

Title: Introduction to quantum groups 2

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Abstract: Hopf-Algebras and their Representations

In order to consolidate the above motivation, we have to introduce Hopf-algebras on a mathematical footing. We define Hopf-algebras, discuss duality and especially we will have a closer look at the question why coproducts induce a multiplication on the dual algebra but not the other way around. With these preparations we close this unit by the discussion of representations and corepresentations - and how these are related for dual Hopf-algebras.

Summary

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 2. deformation of products and coproducts by bilinear operators
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→ Duality:

$$\mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$$

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1. Quantization:

1. no algebra-homomorphism
2. deformation of products and coproducts by bilinear operators ($\star, \star^*, R, R^*$)
3. ordering prescription

2. Duality:

$$\mathcal{F}_h(M) \xleftrightarrow{\text{dual}} \mathcal{F}(M_h)$$

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$$\mathcal{F}_h(\mathfrak{g}), \mu_h \longleftrightarrow U_h(\mathfrak{g}) \Delta_h$$

Summary

1. Quantization:

1. no algebra-homomorphism
2. deformation of products and coproducts by bilinear operators ($\star, \mathcal{F}, \mathcal{L}, R^{\text{ob}}$ and Δ)
3. ordering prescription

2. Duality:

$$\mathcal{F}_h(M) \xrightarrow{\text{dual}} \mathcal{F}(M_h)$$

$$\begin{array}{ccc} \mathcal{F}_h(G), \mu_h & \longleftrightarrow & \mathcal{U}_h(\mathcal{G}) \Delta_h \\ \star_h, R^{\text{ob}} & & \mathcal{L}_h, \mathcal{F}_h \end{array}$$

$\mathbb{R}^n$

$\mathbb{R}^n$

3. Representations: $U(g)$ $g_i = (x_i, x_i)$

$$h, R^0 \subset$$

$$L_h, F_h$$

3. Representations: $U(g)$

$$g_i = (x_i, x_i)$$

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$$g_i = (x_i, x_i) = (g_i, x_i) \cdot x_i + x_i (g_i, x_i)$$

$$H, R^0 \subset$$

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3. Representations. $U(g)$

$$g_i = (x_i, x_i)$$

$$\Delta g_i = g_i \otimes 1 + 1 \otimes g_i$$

$$g_i = (x_i, x_i) = (g_i \otimes x_i) \cdot x_i + x_i (g_i \otimes x_i)$$

$$\Delta g_i = (x_i \otimes x_i)$$

$$\#_h, \mathbb{R}^{ab} \subset \mathbb{R}_h, \mathbb{F}_h$$

3. Representations: $(\mathbb{V}, \rho)$

$$\rho_i = (x_i, x_j)$$

$$\Delta \rho_i = \rho_i \otimes 1 + 1 \otimes \rho_i$$

$$\rho_i(x_r, x_s) = (\rho_i(x_r)) \cdot x_s + x_r(\rho_i(x_s))$$

$$\Delta \rho_i = (x_r, x_s, x_s)$$

2. Introduction to Hopf Algebras

2.1. Overview:

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$(V, +; K)$
vector space

algebra $\mu: V \otimes V \rightarrow V$: product

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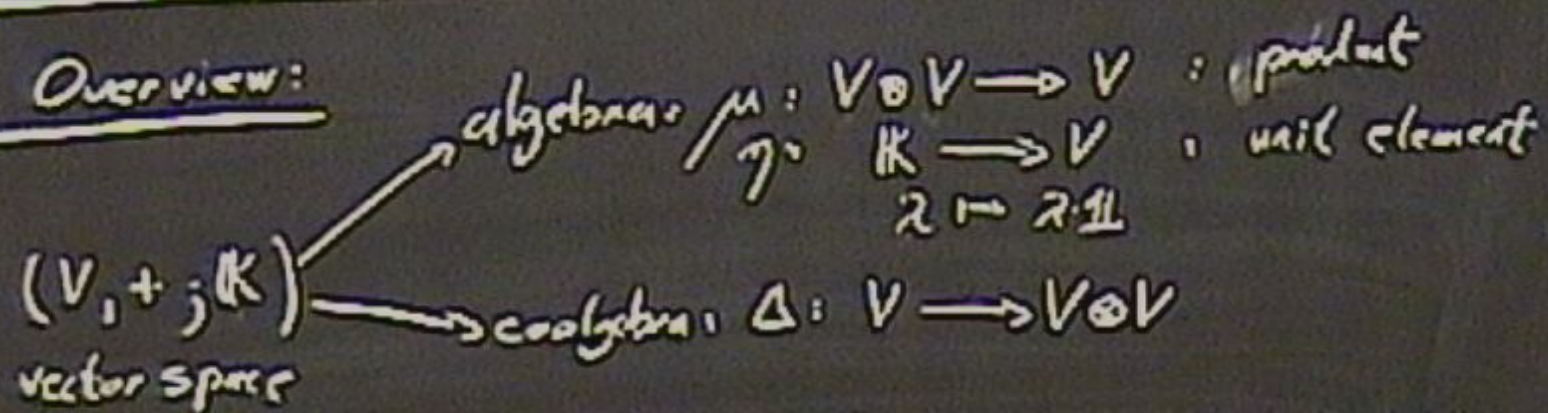
$(V, +, \cdot; K)$
vector space

algebra

$\mu: V \otimes V \rightarrow V$: product
 $\eta: K \rightarrow V$: unit element
 $\lambda \mapsto \lambda \cdot 1$

