Title: Introduction to quantum groups 3

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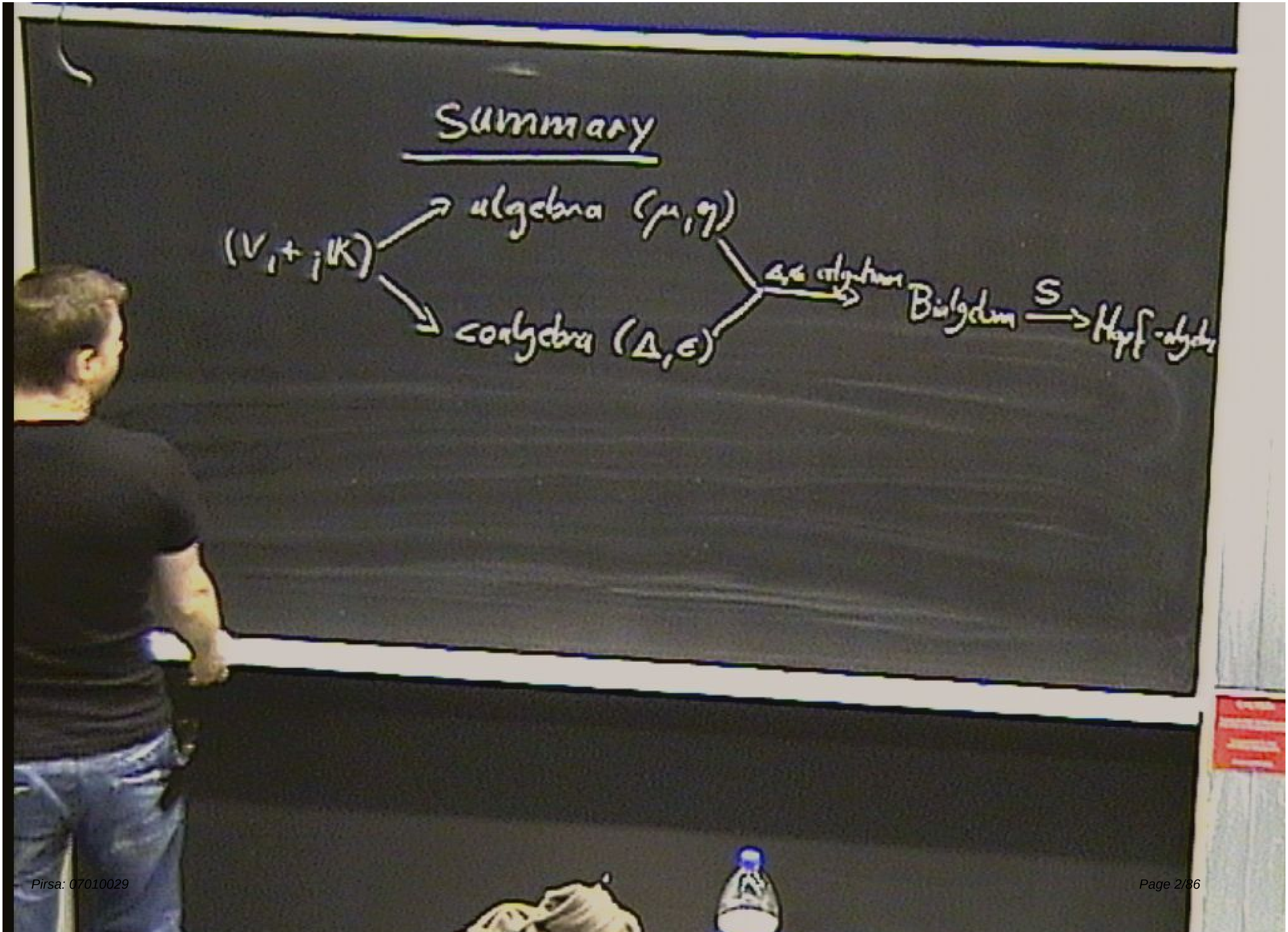
Abstract: Universal Enveloping Algebras and dual Algebras of Functions
The two most relevant types of Hopf-algebras for applications in physics are discussed in this unit. Most central notion will be their duality and representation.

<div id="Cleaner">Motivation: From Quantum Mechanics to Quantum Groups<div id="Cleaner">The notion of 'quantization' commonly used in textbooks of quantum mechanics has to be specified in order to turn it into a defined mathematical operation. We discuss that on the trails of Weyl's phase space deformation, i.e. we introduce the Weyl-Moyal starproduct and the deformation of Poisson-manifolds. Generalizing from this, we understand, why Hopf-algebras are the most genuine way to apply 'quantization' to various other algebraic objects - and why this has direct physical applications.


Summary
(V, +, K)
algebra (μ, η)
coalgebra (Δ, ε)
Bialgebra
S
Hopf-algebra

Summary

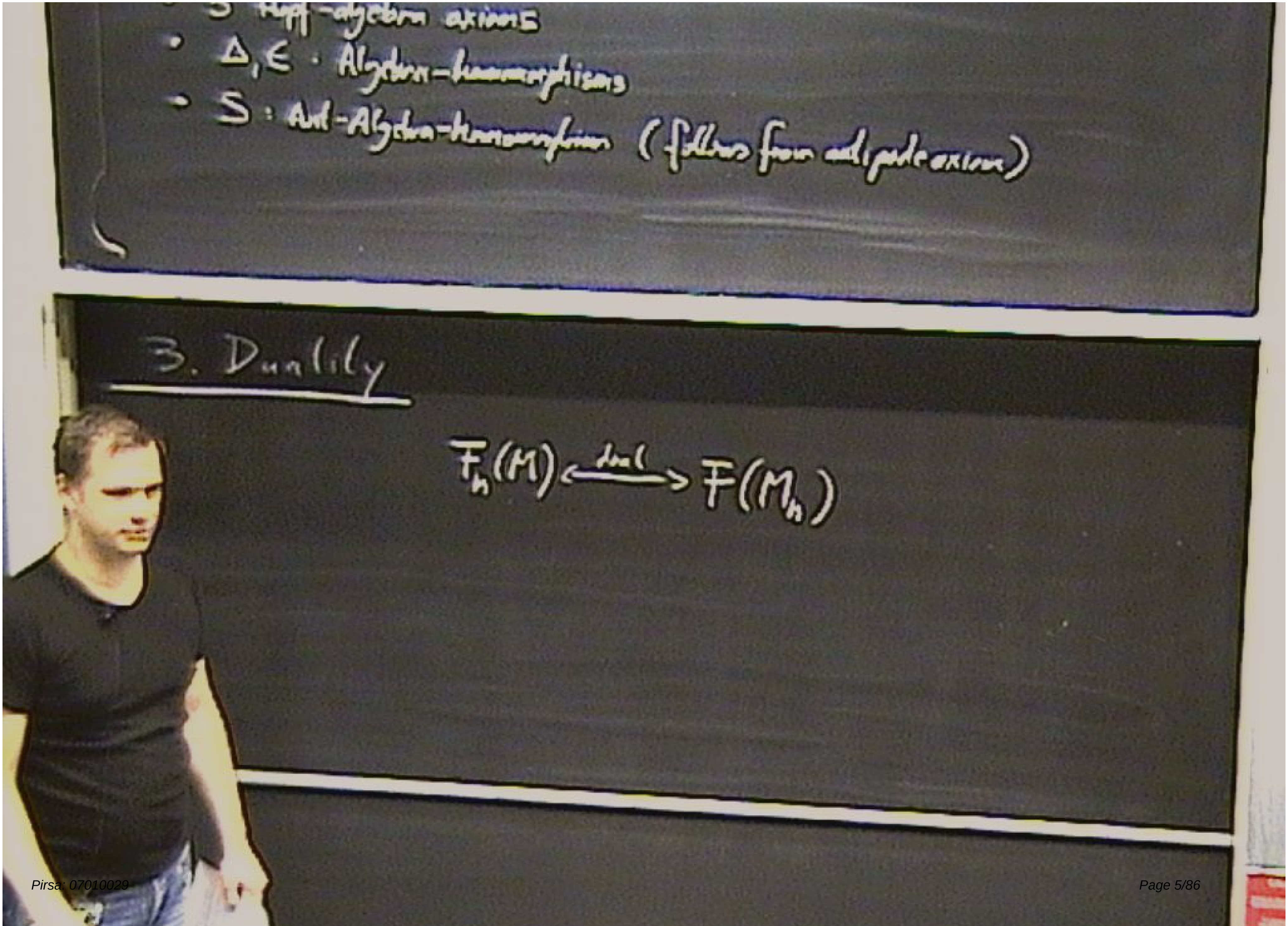
$$
\begin{array}{ccccc}
 & \nearrow & \text{algebra } (\mu, \eta) & & \\
(V, +, \mathbb{K}) & & & \xrightarrow{\text{as algebras}} \text{Bialgebra} \xrightarrow{S} \text{Hopf-algebra} & \\
 & \searrow & \text{coalgebra } (\Delta, \epsilon) & &
\end{array}
$$

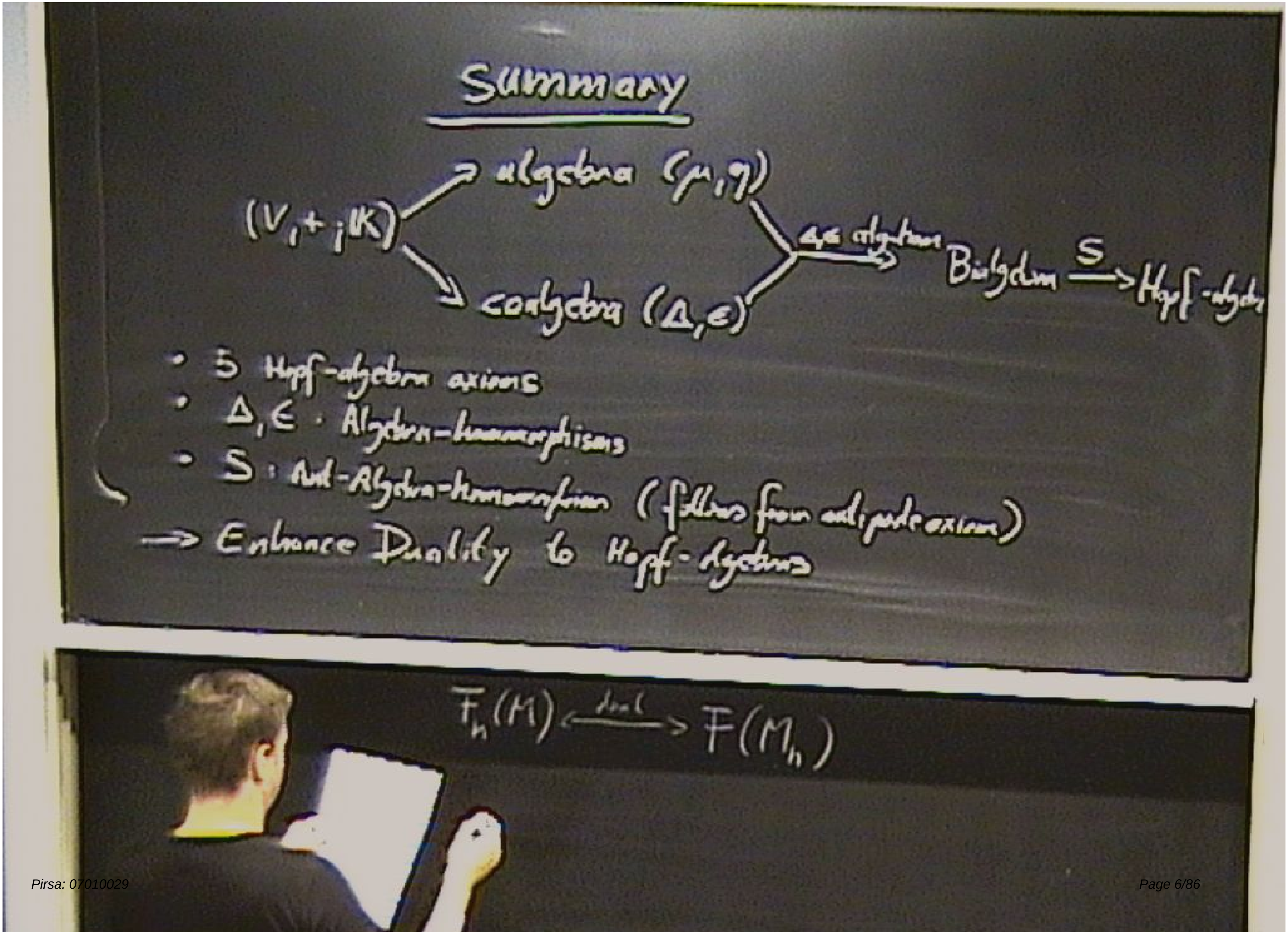
- S Hopf-algebra axioms
- Δ, ϵ : Algebra-homomorphisms
- S : Anti-Algebra-homomorphism

Summary

$$\begin{tikzcd}
 & \text{algebra } (\mu,\eta) \arrow[dr] & & & \\
(V,+,\mathbb{K}) \arrow[ur] \arrow[dr] & & \cdot \arrow[r, "\text{are algebra hom}"] & \text{Bialgebra} \arrow[r, "S"] & \text{Hopf-algebra} \\
 & \text{coalgebra } (\Delta,\epsilon) \arrow[ur] & & &
\end{tikzcd}$$

- S Hopf-algebra axioms
- Δ, ϵ : Algebra-homomorphisms
- S : Anti-Algebra-homomorphism (follows from antipode axiom)


S: Hopf-algebra axioms
• Δ, ε : Algebra-homomorphisms
• S : Anti-Algebra-homomorphism (follows from antipode axiom)
3. Duality
$\mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$


Summary
$(V,+,\mathbb{K})$
algebra (μ,η)
coalgebra (Δ,ϵ)
Δ,ϵ algebra hom
Bialgebra $\xrightarrow{S}$ Hopf-algebra
• S Hopf-algebra axioms
• Δ,ϵ : Algebra-homomorphisms
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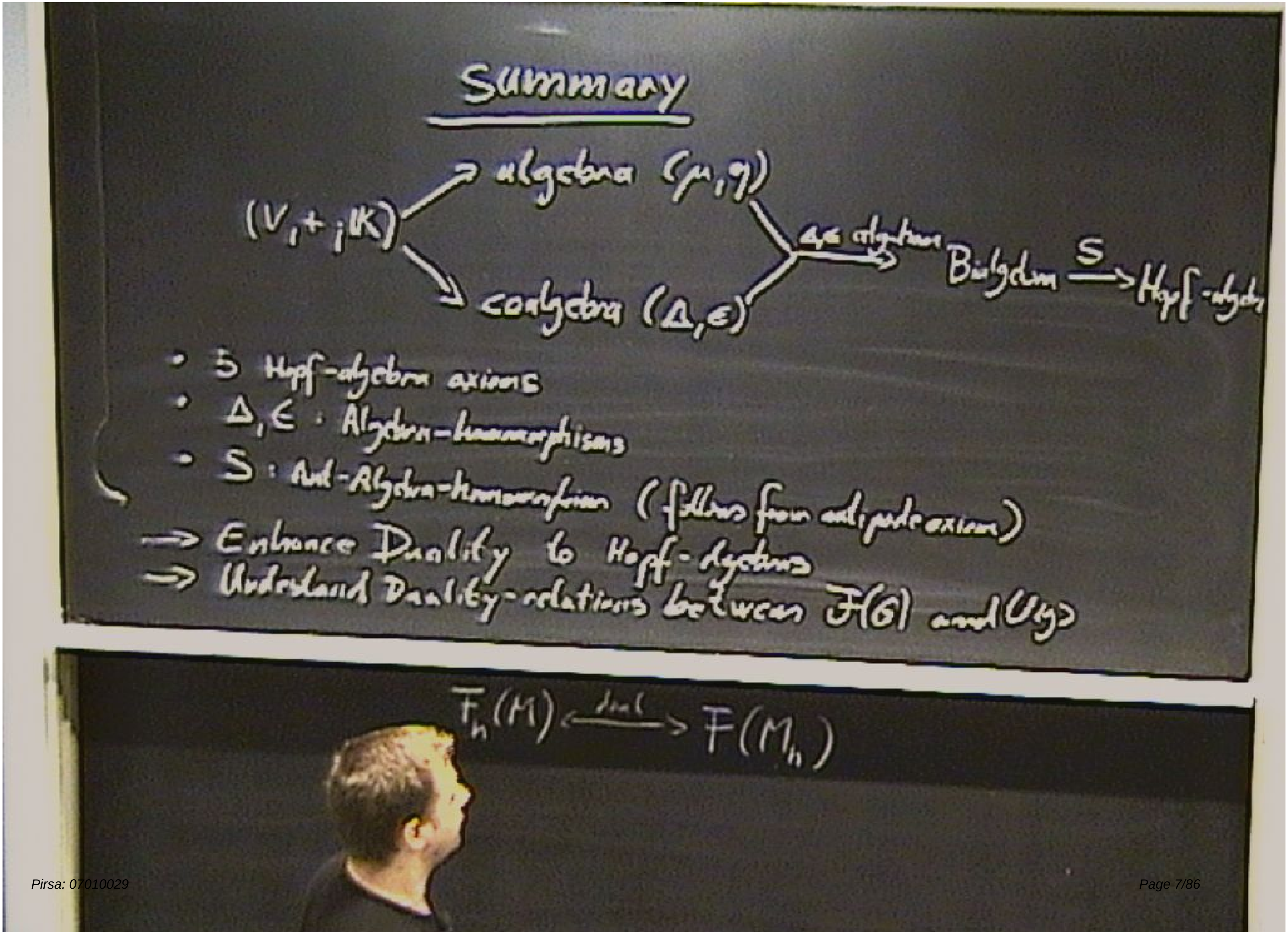
→ Enhance Duality to Hopf-algebras
$F_h(M) \xleftrightarrow{\text{dual}} F(M_h)$


Summary
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Δ, ε algebra-hom
Bialgebra
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Hopf-algebra
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⟹ Enhance Duality to Hopf-algebras
⟹ Understand Duality-relations between F(G) and U(g)
F_h(M) ←dual→ F(M_h)

$$
\begin{tikzcd}
 & \text{algebra } (\mu, \eta) \arrow[dr] & & & \\
(V, +, \mathbb{K}) \arrow[ur] \arrow[dr] & & \cdot \arrow[r, "\text{as algebras}"] & \text{Bialgebra} \arrow[r, "S"] & \text{Hopf-algebra} \\
 & \text{coalgebra } (\Delta, \epsilon) \arrow[ur] & & &
\end{tikzcd}
$$

- 5 Hopf-algebra axioms
- Δ, ϵ : Algebra-homomorphisms
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$\Longrightarrow$ Enhance Duality to Hopf-algebras

$\Longrightarrow$ Understand Duality-relations between $\mathcal{F}_h(G)$ and $\mathcal{U}_h(g)$

3. Duality

$$\mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$$

$$(V,+,\mathbb{K}) \to \text{algebra } (\mu,\eta), \qquad (V,+,\mathbb{K}) \to \text{coalgebra } (\Delta,\epsilon)$$

$$\text{algebra } (\mu,\eta) + \text{coalgebra } (\Delta,\epsilon) \xrightarrow{\text{as algebras}} \text{Bialgebra} \xrightarrow{S} \text{Hopf-algebra}$$

- S Hopf-algebra axioms
- Δ, ϵ : Algebra-homomorphisms
- S : Anti-Algebra-homomorphism (follows from antipode axiom)

$\Rightarrow$ Enhance Duality to Hopf-algebras

$\Rightarrow$ Understand Duality-relations between $\mathcal{F}_h(G)$ and $U_h(g)$

3. Duality

$$\mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$$

$$\mathcal{F}_h(G) \qquad U_h(g)$$

$$\mu_h \qquad \Delta_h$$

Bialgebra $\longrightarrow$ Hopf-algebra

coalgebra (Δ, ϵ)

- S Hopf-algebra axioms
- Δ, ϵ : Algebra-homomorphisms
- S : Anti-Algebra-homomorphism (follows from antipode axiom)

$\Longrightarrow$ Enhance Duality to Hopf-algebras

$\Longrightarrow$ Understand Duality-relations between $\mathcal{F}_h(G)$ and $\mathcal{U}_h(g)$

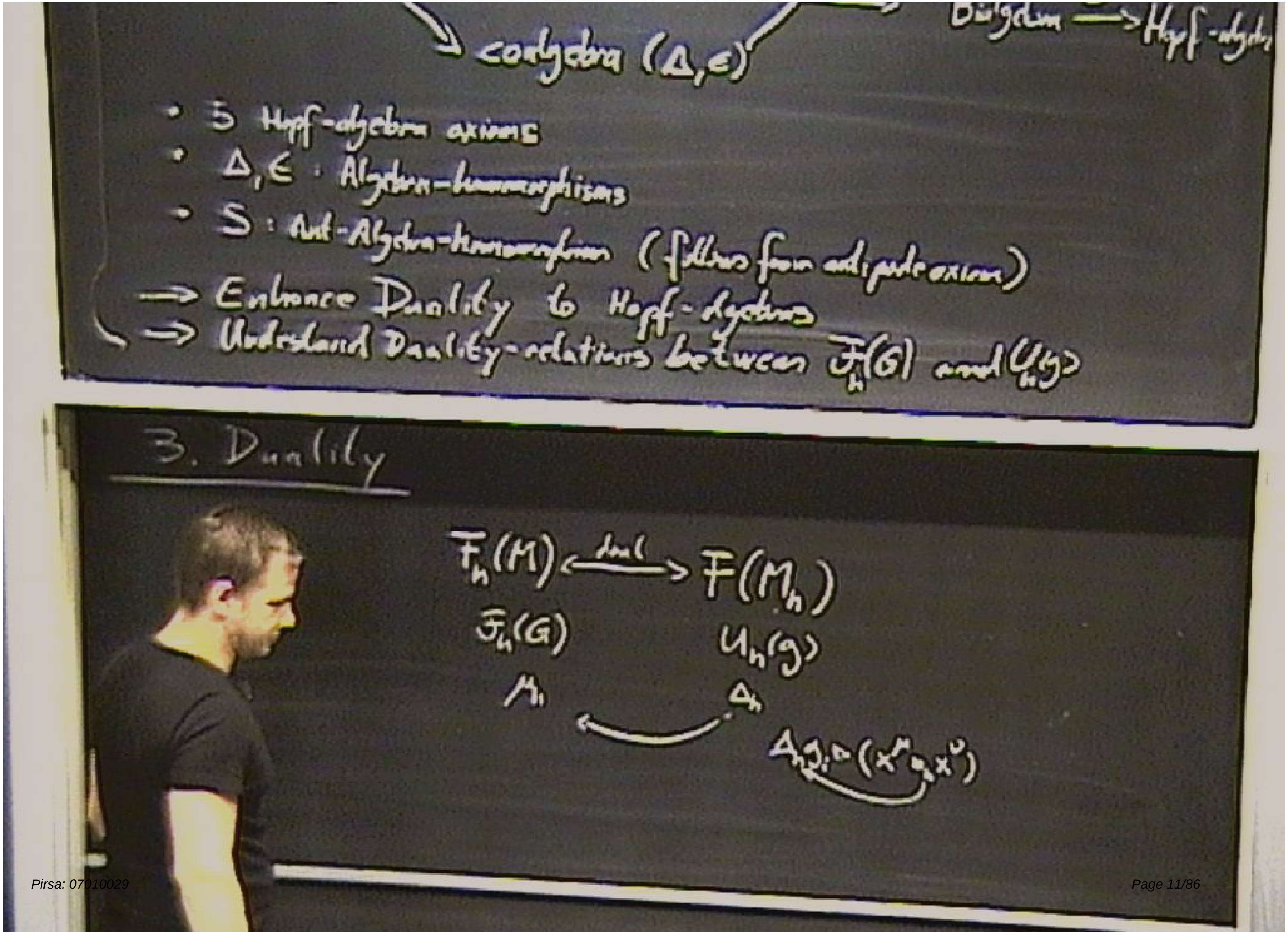
3. Duality

$$\mathcal{F}(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$$

$$\mathcal{F}_h(G) \qquad \mathcal{U}_h(g)$$

$$\mu_h \qquad \Delta_h$$

$$x^\mu *_h x^\nu$$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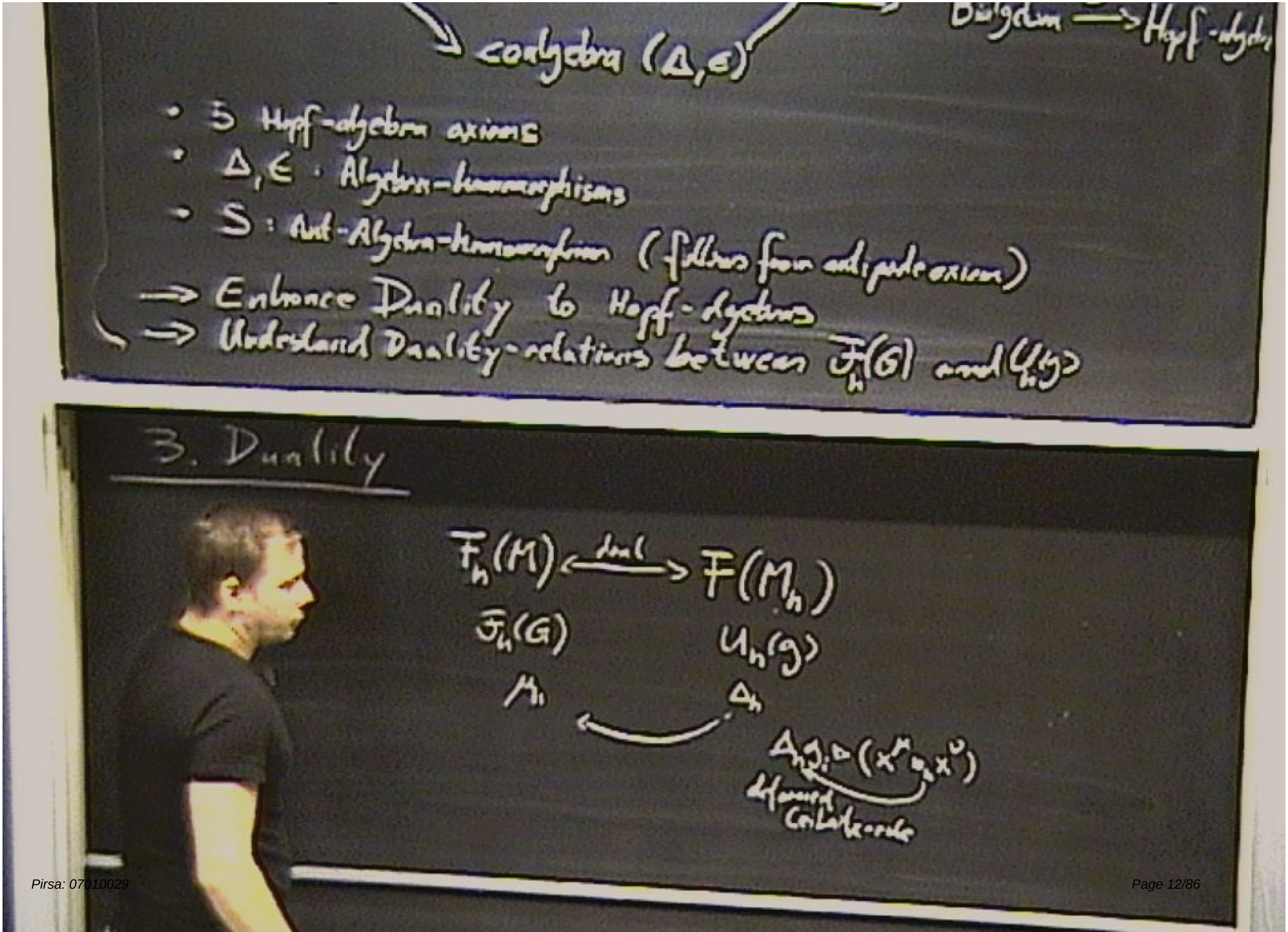
3. Duality
F_h(M) ⟷ F(M_h)
F_h(G)
U_h(g)


