Title: An Information-Theoretic Approach to Quantum Theory

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Abstract: The mathematical formalism of quantum theory has many features whose physical origin remains obscure. In this paper, we attempt to systematically investigate the possibility that the concept of information may play a key role in understanding some of these features. We formulate a set of assumptions, based on generalizations of experimental facts that are representative of quantum phenomena and physically comprehensible theoretical ideas and principles, and show that it is possible to deduce the finite-dimensional quantum formalism from these assumptions. The concept of information, via an information-theoretic invariance principle, plays a central role in the derivation, and gives rise to some of the central structural features of the quantum formalism.

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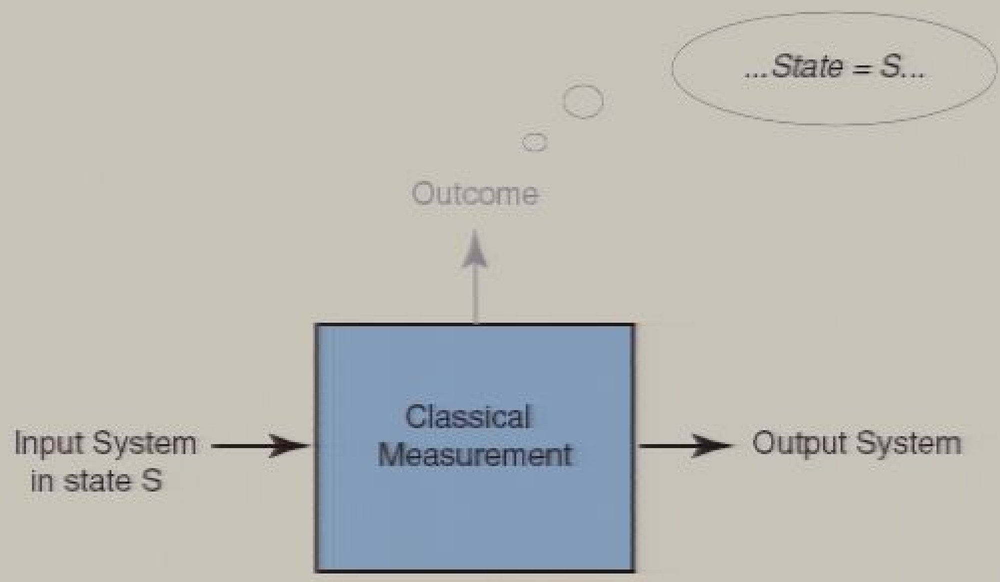
- Is it possible to derive these features using the concept of information? (Wheeler, Wootters, Zeilinger, Fuchs, etc.)
- We will describe a systematic attempt to explore this possibility by formulating a novel information-theoretic principle, and showing how this leads (in conjunction with a set of physical assumptions) to the finite-dimensional quantum formalism.

Why Information?

The concept of information plays a new and fundamental role in quantum physics.

- **Classical Physics.** An ideal measurement provides an experimenter with perfect knowledge about the unknown state of a system....



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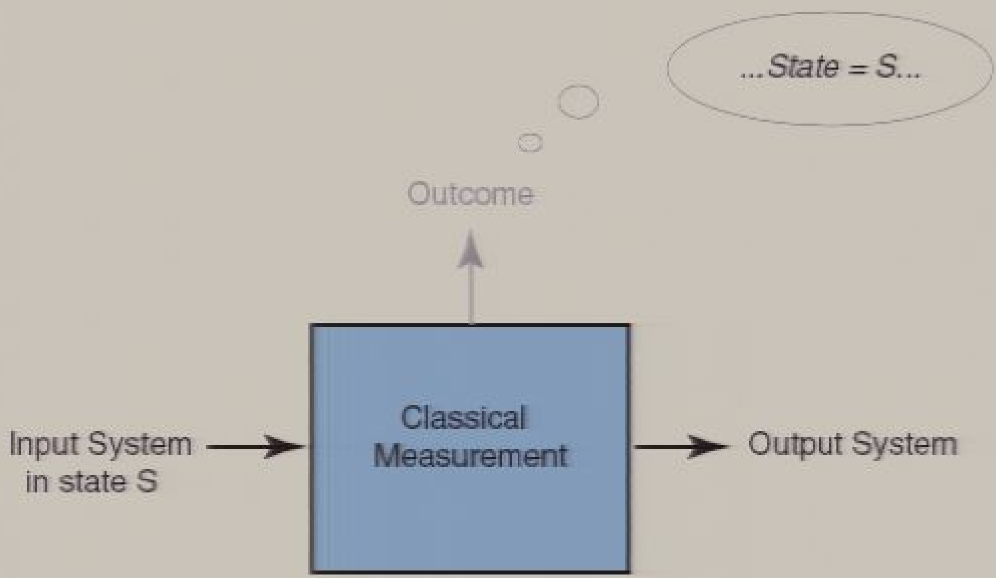

...so there is no distinction between (i) the state itself, and (ii) the ideal experimenter's knowledge of the state.

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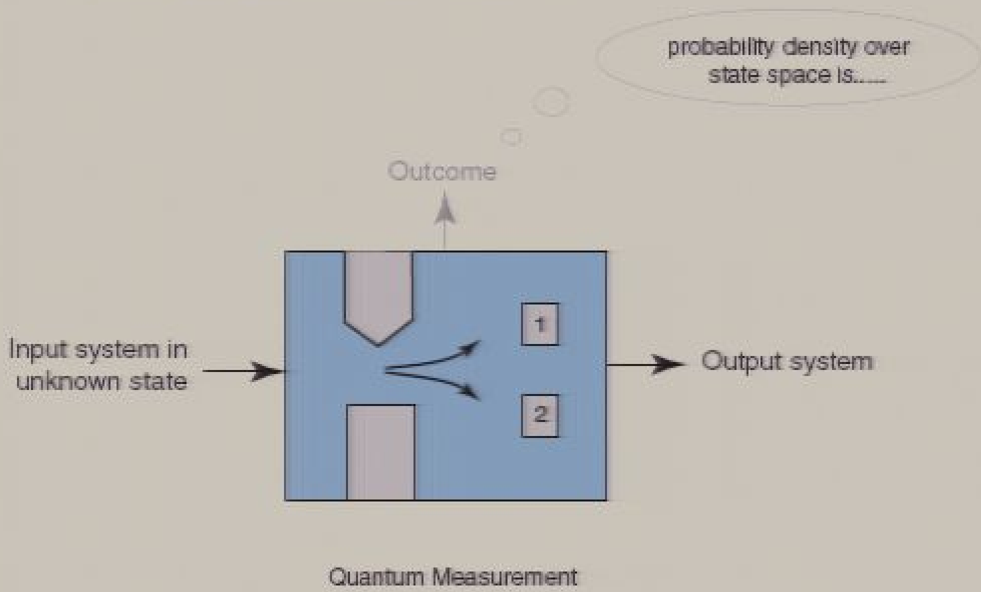

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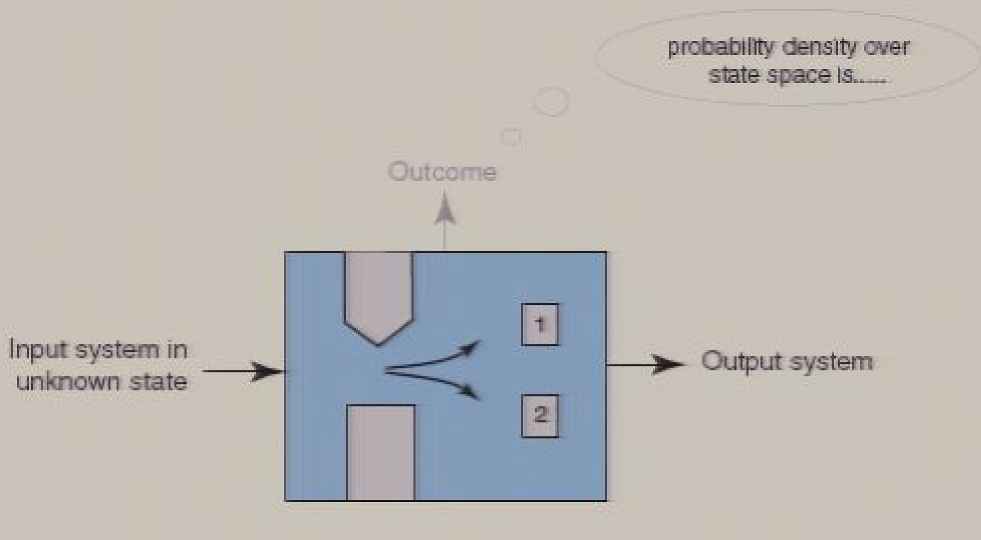

Quantum Measurement

...so a sharp and fundamental distinction is drawn between the state and the experimenter's knowledge of it.

- Information serves as a means to *relate* the two: "How much information has been obtained by the experimenter about the state ?"

Some recent informational approaches

1. Approaches of W. Wootters, C. Brukner, J. Summhammer.
   - Make a few, physically comprehensible assumptions.
   - Impose an informational principle which concerns the amount of information obtained when a measurement is performed.
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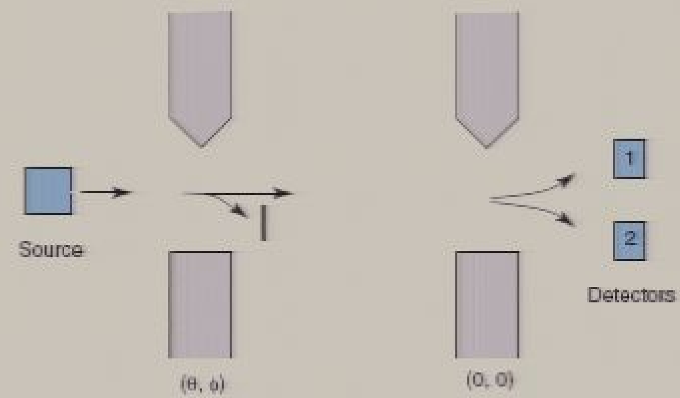
2. Approaches of A. Caticha, Clifton et al., A. Grinbaum, and others.
   - Make abstract assumptions (e.g. introduction of complex numbers) at the outset.
   - But are able to obtain a significant fraction of the quantum formalism.

Wootters' approach

Wootters considers an experimental arrangement where Alice tries to transmit information about θ to Bob using spin-1/2 systems:



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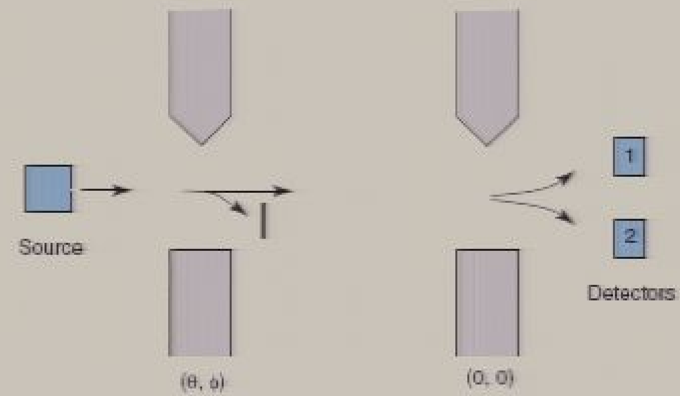

The state of the prepared spin is

$$|\psi\rangle = \sqrt{P_1}e^{i\phi_1}|\uparrow\rangle + \sqrt{P_2}e^{i\phi_2}|\downarrow\rangle,$$

where P_1, P_2 are the outcome probabilities of the measurement.

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Wootters instead asks: "What function $P_1(\theta)$ *maximizes* the amount of Shannon information that Bob gains about θ ?"

Remarkably, he obtains a generalized form of Malus' law,

$$P_1 = \cos^2\left(\frac{m(\theta - \theta_0)}{2}\right),$$

where m is an undetermined integer ($m \neq 0$) and θ_0 is undetermined.

Formulation of the strategy

- The above-mentioned approaches support the view that information has an important role in our understanding of quantum theory.

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- A formulation using only physically comprehensible assumptions has the potential to provide the greater understanding of quantum theory.
- Our objective will be to explore whether is it possible to build up the quantum formalism using the concept of information using only physically comprehensible assumptions.

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  2. can be regarded as based on well-established experimental facts (Michelson-Morley) that are generalized in light of the principle of the uniformity of nature.

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Organization of the Formulation

1. **Basic notions**
   - *Background Assumptions* — definition of elementary terms (*system, background, state, etc.*), which are needed for theoretical modeling to begin.
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At the outset, we adopt from classical physics the following key assumptions:

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- *Time*: In a given frame of reference, one can speak of a physical time which is common to the system and its background, and is represented by a real-valued parameter, t.
- *States*: At any time, the system is in a definite physical state, whose mathematical description is called the state of the system.

Idealizations

1. Measurements are assumed to have the following properties:
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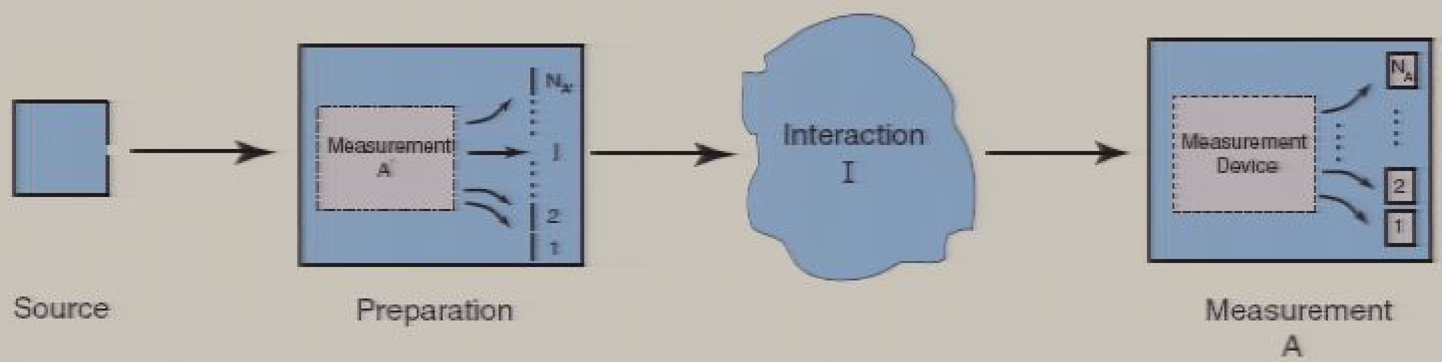

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- An ideal preparation is one that gives us maximal control over the degrees of freedom of the state which encode the property of the system we are studying.
- Classical and quantum physics disagree about what constitutes an ideal preparation.
- Is it possible to find a test or operational procedure that allows us to determine if a preparation is ideal in the quantum setting without reference to the quantum formalism?

