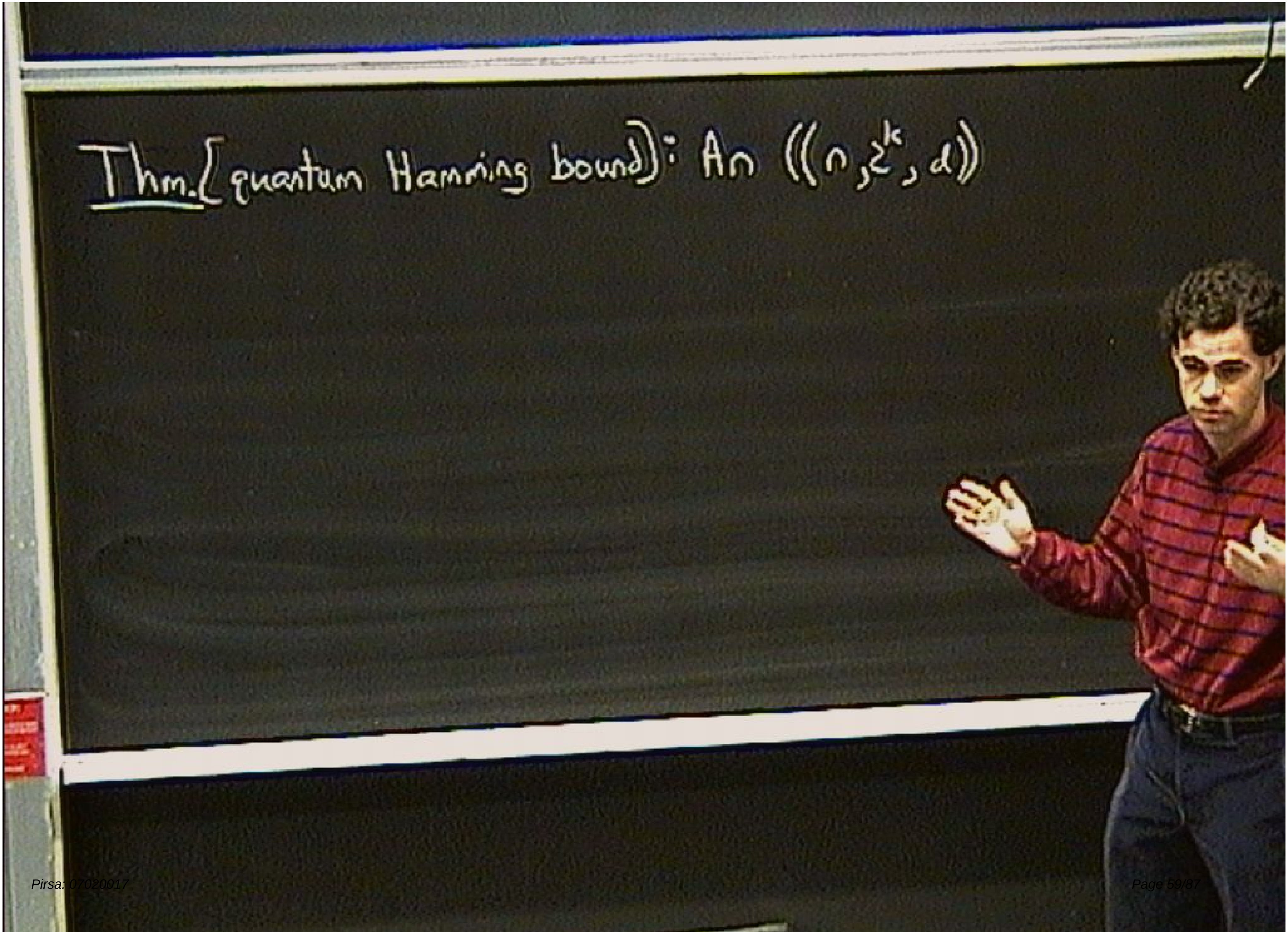
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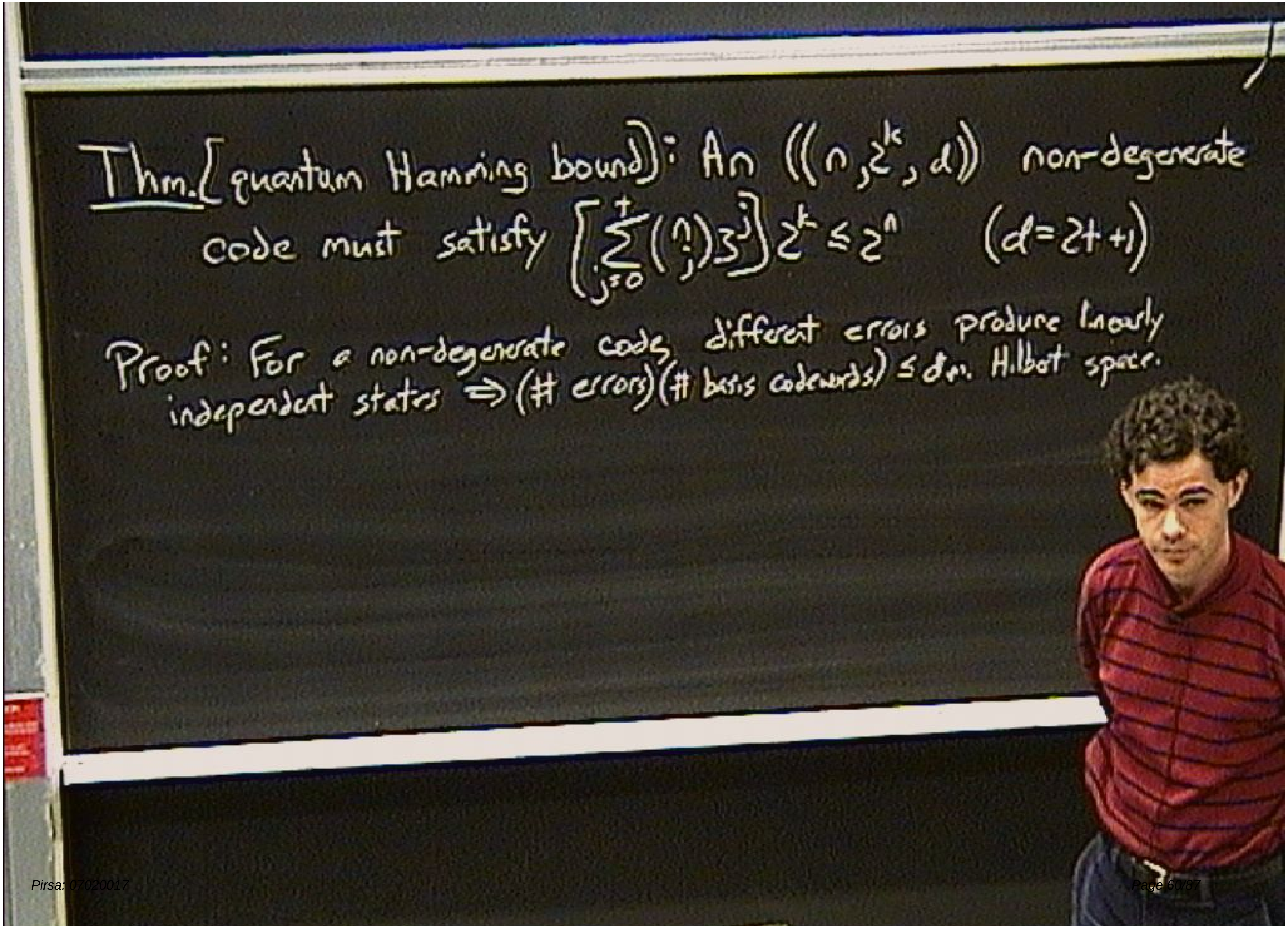
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