coalgebra (Δ, ε)
Bialgebra → Hopf-algebra
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• Δ, ε : Algebra-homomorphisms
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→ Enhance Duality to Hopf-algebras
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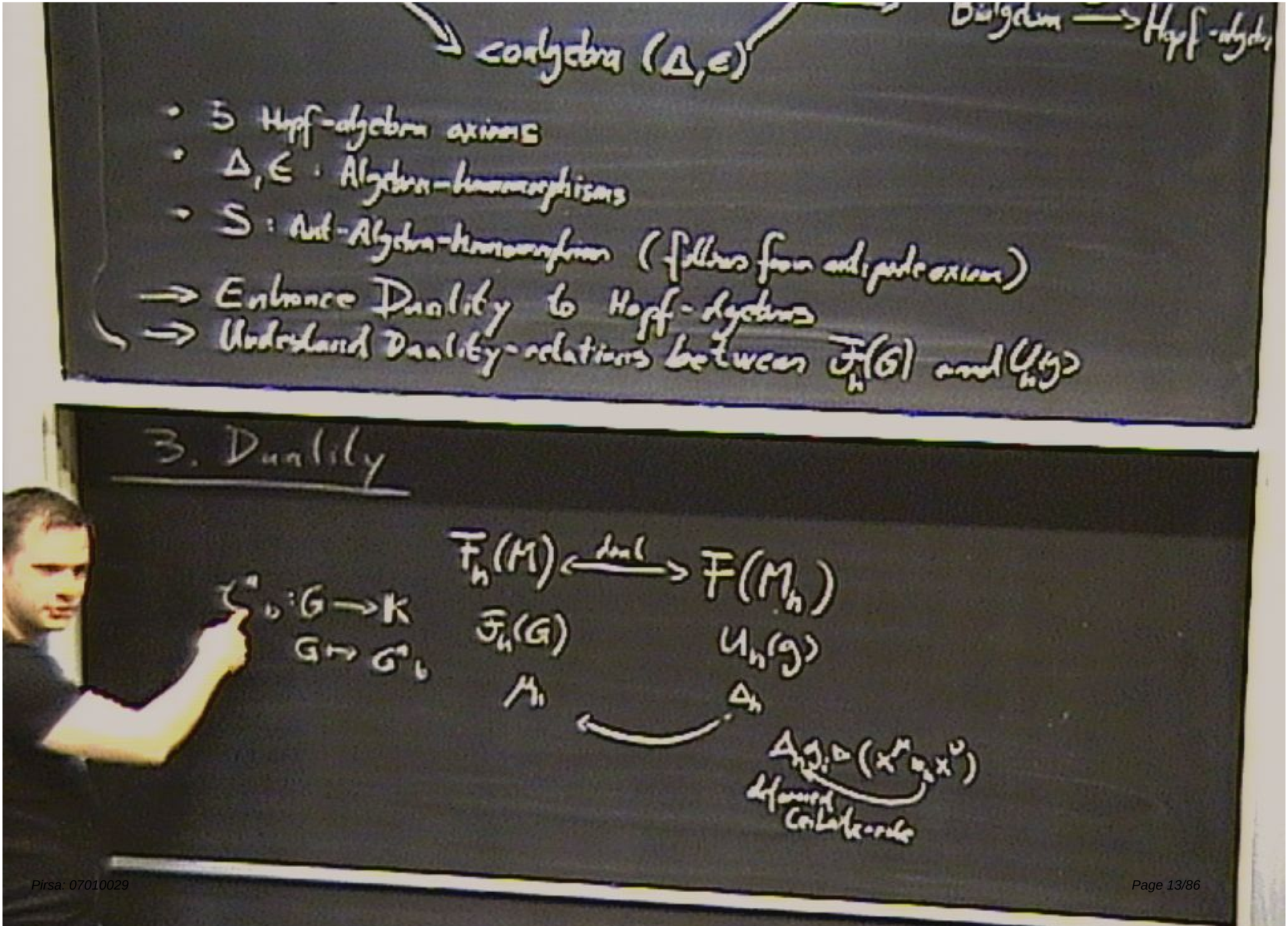
3. Duality
F_h(M) ← dual → F(M_h)
F_h(G)
U_h(g)
μ_h
Δ_h
Δ_h: Δ(x^μ ⊗ x^ν)
deformed Leibniz-rule


coalgebra (Δ, ε)
Bialgebra → Hopf-algebra
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⇒ Enhance Duality to Hopf-algebras
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3. Duality
F_h(M) ←dual→ F(M_h)
F_h(G)
U_h(g)
G → K
Δ_h
deformed
Leibniz-rule

Bialgebra $\longrightarrow$ Hopf-algebra

coalgebra (Δ, ϵ)

- $\exists$ Hopf-algebra axioms
- Δ, ϵ : Algebra-homomorphisms
- S : Anti-Algebra-homomorphism (follows from antipode axiom)

$\longrightarrow$ Enhance Duality to Hopf-algebras

$\Rightarrow$ Understand Duality-relations between $\mathcal{F}_h(G)$ and $\mathcal{U}_h(g)$

3. Duality

$t^a{}_b \cdot t^c{}_d$

$t^a{}_b : G \to K$

$G \mapsto G^a{}_b$

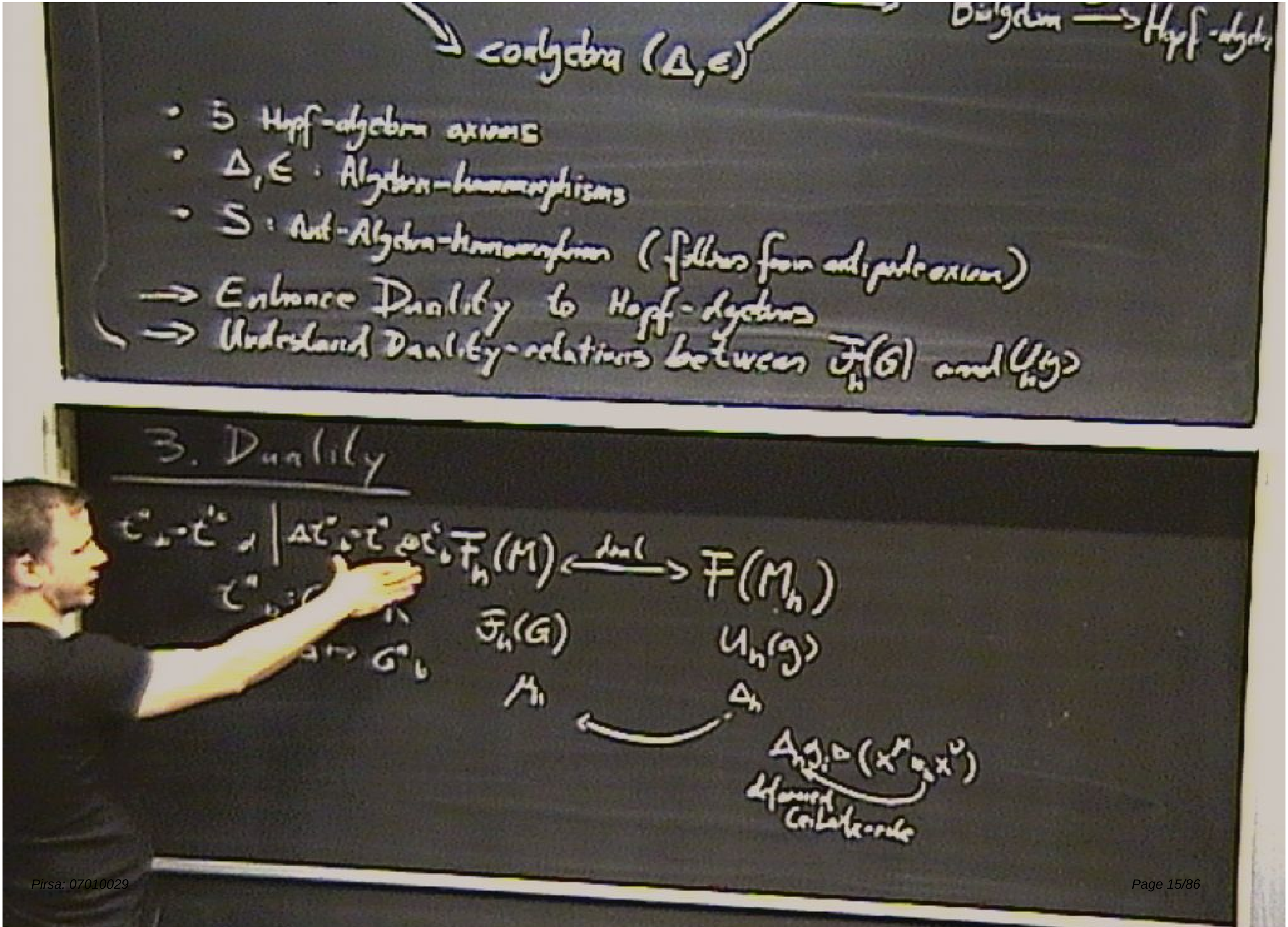
$$\mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$$

$\mathcal{F}_h(G)$ $\quad\quad$ $\mathcal{U}_h(g)$

μ_h $\quad\quad$ Δ_h

$\Delta_h g_i \triangleright (x^\mu \star x^\nu)$

deformed Leibniz-rule


coalgebra (Δ,ε)
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3. Duality
F_h(M) dual F(M_h)
F_h(G)
U_h(g)
deformed Coleibniz-rule

coalgebra (Δ, ϵ) — Bialgebra $\longrightarrow$ Hopf-algebra

- $\exists$ Hopf-algebra axioms
- Δ, ϵ : Algebra-homomorphisms
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$\longrightarrow$ Enhance Duality to Hopf-algebras
$\longrightarrow$ Understand Duality-relations between $\mathcal{F}_h(G)$ and $U_h(g)$

3. Duality

$t^\mu{}_\nu \cdot t^{\nu}{}_\rho \mid \Delta t^\mu{}_\nu = t^\mu{}_\rho \otimes t^\rho{}_\nu$ $\quad \mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$

$t^\mu{}_\nu : G \to K$ $\quad \mathcal{F}_h(G) \quad U_h(g)$

$G \mapsto G^\mu{}_\nu$ $\quad \mu_h \quad \Delta_h \quad \mathcal{F}_h \in U(g) \otimes U(g)$

$\mathcal{F}_h^{\mu\nu}{}_{g\sigma} (x^\sigma x^\rho) \otimes t^g{}_\alpha t^\sigma{}_\rho$ $\quad \Delta_h g_i \triangleright (x^\mu \otimes x^\nu)$

deformed Coleibniz-rule

coalgebra (Δ,ϵ) — Bialgebra $\longrightarrow$ Hopf-algebra

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3. Duality

$t^a{}_b \cdot t^c{}_d$ | $\Delta t^a{}_b = t^a{}_c \otimes t^c{}_b$ $\mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$

$t^a{}_b : G \to K$ $\qquad \mathcal{F}_h(G) \qquad U_h(g)$

$G \mapsto G^a{}_b$

$\mu_h \qquad \Delta_h \qquad \mathcal{F}_h \in U(g) \otimes U(g)$

$\mathcal{F}_h^{\mu\nu}{}_{\rho}(x^\mu x^\nu) \otimes t^\rho = t^\sigma{}_\rho$ $\qquad \Delta_h g_i = \Delta(x^\mu \otimes x^\nu)$

deformed Leibniz-rule

coalgebra (Δ, ϵ) Bialgebra $\longrightarrow$ Hopf-algebra

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$t^a{}_b \cdot t^c{}_d \;\Big|\; \Delta t^a{}_b = t^a{}_c \otimes t^c{}_b \quad \mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$

$t^a{}_b : G \to K$

$G \mapsto G^a{}_b$

$\mathcal{F}_h(G)$ $\qquad U_h(g)$

μ_h $\qquad \Delta_h$ $\qquad \mathcal{F}_h \in U(g) \otimes U(g)$

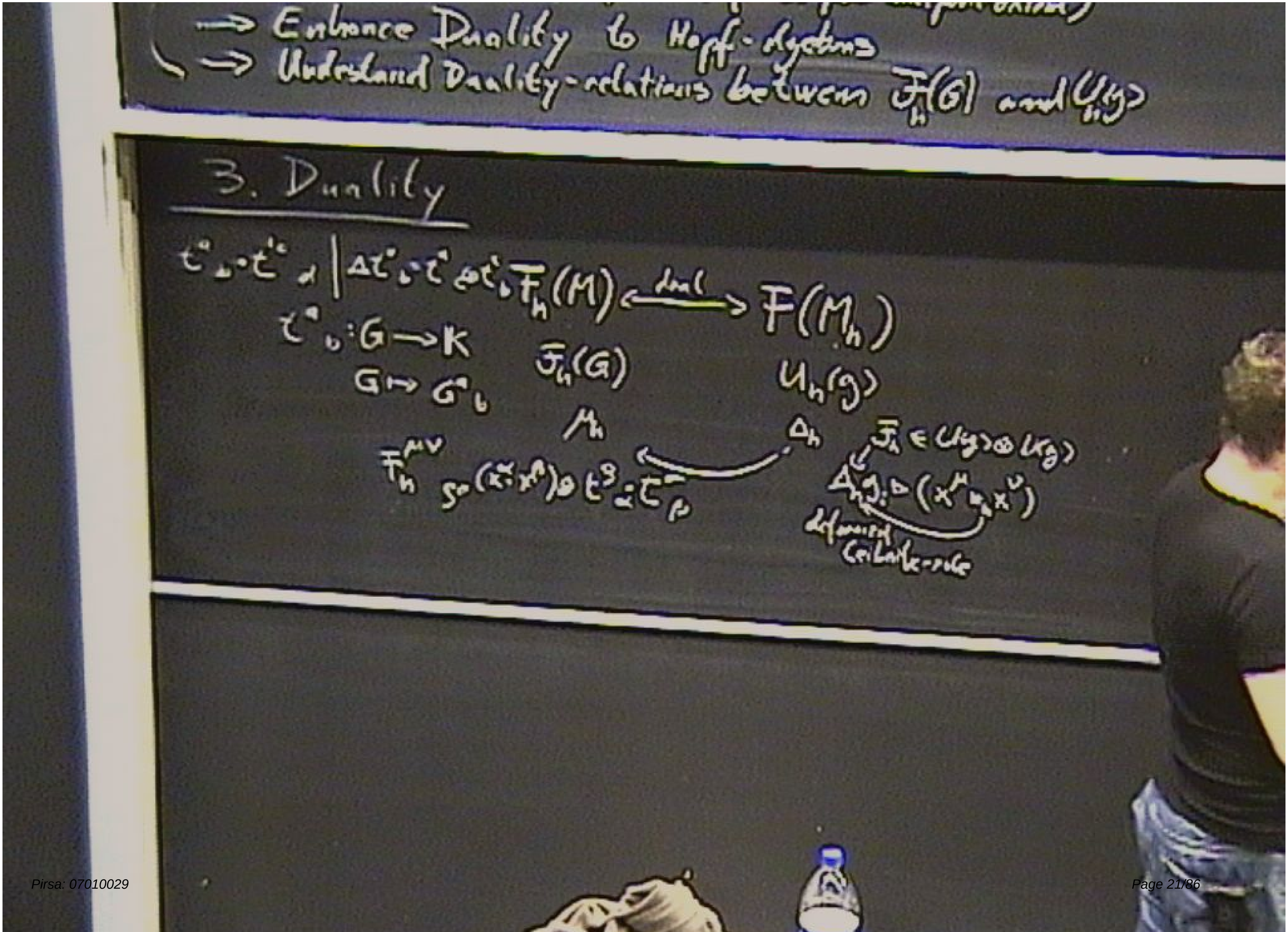
$\mathcal{F}_h^{\mu\nu}$ $\qquad \Delta_h g_i \triangleright (x^\mu \otimes x^\nu)$

deformed Leibniz-rule

3.1 Prelude to Duality

3.1 Prelude to Duality

finite group $(G, \cdot)$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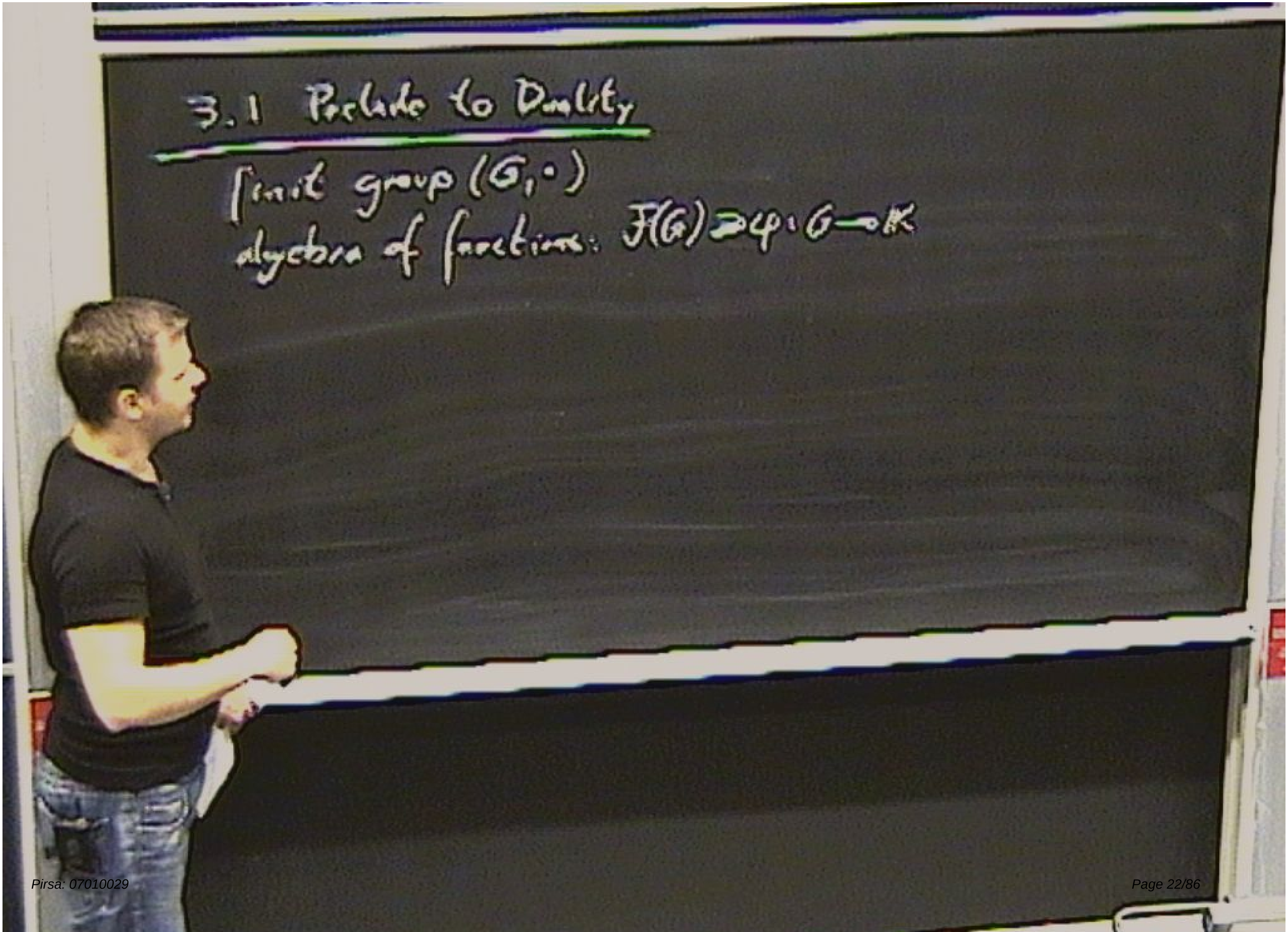
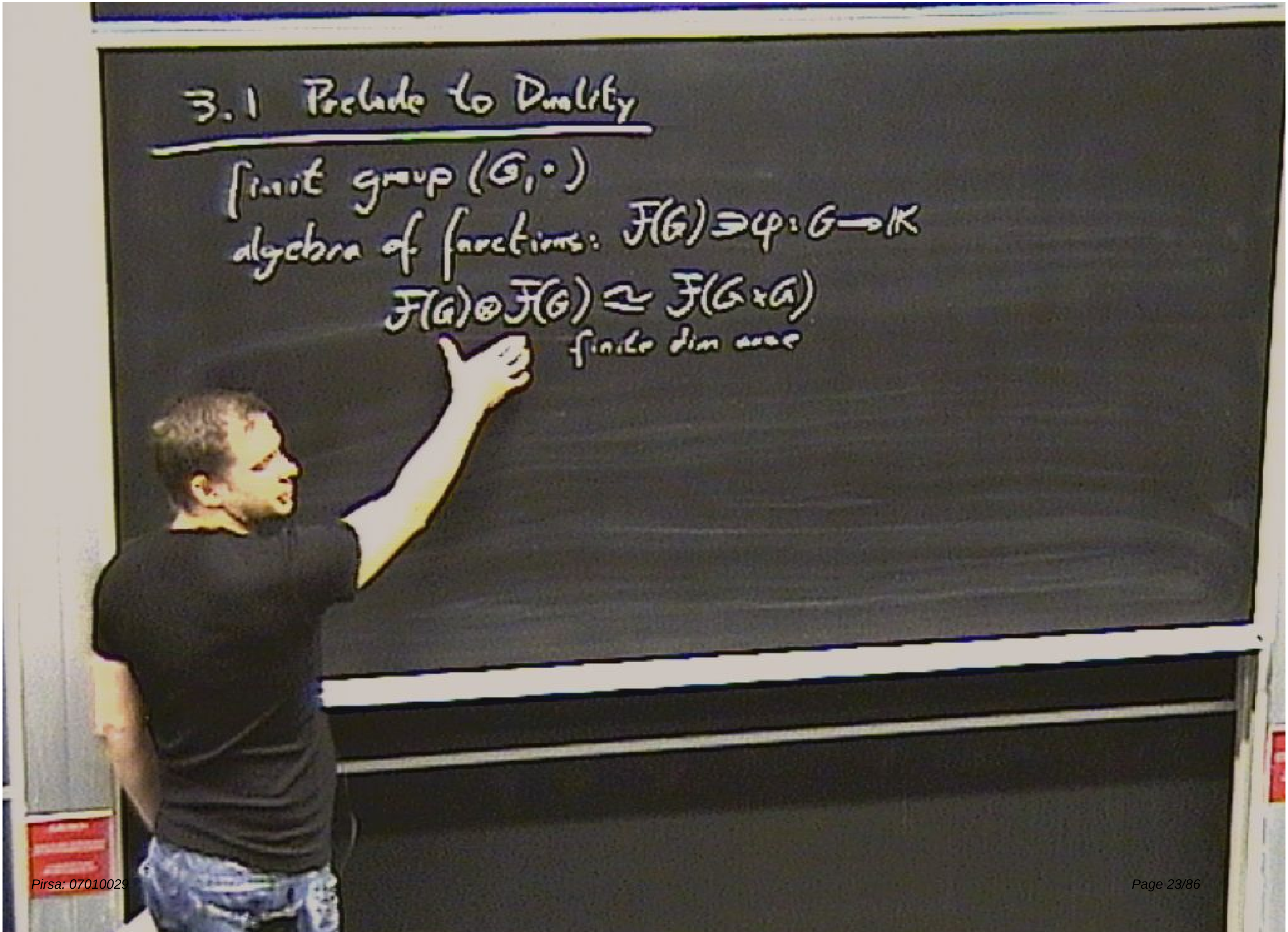
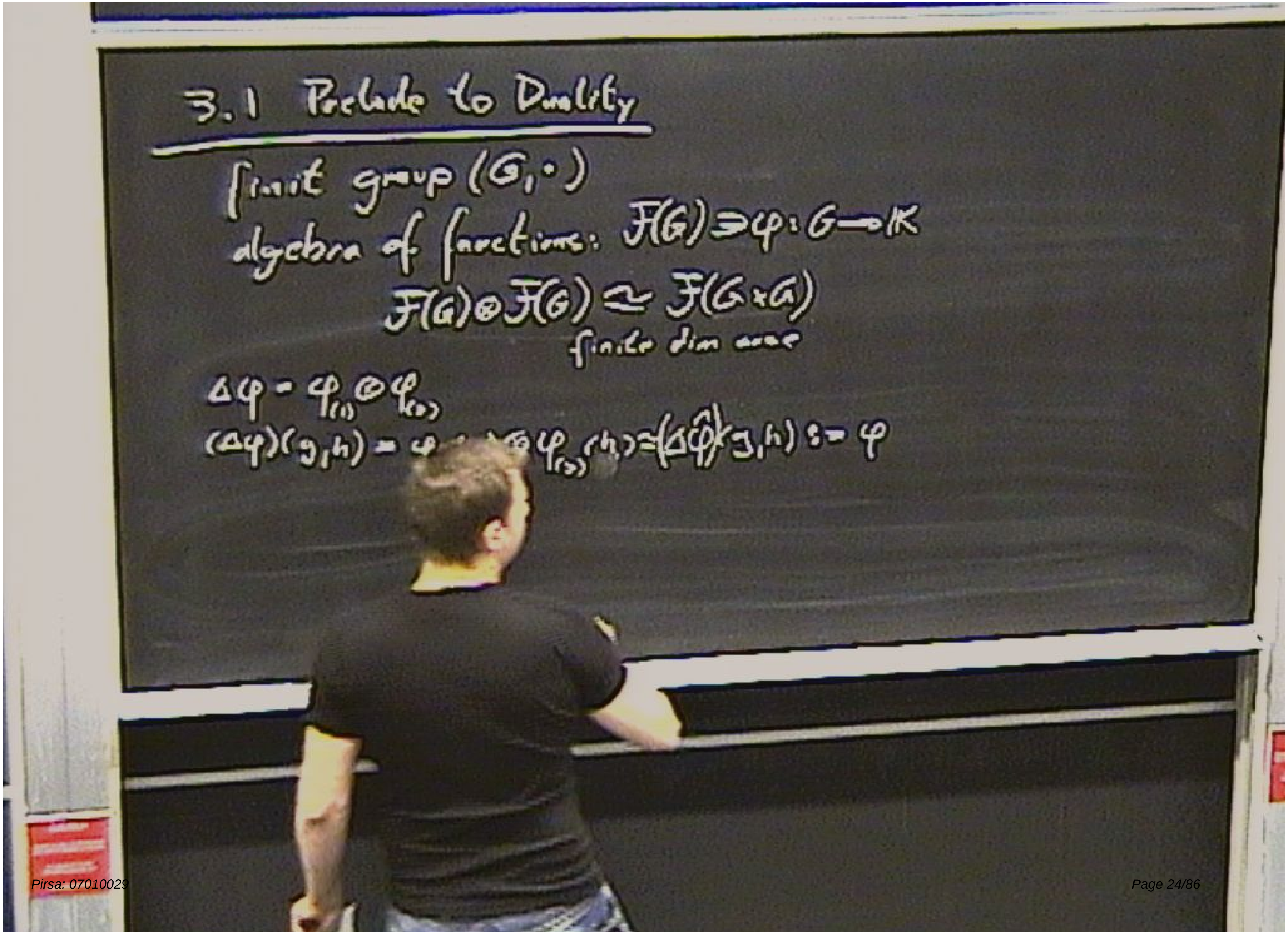

