Title: Quantum Error Correction 3B

Date: Jan 23, 2007 05:00 PM

URL: http://pirsa.org/07010023

Abstract: Clifford group as symplectic group, generators of the Clifford group & encoding circuits for stabilizer codes, efficient simulation of Clifford group circuits, efficient simulation of Pauli measurements

Def: The Clifford group (or normalizer group) C_n on n qubits is $\{U \in U(2^n) \mid UPU^\dagger \in P_n \ \forall P \in P_n\}$

E.g.: Paulis $P, Q \in P_n$ $PQP^\dagger = (-1)^{P,Q} QPP^\dagger$ or $C_n / \{e^{i\theta} I\}$

Change Phases

Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$X \to Z$
$Z \to X$
$Y \to -Y$

$\frac{\pi}{4}$ phase gate $R = R_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$X \to Y$
$Y \to -X$
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CNOT: $X \otimes I \to X \otimes X$
$Z \otimes I \to Z \otimes I$
$I \otimes X \to I \otimes X$
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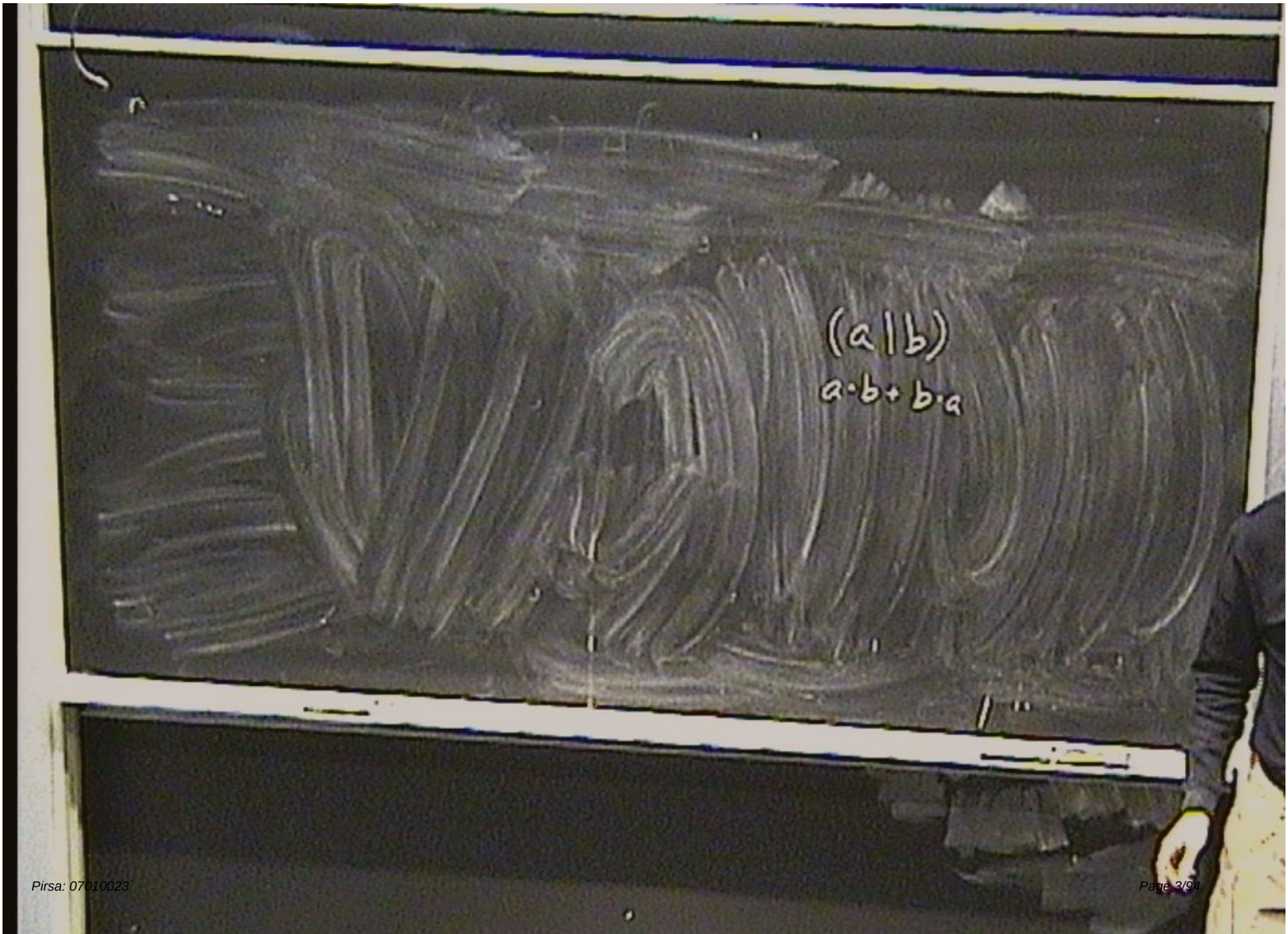
Permutations of qubits


