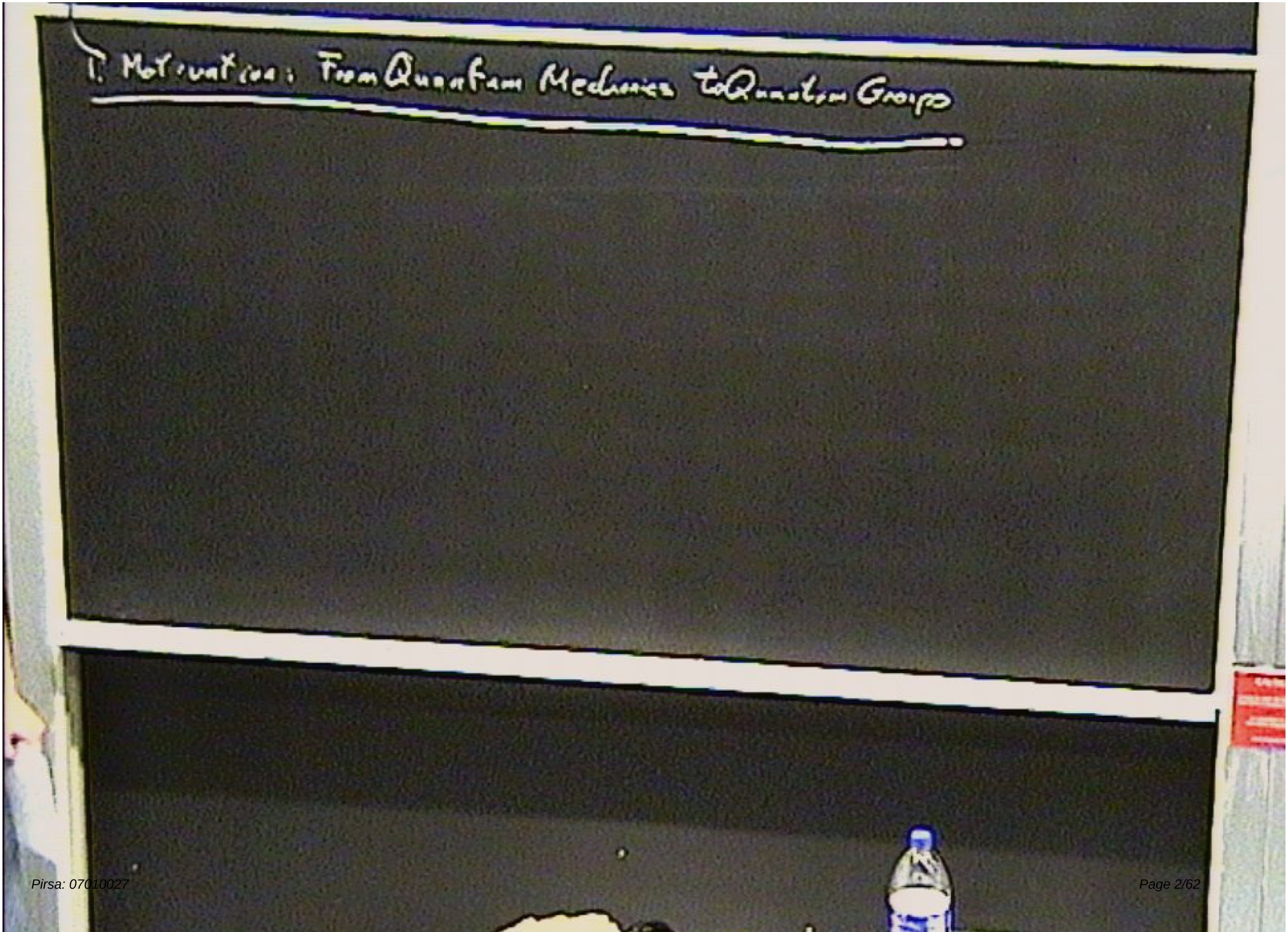
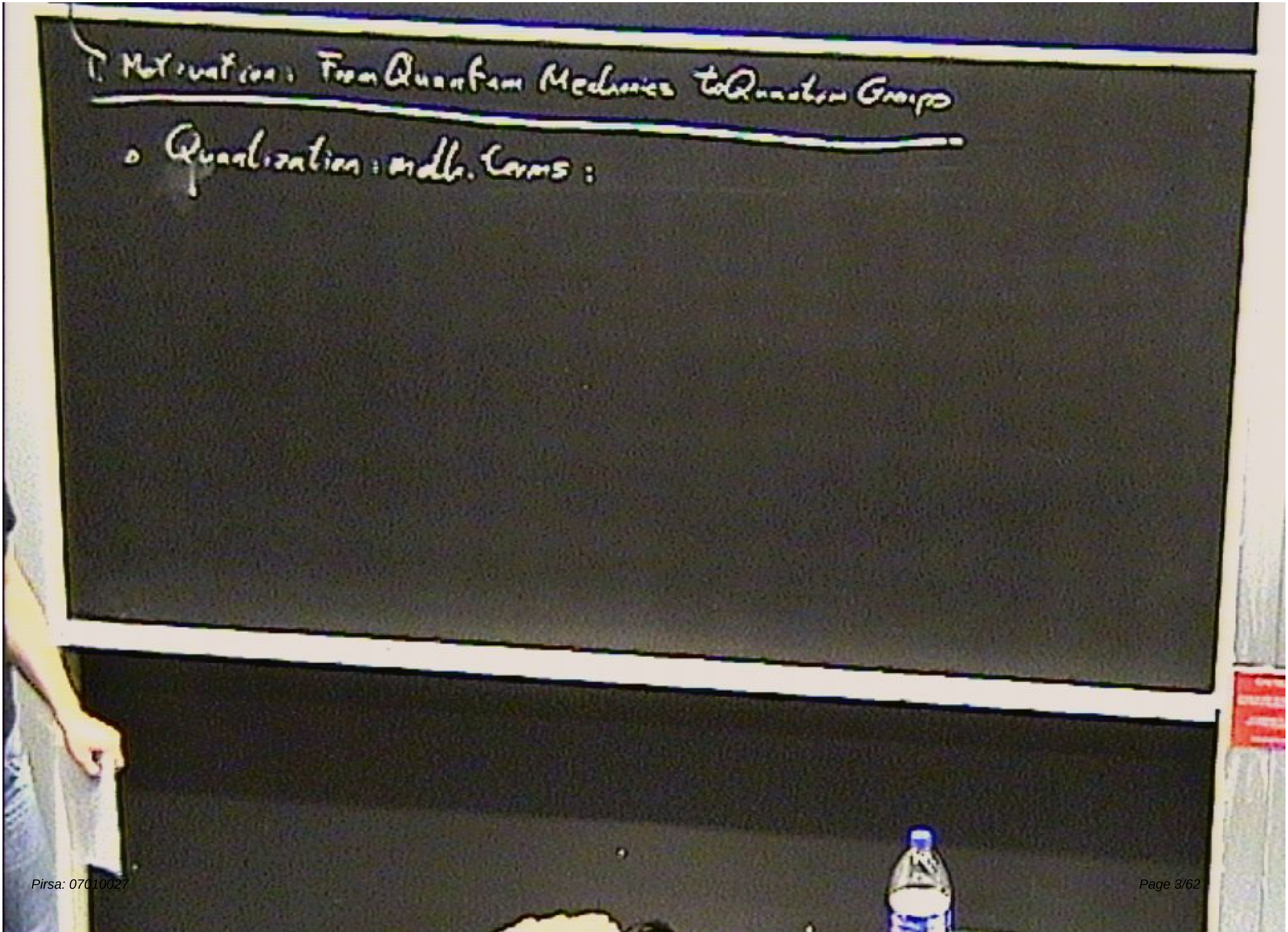
Title: Introduction to quantum groups 1

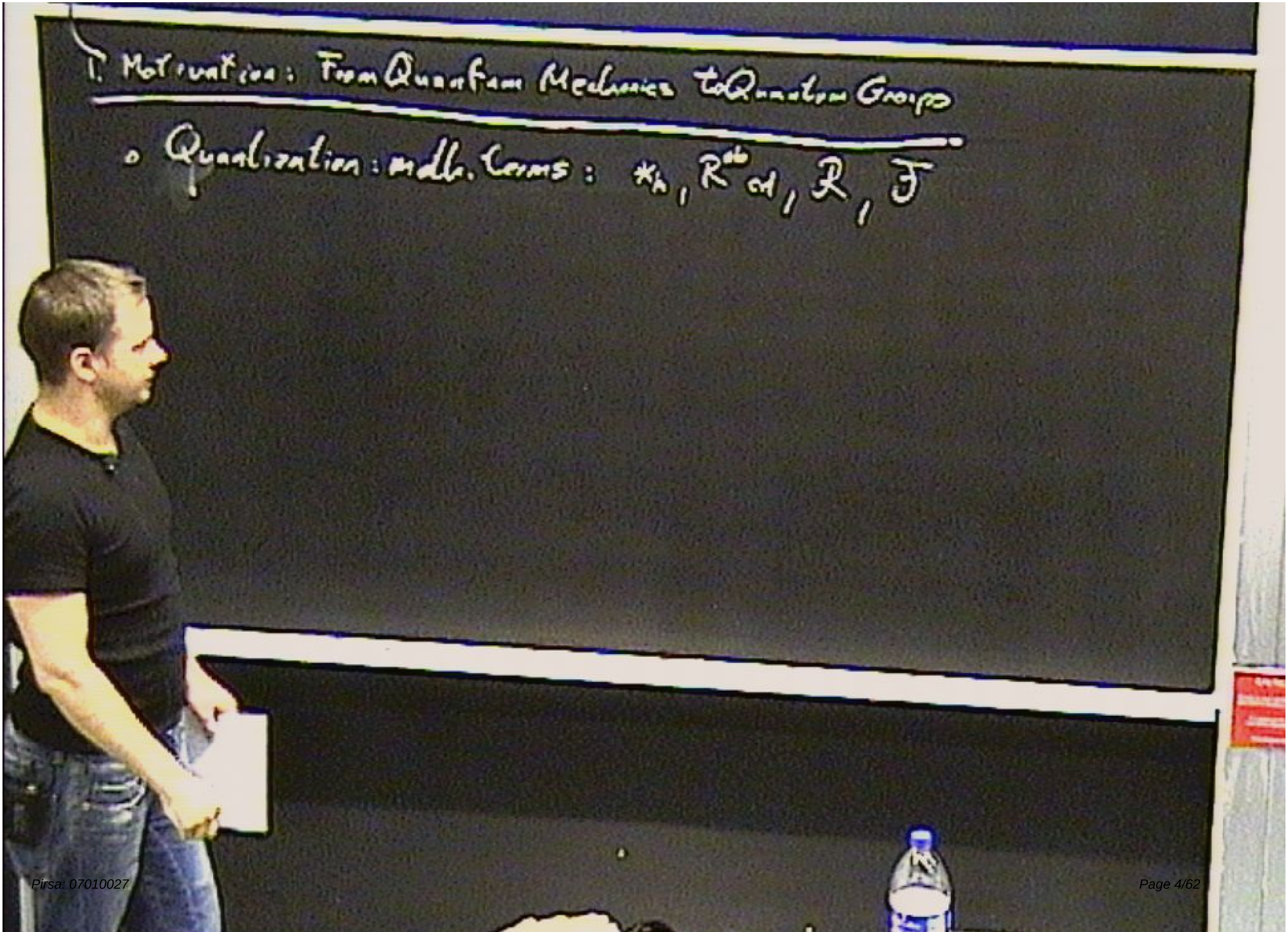
Date: Jan 18, 2007 09:00 AM

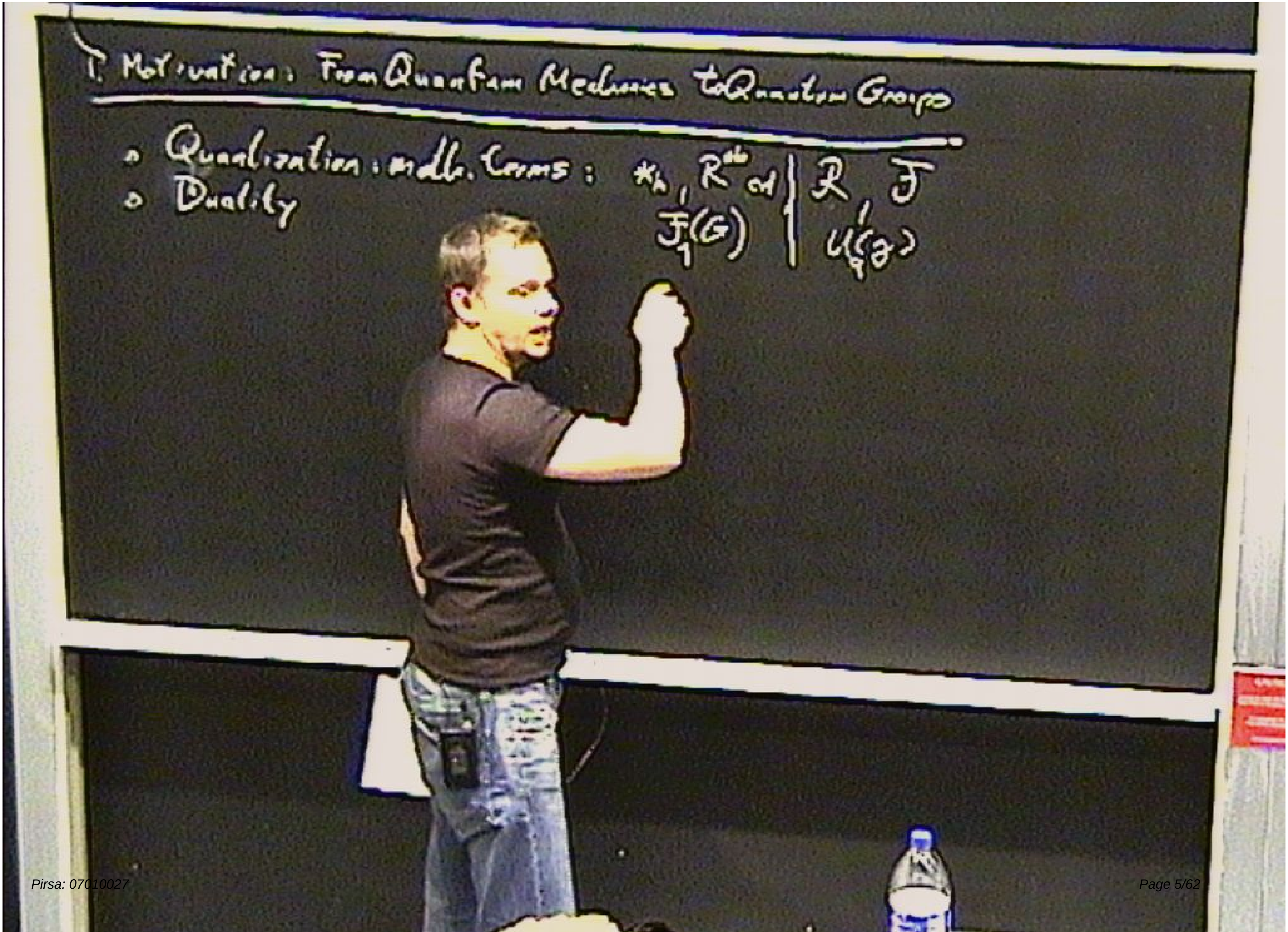
URL: http://pirsa.org/07010027

Abstract: Motivation: From Quantum Mechanics to Quantum Groups
The notion of 'quantization' commonly used in textbooks of quantum mechanics has to be specified in order to turn it into a defined mathematical operation. We discuss that on the trails of Weyl's phase space deformation, i.e. we introduce the Weyl-Moyal starproduct and the deformation of Poisson-manifolds.
Generalizing from this, we understand, why Hopf-algebras are the most genuine way to apply 'quantization'
to various other algebraic objects - and why this has direct physical applications.