3.1 Prelude to Duality
finite group $(G,\cdot)$
algebra of functions: $\mathcal{F}(G) \ni \varphi : G \to \mathbb{K}$


3.1 Prelude to Duality
finite group (G,·)
algebra of functions: F(G) ∋ φ: G → K
F(G)⊗F(G) ≃ F(G×G)
finite dim case


3.1 Prelude to Duality
finite group (G,·)
algebra of functions: F(G) ∋ φ: G → K
F(G) ⊗ F(G) ≃ F(G×G)
finite dim case
Δφ = φ(1) ⊗ φ(2)
(Δφ)(g,h) =

3.1 Prelude to Duality

finite group $(G,\cdot)$

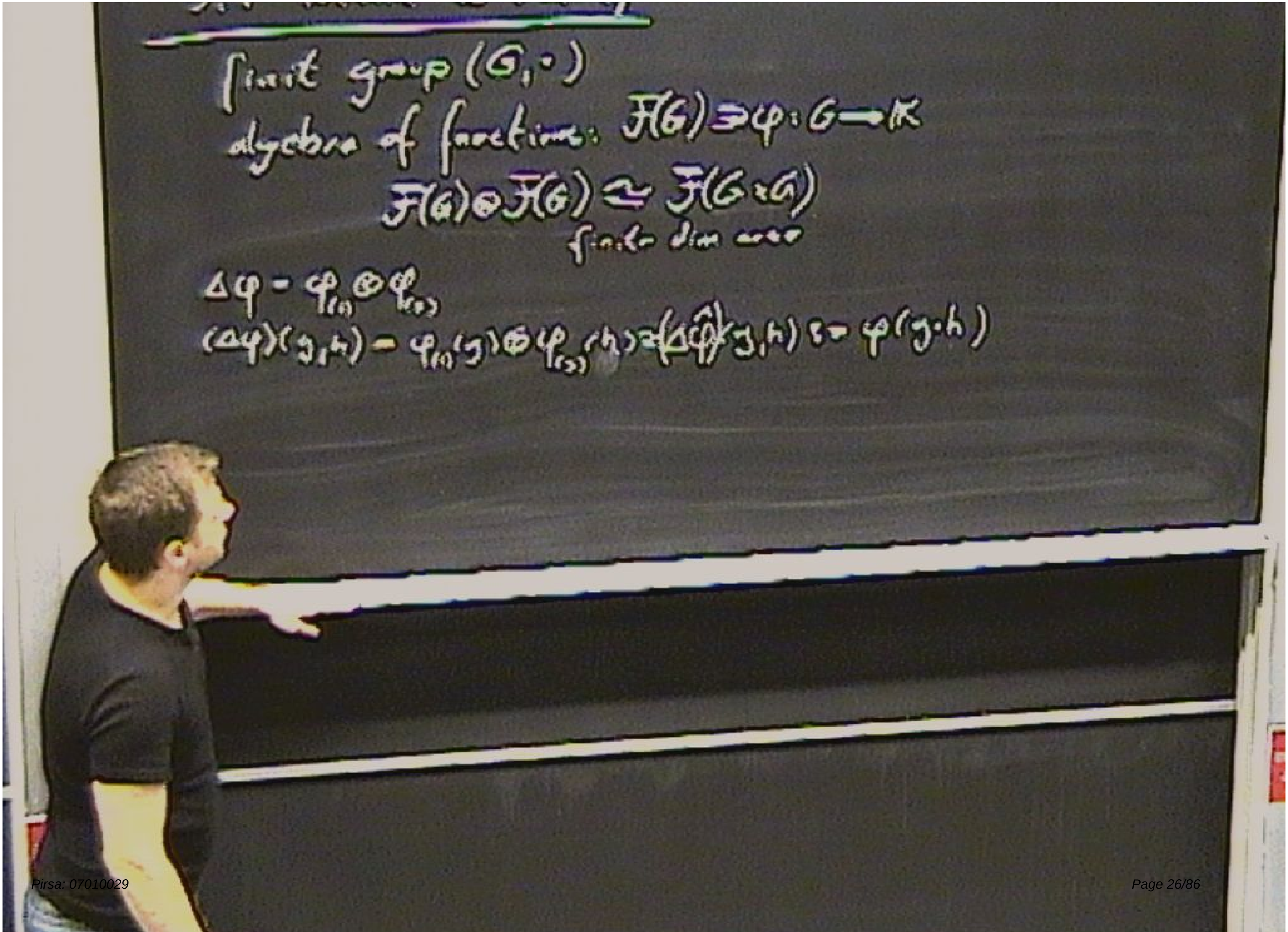
algebra of functions: $\mathcal{F}(G) \ni \varphi : G \to \mathbb{K}$

$$\mathcal{F}(G)\otimes\mathcal{F}(G) \simeq \mathcal{F}(G\times G)$$

finite dim case

$$\Delta\varphi = \varphi_{(1)}\otimes\varphi_{(2)}$$

$$(\Delta\varphi)(g,h) = \varphi_{(1)}(g)\otimes\varphi_{(2)}(h) = (\Delta\varphi)(g,h) := \varphi(g\cdot h)$$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$\mathcal{F}(G)\otimes\mathcal{F}(G)\simeq\mathcal{F}(G\times G)$
finite dim case
$\Delta\varphi=\varphi_{(1)}\otimes\varphi_{(2)}$
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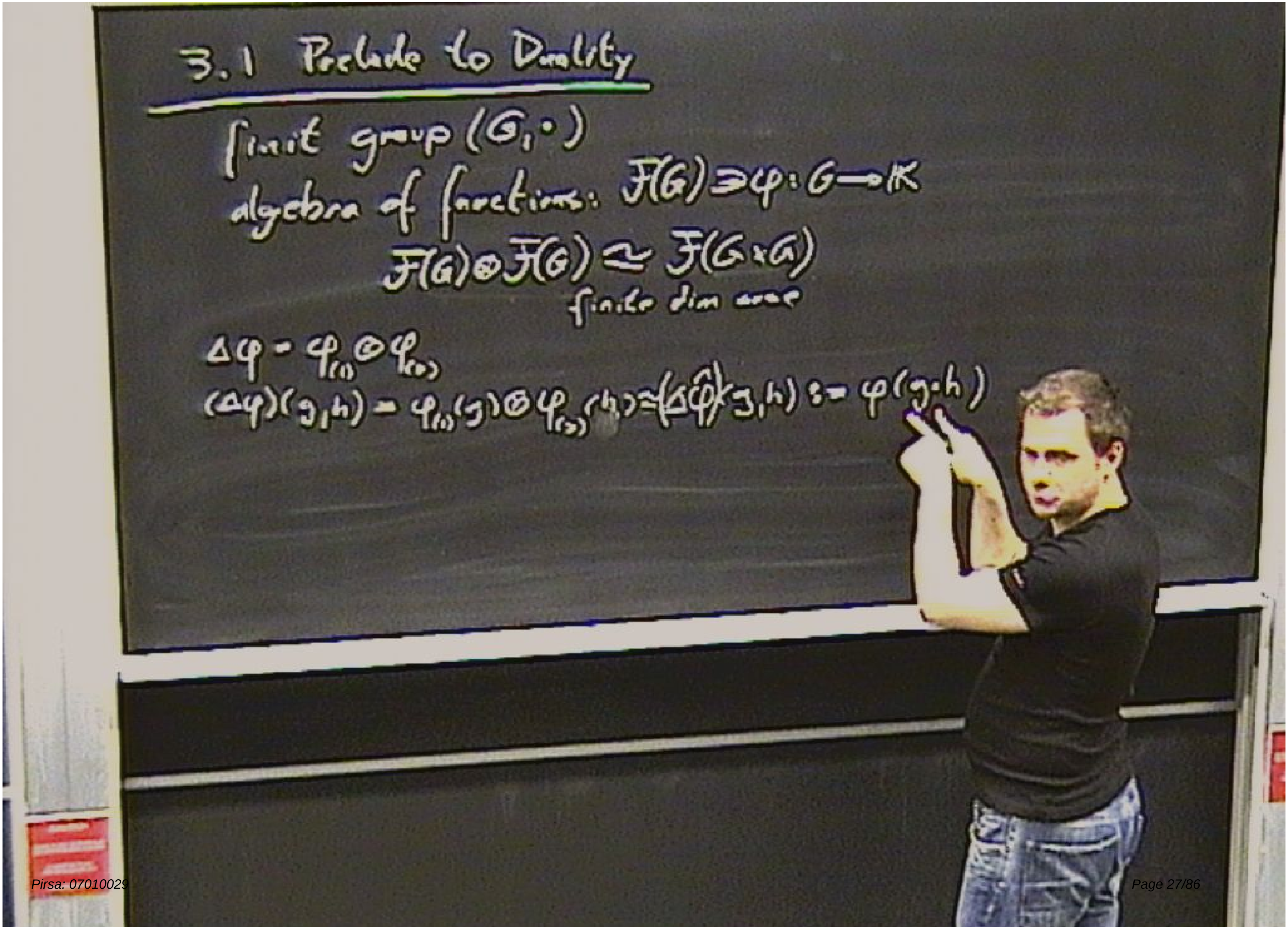
3.1 Prelude to Duality
finite group $(G, \cdot)$
algebra of functions: $\mathcal{F}(G) \ni \varphi : G \to \mathbb{K}$
$\mathcal{F}(G) \otimes \mathcal{F}(G) \simeq \mathcal{F}(G \times G)$
finite dim case
$\Delta\varphi = \varphi_{(1)} \otimes \varphi_{(2)}$
$(\Delta\varphi)(g,h) = \varphi_{(1)}(g) \otimes \varphi_{(2)}(h) = (\Delta\varphi)(g,h) := \varphi(g \cdot h)$

3.1 Prelude to Duality

finit group $(G, \cdot)$

algebra of functions: $\mathcal{F}(G) \ni \varphi : G \to \mathbb{K}$

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finite dim case

$$\Delta \varphi = \varphi_{(1)} \otimes \varphi_{(2)}$$

$$(\Delta\varphi)(g,h) = \varphi_{(1)}(g) \otimes \varphi_{(2)}(h) \simeq (\Delta\hat{\varphi})(g,h) := \varphi(g \cdot h)$$

Duality: Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $\mathcal{F}(G)$

3.1 Prelude to Duality

finite group $(G,\cdot)$

algebra of functions: $\mathcal{F}(G) \ni \varphi : G \to \mathbb{K}$

$$\mathcal{F}(G)\otimes\mathcal{F}(G) \simeq \mathcal{F}(G\times G) \qquad \varphi_1(g)\otimes\varphi_2(h) \longrightarrow \varphi_1(g)\cdot\varphi_2(h)$$

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finite dim case

$$\Delta\varphi = \varphi_{(1)} \otimes \varphi_{(2)}$$

$$(\Delta\varphi)(g,h) = \varphi_{(1)}(g) \otimes \varphi_{(2)}(h) = (\Delta\varphi) \ldots \varphi(g \cdot h)$$

Duality: Check Hopf-Algebra Ax... d see how the group-properties ... d to the coalgebra of $\mathcal{F}(G)$

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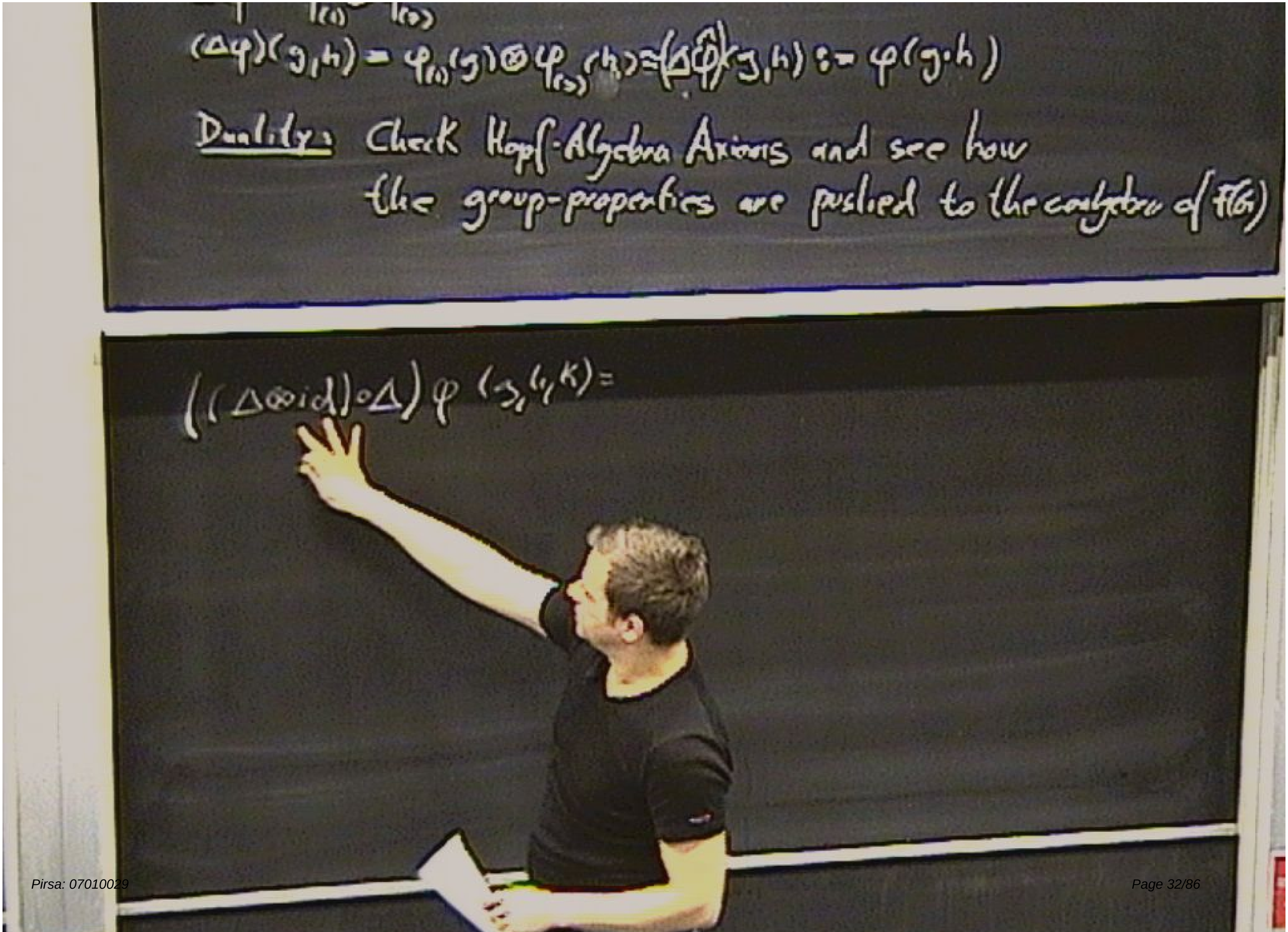
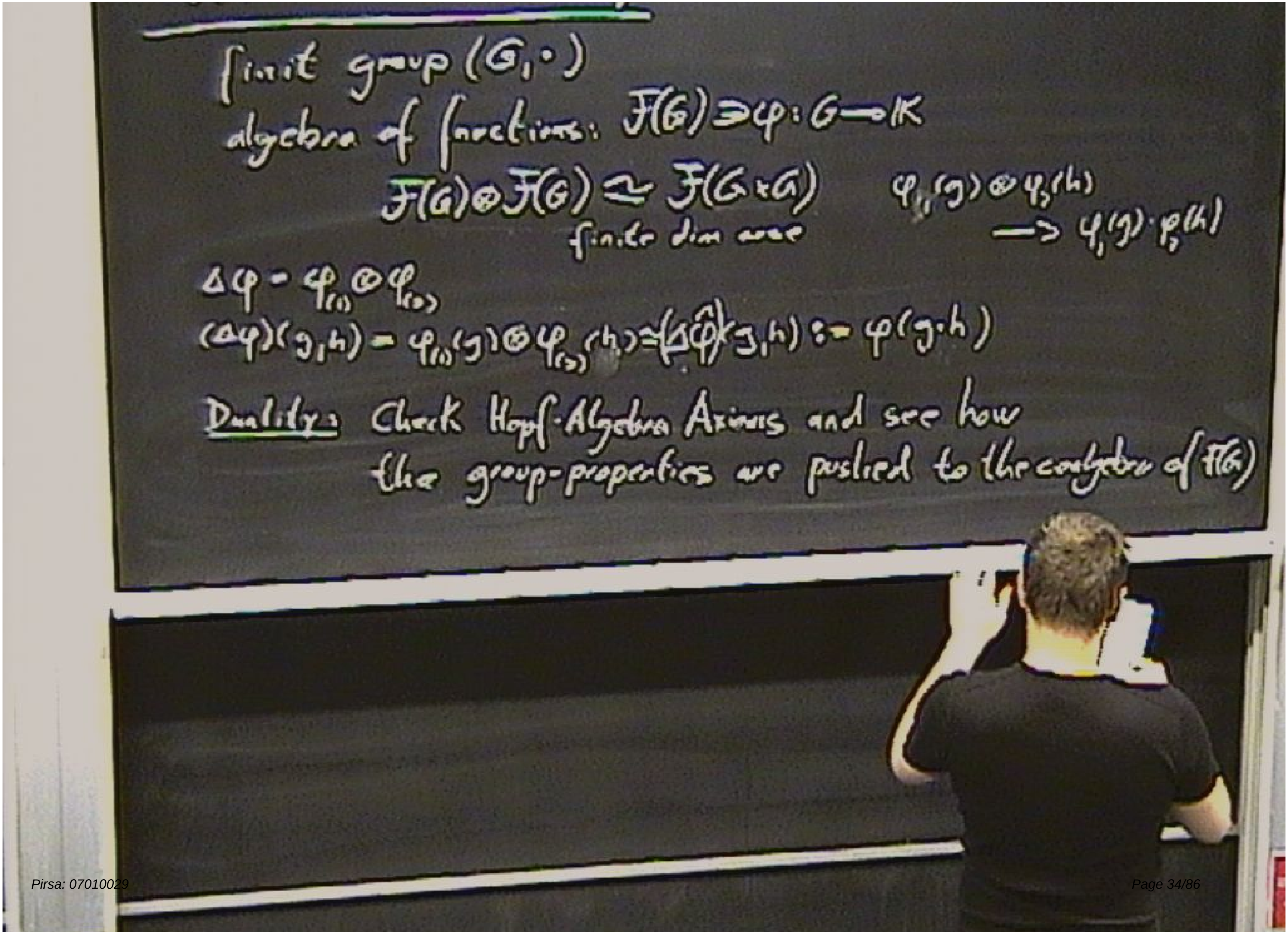
<u>Duality:</u> Check Hopf-Algebra Axioms an[illegible]ee how the group-properties are pusl[illegible] to the coalgebra of $\mathcal{F}(G)$


(Δφ)(g,h) = φ(1)(g) ⊗ φ(2)(h) = (Δφ)(g,h) := φ(g·h)
Duality: Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of F(G)
((Δ⊗id)∘Δ)φ (g,h,k) =

$$(\Delta\varphi)(g,h) = \varphi_{(1)}(g)\otimes\varphi_{(2)}(h) = (\Delta\hat{\varphi})(g,h) := \varphi(g\cdot h)$$

<u>Duality:</u> Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $F(G)$

$$((\Delta\otimes id)\circ\Delta)\varphi(g,h,k) = \varphi((g\cdot h)\cdot k)$$
$$= \varphi(g\cdot(h\cdot k)) = ((id\otimes\Delta)\circ\Delta\varphi)(g,h,k)$$


finit group $(G,\cdot)$
algebra of functions: $\mathcal{F}(G) \ni \varphi : G \to \mathbb{K}$
$\mathcal{F}(G) \otimes \mathcal{F}(G) \simeq \mathcal{F}(G \times G)$
finite dim case
$\varphi_1(g) \otimes \varphi_2(h) \longrightarrow \varphi_1(g) \cdot \varphi_2(h)$
$\Delta\varphi = \varphi_{(1)} \otimes \varphi_{(2)}$
$(\Delta\varphi)(g,h) = \varphi_{(1)}(g) \otimes \varphi_{(2)}(h) \simeq (\Delta\varphi)(g,h) := \varphi(g \cdot h)$
Duality: Check Hopf-Algebra Axioms and see how
the group-properties are pushed to the coalgebra of $\mathcal{F}(G)$

finit group $(G,\cdot)$

algebra of functions: $\mathcal{F}(G) \ni \varphi : G \longrightarrow \mathbb{K}$

$$\mathcal{F}(G)\otimes\mathcal{F}(G) \simeq \mathcal{F}(G\times G) \qquad \varphi_1(g)\otimes\varphi_2(h) \longrightarrow \varphi_1(g)\cdot\varphi_2(h)$$

finite dim case

$$\Delta\varphi = \varphi_{(1)}\otimes\varphi_{(2)}$$

$$(\Delta\varphi)(g,h) = \varphi_{(1)}(g)\otimes\varphi_{(2)}(h) = (\Delta\varphi)(g,h) := \varphi(g\cdot h) \quad , \ \epsilon(\varphi) := \varphi(\mathbb{1})$$

Duality: Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $\mathcal{F}(G)$

$$[((\Delta\otimes id)\circ\Delta)\varphi](g,h,k) = \varphi((g\cdot h)\cdot k)$$
$$= \varphi(g\cdot(h\cdot k)) = [(id\otimes\Delta)\circ\Delta\varphi](g,h,k)$$

$$[(id\otimes\epsilon)\circ\Delta\varphi]$$

finit group $(G,\cdot)$

algebra of functions: $\mathcal{F}(G) \ni \varphi : G \to \mathbb{K}$

$$\mathcal{F}(G) \otimes \mathcal{F}(G) \simeq \mathcal{F}(G \times G) \qquad \varphi_1(g) \otimes \varphi_2(h) \longrightarrow \varphi_1(g) \cdot \varphi_2(h)$$

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__Duality:__ Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $\mathcal{F}(G)$

$$\left[\left((\Delta \otimes id) \circ \Delta\right) \varphi\right](g,h,k) = \varphi((g \cdot h) \cdot k)$$
$$= \varphi(g \cdot (h \cdot k)) = \left[(id \otimes \Delta) \circ \Delta\varphi\right](g,h,k)$$

$$\left[(id \otimes \epsilon) \circ \Delta\varphi\right] = \Delta\varphi(g,1) = \varphi(g)$$
$$\Delta\varphi(1,g) = \left((\epsilon \otimes id) \circ \Delta\varphi\right)(g,h)$$