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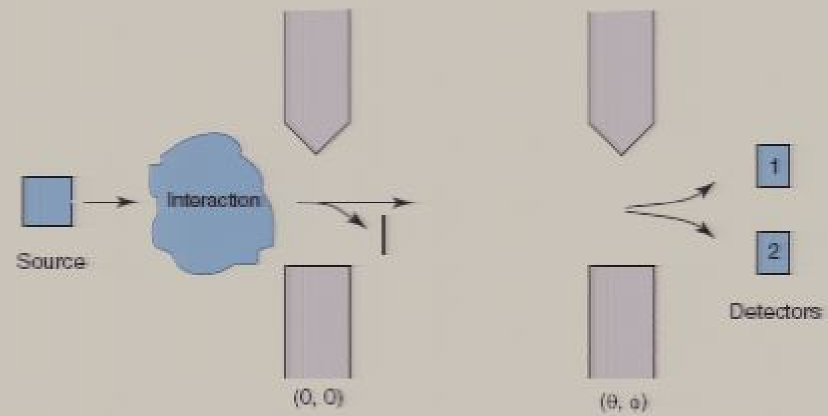

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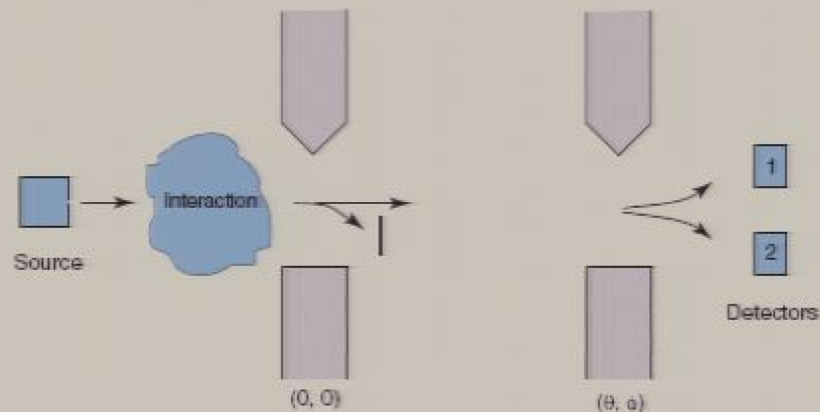

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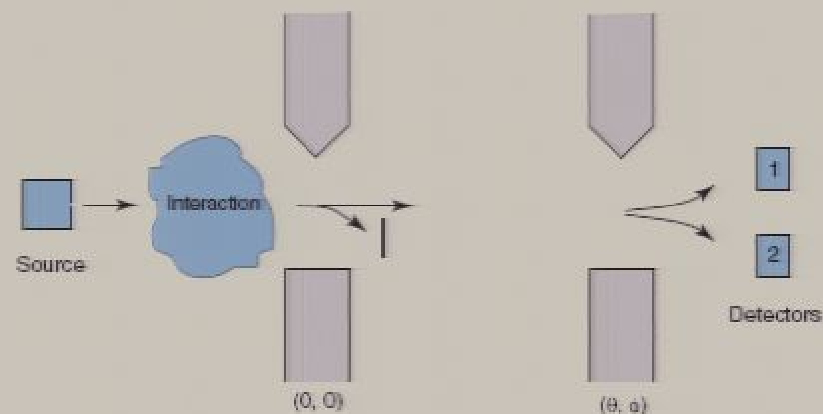

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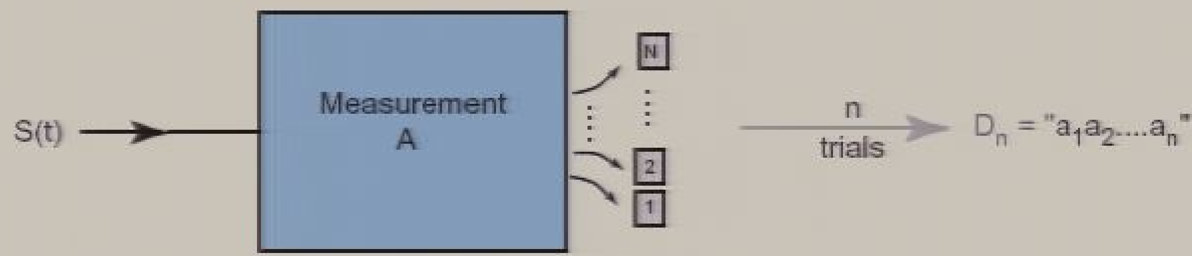
- So, analogously, we can say that the preparation is complete with respect to the measurement.

Operational Definition of Completeness

- Suppose that measurement **A** is performed in n runs of an experiment, and generates the data string $D_n = a_1 a_2 \ldots a_n$, where a_r is the outcome on the rth run.



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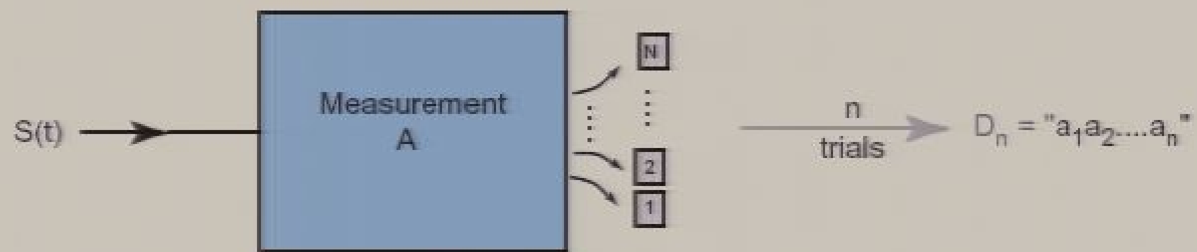
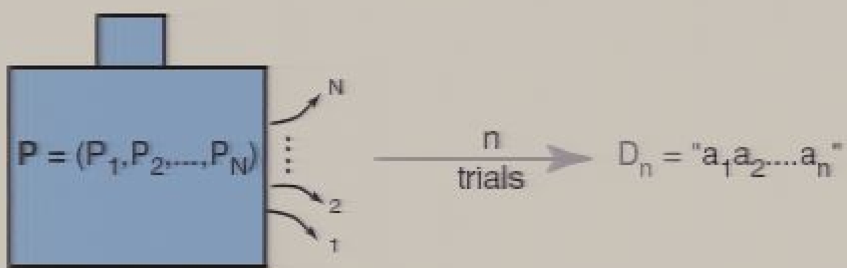


- D_n can be modeled by a probabilistic source with some probability n-tuple $\vec{P} = (P_1, P_2, \ldots, P_N)$.



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- Intuitively, measurements $\mathbf{A}$ and $\mathbf{A}'$ are probing precisely the "same region of property-space" of the system.
- **Example:** Any two Stern-Gerlach measurements form a measurement pair.

Measurement Sets

- **Definition**: The set of measurements generated by $\mathbf{A}$ forms a *measurement set*, $\mathcal{A}$, and is defined as the set of all measurements that

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- Intuitively the measurements in the set are *all* measurements probing a given "region of property space".
- **Example:** The set of all Stern-Gerlach measurements forms a measurement set with respect to a spin-1/2 system. Composites of two or more successive Stern-Gerlach measurements are excluded.

Compatible Interactions

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- Intuitively, the interaction is not coupling the region of property space under study with *other* regions of property space.
- **Example:** Uniform $\vec{B}$-field interactions with a spin-1/2 system are compatible with the Stern-Gerlach measurement set. Non-uniform $\vec{B}$-field interactions are *not* compatible.

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- **Example:** The interaction set for a spin-1/2 system undergoing Stern-Gerlach measurements is the set of all uniform $\vec{B}$-field interactions.

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- **Example:** A source emits a system consisting of two distinguishable spin-1/2 particles on each run of an experiment. If there are two set-ups, with each set-up involving Stern-Gerlach measurements on only one of the particles, then the set-ups are disjoint.

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(c) **Novel Postulates**. Postulates, with no classical analogues, based on experimental facts or novel theoretical principles or ideas.

Organization of the Postulates

1. Measurements.
   1.1 Finite and Probabilistic outcomes.
   1.2 Repetition.
   1.3 Representation of Measurements.
2. States.
   2.1 States.
   2.2 Physical Interpretation of the χ_i.
   2.3 Information Gain.
   2.4 Prior Probabilities.
3. Transformations.
   3.1 One-to-one.
   3.2 Invariance.
   3.3 Parameterized Transformations.
4. Consistency.
5. Composite Systems.

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Postulate 1.1

1 Measurements.

1.1 *Finite and Probabilistic outcomes.* When measurement $\mathbf{A}$ is performed, one of N possible outcomes is observed. Outcome i is obtained with probability P_i ($i = 1, \ldots, N$), with P_i determined by the preparation, interaction, and measurement.

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- Key new idea: All measurements in the measurement set $\mathcal{A}$ have N observable outcomes.
- The postulate is a modification of *determinacy* and *continuum* assumptions of classical physics in the light of experimental facts characteristic of quantum phenomena (such as Stern-Gerlach measurements on silver atoms).

Postulate 1.2

1 Measurements.

1.2 *Repetition.* When measurement **A** is immediately repeated, the same outcome is observed with certainty.

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- Identical to the assumption of *repetition consistency* of classical physics.
- Consistent with experimental findings in many quantum experiments (such as Stern-Gerlach measurements on spin systems).

Postulate 1.3

1 Measurements.

1.3 *Representation of Measurements.* For any given pair of measurements, $\mathbf{A}, \mathbf{A}'$, there exist interactions $\mathbf{I}, \mathbf{I}'$, such that $\mathbf{A}$ can be represented (insofar as the probabilities of the observed outcomes and of the output states are concerned) by an arrangement where $\mathbf{I}$ is immediately followed by $\mathbf{A}'$ which, in turn, is immediately followed by $\mathbf{I}'$.

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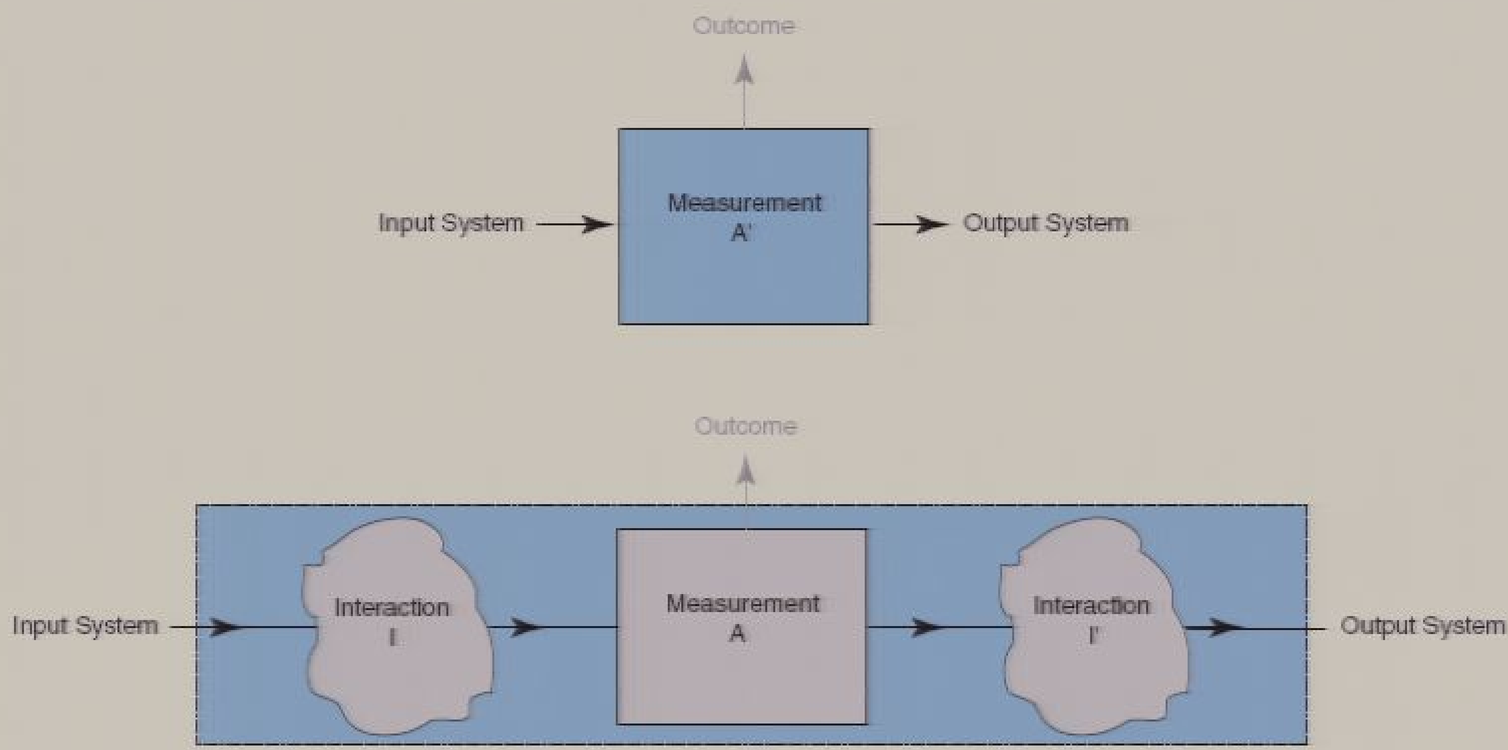
- A novel assumption, generalized from appropriate experimental facts (Stern-Gerlach measurements on silver atoms), with no classical analogue.

Postulate 1.3



Postulate 2.1

2 States.

2.1 *States.* The state, $\mathbf{S}(t)$, at time t, can be represented as $(\vec{P}, \vec{\chi})$, where $\vec{P} = (P_1, \ldots, P_N)$ and $\vec{\chi} = (\chi_1, \ldots, \chi_N)$, where the χ_i are real-valued degrees of freedom.

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(b) If the position measurements are sufficiently high in resolution, we find that the preparation is complete with respect to the subsequent position measurement.

(c) Therefore, we can model the situation using the quantum framework.

Postulate 2.1

(d) As m tends to macroscopic values, the behavior of the particle is well-described by classical physics. Yet the outcomes of the position measurement are still probabilistic.

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(e) Hence, the correct classical model is the discretized Hamilton-Jacobi model, with state

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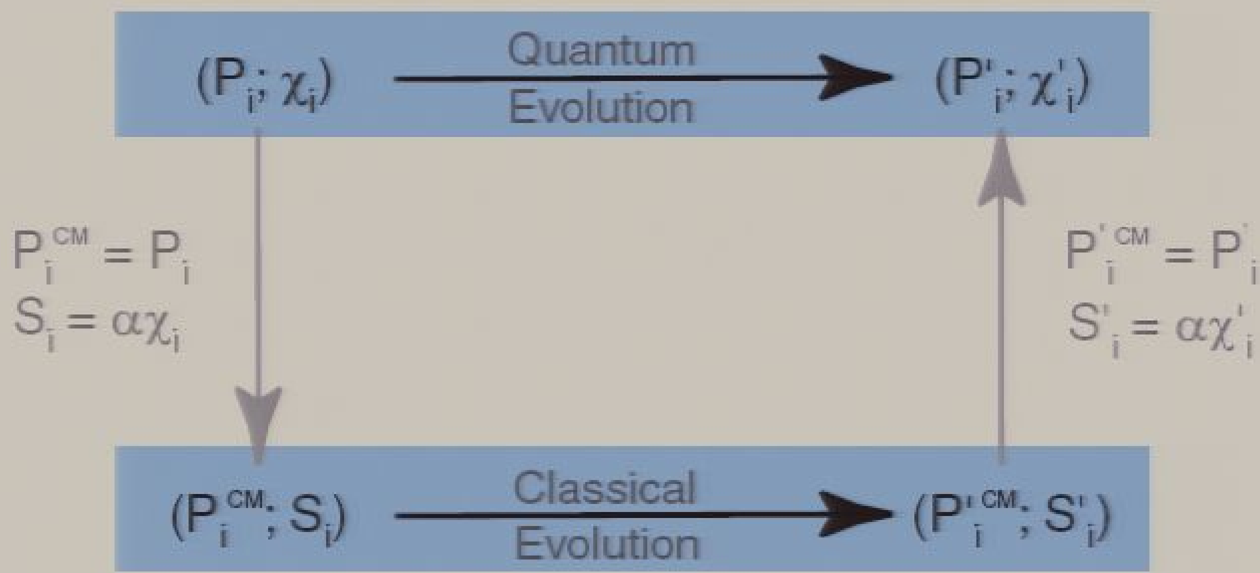
$$(P_i; S_i) = (P_1, \ldots, P_N; S_1, \ldots, S_N).$$

(f) To obtain a one-to-one correspondence between the quantum and classical states, the quantum state must be $(P_i; \chi_i)$, where the χ_i are real-valued degrees of freedom.