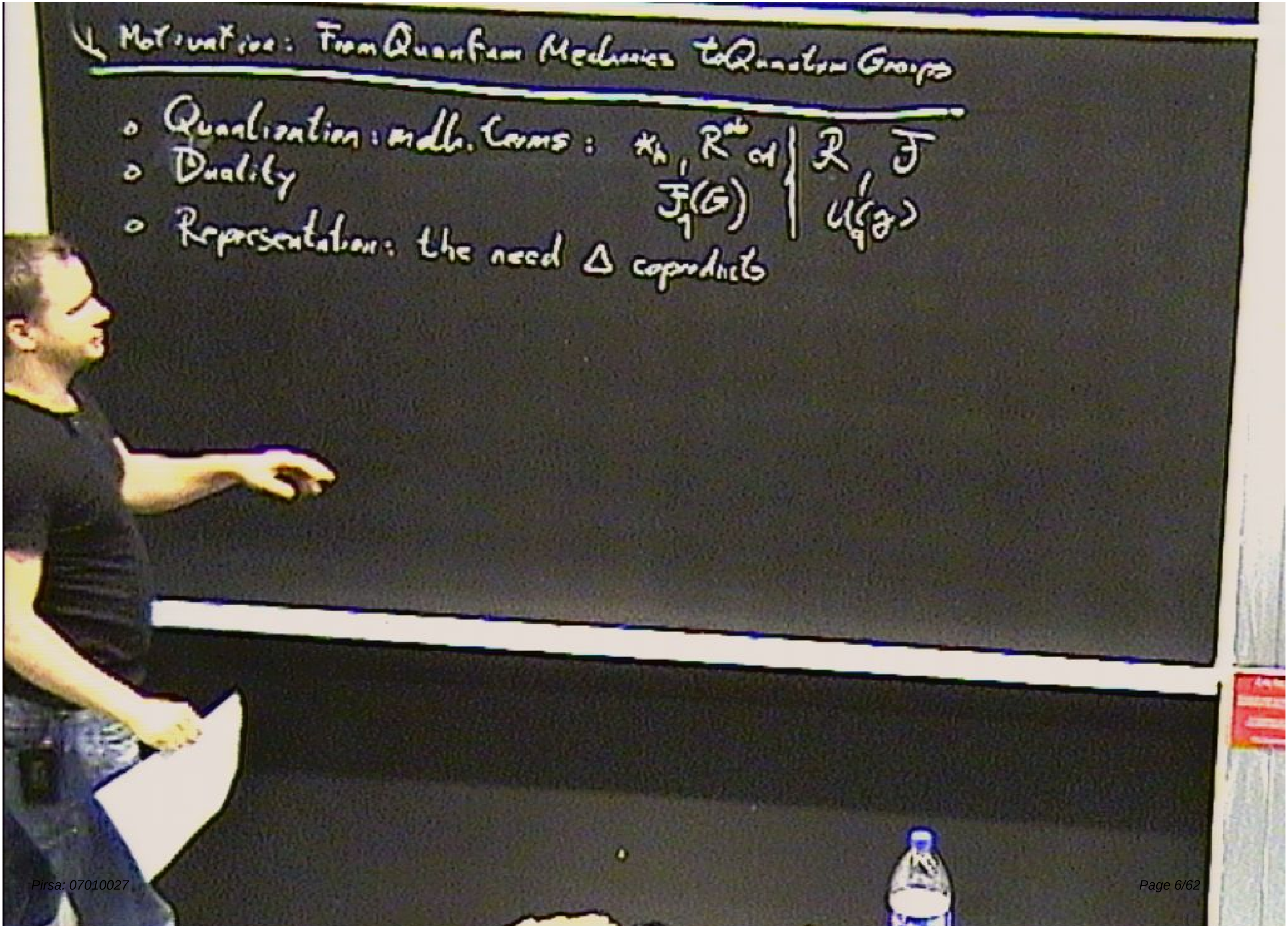

I. Motivation: From Quantum Mechanics to Quantum Groups

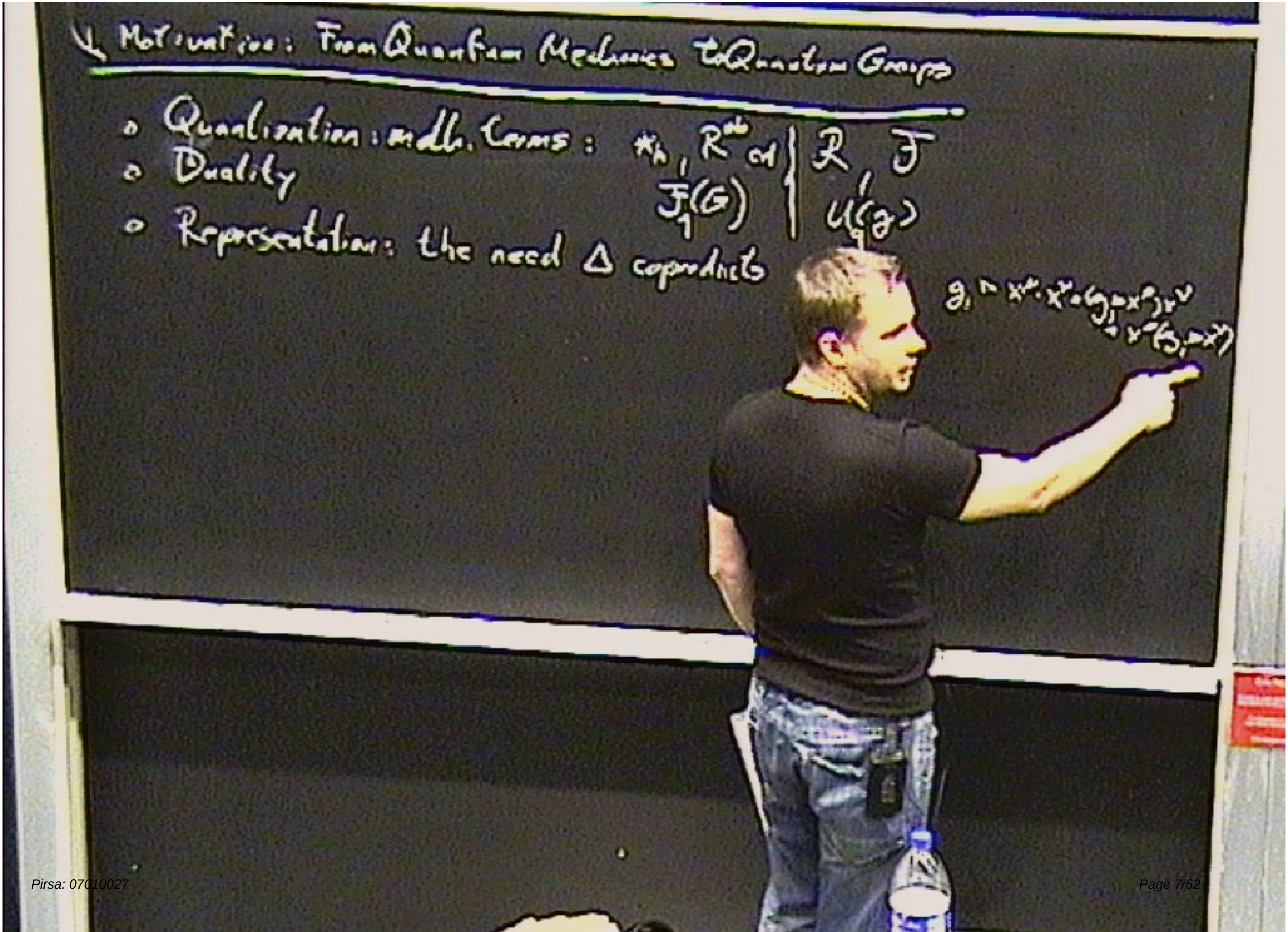

I. Motivation: From Quantum Mechanics to Quantum Groups
Quantization

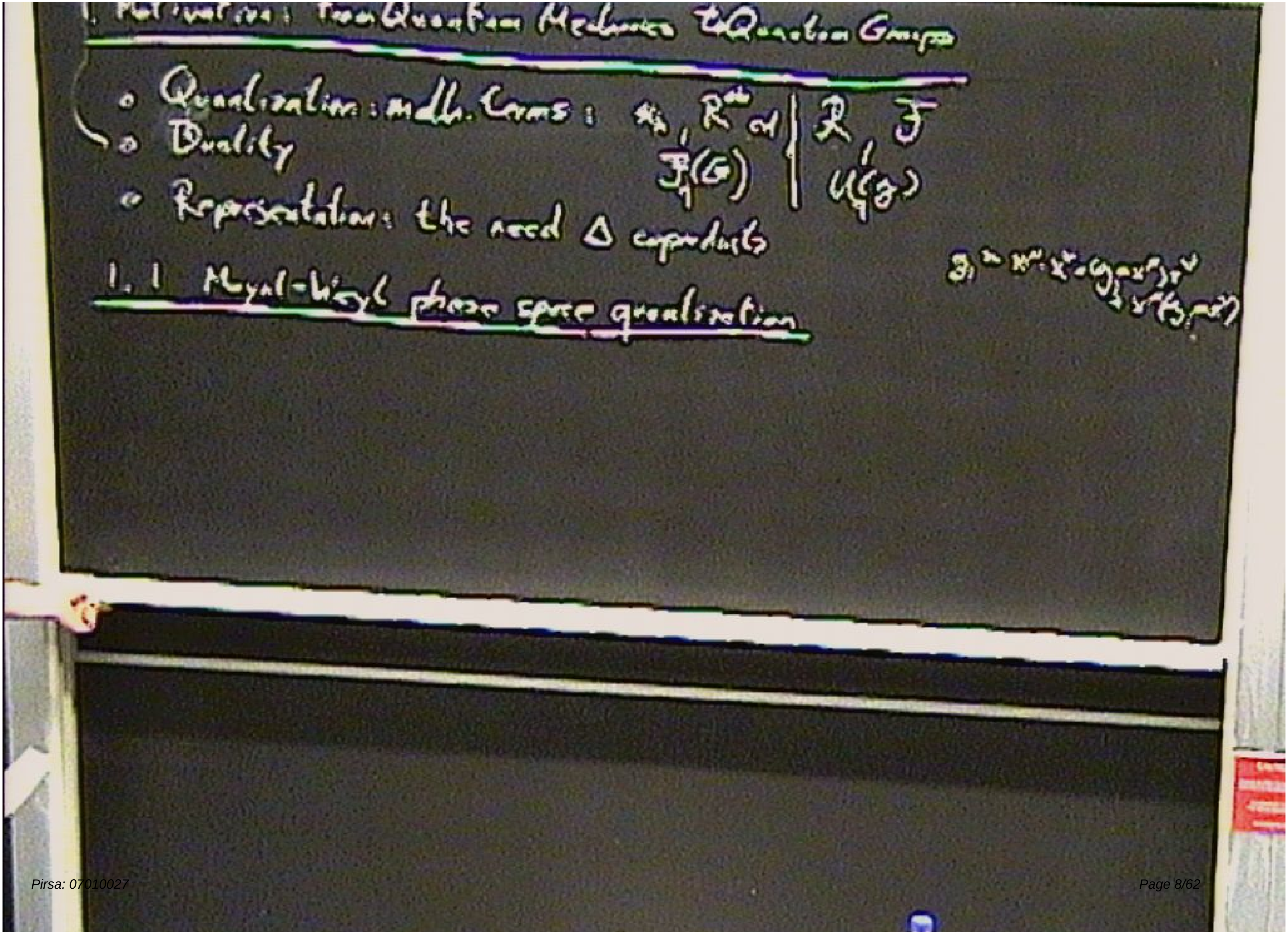

I. Motivation: From Quantum Mechanics to Quantum Groups
Quantization: math. terms:



I. Motivation: From Quantum Mechanics to Quantum Groups
Quantization
Duality


Motivation: From Quantum Mechanics to Quantum Groups
○ Quantization: math. terms: $*_h$, R^{ab} cl | $\mathcal{R}$, $\mathcal{F}$
○ Duality $\mathcal{F}_q(G)$ | $U_q(g)$
○ Representation: the need Δ coproducts


Motivation: From Quantum Mechanics to Quantum Groups
Duality
Representation: the need Δ coproducts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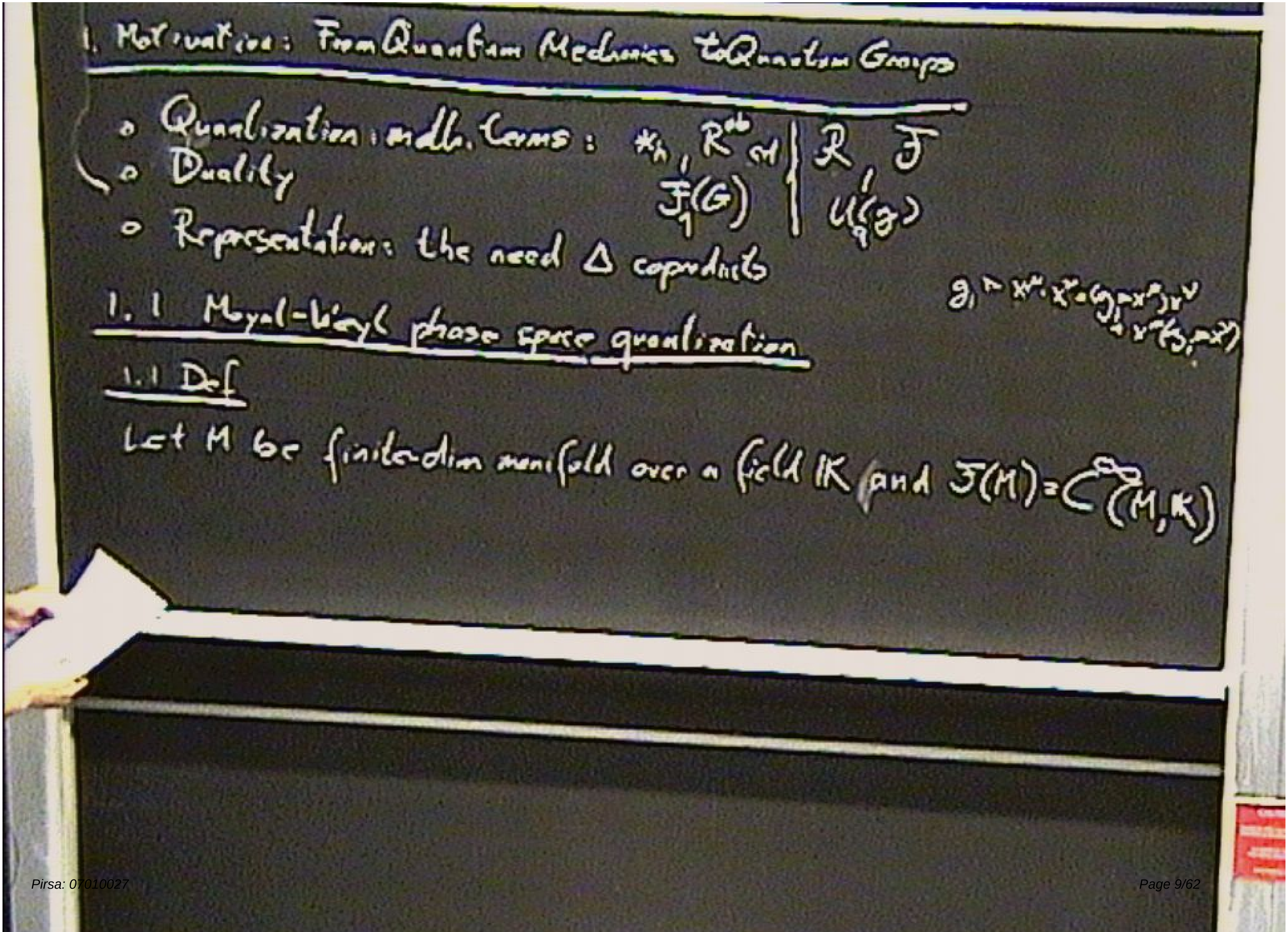
1.1 Moyal-Weyl phase space quantisation


1. Motivation: From Quantum Mechanics to Quantum Groups
∘ Quantization: notations: $*_\hbar$, R^{ab} | $\mathcal{R}$, $\mathcal{F}$
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1.1 Moyal-Weyl phase space quantization
1.1 Def
Let M be finite-dim manifold over a field $\mathbb{K}$ and $\mathcal{F}(M) = C^\infty(M, \mathbb{K})$

1. Motivation: from Quantum Mechanics to Quantum Groups

- Quantization: noncomm. spaces: $\star_h$, R^{ab} ... | $\mathcal{R}$, $\mathcal{F}$
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$g_1 \triangleright x^\mu \cdot x^\nu = (g_{1(1)} \triangleright x^\mu)(g_{1(2)} \triangleright x^\nu)$

1.1 Moyal-Weyl phase space quantization

1.1 Def

Let M be finite-dim manifold over a field $\mathbb{K}$ and $\mathcal{F}(M) = C^\infty(M, \mathbb{K})$
If there exists a bracket $\{\cdot,\cdot\}$

$$\{\cdot,\cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$$

$\{\cdot,\cdot\}$

$$\{\cdot,\cdot\} : \mathcal{F}(M) \times \mathcal{F}(M) \longrightarrow \mathcal{F}(M)$$

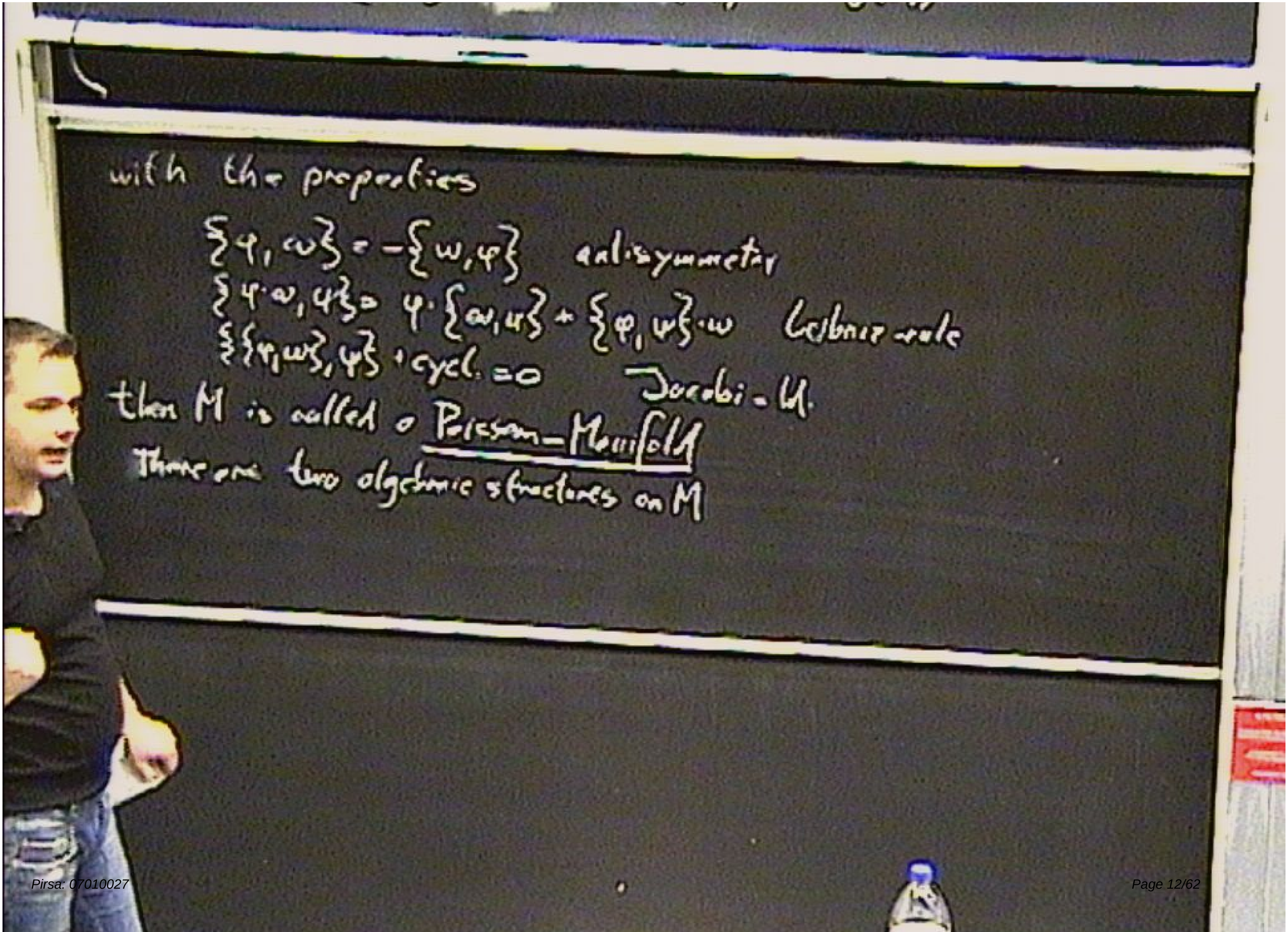
with the properties

$\{\varphi, \omega\} = -\{\omega, \varphi\}$ antisymmetry

$\{\varphi \cdot \omega, \psi\} = \varphi \cdot \{\omega, \psi\} + \{\varphi, \psi\} \cdot \omega$ Leibniz-rule

$\{\{\varphi, \omega\}, \psi\} + \text{cycl.} = 0$ Jacobi-Id.

then M is called a Poisson-Manifold


with the properties
{φ, ω} = −{ω, φ} antisymmetric
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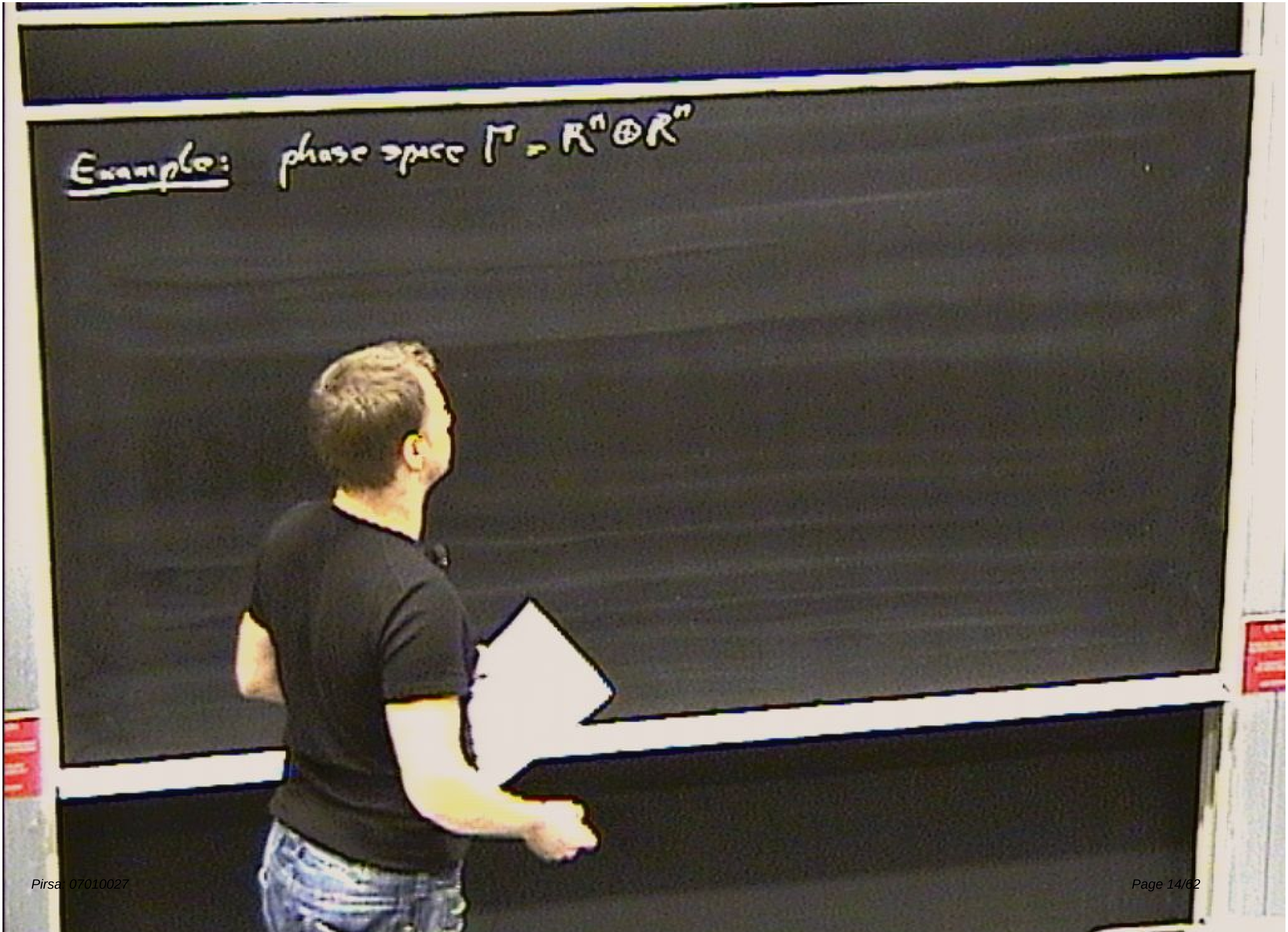
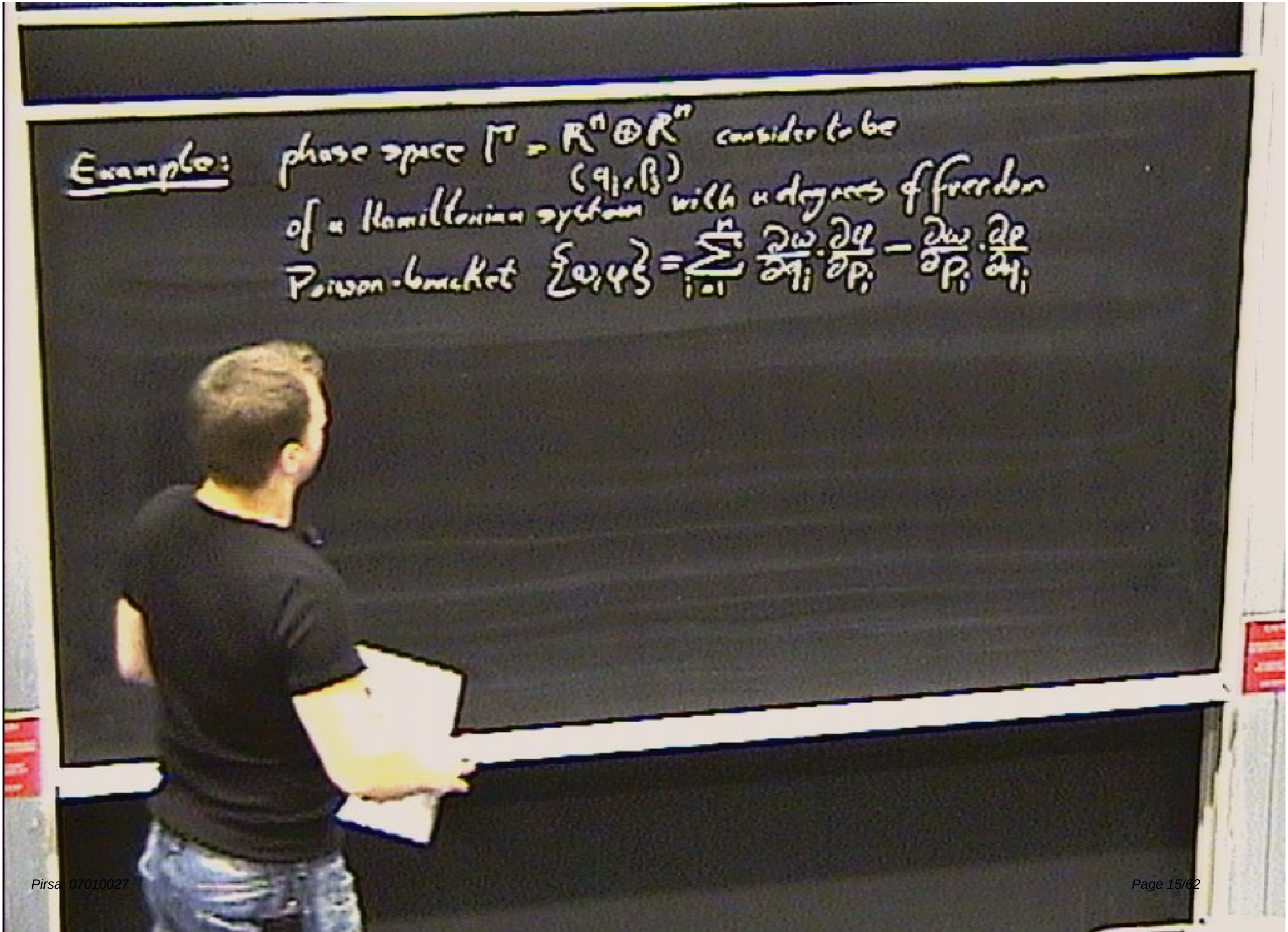
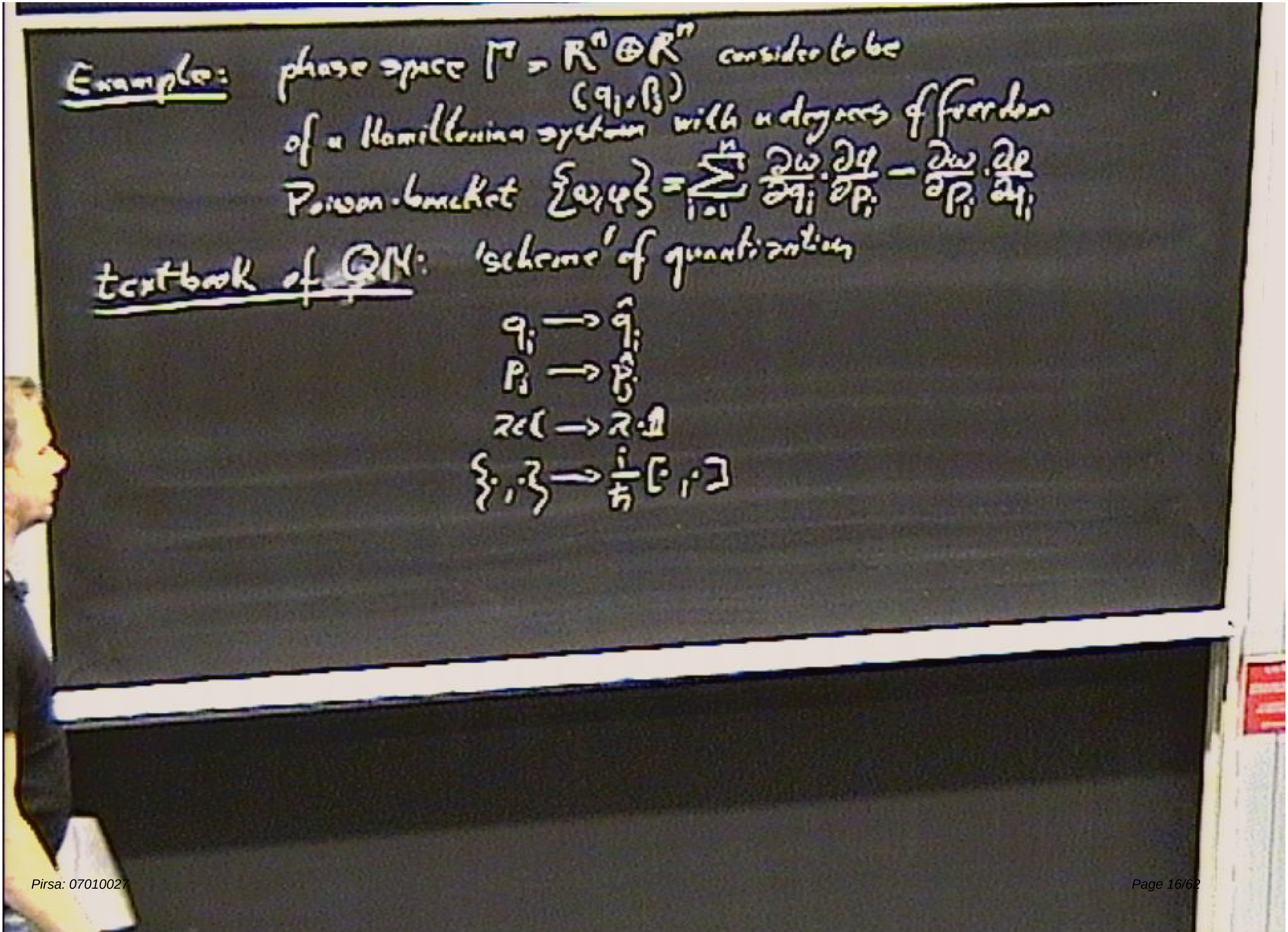
There are two algebraic structures on M