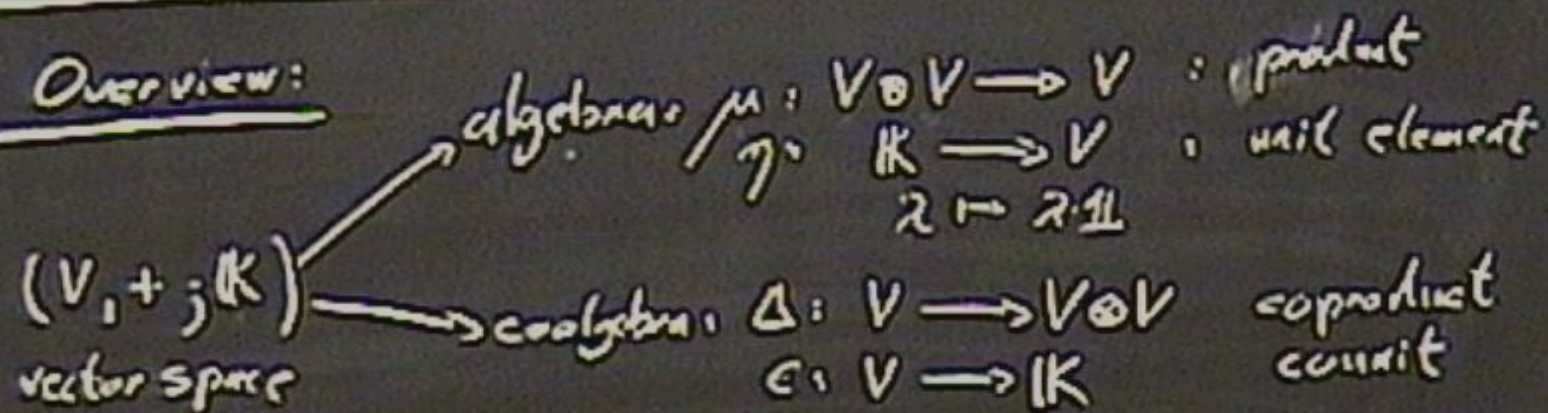
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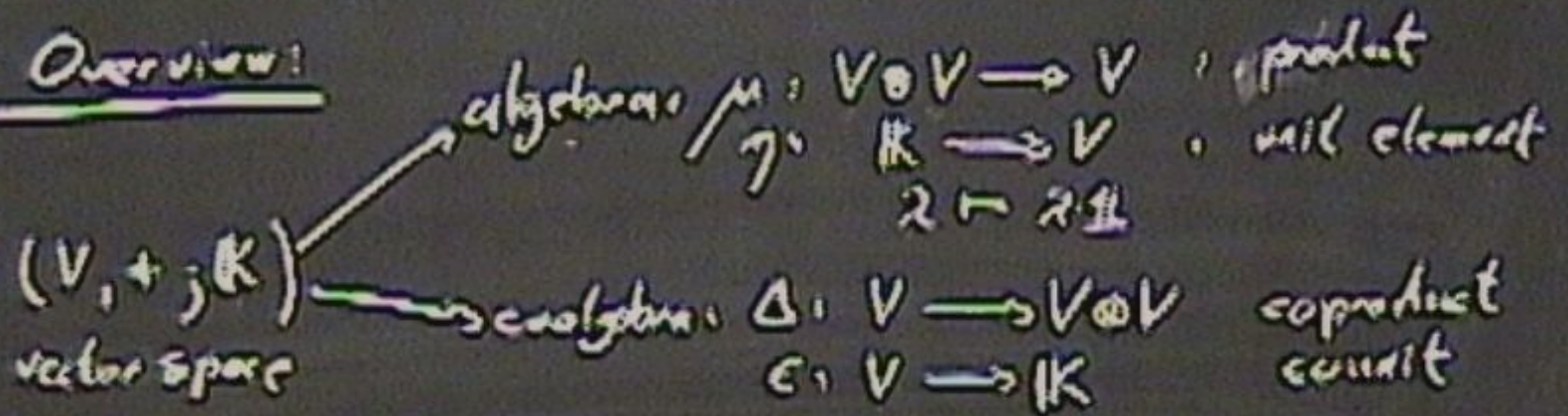
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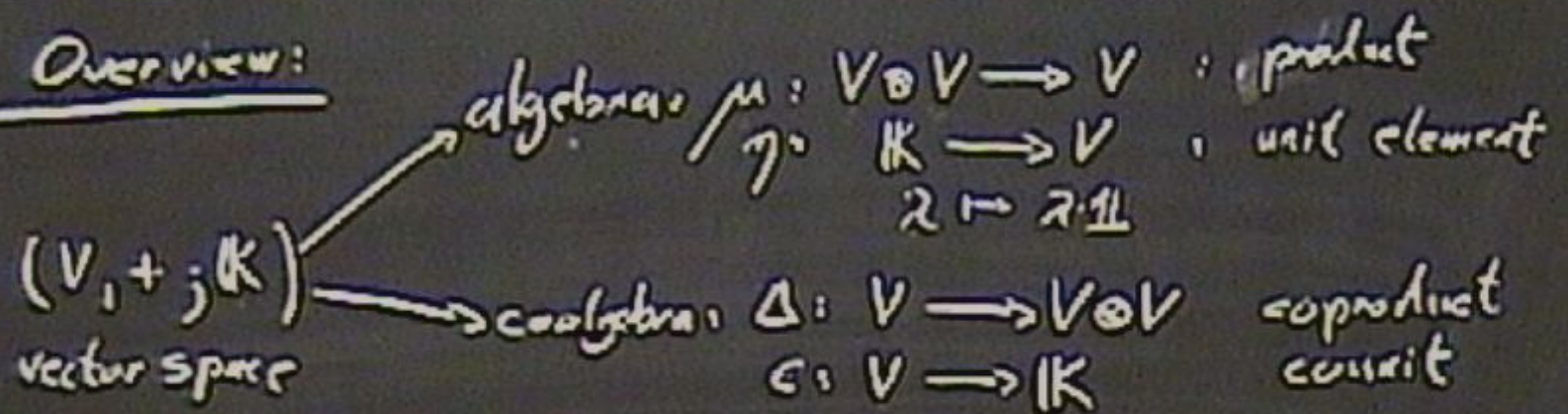
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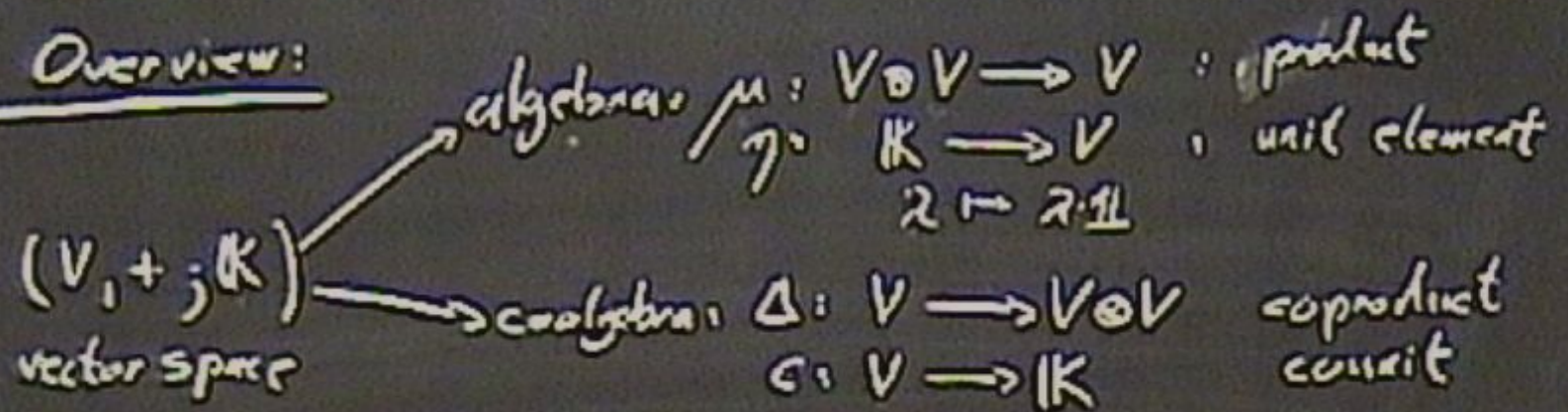
Bialgebras: Algebra and Coalgebra made compatible