Postulate 2.1



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Postulate 2.2

2 States.

2.2 *Physical interpretation of the* χ_i. When measurement $\mathbf{A}$ is performed and the outcome i is obtained, there are additional outcomes that are objectively realized but unobserved:

Postulate 2.1

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
(P_i; \chi_i) \arrow[r, "\text{Quantum}", "\text{Evolution}"'] \arrow[d, "\begin{array}{c} P_i^{CM} = P_i \\ S_i = \alpha\chi_i \end{array}"'] & (P'_i; \chi'_i) \\
(P_i^{CM}; S_i) \arrow[r, "\text{Classical}", "\text{Evolution}"'] & (P'^{CM}_i; S'_i) \arrow[u, "\begin{array}{c} P'^{CM}_i = P'_i \\ S'_i = \alpha\chi'_i \end{array}"']
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(a) one of two outcomes (labeled a and b), obtained with respective probabilities $P_{a|i} = Q^2_{a|i}$ and $P_{b|i} = Q^2_{b|i}$, where $Q_{a|i} = f(\chi_i)$ and $Q_{b|i} = \tilde{f}(\chi_i)$, where f is not a constant function and f and $\tilde{f}$ have range $[-1, 1]$, and

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(b) one of two possible outcomes (with values labeled $+$ and $-$), which is determined by the sign of $Q_{a|i}$ or $Q_{b|i}$ depending upon whether outcome a or b has been realized.

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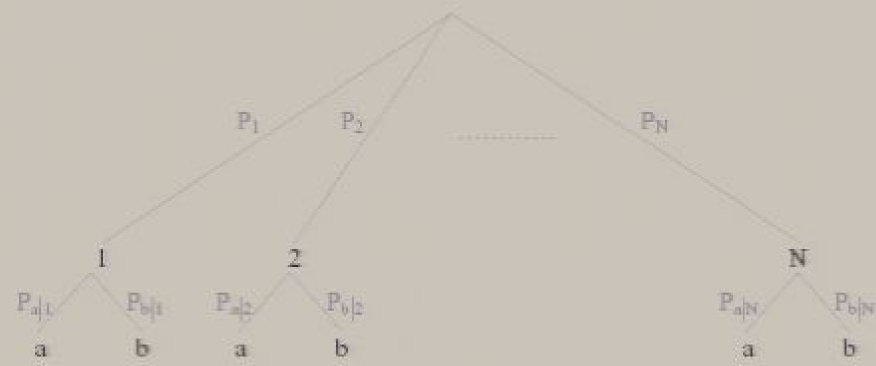
Postulate 2.2



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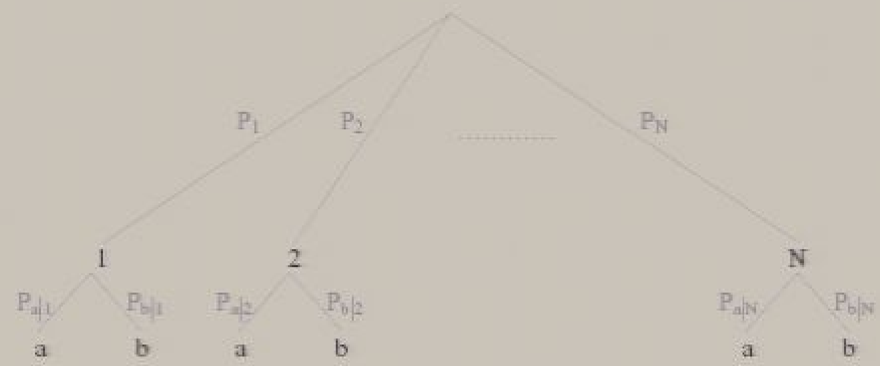

1. When measurement **A** is performed, and outcome i is observed, either a or b is additionally realized (but unobserved).
2. Outcomes a and b are realized probabilistically, with probabilities $P_{a|i} = Q^2_{a|i}$ and $P_{b|i} = Q^2_{b|i}$, respectively.

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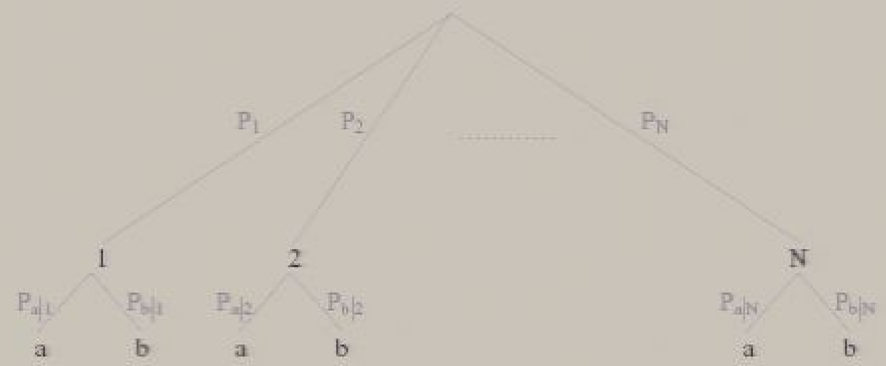

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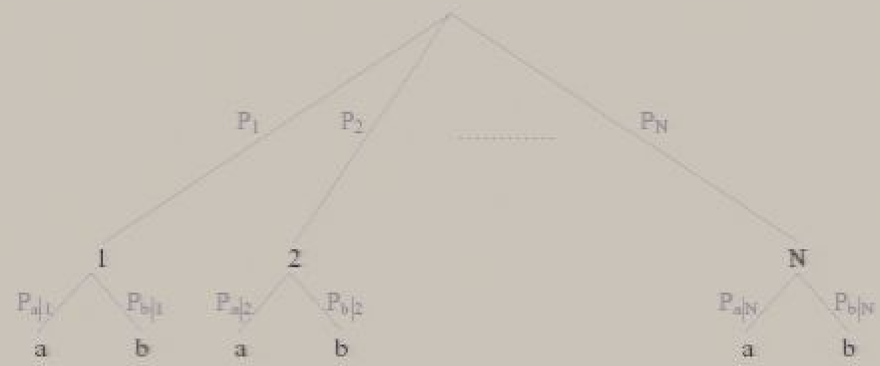

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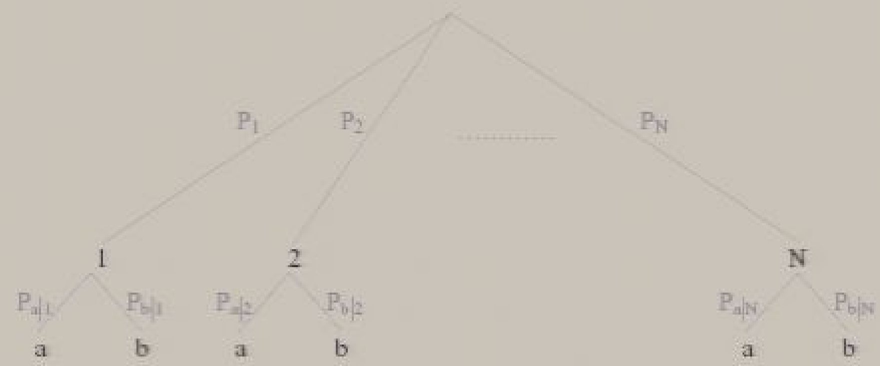

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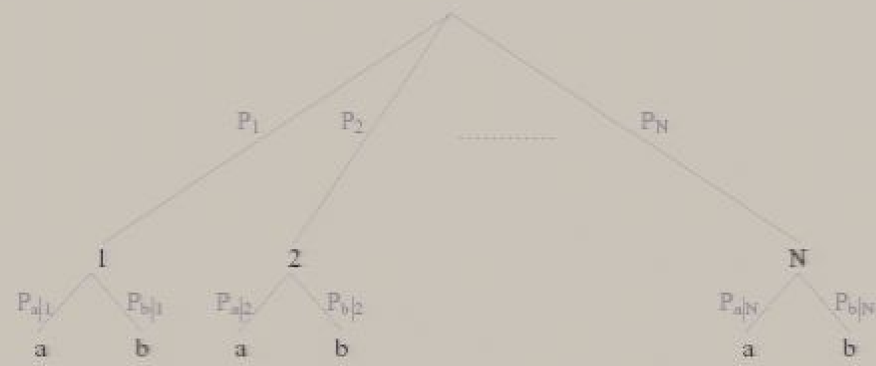

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2. The $Q_{a|i}$ and $Q_{b|i}$ are motivated by the fact that, in the polarization of a photon, the 'state' is $(\cos\theta, \sin\theta)$, which leads to the probabilities $\cos^2\theta, \sin^2\theta$.

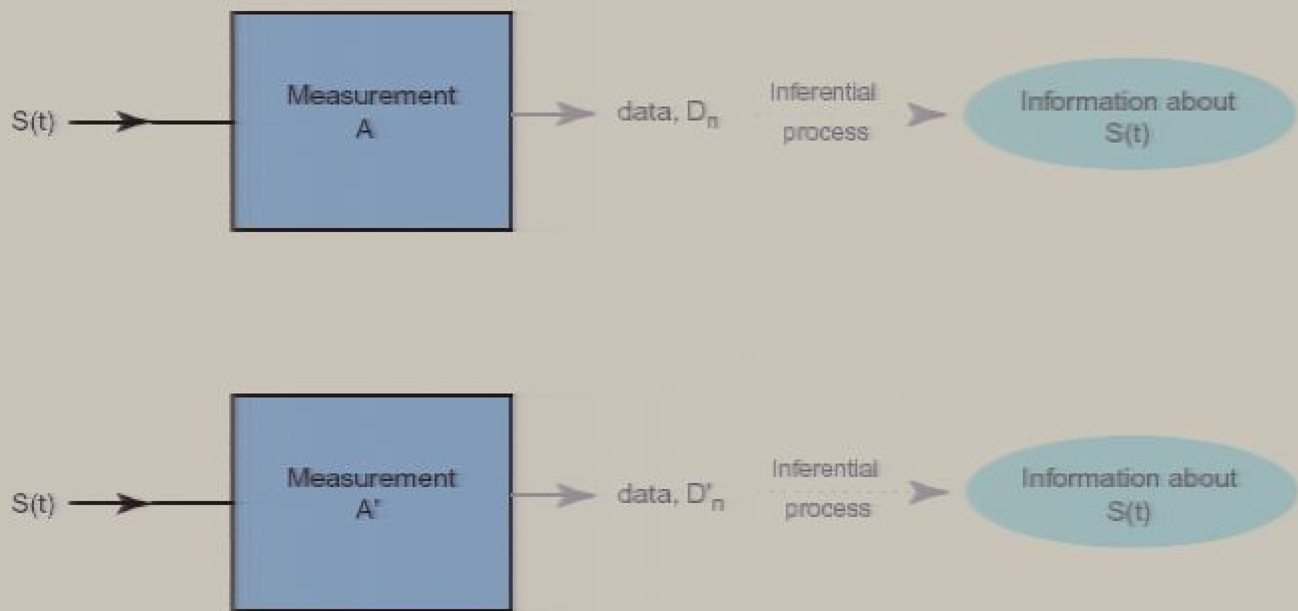
Postulate 2.3

2 States.

2.3 *Information Gain.* In an experiment where measurement $\mathbf{A}$ is performed on a system in any unknown state, $\mathbf{S}(t)$, the amount of Shannon-Jaynes information provided by the probabilistically-determined outcomes about $\mathbf{S}(t)$ in n runs of the experiment is independent of $\mathbf{S}(t)$ in the limit as n tends to infinity.

Postulate 2.3

- Consider two experimental trails, each consisting of n runs.



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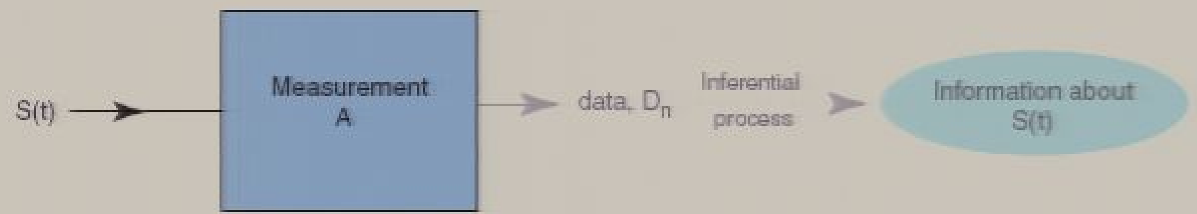
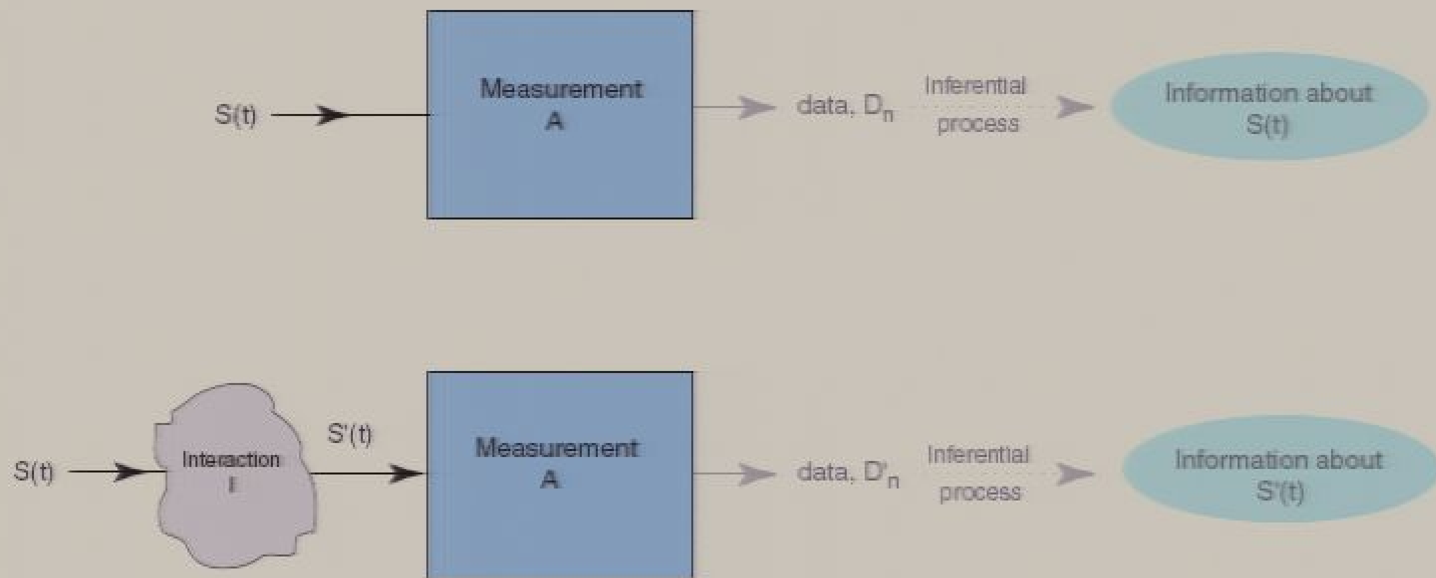
Postulate 2.3