(a|b)
a·b+b·a

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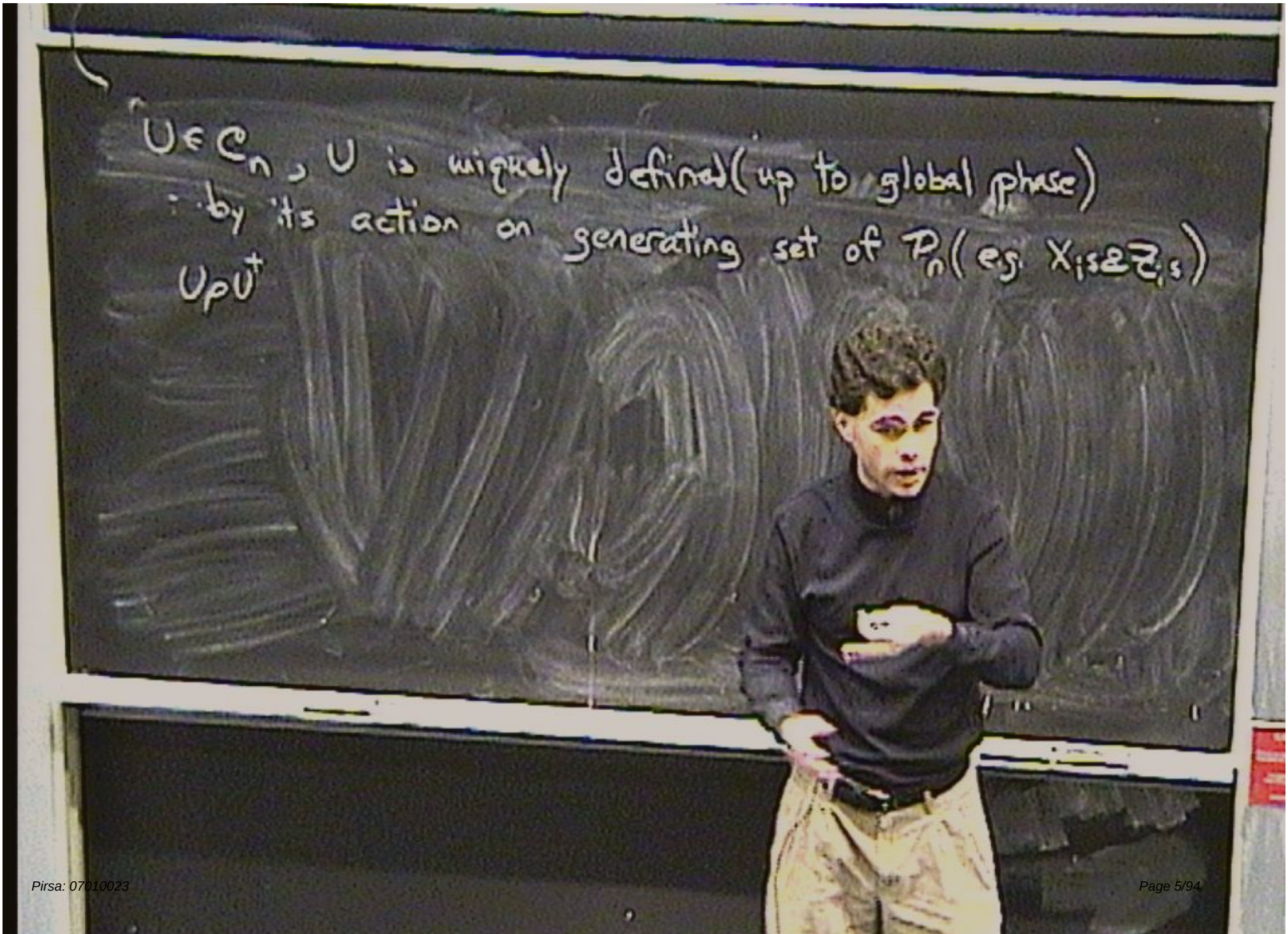
by its action on generating set of $\mathcal{P}_n$ (e.g. X_i's & Z_i's)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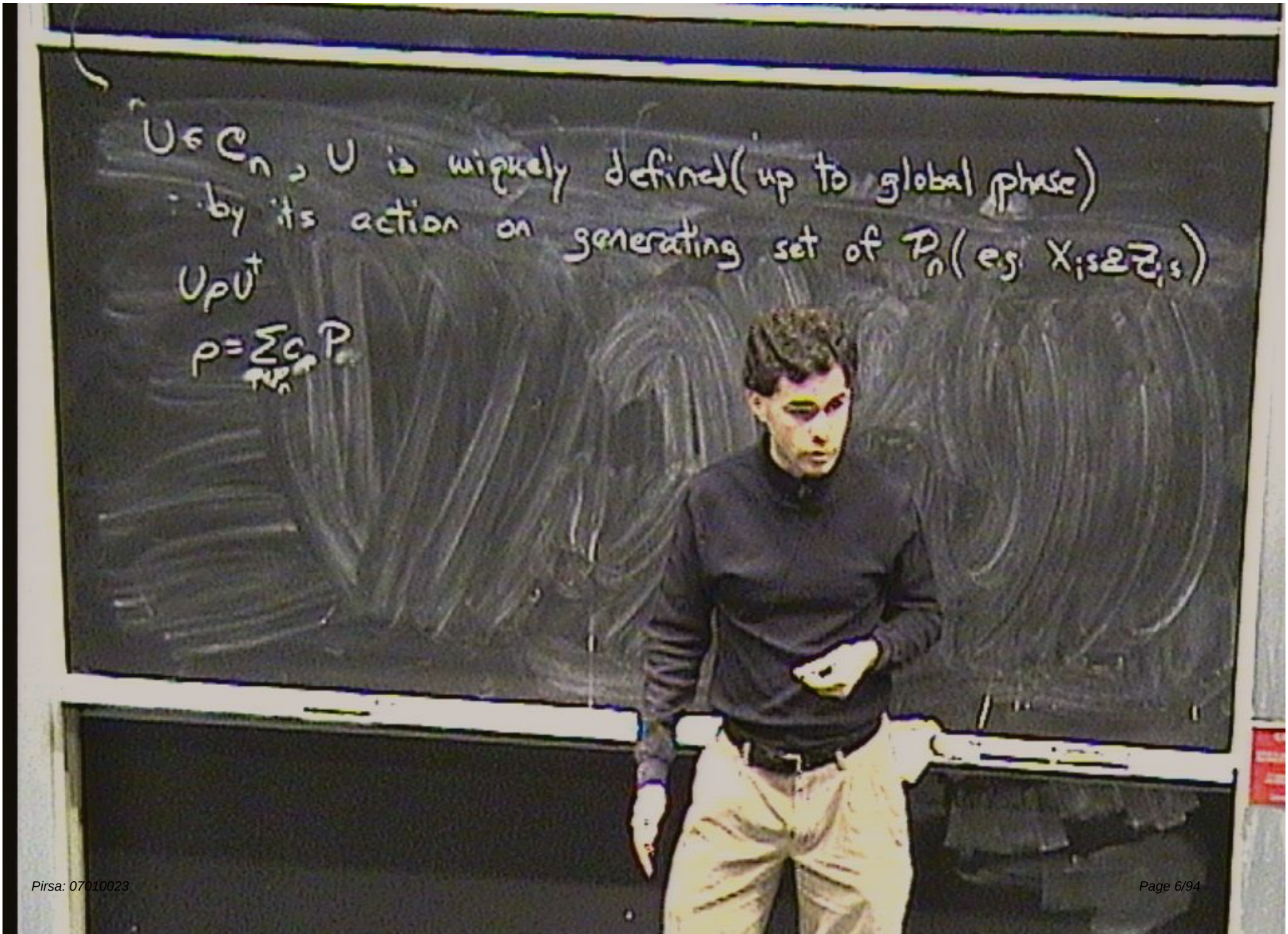
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$\rho = \sum_{P \in \mathcal{P}_n} c_P P$