2.1. Overview:



Bialgebras: Algebra and Coalgebra made compatible
 μ : coalgebra-homomorphism

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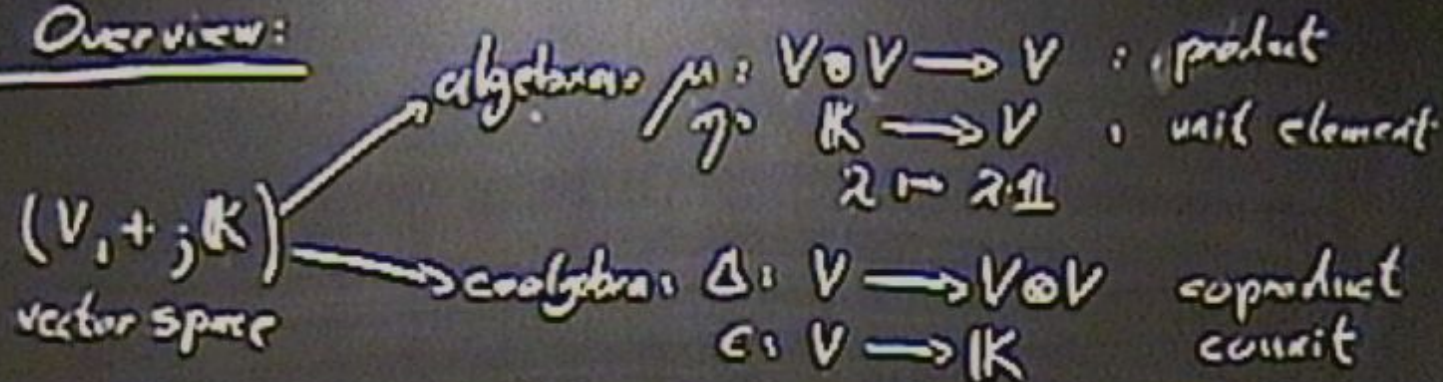


Bialgebras: Algebra and Coalgebra made compatible

μ : coalgebra-homomorphism
 Δ : algebra-homomorphism

2. Introduction to Hopf Algebras

2.1. Overview:



Bialgebras: Algebra and Coalgebra made compatible

μ : coalgebra-homomorphism (Δ, ϵ) :

Δ : algebra-homomorphism $\Delta([x, y]) = [\Delta x, \Delta y]$

Δ : algebra-homomorphism $\Delta(L3, 13, 1) = [43, 43, 1]$

Hopf-Algebra: B. algebra with $S: V \longrightarrow V$



Δ : algebra-homomorphism $\Delta([x_i, y_j]) = [x_i, y_j]$

Hopf-Algebra: Bialgebra with $S: V \rightarrow V$
 $\rightarrow S(v)$: 'inverse element'

1. Quantization:
 1. no algebra-homomorphism
 2. deformation of products and coproducts by bilinear operators ($\star, \mathcal{J}, \mathcal{R}, \mathcal{R}^{\text{op}}$ and)
 3. ordering prescription
2. Duality: $\mathcal{F}_h(M) \xrightarrow{\text{dual}} \mathcal{F}(M_h)$

$$\mathcal{F}_h(G), \mu_h \longleftrightarrow U_h(G), \Delta_h$$

$$\star, \mathcal{R}^{\text{op}}, \mathcal{C} \longleftrightarrow \mathcal{R}_h, \mathcal{F}_h$$

1. Quantization:

1. no algebra-homomorphism
2. deformation of products and coproducts by bilinear operators ($\star, \mathcal{J}, \mathcal{R}, R^{\text{ad}}$)
3. ordering prescription

2. Duality:

$$\mathcal{F}_h(M) \xrightarrow{\text{dual}} \mathcal{F}(M_h)$$

$$S(u) = u^{-1}$$

$$\begin{array}{ccc} \mathcal{F}_h(\mathfrak{g}), \mu_h & \longleftrightarrow & \mathcal{U}_h(\mathfrak{g}) \Delta_h \\ \star, \mathcal{R}^{\text{ad}} & & \mathcal{R}_h, \mathcal{F}_h \end{array}$$

$$S(g_i) \rightarrow -g_i$$

7. Duality:

3. ordering prescription

$$\mathcal{F}_h(M) \xrightarrow{\text{dual}} \mathcal{F}(M_h)$$

$$S(u) = u^{-1}$$

$$\begin{array}{ccc} \mathcal{F}_h(G), \mu_h & \longleftrightarrow & U_h(g) \Delta_h \\ \#_h, \mathbb{R}^b \subset & & \mathbb{R}_h, \mathcal{F}_h \end{array}$$

$$S(g_i) \rightarrow -g_i$$

$$g_i = (x_i, x_3) = (g_i \circ x_r) \cdot x_3 + y_r (g_i \circ x_s)$$

$$\Delta g_i \circ (x_r, x_s, x_3)$$

$$U_h(g) \Delta_h \xrightarrow{S(g_i)} -g_i$$

3. Representations: $U(g)$ $g_i \triangleright (x_i, x_j)$

$$\Delta g_i = g_i \otimes 1 + 1 \otimes g_i$$

$$g_i \triangleright (x_r, x_s) = (g_i \triangleright x_r) \cdot x_s + x_r (g_i \triangleright x_s)$$

$$\Delta g_i \triangleright (x_r, x_s)$$

Hopf-Algebra: B. algebra with $S: V \rightarrow V$
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2.2. Algebras

$(A, +, \cdot, K)$ enhanced by (μ, η)

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$(A, +, \cdot, K)$ enhanced by (μ, η)

associativity: $a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_2) \cdot a_3$

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associativity: $a_1 \cdot (a_2 \cdot a_3) = (a_1 \cdot a_2) \cdot a_3$

$$\mu \circ (\text{id} \otimes \mu) = \mu \circ (\mu \otimes \text{id})$$

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associativity axiom

transposition maps $\sigma: ABA \rightarrow ABA$
 $abb \mapsto baa$

commutativity: $\mu = \mu \circ \sigma$

transposition map: $\sigma: A \otimes A \rightarrow A \otimes A$
 $ab \mapsto ba$

coassociativity: $\mu \circ \mu \circ \sigma = \mu \circ \mu$

unit element: $\eta: K \rightarrow A$
 $\lambda \mapsto \lambda \cdot 1$

$$\mu \circ (\text{id} \otimes \eta) = \text{id} = \mu \circ (\eta \otimes \text{id})$$
$$a \cdot 1 = a = 1 \cdot a$$

transposition map: $\sigma: A \otimes A \rightarrow A \otimes A$
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$(\text{id} \otimes \eta) a \otimes b$

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transposition maps: $\sigma: A \otimes A \rightarrow A \otimes A$
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$$V \otimes K \cong V \cong K \otimes V$$

associativity: $\mu = \mu \circ \sigma$

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$\lambda \cdot 1$

transposition maps: $\sigma: A \otimes A \rightarrow A \otimes A$
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transposition maps $\sigma: A \otimes A \rightarrow A \otimes A$
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commutativity: $\mu \circ \sigma = \mu$