- Now, by Postulate 1.3 (*Representation of Measurements*), we can replace measurement $\mathbf{A}'$ in trial 2 as follows:



Postulate 2.3

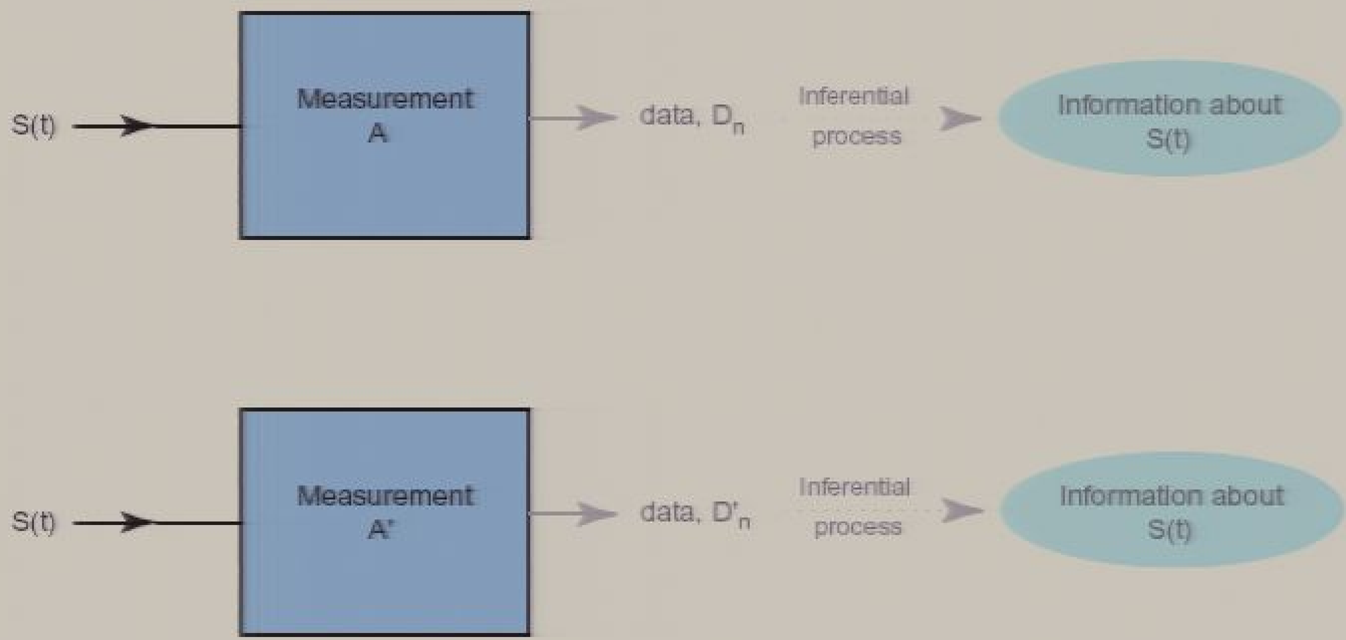
- Grouping together $\mathbf{S}(t)$ and $\mathbf{I}$, we have, equivalently:



...where $S'(t) = \mathcal{M}(S(t))$, where $\mathcal{M}$ represents the effect of interaction $\mathbf{I}$.

Postulate 2.3

- Suppose that trials 1 and 2 yield different amounts of information about $\mathbf{S}(t)$.



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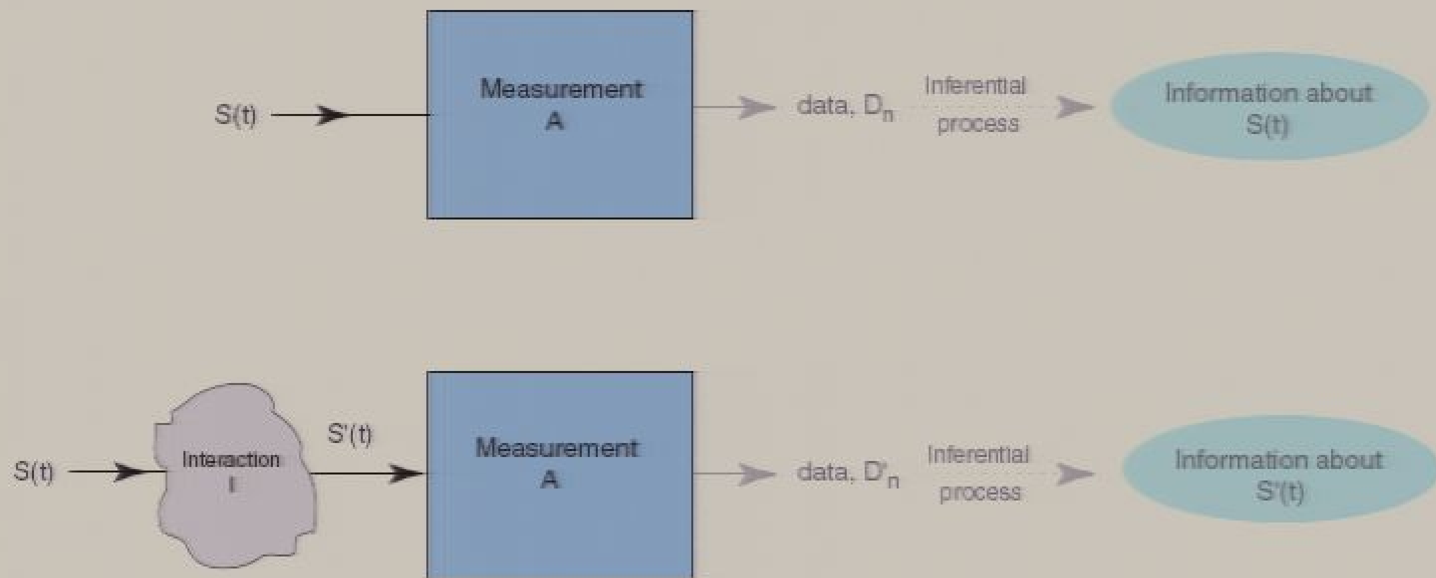

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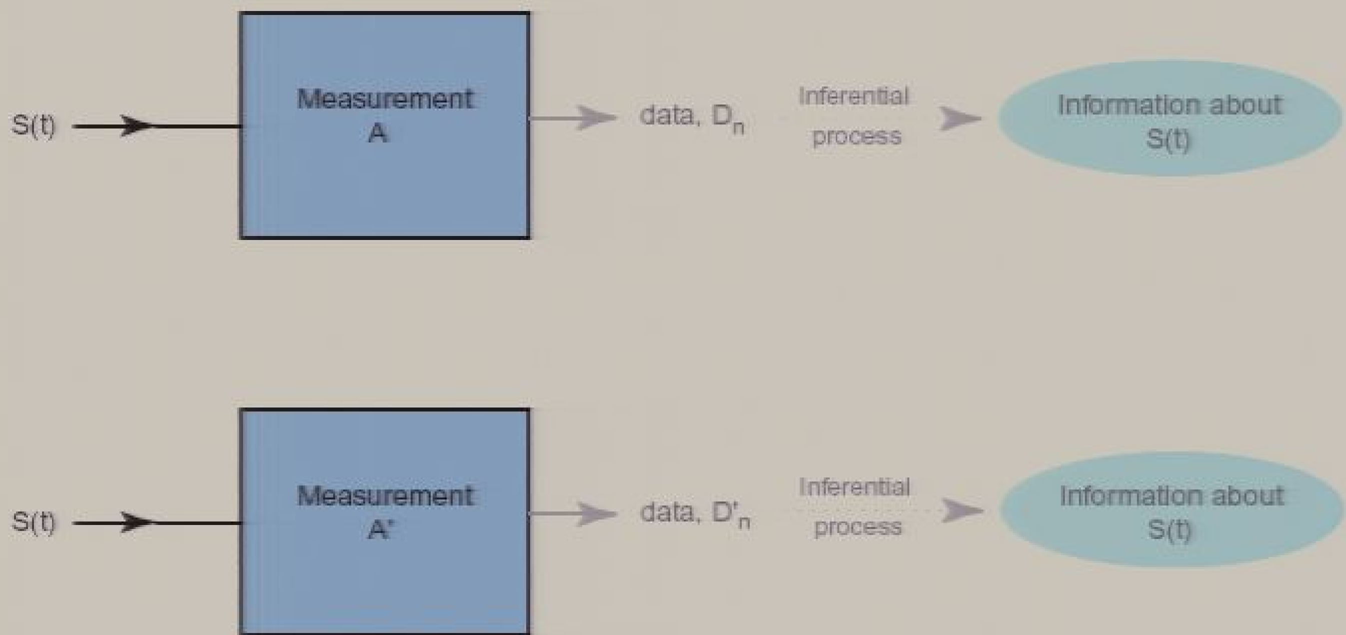

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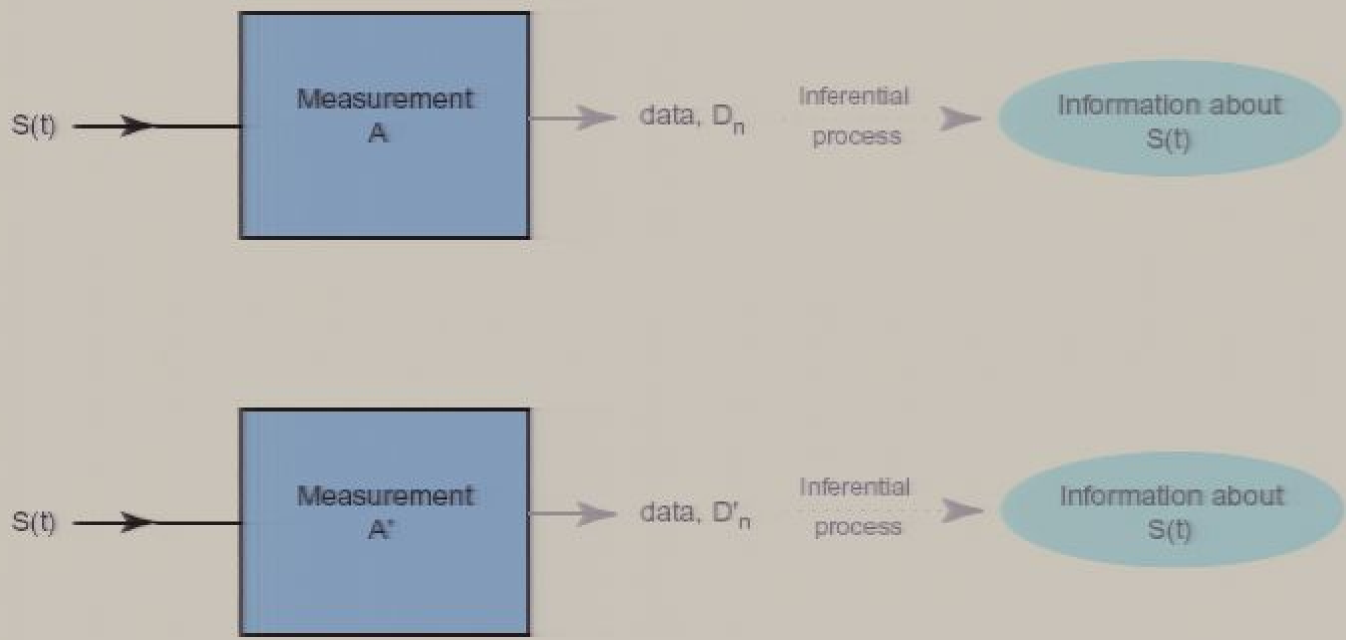

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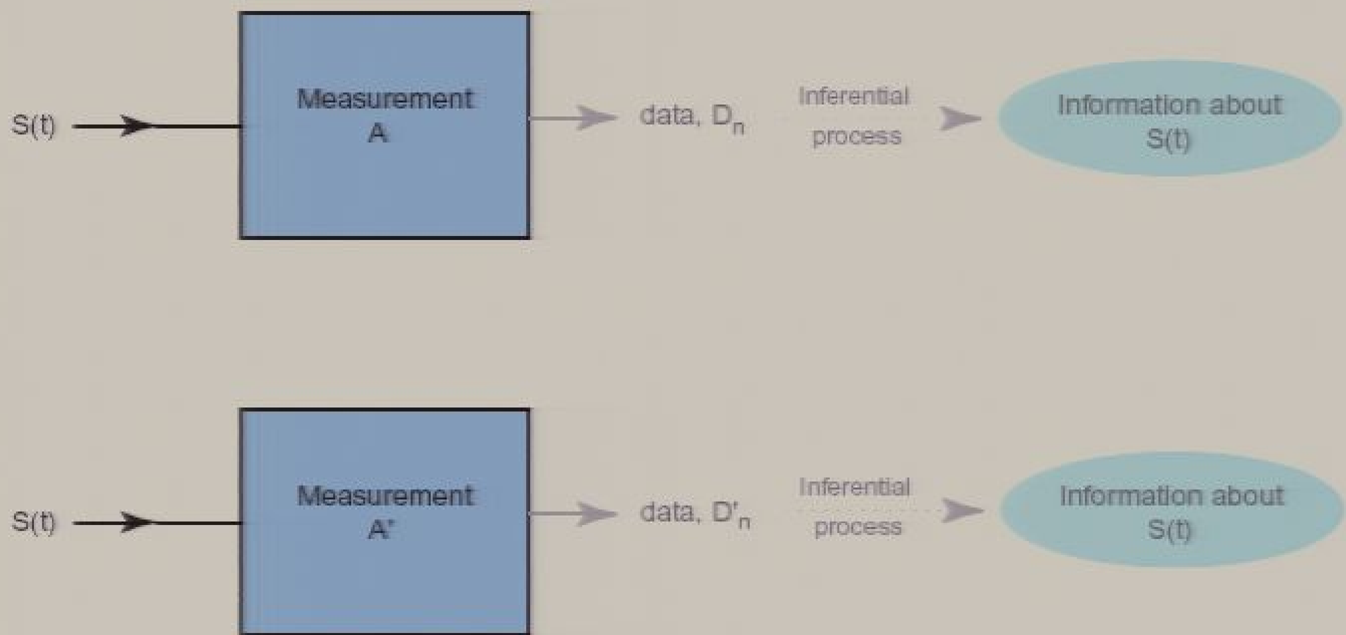
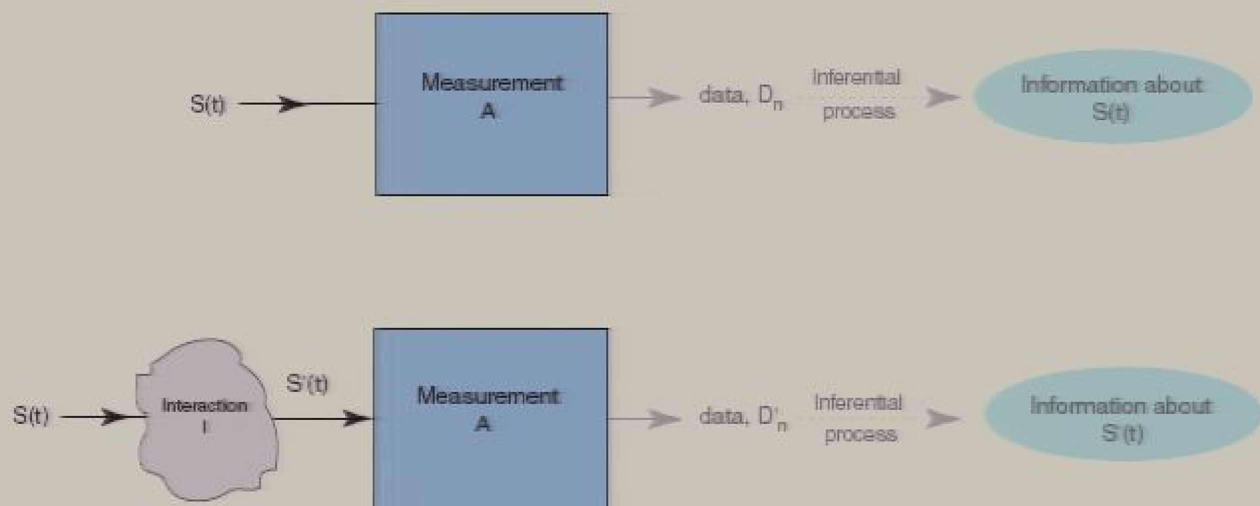


- This implies that one of the measurements $\mathbf{A}$ and $\mathbf{A}'$ is privileged insofar as the amount of information it yields about $\mathbf{S}(t)$.
- We make the intuitively plausible assertion that this is not possible — that no measurement in $\mathcal{A}$ is informationally privileged.

