Title: An extended, quartic quantum theory and a generalised theory of quantum information processing

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Abstract: We propose an extended quantum theory, in which the number of degrees of freedom K behaves as FOURTH power the number N of distinguishable states. As the simplex of classical N--point probability distributions can be embedded inside a higher dimensional convex body of mixed quantum states, one can further increase the dimensionality constructing the set of extended quantum states. The embedding proposed corresponds to an assumption that the physical system described in N dimensional Hilbert space is coupled with an auxiliary subsystem of the same dimensionality. The extended theory is shown to be a non-trivial generalisation of the standard quantum theory for which K=N^2. Imposing certain restrictions on initial conditions and dynamics allowed in the quartic theory one obtains quadratic theory as a special case. We discuss the question, how the theory of information processing looks like in the framework of the generalised quantum theory. In particular we propose a scheme of extended dense coding, in which one transmits two qubits by sending one extended bit, provided it was initially entangled with the extended bit of the receiver.

A generalised, quartic Quantum Mechanics and a generalised theory of quantum information

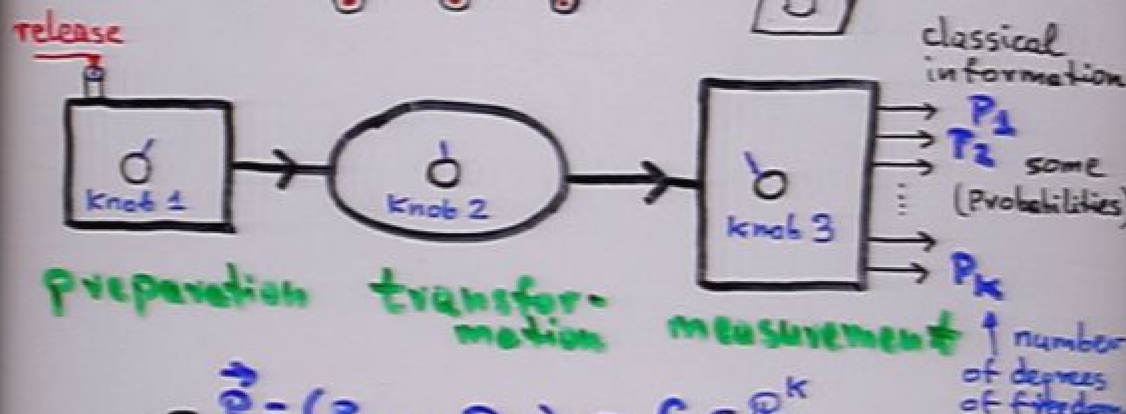
Karol Życzkowski

Jagiellonian University (Cracow),

Polish Academy of Scienes (Warsaw)

General scheme:

3 Knobs + release button



$\vec{p} = (p_1, \dots, p_k) \in S \subset \mathbb{R}^k$ ← suitable set

state $\equiv$ **thing**, represented by any mathematical object, useful to calculate probability of any measurement: e.g. vector of probability, density matrix ρ.
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N - maximal number of ...

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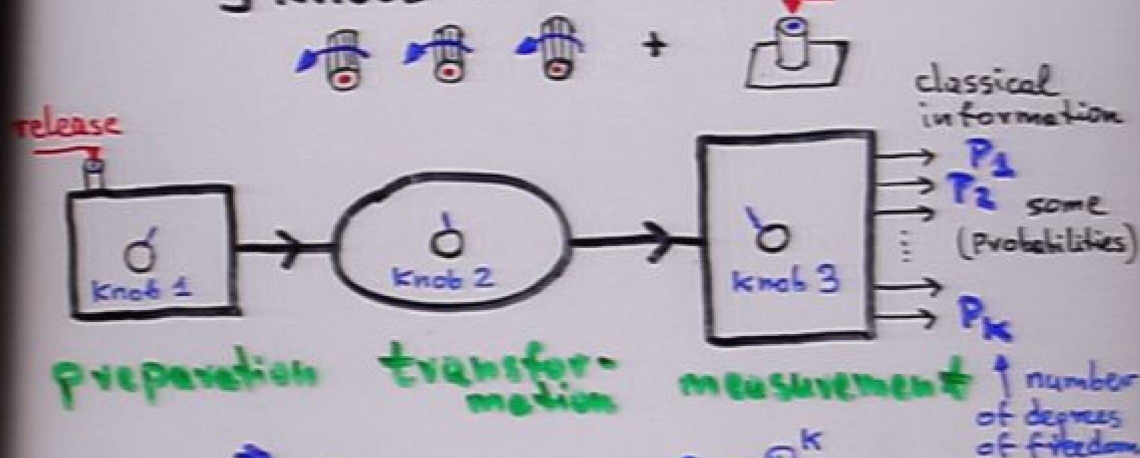

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N - maximal number of states that can be **distinguished** in a single-shot experiment.