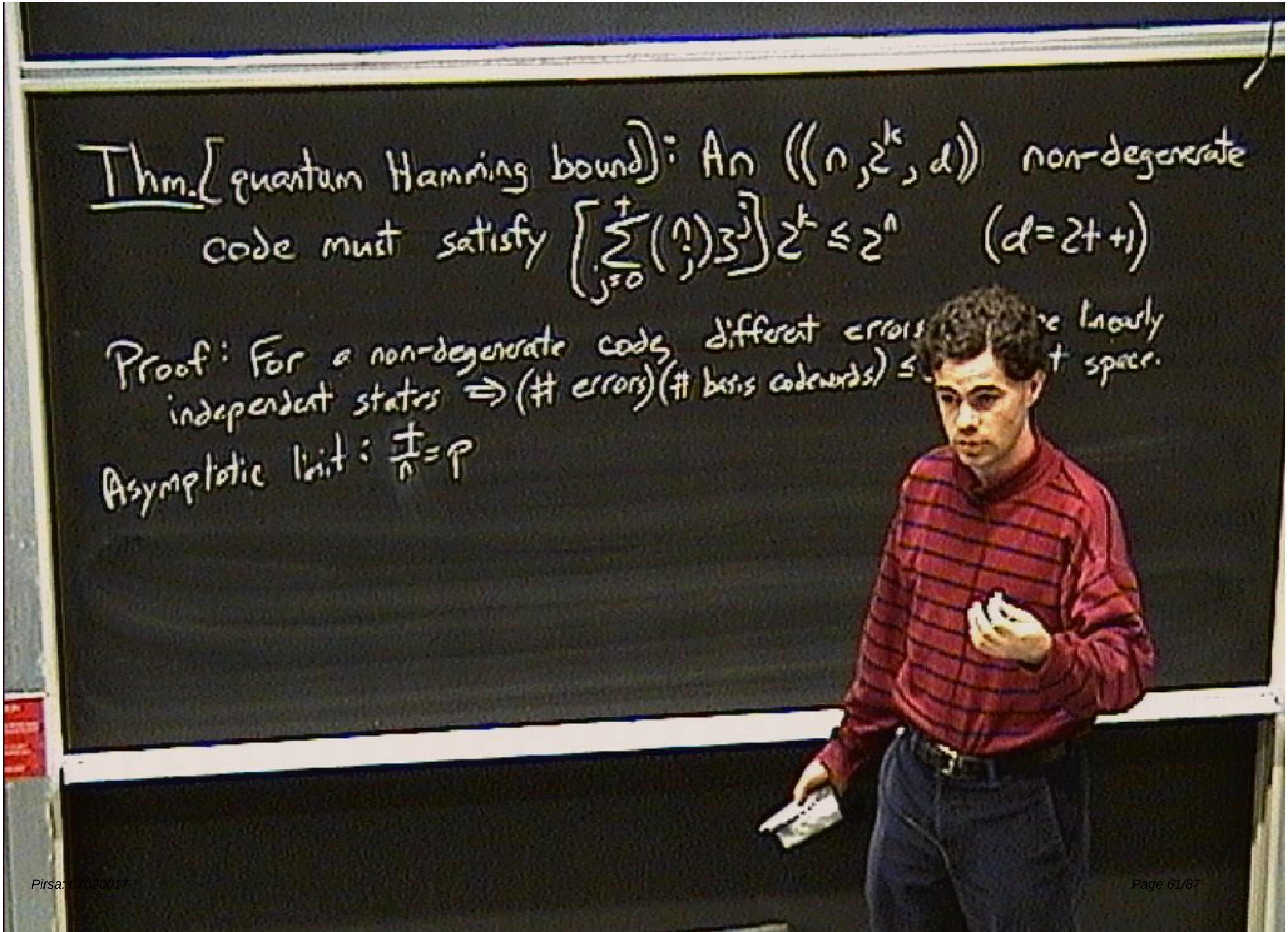
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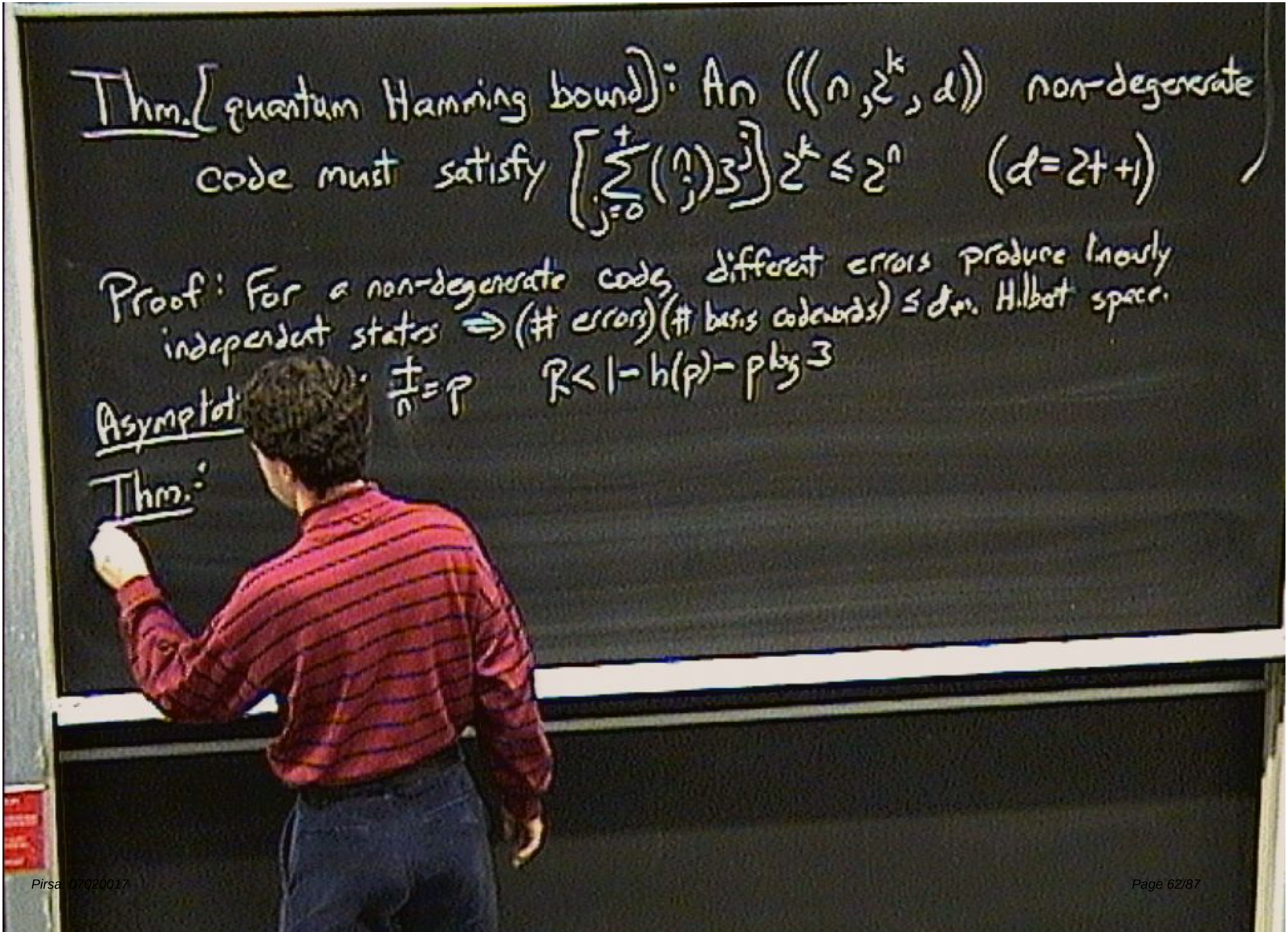
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