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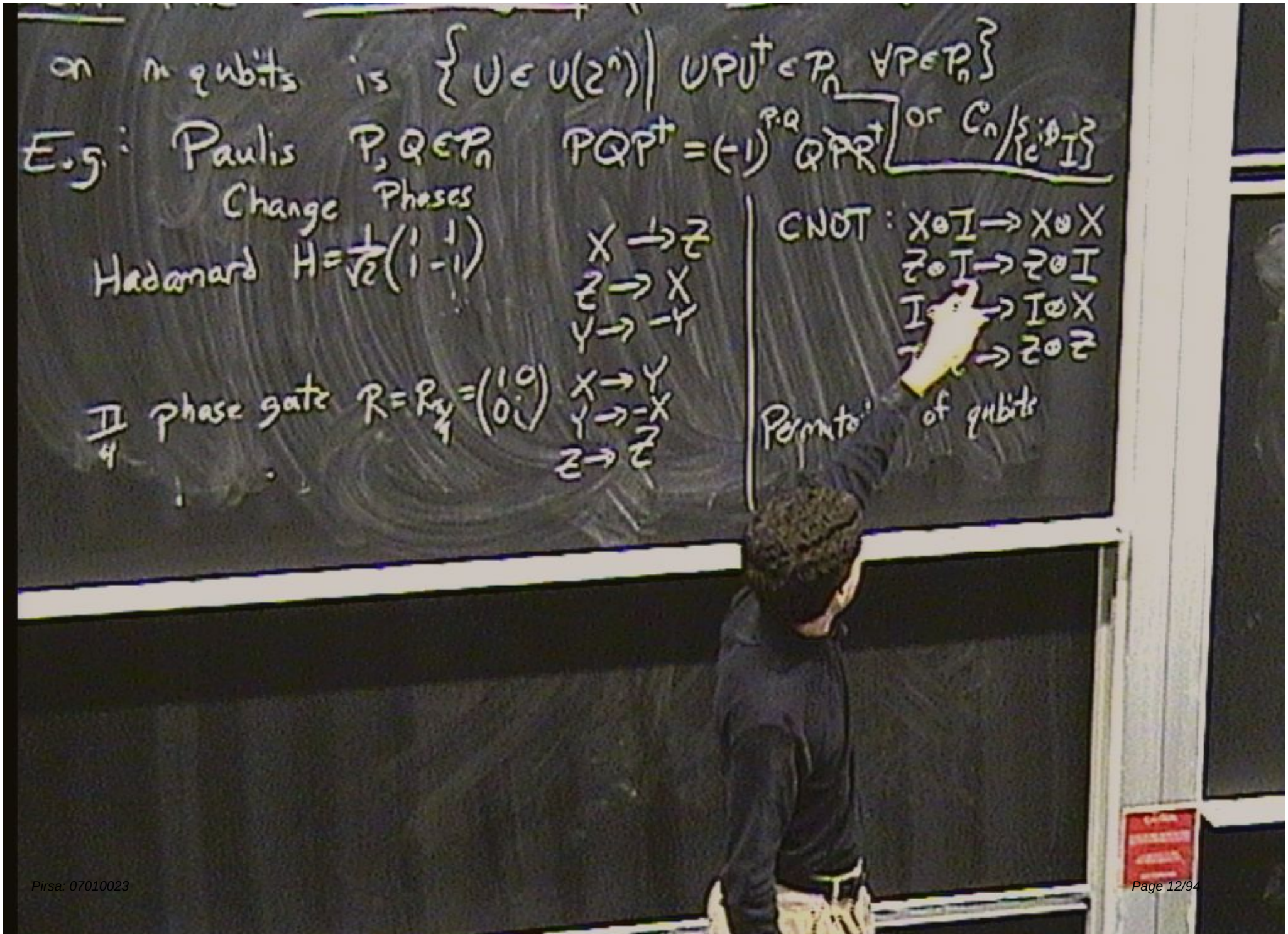
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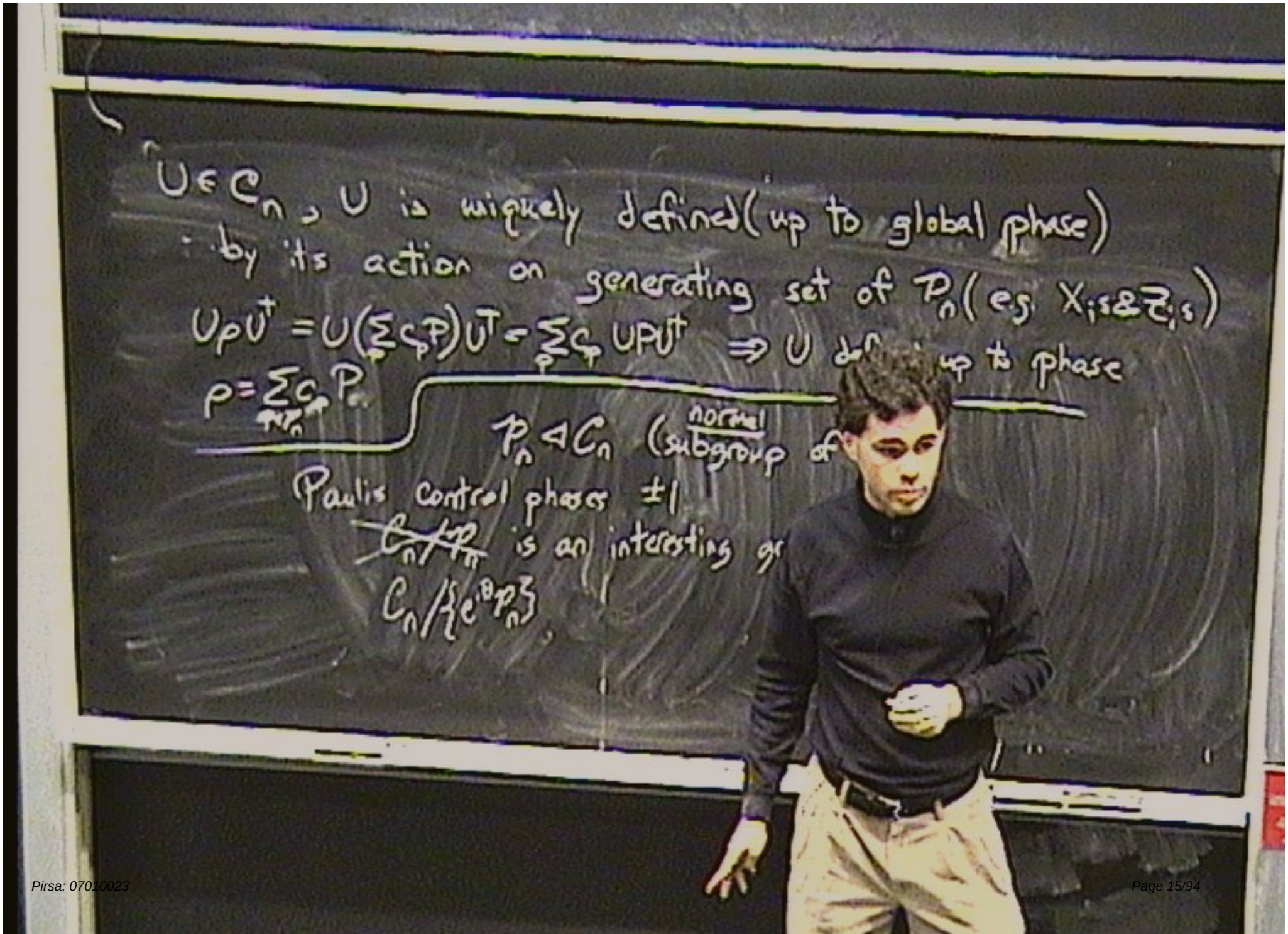
C_n / P_n is an interesting group too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Pauli's control phases ± 1