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examples:

1. Classical theory, $K = N$, $\omega = (P_1 \cdots P_N)$, $\sum_{i=1}^{N} P_i \le 1$ (unnormalised vector)

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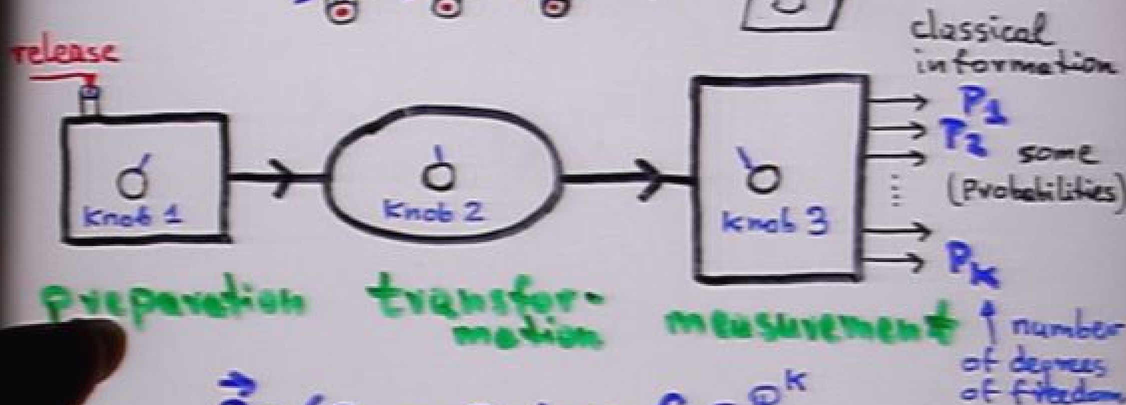

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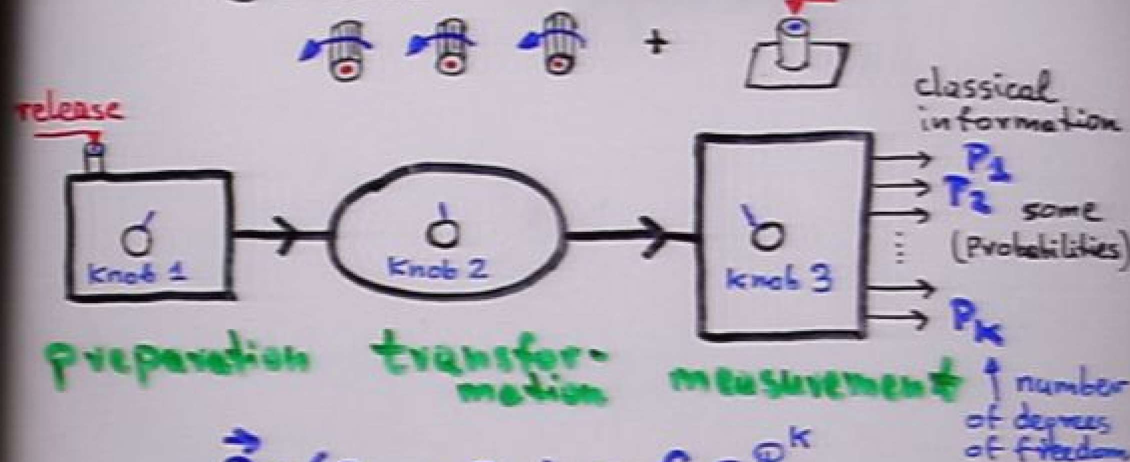

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Quantum Theory from $\overline{V}$ reasonable axioms [5]

Lucien Hardy quant-ph/0101012
(2001) quant-ph/0111068

Axioms of Hardy:

1. **Probabilities.** Frequencies of measurements outcomes for an ensemble of n systems for a given set-up converge to the same value in the limit $n \to \infty$.

2. **Subspaces.** There exist systems for which $N = 1, 2, \ldots$ and all the systems with a fixed N have the same properties.

3. **Composite Systems.** For a system composed of subsystems A and B
$N = N_A \cdot N_B$ and $K = K_A \cdot K_B$ (Product rule)

4. **Continuity.** Between any two pure states of the system there exist a continuous, reversible transformation

5. **Simplicity.** For each number N the number K takes minimal value consistent with other axioms

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Some implications

2. Subspaces
3. Composite syst. $\Big\} \Rightarrow$

$$K(N+1) > K(N)$$
$$K(NM) = K(N) \cdot K(M)$$

(*) $K(N) = N^{r}$

[rules out quantum theory based on real density matrices, for which $K = \frac{N(N+1)}{2}$]

Special case

a) $r = 1$, $K = N$

Classical theory (linear)

N=3 probability simplex

4. Continuity –
rules out classical theory with $r = 1$!

No continuous transformation $|1\rangle \leftrightarrow |2\rangle$

5. Simplicity: implies

b) $r = 2$, $K = N^2$

Quantum theory (quadratic)

N=2

continuous transform.

$|2\rangle$ U

$|\psi\rangle = U|2\rangle = \cos\vartheta\,|2\rangle + \sin\vartheta\,|1\rangle$.