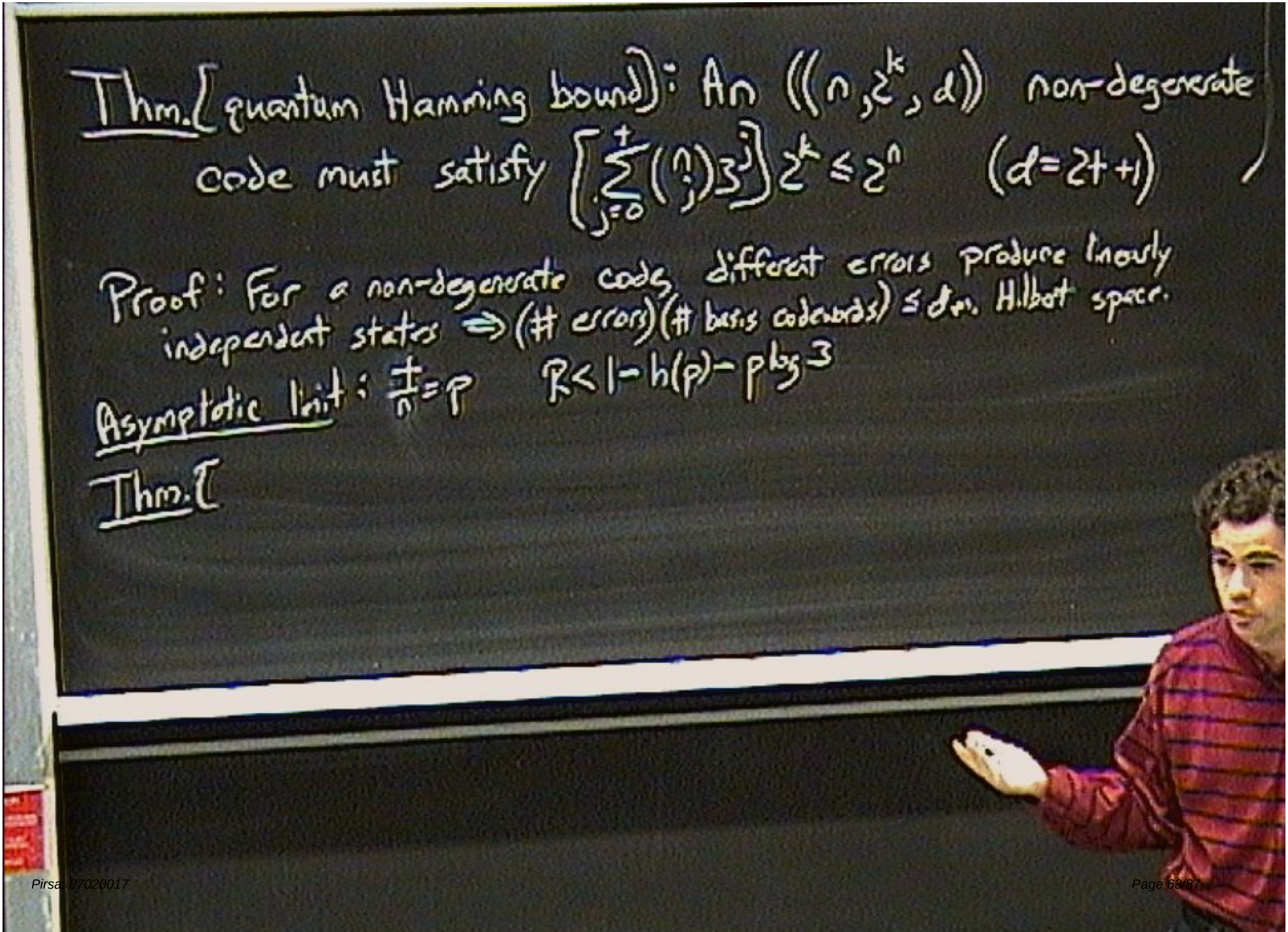
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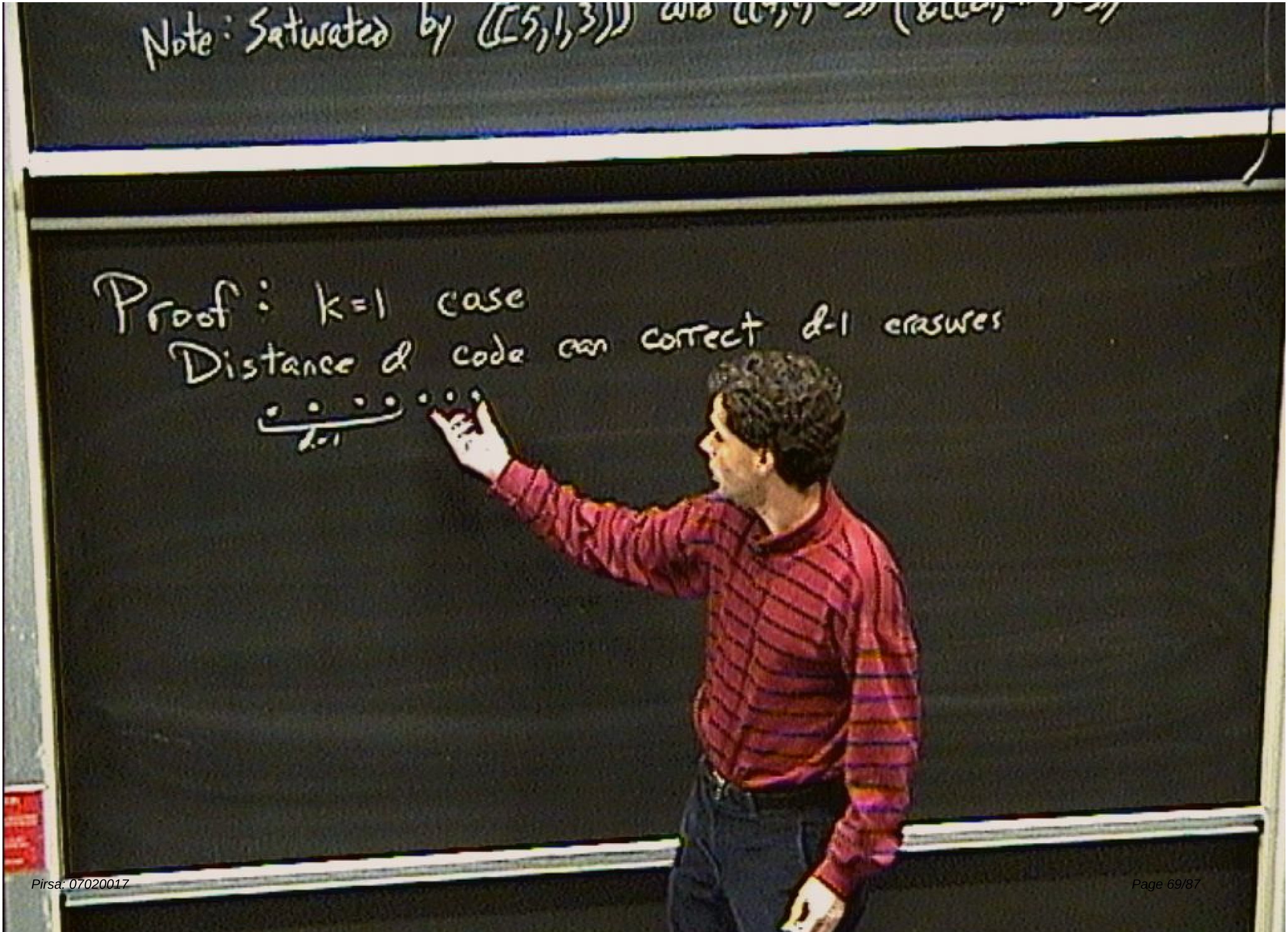
Distance d code can correct $d-1$ erasures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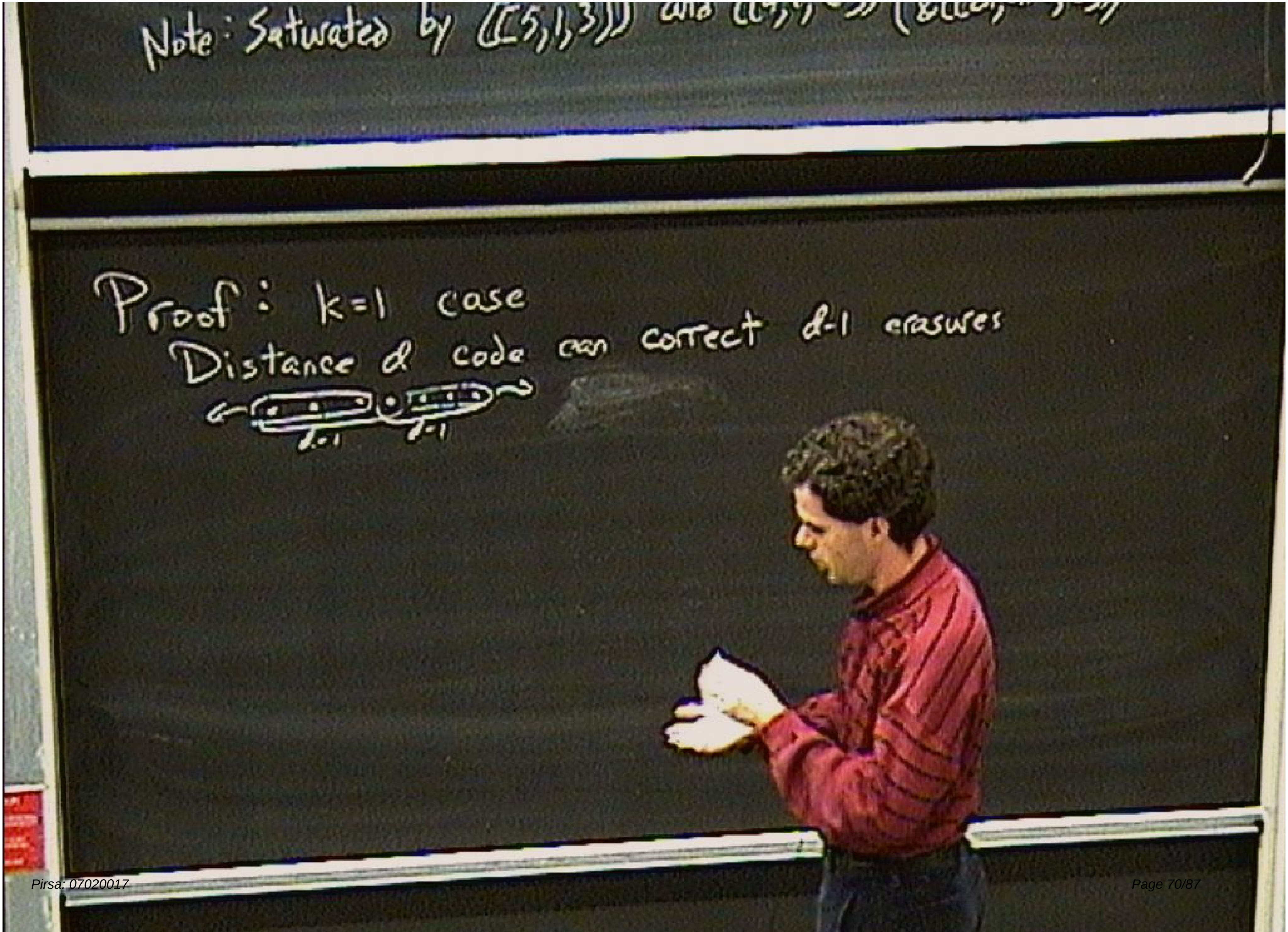
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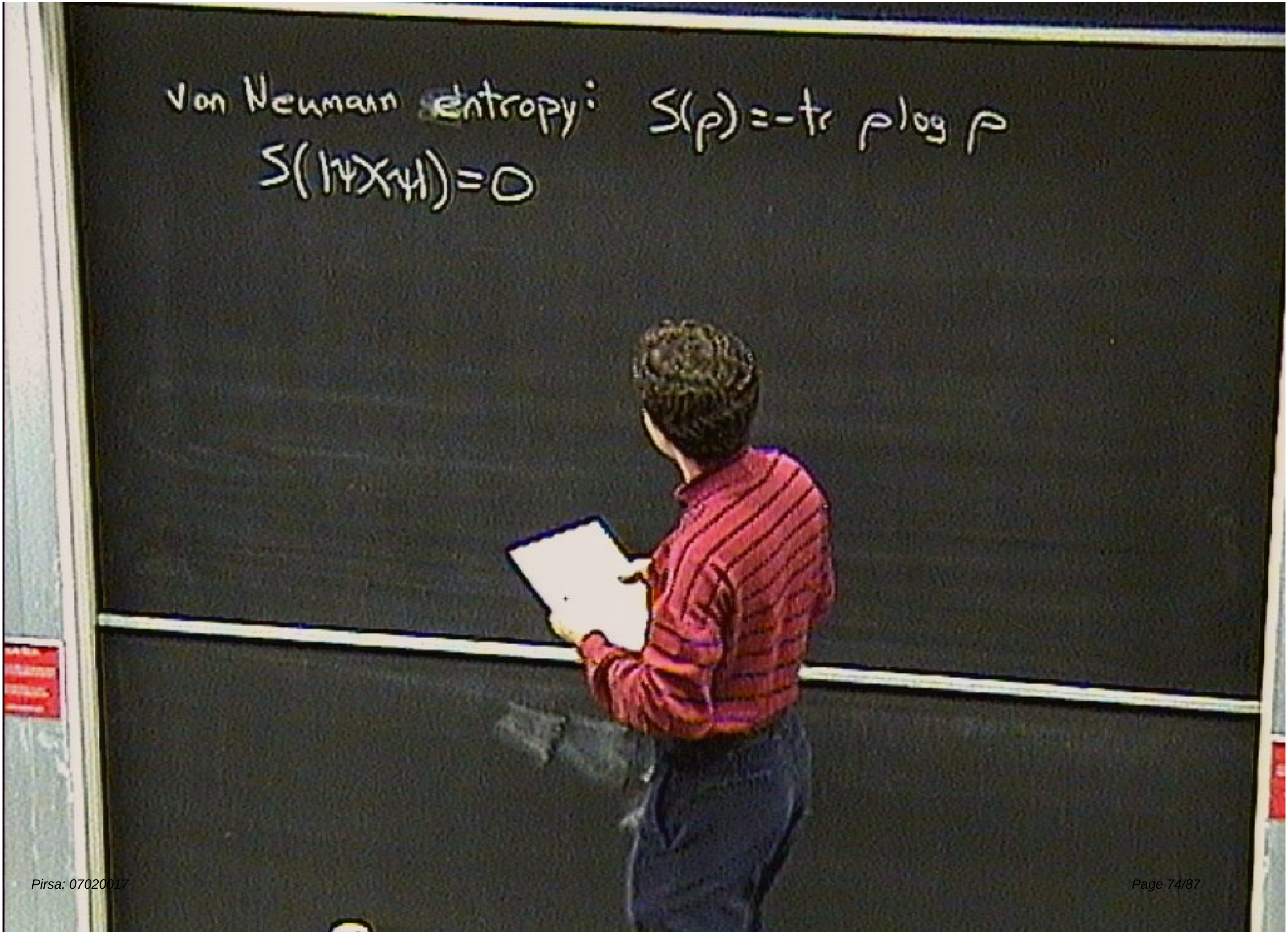