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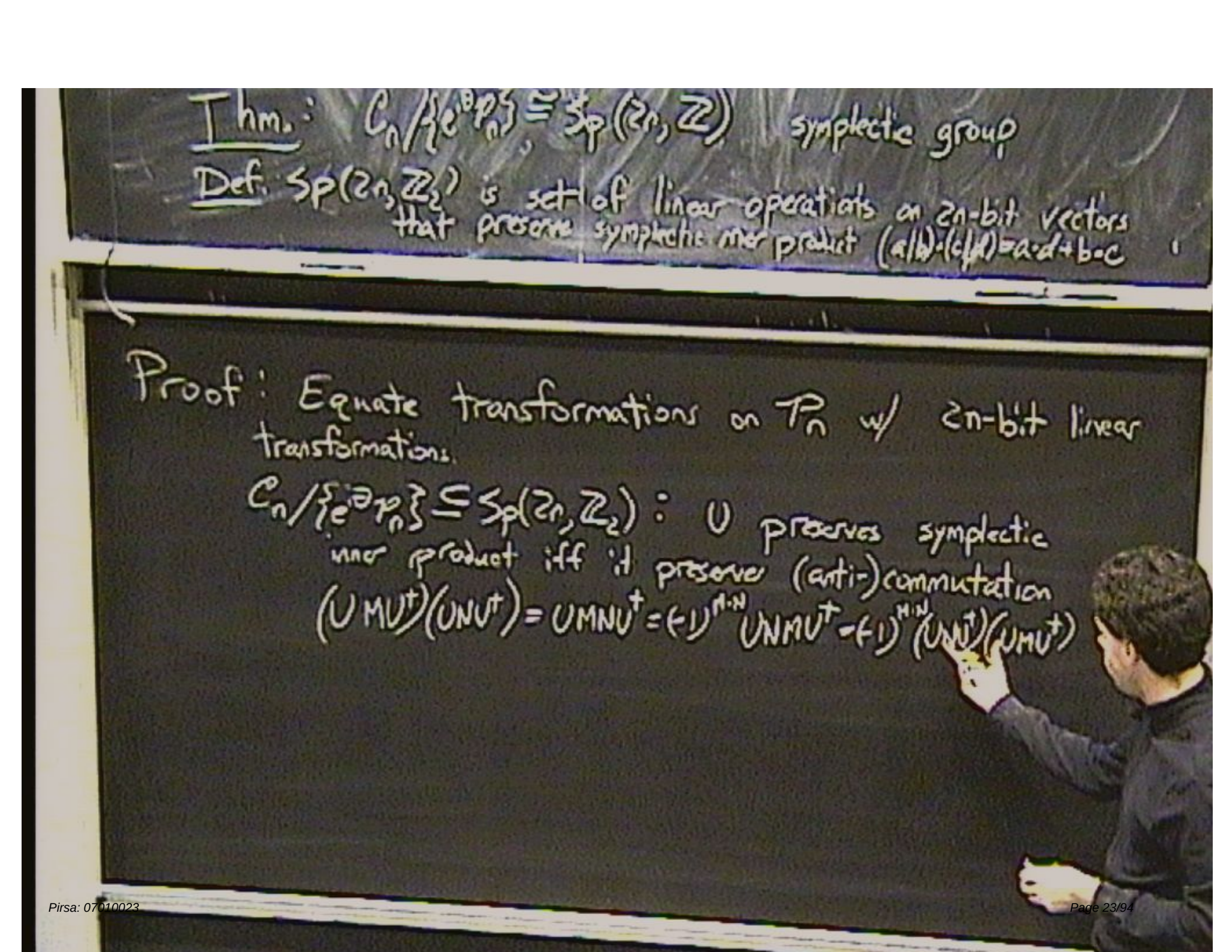