Postulate 2.3

- But this implies that measurement **A** yields the same amount of information about states $\mathbf{S}(t)$ and $\mathbf{S}'(t)$.



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Postulate 2.4

2 States.

2.4 *Prior probabilities.* The prior probability, $\Pr(\chi_i|I)$, where I is the background knowledge of the experimenter prior to performing measurement $\mathbf{A}$, is uniform.

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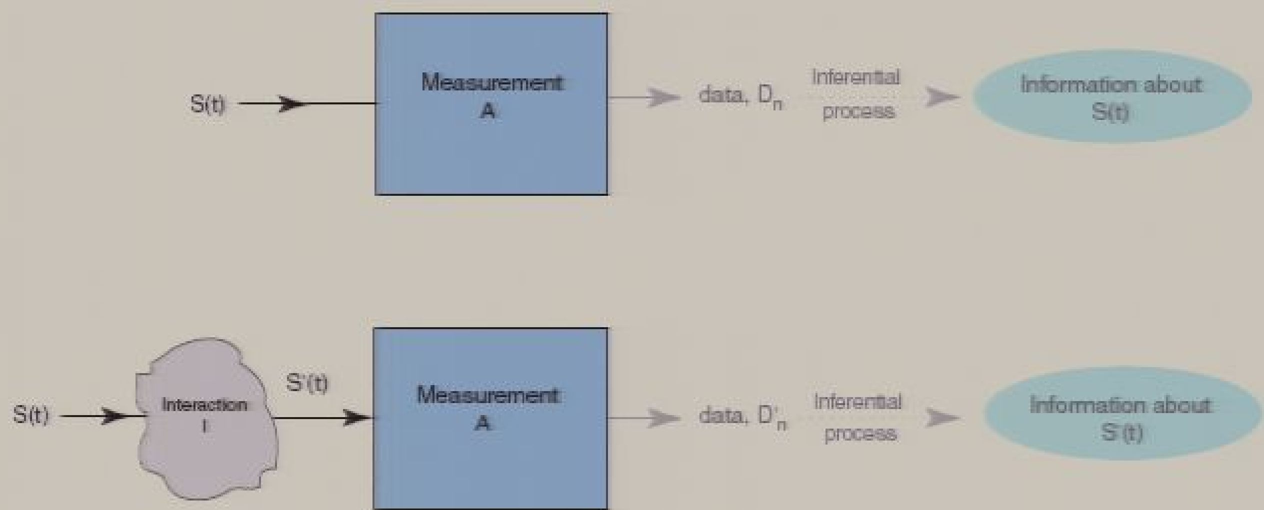

- Postulate 2.3 ensures that this is the case for all $\mathbf{S}(t)$ and $\mathbf{S}'(t)$.

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- This implies that $\Pr(S_1, \ldots, S_N) = \Pr(S_1 + S_0, \ldots, S_N + S_0)$ for all S_0 and all S_i.
- Therefore, $\Pr(S_i|I)$ is uniform.

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- Therefore, $\Pr(S_i|I)$ is uniform.
- Under the classical–quantum correspondence $\chi_i = S_i/\alpha$, it follows that $\Pr(\chi_i|I)$ is uniform.

Postulates 3 and 3.1

3 Transformations. Any transformation of a physical system, whether due to temporal evolution of the system, or due to a passive change of frame of reference (a symmetry transformation), is represented by a map, $\mathcal{M}$, over the state space of the system.

3.1 *One-to-one.* The map, $\mathcal{M}$, is one-to-one.

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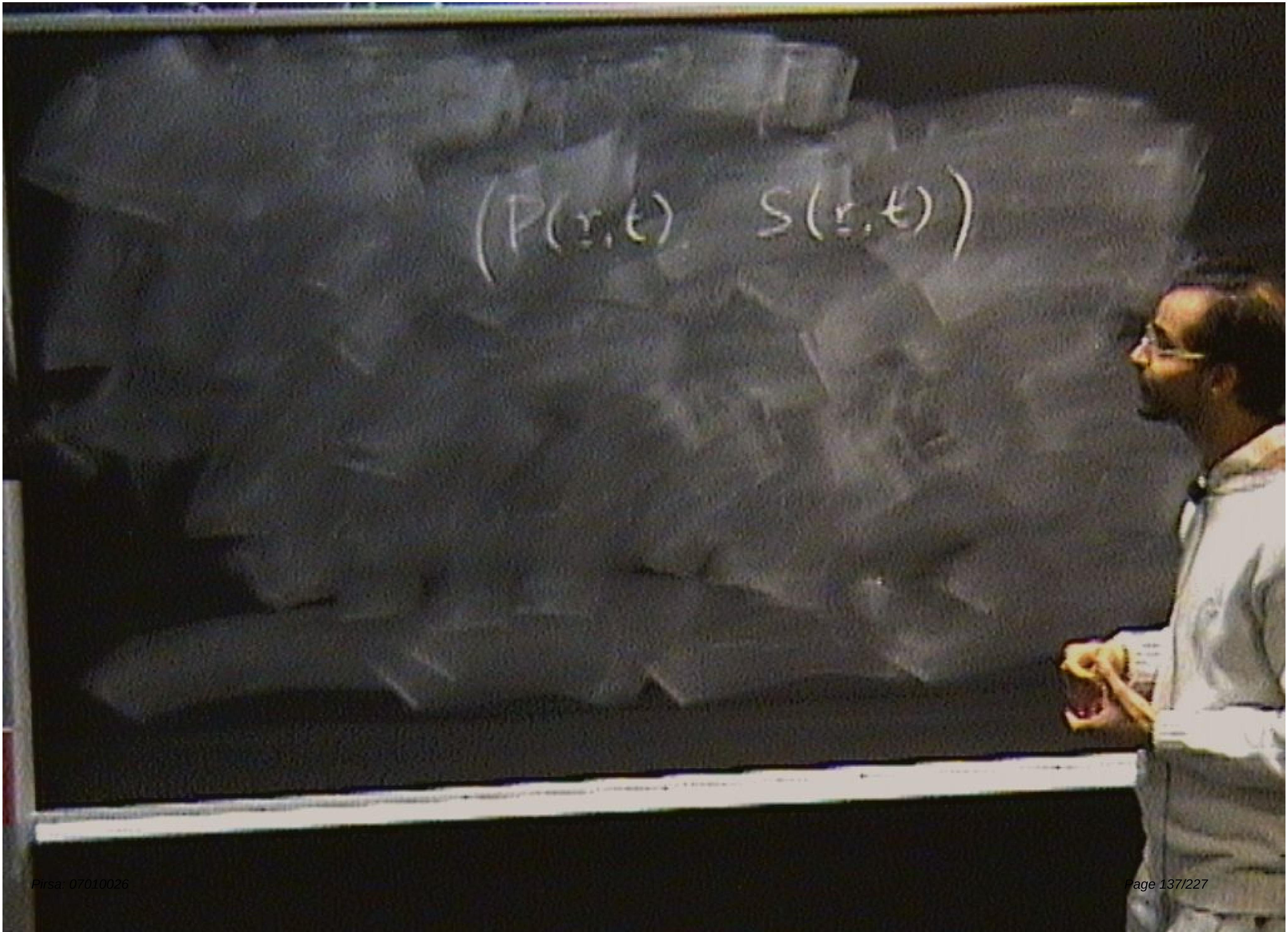
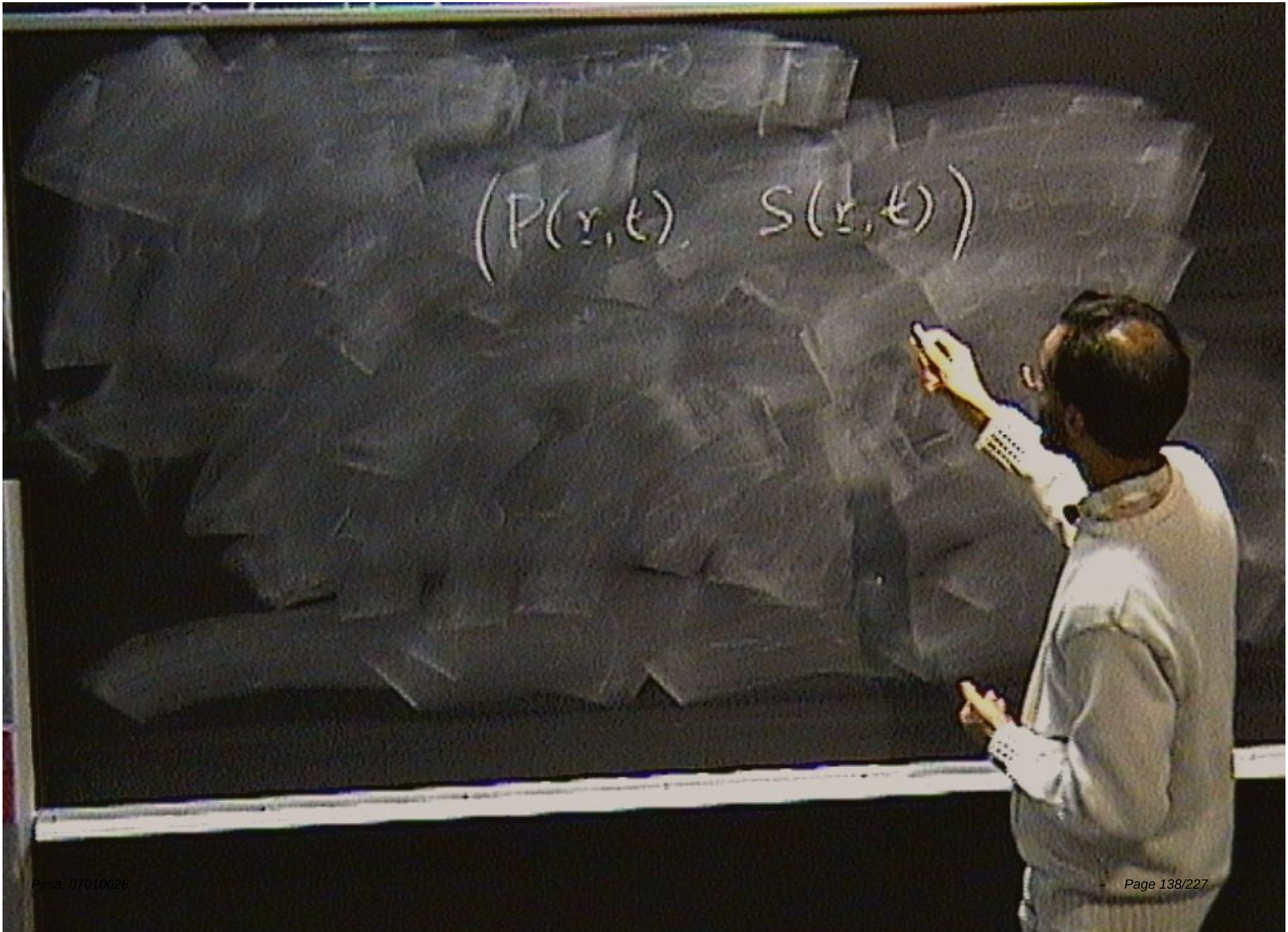