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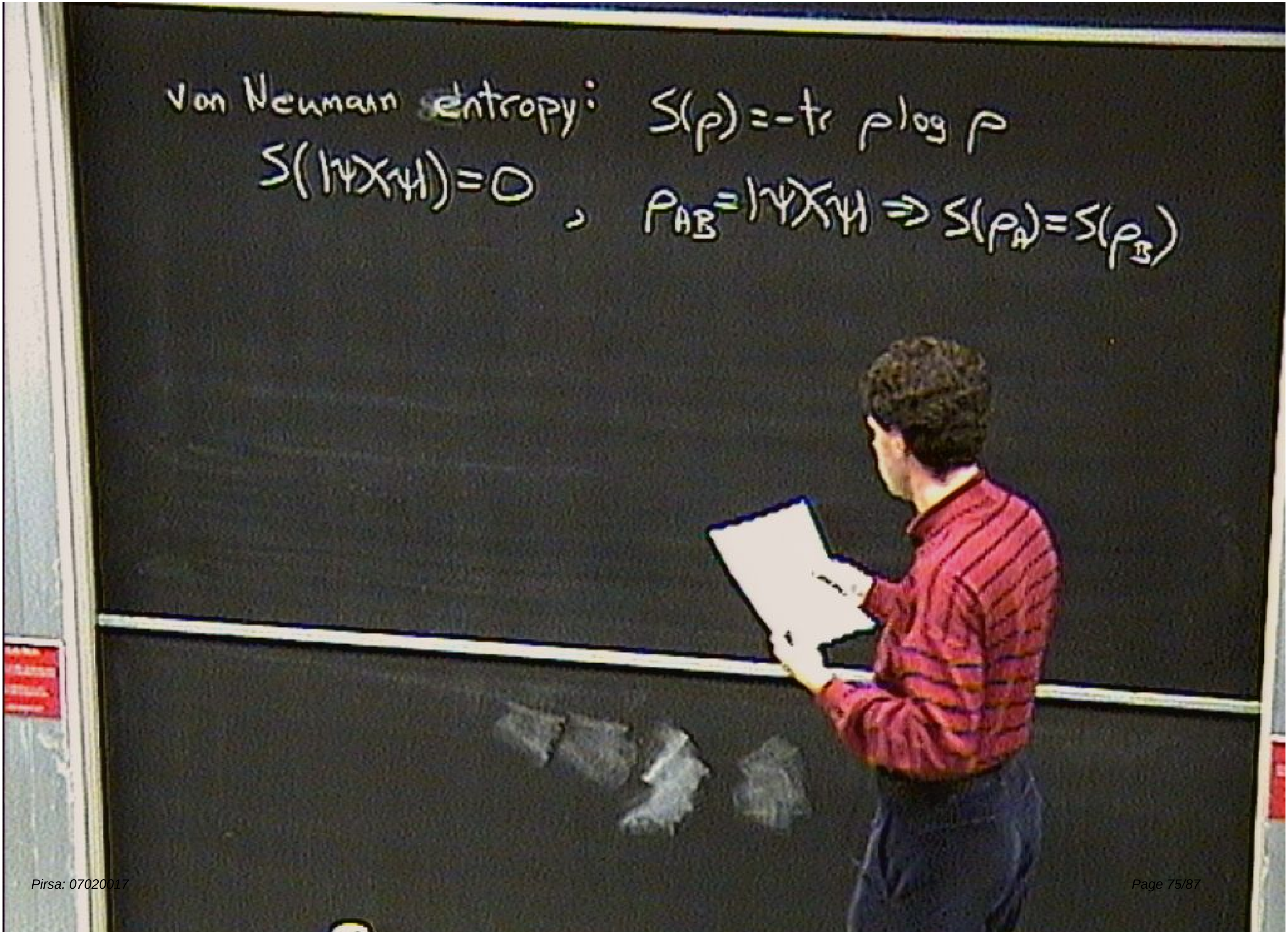
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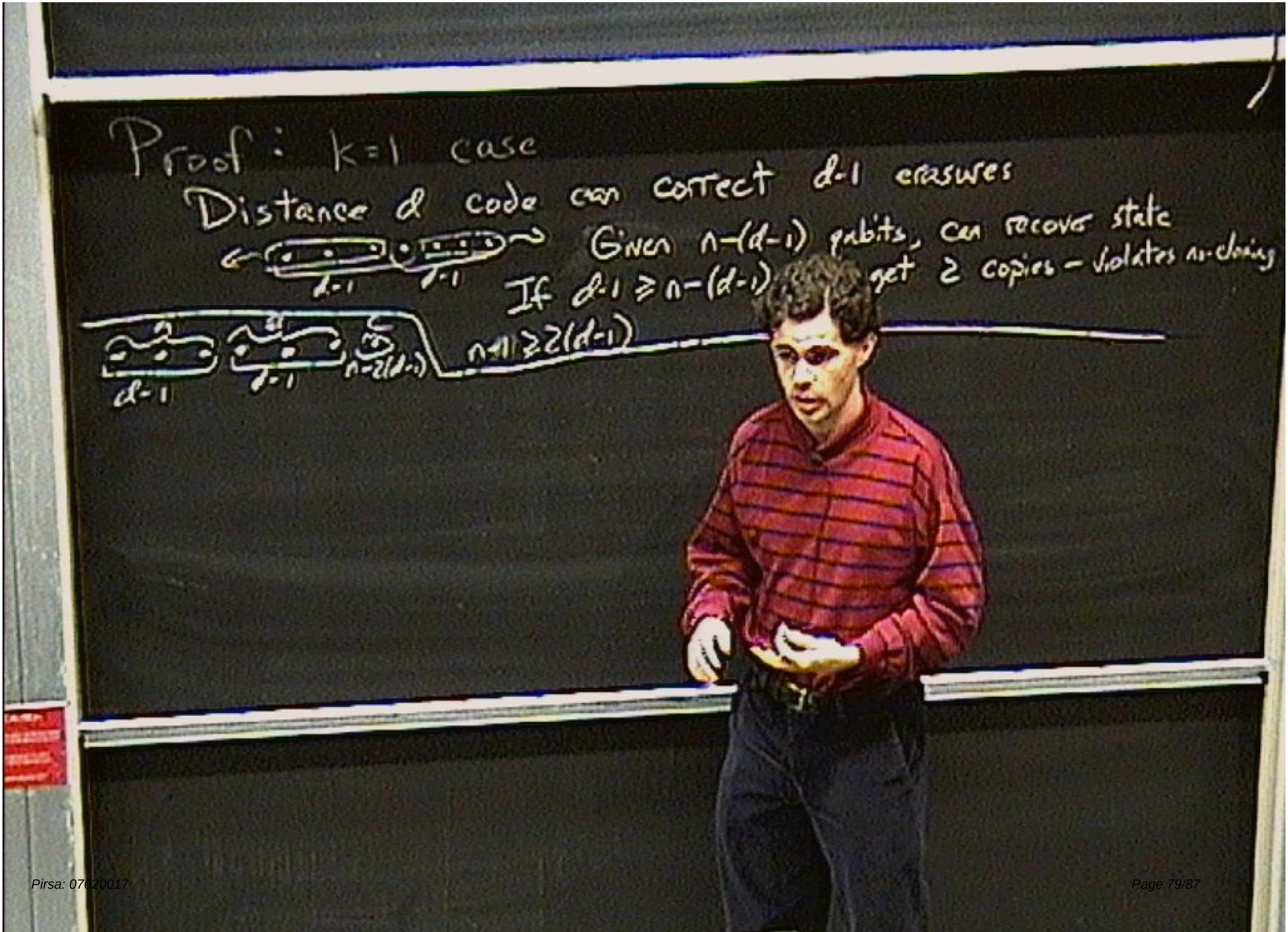