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unit axiom

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2.1. Def

transposition maps $\sigma: A \otimes A \rightarrow A \otimes A$
 $ab \mapsto ba$

$$\forall v \in V, \forall k \in K, v \otimes k = v \otimes k \otimes 1$$

coassociativity: $\mu \circ \mu \circ \sigma = \mu \circ \mu$

unit element:

$$\eta: K \rightarrow A$$
$$k \mapsto k \cdot 1$$

$$\mu \circ (\text{id} \otimes \eta) = \text{id} = \mu \circ (\eta \otimes \text{id})$$

$$a \cdot 1 \otimes b = a \otimes b \cdot 1 = ab$$

unit axiom

2.1. Def Algebra is a vector space $(A, +, \mu, \eta; K)$ that satisfies associativity and unit axiom.

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2.2. Def $\mathbb{C}$ -Algebra: finite dim. vector space

unit axiom

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2.2. Def $\mathcal{A}$ Lie-Algebra: finite dim. vector space
 $(\mathfrak{g}, +, [\cdot, \cdot]; K)$ $[\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$

unit axiom

2.1. Def Algebra is a vector space $(A, +, \mu, \gamma; K)$ that satisfies associativity and unit axiom.

2.2. Def $\mathcal{A}$ Lie-Algebra: finite dim. vector space

$$(\mathfrak{g}, +, [\cdot, \cdot], K) \quad [\cdot, \cdot]: \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$$

$$[\mathfrak{g}, \mathfrak{h}]_{\mathfrak{g}} = -[\mathfrak{h}, \mathfrak{g}]_{\mathfrak{g}} \quad \text{antisymmetry}$$

unit axiom

2.1. Def Algebra is a vector space $(A, +, \mu, \gamma; K)$ that satisfies associativity and unit axiom.

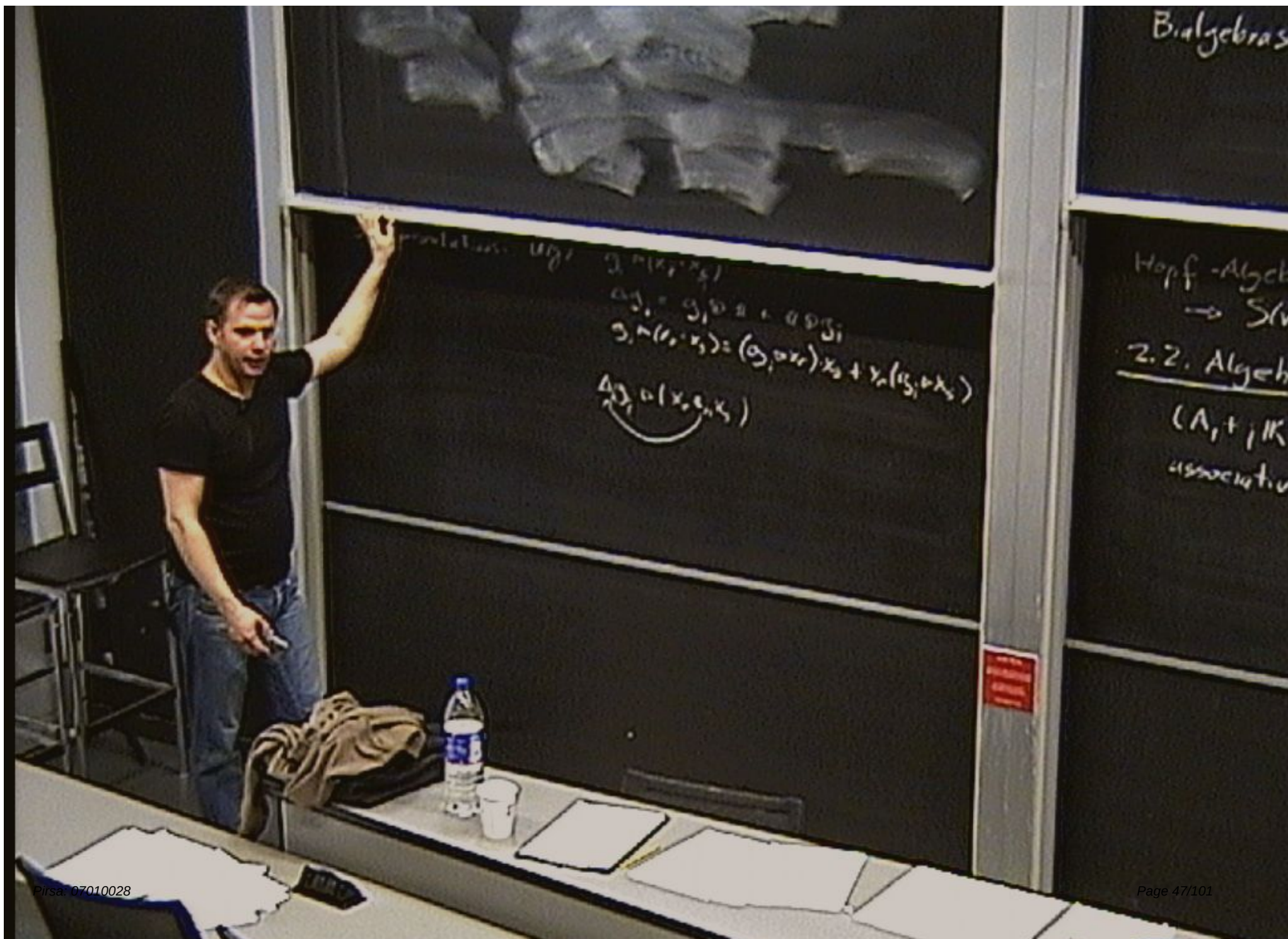
2.2. Def Lie-Algebra: finite dim. vector space