* pure state

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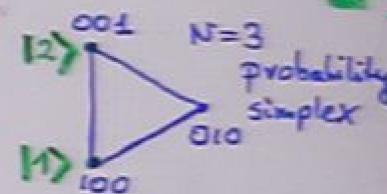

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example: Qubit + Bloch sphere

the classical states - mixtures

7

An attempt to explore theories beyond **Quantum Mechanics.**

Higher order theories

a) cubic, $r=3$, $K=N^3$, $\left\{ \begin{array}{l} \text{density} \\ \text{tensors} \end{array} \right.$ ς_{abc}

b) quartic, $r=4$, $K=N^4$ ς_{ijkl}

$\underbrace{\varsigma_{\alpha_1 \cdots \alpha_r}}_{r\text{- indeces}}$ – tensor-like objects, in some special cases they should reduce to density matrices ς_{mn} to give the standard QM.

possibilities:

i) higher tensor calculus
(delicate issue: e.g. – no eigendecomposition of a tensor
– even rank of a tensor is not defined uniquely...

ii) standard QM for composite systems with additional restrictions

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$\vec{p}=\{p_1 \cdots p_N\}$	$\varrho=\varrho^+$	$\sigma=\sigma^+$

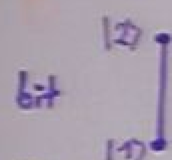
extension → $\varrho = \mathrm{diag}(\vec{p})$ → $\sigma = \varrho \otimes \frac{1}{N}\mathbb{1}$ ← ancilla ("ghost")

„decoherence" $\vec{p} = \mathrm{diag}(\varrho)$ ← $\varrho = \mathrm{tr}_B\, \sigma$ ←

„hyperdecoherence"

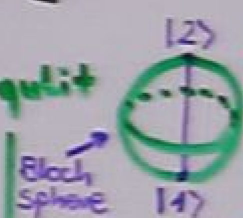
example: $N=2$

bit



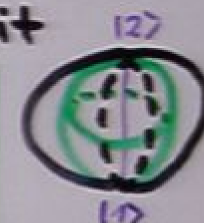
$\vec{p}=(p,1-p)\in\Delta_1\subset\mathbb{R}^1$

qubit



$\varrho\in M_2\subset\mathbb{R}^3$

extended bit



„super..."
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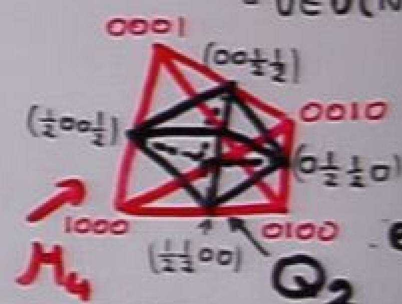
$\sigma\in Q_2\subset M_4\subset\mathbb{R}^{15}$

„exbit" $Q_N \equiv \mathrm{conv\,hull}\left[U\,|1\rangle\langle 1|\otimes \frac{\mathbb{1}}{N}\,U^+\right] =$

↑ $U\in U(N^2)$

= truncated M_{N^2} such that only N states are distinguishable



classical, $\vec p$ | quantum, ϱ | σ

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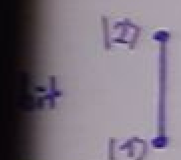

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$\varrho \in M_2 = \mathbb{R}^3$

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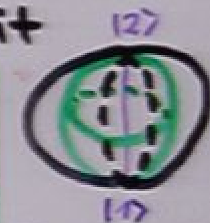

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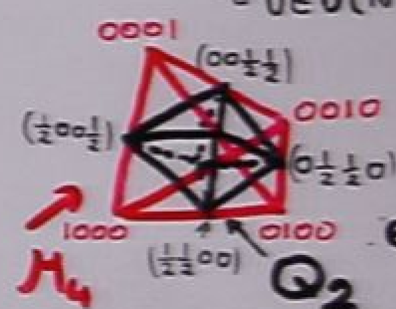

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Bloch sphere

$\varrho \in \mathcal{M}_2 \subset \mathbb{R}^3$

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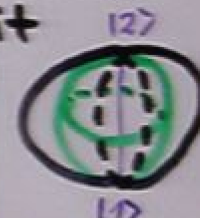

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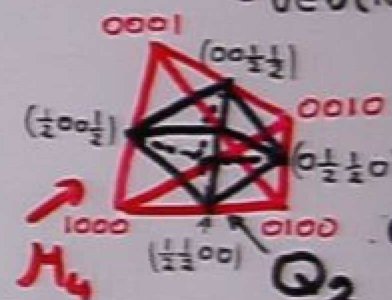

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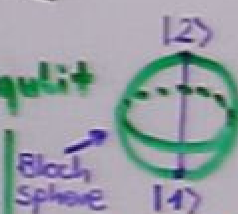

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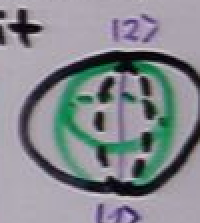

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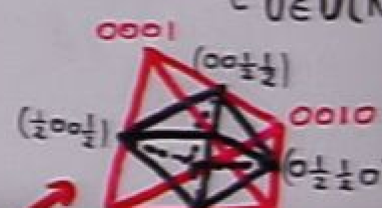

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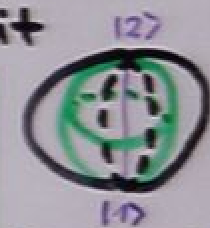

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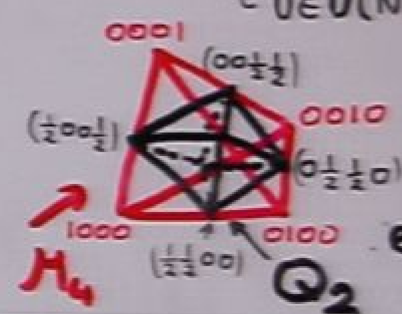


<u>Lemma</u> (D. Markham, KŻ)

Let $\{\varrho_i\}_{i=1}^{k}$ be a set of k mutually distinguishable states on $\mathcal{H}_D$.