(P(r,t) S(r,t))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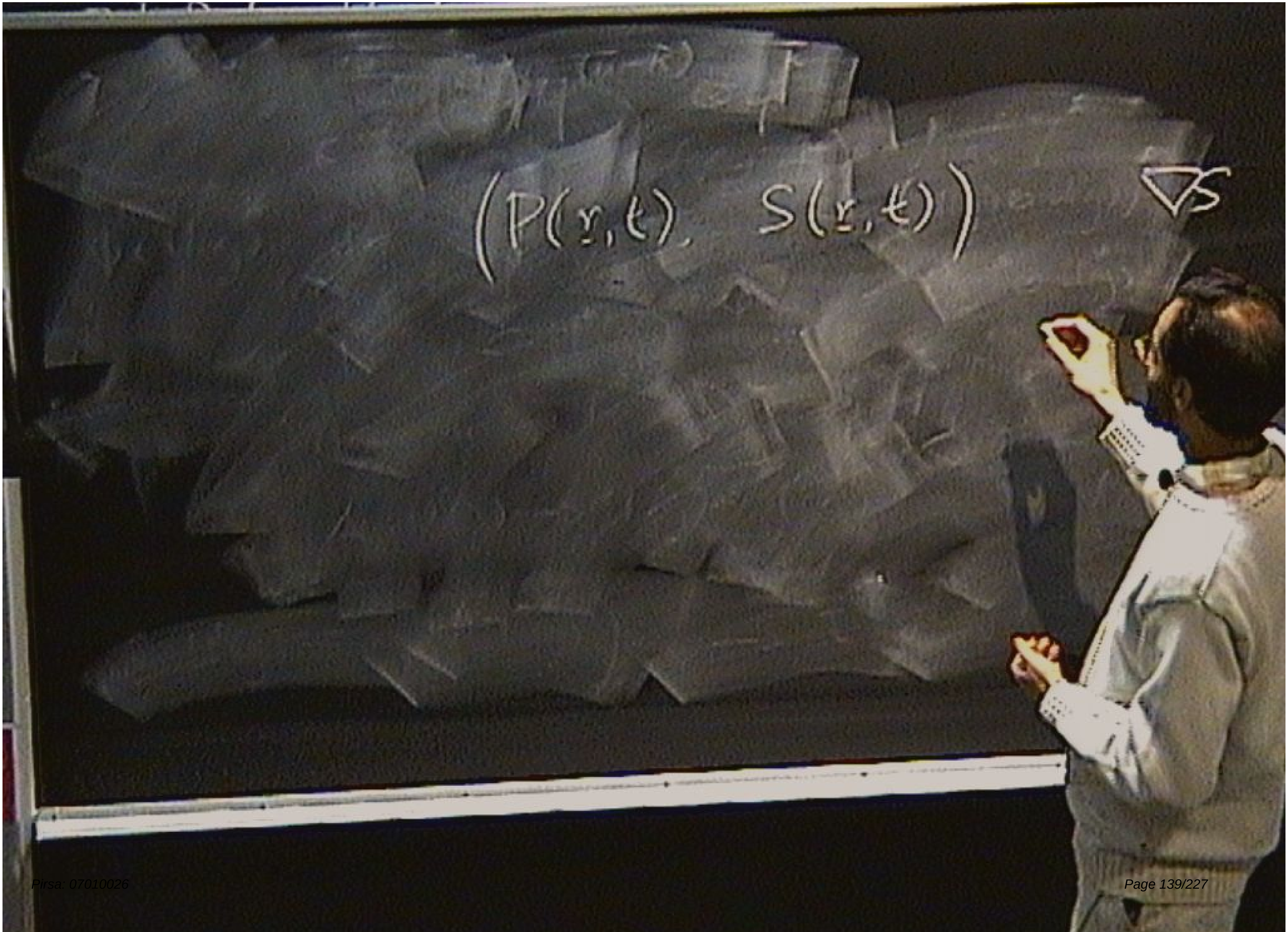
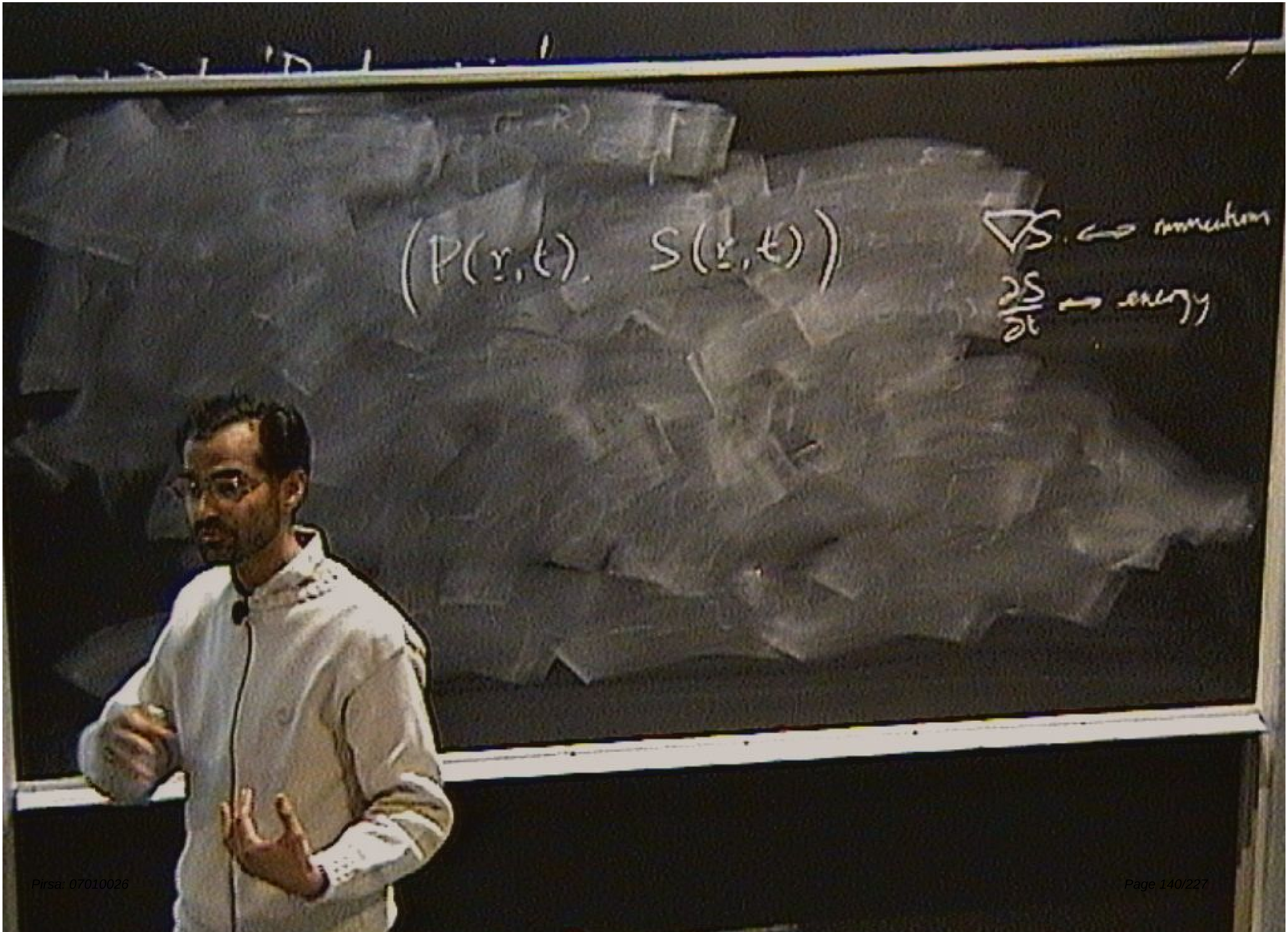

$\left(P(\underline{r},t), \quad S(\underline{r},t) \right)$
∇S


(P(r,t)   S(r,t))
∇S ↔ momentum
∂S/∂t ↔ energy

Postulate 2.4

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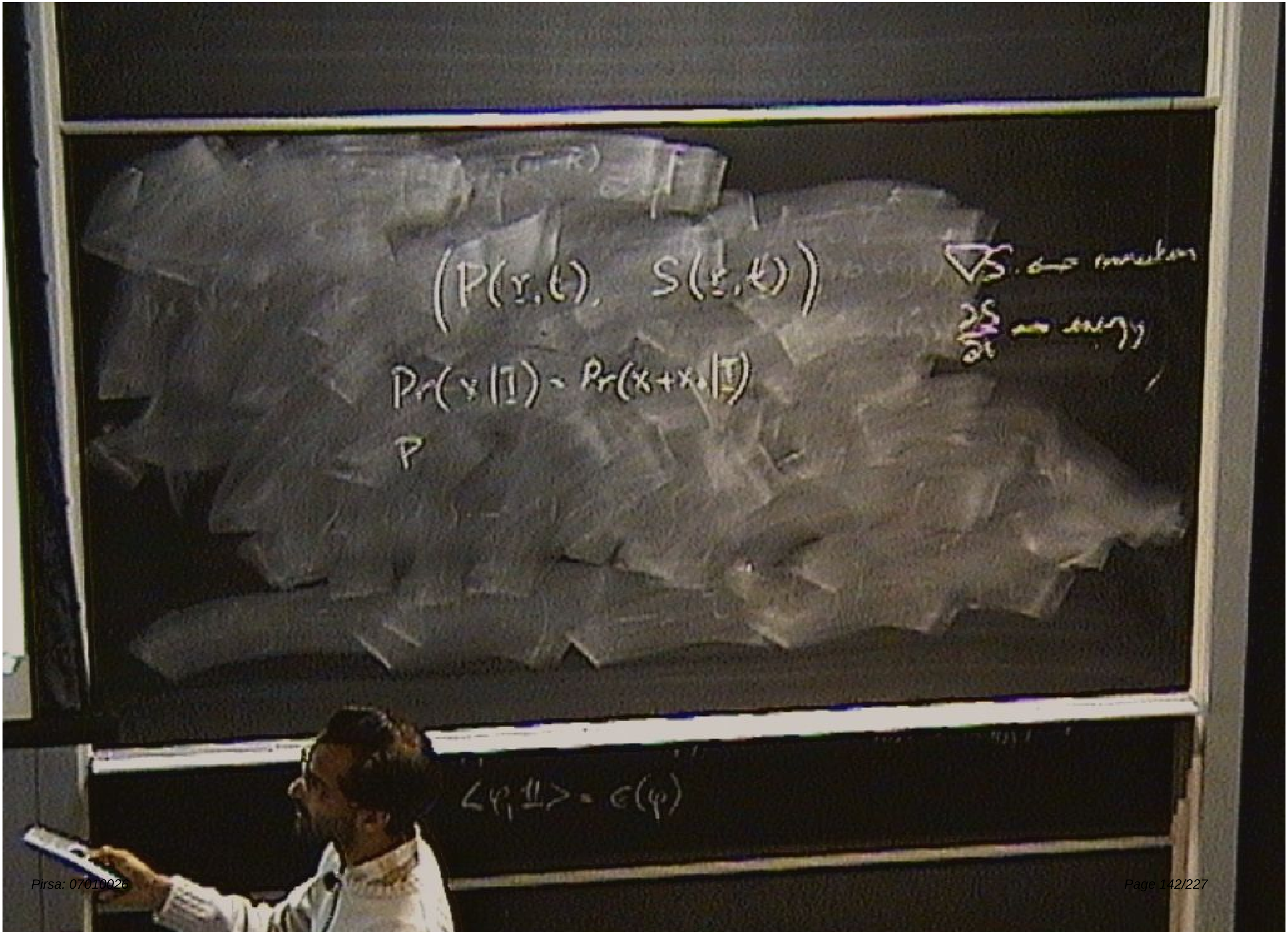
$(P(x,t),\ S(x,t))$
∇S ~ momentum
$\frac{\partial S}{\partial t}$ ~ energy
$Pr(x|I) = Pr(x+x_0|I)$
P
$\langle \varphi, 1\!\!1 \rangle = \epsilon(\varphi)$

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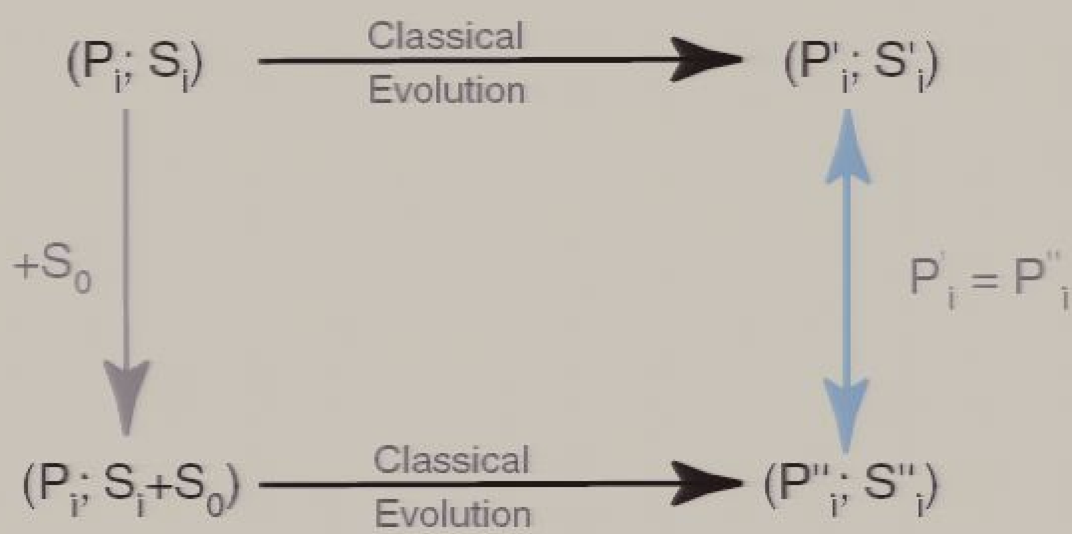
2.4 *Prior probabilities.* The prior probability, $\Pr(\chi_i|I)$, where I is the background knowledge of the experimenter prior to performing measurement **A**, is uniform.

- In the discretized Hamilton-Jacobi model with state (P_i, S_i), the zero-value of the S_i is physically irrelevant.
- This implies that $\Pr(S_1, \ldots, S_N) = \Pr(S_1 + S_0, \ldots, S_N + S_0)$ for all S_0 and all S_i.
- Therefore, $\Pr(S_i|I)$ is uniform.
- Under the classical–quantum correspondence $\chi_i = S_i/\alpha$, it follows that $\Pr(\chi_i|I)$ is uniform.

Postulate 3.2

- In the discretized Hamilton-Jacobi model with state (P_i, S_i), one can add S_0 to each of the S_i without changing the predictions of the model.



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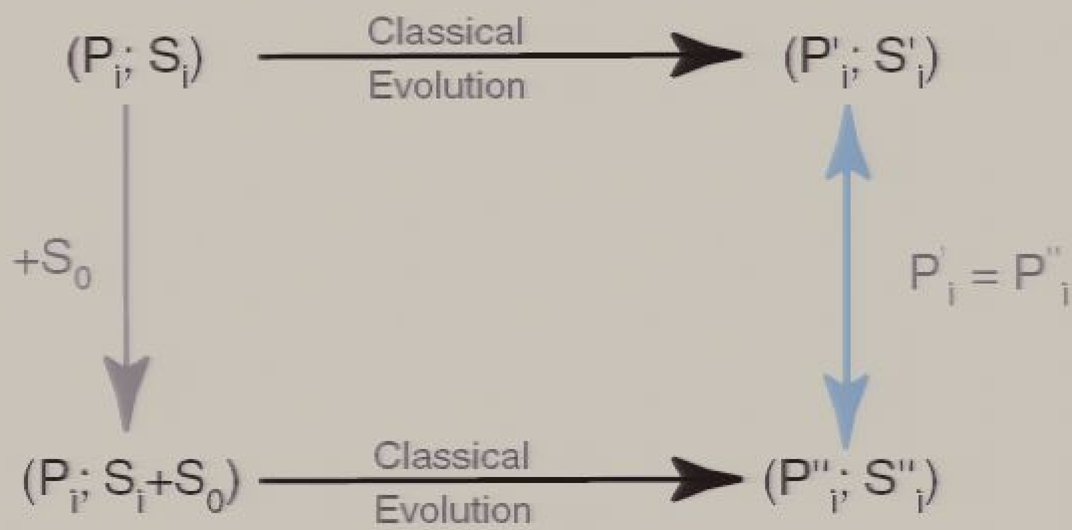
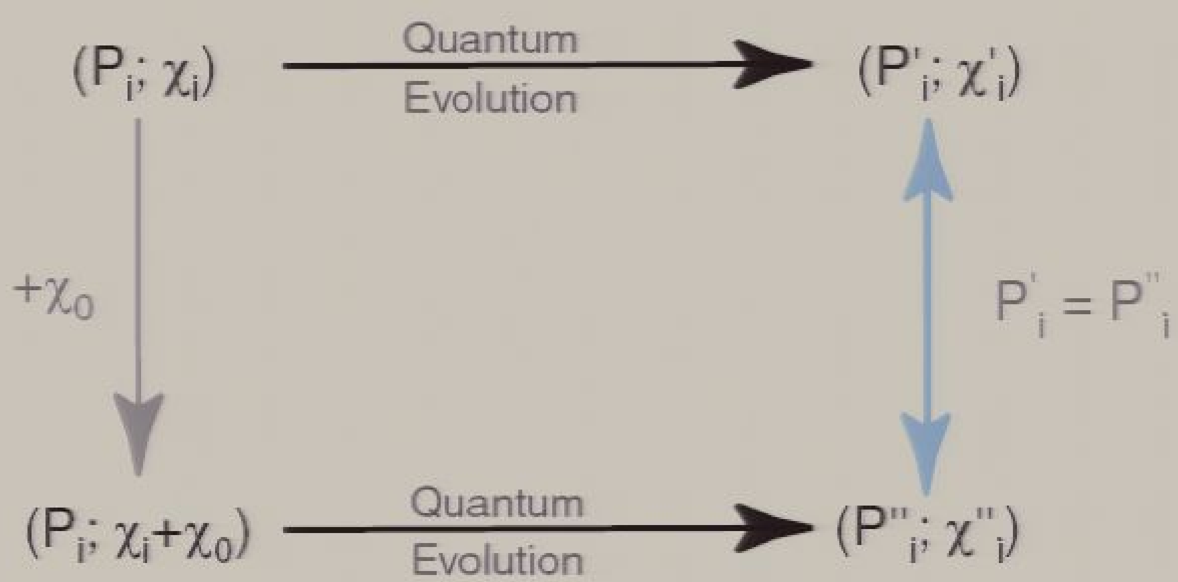


Postulate 3.2

- Under the classical–quantum correspondence $\chi_i = S_i/\alpha$, this implies that one can add χ_0 to each of the χ_i in the quantum state $(P_i; \chi_i)$ without changing the predictions.