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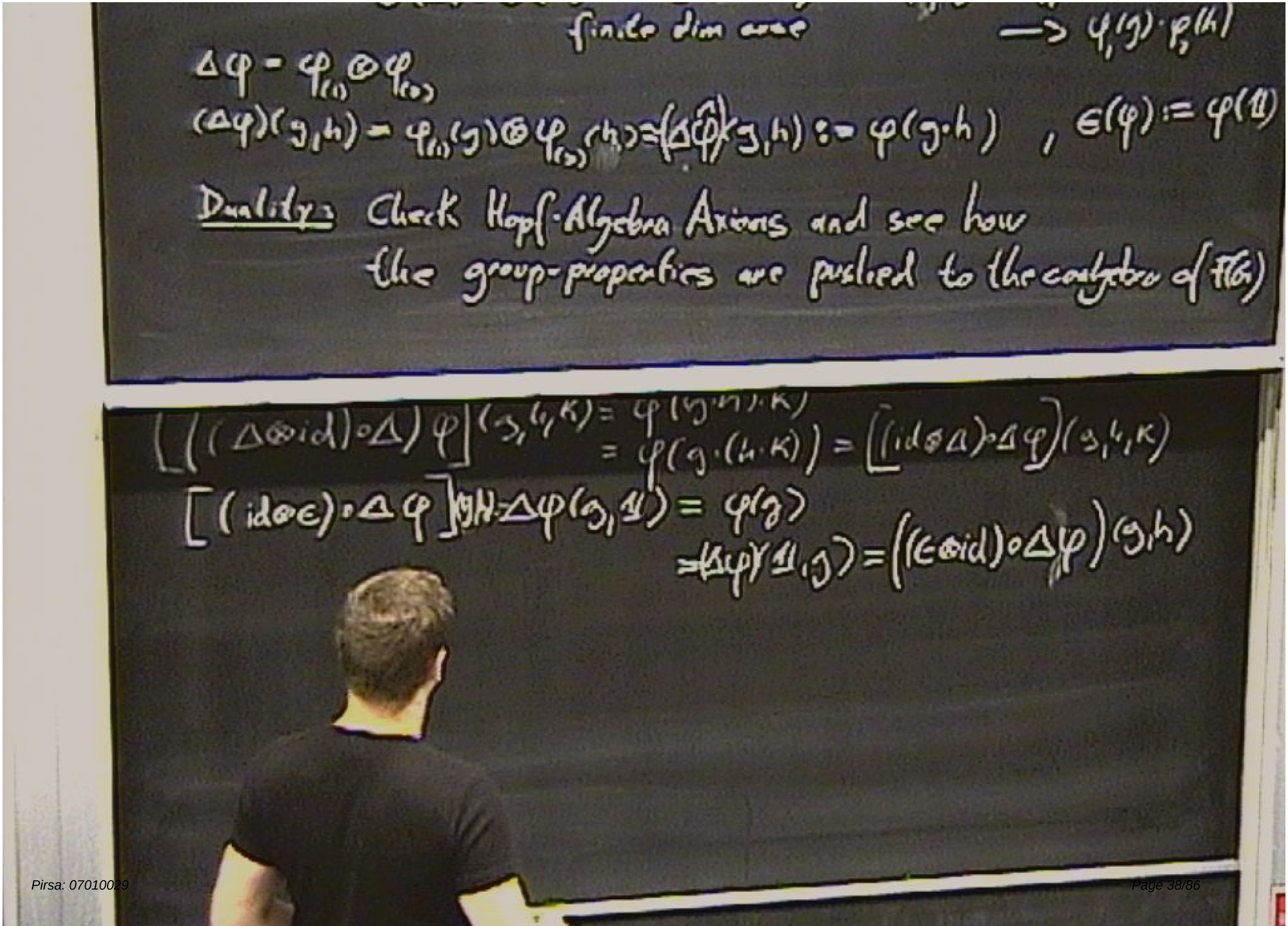
<u>Duality:</u> Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $\mathcal{F}(G)$

$$\left[\left((\Delta \otimes id) \circ \Delta\right) \varphi\right](g,h,k) = \varphi((g \cdot h) \cdot k)$$
$$= \varphi(g \cdot (h \cdot k)) = \left[\left((id \otimes \Delta) \circ \Delta \varphi\right)\right](g,h,k)$$

$$\left[(id \otimes \epsilon) \circ \Delta \varphi\right] = \Delta\varphi(g, \mathbb{1}) = \varphi(g)$$
$$= (\Delta\varphi)(\mathbb{1}, g) = \left((\epsilon \otimes id) \circ \Delta \varphi\right)(g,h)$$


finite dim case $\longrightarrow \varphi_1(g)\cdot\varphi_2(h)$
$\Delta\varphi = \varphi_{(1)} \otimes \varphi_{(2)}$
$(\Delta\varphi)(g,h) = \varphi_{(1)}(g) \otimes \varphi_{(2)}(h) = (\Delta\varphi)(g,h) := \varphi(g\cdot h)$, $\epsilon(\varphi) := \varphi(1)$
Duality: Check Hopf-Algebra Axioms and see how
the group-properties are pushed to the coalgebra of F(G)
$[((\Delta\otimes id)\circ\Delta)\varphi](g,h,k) = \varphi((g\cdot h)\cdot k)$
$= \varphi(g\cdot(h\cdot k)) = [(id\otimes\Delta)\circ\Delta\varphi](g,h,k)$
$[(id\otimes\epsilon)\circ\Delta\varphi](g) = \Delta\varphi(g,1) = \varphi(g)$
$= (\Delta\varphi)(1,g) = ((\epsilon\otimes id)\circ\Delta\varphi)(g,h)$

finite dim rep $\longrightarrow \varphi_1(g)\cdot\varphi_2(h)$

$$\Delta\varphi = \varphi_{(1)}\otimes\varphi_{(2)}$$

$$(\Delta\varphi)(g,h) = \varphi_{(1)}(g)\otimes\varphi_{(2)}(h) = (\Delta\varphi)(g,h) := \varphi(g\cdot h)\ , \quad \epsilon(\varphi) := \varphi(1)$$

<u>Duality:</u> Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $F(G)$

$$\big[\big((\Delta\otimes id)\circ\Delta\big)\varphi\big](g,h,k) = \varphi((g\cdot h)\cdot k)$$
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$$= (\Delta\varphi)(1,h) = \big((\epsilon\otimes id)\circ\Delta\varphi\big)(g,h)$$

$$(id\otimes\epsilon)(\Delta\varphi)(g,h) = (id\otimes\epsilon)\varphi(g\cdot h)$$

finite dim case $\longrightarrow \varphi_1(g)\cdot\varphi_2(h)$

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<u>Duality:</u> Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $F(G)$

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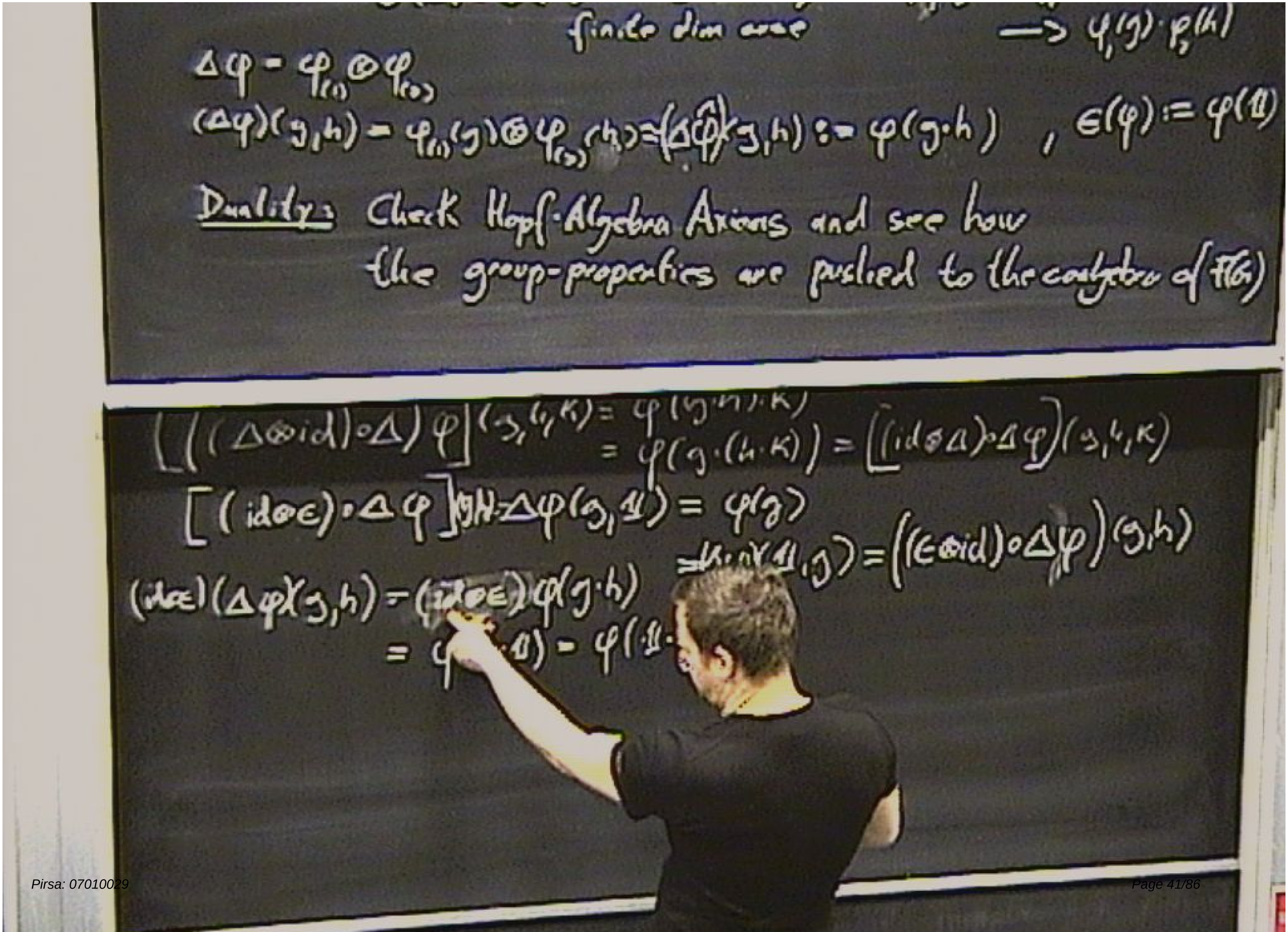
$$= \varphi(g\cdot\mathbb{1})$$


finite dim case → φ1(g)·φ2(h)
Δφ = φ(1) ⊗ φ(2)
(Δφ)(g,h) = φ(1)(g) ⊗ φ(2)(h) = (Δφ)(g,h) := φ(g·h) , ε(φ) := φ(1)
Duality: Check Hopf-Algebra Axioms and see how
the group-properties are pushed to the coalgebra of F(G)
[((Δ⊗id)∘Δ)φ](g,h,k) = φ((g·h)·k)
= φ(g·(h·k)) = [((id⊗Δ)∘Δφ](g,h,k)
[(id⊗ε)∘Δφ](g) = Δφ(g,1) = φ(g)
= ((ε⊗id)∘Δφ)(g,h)
(id⊗ε)(Δφ)(g,h) = (id⊗ε)φ(g·h)
= φ(g·1) = φ(1·

finite dim case $\longrightarrow \varphi_1(g)\cdot\varphi_2(h)$

$$\Delta\varphi = \varphi_{(1)}\otimes\varphi_{(2)}$$

$$(\Delta\varphi)(g,h) = \varphi_{(1)}(g)\otimes\varphi_{(2)}(h) = (\Delta\varphi)(g,h) := \varphi(g\cdot h)\ , \quad \epsilon(\varphi) := \varphi(\mathbb{1})$$

__Duality:__ Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $F(G)$

$$\left[\left((\Delta\otimes id)\circ\Delta\right)\varphi\right](g,h,k) = \varphi((g\cdot h)\cdot k)$$
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$$(id\otimes\epsilon)(\Delta\varphi)(g,h) = (id\otimes\epsilon)\varphi(g\cdot h)$$
$$= \varphi(g\cdot\mathbb{1}) = \varphi(\mathbb{1}\cdot g) = (\epsilon\otimes id)\varphi(h\cdot g)$$

$\mathcal{F}(G)\otimes\mathcal{F}(G) \cong \mathcal{F}(G\times G)$ $\qquad \varphi_1(g)\otimes\varphi_2(h)$

finite dim case $\qquad \longrightarrow \varphi_1(g)\cdot\varphi_2(h)$

$\Delta\varphi = \varphi_{(1)}\otimes\varphi_{(2)}$

$(\Delta\varphi)(g,h) = \varphi_{(1)}(g)\otimes\varphi_{(2)}(h) = (\Delta\varphi)(g,h) := \varphi(g\cdot h)$, $\epsilon(\varphi) := \varphi(\mathbb{1})$

Duality: Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $\mathcal{F}(G)$

$= \varphi(g\cdot(h\cdot k)) = [((id\otimes\Delta)\circ\Delta)\varphi](g,h,k)$

$[(id\otimes\epsilon)\circ\Delta\varphi](g) = \Delta\varphi(g,\mathbb{1}) = \varphi(g)$

$= (\Delta\varphi)(\mathbb{1},g) = ((\epsilon\otimes id)\circ \ldots)(g,h)$

$(id\otimes\epsilon)(\Delta\varphi)(g,h) = (id\otimes\epsilon)\varphi(g\cdot h)$

$= \varphi(g\cdot\mathbb{1}) = \varphi(\mathbb{1}\cdot g) = (\epsilon\otimes id)\varphi(h\cdot g) = [(\ldots)](g,h)$

$\Delta\varphi = \varphi_{(1)}\otimes\varphi_{(2)}$

$[\mu\circ(S\otimes id)\circ\Delta\varphi](g) = \underbrace{[S(\varphi_{(1)})](g)}_{\varphi_{(1)}(g^{-1})}\cdot\varphi_{(2)}(g) = \varphi_{(1)}(g^{-1})\cdot \ldots$

$\Delta\varphi = \varphi_{(1)} \otimes \varphi_{(2)}$

$(\Delta\varphi)(g,h) = \varphi_{(1)}(g) \otimes \varphi_{(2)}(h) = (\Delta\varphi)(g,h) := \varphi(g\cdot h)$, $\epsilon(\varphi) := \varphi(\mathbb{1})$

Duality: Check Hopf-Algebra Axioms and see how the group-properties are pushed to the coalgebra of $\mathcal{F}(G)$

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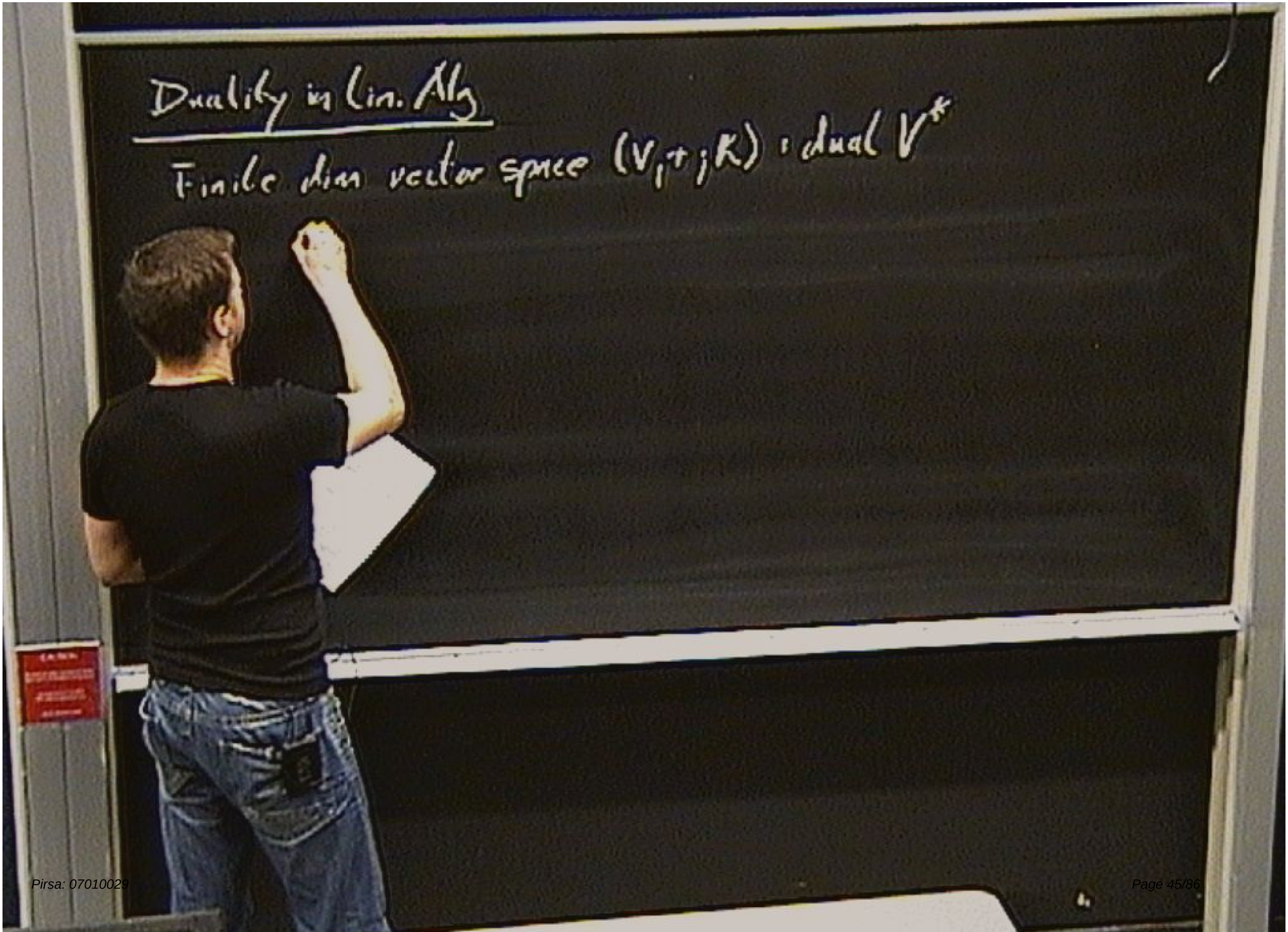
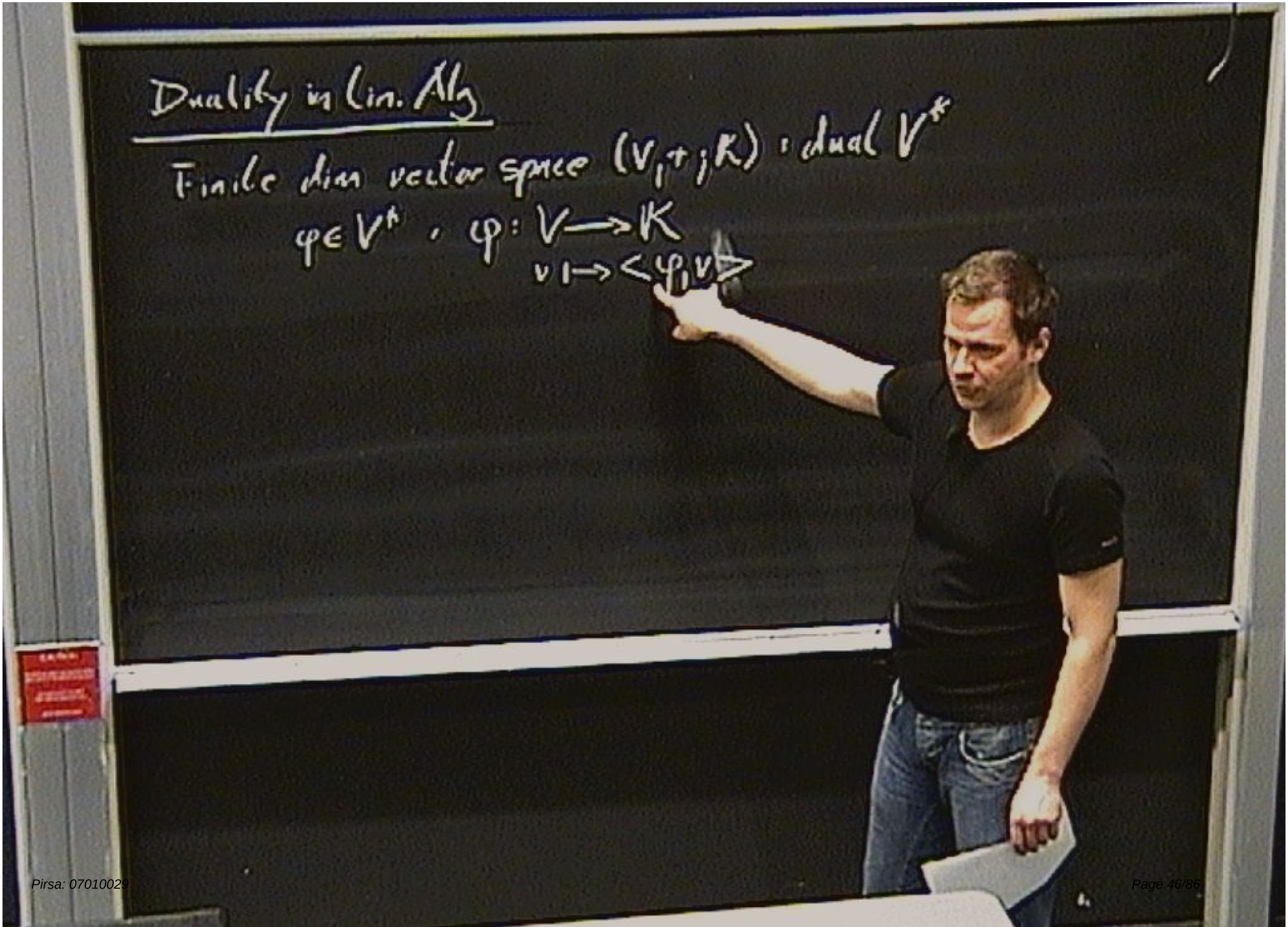
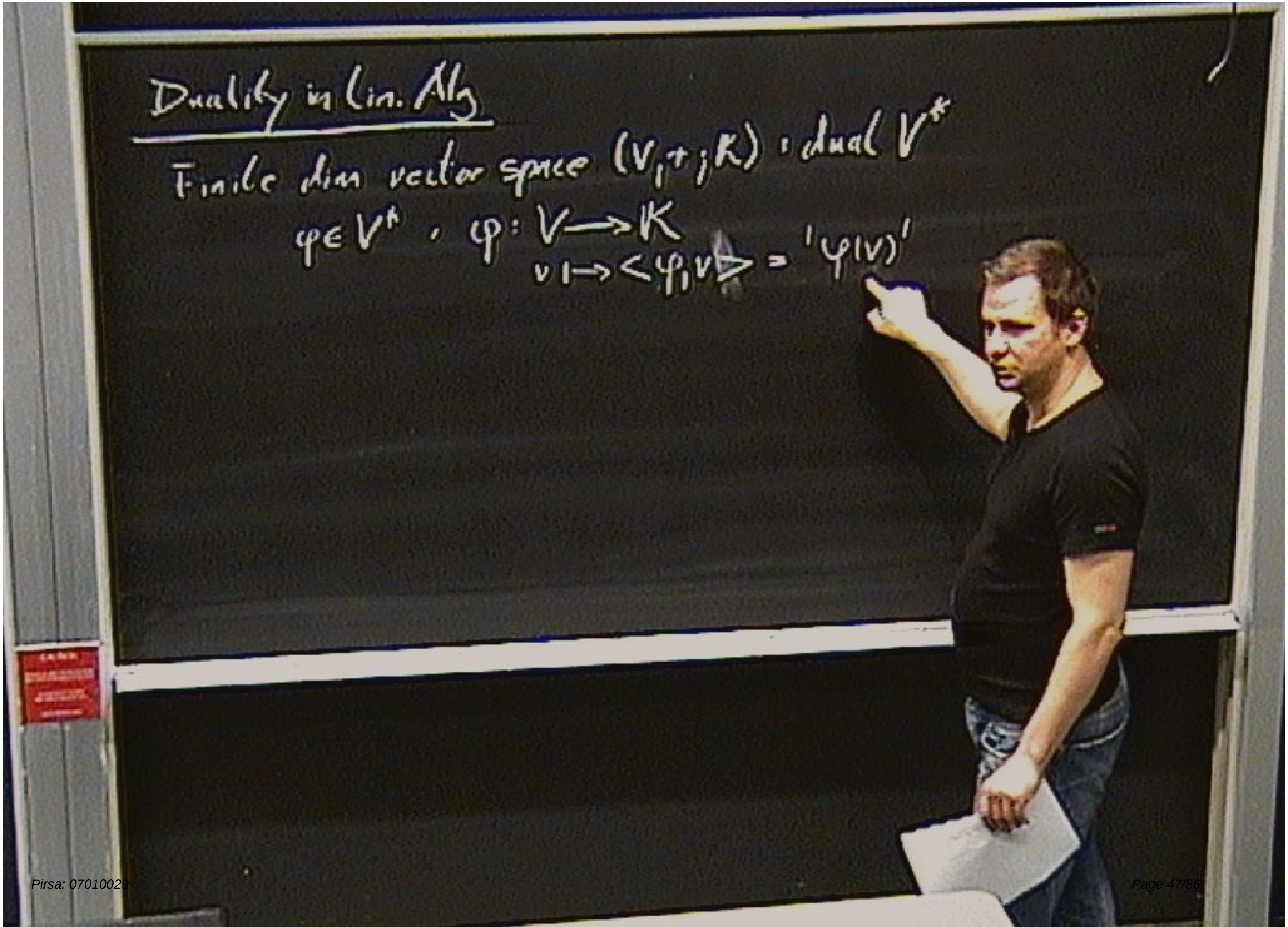
$\Delta\varphi = \varphi_{(1)} \otimes \varphi_{(2)}$

$\left[\mu\circ(S\otimes id)\circ\Delta\varphi\right](g) = \underbrace{\left[S(\varphi_{(1)})\right](g)}_{\varphi_{(1)}(g^{-1})} \cdot \varphi_{(2)}(g) = \varphi_{(1)}(g^{-1})\cdot\varphi_{(2)}(g)$
$= \varphi(g^{-1}\cdot g) = \varphi(\mathbb{1})$


Duality in Lin. Alg
Finite dim vector space $(V,+,\cdot,\mathbb{R})$: dual V^*


Duality in Lin. Alg
Finite dim vector space $(V, +; K)$: dual V^*
$\varphi \in V^*$, $\varphi : V \longrightarrow K$
$v \longmapsto \langle \varphi, v \rangle$


Duality in Lin. Alg
Finite dim vector space $(V,+;K)$: dual V^*
$\varphi \in V^*$, $\varphi : V \longrightarrow K$
$v \longmapsto \langle \varphi, v \rangle = '\varphi(v)'$