Then $$\sum_{i=1}^{K} \mathrm{rank}(\varrho_i) \leq D$$

$\Downarrow$

set Q_N contains N distinguishable states, e.g.

$$\varrho_1 = \mathrm{diag}(\underbrace{\tfrac{1}{N} \cdots \tfrac{1}{N}}_{N}, \overbrace{0 \cdots \qquad 0}^{N^2-N})$$

$$\varrho_2 = \mathrm{diag}(0 \cdots 0, \underbrace{\tfrac{1}{N} \cdots \tfrac{1}{N}}_{N}, 0 \cdots 0)$$

$$\vdots$$

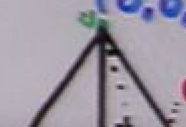
$$\varrho_N = \mathrm{diag}(0 \cdots \quad \cdots 0, \underbrace{\tfrac{1}{N} \cdots \tfrac{1}{N}}_{N})$$

example: <u>exbit</u>

$\varrho_1 = |1\rangle\langle 1| \otimes \mathbb{1}/2$

$Q_2 = \{\sigma \in M_4 : \mathrm{eig}(\sigma) \in T_2\}$

$T_2 = \mathrm{convhull}[\mathrm{Perm}(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0)]$



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example: qubit

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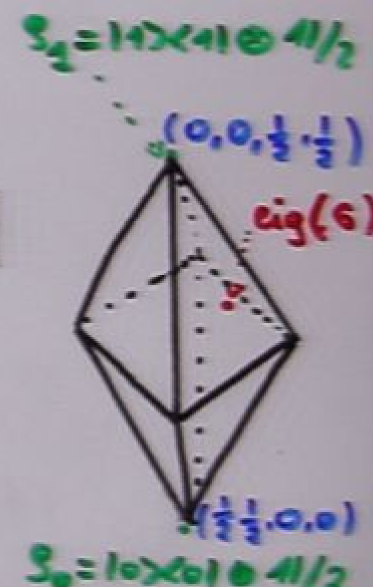
permutohedron

in this case

$T_2 =$ octahedron $=$

$=$ "truncated" tetrahedron



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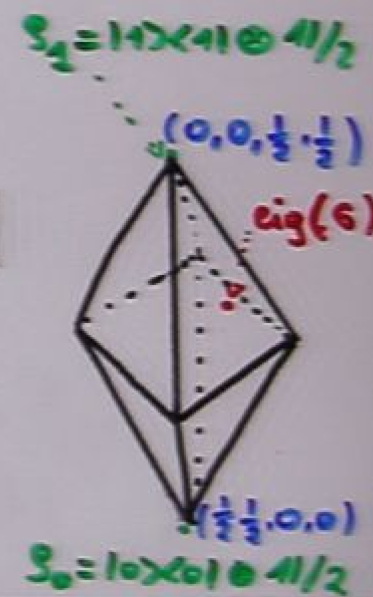
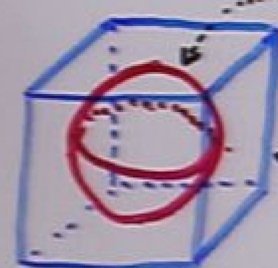


Analogy : quantum $\Leftrightarrow$ classical transition



Bloch ball = M_2 = qubit

is embedded inside

cube of 3 bits

$$M_2 \subset C_2 \times C_2 \times C_2$$

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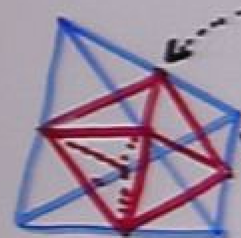
extended theory $\longleftrightarrow$ standard quantum transition



Exbit = Q_2

is embedded inside

ququart = 2 qubits

$$Q_2 \subset M_4$$

Measurements

(positive, operator valued measures)

a) Quantum theory

$$\mathrm{POVM} = \{E_i\}_{i=1}^{k} : E_i \in \mathcal{E}_N \;\; \sum_{i=1}^{k} E_i = \mathbb{1}.$$

$$\mathcal{E}_N = \{ E_i : E_i = E_i^{+} ,\; E_i \geq 0 \}$$

$$\Updownarrow$$

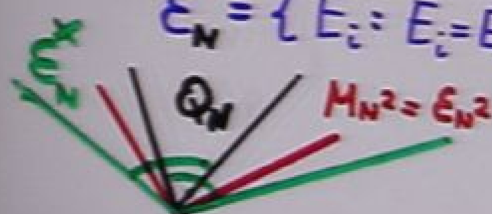
$$\mathrm{Tr}\, E_i \rho \geq 0 \quad \forall \rho \in M_N \qquad \rho \geq 0$$