Postulate 3.3

3 Transformations.

3.3 *Parameterized Transformations.* If a map, $\mathcal{M}_{\boldsymbol{\pi}}$, represents a physical transformation that is continuously dependent upon the real-valued parameter vector $\boldsymbol{\pi}$, then $\mathcal{M}_{\boldsymbol{\pi}}$ is continuously dependent upon $\boldsymbol{\pi}$. If the physical transformation is a continuous transformation, then, for some value of $\boldsymbol{\pi}$, $\mathcal{M}_{\boldsymbol{\pi}}$ reduces to the identity.

- Adopted unchanged from classical physics.

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- **Example 1:** A reflection-rotation of a frame of reference through an angle θ is continuously dependent upon θ. The map, $\mathcal{M}_\theta$, that represents the transformation is therefore continuously dependent upon θ.
- **Example 2:** A rotation of a frame of reference through an angle θ is a continuous transformation. For the value $\theta = 0$, the map, $\mathcal{M}_\theta$, representing the transformation, reduces to the identity.

Postulate 4

4. Consistency. The posterior probability distributions over state space that result from the following two processes coincide in the limit as n tends to infinity:
   (i) inferring a posterior over state space based upon the objectively realized outcomes when the measurement $\mathbf{A}$ is performed upon n copies of a system in state $\mathbf{S}$, and then transforming the posterior under the map $\mathcal{M}$, or
   (ii) inferring a posterior over state space based upon the objectively realized outcomes when the measurement $\mathbf{A}$ is performed upon n copies of a system in state $\mathcal{M}(\mathbf{S})$.

Postulate 4

$$
\begin{tikzcd}
S \arrow[r, "\text{Map M}"] \arrow[d, "\text{measurement}"'] & \mathrm{M}(S) \arrow[d, "\text{measurement}"] \\
\mathrm{p}(S) \arrow[r, "\text{Map M}^*"'] & \mathrm{p}'(S)
\end{tikzcd}
$$

Postulate 5

5. Composite Systems. Suppose that a system admits a quantum model with respect to the measurement set $\mathcal{A}^{(1)}$ whose measurements have $N^{(1)}$ possible observable outcomes, and admits a quantum model with respect to the measurement set $\mathcal{A}^{(2)}$ whose measurements have $N^{(2)}$ possible observable outcomes, where the sets $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ are disjoint.

   Consider the quantum model of the system with respect to the measurement set $\mathcal{A} = \mathcal{A}^{(1)} \times \mathcal{A}^{(2)}$. If the states of the sub-systems are represented as $(P_i^{(1)}; \chi_i^{(1)})$ $(i = 1, 2, \ldots, N^{(1)})$ and $(P_j^{(2)}; \chi_j^{(2)})$ $(j = 1, 2, \ldots, N^{(2)})$, respectively, then the state of the composite system can be represented as $(P_{ij}; \chi_{ij})$, where

$$P_{ij} = P_i^{(1)} P_j^{(2)},$$

$$\chi_{ij} = \chi_i^{(1)} + \chi_j^{(2)}.$$

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Postulate 5

- Suppose that, in the discretized Hamilton-Jacobi model of a particle, the state of a particle with respect to x- and y-position measurements is (P_i^x, S_i^x) and (P_i^y, S_i^y), respectively.

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$$(P_{ij}; S_{ij}) = (P_i P_j; S_i^x + S_j^y).$$

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- Under the classical–quantum correspondence $\chi_i = S_i/\alpha$, this implies, if the states of two disjoint models of a system are $(P_i^{(1)}; \chi_i^{(1)})$ and $(P_i^{(2)}; \chi_i^{(2)})$, then the state of the composite system is

$$(P_{ij}; \chi_{ij}) = (P_i^{(1)} P_j^{(2)}; \chi_i^{(1)} + \chi_j^{(2)}).$$

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Main steps in the deduction of the formalism (1)

Step 1. Implement the Principle of Information Gain.

Step 2. Represent $\mathbf{S}(t)$ as a unit vector in a $2N$-dimensional Euclidean space.

Step 3. Find the set of possible mappings of state space which represent physical transformations.

Step 4. Obtain a representation of measurements as Hermitian operators.

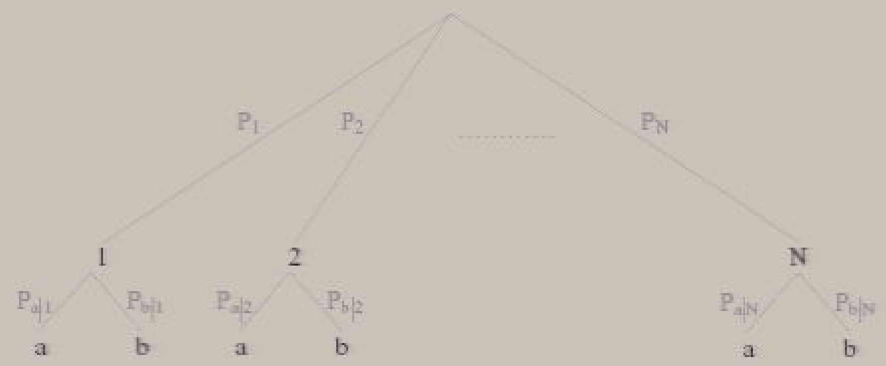
Step 5. Obtain the tensor product rule.

Main steps in the deduction of the formalism (2)

In later steps (not discussed here):

- Formulate the "Average-Value Correspondence Principle" (AVCP)
- Derive the temporal evolution operator.
- Derive the formal rules (such as operator rules and commutation relations) needed to apply the abstract quantum formalism.

Step 1: Implement the Principle of Information Gain



The probabilistic source which models measurement **A** being performed has the probability n-tuple

$$
\begin{aligned}
\tilde{\mathbf{P}} &= (\tilde{P}_1, \tilde{P}_2, \ldots, \tilde{P}_{2N-1}, \tilde{P}_{2N}) \\
&= (P_1 P_{a|1}, P_1 P_{b|1}, \ldots, P_N P_{a|N}, P_N P_{b|N})
\end{aligned}
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Step 1. Implement the Principle of Information Gain.

- Postulate 2.3 (Information Gain) implies that, for any $\tilde{\mathbf{P}}$ the amount of information obtained about $\tilde{\mathbf{P}}$ in n runs is independent of $\tilde{\mathbf{P}}$ in the limit as $n \to \infty$.

Step 1. Implement the Principle of Information Gain.

- Postulate 2.3 (Information Gain) implies that, for any $\tilde{\mathbf{P}}$ the amount of information obtained about $\tilde{\mathbf{P}}$ in n runs is independent of $\tilde{\mathbf{P}}$ in the limit as $n \to \infty$.
- To implement this condition, we need to examine the process by which information is gained about a probabilistic source.

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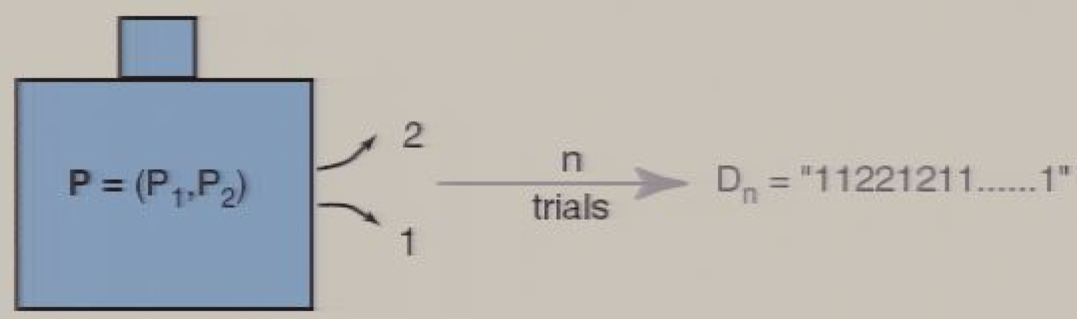
Gaining information from a probabilistic source

- A probabilistic source with two possible outcomes is interrogated n times, and a data string, D_n, is obtained.



Gaining information from a probabilistic source

1. From D_n, obtain the frequency, f_1, of outcome 1.

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Gaining information from a probabilistic source

1. From D_n, obtain the frequency, f_1, of outcome 1.
2. Using Bayes rule, calculate the posterior, $\Pr(P_1|f_1, n, I)$,

$$\Pr(P_1|f_1, n, I) = \frac{\Pr(f_1|P_1, n, I)\Pr(P_1|n, I)}{\Pr(f_1|n, I)},$$

where "I" symbolizes the knowledge of the experimenter before obtaining D_n.

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3. Calculate the information gain, ΔK:

$$\Delta K = (\text{Initial uncertainty about } P_1) - (\text{Final uncertainty about } P_1)$$

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The Shannon-Jaynes entropy

- We quantify the uncertainty using the Shannon-Jaynes entropy: the uncertainty in P_1 if ones knowledge about P_1 is given by the probability density function $f(P_1)$ is

$$\mathrm{H}\left[f(P_1)\right] = -\int f(P_1) \ln \frac{f(P_1)}{\Pr(P_1|I)}\, dP_1$$

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- Hence, the information gain is given by

$$\begin{aligned}\Delta K &= \mathrm{H}\left[\Pr(P_1|I)\right] - \mathrm{H}\left[\Pr(P_1|f_1, n, I)\right] \\ &= \int \Pr(P_1|f_1, n, I) \ln \frac{\Pr(P_1|f_1, n, I)}{\Pr(P_1|I)}\, dP_1\end{aligned}$$

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- The information gain is, as we would expect, dependent upon the prior, $\Pr(P_1|I)$. But what is $\Pr(P_1|I)$?

A uniform prior

- Let's try a uniform prior, $\Pr(P_1|I) = 1$. In that case, in the limit of large n, we find

$$\begin{aligned}\Delta K &= -\ln(\sigma\sqrt{2\pi e}) \\ &= \frac{1}{2}\ln\left(\frac{n}{2\pi e}\right) - \frac{1}{2}\ln\left(P_1\left(1 - P_1\right)\right).\end{aligned}$$

where $\sigma^2 = f_1(1 - f_1)/n$.

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- Hence, with the uniform prior, $\Pr(P_1|I) = 1$, the information gain depends upon the value of P_1.

- Is there a choice of $\Pr(P_1|I)$ such that the information gain is independent of P_1?

A special prior

- One finds that the information gain is independent of P_1 if and only if

$$\Pr(P_1|I) = \frac{1}{\pi}\frac{1}{\sqrt{P_1(1-P_1)}}.$$

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- Readily generalizes to an M-outcome probabilistic source with probability n-tuple $\vec{P} = (P_1, \ldots, P_M)$:

$$\Pr(\vec{P}|I) = \frac{2}{A_{M-1}}\frac{1}{\sqrt{P_1 \ldots P_M}}\,\delta\left(1 - \sum_i P_i\right),$$

where A_{M-1} is the surface area of a unit M-ball.

Visualization of the special prior

- If we change variables to $(Q_1, Q_2) = (\sqrt{P_1}, \sqrt{P_2})$, we find that the prior $\Pr(Q_1, Q_2|I)$ is constant on $(Q_1^2 + Q_2^2) = 1$ for $Q_1 \geq 0$ and $Q_2 \geq 0$.

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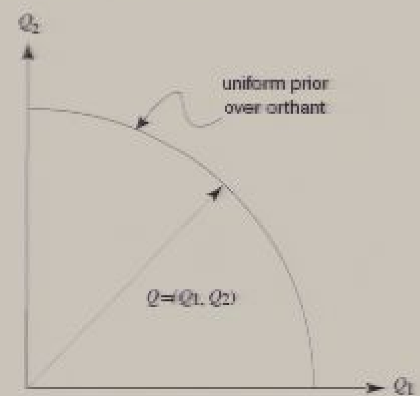
- If we define a two-dimensional Euclidean space, Q^2, then we can equivalently represent and visualize the prior as:



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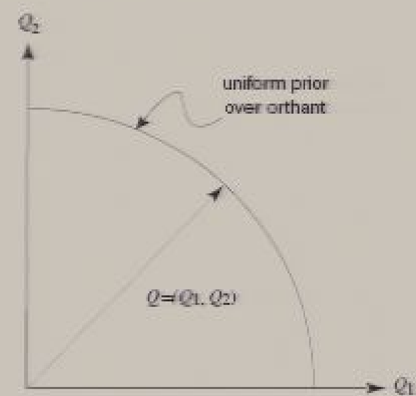

- The posterior, for large n, is a Gaussian with standard deviation $1/2\sqrt{n}$ over the positive quadrant.

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- Similarly, for an M-outcome probabilistic source, we define Q-space to be an M-dimensional Euclidean space. Then we can represent the prior $\Pr(\mathbf{Q}|I)$ as a uniform prior over the positive orthant of the unit hypersphere.