Duality in Lin. Alg

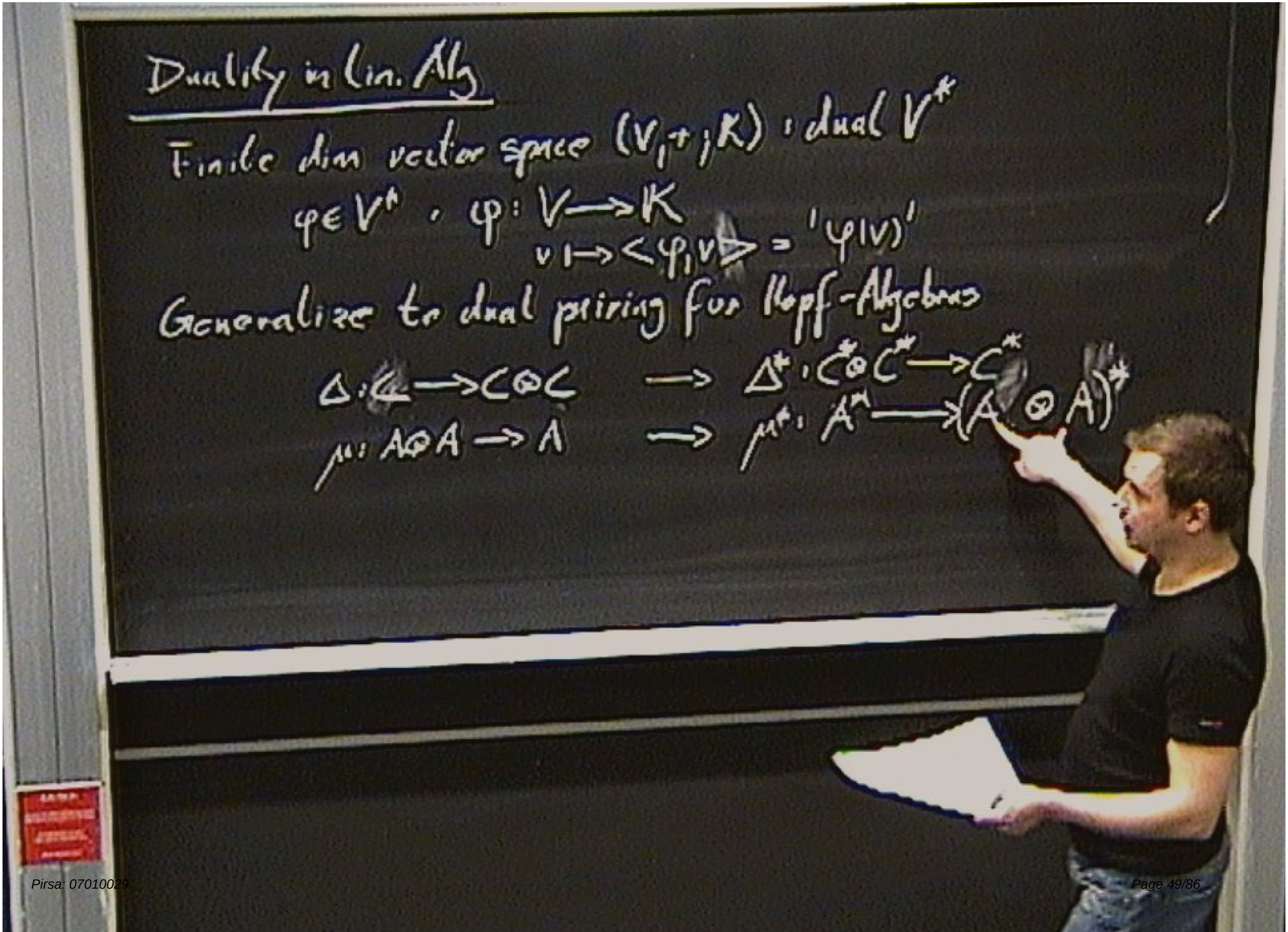
Finite dim vector space $(V,+;K)$: dual V^*

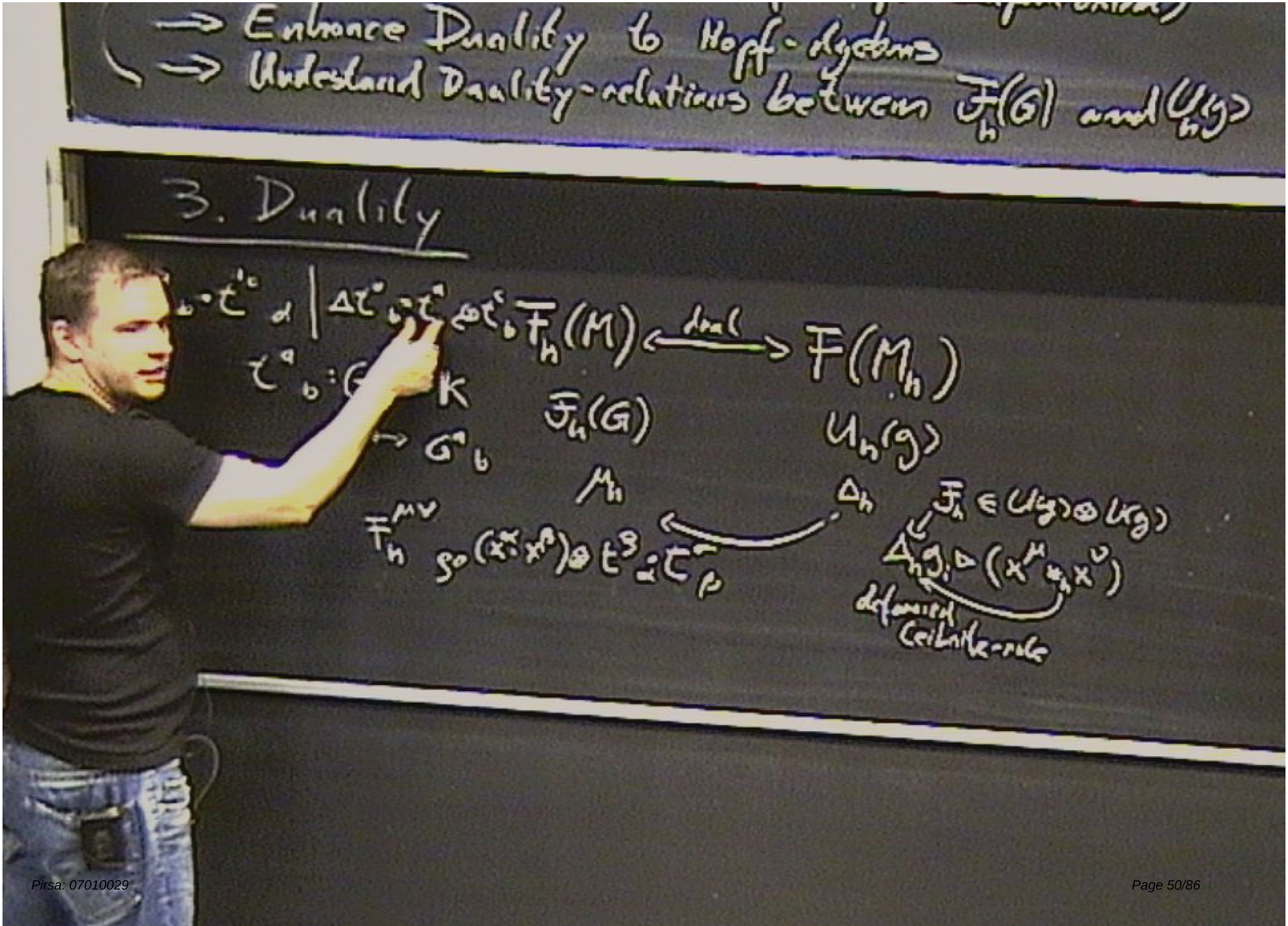
$$\varphi \in V^* , \quad \varphi : V \longrightarrow K$$

$$v \longmapsto \langle \varphi, v \rangle = \text{'}\varphi(v)\text{'}$$

Generalize to dual pairing for Hopf-Algebras

$$\Delta : C \longrightarrow C \otimes C \quad \longrightarrow \quad \Delta^* : C \otimes C \longrightarrow C$$


Duality in Lin. Alg
Finite dim vector space $(V,+;K)$: dual V^*
$\varphi \in V^*$, $\varphi : V \longrightarrow K$
$v \longmapsto \langle \varphi, v \rangle = \text{'}\varphi(v)\text{'}$
Generalize to dual pairing for Hopf-Algebras
$\Delta : C \longrightarrow C \otimes C \longrightarrow \Delta^* : C^* \otimes C^* \longrightarrow C^*$
$\mu : A \otimes A \rightarrow A \longrightarrow \mu^* : A^* \longrightarrow (A \otimes A)^*$


⇒ Enhance Duality to Hopf-algebras
⇒ Understand Duality-relations between F_h(G) and U_h(g)
3. Duality
F_h(M)
dual
F(M_h)
F_h(G)
U_h(g)
M_h
Δ_h
deformed
Leibniz-rule

Duality in Lin. Alg

Finite dim vector space $(V, +; K)$: dual V^*

$\varphi \in V^*$, $\varphi : V \longrightarrow K$

$v \longmapsto \langle \varphi, v \rangle = \text{'}\varphi(v)\text{'}$

Generalize to dual pairing for Hopf-Algebras

$\Delta : C \longrightarrow C \otimes C \quad \longrightarrow \quad \Delta^* : C^* \otimes C^* \longrightarrow C^*$

$\mu : A \otimes A \longrightarrow A \quad \longrightarrow \quad \mu^* : A^* \longrightarrow (A \otimes A)^*$

$A^* \otimes A^*$

$A^* \otimes A^*$

3.1 Def 'Dual pairing'

Two Hopf-algebras H, B are called dually paired if there exists a bilinear map

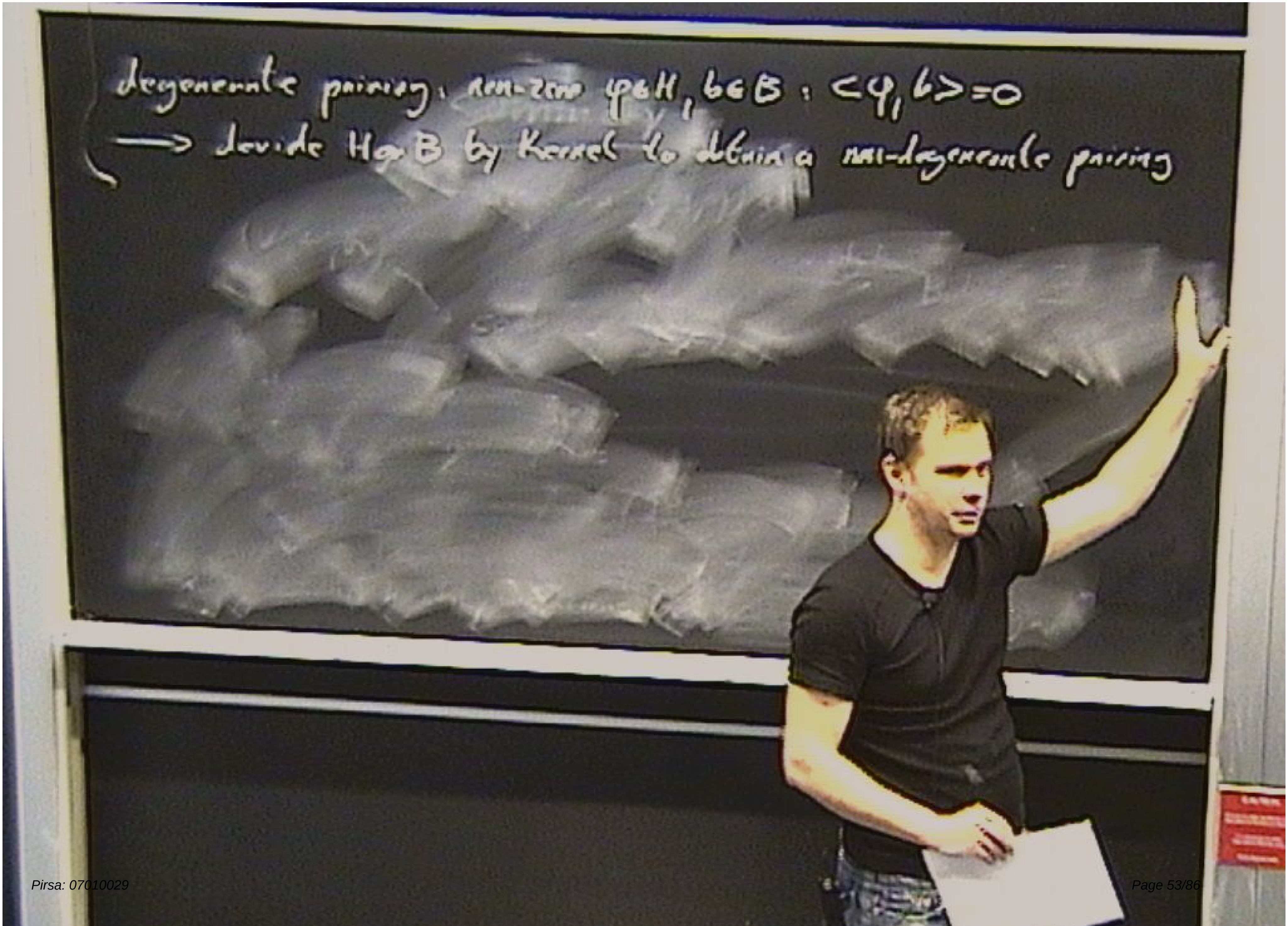
$$\langle \cdot , \cdot \rangle : H \otimes B \longrightarrow \mathbb{K}$$

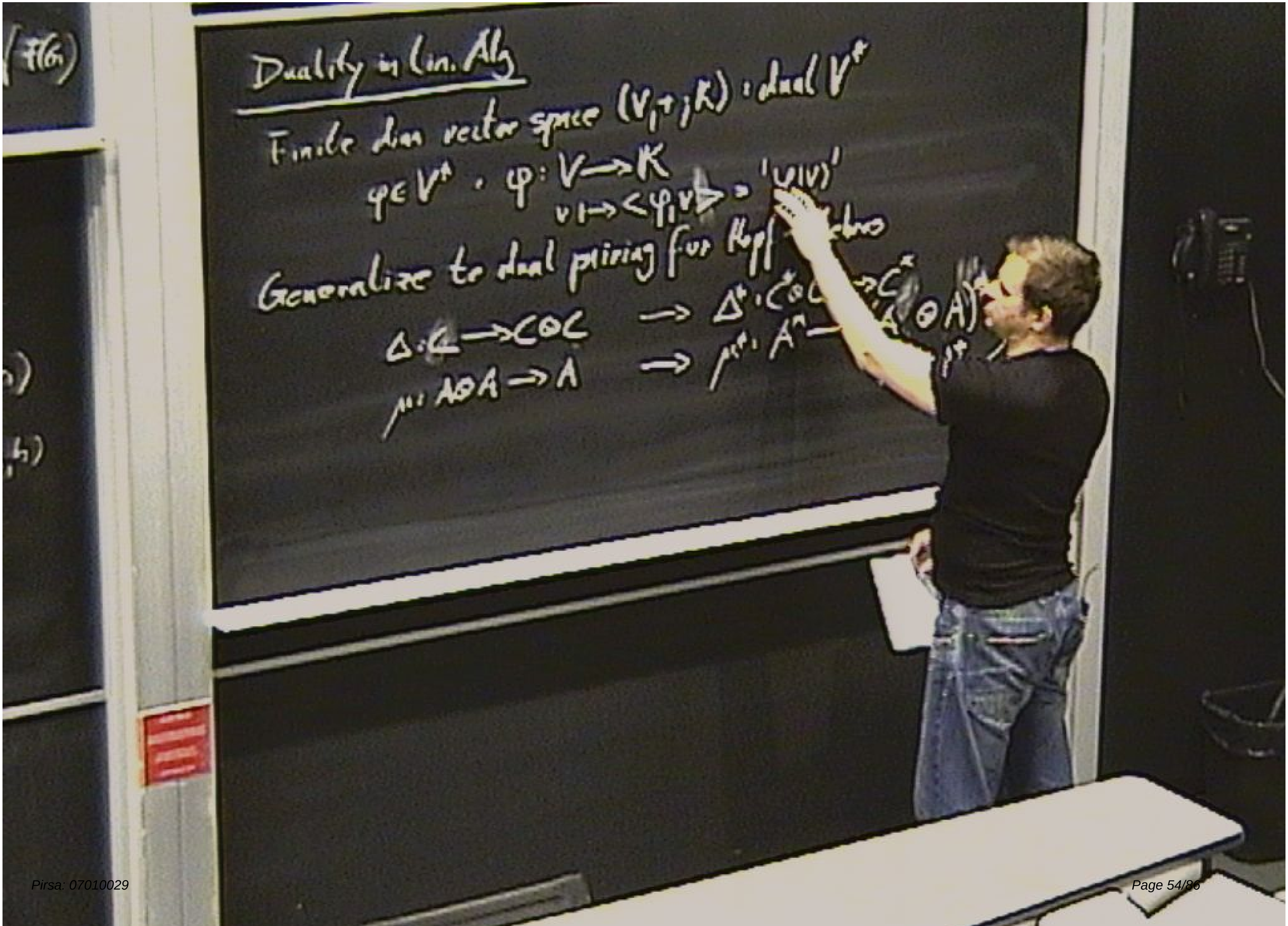
such that: $\langle \varphi \cdot \psi, b \rangle = \langle \varphi \otimes \psi, \Delta b \rangle = \langle \varphi, b_{(1)} \rangle \langle \psi, b_{(2)} \rangle$

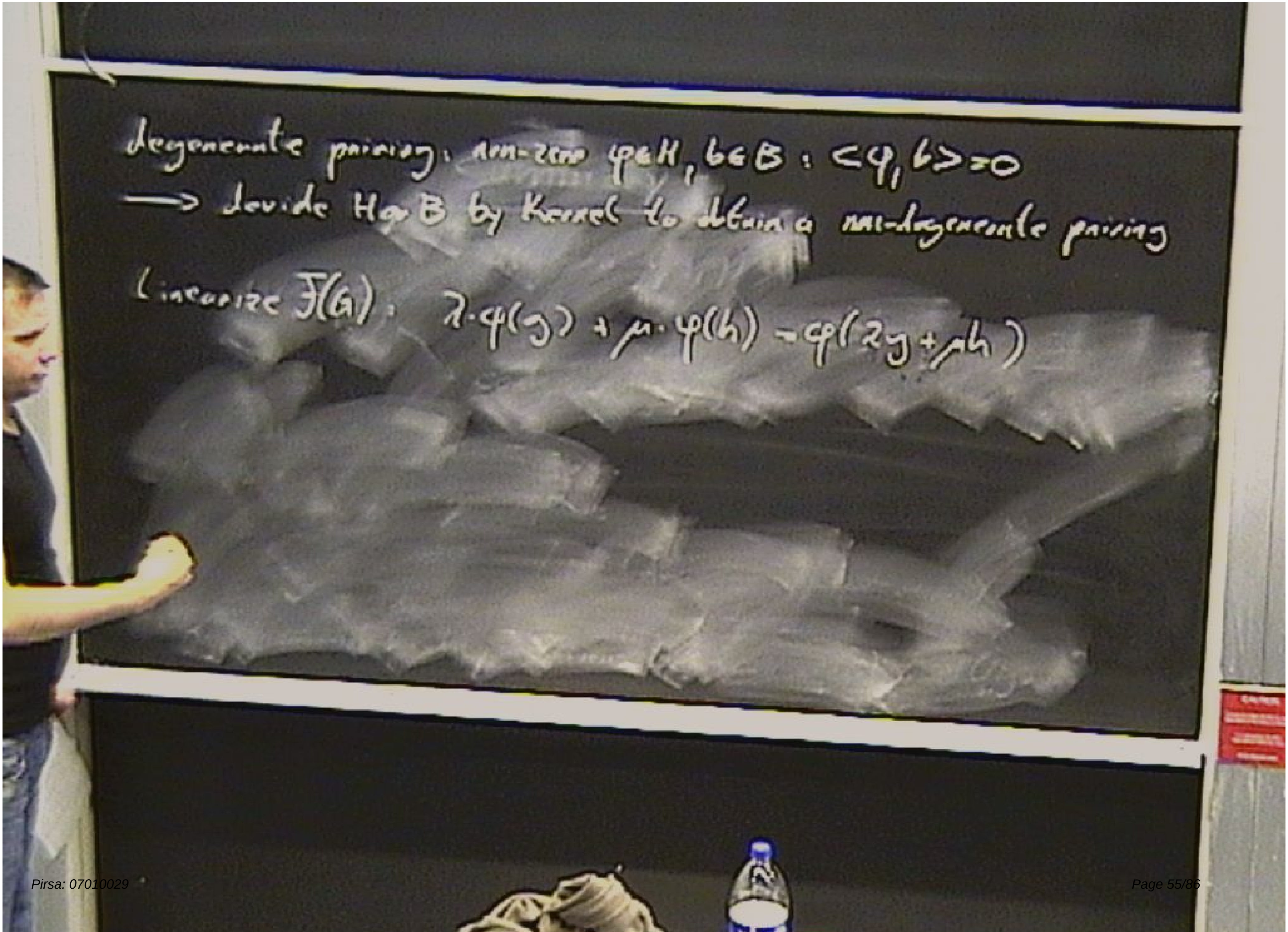
$$\langle \mathbb{1}, b \rangle = \epsilon(b)$$

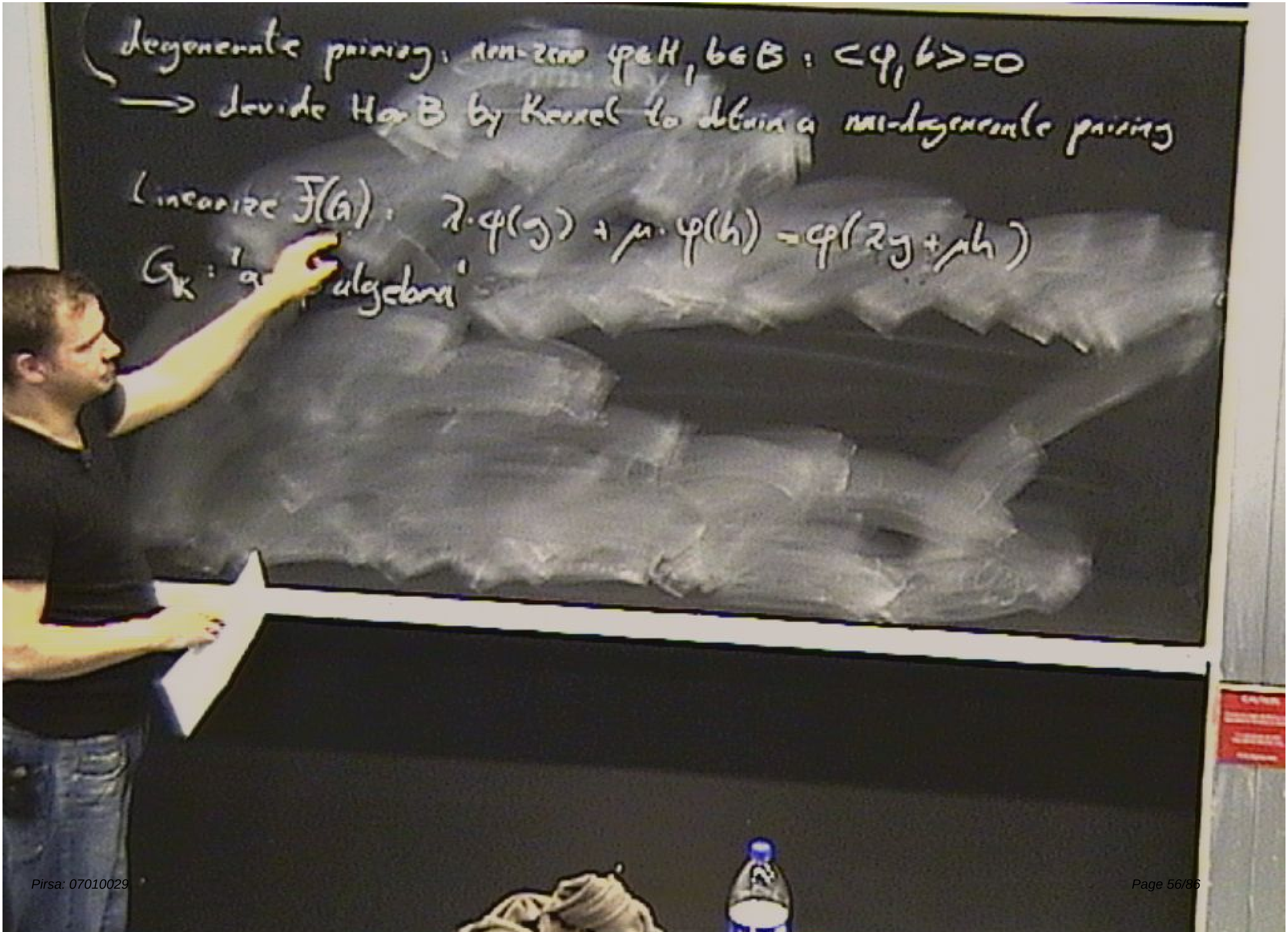
$$\langle \varphi, b \cdot c \rangle = \langle \Delta \varphi, b \otimes c \rangle = \langle \varphi_{(1)}, b \rangle \langle \varphi_{(2)}, c \rangle$$