1. $(\varphi\cdot\omega)(m) = \varphi(m)\cdot\omega(m)$
2. Poisson-structure


Example: phase space $\Gamma = R^n \oplus R^n$


Example: phase space $\Gamma = \mathbb{R}^n \oplus \mathbb{R}^n$ consider to be
(q_i, p_i)
of a Hamiltonian system with n degrees of freedom
Poisson-bracket $\{\omega, \varphi\} = \sum_{i=1}^{n} \frac{\partial \omega}{\partial q_i} \cdot \frac{\partial \varphi}{\partial p_i} - \frac{\partial \omega}{\partial p_i} \cdot \frac{\partial \varphi}{\partial q_i}$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textbook of QM: 'scheme' of quantization
q_i → q̂_i
p_j → p̂_j
λ·1 → λ·𝟙
{·,·} → i/ħ [·,·]

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textbook of QM: 'scheme' of quantization

$$q_i \longrightarrow \hat{q}_i \qquad \varphi \longrightarrow \hat{\varphi}$$

$$p_j \longrightarrow \hat{p}_j$$

$$\lambda \in \mathbb{C} \longrightarrow \lambda \cdot \mathbb{1}$$

$$\{\cdot,\cdot\} \longrightarrow \frac{i}{\hbar}[\cdot,\cdot]$$

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problems: 1. this is no homomorphism

2. ordering prescription

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then M is called a <u>Poisson-Manifold</u>

There are two algebraic structures on M

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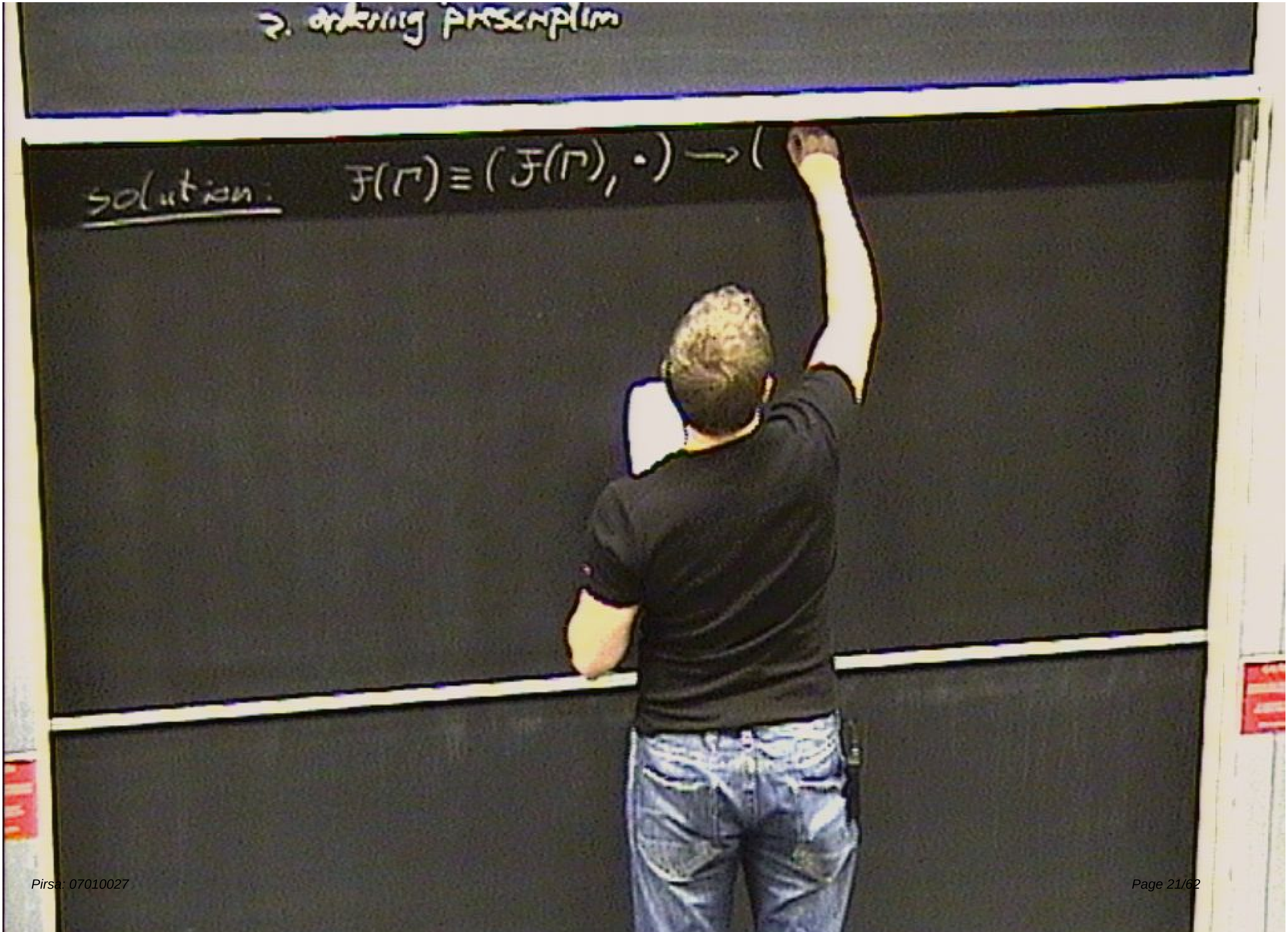
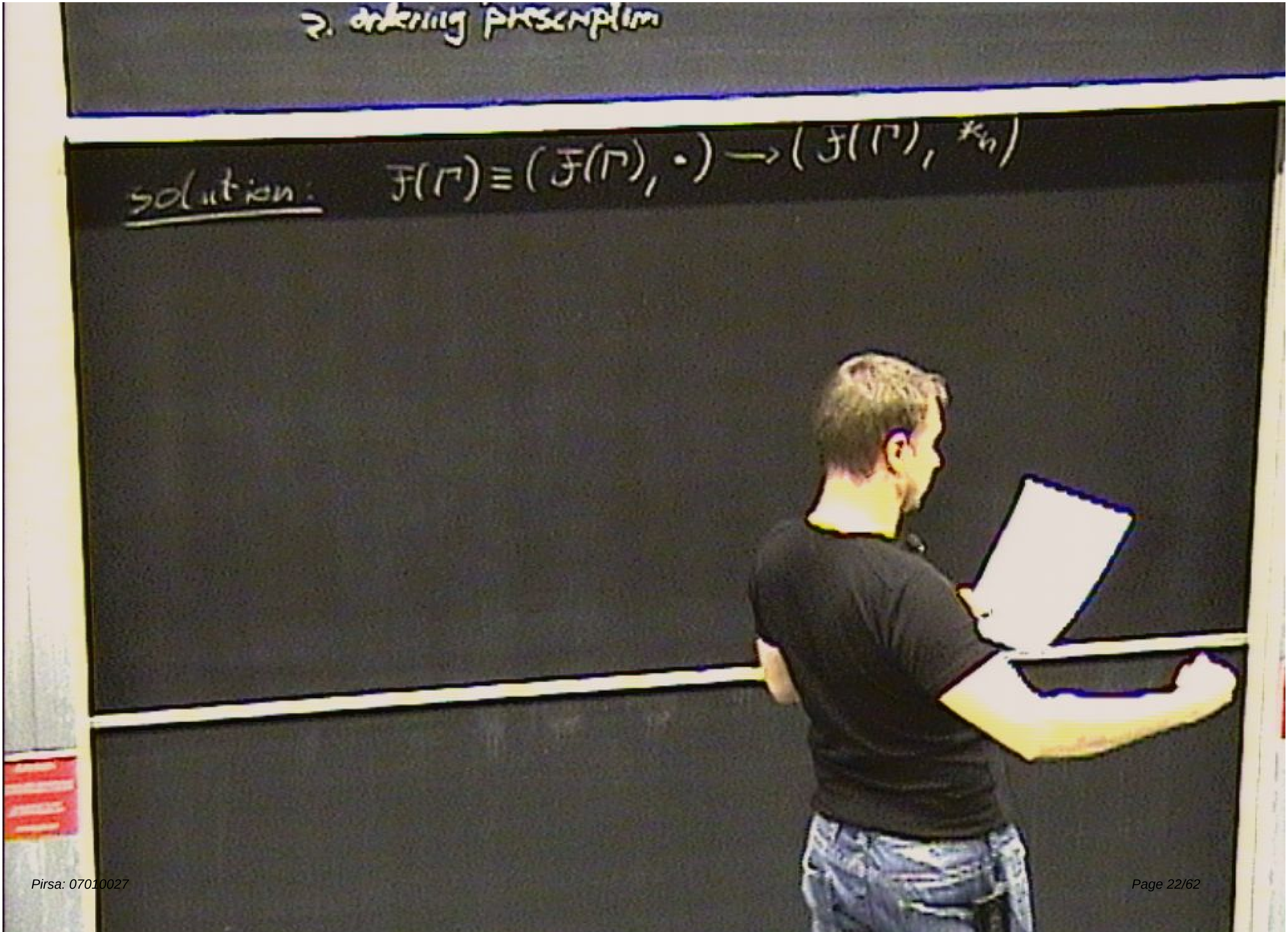

2. ordering prescription
solution:


2. ordering prescription
solution:
F(Γ) ≡ (F(Γ), •) → (F(Γ), *n)

2. ordering prescription

solution: $\mathcal{F}(\Gamma) \equiv (\mathcal{F}(\Gamma), \cdot) \longrightarrow (\mathcal{F}(\Gamma), *_h) \equiv \mathcal{F}_h(\Gamma)$

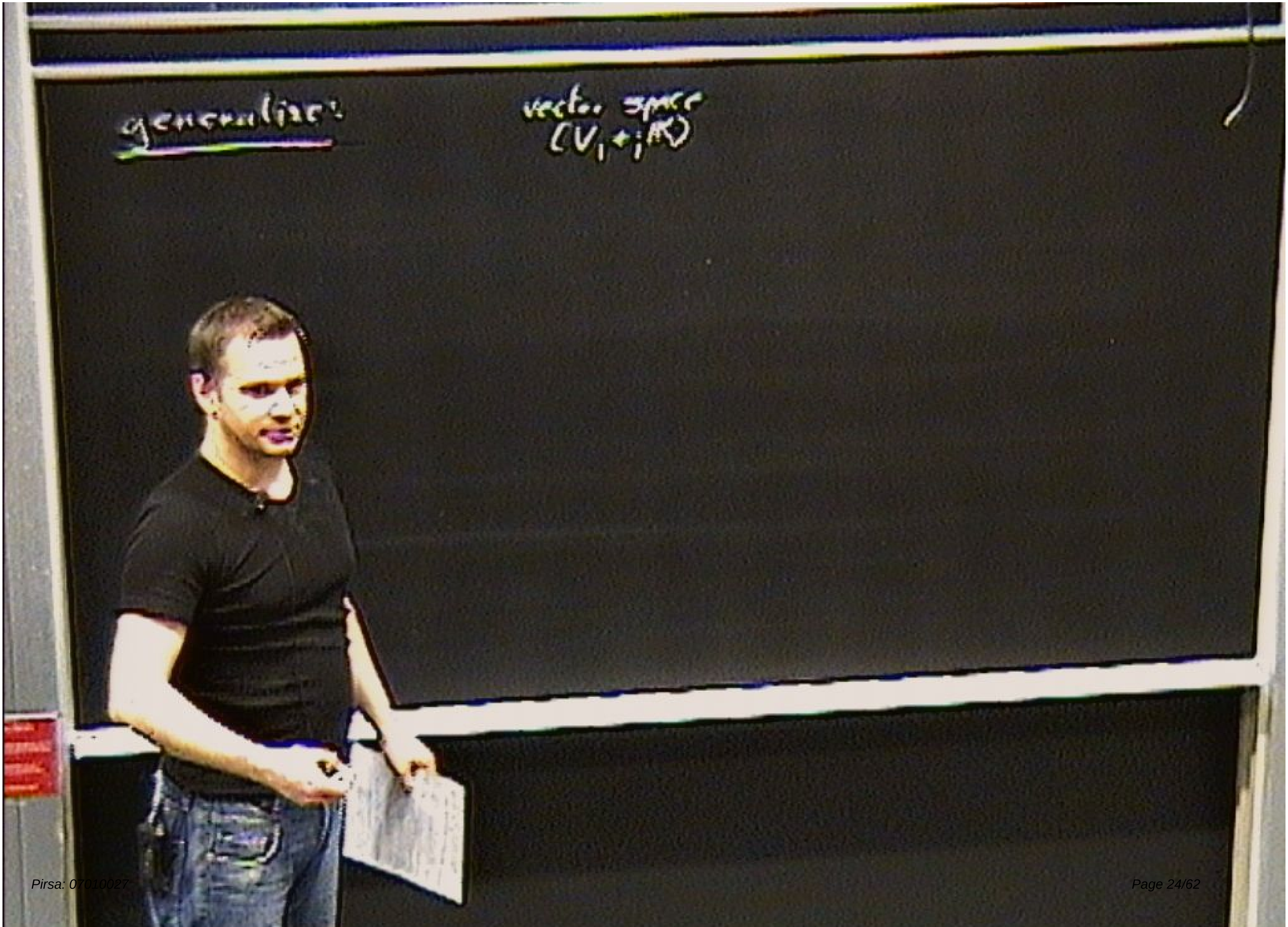
1.2. Def

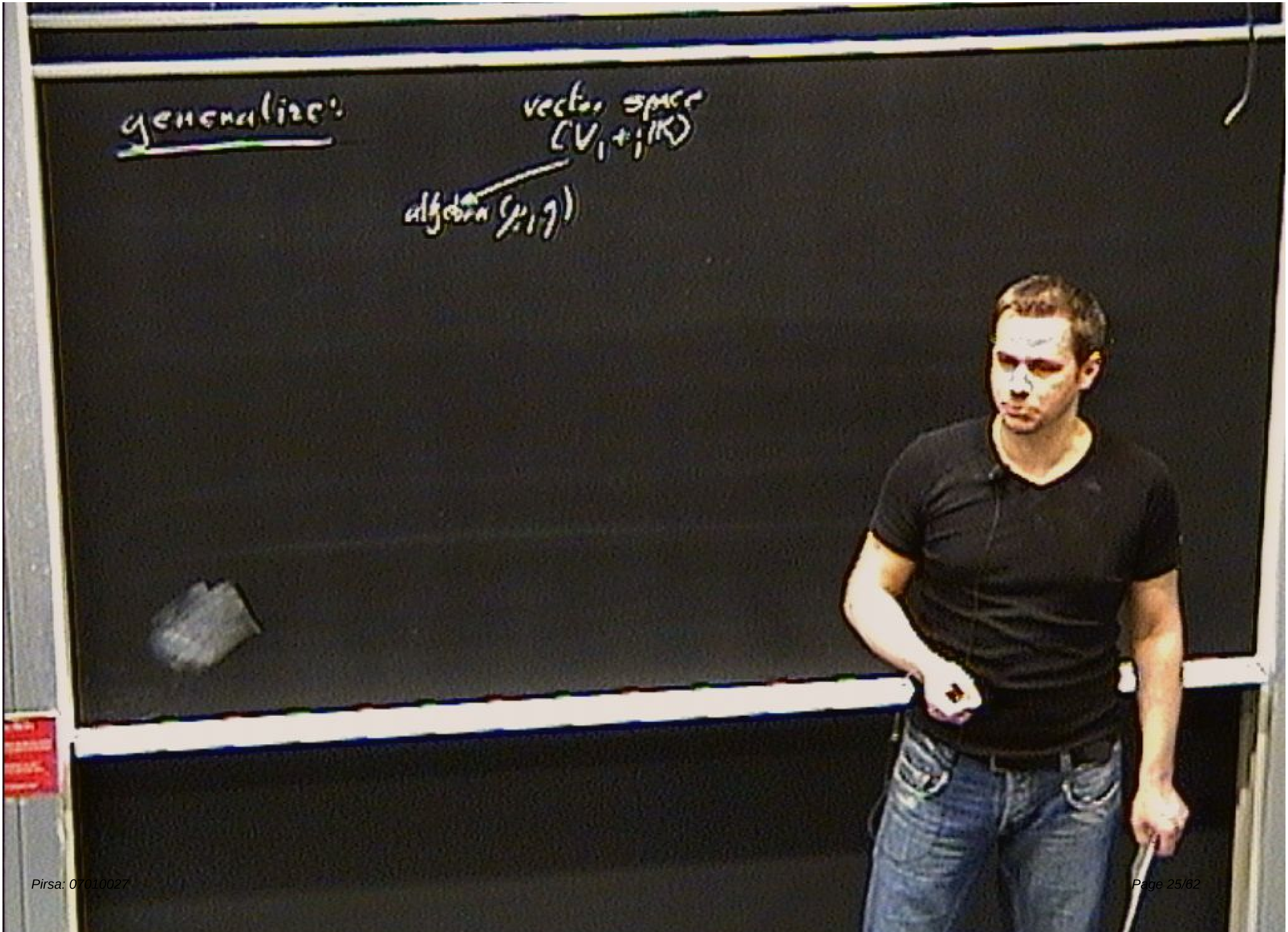
Let $(M, \{\ ;\ \}, K)$ be a Poisson manifold

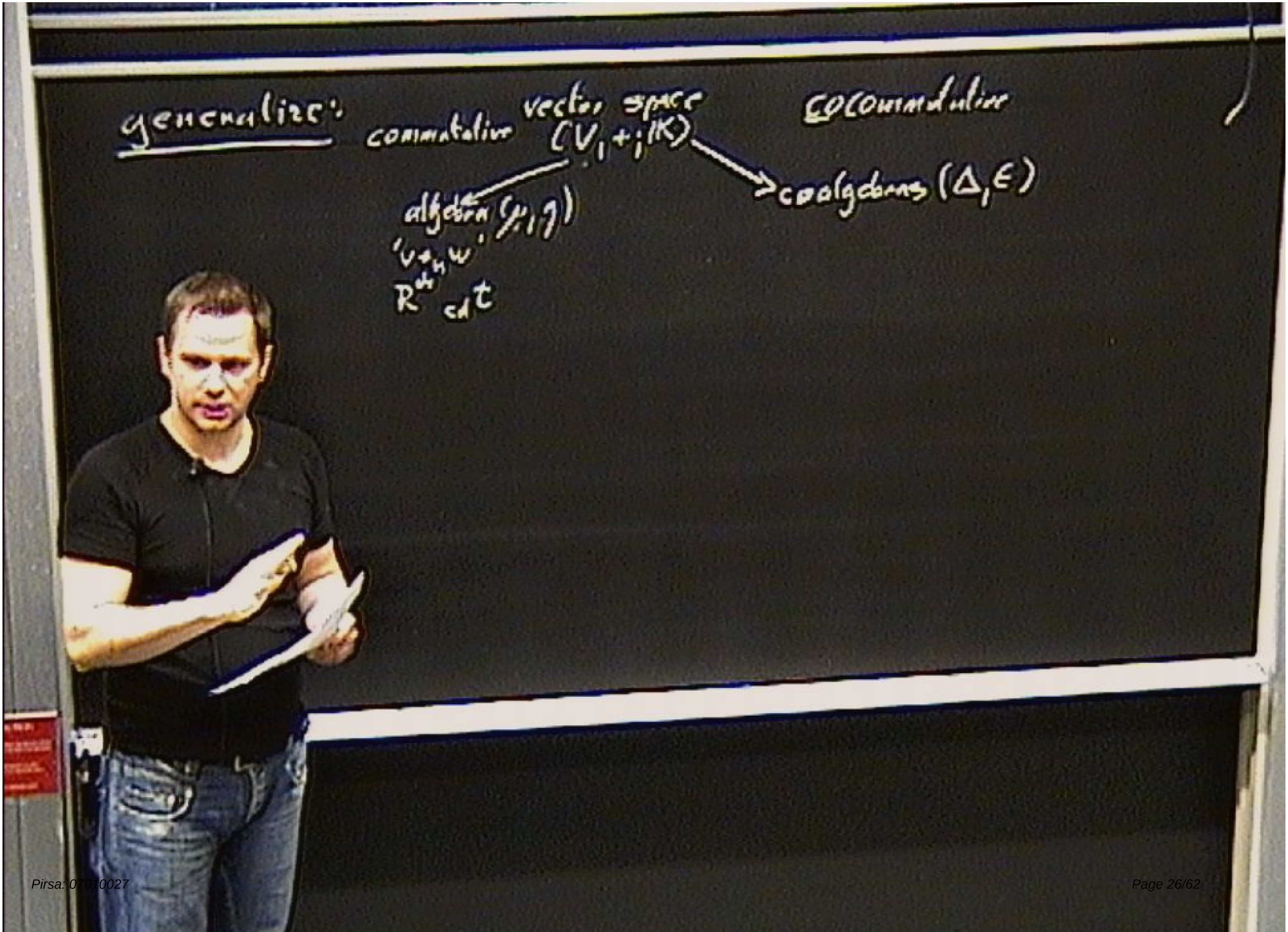
For $h \in K$ $M_h = (M, [\ ;\ ], K)$ is called a quantization of M