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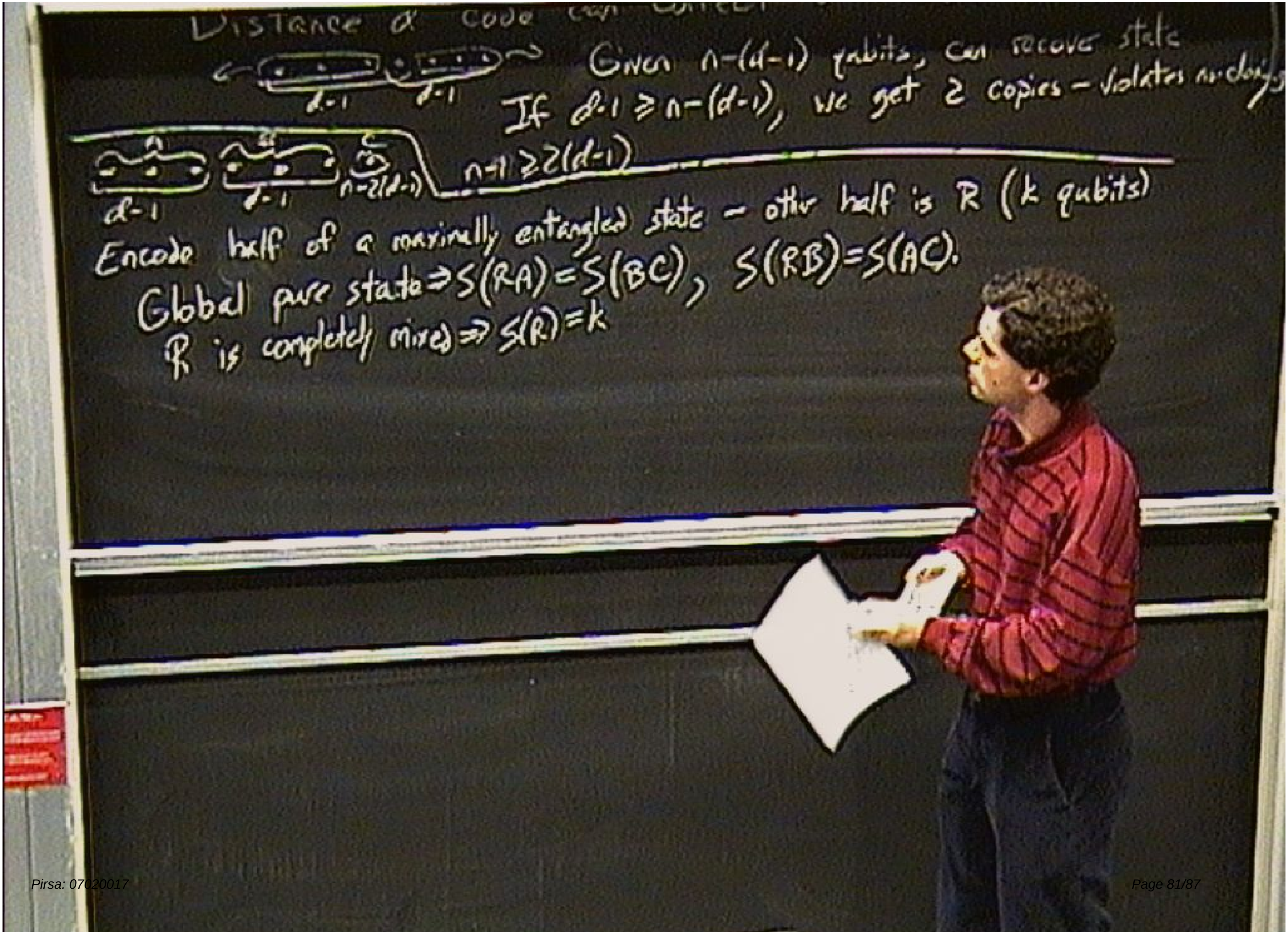
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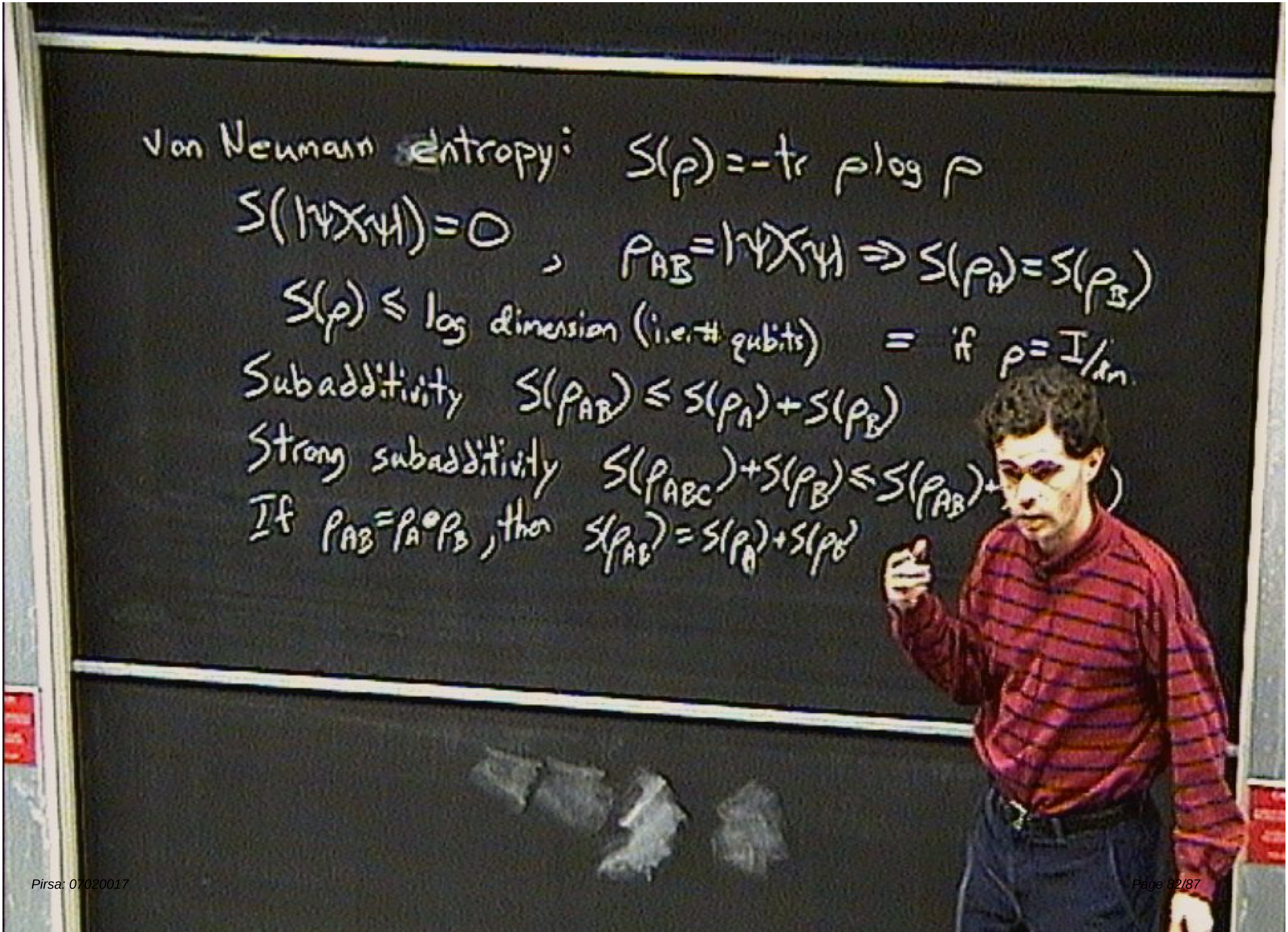
Global pure state ⇒ S(RA)=S(BC), S(RB)=S(AC).
R is completely mixed ⇒ S(R)=k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Subadditivity $S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B)$