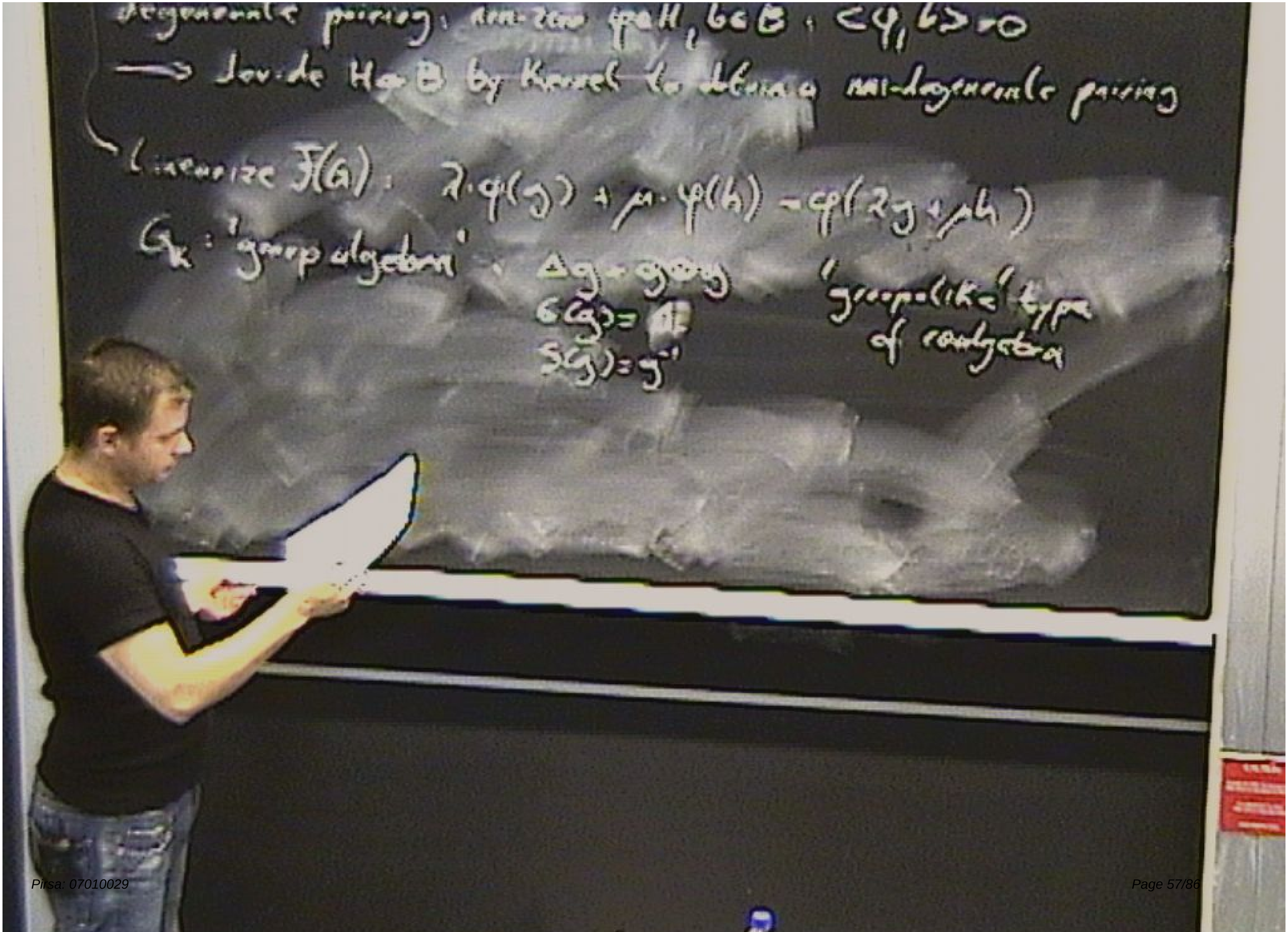
$$\langle \varphi, \mathbb{1} \rangle = \epsilon(\varphi)$$


degenerate pairing: non-zero $\varphi \in H$, $b \in B$: $\langle \varphi, b \rangle = 0$
$\longrightarrow$ divide $H \otimes B$ by kernel to obtain a non-degenerate pairing


Duality in Lin. Alg
Finite dim vector space (V,+,K) : dual V*
φ ∈ V* , φ : V → K
v ↦ <φ,v> = 'φ(v)'
Generalize to dual pairing for Hopf
Δ : C → C⊗C
μ : A⊗A → A


degenerate pairing: non-zero φ∈H, b∈B : <φ, b>=0
⟶ devide H⊗B by kernel to obtain a non-degenerate pairing
linearize F(G): λ·φ(g) + μ·φ(h) − φ(λg+μh)


degenerate pairing: non-zero φ∈H, b∈B : <φ, b>=0
→ divide H or B by kernel to obtain a non-degenerate pairing
Linearize F(G): λ·φ(g) + μ·φ(h) = φ(λg+μh)
algebra'


degenerate pairing: non-zero φ ∈ H, b ∈ B : ⟨φ, b⟩ = 0
⟶ divide H ⊗ B by kernel to obtain a non-degenerate pairing
linearize F(G) : λ·φ(g) + μ·φ(h) = φ(λg + μh)
G_K : 'group algebra' : Δg = g ⊗ g
ε(g) = 1
S(g) = g⁻¹
'grouplike' type
of coalgebra

degenerate pairing: non-zero $\varphi \in H$, $b \in B$: $\langle \varphi, b \rangle = 0$

$\longrightarrow$ divide $H \otimes B$ by kernel to obtain a non-degenerate pairing

Linearize $\mathcal{F}(G)$: $\lambda \cdot \varphi(g) + \mu \cdot \varphi(h) = \varphi(\lambda g + \mu h)$

G_K : 'group algebra' : $\Delta g = g \otimes g$, $\epsilon(g) = 1$, $S(g) = g^{-1}$ — 'group-like' type of coalgebra

$\varphi \in \mathcal{F}(G_K)$, $h = \sum h(g_i) \cdot g_i \in G_K$

$$\boxed{\langle \varphi, h \rangle = \varphi(h)}$$

choose basis: $\langle \varphi_i, h_j \rangle = \delta_{ij}$

G_k group algebra $\Delta(g) = g \otimes g$ group-like type of coalgebra

$\epsilon(g) = 1$

$S(g) = g^{-1}$

$\varphi \in \mathcal{F}(G_k)$, $h = \sum h(g_i) \cdot g_i \in G_k$

$$\boxed{\langle \varphi, h \rangle = \varphi(h)}$$ character basis: $\langle \varphi_i, h_j \rangle = \delta_{ij}$

3.3 Dual pairing for $U(g)$

Duality in Lin. Alg

Finite dim vector space $(V,+,K)$: dual V^*

$$\varphi \in V^* \ , \quad \varphi : V \longrightarrow K$$

$$v \longmapsto \langle \varphi, v \rangle = \text{'}\varphi(v)\text{'}$$

Generalize to dual pairing for Hopf-Algebras

$$\Delta : C \longrightarrow C \otimes C \quad \longrightarrow \quad \Delta^* : C^* \otimes C^* \longrightarrow C^*$$

$$\mu : A \otimes A \longrightarrow A \quad \longrightarrow \quad \mu^* : A^* \longrightarrow (A \otimes A)^* \supset A^* \otimes A^* \ \text{dense}$$

$$\langle \mathbb{1}, b \rangle = \epsilon(b)$$

$$\langle \varphi, b \cdot c \rangle = \langle \Delta\varphi, b \otimes c \rangle = \ldots \langle \varphi_{(2)}, c \rangle$$

$$\langle \varphi, \mathbb{1} \rangle = \epsilon(\varphi)$$

$$\Delta: C \longrightarrow C\otimes C \quad \longrightarrow \quad \Delta^*: C^*\otimes C^* \longrightarrow C^*$$

$$\mu: A\otimes A \longrightarrow A \quad \longrightarrow \quad \mu^*: A^* \longrightarrow (A\otimes A)^* \hookleftarrow A^*\otimes A^*$$

dense

3.1 Def. Dual pairing

Two Hopf-algebras H, B are called dually paired if there exists a bilinear map

$$\langle\cdot,\cdot\rangle: H\otimes B \longrightarrow \mathbb{K}$$

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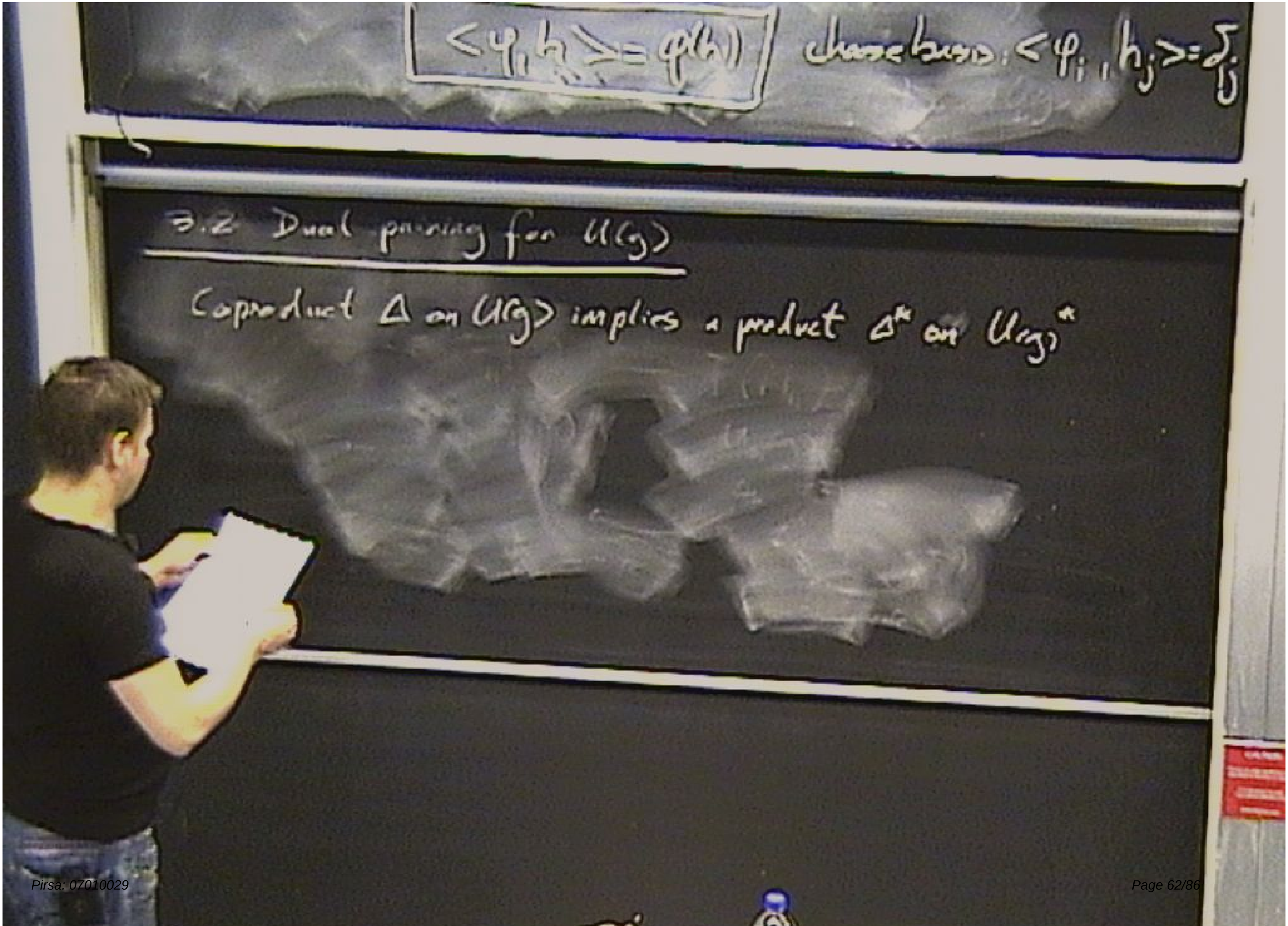
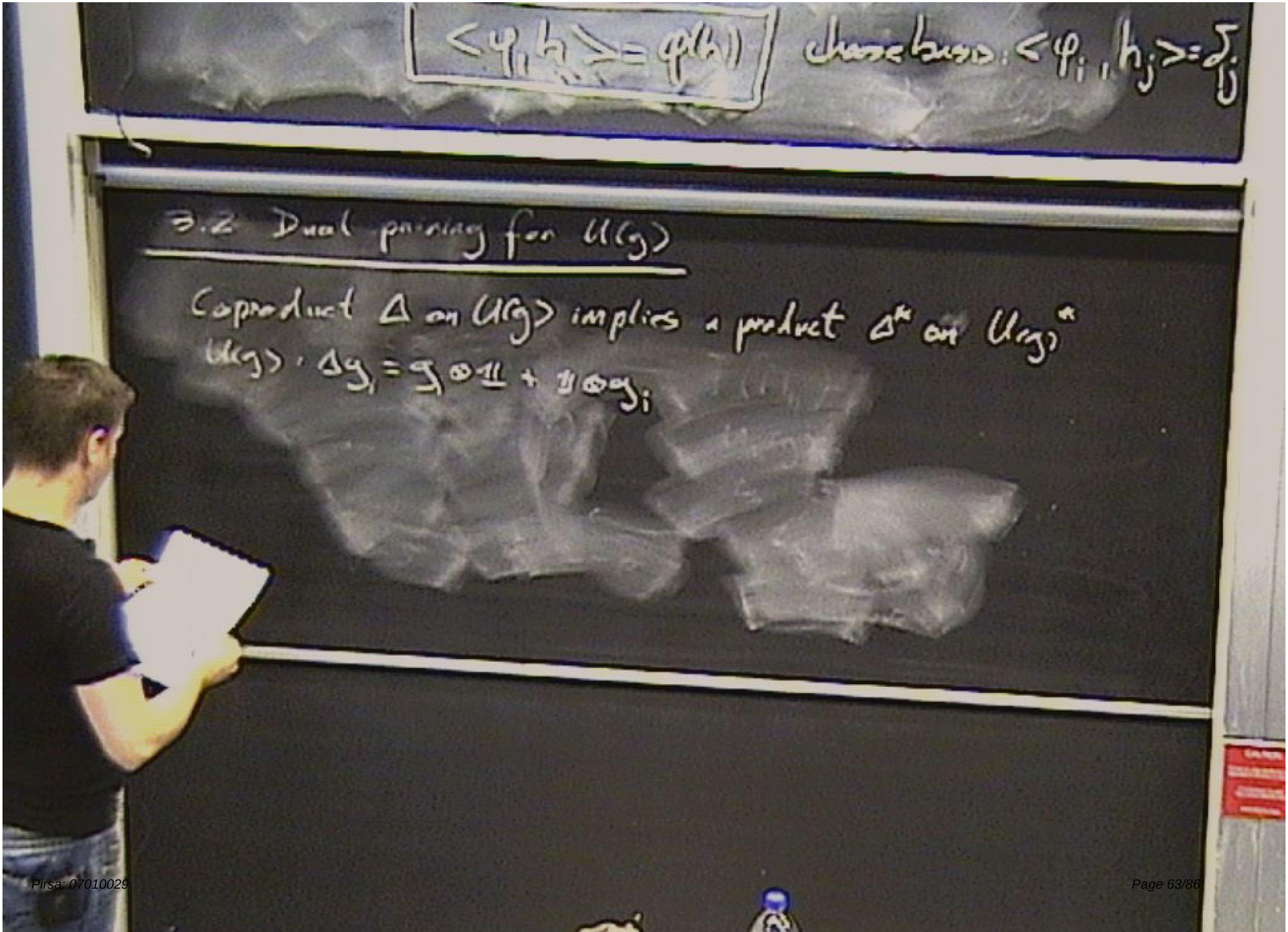
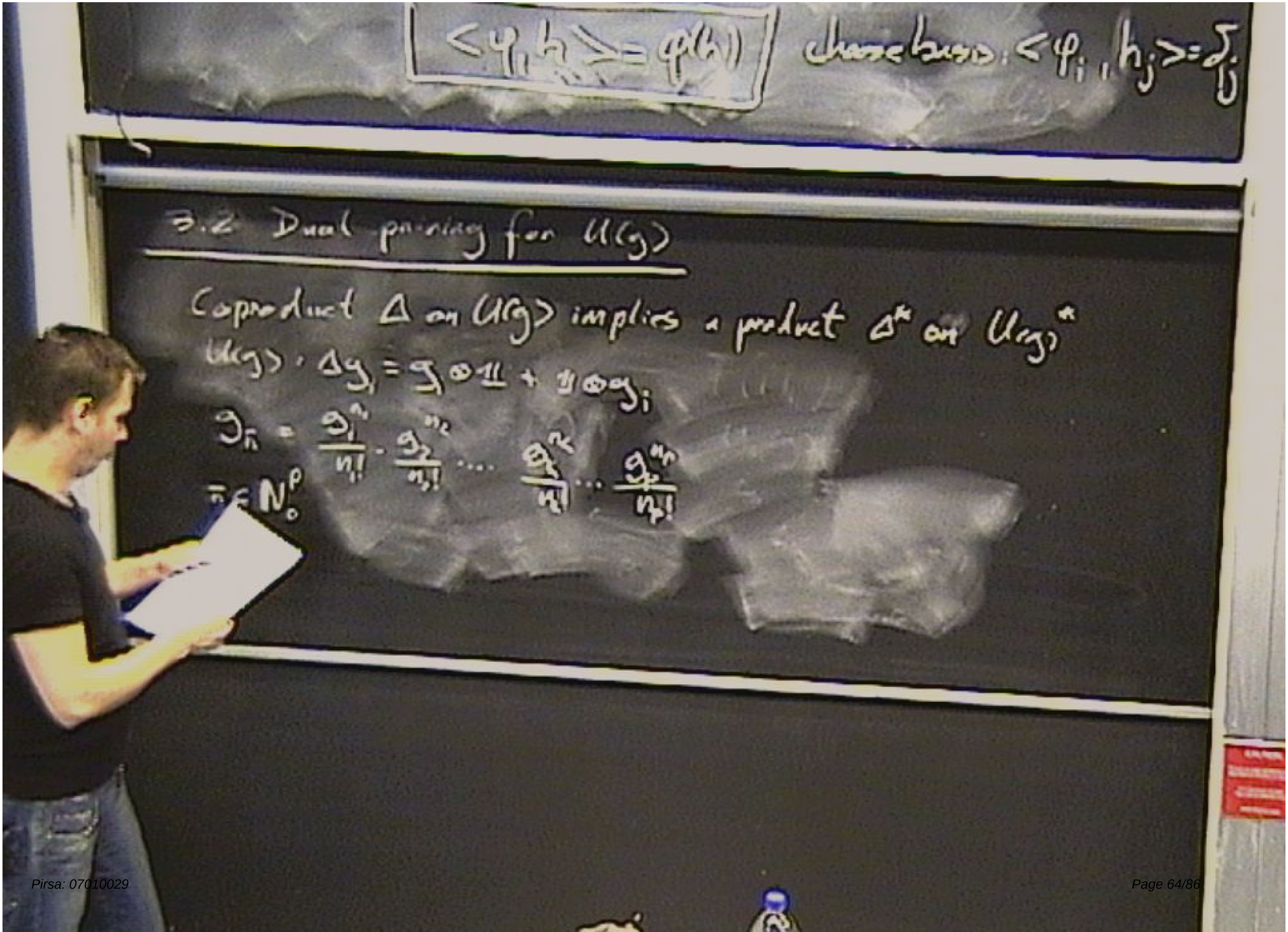

$\boxed{\langle \varphi, h \rangle = \varphi(h)}$
choose basis: $\langle \varphi_i, h_j \rangle = \delta_{ij}$
3.2 Dual pairing for $U(g)$
Coproduct Δ on $U(g)$ implies a product Δ^* on $U(g)^*$


$\boxed{\langle \varphi, h \rangle = \varphi(h)}$ choose basis: $\langle \varphi_i , h_j \rangle = \delta_{ij}$
3.2 Dual pairing for $U(g)$
Coproduct Δ on $U(g)$ implies a product Δ^* on $U(g)^*$
$U(g)$: $\Delta g_i = g_i \otimes 1\!\!1 + 1\!\!1 \otimes g_i$


$\boxed{\langle \varphi, h \rangle = \varphi(h)}$
choose basis: $\langle \varphi_i, h_j \rangle = \delta_{ij}$
3.2 Dual pairing for $U(g)$
Coproduct Δ on $U(g)$ implies a product Δ^* on $U(g)^*$
$U(g)$: $\Delta g_i = g_i \otimes 1 + 1 \otimes g_i$
$g_{\vec{n}} = \frac{g_1^{n_1}}{n_1!} \cdot \frac{g_2^{n_2}}{n_2!} \cdots \frac{g_k^{n_k}}{n_k!} \cdots \frac{g_p^{n_p}}{n_p!}$
$\vec{n} \in \mathbb{N}_0^p$

$\boxed{\langle \varphi, h \rangle = \varphi(h)}$ choose basis: $\langle \varphi_i, h_j \rangle = \delta_{ij}$

3.2 Dual pairing for $U(g)$

Coproduct Δ on $U(g)$ implies a product Δ^* on $U(g)^*$

$U(g)$: $\Delta g_i = g_i \otimes \mathbb{1} + \mathbb{1} \otimes g_i$

$g_{\bar{n}} = \frac{g_1^{n_1}}{n_1!} \cdot \frac{g_2^{n_2}}{n_2!} \cdots \frac{g_k^{n_k}}{n_k!} \cdots \frac{g_p^{n_p}}{n_p!}$

$\bar{n} \in \mathbb{N}_0^p$ $\longrightarrow \Delta(g_{\bar{n}}) = \sum_{\bar{k}=0}^{\bar{n}} g_{\bar{k}} \otimes g_{(\bar{n}-\bar{k})}$

$\boxed{\langle \varphi, h \rangle = \varphi(h)}$ choose basis: $\langle \varphi_i, h_j \rangle = \delta_{ij}$

3.2 Dual pairing for $U(g)$

Coproduct Δ on $U(g)$ implies a product Δ^* on $U(g)^*$

$U(g)$: $\Delta g_i = g_i \otimes 1\!\!1 + 1\!\!1 \otimes g_i$

$$g_{\bar n} = \frac{g_1^{n_1}}{n_1!} \cdot \frac{g_2^{n_2}}{n_2!} \cdots \frac{g_k^{n_k}}{n_k!} \cdots \frac{g_p^{n_p}}{n_p!} \longrightarrow \Delta(g_{\bar n}) = \sum_{\bar k = 0}^{\bar n} g_{\bar k} \otimes g_{(\bar n - \bar k)}$$

$\bar n \in \mathbb{N}_0^p$

Basis for $U(g)^*$: $(\varphi^{\bar m})_{\bar m \in \mathbb{N}_0^p}$ such that $\langle \varphi^{\bar m}, g_{\bar n} \rangle = \delta^{\bar m}_{\bar n}$

$\boxed{\langle \varphi, h \rangle = \varphi(h)}$ choose basis: $\langle \varphi_i , h_j \rangle = \delta_{ij}$

3.2 Dual pairing for $U(g)$

Coproduct Δ on $U(g)$ implies a product Δ^* on $U(g)^*$

$U(g)$: $\Delta g_i = g_i \otimes 1\!\!1 + 1\!\!1 \otimes g_i$

$[\Delta g_i , \Delta g_j] = \Delta([g_i , g_j])$

$\Delta(g_1 \cdot g_2) = \Delta(g_1)\cdot\Delta(g_2)$

$$g_{\bar n} = \frac{g_1^{n_1}}{n_1!}\cdot\frac{g_2^{n_2}}{n_2!}\cdots\frac{g_k^{n_k}}{n_k!}\cdots\frac{g_p^{n_p}}{n_p!} \longrightarrow \Delta(g_{\bar n}) = \sum_{\bar k = 0}^{\bar n} g_{\bar k} \otimes g_{(\bar n - \bar k)}$$

$\bar n \in \mathbb{N}_0^p$

Basis for $U(g)^*$: $(\varphi^{\bar m})_{\bar m \in \mathbb{N}_0^p}$ such that $\langle \varphi^{\bar m} , g_{\bar n} \rangle = \delta^{\bar m}_{\bar n}$

7.1 Def. Dual pairing

Two Hopf-algebras H, B are called dually paired if there exists a bilinear map

$$\langle \cdot , \cdot \rangle : H \otimes B \longrightarrow \mathbb{K}$$

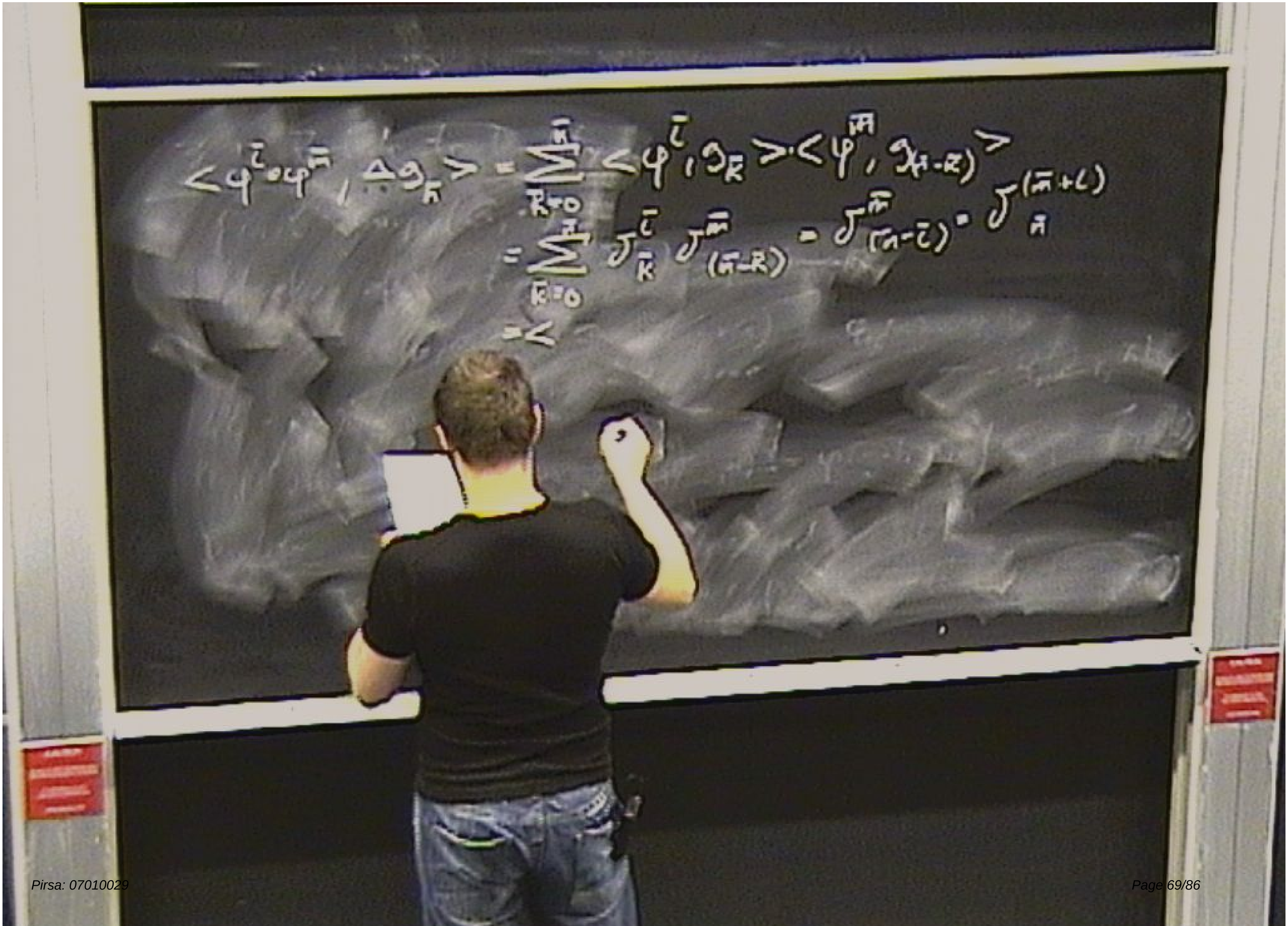
such that:

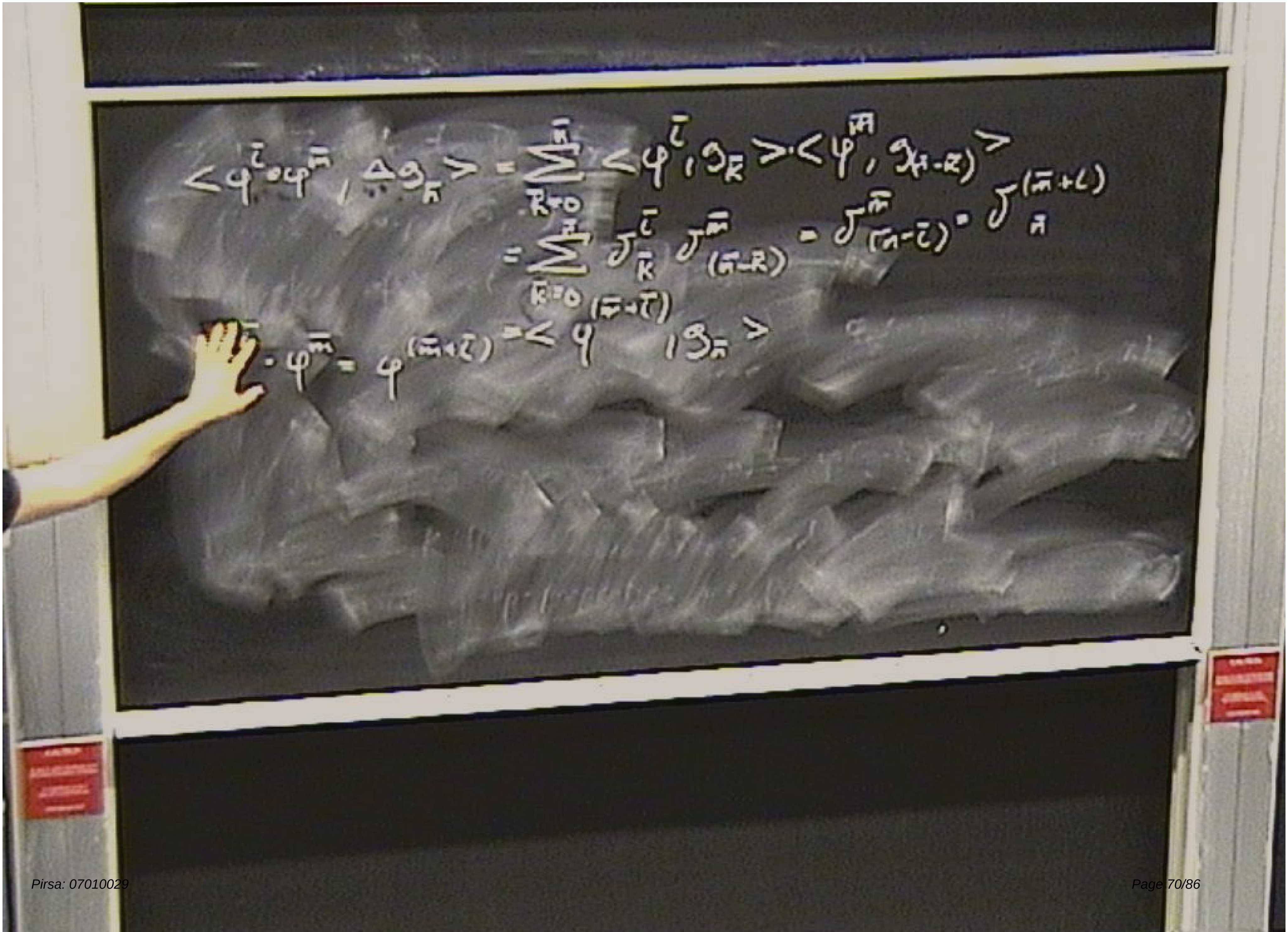
$$\langle \varphi \cdot \psi, b \rangle = \langle \varphi \otimes \psi, \Delta b \rangle = \langle \varphi, b_{(1)} \rangle \langle \psi, b_{(2)} \rangle$$

$$\langle \mathbb{1}, b \rangle = \epsilon(b)$$

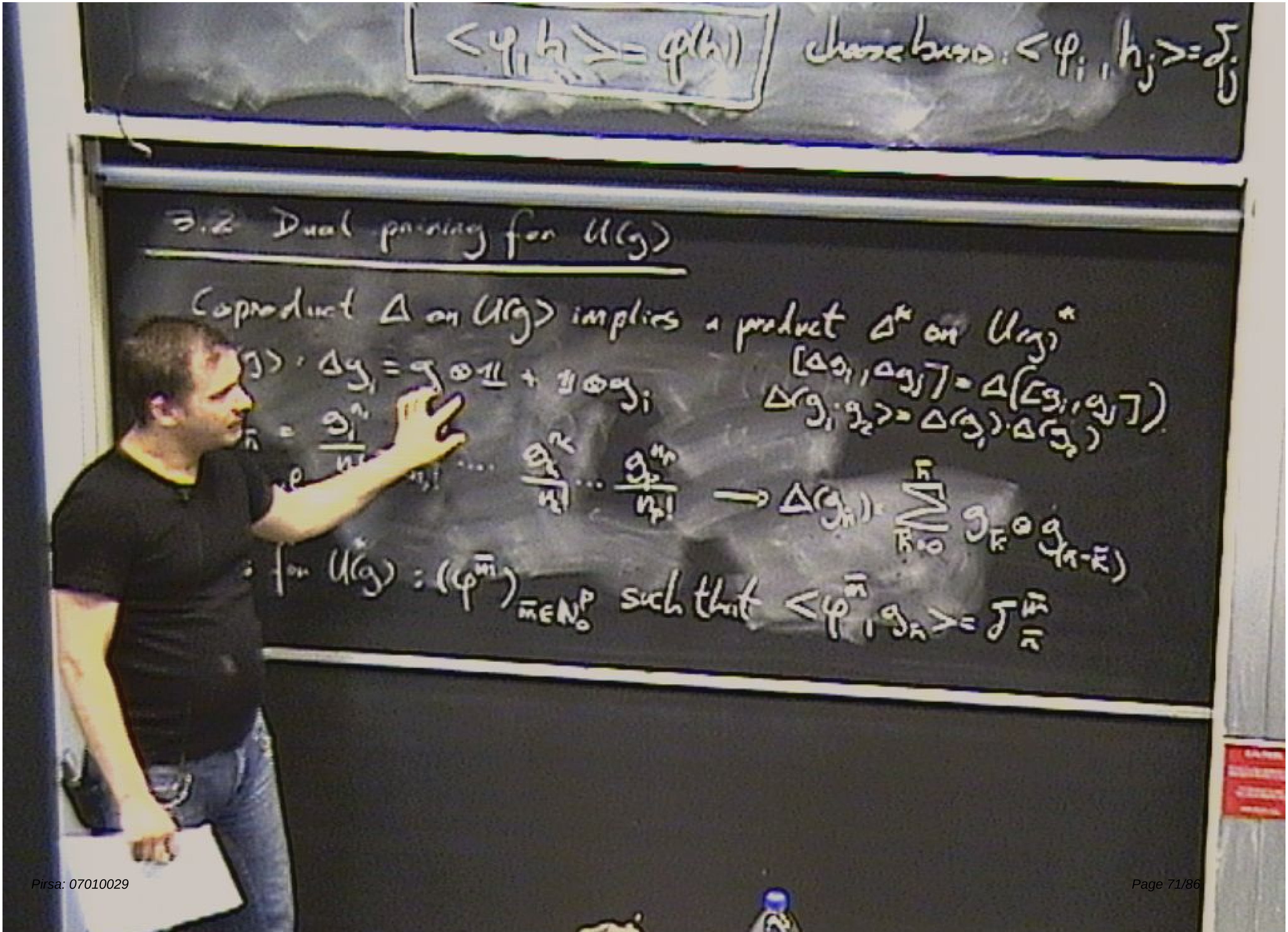
$$\langle \varphi, b \cdot c \rangle = \langle \Delta \varphi, b \otimes c \rangle = \langle \varphi_{(1)}, b \rangle \langle \varphi_{(2)}, c \rangle$$

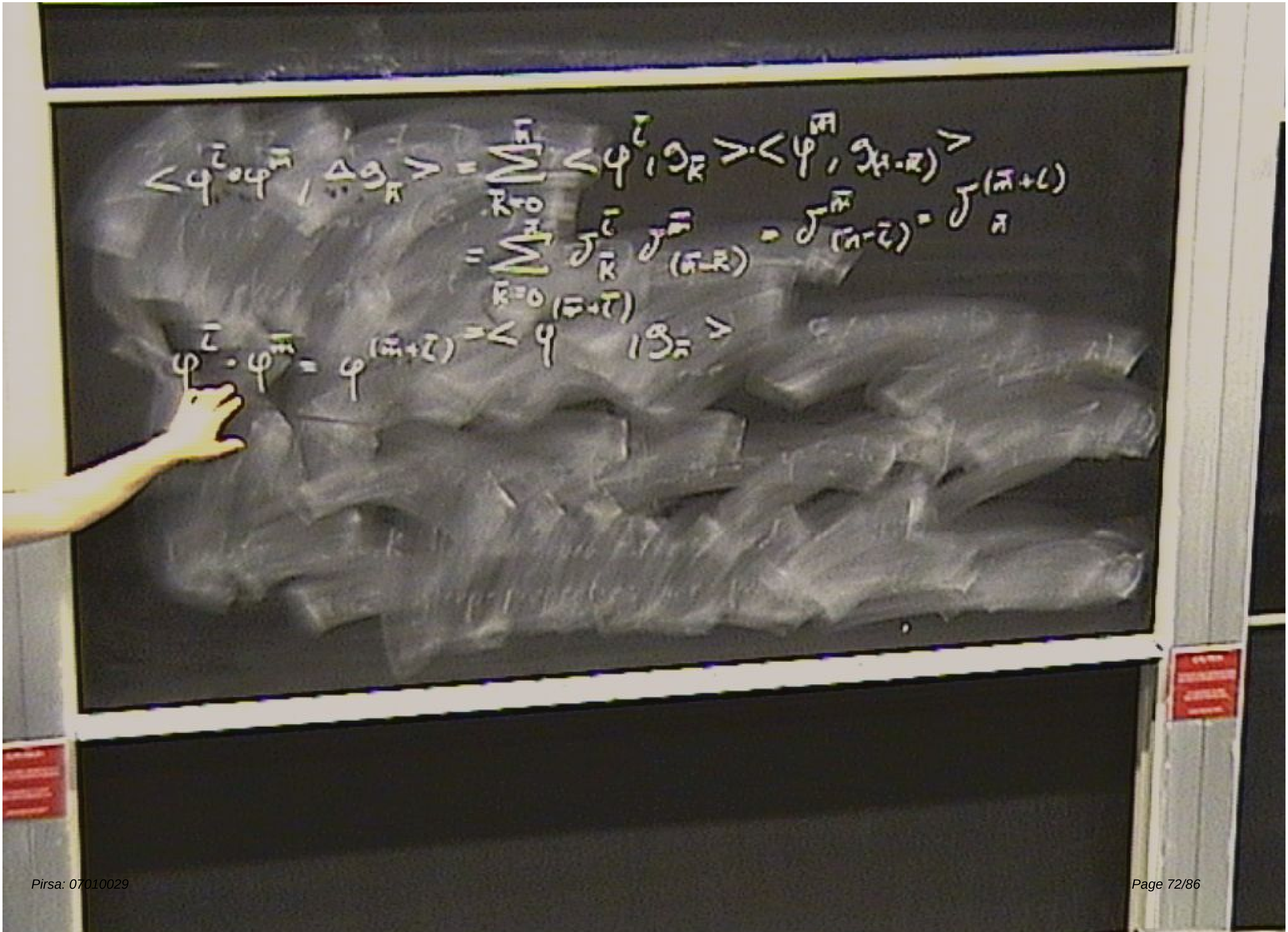
$$\langle \varphi, \mathbb{1} \rangle = \epsilon(\varphi)$$

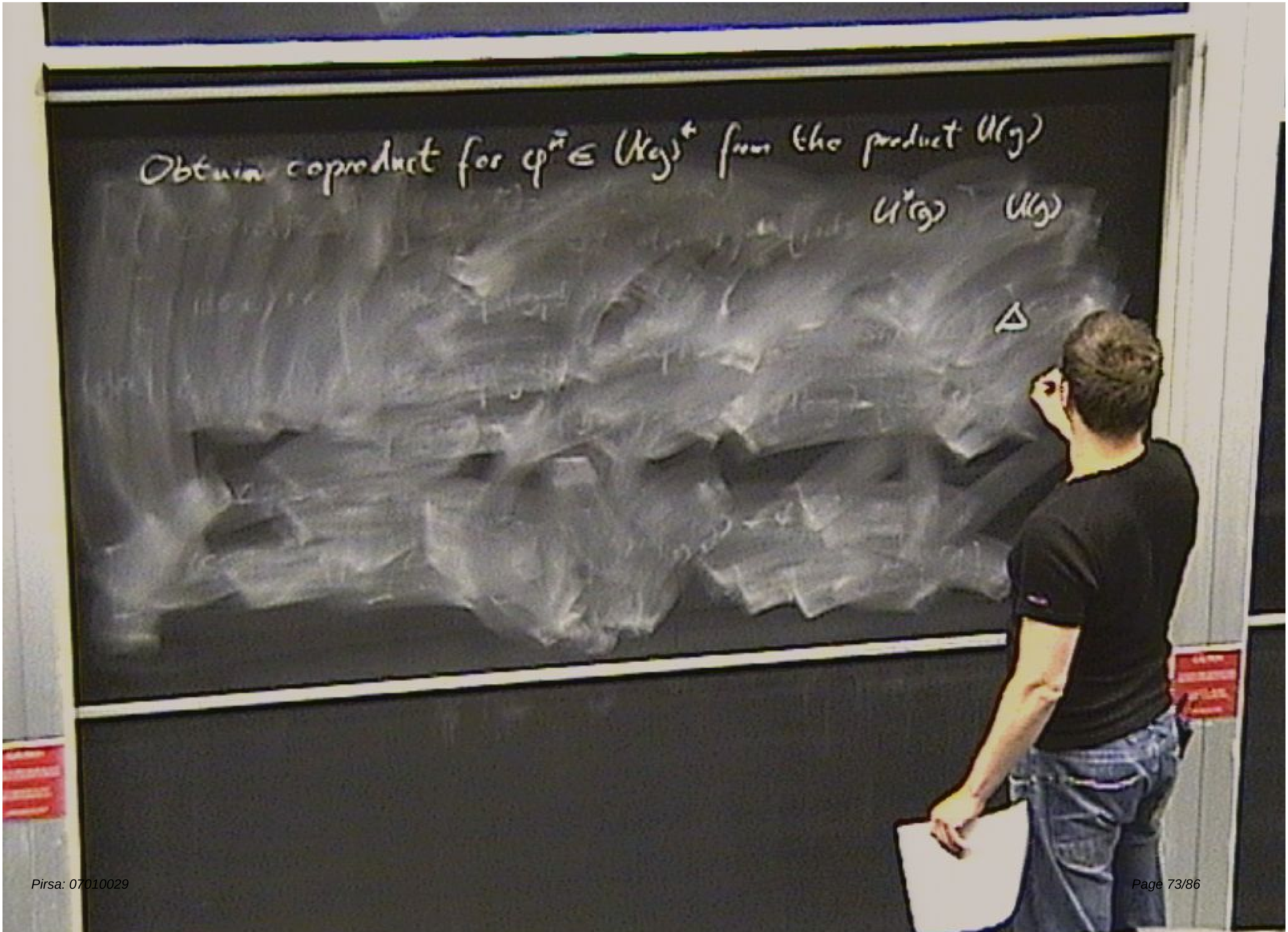

$<\varphi^{\bar{i}} \circ \varphi^{\bar{m}}, \Delta g_{\bar{n}}> = \sum_{\bar{k}=0}^{\bar{n}} <\varphi^{\bar{i}}, g_{\bar{k}}> \cdot <\varphi^{\bar{m}}, g_{(\bar{n}-\bar{k})}>$
$= \sum_{\bar{k}=0}^{\bar{n}} \delta^{\bar{i}}_{\bar{k}} \, \delta^{\bar{m}}_{(\bar{n}-\bar{k})} = \delta^{\bar{m}}_{(\bar{n}-\bar{i})} = \delta^{(\bar{m}+\bar{i})}_{\bar{n}}$
$=<$

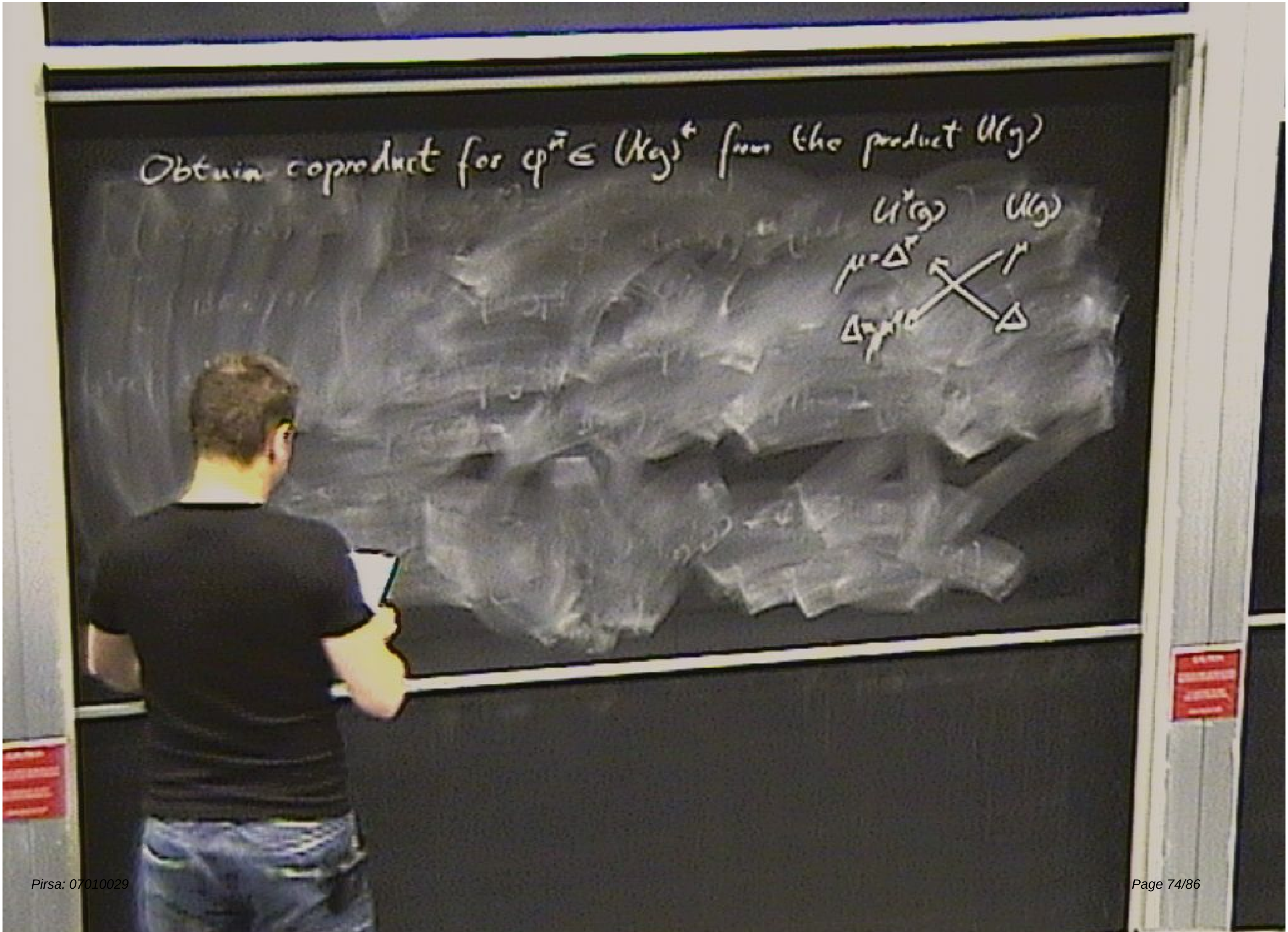


$\langle \varphi^{\bar{l}} \circ \varphi^{\bar{m}}, \Delta g_{\bar{n}} \rangle = \sum_{\bar{k}=0}^{\bar{n}} \langle \varphi^{\bar{l}}, g_{\bar{k}} \rangle \langle \varphi^{\bar{m}}, g_{(\bar{n}-\bar{k})} \rangle$
$= \sum_{\bar{k}=0}^{\bar{n}} \delta^{\bar{l}}_{\bar{k}} \delta^{\bar{m}}_{(\bar{n}-\bar{k})} = \delta^{\bar{m}}_{(\bar{n}-\bar{l})} = \delta^{(\bar{m}+\bar{l})}_{\bar{n}}$
$\bar{l} \circ \varphi^{\bar{m}} = \varphi^{(\bar{m}+\bar{l})} = \langle \varphi^{(\bar{m}+\bar{l})}, g_{\bar{n}} \rangle$


$\boxed{\langle \varphi, h \rangle = \varphi(h)}$ choose basis: $\langle \varphi_i , h_j \rangle = \delta_{ij}$
3.2 Dual pairing for $U(g)$
Coproduct Δ on $U(g)$ implies a product Δ^* on $U(g)^*$
$\Delta g_i = g_i \otimes 1 + 1 \otimes g_i$
$[\Delta g_i , \Delta g_j] = \Delta([g_i , g_j])$
$\Delta(g_1 \cdot g_2) = \Delta(g_1) \cdot \Delta(g_2)$
$\longrightarrow \Delta(g_{\bar n}) = \sum_{\bar k = 0}^{\bar n} g_{\bar k} \otimes g_{(\bar n - \bar k)}$
for $U(g)^*$: $(\varphi^{\bar m})_{\bar m \in \mathbb{N}_0^p}$ such that $\langle \varphi^{\bar m} , g_{\bar n} \rangle = \delta^{\bar m}_{\bar n}$


$\langle \varphi^{\bar{l}} \circ \varphi^{\bar{m}}, \Delta g_{\bar{n}} \rangle = \sum_{\bar{k}=0}^{\bar{n}} \langle \varphi^{\bar{l}}, g_{\bar{k}} \rangle \langle \varphi^{\bar{m}}, g_{(\bar{n}-\bar{k})} \rangle$
$= \sum_{\bar{k}=0}^{\bar{n}} \delta^{\bar{l}}_{\bar{k}} \, \delta^{\bar{m}}_{(\bar{n}-\bar{k})} = \delta^{\bar{m}}_{(\bar{n}-\bar{l})} = \delta^{(\bar{m}+\bar{l})}_{\bar{n}}$
$\varphi^{\bar{l}} \cdot \varphi^{\bar{m}} = \varphi^{(\bar{m}+\bar{l})} = \langle \varphi^{(\bar{m}+\bar{l})}, g_{\bar{n}} \rangle$