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Implications of the Principle of Information Gain

- Let us represent $\tilde{\mathbf{P}} = (\tilde{P}_1, \ldots, \tilde{P}_{2N})$ as a vector in a $2N$ dimensional Euclidean space:

$$\begin{aligned} \mathbf{Q} &= (Q_1, Q_2, \ldots, Q_{2N}) \\ &= (\sqrt{\tilde{P}_1}, \sqrt{\tilde{P}_2}, \ldots, \sqrt{\tilde{P}_{2N}}). \end{aligned}$$

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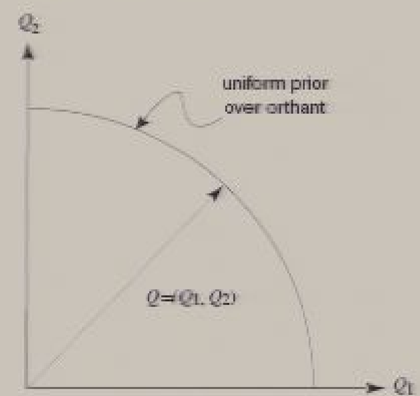

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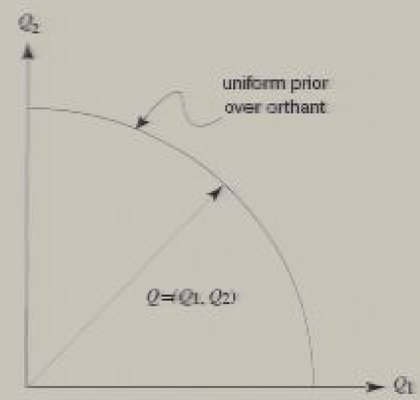

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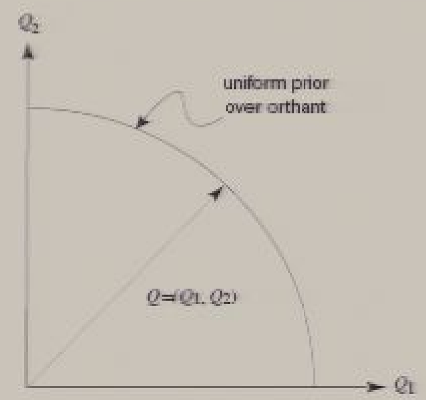

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Step 2. Represent $\mathbf{S}(t)$ in a $2N$-dimensional Euclidean space.

- If we allow *all* $\mathbf{Q}$ on the unit hypersphere, then the $2N$ signs of the $Q_{a|i}$ and the $Q_{b|i}$ are encoded by the *orthant* containing $\mathbf{Q}$.

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- Using Postulate 2.4, namely $Pr(\chi_i|I)$ is uniform, and using the relations

$$Q_{a|i} = f(\chi_i) \quad \text{and} \quad Q_{b|i} = \tilde{f}(\chi_i),$$

we find that

$$f(\chi_i) = \cos\chi_i \quad \text{and} \quad \tilde{f}(\chi_i) = \sin\chi_i.$$

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- The prior is uniform over S^{2N-1}.
- The posterior consists of a symmetric Gaussian, with standard deviation $\sigma = 1/2\sqrt{n}$, over one orthant, and zero in all other orthants.

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2. **Postulate 4**: the possible transformations are orthogonal transformations.

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2. **Postulate 4**: the possible transformations are orthogonal transformations.
3. **Postulate 3.2**: the possible transformations are a particular subset of the orthogonal transformations.

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2. **Postulate 4**: the possible transformations are orthogonal transformations.
3. **Postulate 3.2**: the possible transformations are a particular subset of the orthogonal transformations.
4. **Represent state space in complex form**: the set of possible transformations is the set of unitary and antiunitary transformations.

Implementation of Postulate 4

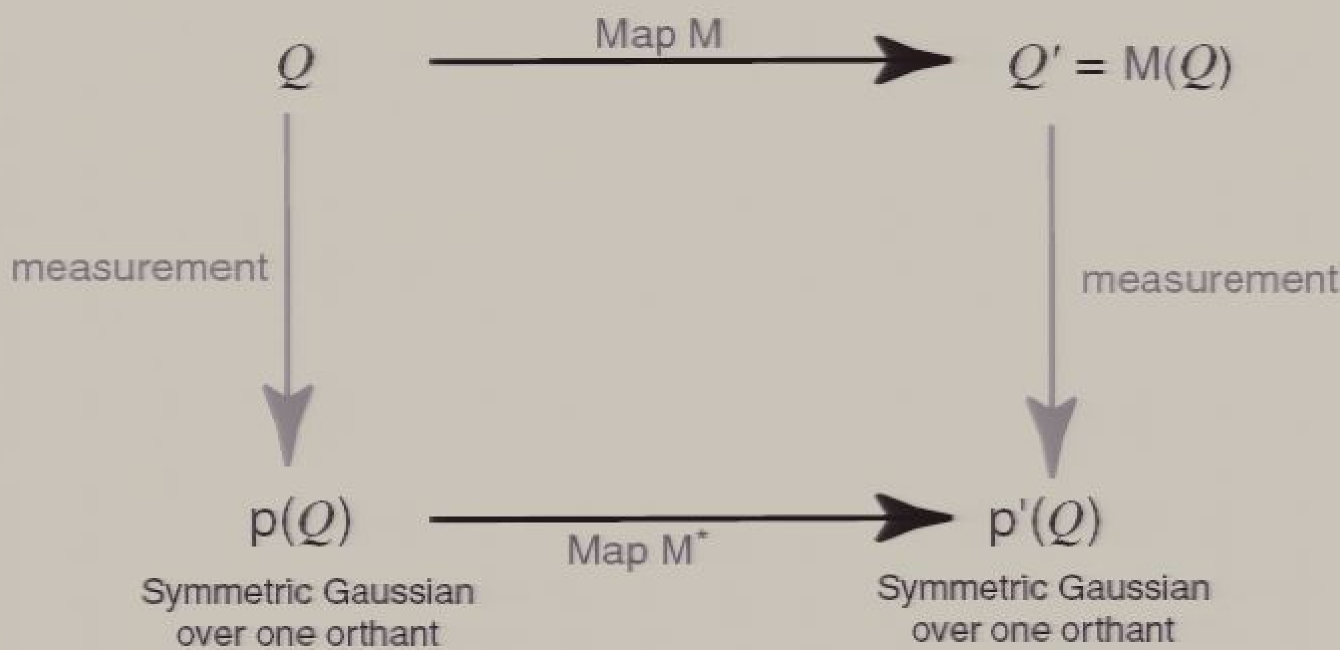
- Under the map $\mathcal{M}$, over S^{2N-1}, the uniform prior transforms into the probability density function $\tilde{p}(\mathbf{Q}')$ given by

$$\tilde{p}(\mathbf{Q}') = \Pr(\mathbf{Q}|\mathrm{I}) \left| \frac{\partial(Q'_1, \ldots, Q'_{2N})}{\partial(Q_1, \ldots, Q_{2N})} \right|^{-1}, \tag{1}$$

  where $\mathbf{Q} = (Q_1, \ldots, Q_{2N})$ and $\mathbf{Q}' = (Q'_1, \ldots, Q'_{2N})$ and $\mathbf{Q}' = \mathcal{M}(\mathbf{Q})$.

- However, since no measurement has been performed, the prior must remain unchanged. Therefore, the Jacobian is unity.

Implementation of Postulate 4



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Implementation of Postulate 4

$$\begin{tikzcd}[column sep=huge, row sep=huge]
Q \arrow[r, "\text{Map M}"] \arrow[d, "\text{measurement}"'] & Q' = \mathrm{M}(Q) \arrow[d, "\text{measurement}"] \\
\mathrm{p}(Q) \arrow[r, "\text{Map M}^*"'] & \mathrm{p}'(Q)
\end{tikzcd}$$

Symmetric Gaussian over one orthant (under $\mathrm{p}(Q)$)

Symmetric Gaussian over one orthant (under $\mathrm{p}'(Q)$)

One finds that any given map $\mathcal{M}$ must be an orthogonal transformation, M, of S^{2N-1}.

Implementation of Postulate 3.2

- Postulate 3.2 requires that the outcome probabilities $P'_1, \ldots, P'_N$ of measurement $\mathbf{A}$ performed on a system in state $\mathbf{Q}' = M\mathbf{Q}$ are unaffected by the addition of an arbitrary real constant to the χ_i where

$$\mathbf{Q} = (\sqrt{P_1}\cos\chi_1, \sqrt{P_1}\sin\chi_1, \ldots, \sqrt{P_N}\cos\chi_1, \sqrt{P_N}\sin\chi_1).$$

Implementation of Postulate 3.2

- One finds that this leads to the constraint that M has the form

$$M = \begin{pmatrix} T^{(11)} & T^{(12)} & \dots & T^{(1N)} \\ T^{(21)} & T^{(22)} & \dots & T^{(2N)} \\ \dots & \dots & \dots & \dots \\ T^{(N1)} & T^{(N2)} & \dots & T^{(NN)} \end{pmatrix},$$

where

$$T^{(ij)} = \sqrt{\alpha_{ij}} \begin{pmatrix} \cos\varphi_{ij} & -\sigma_{ij}\sin\varphi_{ij} \\ \sin\varphi_{ij} & \sigma_{ij}\cos\varphi_{ij} \end{pmatrix}$$

and where all the non-zero T sub-matrices are either all scale-rotation or all scale-reflection-rotation matrices.

Complex representation of state space

- Let us represent $\mathbf{Q} = (Q_1, \ldots, Q_{2N})$ as a unit vector v,

$$\mathsf{v} = \begin{pmatrix} Q_1 + iQ_2 \\ Q_3 + iQ_4 \\ \cdots \\ Q_{2N-1} + iQ_{2N} \end{pmatrix}$$

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$$\mathsf{v}' = \mathsf{V}\mathsf{v}.$$

- If we set $\mathsf{V}_{ij} = \sqrt{\alpha_{ij}} \exp(i\varphi_{ij})$, then the resulting transformation is identical to that produced by M with the non-zero $T^{(ij)}$ being scale-rotations.

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- Similarly, the transformation produced by VK, where K is the complex conjugation operation, is identical to that produced by M with the non-zero $T^{(ij)}$ being scale-reflection-rotations.

Complex representation of state space

- Similarly, the transformation produced by VK, where K is the complex conjugation operation, is identical to that produced by M with the non-zero $T^{(ij)}$ being scale-reflection-rotations.
- The converse is also true: every unitary or antiunitary transformation can be cast in the form of M satisfying Postulate 3.2.

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Complex representation of state space

N - dimensional complex vector space		$2N$ - dimensional real Q-space
States are unit vectors, $\mathbf{v}$. Transformations are the set of all **unitary** or **antiunitary** transformations of the unit hypersphere	⟷	States are unit vectors, Q. Transformations are a subset of the **orthogonal** transformations of the unit hypersphere

Step 4: Obtain a representation of measurements

- After measurement **A** has been performed and outcome i observed, what is the state of the system?

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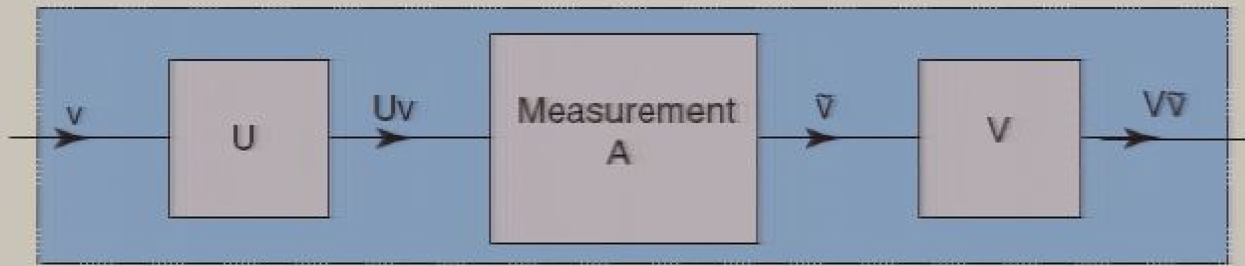
Step 4: Obtain a representation of measurements

- From Postulate 1.3, a measurement $\mathbf{A}'$ can be represented in terms of measurement $\mathbf{A}$ as follows:



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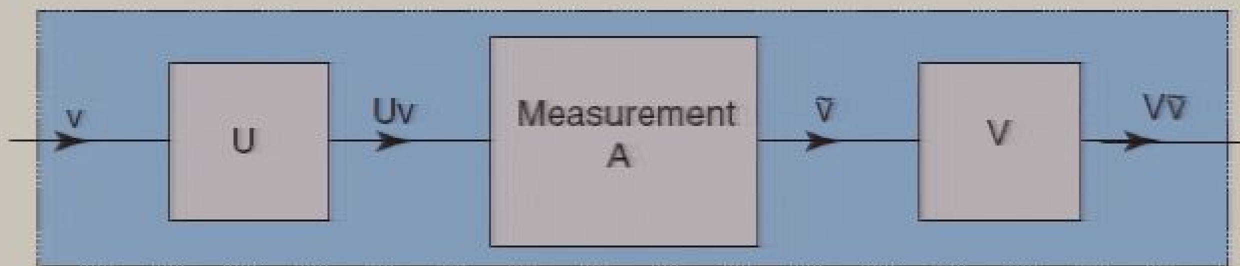

where U and V are unitary transformations.

- Suppose that measurement $\mathbf{A}'$ performed on state v'_i yields outcome i with certainty. Then, in the above arrangement, we require that, for all i,

$$\mathsf{U}\mathsf{v}'_i = \mathsf{v}_i e^{i\xi_i},$$

where ξ_i is arbitrary.

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- Since U is unitary, the v'_i also form an orthonormal basis.

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- An input state $\mathsf{v} = \sum_i c_i \mathsf{v}'_i$ therefore gets transformed into $\mathsf{U}\mathsf{v} = \sum_i c_i e^{i\xi_i} \mathsf{v}_i$.
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- Hence, measurement $\mathbf{A}$ yields outcome i with probability c_i^2.
- Hence, the v_i, together with the corresponding outcome values, a_i, of $\mathbf{A}'$, characterize measurement $\mathbf{A}'$, and can be represented by the Hermitian operator, $\mathsf{A} = \sum_i a_i \mathsf{v}'_i \mathsf{v}'^{\dagger}_i$.

Step 5: Obtain the tensor product rule

- Consider a composite system with two sub-systems with abstract models $\mathbf{q}(N^{(1)})$ and $\mathbf{q}(N^{(2)})$, respectively, where the composite system has the abstract model $\mathbf{q}(N)$.

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$$P_{ij} = P_i^{(1)} P_j^{(2)}$$
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- If we write the states of the sub-systems in complex form, then it follows that v can simply be written as $\mathsf{v}^{(1)} \otimes \mathsf{v}^{(2)}$.
- This is easily generalized to a composite system containing d sub-systems.

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Some General Remarks

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- We obtain a mathematical structure that is neither more nor less general than the finite-dimensional quantum formalism. Consequently, the derivation provides an excellent 'laboratory' for investigating proposed modifications or novel applications of the quantum formalism.
- Complex numbers in the quantum formalism appear to be directly connected with the fact that all possible physical transformations can be represented by unitary or antiunitary transformations. Both stem from the invariance postulate (Postulate 3.2).
- The concept of information plays a central role in the emergence of the quantum formalism. Specifically, it leads to:
  1. Q–space, which introduces real amplitudes,
  2. the sinusoidal functions f and $\tilde{f}$.
  3. the restriction that $\mathcal{M}$ is an orthogonal transformation of Q–space.

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- The derivation highlights the physical importance of the notion of a prior over a continuous parameter, which enters primarily via the Shannon-Jaynes entropy.

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- One can see rather clearly which assumptions quantum theory shares with classical physics, which are modifications of classical ideas, and which are novel. Information is the key new ingredient.

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- A measurement able to resolve a and b must have temporal resolution $\Delta t < \hbar/E$.
- But, using $\Delta E \Delta t \geq \hbar/2$, the energy associated with the measurement has uncertainty $\Delta E \geq \hbar/2\Delta t$. Hence, $\Delta E \geq E/2$.
- From $E = mc^2$, it follows that ΔE must be of the order of the rest energy of the system.
- But such a measurement would probably not preserve the identity of the system, as is required by the idealizations.
- Conversely, an acceptable measurement will have insufficient temporal resolution to resolve the outcomes a and b.

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