$$(g, +, [\cdot, \cdot], K) \quad [\cdot, \cdot]: g \times g \rightarrow g$$

$$[g, h]_g = -[h, g]_g \quad \text{antisymmetry}$$

$$[g, h, k]_g = [g, k]_g + [h, k]_g \quad \text{bilinearity}$$

$$[[g, h], k] + \text{cycl.} = 0 \quad \text{Jacobi-identity}$$



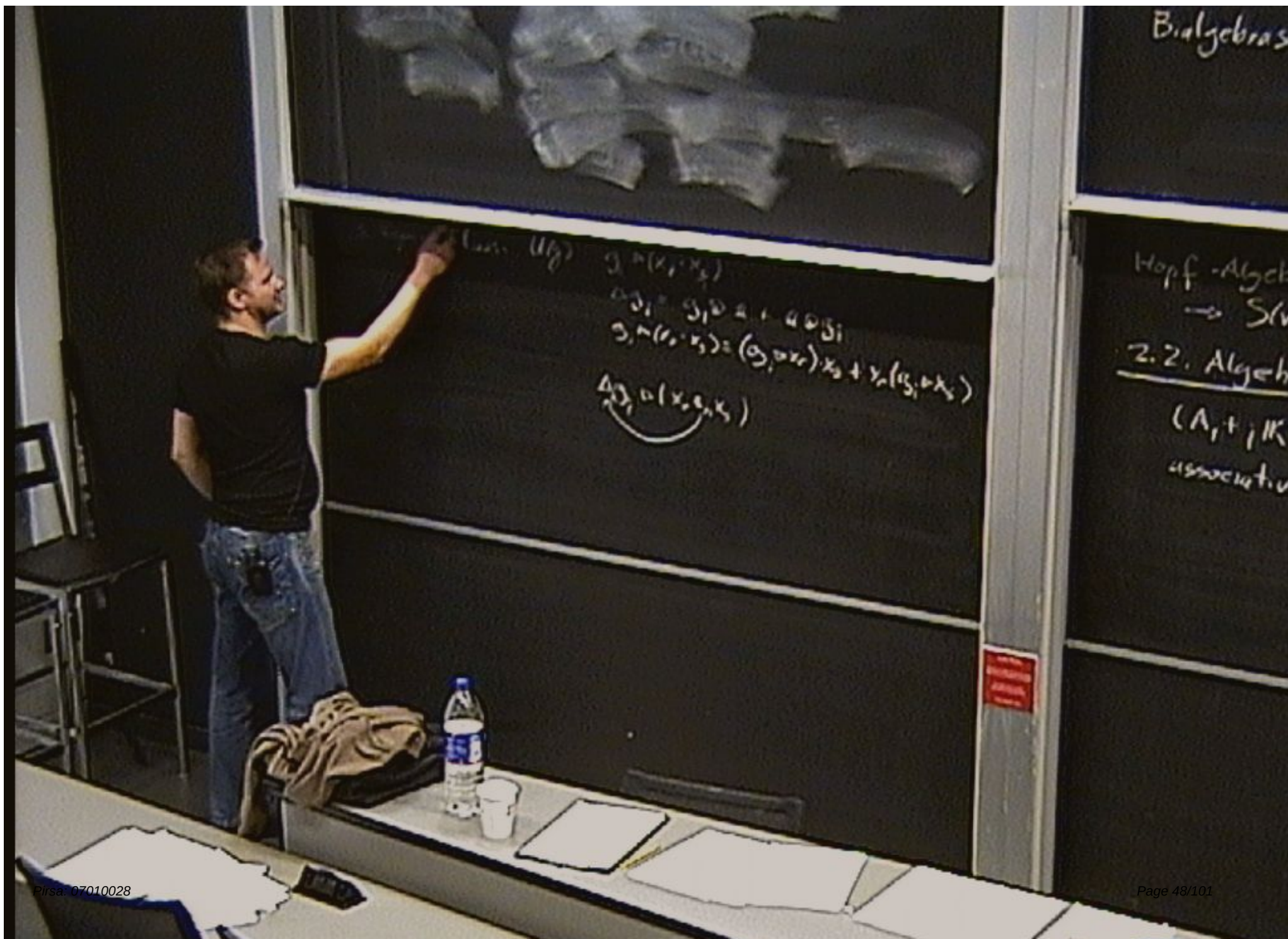
Bialgebras

Hopf-Algeb
→ Str

2.2. Algeb

$(A, +, \cdot, K)$
associativ

...relation: $u(g) = g^{-1}(x_1 \cdot x_2)$
 $\Delta g_i = g_i \otimes 1 + u(g) \otimes g_i$
 $g_i \cdot (x_1 \cdot x_2) = (g_i \cdot x_1) \cdot x_2 + x_1 \cdot (g_i \cdot x_2)$
 $\Delta g_i \cdot (x_1 \cdot x_2)$



Bialgebras

Hopf-Algebr
→ Str

2.2. Algebr

$(A, +, \cdot, K)$
assoziativ

$$\begin{aligned} & \text{Lieb. Alg.} \quad g \cdot (x, y) \\ & \Delta g_i = g_i \otimes 1 + 1 \otimes g_i \\ & g \cdot (x, y) = (g, x) \cdot y + x \cdot (g, y) \\ & \Delta(g \cdot (x, y)) \end{aligned}$$

$$(V, +, \cdot; K), (W, +, \cdot; K)$$

direct sum $V \oplus W$

$$V \times W$$

tensor product

$$(V, +, \cdot, K), (W, +, \cdot, K)$$

$$V \times W$$

direct sum $V \oplus W$

$$(v, w) + (v', w') = (v + v', w + w')$$

$$2(v, w) = (2v, 2w)$$

tensor product $V \otimes W$

$$(V, +, \cdot; K), (W, +, \cdot; K)$$

$$V \times W \begin{cases} \rightarrow \text{direct sum } V \oplus W \\ (v, w) + (v', w') = (v + v', w + w') \\ \lambda(v, w) = (\lambda v, \lambda w) \\ \rightarrow \text{tensor product } V \otimes W \end{cases}$$

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$$(V, +, \cdot; K), (W, +, \cdot; K)$$

$$V \times W$$

direct sum $V \oplus W$

$$(v, w) + (v', w') = (v + v', w + w')$$

$$\lambda(v, w) = (\lambda v, \lambda w)$$

tensor product $V \otimes W$

$$(v, w) + (v', w) = (v + v', w)$$

$$\lambda(v, w) = (\lambda v, w) = (v, \lambda \cdot w)$$

$$K \otimes V \cong V \cong V \otimes K$$

$$(V, +, \cdot; K), (W, +, \cdot; K)$$

$V \times W$ → direct sum $V \oplus W$

$$(v, w) + (v', w') = (v + v', w + w')$$

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→ tensor product $V \otimes W$

$$(v, w) + (v', w') = (v + v', w)$$

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$$K \otimes V \cong V \cong V \otimes K$$

$$T(V) = K \otimes V \otimes V \otimes V \otimes V \otimes \dots \otimes V \otimes K \otimes \dots$$

2.3 Def Universal Enveloping Algebra

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Lie-Algebra $(\mathfrak{g}, \mathcal{L}, \mathcal{I}_{\mathfrak{g}}, \mathbb{K})$

$U(\mathfrak{g})$

2.3 Def Universal Enveloping Algebra

Lie-Algebra $(\mathfrak{g}, L, I_{\mathfrak{g}}, K)$

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{I_{\mathfrak{g}}}$$

$I_{\mathfrak{g}}$ generated by $x \otimes y - y \otimes x - [x, y]_{\mathfrak{g}}$

$V \times W$

$$(v, w) + (v', w') = (v + v', w + w')$$

$$\lambda(v, w) = (\lambda v, \lambda w)$$

tensor product $V \otimes W$

$$(v, w) + (v', w') = (v + v', w)$$

$$\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$$

$$K \otimes V \cong V \cong V \otimes K$$

$$T(V) = K \oplus V \oplus V \otimes V \oplus V^{\otimes 3} \oplus \dots \oplus V^{\otimes n} \oplus \dots$$

Lie-Algebra $(\mathfrak{g}, L, I_{\mathfrak{g}}, K)$

$$\frac{(\mathfrak{g})}{I_{\mathfrak{g}}}$$

$I_{\mathfrak{g}}$: generated by $v \otimes v_j - v_j \otimes v - [v_i, v_j]_g$

2.3 Def Universal Enveloping Algebra

Lie-Algebra $(\mathfrak{g}, L, I_{\mathfrak{g}}, K)$

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{I_{\mathfrak{g}}}$$

$I_{\mathfrak{g}}$ generated by

$$x_j \otimes x_i - [x_i, x_j]_{\mathfrak{g}}$$

2.3 Def Universal Enveloping Algebra

Lie-Algebra $(\mathfrak{g}, L, I_{\mathfrak{g}}, k)$

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$I_{\mathfrak{g}}$ generated by $x_i x_j - x_j x_i - [x_i, x_j]_{\mathfrak{g}}$

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$I_{\mathfrak{g}}$ generated by $x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j]_k$
 $\varphi \in T(V) \quad \varphi \otimes (x_i \otimes x_j)$

2.3 Def Universal Enveloping Algebra

Lie-Algebra $(\mathfrak{g}, \mathfrak{I}_{\mathfrak{g}}, k)$

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{\mathfrak{I}_{\mathfrak{g}}}$$

$\mathfrak{I}_{\mathfrak{g}}$: generated by $x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j]_k$
 $\varphi \in T(\mathfrak{g}) \quad \varphi \otimes (x_i \otimes x_j)$