Obtain coproduct for φ^a ∈ U(g)* from the product U(g)
U*(g) U(g)
Δ


Obtain coproduct for φ^μ ∈ U(g)* from the product U(g)
U*(g)
U(g)
μ=Δ*
μ
Δ

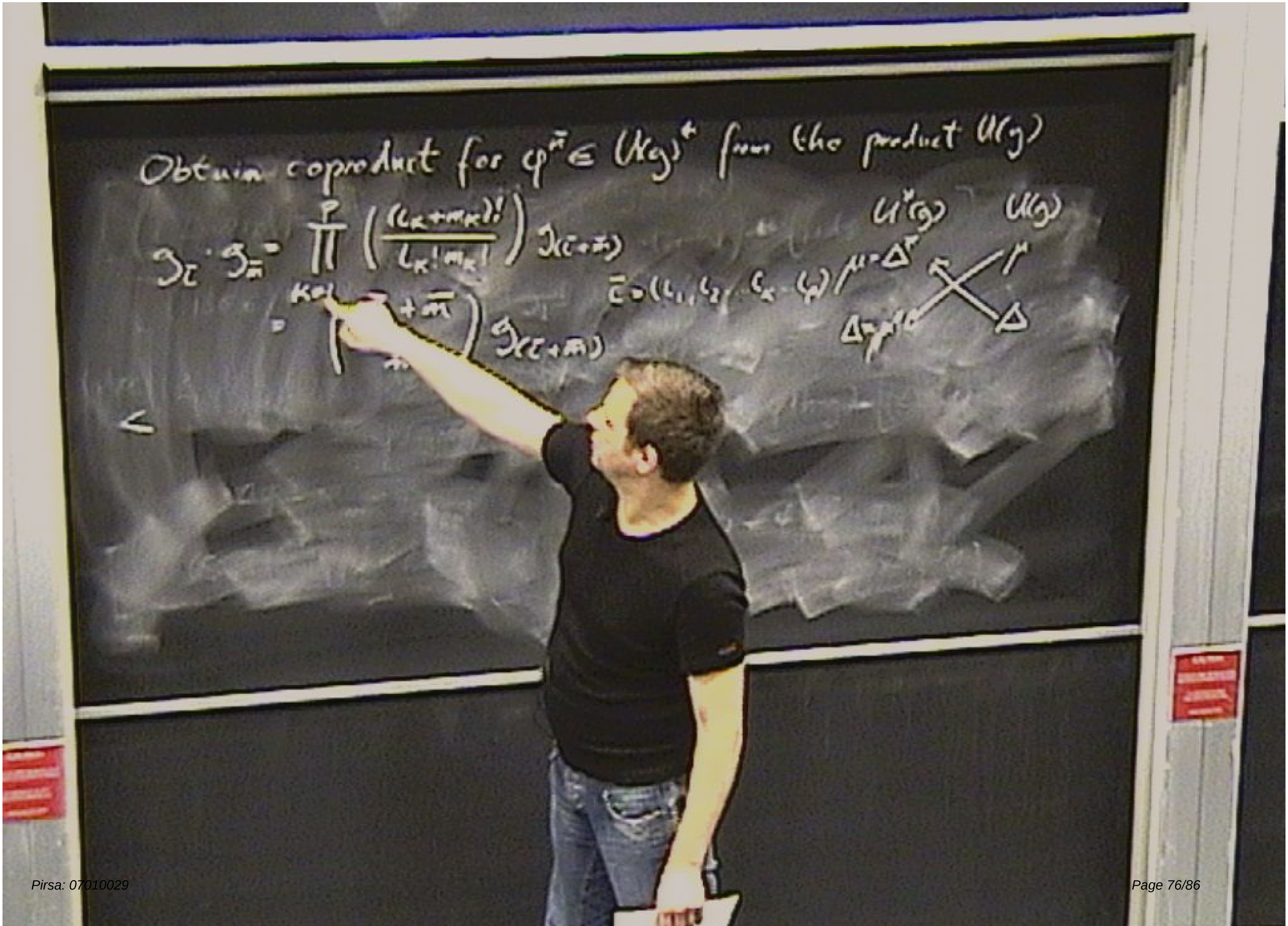
Obtain coproduct for $\varphi^{\bar m} \in (U\mathfrak{g})^*$ from the product $U(\mathfrak{g})$

$$\mathcal{J}_{\bar l} \cdot \mathcal{J}_{\bar m} = \prod_{k=1}^{p} \left(\frac{(l_k+m_k)!}{l_k!\, m_k!}\right) \mathcal{J}_{(\bar l+\bar m)} = \binom{\bar l+\bar m}{\bar m} \mathcal{J}_{(\bar l+\bar m)}$$

$U^*(\mathfrak{g})$ $\quad$ $U(\mathfrak{g})$

$\mu = \Delta^*$ $\quad$ μ

$\Delta = \mu^*$ $\quad$ Δ


Obtain coproduct for $\varphi^{\bar n} \in U(g)^*$ from the product $U(g)$
$U^*(g)$
$U(g)$

Obtain coproduct for $\varphi^{\bar{m}} \in (U(g))^*$ from the product $U(g)$

g: Abelian

$$\mathcal{J}_{\bar{\ell}} \cdot \mathcal{J}_{\bar{m}} = \prod_{k=1}^{p} \left(\frac{(\ell_k + m_k)!}{\ell_k!\, m_k!} \right) \mathcal{J}_{(\bar{\ell}+\bar{m})}$$

$$= \binom{\bar{\ell} + \bar{m}}{\bar{m}} \mathcal{J}_{(\bar{\ell}+\bar{m})}$$

$\bar{\ell} = (\ell_1, \ell_2, \ldots \ell_k \ldots \ell_p)$

$U^*(g)$ $\quad$ $U(g)$

$\mu = \Delta^T$ $\quad$ μ

$\Delta = \mu^T$ $\quad$ Δ

<

Obtain coproduct for $\varphi^{\bar n} \in U(g)^*$ from the product $U(g)$

g : Abelian

$$g_{\bar l} \cdot g_{\bar m} = \prod_{k=1}^{p} \left(\frac{(l_k+m_k)!}{l_k!\, m_k!} \right) g_{(\bar l+\bar m)} = \begin{pmatrix} \bar l + \bar m \\ \bar m \end{pmatrix} g_{(\bar l+\bar m)}$$

$\bar l = (l_1, l_2, \ldots l_k \ldots l_p)$

$U^*(g)$ $\quad$ $U(g)$

$\mu = \Delta^*$

$\Delta = \mu^*$

$$\langle \varphi^{\bar n}, g_{\bar l}\, g_{\bar m} \rangle = \begin{pmatrix} \bar l + \bar m \\ \bar m \end{pmatrix} \langle \varphi^{\bar n}, g_{(\bar l+\bar m)} \rangle = \begin{pmatrix} \bar l + \bar m \\ \bar m \end{pmatrix} \delta^{\bar n}_{(\bar l+\bar m)}$$

Obtain coproduct for $\varphi^{\bar{n}} \in U(g)^*$ from the product $U(g)$

$U(g)$: Abelian

$$g_{\bar{l}} \cdot g_{\bar{m}} = \prod_{k=1}^{p} \left(\frac{(l_k+m_k)!}{l_k!\, m_k!} \right) g_{(\bar{l}+\bar{m})}$$

$$= \begin{pmatrix} \bar{l}+\bar{m} \\ \bar{m} \end{pmatrix} g_{(\bar{l}+\bar{m})}$$

$\bar{l} = (l_1, l_2, \ldots l_k \ldots l_p)$

$U^*(g)$ $U(g)$

$\mu = \Delta^*$ μ

$\Delta = \mu^*$ Δ

$$<\varphi^{\bar{n}}, g_{\bar{l}}\, g_{\bar{m}}> = \begin{pmatrix} \bar{l}+\bar{m} \\ \bar{m} \end{pmatrix} <\varphi^{\bar{n}}, g_{(\bar{l}+\bar{m})}> = \begin{pmatrix} \bar{l}+\bar{m} \\ \bar{m} \end{pmatrix} \delta^{\bar{n}}_{(\bar{l}+\bar{m})}$$

Obtain coproduct for $\varphi^{\bar n} \in U(g)^*$ from the product $U(g)$

$U(g)$: Abelian

$[g_i, g_j] = 0$. $\quad g_{\bar l} \cdot g_{\bar m} = \prod_{k=1}^{p} \left(\frac{(l_k + m_k)!}{l_k! \, m_k!} \right) g_{(\bar l + \bar m)}$

$$= \binom{\bar l + \bar m}{\bar m} g_{(\bar l + \bar m)}$$

$\bar l = (l_1, l_2, \ldots l_k \ldots l_p)$

$U^*(g) \qquad U(g)$

$\mu = \Delta^T \qquad \mu$

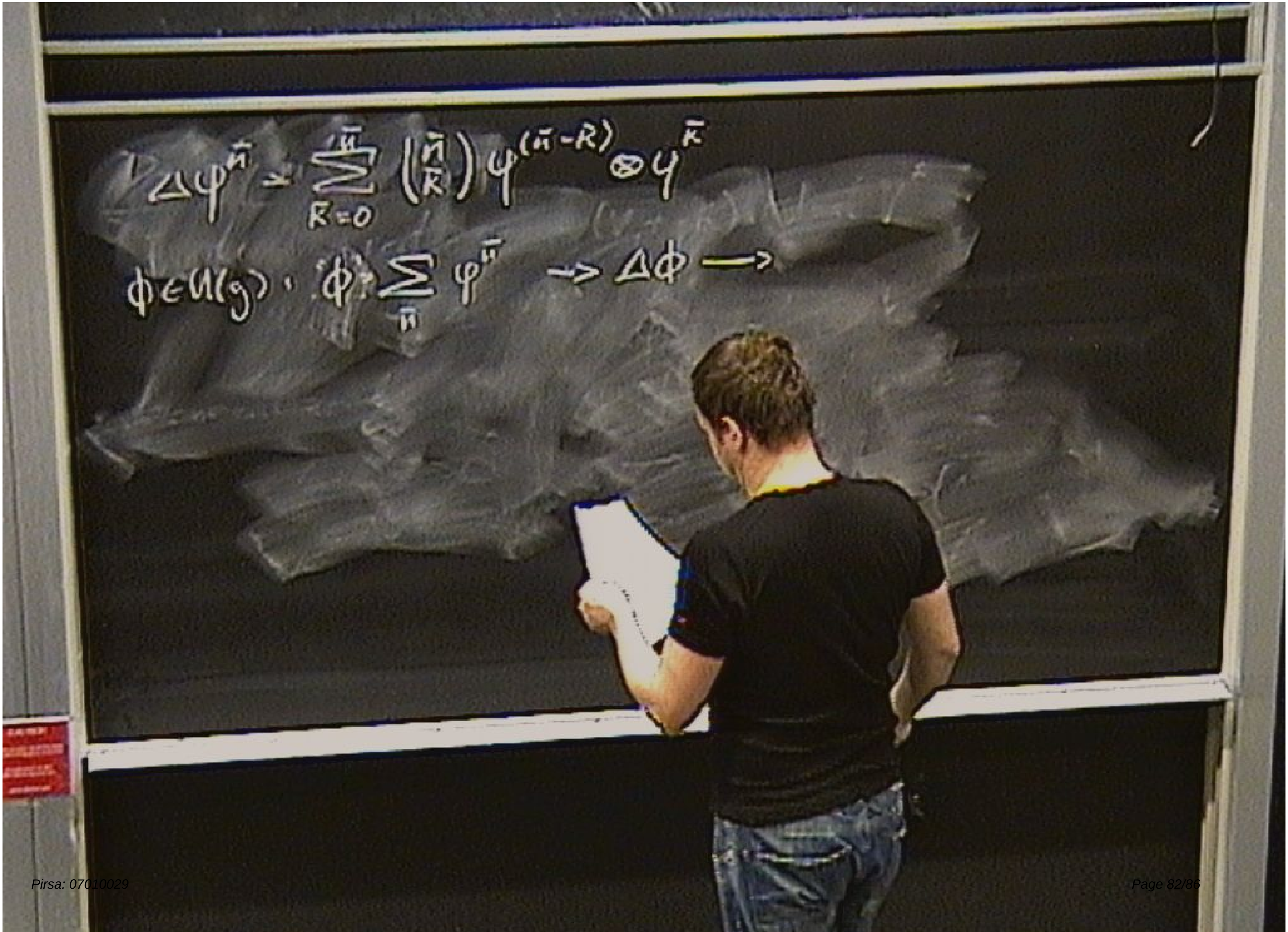
$\Delta = \mu^T \qquad \Delta$

$$\langle \varphi^{\bar n}, g_{\bar l} \cdot g_{\bar m} \rangle = \binom{\bar l + \bar m}{\bar m} \langle \varphi^{\bar n}, g_{(\bar l + \bar m)} \rangle = \binom{\bar l + \bar m}{\bar m} \delta^{\bar n}_{(\bar l + \bar m)}$$

$$= \sum_{\bar k = 0}^{\bar n} \binom{\bar l + \bar k}{\bar k} \delta^{\bar n}_{(\bar l + \bar k)} \delta^{\bar k}_{\bar m} = \sum_{\bar k = 0}^{\bar n} \binom{\bar n}{\bar k} \delta^{(\bar n - \bar k)}_{\bar l} \delta^{\bar k}_{\bar m}$$

$$= \sum_{\bar k = 0}^{\bar n} \binom{\bar n}{\bar k} \langle \varphi^{(\bar n - \bar k)}, g_{\bar l} \rangle \langle \varphi^{\bar k}, g_{\bar m} \rangle$$

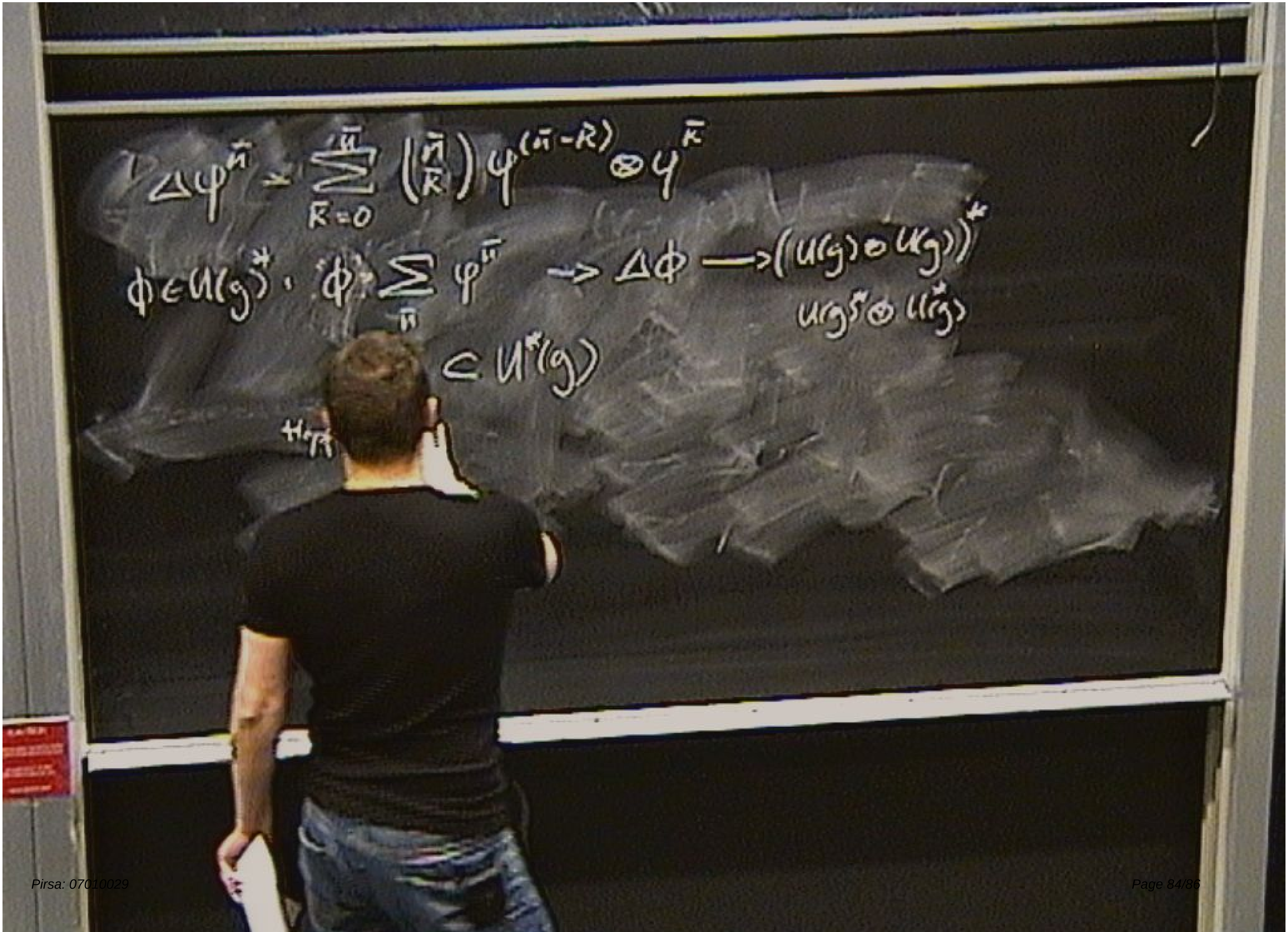
$$\Delta \varphi^{\bar{n}} = \sum_{\bar{k}=0}^{\bar{n}} \binom{\bar{n}}{\bar{k}} \varphi^{(\bar{n}-\bar{k})} \otimes \varphi^{\bar{k}}$$

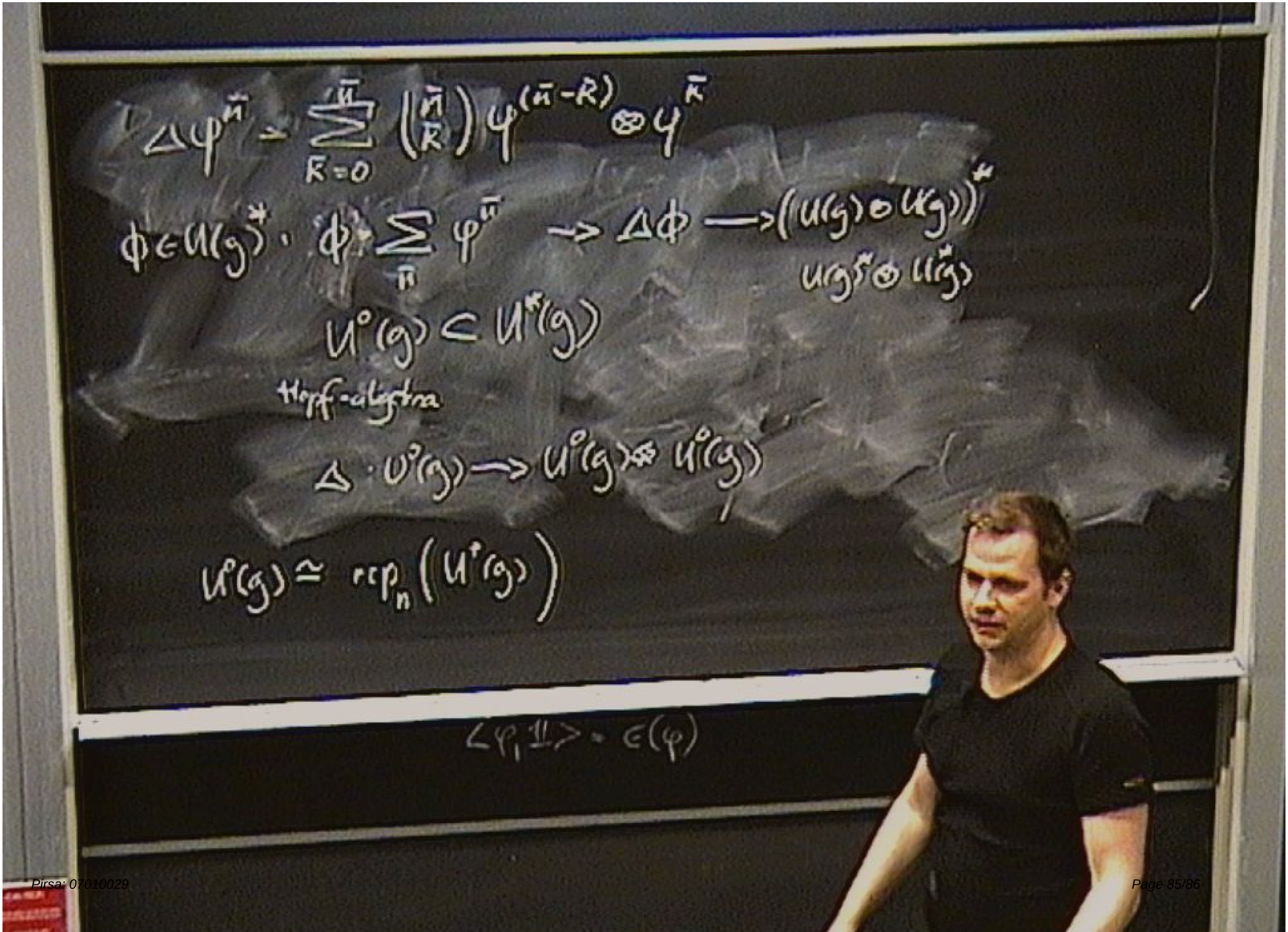

$\Delta\varphi^{\bar n} = \sum_{\bar k=0}^{\bar n} \binom{\bar n}{\bar k} \varphi^{(\bar n-\bar k)} \otimes \varphi^{\bar k}$
$\phi \in U(g), \phi = \sum_{\bar n} \varphi^{\bar n} \to \Delta\phi \longrightarrow$

$$\Delta \varphi^{\bar{n}} = \sum_{\bar{k}=0}^{\bar{n}} \binom{\bar{n}}{\bar{k}} \varphi^{(\bar{n}-\bar{k})} \otimes \varphi^{\bar{k}}$$

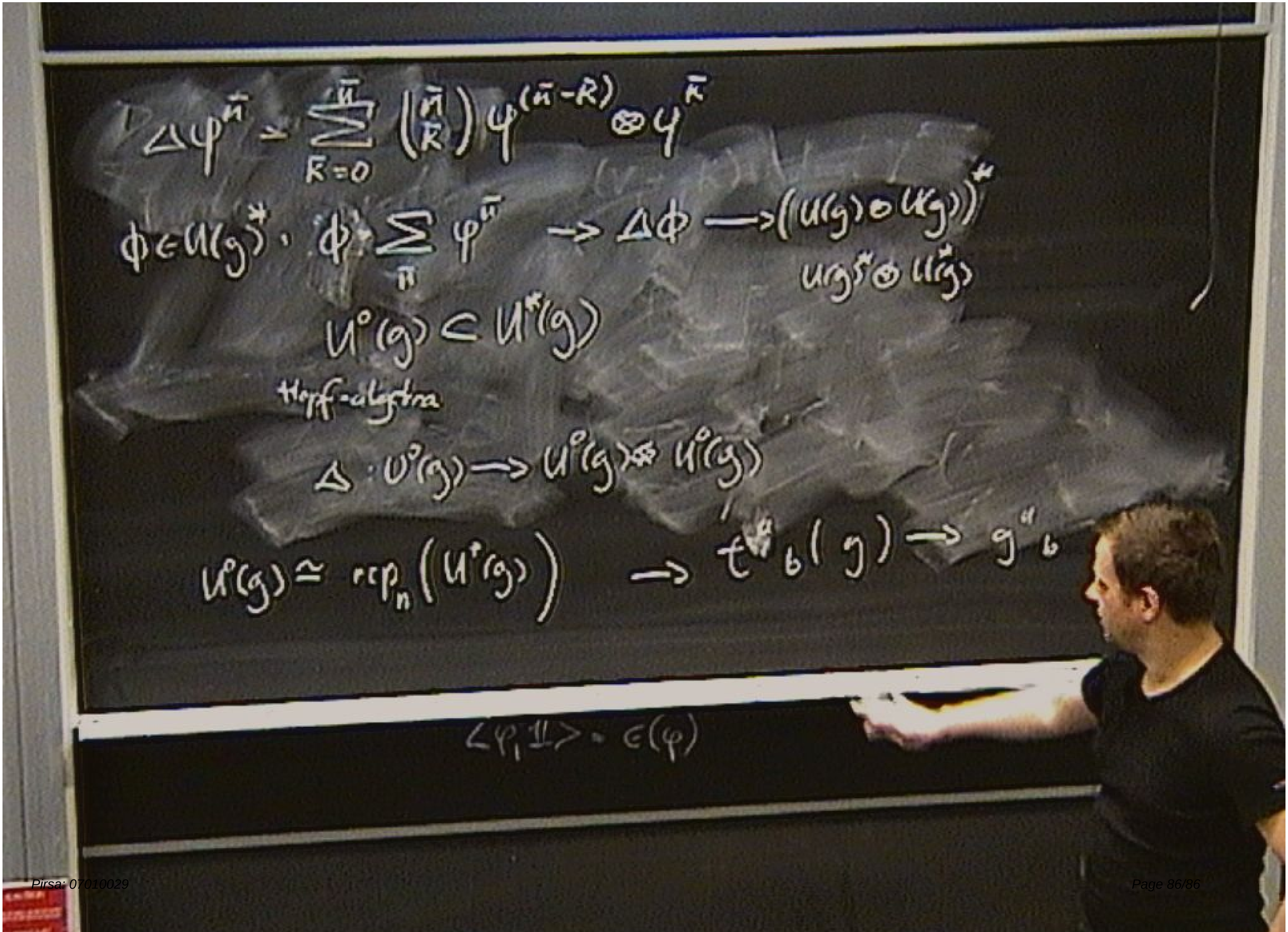
$$\phi \in U(g), \quad \phi = \sum_{\bar{n}} \varphi^{\bar{n}} \longrightarrow \Delta\phi \longrightarrow (U(g) \otimes U(g))^{*}$$

$$U(g) \hat{\otimes} U(g)$$


Δφ^n = Σ_{k=0}^{n} (n k) φ^(n−k) ⊗ φ^k
φ ∈ U(g)*, φ = Σ_n φ^n → Δφ → (U(g) ⊗ U(g))*
U(g)* ⊗ U(g)*
⊂ U*(g)



$\Delta\varphi^{\bar{n}} = \sum_{\bar{k}=0}^{\bar{n}} \binom{\bar{n}}{\bar{k}} \varphi^{(\bar{n}-\bar{k})} \otimes \varphi^{\bar{k}}$
$\phi \in U(g)^* \cdot \phi = \sum_{\bar{n}} \varphi^{\bar{n}} \rightarrow \Delta\phi \longrightarrow (U(g) \otimes U(g))^*$
$U(g)^* \otimes U(g)^*$
$U^\circ(g) \subset U^*(g)$
Hopf-algebra
$\Delta \cdot U^\circ(g) \longrightarrow U^\circ(g) \otimes U^\circ(g)$
$U(g) \simeq \mathrm{rep}_n(U^\circ(g))$
$\langle \varphi, 1\!\!1 \rangle = \epsilon(\varphi)$


$\Delta\varphi^{\bar n} = \sum_{\bar k=0}^{\bar n} \binom{\bar n}{\bar k} \varphi^{(\bar n-\bar k)} \otimes \varphi^{\bar k}$
$\phi \in U(g)^*,\ \phi = \sum_{\bar n} \varphi^{\bar n} \to \Delta\phi \to (U(g)\otimes U(g))^*$
$U(g)^* \otimes U(g)^*$
$U^0(g) \subset U^*(g)$
Hopf-algebra
$\Delta : U^0(g) \to U^0(g) \otimes U^0(g)$
$U^0(g) \simeq \mathrm{rep}_n(U^*(g)) \to t^a{}_b(g) \to g^a{}_b$
$\langle \varphi, \mathbb{1} \rangle = \epsilon(\varphi)$