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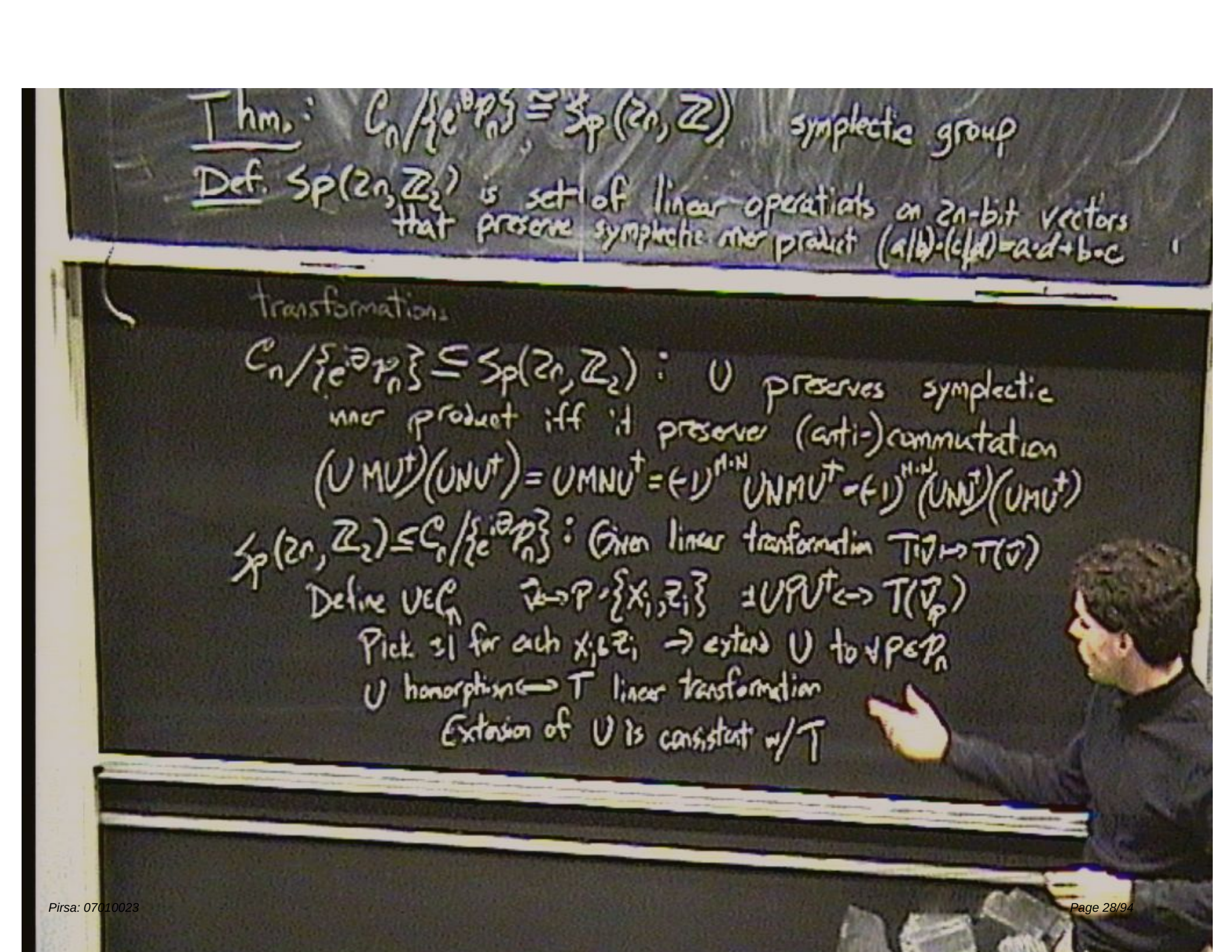
U homomorphism $\leftrightarrow$ T linear transformation