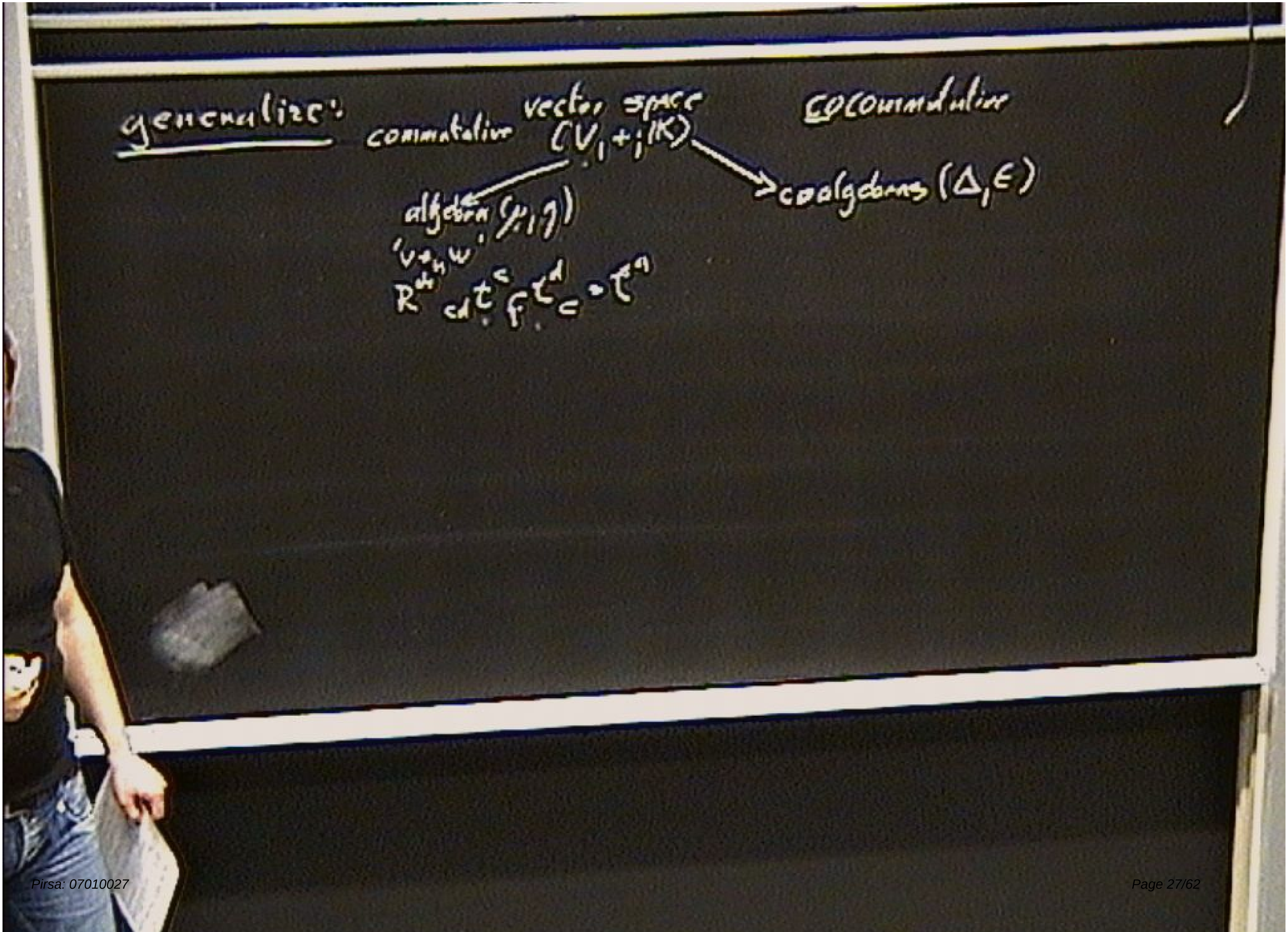
iff

$$\forall f_1, f_2 \in \mathcal{F}(M): \frac{[f_1 ; f_2]}{h} = \{f_1, f_2\} \ \mathrm{mod}(h)$$


generalize:
vector space
(V, +, ℝ)


generalize:
vector space
algebra


generalize:
commutative
vector space
(V, +, K)
cocommutative
algebras
coalgebras (Δ, ε)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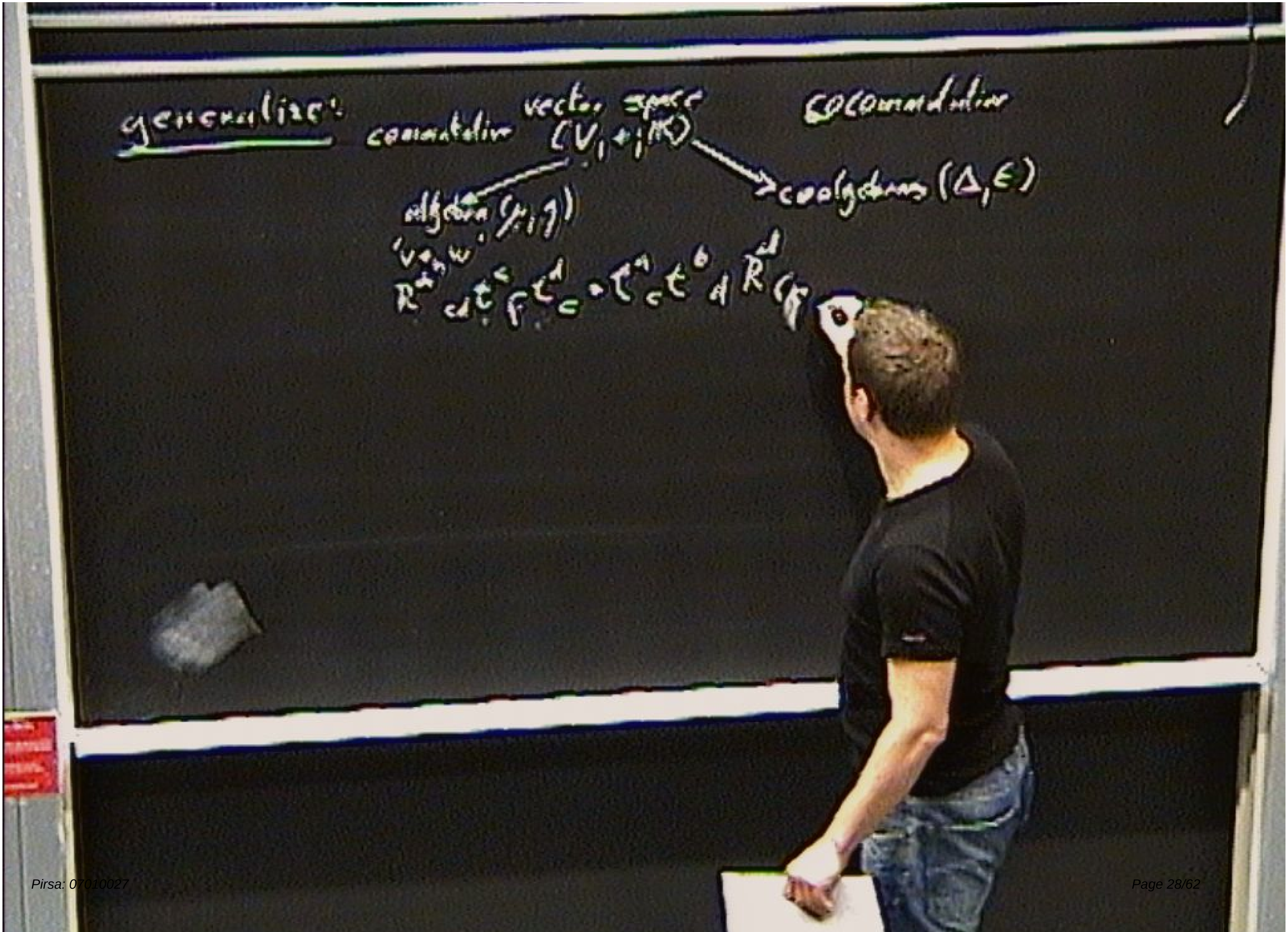
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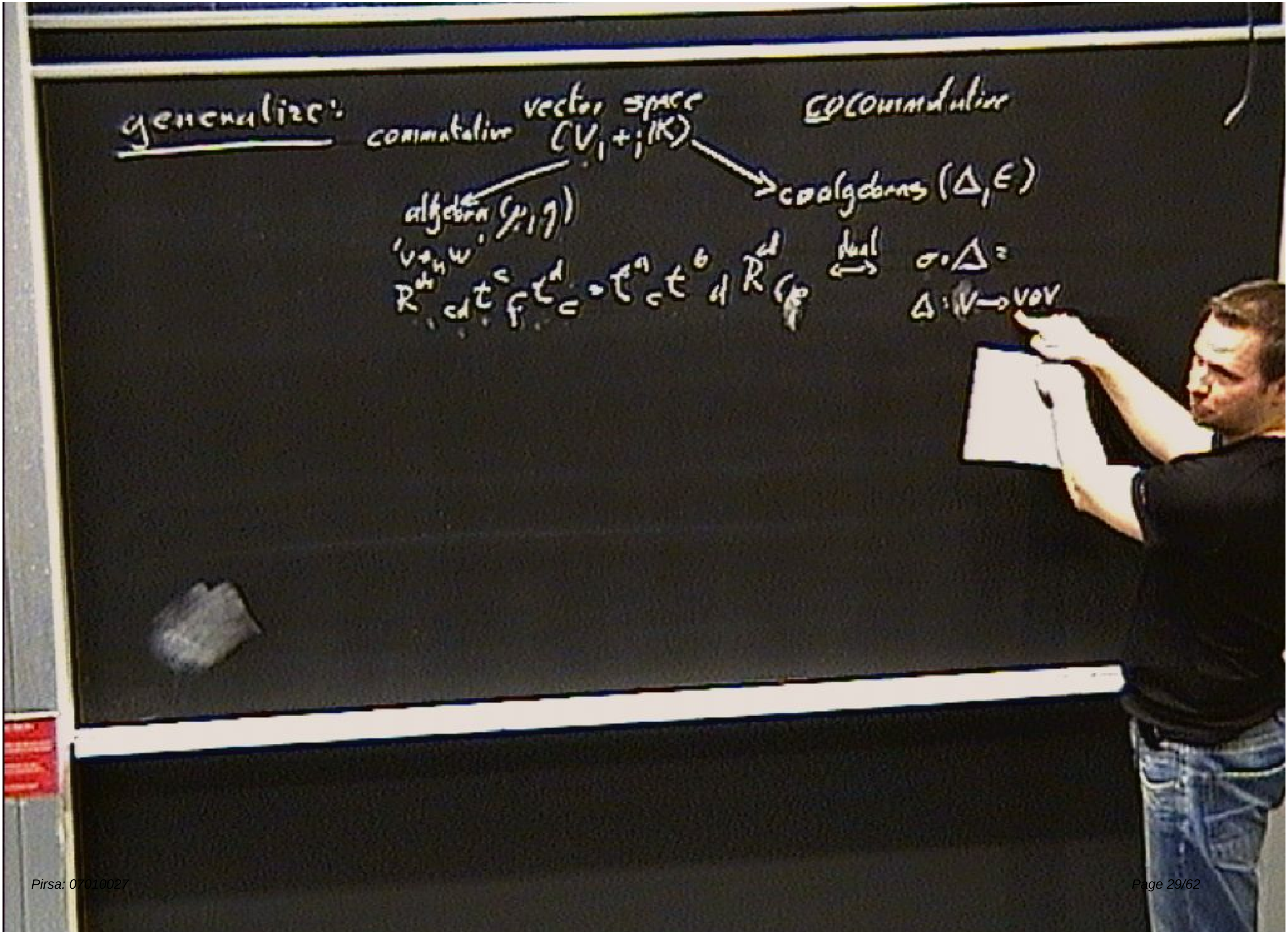
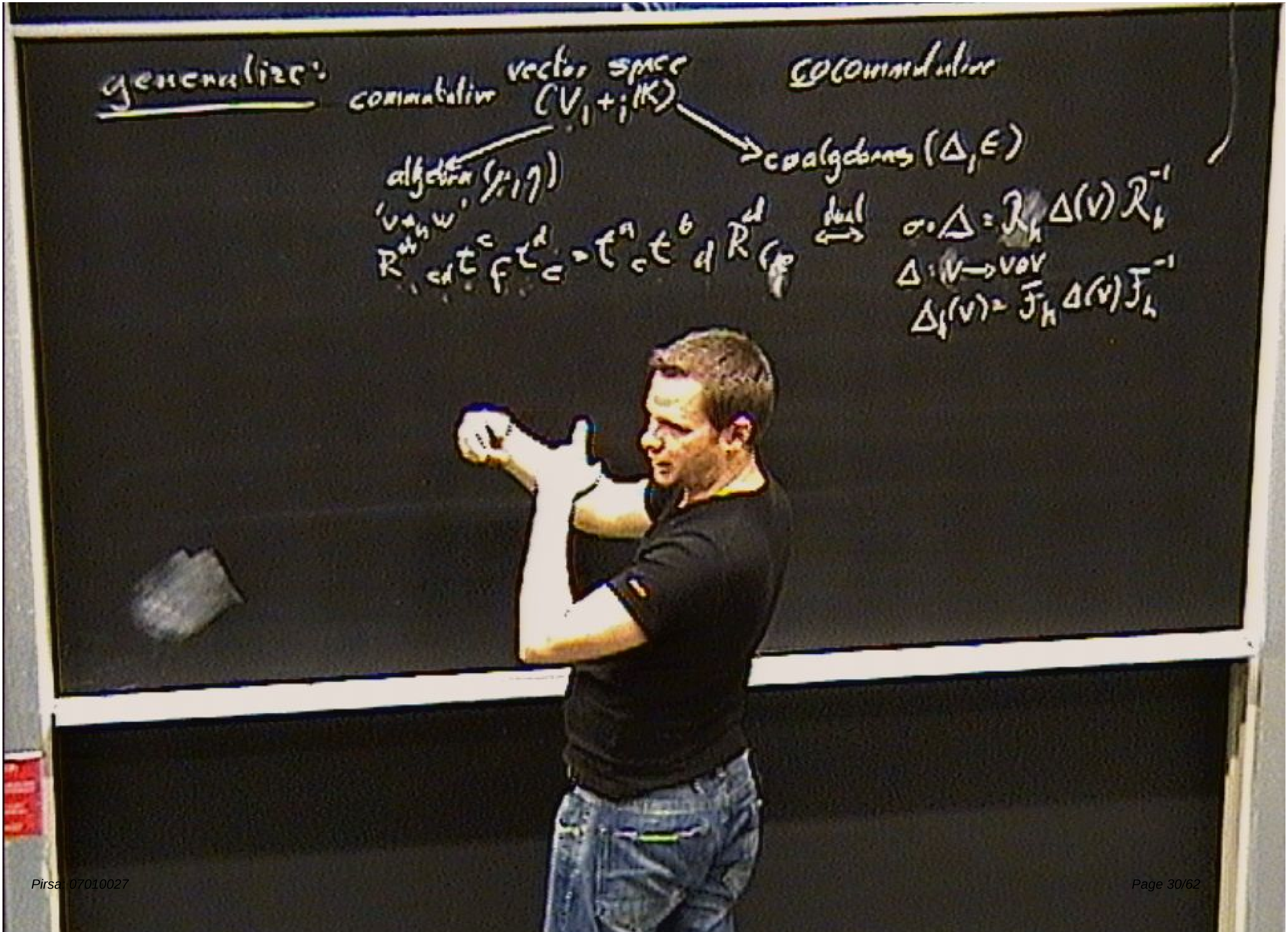

generalize:
vector space
commutative
cocommutative
algebra
coalgebra
dual


generalize:
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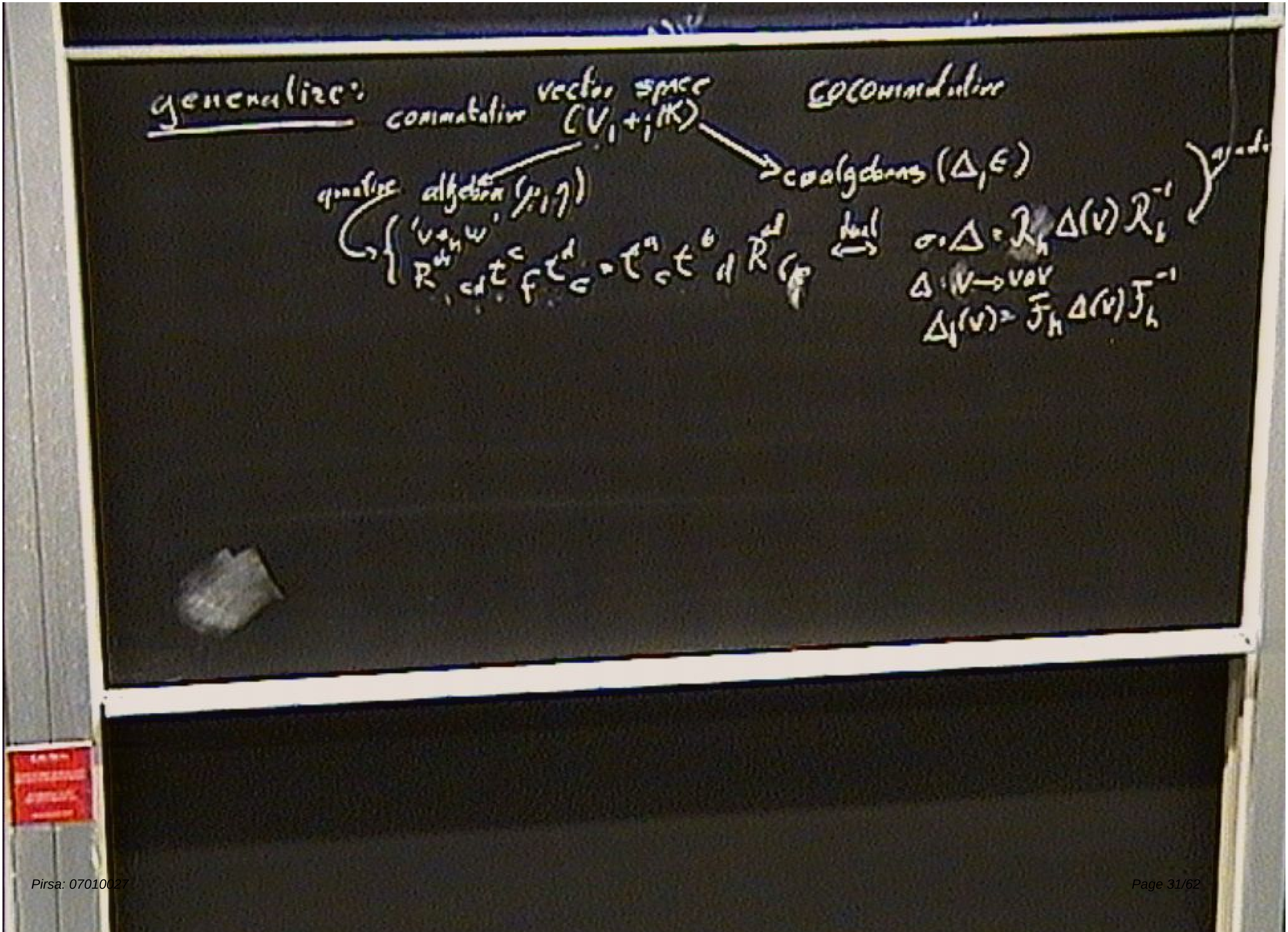
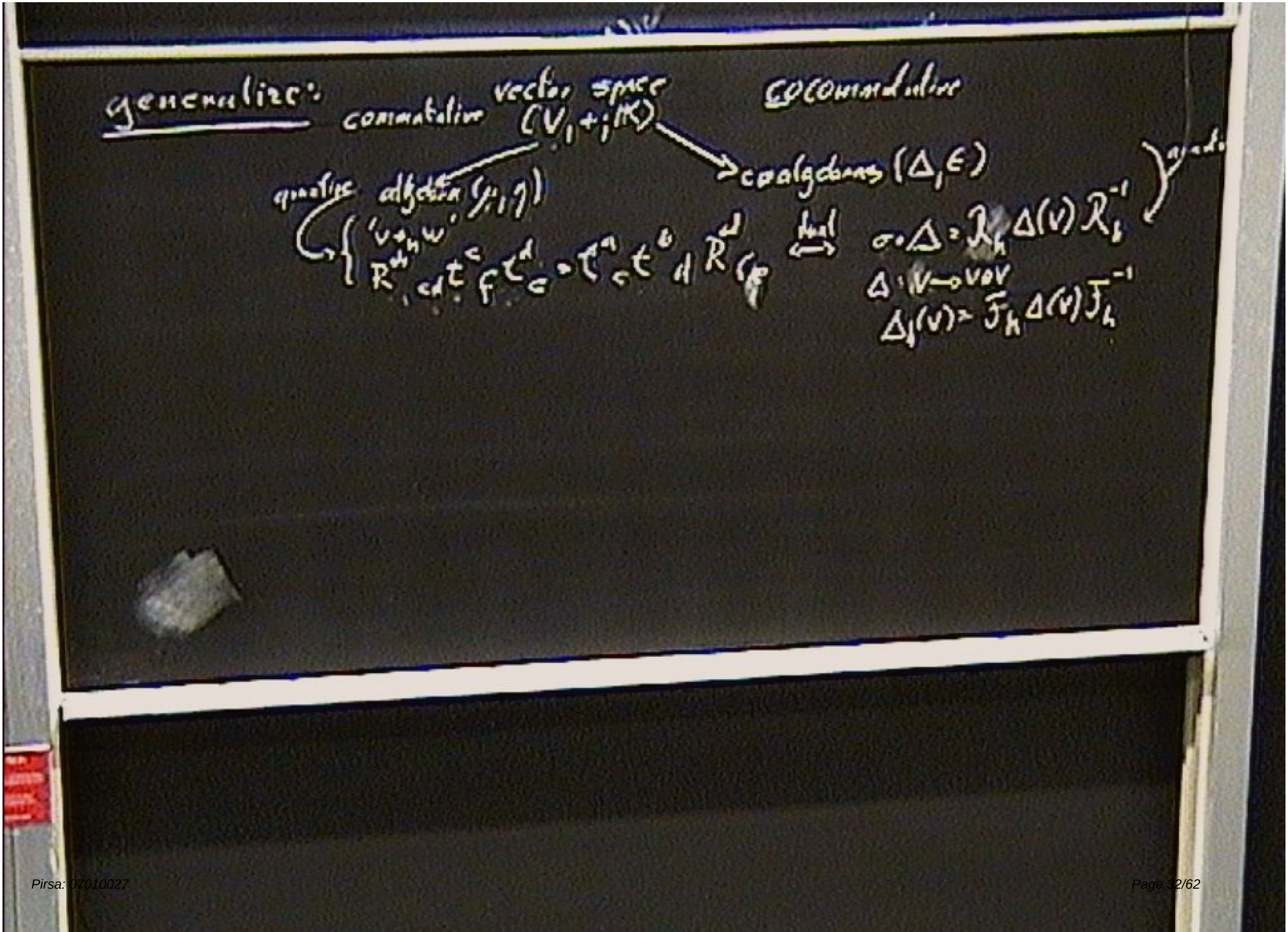
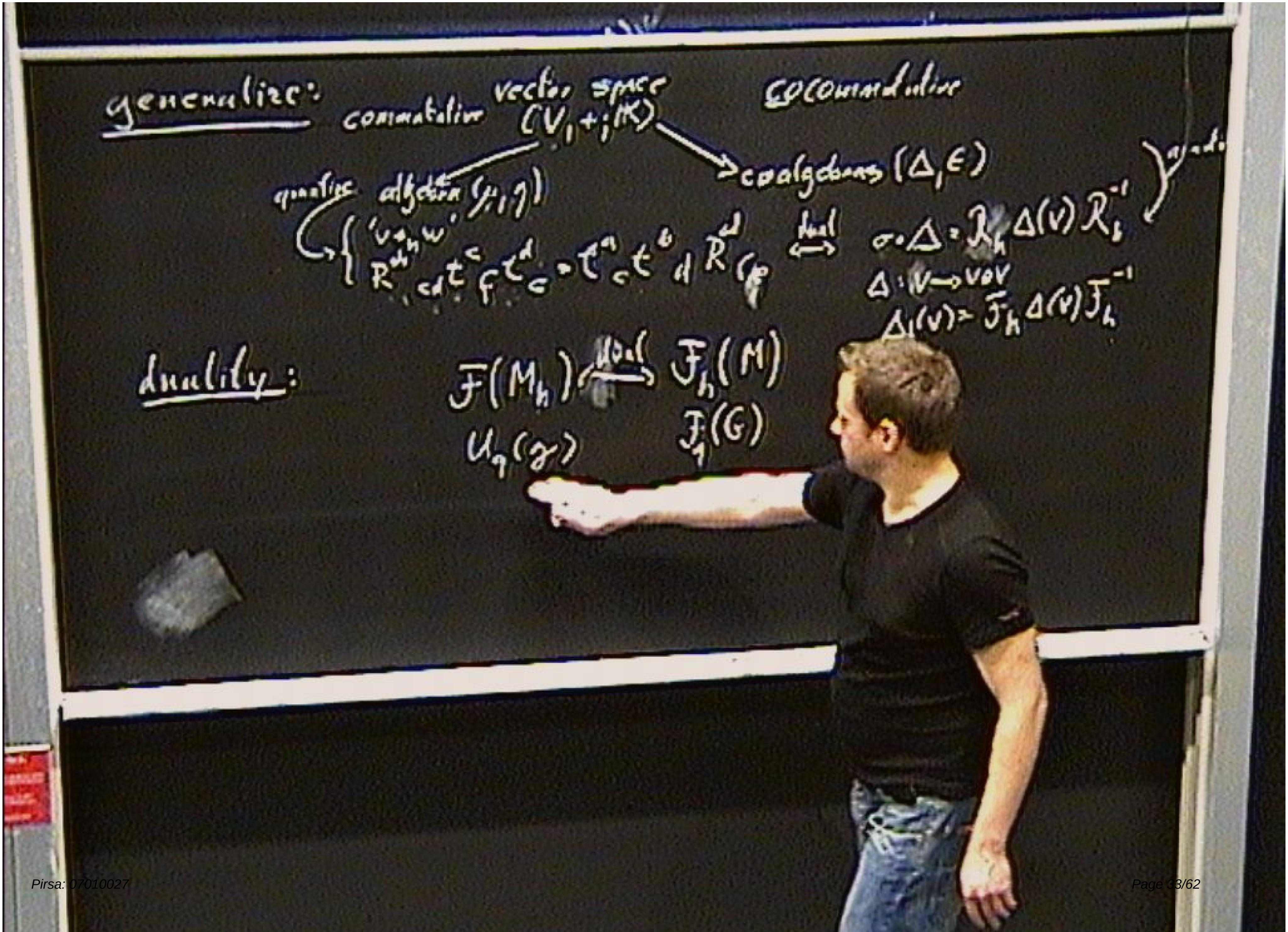