$V \times W$

$$(v, w) + (v', w') = (v + v', w + w')$$

$$\lambda(v, w) = (\lambda v, \lambda w)$$

tensor product $V \otimes W$

$$(v, w) + (v', w') = (v + v', w')$$

$$\lambda(v, w) = (\lambda v, w) = (v, \lambda w)$$

$$K \otimes V \cong V \cong V \otimes K$$

$$T(V) = K \otimes V \otimes V \otimes V \otimes \dots \otimes V \otimes \dots$$

$$K \otimes V \cong V \otimes K$$

$$(v, w) + (v, w) = (v, w + w)$$

$$2(v, w) = (2v, w) = (v, 2 \cdot w)$$

$$T(V) = K \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \dots \oplus V^{\otimes k} \oplus \dots$$



$$I_g = \frac{g^2}{2}$$

I_g generated by $g_i \otimes g_j - g_j \otimes g_i - [g_i, g_j]g$

$$\varphi \in T(V) \quad \varphi \otimes (g_i \otimes g_j)$$

$$T(V) = K \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \dots$$

Lie-Algebra $(\mathfrak{g}, L, \mathcal{I}_{\mathfrak{g}}; K)$

$$U(\mathfrak{g}) = T(V)$$

$\mathcal{I}_{\mathfrak{g}}$ generated by $x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j]_K$

$$\varphi \in T(V) \quad \varphi \otimes (x_i \otimes x_j)$$

$$K \otimes V \cong V \cong V \otimes K$$

$$(v, w) + (v', w) = (v + v', w)$$

$$2(v, w) = (2v, w) = (v, 2w)$$

$$T(V) = K \oplus V \oplus V \otimes V \oplus V^{\otimes 2} \oplus \dots \oplus V^{\otimes k} \oplus \dots$$

$$\varphi \in T(V)$$

I_2

I_2 generated by $v_i \otimes v_j - v_j \otimes v_i - [v_i, v_j]_2$

$$\varphi \in T(V) \quad \varphi \otimes (v_i \otimes v_j)$$

$$T(v) = K \oplus V \oplus V \oplus V \oplus V \oplus \dots \oplus V \oplus K$$

$$\varphi \in T(v)$$

$$\omega \in T(v)$$

$$\varphi \neq \omega$$

$$L \in \text{alg com } (\gamma_1, \gamma_2, \dots)$$

$$U(g)$$

$$I_g \text{ generated by } \gamma_i \gamma_j - \gamma_j \gamma_i - [\gamma_i, \gamma_j]_g$$

$$\varphi \in T(v)$$

$$\varphi \in (\gamma_i \gamma_j)_g$$

$$T(V) = K \oplus V \oplus V \oplus V \oplus V \oplus \dots \oplus V \oplus \dots$$

$$\varphi \in T(V)$$

$$\omega \in T(V)$$

$$\varphi \neq \omega$$

$$U(g) = \frac{T(g)}{I_g}$$

$$I_g : \text{gener.}$$

$$\varphi \in T(V)$$

$$g_i \circ g_j - g_j \circ g_i = [g_i, g_j]_g$$

$$\varphi(g_i \circ g_j) = \varphi(g_j \circ g_i + [g_i, g_j]_g)$$

$$T(V) = K \oplus V \oplus V \oplus V \oplus V \oplus \dots$$

$$\varphi \in T(V)$$

$$\omega \in T(V) \quad \varphi \not\rightarrow \omega$$

$$L(\mathfrak{g}) = \text{Lie algebra } (\mathfrak{g}, [\cdot, \cdot])$$

$$U(\mathfrak{g}) = T(\mathfrak{g})$$

$$T_{\mathfrak{g}} \text{ generated by } x_i \otimes y_j - y_j \otimes x_i - [x_i, y_j]_{\mathfrak{g}}$$

$$\varphi \in T(V)$$

$$\varphi \otimes (x_i \otimes y_j) = \varphi \otimes (y_j \otimes x_i + [x_i, y_j]_{\mathfrak{g}})$$

$$T(V) = K \oplus V \oplus V \oplus V \oplus V \oplus \dots$$

$$\phi \in T(V)$$

$$\omega \in T(V)$$

$$\phi \neq \omega$$

Lie-Algebra

$$U(g) = \frac{T(g)}{I}$$

I

ind by

$$x_i x_j - x_j x_i - [x_i, x_j]$$

$$\varphi(x_i x_j) = \varphi(x_j x_i + [x_i, x_j])$$

$$T(V) = K \oplus V \oplus V \oplus V \oplus V \oplus \dots$$

$$\phi \in T(V)$$

$$\omega \in T(V)$$

$$\phi \mapsto \omega$$

Lie-Algebra $(\mathfrak{g}, [\cdot, \cdot], K)$

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{I_{\mathfrak{g}}}$$

$I_{\mathfrak{g}}$ generated by $x_i x_j - x_j x_i - [x_i, x_j]_{\mathfrak{g}}$

$$\varphi \in T(V)$$

$$\underbrace{\varphi(x_i x_j)}_{\phi} = \underbrace{\varphi(x_j x_i + [x_i, x_j]_{\mathfrak{g}})}_{\omega}$$

$$T(V) = K \oplus V \oplus V \oplus V \oplus V \oplus \dots$$

$$\phi \in T(V)$$

$$\omega \in T(V)$$

$$\phi \neq \omega$$

$$Lie-Algebra (g, [\cdot, \cdot], K)$$

$$I_g = \frac{T(g)}{I_g}$$

$$I_g \text{ generated by } g_i g_j - g_j g_i - [g_i, g_j]_g$$

$$\phi \in T(V)$$

$$\underbrace{\phi(g_i g_j)}_{\phi} = \underbrace{\phi(g_j g_i + [g_i, g_j]_g)}_{\omega}$$

$$T(V) = K \oplus V \oplus V \otimes V \oplus V \otimes V \otimes V \oplus \dots$$

$$\phi \in T(V)$$

$$\omega \in T(V)$$

$$\phi \mapsto \omega$$

Lie-Algebra $(\mathfrak{g}, [\cdot, \cdot], K)$

$$U(\mathfrak{g}) = \frac{T(\mathfrak{g})}{I_{\mathfrak{g}}}$$

$I_{\mathfrak{g}}$ generated by

$$g_i g_j - g_j g_i - [g_i, g_j]$$

commutation relations

$$\phi \in T(V)$$

$$\underbrace{\phi(g_i g_j)}_{\phi} = \underbrace{\phi(g_i g_j + [g_i, g_j])}_{\omega}$$

$$U(\mathfrak{g}) \otimes \rightarrow$$

7.3 Coalgebras

coalgebra: (C, Δ, ϵ) that is counit ϵ and comultiplication Δ

7.3 Coalgebras

coalgebra : $(C, +, K)$ that is equipped coproduct ' Δ ' and counit ' ϵ '