Extension of U is consistent w/ T



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$U \sim T$

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$U \sim T$ (same T)

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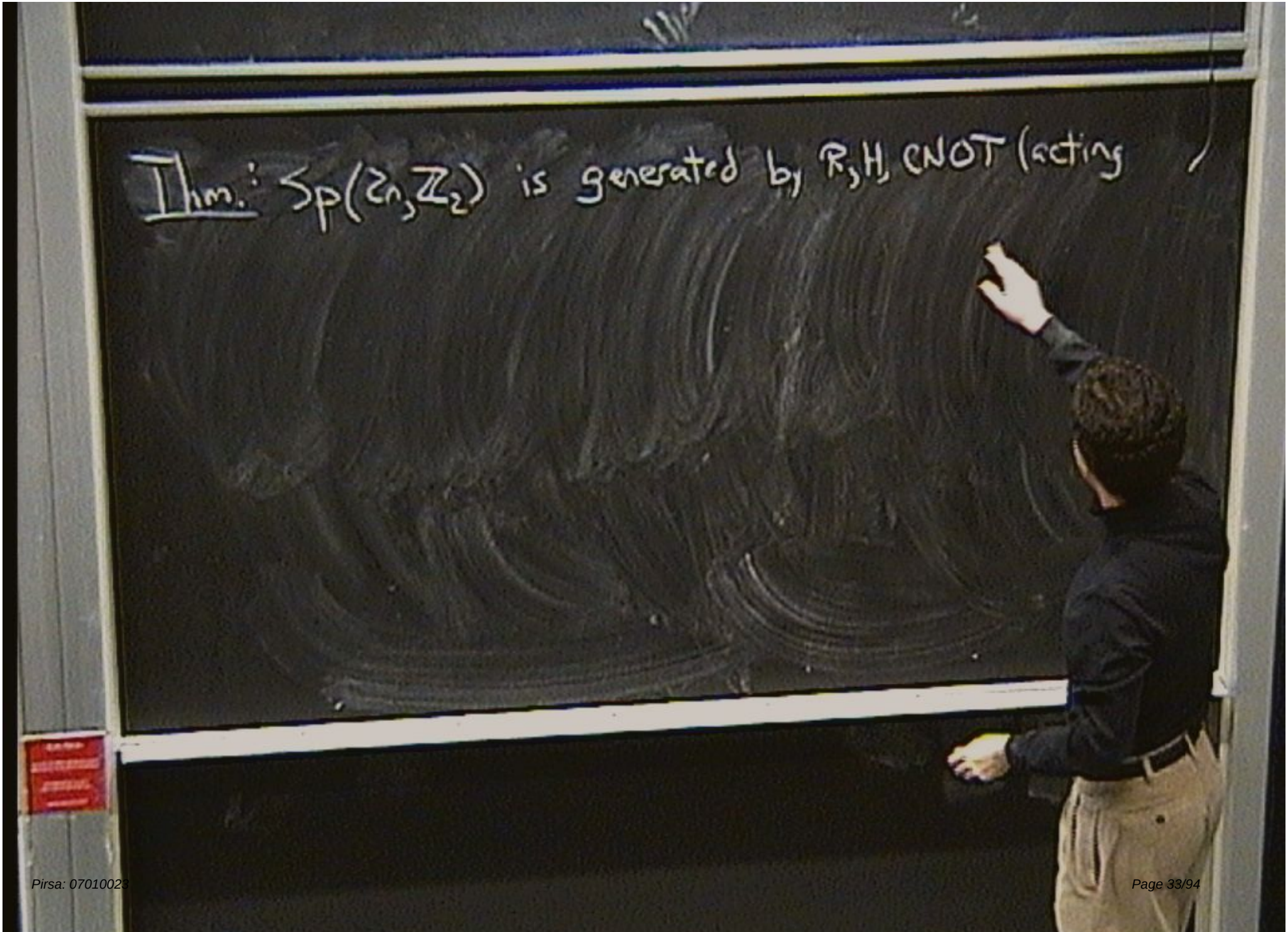
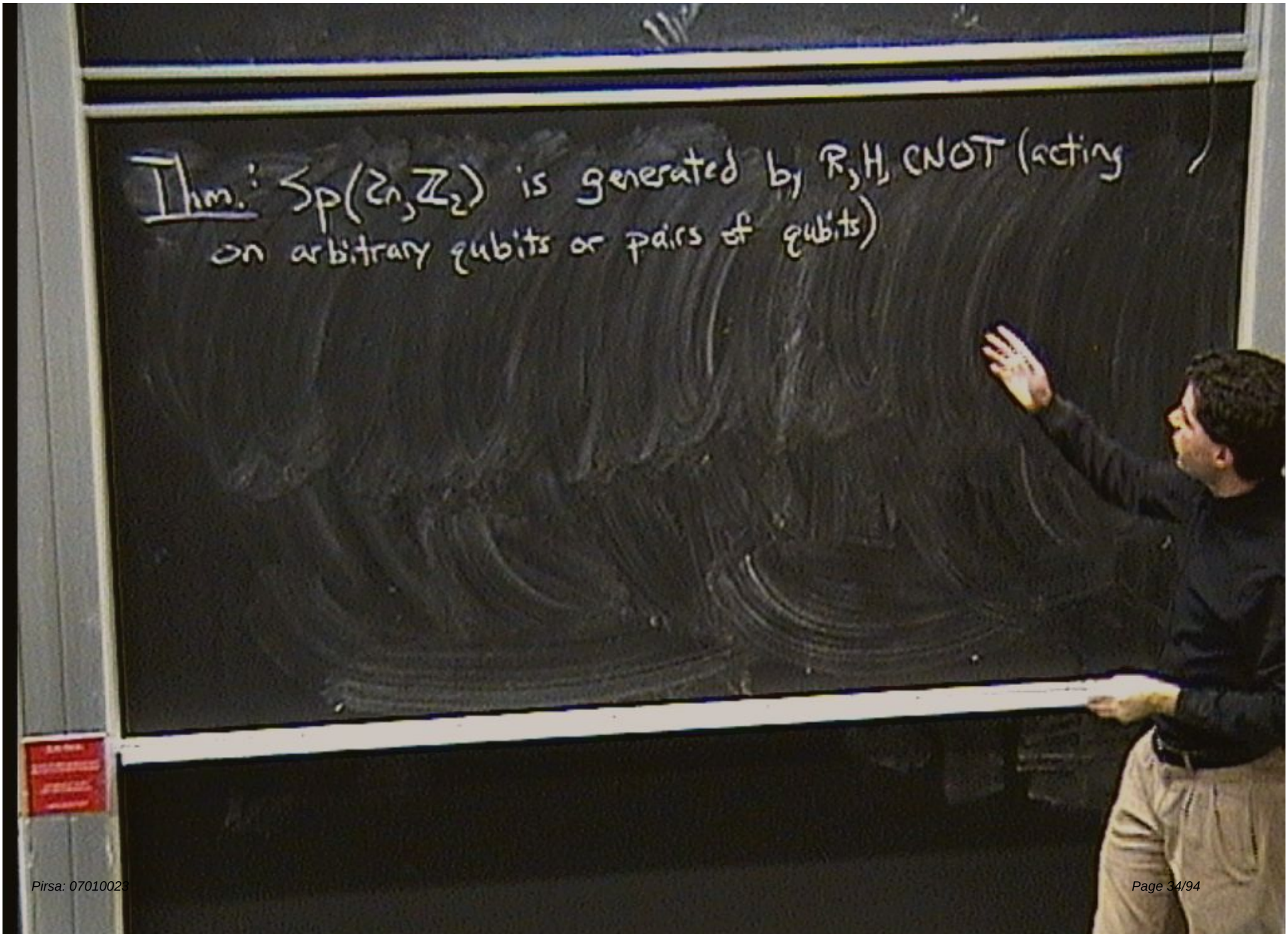
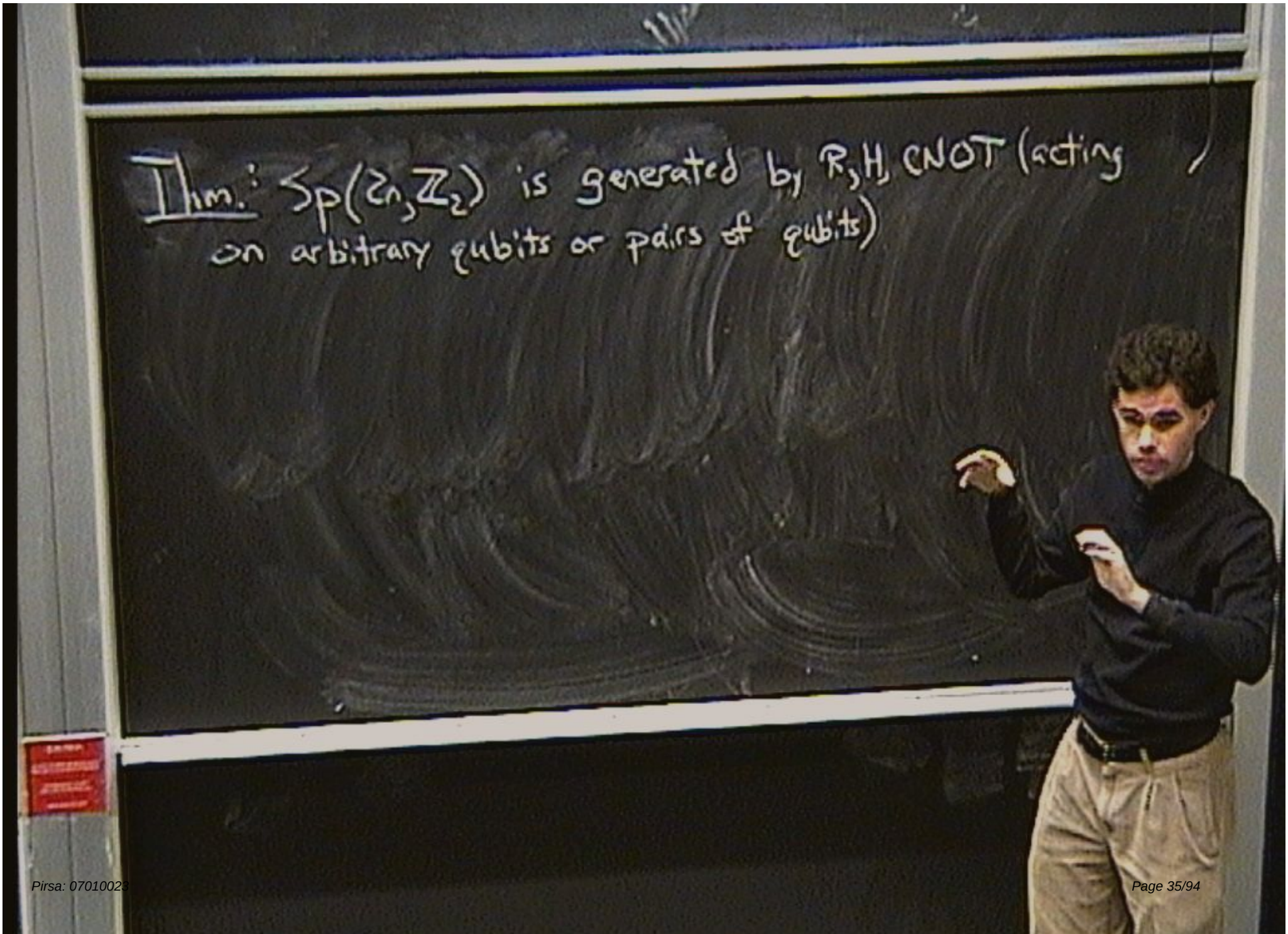