self-dual cones

b) Extended Quantum theory

$$\mathrm{XPOVM} = \{E_i\}_{i=1}^{k} : E_i \in \mathcal{E}^{\times}, \; \sum_{i=1}^{k} E_i = \mathbb{1}.$$

$$\mathcal{E}_N^{\times} = \{ E_i : E_i = E_i^{+},\; \mathrm{Tr}\, E_i \sigma \geq 0 \;\; \forall \sigma \in Q_N \}$$



dual cones

$$\mathcal{E}_N^{\times} = Q_N^{o}.$$

for $N=2$: octahedron

example: $N=2$

Measurements

(positive, operator valued measures)

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$$\text{POVM} = \{E_i\}_{i=1}^{K} : E_i \in \mathcal{E}_N \quad \sum_{i=1}^{K} E_i = \mathbb{1}.$$

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$$\Updownarrow$$

self-dual cones

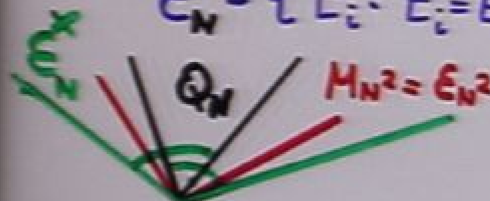
$\varrho \geq 0$

$$\text{Tr}\, E_i \varrho \geq 0 \quad \forall \varrho \in M_N$$

b) Extended Quantum theory

$$\text{XPOVM} = \{E_i\}_{i=1}^{K} : E_i \in \mathcal{E}^{*},\ \sum_{i=1}^{K} E_i = \mathbb{1}.$$

$$\mathcal{E}_N^{*} = \{E_i : E_i = E_i^{+},\ \text{Tr}\, E_i \sigma \geq 0\ \forall \sigma \in Q_N\}$$



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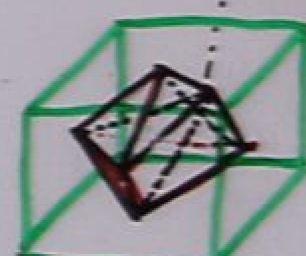
for N=2

octahedron Q_2



$\leftarrow$ cube $\mathcal{E}_2^{*}$

example : N=2

$E_1 = \text{diag}\left(-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$E_2 = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

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$E_4 = \text{diag}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$

$$\sum_{i=1}^{4} E_i = \mathbb{1},\quad \text{Tr}\, E_i \sigma \geq 0 \Rightarrow E_i \in \mathcal{E}_2^{*}$$

$$\text{POVM} = \{E_i\}_{i=1}^{k} : E_i \in \mathcal{E}_N \ \sum_{i=1}^{k} E_i = \mathbb{1}.$$

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6) Extended Quantum theory

$$\text{XPOVM} = \{E_i\}_{i=1}^{k} : E_i \in \mathcal{E}^{x},\ \sum_{i=1}^{k} E_i = \mathbb{1}.$$

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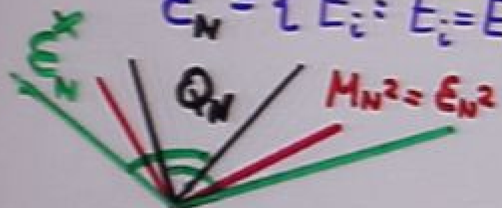

dual cones

$$\mathcal{E}_N^{x} = Q_N^{\circ}$$

for $N=2$

octahedron Q_2

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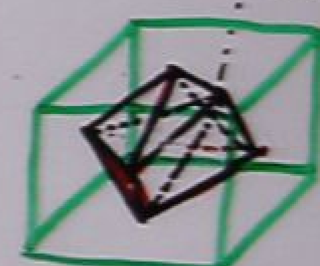

cube $\mathcal{E}_2^{x}$

$\sum_{i=1}^{4} E_i = \mathbb{1}$, $\operatorname{Tr} E_i \sigma \geq 0 \Rightarrow E_i \in \mathcal{E}_2^{x}$

XT := eXtended Quantum Theory:

1. EXtended set of states $Q_N \subset \mathcal{H}_{N^2} \subset \mathbb{R}^{N^4}$

quartic theory

2. Measurements: XPOVM

dual set $\mathcal{E}^X_N = Q^\circ_N \supset \mathcal{E}_{N^2}$

(larger than the set of POVM)

3. Composite systems: standard tensor product structure,

$\sigma_A \in Q_N,\ \sigma_B \in Q_K \Rightarrow \sigma_{AB} \in Q_{NK}$.

4. discrete linear dynamics:

supermaps $\sigma' = \Gamma(\sigma)$, $\Gamma: Q_N \to Q_N$

completely preserving maps (not necessarily completely positive!)

$$(\Gamma \otimes \mathbb{I})\sigma \in Q_{N^2} \quad \forall \sigma \in Q_{N^2}$$

Proposition 1.

Extended quartic theory covers entire quantum (quadratic) theory.