generalize:
commutative
vector space
(V, +, K)
cocommutative
coalgebras (Δ, ε)
quantize
algebra
dual


generalize:
commutative
vector space
$(V, +, \mathbb{K})$
cocommutative
coalgebras (Δ, ϵ)
algebra
$\Delta : V \to V \otimes V$
$\Delta_J(v) = \mathcal{F}_h \Delta(v) \mathcal{F}_h^{-1}$


generalize:
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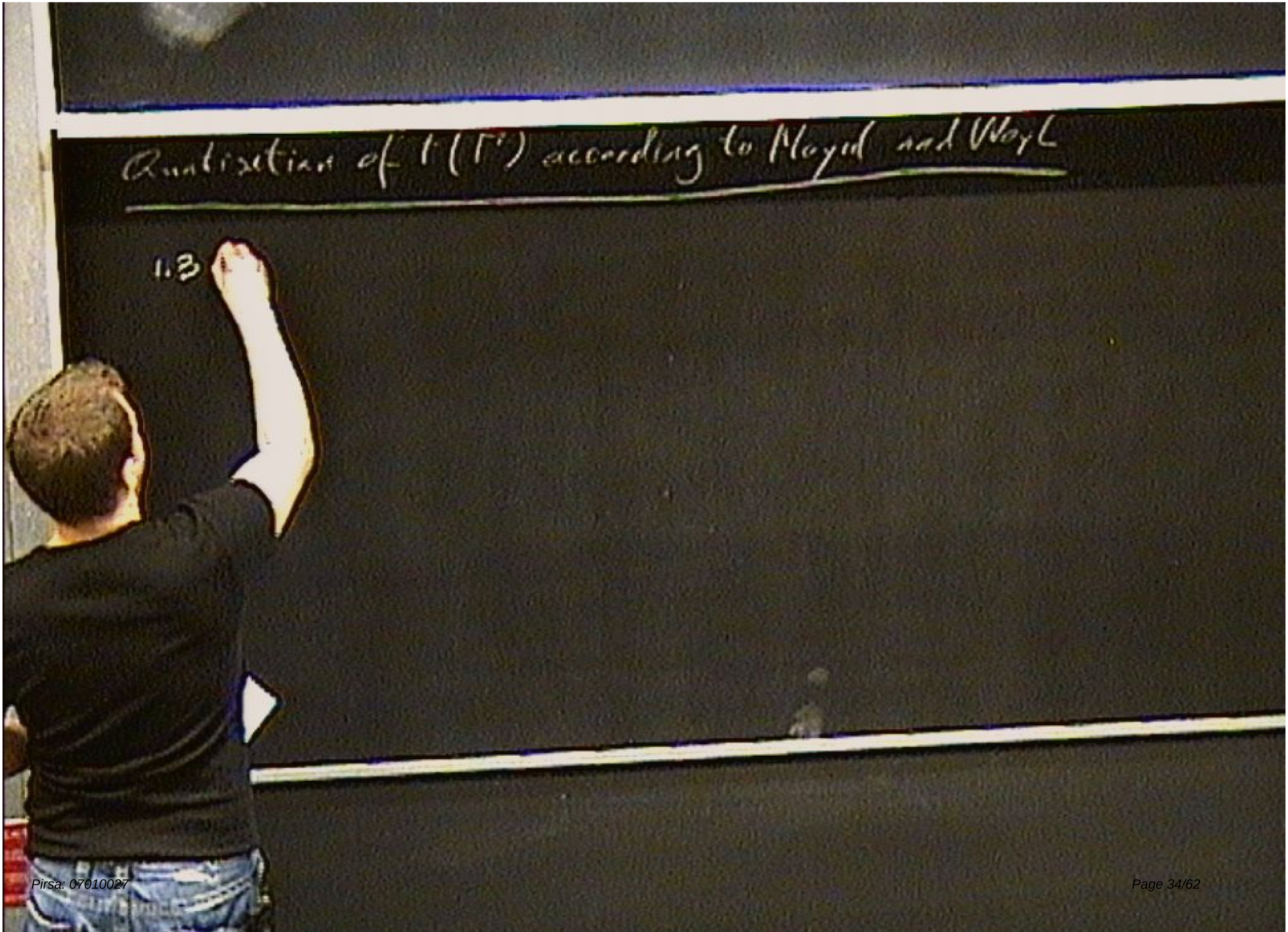
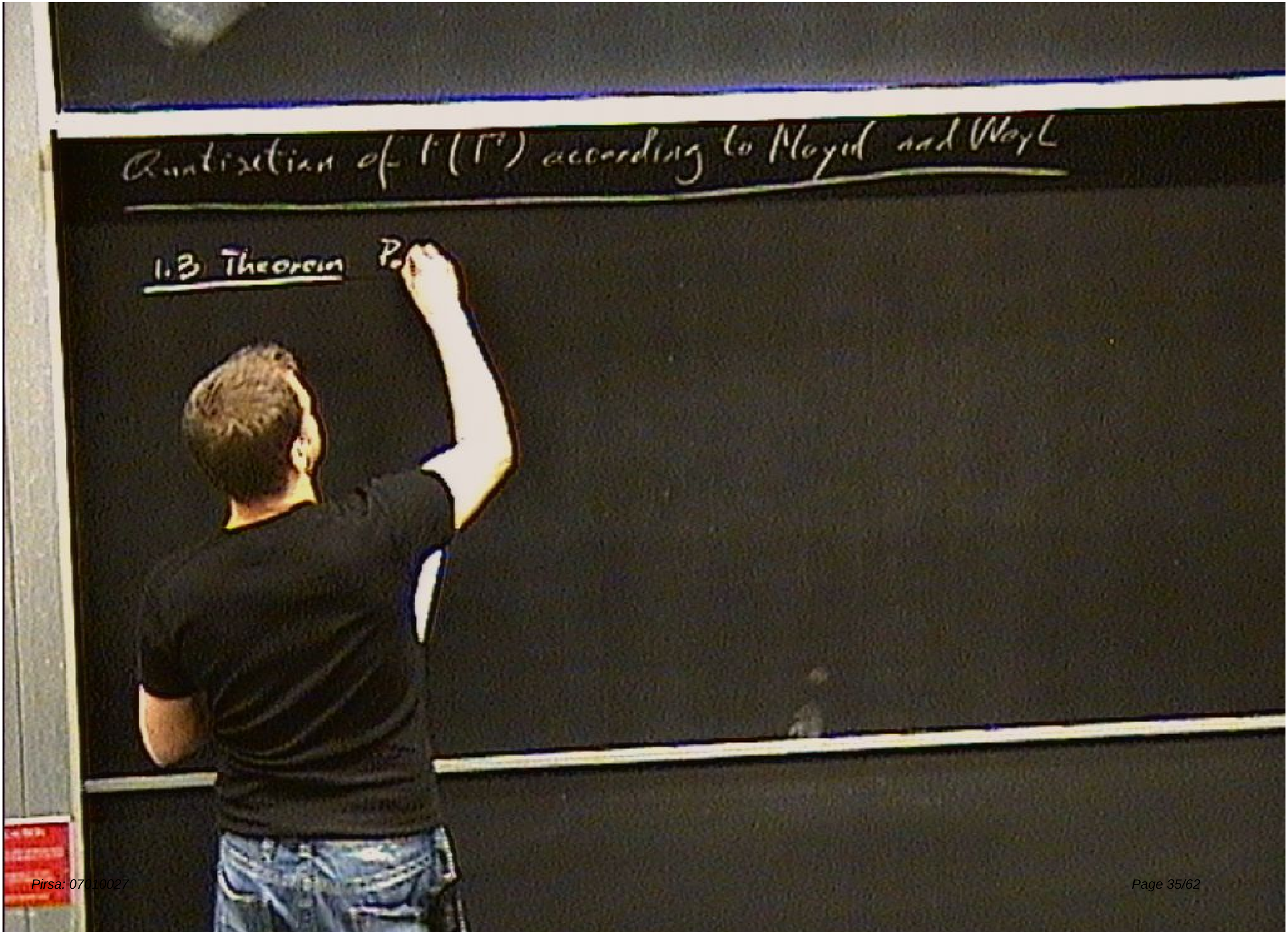
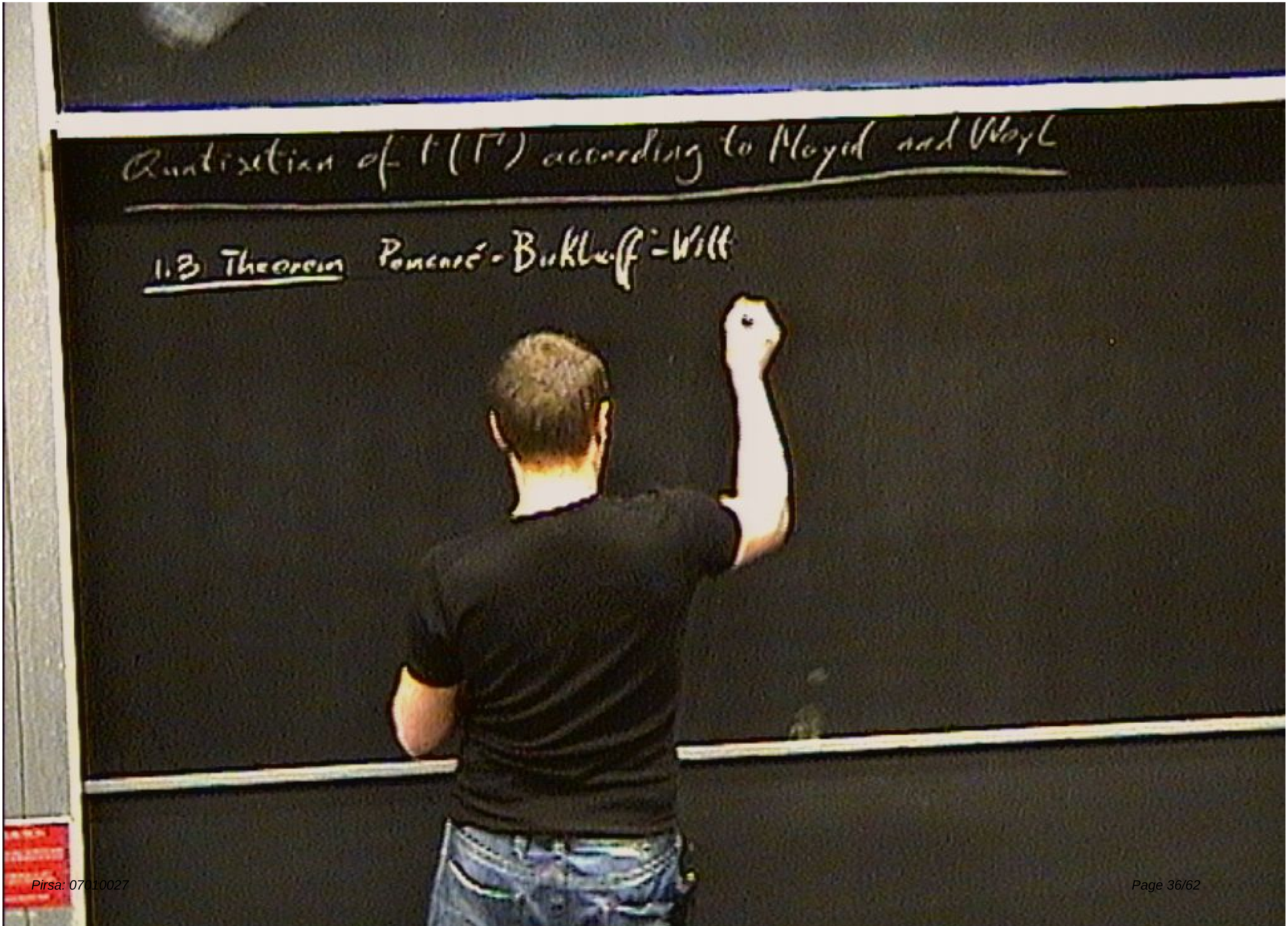
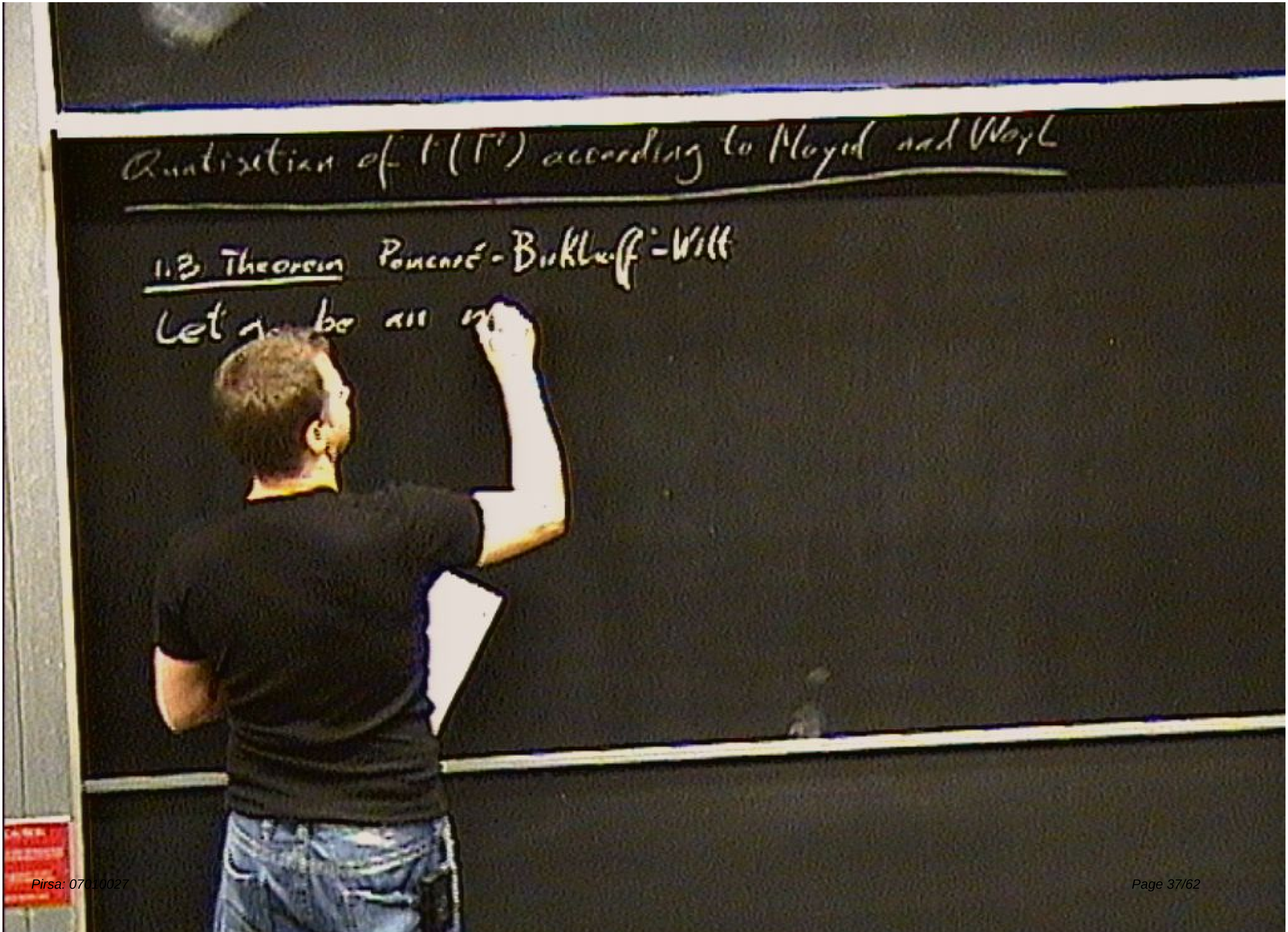
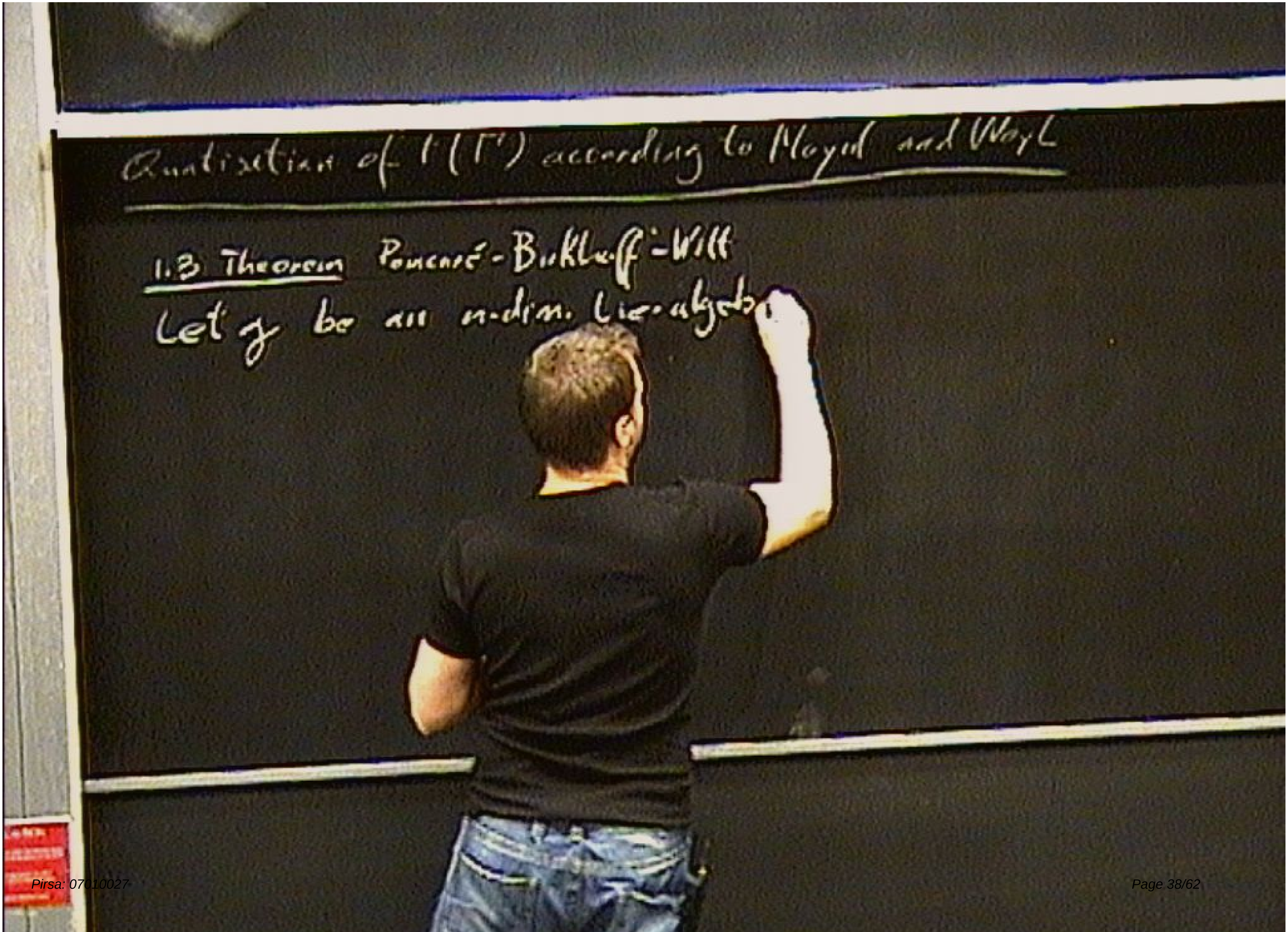
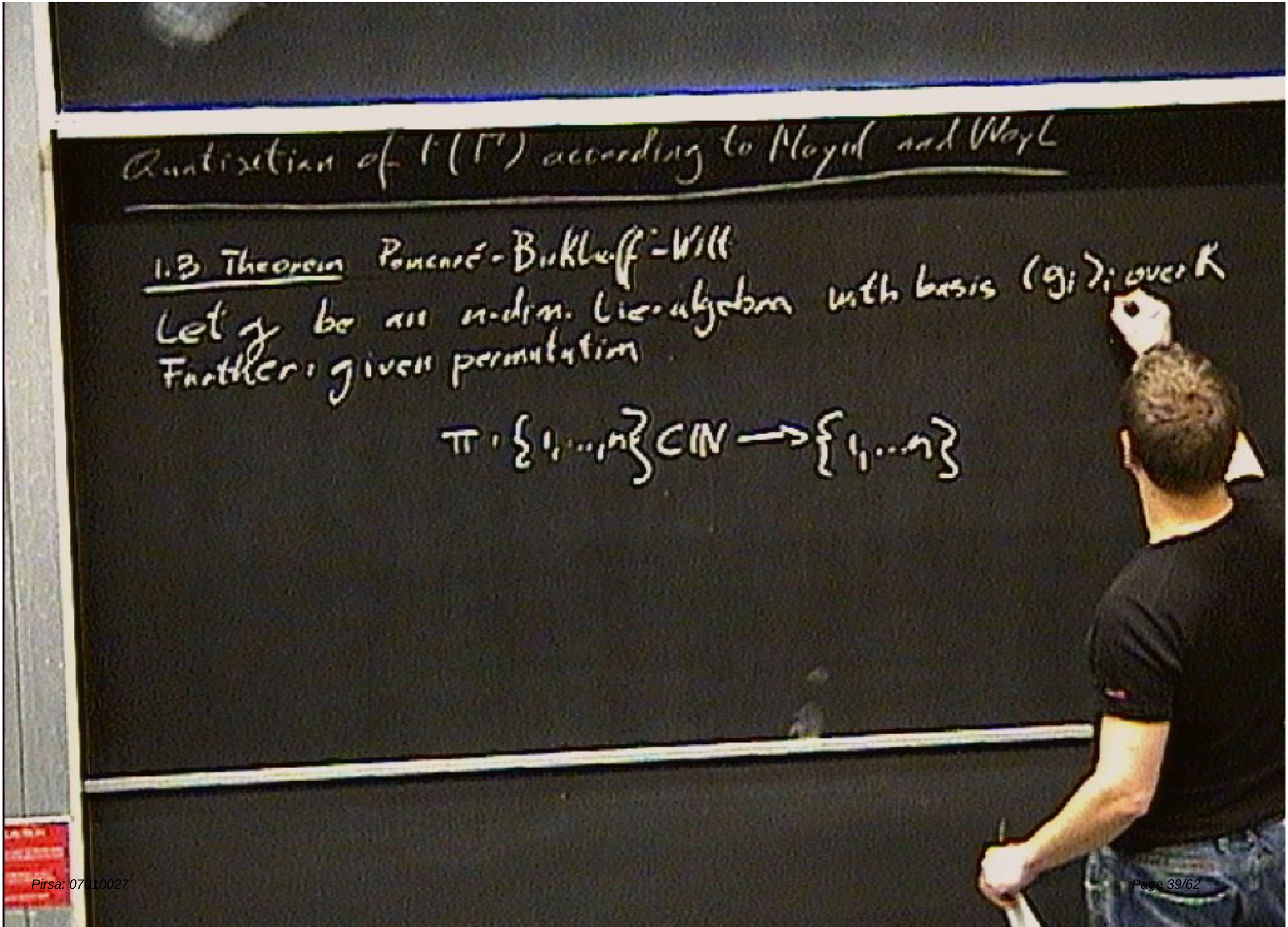