$$\Delta : C \rightarrow C \otimes C$$

$$c \mapsto \sum c_{(1)} \otimes c_{(2)}$$

7.3 Coalgebras

coalgebra: $(C, +, K)$ that is equipped coproduct Δ and counit ϵ

$$\Delta: C \rightarrow C \otimes C$$

$$c \mapsto \sum c_{(1)} \otimes c_{(2)}$$

Sweedler notation

2.3 Coalgebras

coalgebra : (C, Δ, ϵ) that is cocommutative coproduct Δ and counit ϵ

$$\Delta : C \rightarrow C \otimes C$$

$$c \mapsto \sum c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}$$

Sweedler notation

7.3 Coalgebras

coalgebra : (C, Δ, ϵ) that is enhanced coproduct Δ and counit ϵ

$$\Delta : C \rightarrow C \otimes C$$

$$c \mapsto \sum c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}$$

Sweedler notation

$$(\Delta \circ \Delta) \circ \Delta$$

$$\rightarrow c_{(1)} \otimes c_{(2)}$$

7.3 Coalgebras

coalgebra: (C, Δ, ϵ) that is cocommutative coproduct Δ and counit ϵ

$$\Delta: C \rightarrow C \otimes C$$

$$c \mapsto \sum c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}$$

Sweedler notation

$$(\Delta \otimes \Delta) \circ \Delta = \Delta \circ (\Delta \otimes \Delta)$$

$$1. c \rightarrow c_{(1)} \otimes c_{(2)}$$

$$2. c_{(1)} \otimes c_{(2)} \rightarrow$$

7.3 Coalgebras

coalgebra: (C, Δ) that is cocommutative coproduct Δ and counit ϵ

$$\Delta: C \rightarrow C \otimes C$$

$$c = \sum c_{1j} \otimes c_{2j} = c_{10} \otimes c_{20} + \dots$$

Sweedler notation

$$(\Delta \otimes \Delta) \circ \Delta$$

$$1. c \rightarrow c_{1j} \otimes c_{2j}$$

$$2. c_{1j} \otimes c_{2j} \rightarrow c_{1j} \otimes c_{2k} \otimes c_{3l}$$

7.3 Coalgebras

coalgebra : (C, Δ, ϵ) that is cocommutative coproduct ' Δ ' and counit ' ϵ '

$$\Delta : C \rightarrow C \otimes C$$

$$c \mapsto \sum c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}$$

Sweedler notation

$$(\Delta \otimes \Delta) \circ \Delta$$

$\equiv$

$$(\Delta \otimes \text{id}) \circ \Delta$$

$$1. c \mapsto c_{(1)} \otimes c_{(2)}$$

$$2. c_{(1)} \otimes c_{(2)} \mapsto c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$$

$$1. c \mapsto c_{(1)} \otimes c_{(2)}$$

$$2. c_{(1)} \otimes c_{(2)} \mapsto c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)}$$

7.3 Coalgebras

coalgebra: $(C, +, K)$ that is equipped coproduct ' Δ ' and counit ' ϵ '

$$\Delta: C \rightarrow C \otimes C$$

$$c \mapsto \sum c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}$$

Sweedler notation

$$\begin{aligned} & ((\Delta \circ \Delta) \circ \Delta) \\ & \left(\begin{array}{l} 1. c \rightarrow c_{(1)} \otimes c_{(2)} \\ 2. c_{(1)} \otimes c_{(2)} \rightarrow c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)} \end{array} \right) \\ & ((\Delta \circ \Delta) \circ \Delta) \\ & \left(\begin{array}{l} 1. c \rightarrow c_{(1)} \otimes c_{(2)} \\ 2. c_{(1)} \otimes c_{(2)} \rightarrow c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} \end{array} \right) \end{aligned}$$

7.3 Coalgebras

coalgebra: $(C, +, K)$ that is enhanced coproduct Δ and counit ϵ

$$\Delta: C \rightarrow C \otimes C$$

$$c \mapsto \sum c_{(1)} \otimes c_{(2)} = c_{(1)} \otimes c_{(2)}$$

Sweedler notation

$$U(g)$$

$$\Delta g_i = \sum (g_i)_{(1)} \otimes (g_i)_{(2)} = g_i \otimes 1 + 1 \otimes g_i$$

$$(\Delta \otimes \Delta) \circ \Delta = \Delta \circ (\Delta \otimes \Delta)$$

$$\begin{aligned} & \left(\begin{array}{l} 1. c \rightarrow c_{(1)} \otimes c_{(2)} \\ 2. c_{(1)} \otimes c_{(2)} \rightarrow c_{(1)} \otimes c_{(1)(1)} \otimes c_{(1)(2)} \end{array} \right) \\ & \left(\begin{array}{l} 1. c \rightarrow c_{(1)} \otimes c_{(2)} \\ 2. c_{(1)} \otimes c_{(2)} \rightarrow c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} \end{array} \right) \end{aligned}$$

$$(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$$

$$c_{1n} \otimes c_{m1} \otimes c_{21} = c_{(1)n} \otimes c_{(1)2} \otimes c_{(2)}$$

associativity
axiom

$$\begin{aligned}
 (\text{id} \otimes \Delta) \circ \Delta &= (\Delta \otimes \text{id}) \circ \Delta \\
 \zeta_{(1)} \otimes \zeta_{(2)} \otimes \zeta_{(3)} &= \zeta_{(1)} \otimes \zeta_{(2)} \otimes \zeta_{(3)} \\
 &\text{coassociativity} \\
 &\text{axiom}
 \end{aligned}$$

cocommutativity: $\Delta = \sigma \circ \Delta$

$$\begin{aligned}
 (\text{id} \otimes \Delta) \circ \Delta &= (\Delta \otimes \text{id}) \circ \Delta \\
 c_{1n} \otimes c_{m1} \otimes c_{12} &= c_{(1)n} \otimes c_{(1)2} \otimes c_{(1)} \\
 &\text{coassociativity} \\
 &\text{axiom}
 \end{aligned}$$

cocommutativity :
 counit element :

$\rightarrow \mathbb{K}$

$$(\text{id} \otimes \epsilon) \circ \Delta = \text{id} = (\epsilon \otimes \text{id}) \circ \Delta$$

$$c_n \otimes c_n$$

$$\begin{aligned}
 (\text{id} \otimes \Delta) \circ \Delta &= (\Delta \otimes \text{id}) \circ \Delta \\
 c_{11} \otimes c_{22} \otimes c_{33} &= c_{(1)11} \otimes c_{(1)22} \otimes c_{(1)33} \\
 \text{associativity} \\
 \text{axiom}
 \end{aligned}$$

cocomutative
counit

$$\Delta = \sigma \circ \Delta$$

$$\epsilon: C \rightarrow K$$

$$(\text{id} \otimes \epsilon) \circ \Delta = \text{id} = (\epsilon \otimes \text{id}) \circ \Delta$$

$$c_{11} \otimes c_{11}$$

$$\begin{aligned}
 (\text{id} \otimes \Delta) \circ \Delta &= (\Delta \otimes \text{id}) \circ \Delta \\
 c_{11} \otimes c_{22} \otimes c_{33} &= c_{(1)11} \otimes c_{(1)22} \otimes c_{(1)33} \\
 \text{coassociativity} \\
 \text{axiom}
 \end{aligned}$$

cocommutativity: $\Delta = \sigma \circ \Delta$
 counit element: $\epsilon: C \rightarrow K$

$$\begin{aligned}
 (\text{id} \otimes \epsilon) \circ \Delta &= \text{id} = (\epsilon \otimes \text{id}) \circ \Delta \\
 c_{11} \otimes c_{12} &= c = \epsilon(c_{11}) c_{12} \\
 \text{counit axiom}
 \end{aligned}$$