Proof: use product operators / states

$$\sigma = \rho \otimes 1\!\!1/n \, ; \quad Y_i = X_i \otimes 1\!\!1$$

Kraus operators

then partial trace

$$\rho = \mathrm{Tr}_B \, \sigma$$ gives back QT.

Proposition 2.

Extended theory is not equivalent to QT.

Proof: use generic (non product) states and evolve them by generic supermap Γ

$$\begin{array}{lccccl}
XT: & \sigma & \xrightarrow{\quad \Gamma \quad} & & \sigma' = \Gamma(\sigma) & \text{quartic theory} \\
\text{hyper-decoherence} & \downarrow & & & \downarrow & \downarrow \\
QT: & \mathrm{Tr}_B \sigma = \rho & \xrightarrow{\Psi(\Gamma)} & \rho'' \neq & \rho' = \mathrm{Tr}_B \sigma' , & \text{quadratic theory}
\end{array}$$

where $\Psi(\Gamma)$ is a CP map in QT

such that its Ch...

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where $\Psi(\Gamma)$ is a CP map in QT such that its Choi matrix is obtained by partial trace of Choi matrix for Γ.

for an updated Catalogue consult
http://chaos.if.uj.edu/~Karol/Hadamard

1-d $B_6^{(1)}(y)$ family

K. Beauchamp and R. Nicoara, April 2006

W. Bruzda, May 2006

$$B_6^{(1)}(\mathbf{y}) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1/x & -\mathbf{y} & \mathbf{y} & 1/x \\ 1 & -x & 1 & \mathbf{y} & 1/z & -1/t \\ 1 & -1/\mathbf{y} & 1/\mathbf{y} & -1 & -1/t & 1/t \\ 1 & 1/\mathbf{y} & z & -t & 1 & -1/x \\ 1 & x & -t & t & -x & -1 \end{bmatrix}$$

where

$$x(y) = \frac{1 + 2y + y^2 \pm \sqrt{2}\sqrt{1 + 2y + 2y^3 + y^4}}{1 + 2y - y^2}$$

$$z(y) = \frac{1 + 2y - y^2}{y(-1 + 2y + y^2)}$$

$$t(y) = xyz$$

and y is a free parameter $\mathbf{y} = \exp(i\, \mathbf{s})$ and $\mathbf{s}$ varies in the interval:

$$\frac{\arccos(\frac{1}{2}(\sqrt{3}-1))}{2\,\pi} < \mathbf{s} < 1 - \frac{\arccos(\frac{1}{2}(\sqrt{3}-1))}{2\,\pi}$$

a non-affine

eXtended theory of Quantum Information

assume, we can use exbits for information processing.

What could you do with them ?

example : dense coding

a) standard quantum dense coding

Alice

coding of 2 bits $\{1\!\!1, \sigma_x, \sigma_y, \sigma_z\}$

sending of 1 qubit

$|\psi^+\rangle$

Bob

initial entanglement

joint measurement in Bell basis

2 bits sent

saldo

entanglement + transfer of 1 qubit $\Rightarrow$ sends 2 bits

b) eXtended quantum dense coding

coding of 2 qubits

Alice

1 3

2 4

sending of 1 exbit

initial entanglement

Bob

joint measurement (partial trace)

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extended theory of quantum [illegible]

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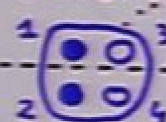
b) extended quantum dense coding

coding of 2 qubits

Alice — sending of 1 exbit



initial entanglement

Bob — joint measurement (partial trace)

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saldo

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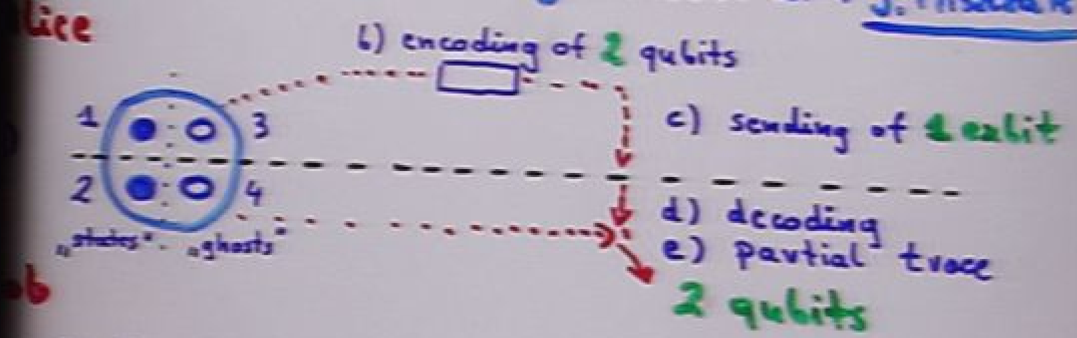
Extended quantum dense coding

joint work with J. Miszczak



initial entangled state of 2 exbits:

$$\rho_a = \frac{1}{4}\left[|0000\rangle\langle 0000| + |1111\rangle\langle 1111| + |1100\rangle\langle 1100| + |0011\rangle\langle 0011|\right]$$

$$\rho_a \in Q_4$$

encoding of 2 qubits by Alice

$$\rho_b = U \rho_a U^{+}, \quad U = U_A \otimes 1\!\!1_B = W_1 \otimes W_2 \otimes 1\!\!1_4 ,$$

$$W_1 = \left(|\psi_1\rangle, -|\psi_1^{*}\rangle\right), \; W_2 = \left(|\psi_2\rangle, -|\psi_2^{*}\rangle\right)$$

sending of 1 exbit to Bob

decoding = global unitary done by Bob

$$\rho_d = D \rho_b D^{+}, \text{ where } D = S^{2\leftrightarrow 3}(X \otimes X)$$