Quantisation of F(T²) according to Moyal and Weyl
1.3


Quantization of
according to Moyal and Weyl
1.3 Theorem


Quantization of F(Γ) according to Moyal and Weyl
1.3 Theorem Poincaré-Birkhoff-Witt


Quantisation of
according to Moyal and Weyl
Theorem
Poincaré-Birkhoff-Witt
Let
be an


Quantization of F(Γ*) according to Moyal and Weyl
1.3 Theorem Poincaré-Birkhoff-Witt
Let g be an n-dim. Lie-algebra


Quantization of F(Γ) according to Moyal and Weyl
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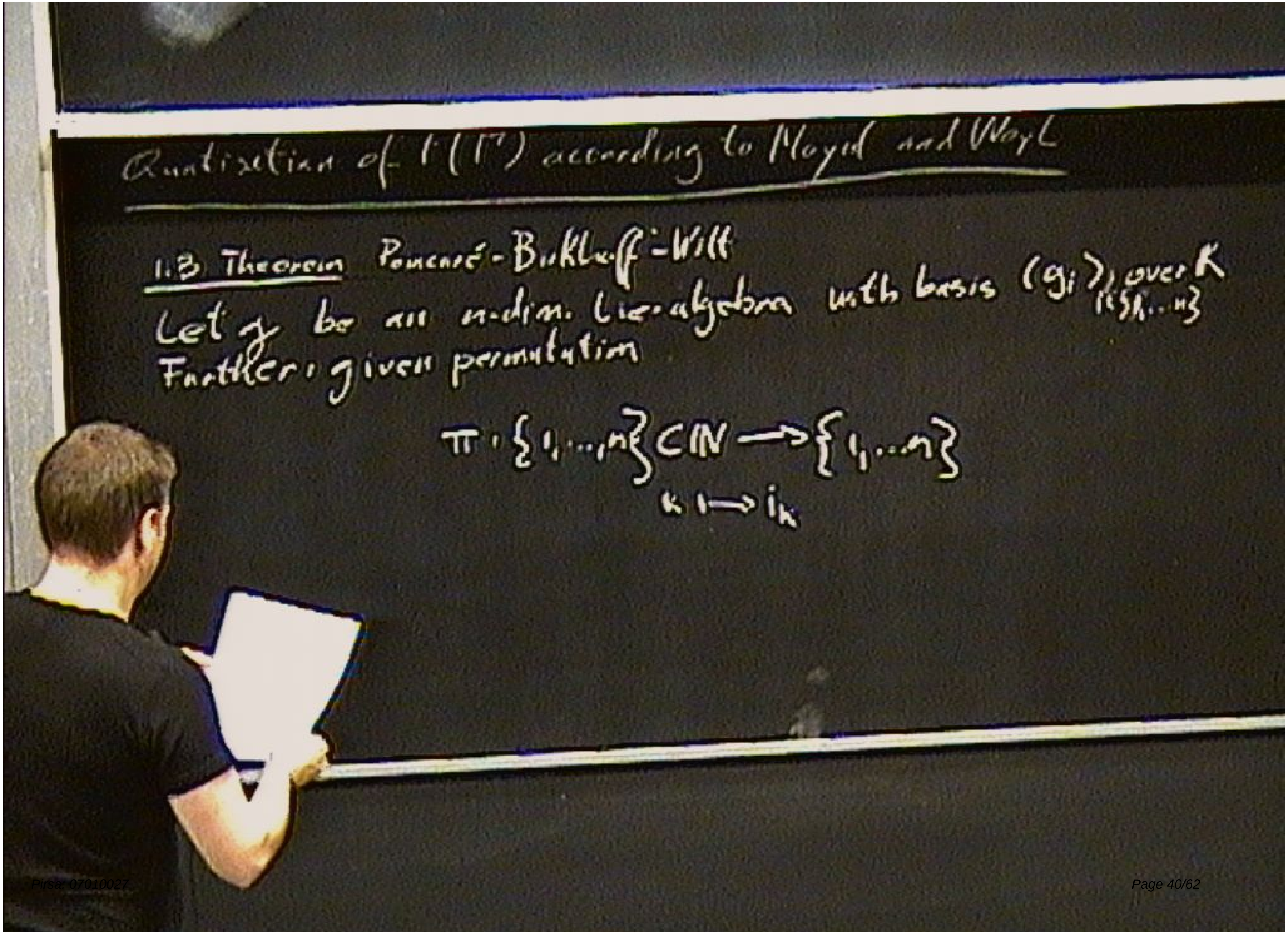
Let g be an n-dim. Lie-algebra with basis (gi)i over K
Further, given permutation
π: {1,...,n} ⊂ ℕ → {1,...n}


Quantization of F(Γ) according to Moyal and Weyl
1.13 Theorem Poincaré-Birkhoff-Witt
Let g be an n-dim. Lie algebra with basis (g_i)_{1≤i≤n} over K
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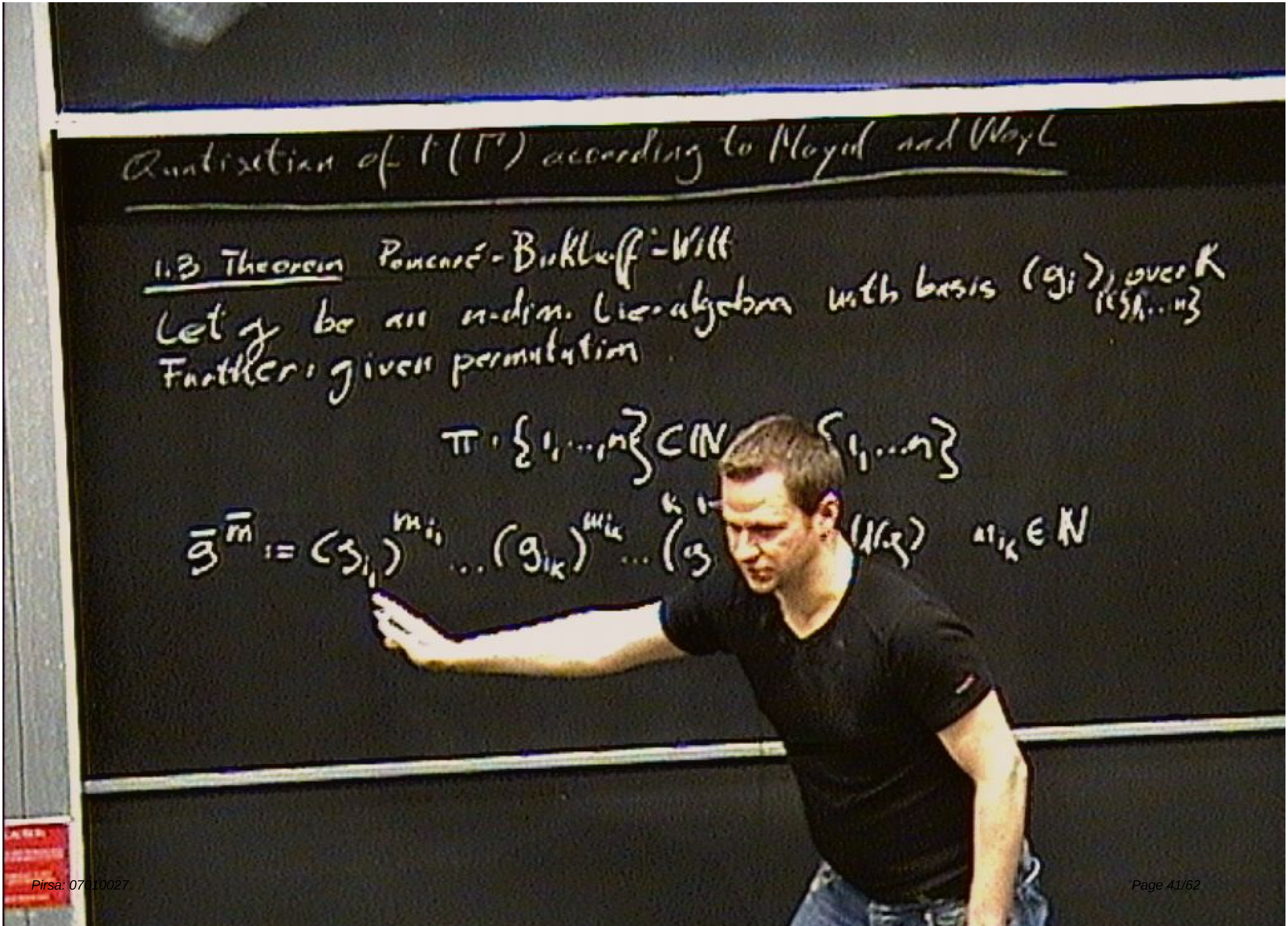
π : {1,...,n} ⊂ ℕ → {1,...,n}
k ↦ i_k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Let g be an n-dim. Lie-algebra with basis (g_i)_{i∈{1,...,n}} over K
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π : {1,...,n} ⊂ ℕ → {1,...,n}
ḡ^m̄ := (g_{i_1})^{m_{i_1}} ... (g_{i_k})^{m_{i_k}} ... (g ... U(g) m_{i_k} ∈ ℕ

Quantisation of $F(\Gamma)$ according to Moyal and Weyl

1.3 Theorem Poincaré-Birkhoff-Witt

Let $\mathfrak{g}$ be an n-dim. Lie-algebra with basis $(g_i)_{i\in\{1,\dots,n\}}$ over K

Further, given permutation

$$\pi : \{1,\dots,n\}\subset\mathbb{N}\longrightarrow\{1,\dots,n\}$$

$$k\longmapsto i_k$$

$$\bar{g}^{\bar{m}} := (g_{i_1})^{m_{i_1}}\dots(g_{i_k})^{m_{i_k}}\dots(g_{i_n})^{m_{i_n}}\in U(\mathfrak{g})\qquad m_{i_k}\in\mathbb{N}$$

this constitutes a basis

Quantization of $F(\Gamma)$ according to Moyal and Weyl

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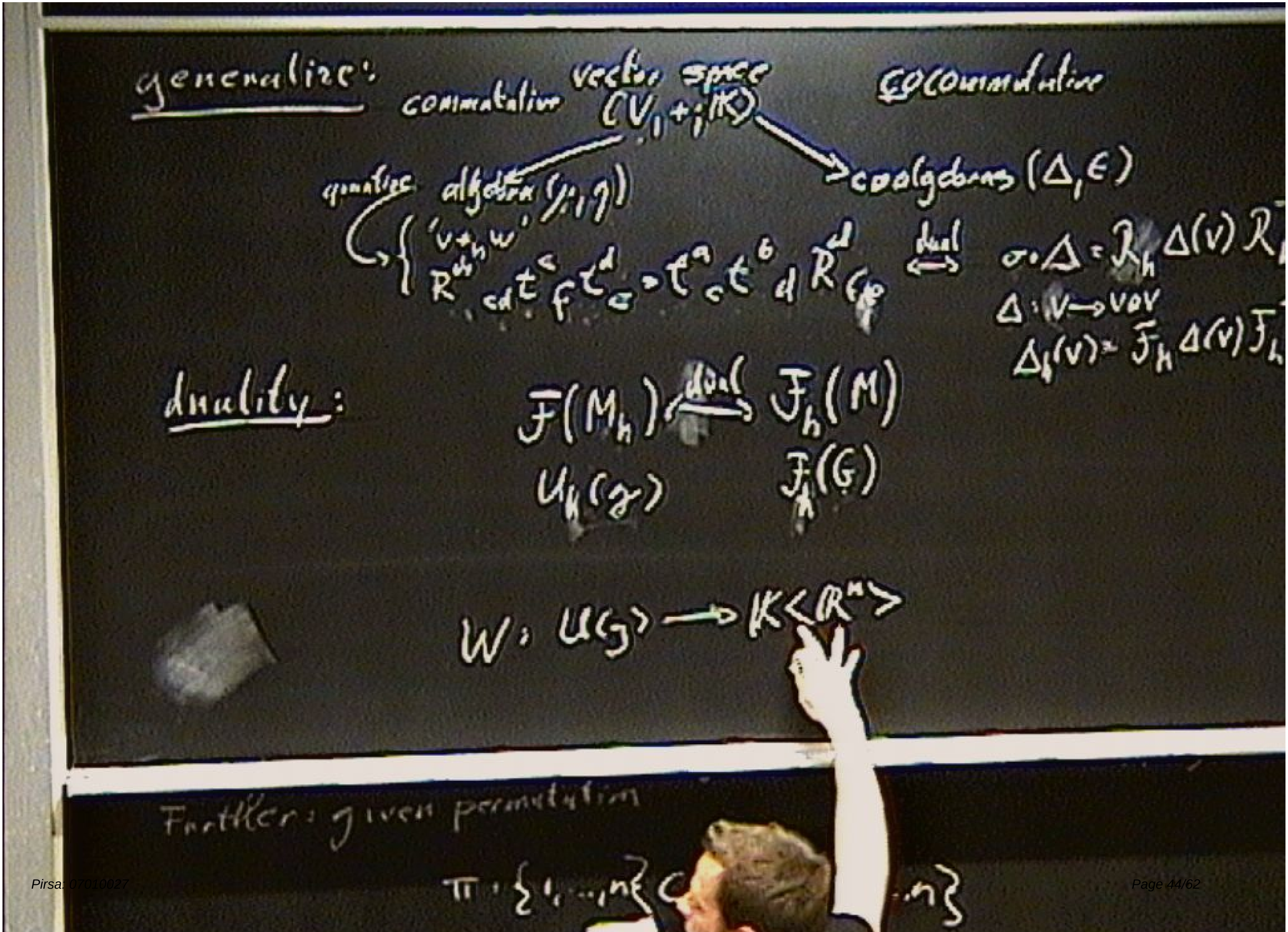
this constitutes a basis of $U(\mathfrak{g})$ and there exist isomorphism of vector-spaces W


generalize:
vector space
commutative
(V, +, K)
cocommutative
coalgebras (Δ, ε)
quantize
'v + w'
dual
σ∘Δ = R_h Δ(v) R
Δ : V → V⊗V
Δ_f(v) = F_h Δ(v) F_h
duality:
F(M_h) ← dual → F_h(M)
U_h(g)
F_h(G)
W : U(g) → K<R^n>
Further: given permutation
π : {1, ..., n}
n}

duality: $\mathcal{F}(M_h) \overset{\text{dual}}{\longleftrightarrow} \mathcal{F}_h(M)$

$U_h(\mathfrak{g}) \qquad \mathcal{F}_h(G)$

$$W : U(\mathfrak{g}) \longrightarrow K\langle \mathbb{R}^n \rangle$$
$$\bar{g}^{\bar{m}} \longmapsto \bar{x}^{\bar{m}}$$

1.3 Theorem Poincaré-Birkhoff-Witt

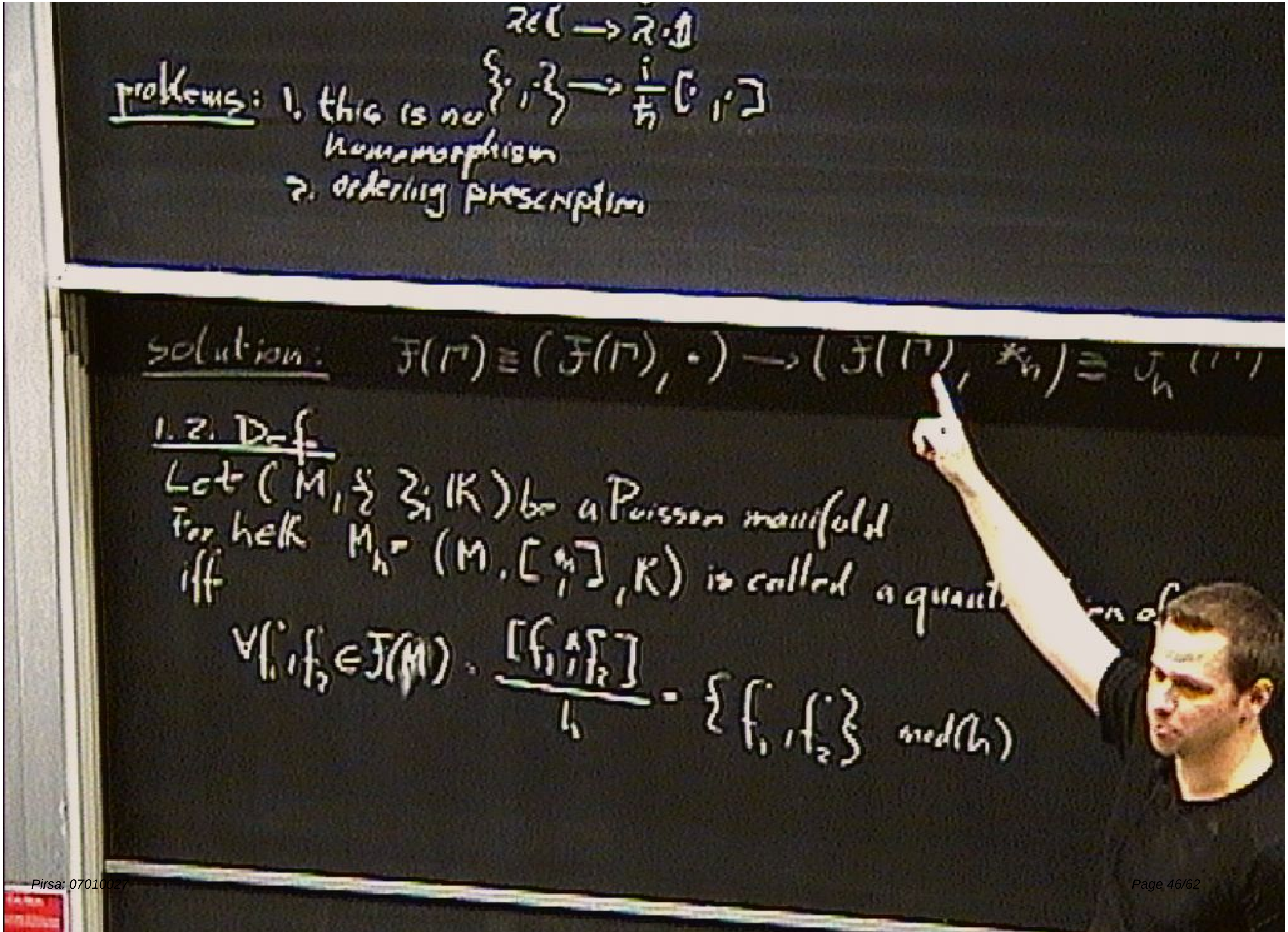
Let $\mathfrak{g}$ be an n-dim. Lie-algebra with basis $(g_i)_{i\in\{1,\dots,n\}}$ over K