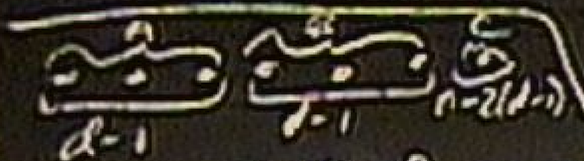
Strong subadditivity $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$

If $\rho_{AB} = \rho_A \otimes \rho_B$, then $S(\rho_{AB}) = S(\rho_A) + S(\rho_B)$

Distance d code can correct

Given $n-(d-1)$ qubits, can recover state

If $d-1 \geq n-(d-1)$, we get 2 copies – violates no-cloning



$n-1 \geq 2(d-1)$

Encode half of a maximally entangled state – other half is R (k qubits)

Global pure state $\Rightarrow S(RA)=S(BC)$, $S(RB)=S(AC)$.

R is completely mixed $\Rightarrow S(R)=k$

Can correct $d-1$ erasures $\Rightarrow$ A & R are tensor product $S(AR)=S(A)+S(R)$

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Given $n-(d-1)$ qubits, can recover state

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$d-1$ $\quad$ $d-1$ $\quad$ $n-2(d-1)$ $\quad$ $n-1 \geq 2(d-1)$

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By subadditivity, $S(A)+S(R)=S(RA)=S(BC)\leq S(B)+S(C)$

Distance d code can correct $d-1$ erasures

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Can correct $d-1$ erasures $\Rightarrow$ A & R are tensor product $S(AR)=S(A)+S(R)$

By subadditivity, $S(A)+S(R)=S(RA)=S(BC)\leq S(B)+S(C) \Rightarrow k \leq S(C)+[S(B)-S(A)]$

Similarly, $k \leq S(C)+[S(A)-S(B)]$

Distance d code can correct ...

Given $n-(d-1)$ qubits, can recover state

If $d-1 \geq n-(d-1)$, we get 2 copies – violates no-cloning

A, B, C; $d-1$, $d-1$, $n-2(d-1)$

$n-1 \geq 2(d-1)$

Encode half of a maximally entangled state – other half is R (k qubits)

Global pure state $\Rightarrow S(RA)=S(BC)$, $S(RB)=S(AC)$.

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Can correct $d-1$ erasures $\Rightarrow$ A & R are tensor product $S(AR)=S(A)+S(R)$

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Similarly, $k \leq S(C)+[S(A)-S(B)] \Rightarrow \boxed{k \leq S(C) \leq n-2(d-1)}$