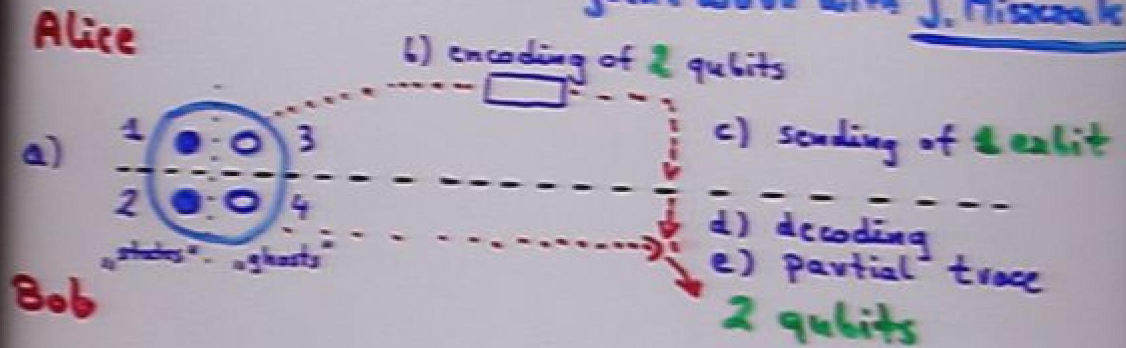
$$X = (\sigma_z \sigma_x) \otimes |1\rangle\langle 1| + 1\!\!1 \otimes |0\rangle\langle 0|$$

$S^{2\leftrightarrow 3}$ = SWAP operation exchanging „2" & „3"

measurement = partial trace (over both „ghosts"

Extended quantum dense coding

joint work with J. Miszczak



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e) measurement = partial trace ...

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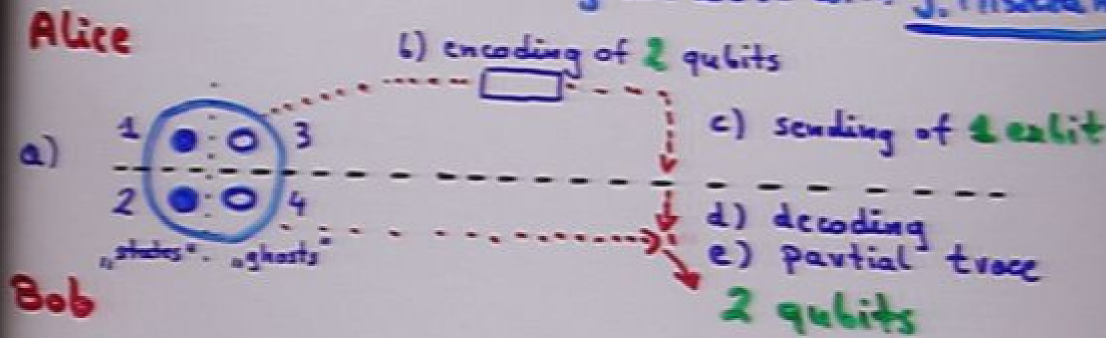

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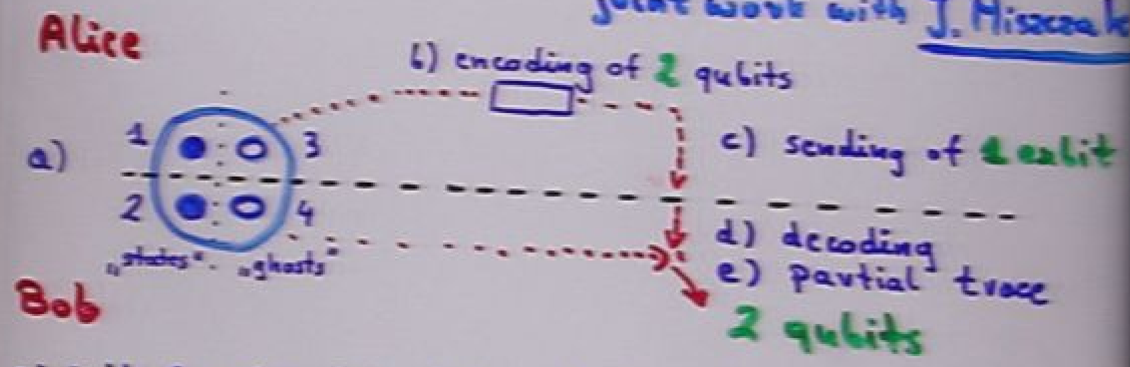
joint work with J. Miszczak