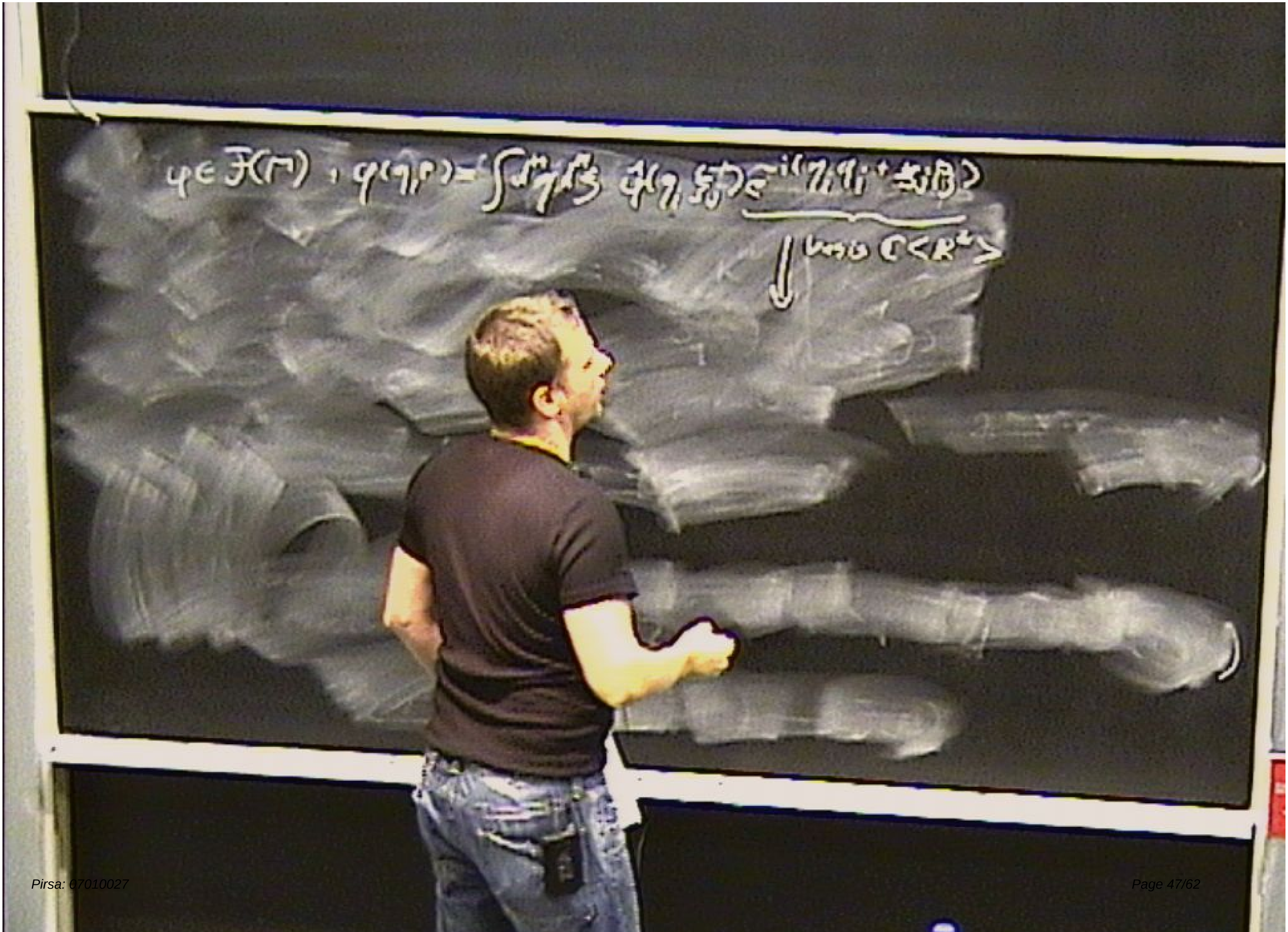
Further, given permutation

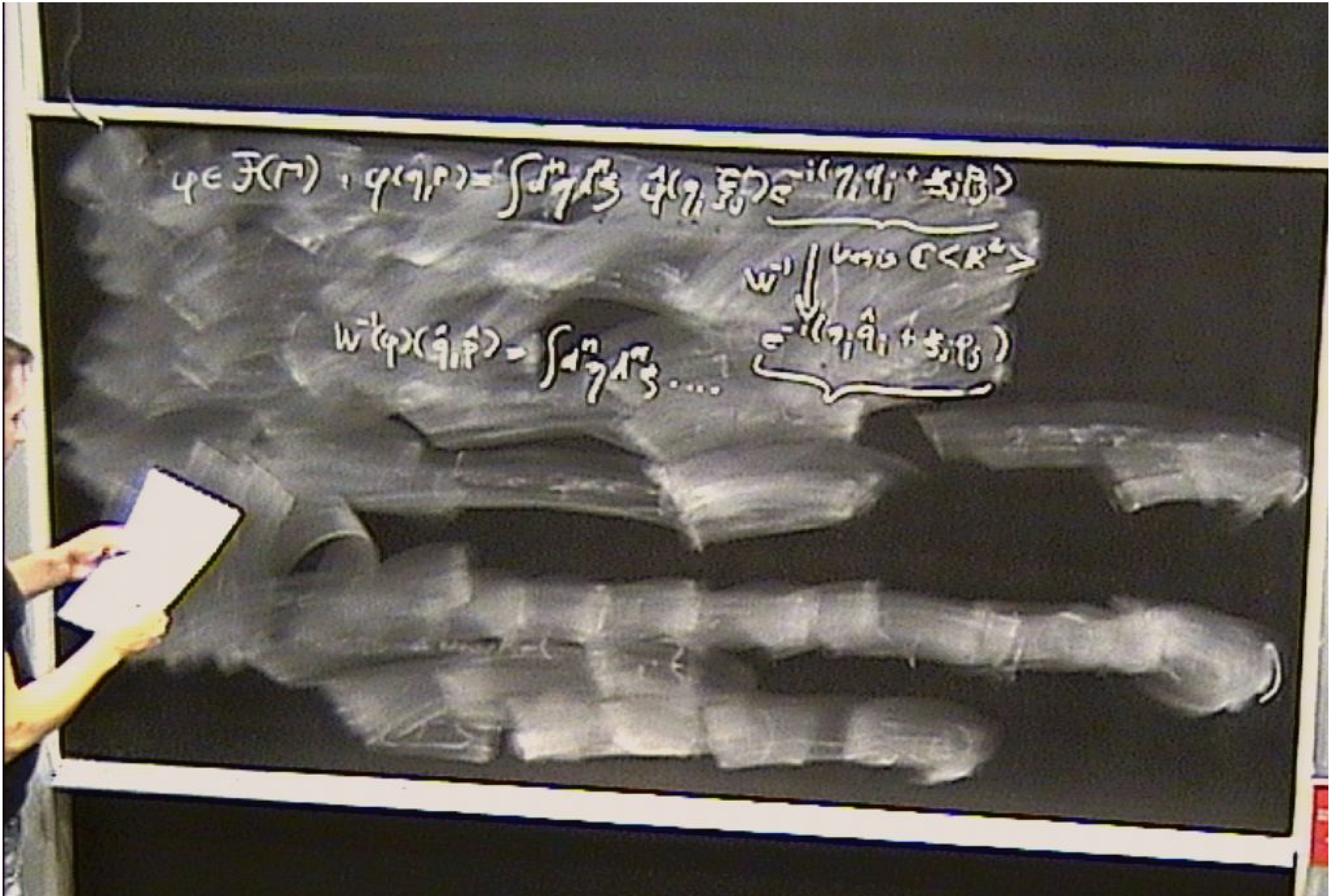
$$\pi : \{1,\dots,n\} \subset \mathbb{N} \longrightarrow \{1,\dots,n\}$$
$$k \longmapsto i_k$$

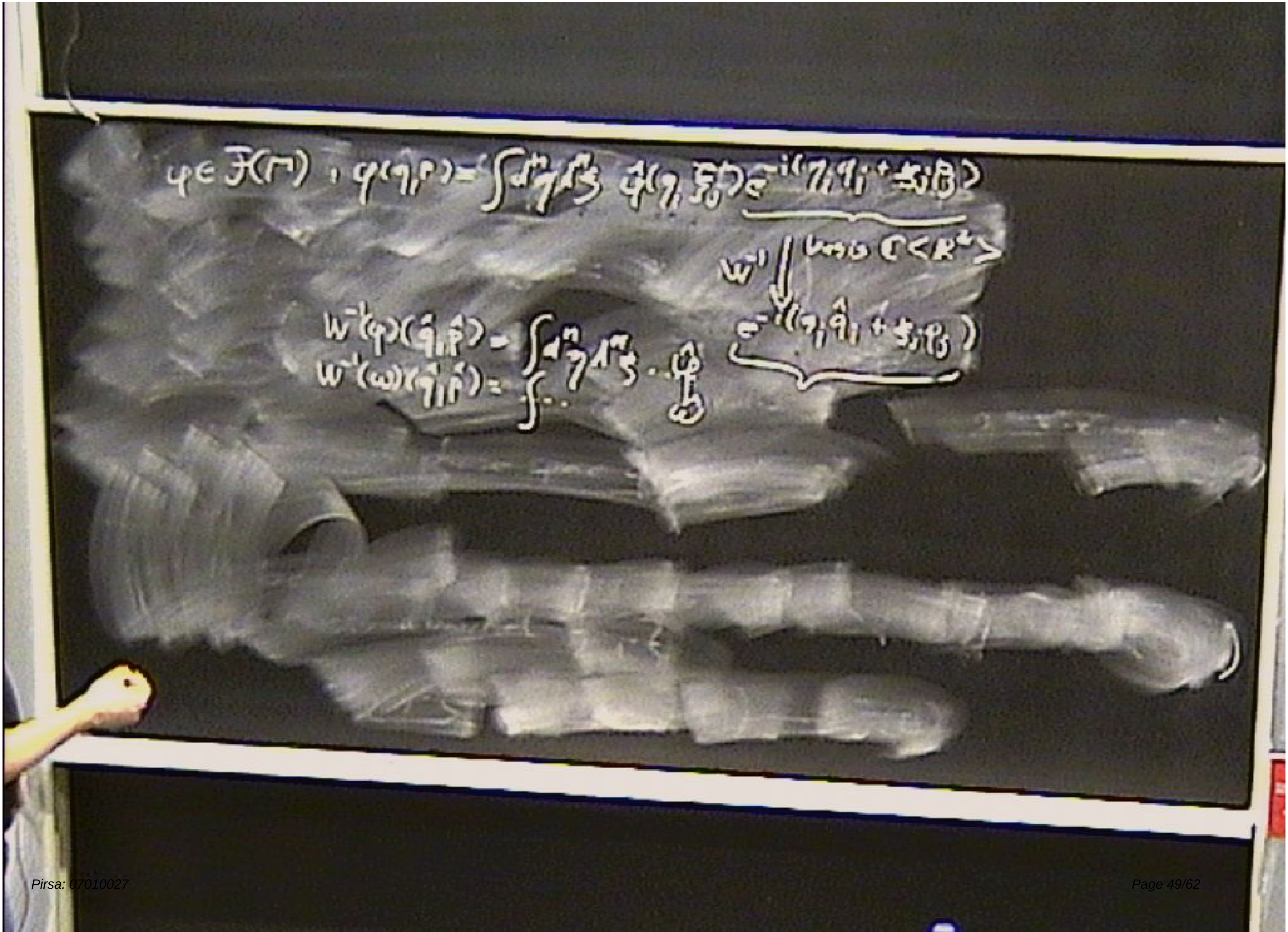
$$\bar{g}^{\bar{m}} := (g_{i_1})^{m_{i_1}} \dots (g_{i_k})^{m_{i_k}} \dots (g_{i_n})^{m_{i_n}} \in U(\mathfrak{g}) \qquad m_{i_k} \in \mathbb{N}_0$$

this constitutes a basis of $U(\mathfrak{g})$ and there exist isomorphism of vector spaces W

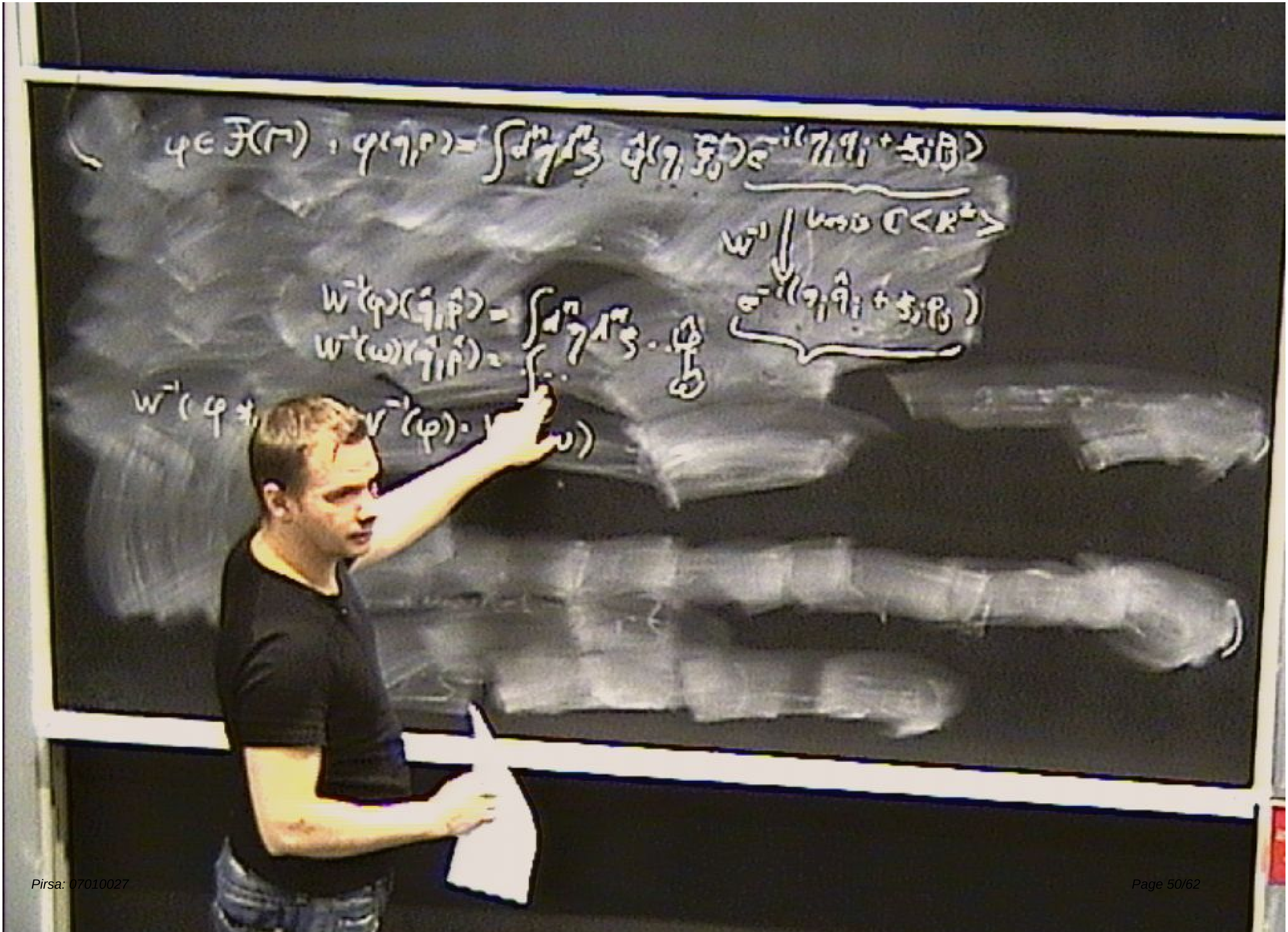

ℝ·1 → ℝ·1
problems: 1. this is no {,} → i/ħ [,]
homomorphism
2. ordering prescription
solution: F(Γ) ≡ (F(Γ), ·) → (F(Γ), *ħ) ≡ Fħ(Γ)
1.2. Def.
Let (M, { , }, K) be a Poisson manifold
For ħ∈K Mħ = (M, [*,], K) is called a quantization of
iff
∀ f1, f2 ∈ F(M): [f1 *, f2] / ħ = {f1, f2} mod(ħ)

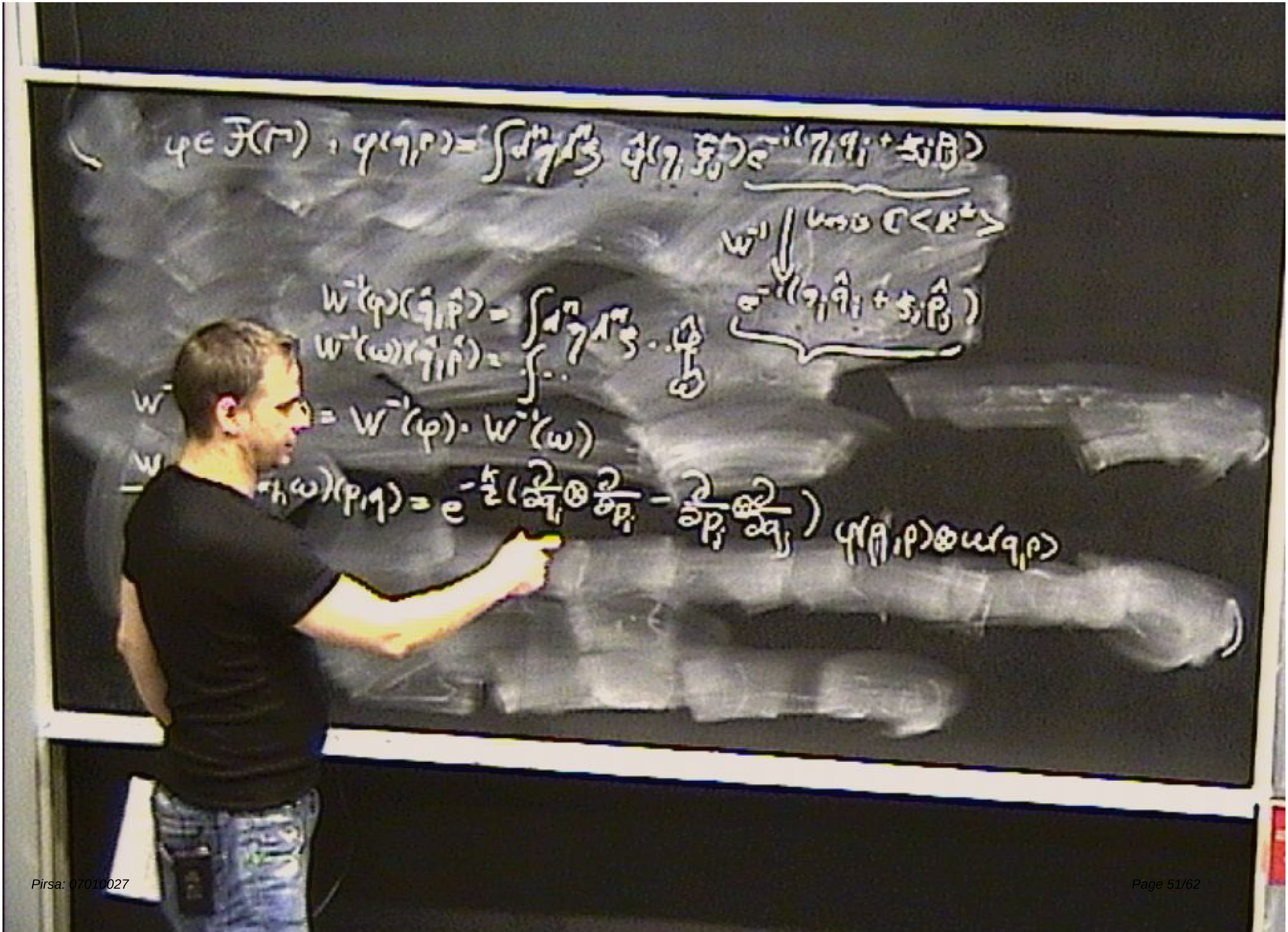


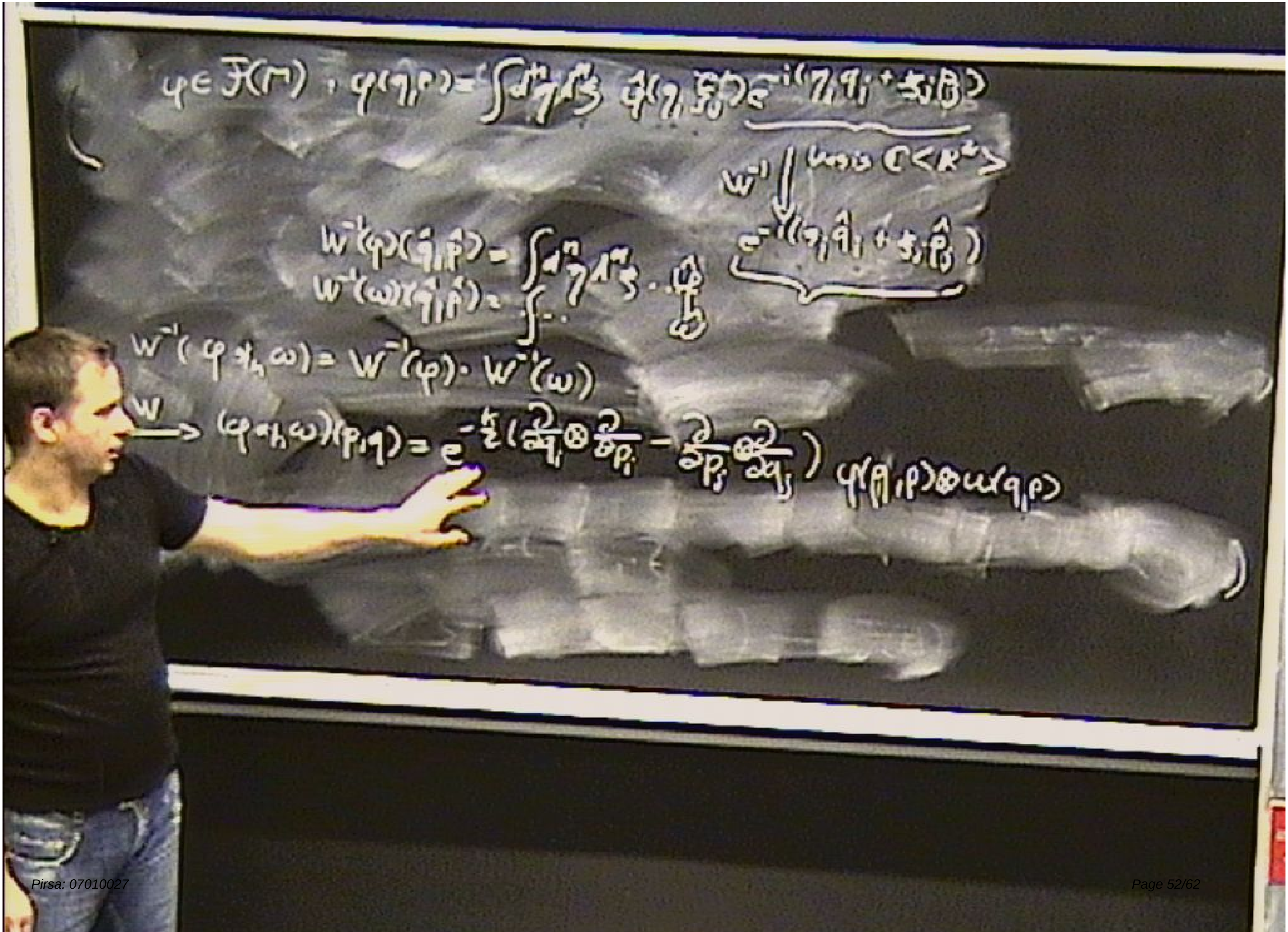

$\varphi \in \mathcal{F}(\Gamma)$, $\varphi(q,p) = \int d^n\eta\, d^n\xi\, \tilde{\varphi}(\eta,\xi)\, e^{i(\eta_i q_i + \xi_i p_i)}$
W^{-1}
lives in $C\langle R^2\rangle$
$W^{-1}(\varphi)(\hat{q},\hat{p}) = \int d^n\eta\, d^n\xi \ldots\, e^{i(\eta_i \hat{q}_i + \xi_i \hat{p}_i)}$