$$\begin{aligned}
 (\text{id} \otimes \Delta) \circ \Delta &= (\Delta \otimes \text{id}) \circ \Delta \\
 c_{11} \otimes c_{22} \otimes c_{33} &= c_{(1)11} \otimes c_{(1)22} \otimes c_{(1)33} \\
 &\text{coassociativity} \\
 &\text{axiom}
 \end{aligned}$$

cocommutativity: $\Delta = \sigma \circ \Delta$

counit element: $\epsilon: C \rightarrow K$

7.6 Def Coalgebra: $(C, +, \Delta, \epsilon, K)$
that satisfies coassociativity
and counit axiom.

$$(\text{id} \otimes \epsilon) \circ \Delta = \text{id} = (\epsilon \otimes \text{id}) \circ \Delta$$

$$c_{11} \epsilon(c_{12}) = c = \epsilon(c_{11}) c_{12}$$

counit axiom

2.4. Bialgebras and Hopf Algebras

2.5 Def A Bialgebra $(B, \mu, \gamma, \Delta, \epsilon; K)$ is an algebra and a coalgebra made compatible

2.4. Bialgebras and Hopf Algebras

2.5 Def A Bialgebra $(B, \mu, \eta, \Delta, \epsilon; K)$ is an algebra and a coalgebra made compatible

$$\begin{aligned}\Delta(bc) &= (\Delta b)(\Delta c) & \Delta 1 &= 1 \\ \epsilon(bc) &= \epsilon(b) \cdot \epsilon(c)\end{aligned}$$

2.4. Bialgebras and Hopf Algebras

2.5 Def A Bialgebra $(B, \mu, \eta, \Delta, \epsilon; K)$ is an algebra and a coalgebra made compatible

$$(\mu \otimes b) \cdot (c \otimes 1) = \mu \otimes b \cdot c$$

$$\Delta(bc) = (\Delta b)(\Delta c) \quad \Delta 1 = 1 \otimes 1$$

$$\epsilon(bc) = \epsilon(b) \cdot \epsilon(c)$$

2.4. Bialgebras and Hopf Algebras

2.5 Def A Bialgebra $(B, \mu, \eta, \Delta, \epsilon; K)$ is an algebra and a coalgebra made compatible

$$(\mu \circ b) \cdot (c \circ d) = a \circ (b \cdot c \cdot d)$$

$$\Delta(bc) = (\Delta b)(\Delta c) \quad , \quad \Delta 1 = 1 \otimes 1$$
$$\epsilon(bc) = \epsilon(b) \cdot \epsilon(c) \quad , \quad \epsilon(1) = 1 \in K$$

2.4. Bialgebras and Hopf Algebras

2.5 Def. A Bialgebra $(B, \mu, \eta, \Delta, \epsilon, K)$ is an algebra and a coalgebra made compatible

$$(a \otimes b) \cdot (c \otimes d) = a \otimes bcd$$

$$\Delta(bc) = (\Delta b)(\Delta c), \quad \Delta 1 = 1 \otimes 1$$

$$\epsilon(bc) = \epsilon(b) \cdot \epsilon(c), \quad \epsilon(1) = 1 \in K$$

antipode map: $S: B \rightarrow B$

2.5 Def A Bialgebra $(B, \mu, \eta, \Delta, \epsilon; K)$ is an algebra and a coalgebra made compatible

$$(\mu \circ b) \cdot (c \circ d) = a \circ b \cdot d$$

$$\Delta(bc) = (\Delta b)(\Delta c), \quad \Delta 1 = 1 \otimes 1$$
$$\epsilon(bc) = \epsilon(b) \cdot \epsilon(c), \quad \epsilon(1) = 1 \in K$$

antipode map: $S: B \rightarrow B$

$$\mu \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon$$
$$= \mu \circ (\text{id} \otimes S) \circ \Delta$$

antipode axiom

antipode map: $S: B \rightarrow B$

$$\mu(S \otimes \text{id}) \circ \Delta = \mu(\text{id} \otimes S) \circ \Delta$$

antipode axiom

7.6 Def A Hopf-Algebra $(H, \mu, \eta, \Delta, \epsilon, S; K)$
is a Bialgebra with antipode S
that satisfies the antipode axiom

7.6 Def A Hopf-Algebra $(H, \mu, \eta, \Delta, \epsilon, S, 1, K)$
is a Bialgebra with antipode S
that satisfies the antipode axiom

Ex

$U(g)$

$$\Delta g_i = g_i \otimes 1 + 1 \otimes g_i$$

$$\epsilon(g_i) = 0$$

$$S(g_i) = -g_i$$

7.6 Def A Hopf-Algebra $(H, \mu, \eta, \Delta, \epsilon, S, 1_K)$
is a Bialgebra with antipode S
that satisfies the antipode axiom

Exercise: $(U(g))$ $\Delta g_i = g_i \otimes 1 + 1 \otimes g_i$
 $\epsilon(g_i) = 0$
 $S(g_i) = -g_i$

1.) Check Hopf-Algebra axioms

7.6 Def A Hopf-Algebra $(H, \mu, \eta, \Delta, \epsilon, S, 1, K)$
is a Bialgebra with antipode S
that satisfies the antipode axiom

Exercise: $U(g)$ $\Delta g_i = g_i \otimes 1 + 1 \otimes g_i$
 $\epsilon(g_i) = 0$
 $S(g_i) = -g_i$

- 1.) Check Hopf-Algebra axioms
- 2.) Check homomorphism of Δ, ϵ
(algebra)

2.1. Bialgebras and Hopf Algebras

2.5 Def. A Bialgebra $(B, \mu, \eta, \Delta, \epsilon; K)$ is an algebra and a coalgebra made compatible

$$(\mu \otimes \text{id}) \circ (\text{id} \otimes \Delta) = \mu \otimes \Delta$$

$$\Delta(bc) = (\Delta b)(\Delta c), \quad \Delta 1 = 1 \otimes 1$$

$$\epsilon(bc) = \epsilon(b) \cdot \epsilon(c), \quad \epsilon(1) = 1 \in K$$

antipode map: $S: B \rightarrow B$

$$\mu(S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon$$

$$= \mu(\text{id} \otimes S) \circ \Delta$$

antipode axiom

$$S(g_i) = -g_i$$

- 1) Check Hopf-Algebra axioms
- 2) Check bijectivity of Δ, ϵ (algebra)

7.5 Def A Bialgebra $(B, \mu, \eta, \Delta, \epsilon; K)$ is an algebra and a coalgebra made compatible

$$(\mu \circ b) \cdot (\epsilon \circ d) = \eta \circ b \circ d$$

$$\Delta(b \cdot c) = (\Delta b)(\Delta c), \quad \Delta 1 = 1 \otimes 1$$

$$\epsilon(b \cdot c) = \epsilon(b) \cdot \epsilon(c), \quad \epsilon(1) = 1 \in K$$

antipode map: $S: B \rightarrow B$

$$\mu(S \otimes \text{id}) \circ \Delta = \eta \circ \epsilon$$

$$= \mu(\text{id} \otimes S) \circ \Delta$$

antipode axiom

$$S(ab) = S(b) \cdot S(a)$$

$$\epsilon(g_i) = 0$$

$$S(g_i) = -g_i$$

- 1) Check Hopf-Algebra axioms
- 2) Check homomorphism of Δ, ϵ
(algebra)