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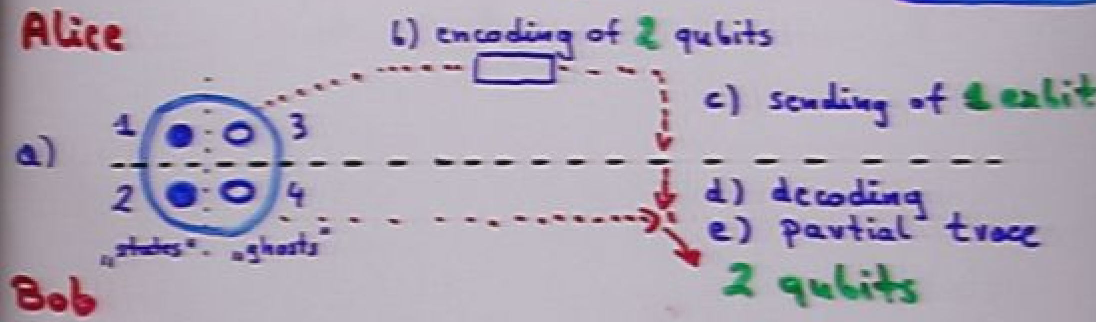
Extended quantum dense coding

joint work with J. Miszczak



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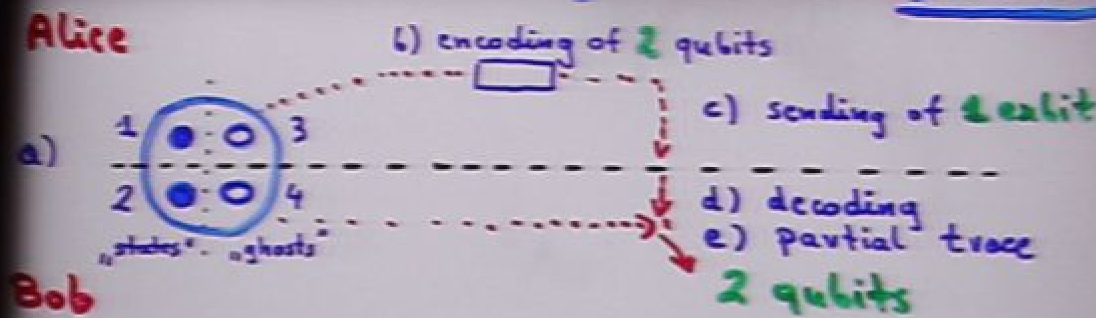

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$$\varrho_{fin} = \mathrm{Tr}_{3,4}\, \varrho_d = |\psi_1\rangle\langle\psi_1| \otimes |\psi_2\rangle\langle\psi_2| \quad !$$

2 qubits sent!

Higher-order-quantum theories

- more „ghosts" ≡ larger ancillas...

$K = N^r$, $r = 2^m$

further embedding of Bloch ball into higher dimensions...

$$Q_N^{(m)} := \{ G \in M_N : G \in \text{convhull}\left[U(|1\rangle\langle 1| \otimes \tfrac{\mathbb{1}}{N^m}) U^\dagger \right]$$

set of even-more-extended-states.

where $U \in U(N^{m+1})$

- more faces of permutohedron
- „more truncated" simplex of eigenvalues
- even larger set of extended POVMs.

$r = 1$ (linear) classical
$r = 2$ (quadratic) quantum
$r = 4$ (quartic) extended
$r = 8$ (octonic)
⋮
$r = 2^m$ (two-em-tic) more -11- theory

infinite hierarchy of embedded quantum theories

we obtain an infinite set of embedded theories of

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⇓

we obtain an infinite set of embedded theories of generalised quantum

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$\Downarrow$

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CONCLUSIONS

1. By extending the set of states and operations we proposed a generalised, quartic quantum theory.

2. EXtended theory covers standard QT, but is different from QT.

3. Conclusion for V axioms approach by L. Hardy:
 simplicity axiom is necessary.

Challange

4. (physics) Design an experimental scheme for which probabilities computed with respect to XT and QT do differ !
 [entanglement with „ghosts"]
 i) If these differencies cannot be observed estimate upper bound for the time of hyperdecoherence

5. (Information theory)
 a) find a useful extended operation (not admissible in QT).
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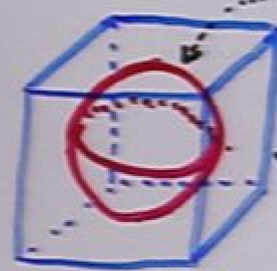
Analogy : quantum $\leftrightarrow$ classical transition

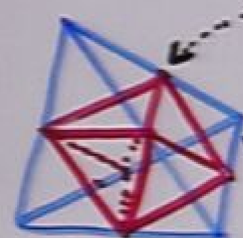


Bloch ball = M_2 = qubit
is embedded inside
cube of 3 bits

$$M_2 \subset C_2 \times C_2 \times C_2$$

---

extended theory $\longleftrightarrow$ standard quantum transition



Exbit = Q_2
is embedded inside
ququart = 2 qubits

$$Q_2 \subset M_4$$