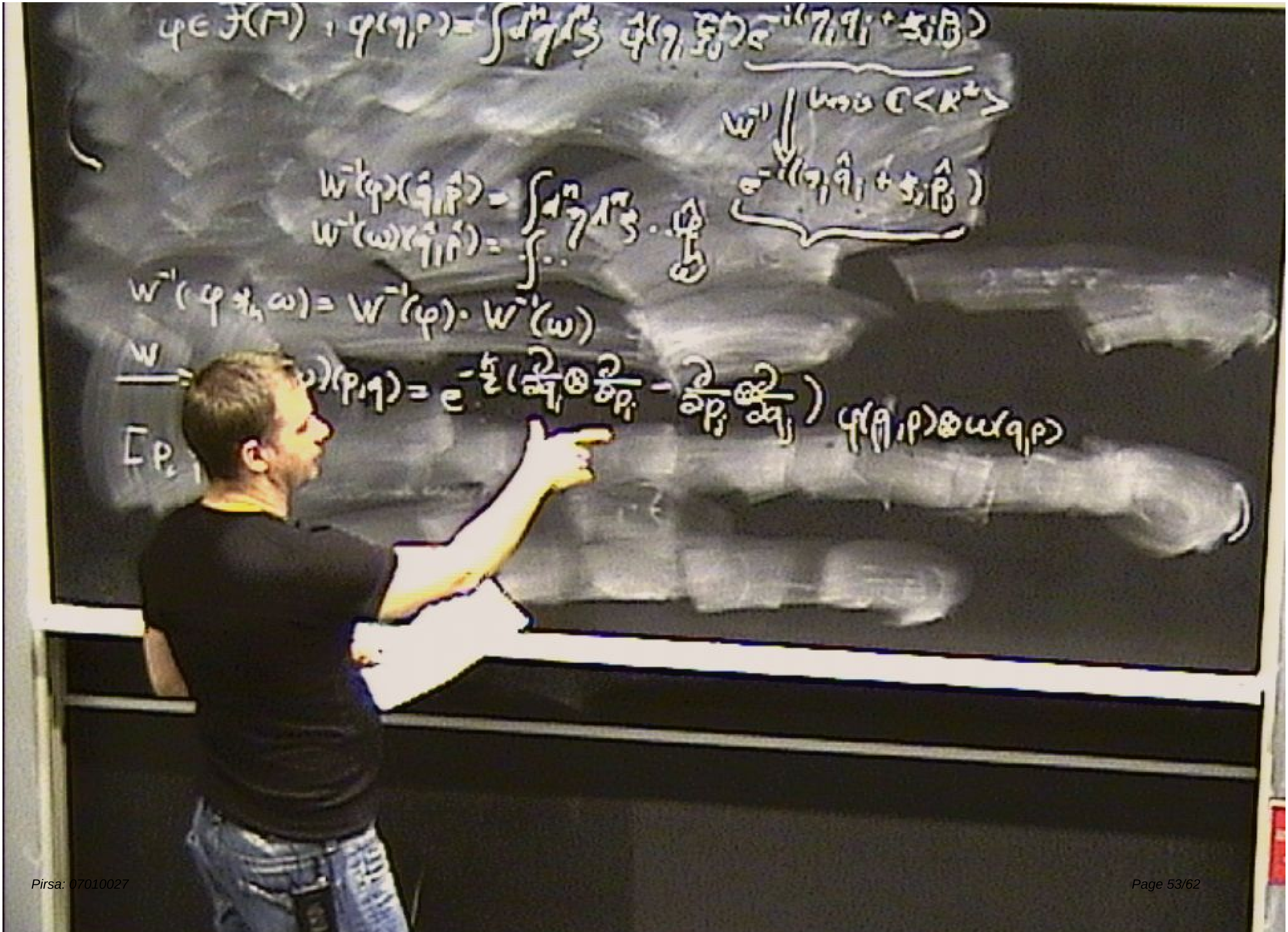


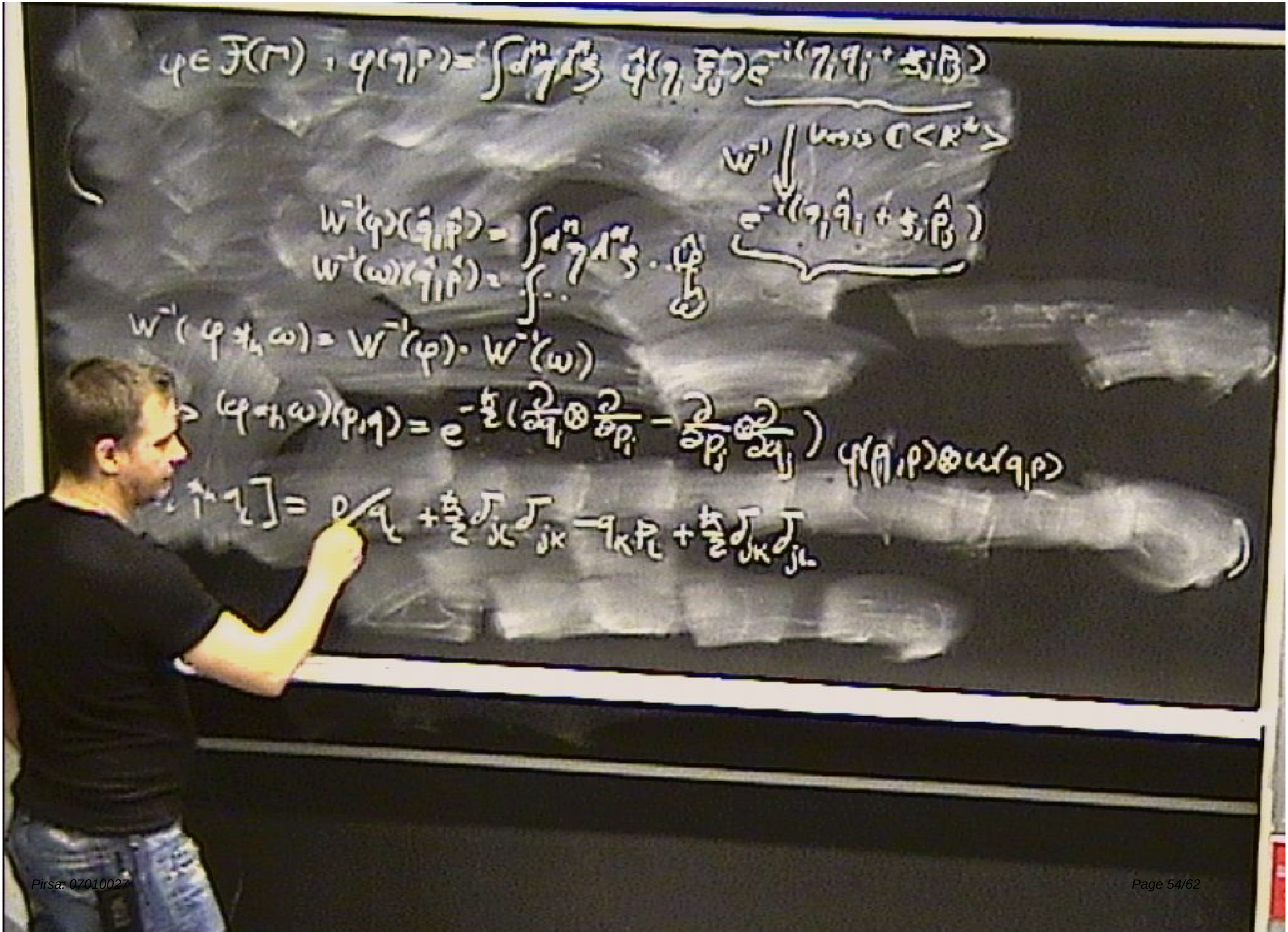
φ ∈ F(Γ) , φ(q,p) = ∫dⁿη dⁿξ φ̃(η,ξ) e^{i(η_i q_i + ξ_i p_i)}
W⁻¹
e^{-i(η_i q̂_i + ξ_i p̂_i)}
W⁻¹(φ)(q̂,p̂) = ∫dⁿη dⁿξ ..
W⁻¹(ω)(q̂,p̂) = ∫..




= W^{-1}(φ) · W^{-1}(ω)


φ ∈ F(M) , φ(q,p) = ∫ d^n η d^n ξ φ̃(η, ξ) e^{i(η_i q_i + ξ_i p_i)}
W⁻¹
W⁻¹(φ)(q̂, p̂) = ∫ d^n η d^n ξ ... φ̃
W⁻¹(ω)(q̂, p̂) = ∫ ...
W⁻¹(φ ⋆ℏ ω) = W⁻¹(φ) · W⁻¹(ω)
W
(φ ⋆ℏ ω)(p,q) = e^{-iℏ/2 (∂/∂q_i ⊗ ∂/∂p_i − ∂/∂p_i ⊗ ∂/∂q_i)} φ(q,p) ⊗ ω(q,p)





$$\varphi \in \mathcal{F}(\Gamma), \quad \varphi(q,p) = \int d^n\eta\, d^n\xi\; \tilde{\varphi}(\eta_i, \xi_j)\, e^{i(\eta_i q_i + \xi_j p_j)}$$

W^{-1} $\downarrow$ in $\mathbb{C}\langle R^2 \rangle$

$$e^{i(\eta_i \hat{q}_i + \xi_j \hat{p}_j)}$$

$$W^{-1}(\varphi)(\hat{q}, \hat{p}) = \int d^n\eta\, d^n\xi \cdot \tilde{\varphi}$$

$$W^{-1}(\omega)(\hat{q}, \hat{p}) = \int \cdots \tilde{\omega}$$

$$W^{-1}(\varphi *_\hbar \omega) = W^{-1}(\varphi) \cdot W^{-1}(\omega)$$

$$\xrightarrow{W} (\varphi *_\hbar \omega)(p,q) = e^{-\frac{i\hbar}{2}\left(\frac{\partial}{\partial q_i} \otimes \frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j} \otimes \frac{\partial}{\partial q_j}\right)} \varphi(q,p) \otimes \omega(q,p)$$

$$[p_i \,{*_\hbar}\, q_k] = p_i q_k + \frac{i\hbar}{2}\delta_{jk}\delta_{jk} - q_k p_i + \frac{i\hbar}{2}\delta_{jk}\delta_{jk}$$

$$= i\hbar\, \delta_{jk} \longleftrightarrow \{p, q\} = \delta_{jk}$$

$\varphi \in \mathcal{F}(\Gamma)$, $\varphi(q,p) = \int d^n\eta\, d^n\xi\, \tilde{\varphi}(\eta,\xi)\, e^{i(\eta_i q_i + \xi_j p_j)}$

$W^{-1} \downarrow$ basis $C<R^2>$

$e^{i(\eta_i \hat{q}_i + \xi_j \hat{p}_j)}$

$W^{-1}(\varphi)(\hat{q},\hat{p}) = \int d^n\eta\, d^n\xi \cdot \tilde{\varphi}$

$W^{-1}(\omega)(\hat{q},\hat{p}) = \int \cdots \tilde{\omega}$

$W^{-1}(\varphi *_\hbar \omega) = W^{-1}(\varphi)\cdot W^{-1}(\omega)$

$$\xrightarrow{W} (\varphi *_\hbar \omega)(p,q) = e^{-\frac{i\hbar}{2}\left(\frac{\partial}{\partial q_i}\otimes\frac{\partial}{\partial p_i} - \frac{\partial}{\partial p_j}\otimes\frac{\partial}{\partial q_j}\right)} \varphi(q,p)\otimes\omega(q,p)$$

$$[p_l *_\hbar q_l] = p_k q_l + \frac{i\hbar}{2}\delta_{jk}\delta_{jk} - q_k p_l + \frac{i\hbar}{2}\delta_{jk}\delta_{jk}$$

$$= i\hbar\,\delta_{jk} \longleftrightarrow \{p,q\} = \delta_{jk}$$

$$[p_k, q_l] = p_k q_l + \tfrac{i\hbar}{2}\delta_{jl}\delta_{jk} - q_l p_k + \tfrac{i\hbar}{2}\delta_{jk}\delta_{jl}$$

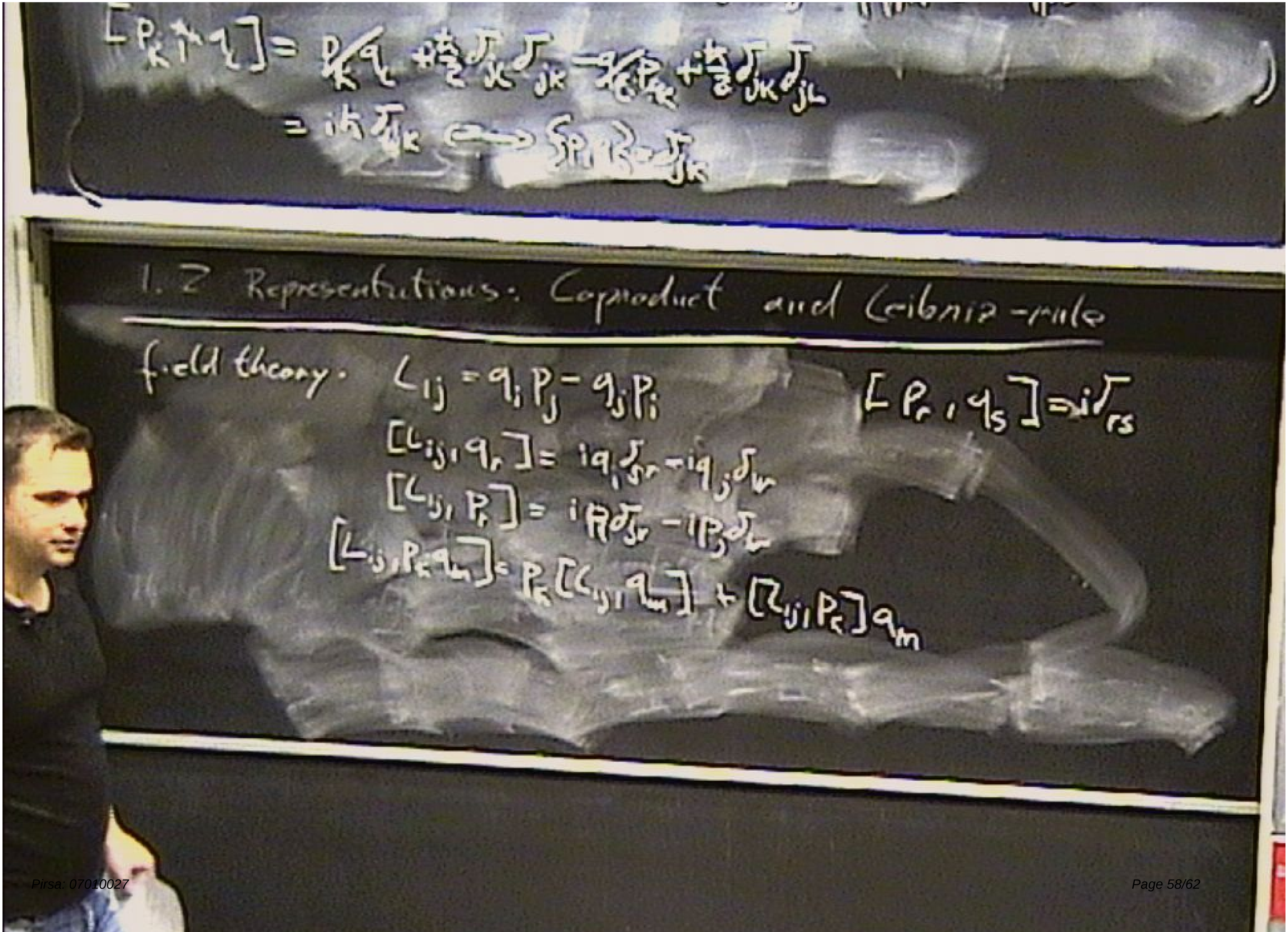
$$= i\hbar\,\delta_{lk} \longleftrightarrow \{p_j, q_k\} = \delta_{jk}$$

1.2 Representations: Coproduct and Leibniz-rule

field theory. $L_{ij} = q_i p_j - q_j p_i$ $\qquad [p_r, q_s] = i\delta_{rs}$

$$[L_{ij}, q_r] = i q_i \delta_{jr} - i q_j \delta_{ir}$$

$$[L_{ij}, p_r] = i p_i \delta_{jr} - i p_j \delta_{ir}$$


= iħ δjk ⟷ {pj, qk} = δjk
1.2 Representations: Coproduct and Leibniz-rule
field theory. Lij = qi pj − qj pi
[pr, qs] = iδrs
[Lij, qr] = iqi δjr − iqj δir
[Lij, pr] = ipi δjr − ipj δir
[Lij, pk qm] = pk [Lij, qm] + [Lij, pk] qm

$$[p_k, q_l] = p_k q_l + \tfrac{i\hbar}{2}\delta_{jk}\delta_{jk} - q_l p_k + \tfrac{i\hbar}{2}\delta_{jk}\delta_{jl}$$

$$= i\hbar\,\delta_{jk} \Longleftrightarrow \{p_j, q_k\} = \delta_{jk}$$

1.2 Representations: Coproduct and Leibniz-rule

field theory: $L_{ij} = q_i p_j - q_j p_i$

$$[L_{ij}, q_r] = i q_i \delta_{jr} - i q_j \delta_{ir}$$

$$[L_{ij}, p_r] = i p_i \delta_{jr} - i p_j \delta_{ir}$$

$$[L_{ij}, p_k q_m] = p_k [L_{ij}, q_m] + [L_{ij}, p_k] q_m$$

Leibniz-rule: $\Delta L_{ij} = L_{ij} \otimes \mathbb{1} + \mathbb{1} \otimes L_{ij}$

$$[p_r, q_s] = i\delta_{rs}$$

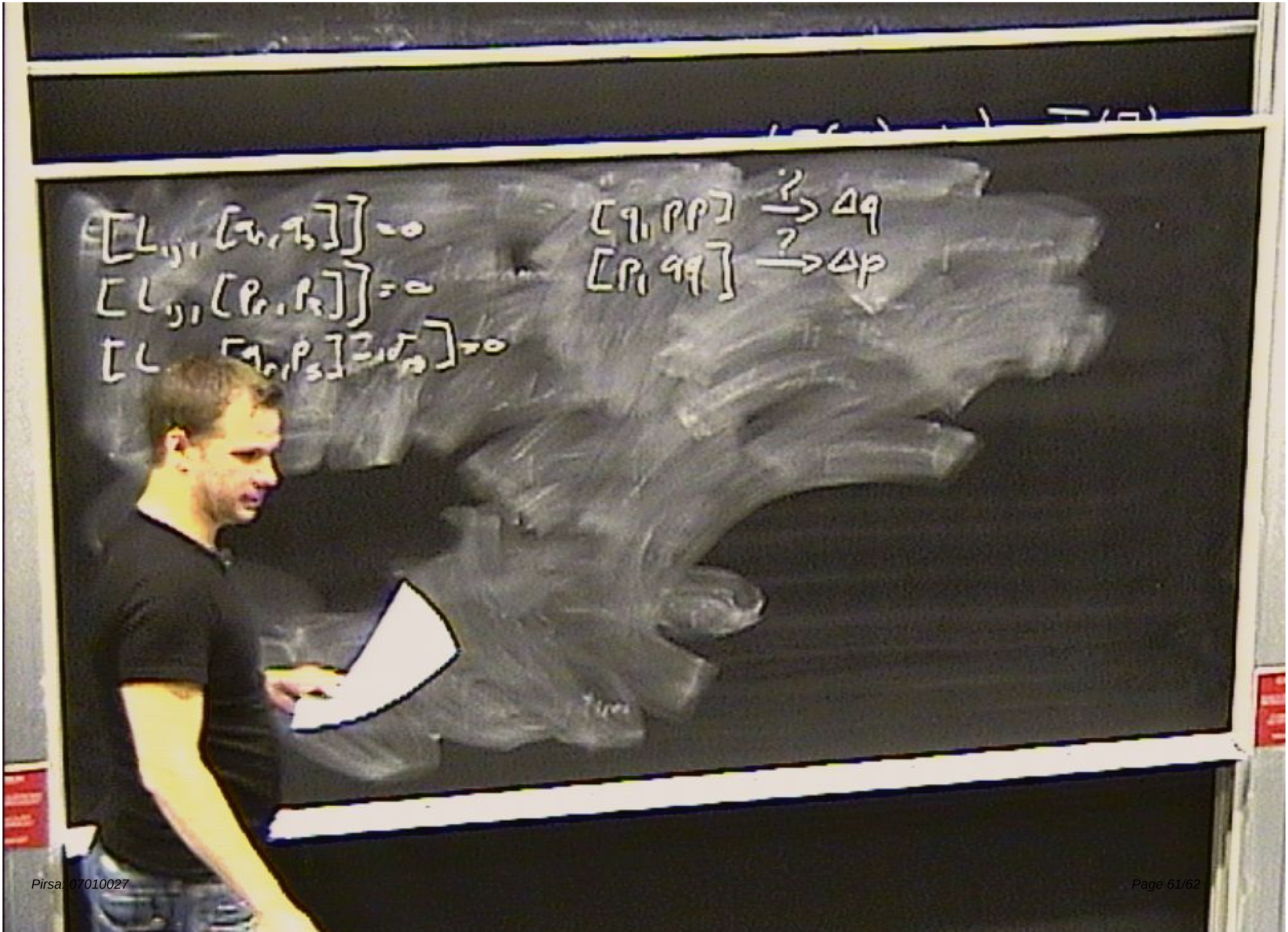
$$[q_r, q_s] = 0$$

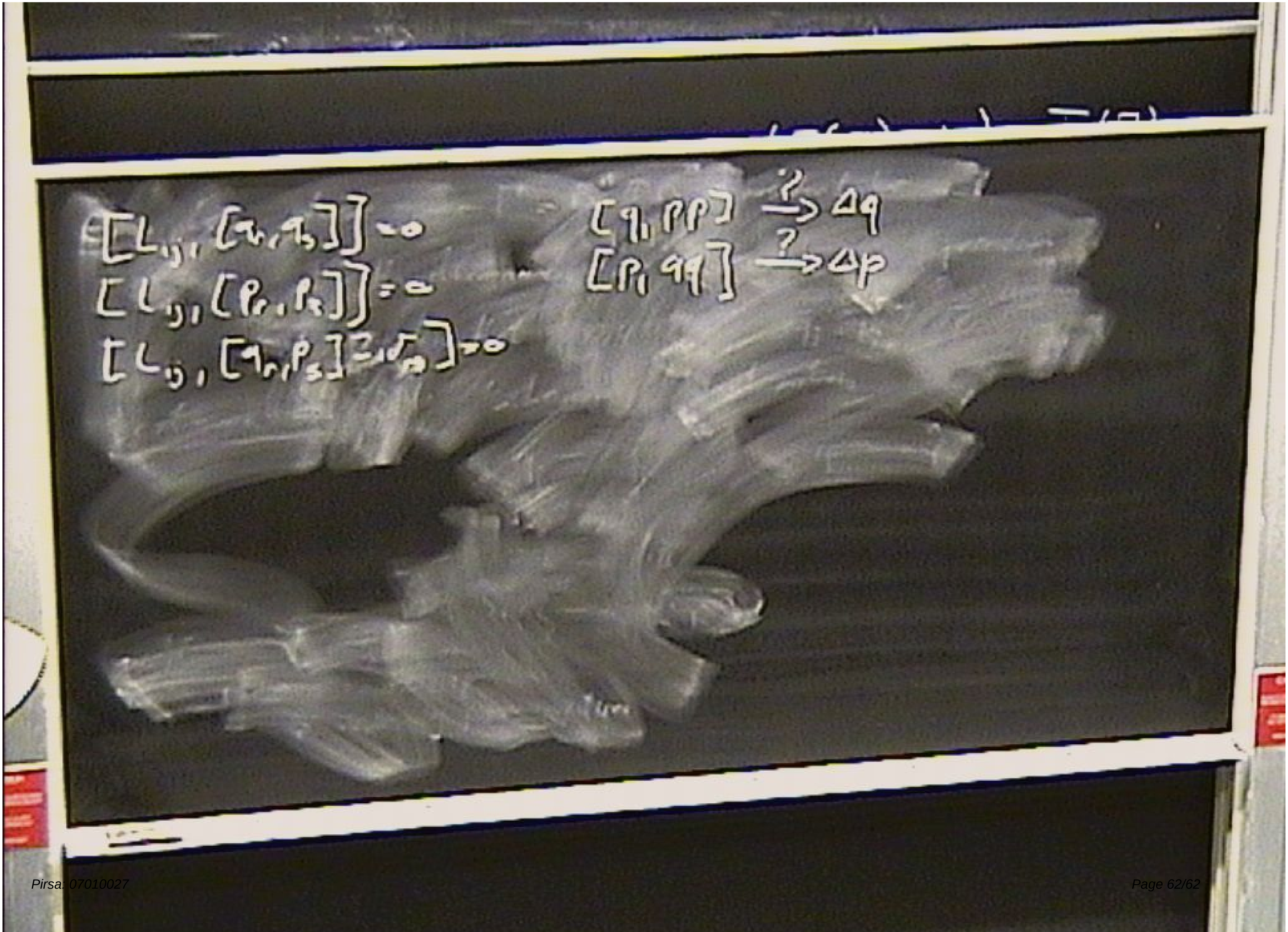
$$[p_r, p_s] = 0$$

$$[L_{ij}, [q_r, q_s]] = 0$$

$$[L_{ij}, [p_r, p_s]] = 0$$

$$[L_{ij}, [q_r, p_s] - i\delta_{rs}] = 0$$


[q, pp] ?→ Δq
[p, qq] ?→ Δp


[Lij, [qr, qs]] = 0
[Lij, [pr, ps]] = 0
[Lij, [qr, ps] = iδrs] = 0
[q, pp] ?→ Δq
[p, qq] ?→ Δp