Thm.: $Sp(2n, \mathbb{Z}_2)$ is generated by R, H, CNOT (acting


Thm.: Sp(2n, Z2) is generated by R, H, CNOT (acting
on arbitrary qubits or pairs of qubits)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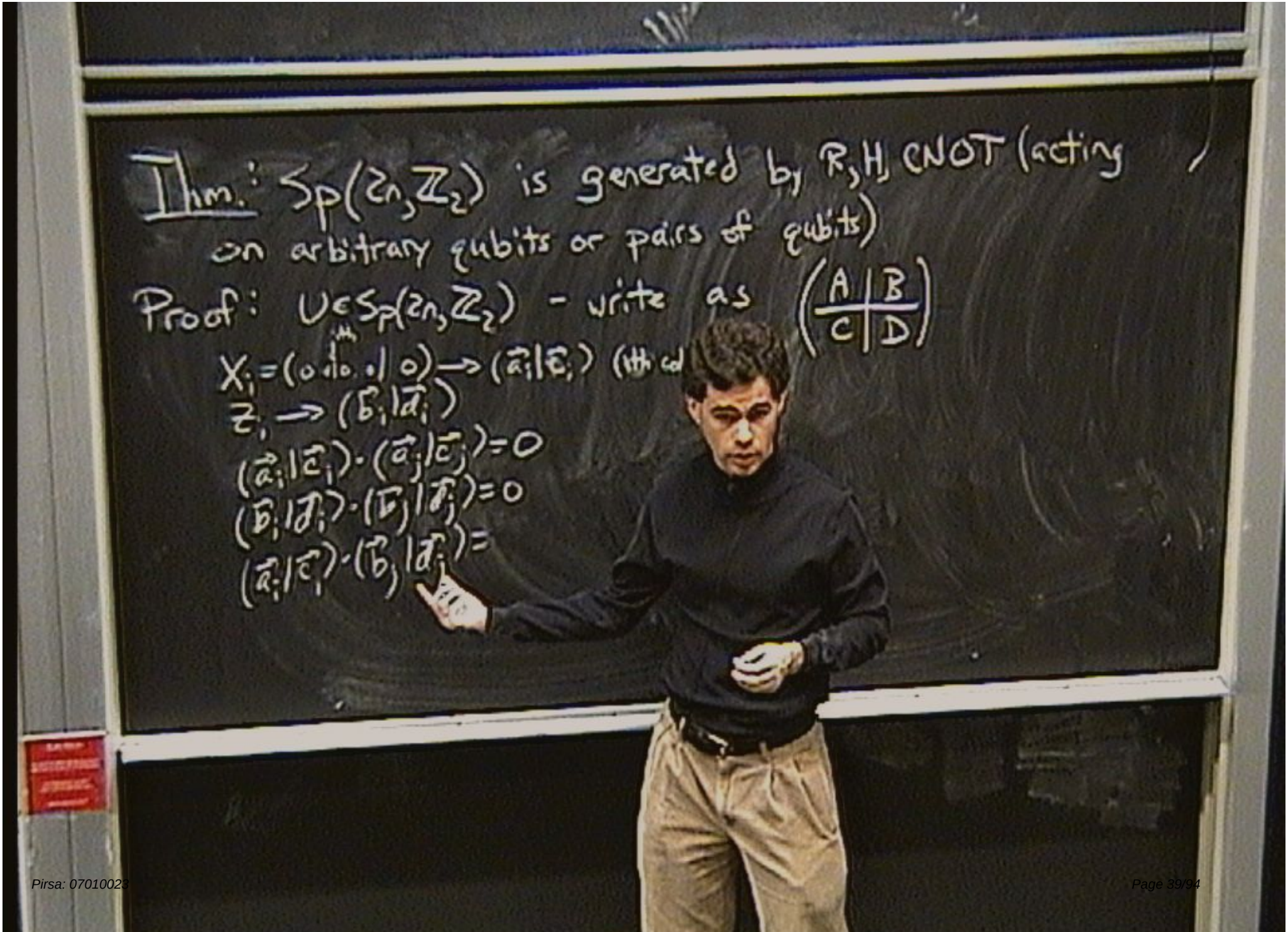
$X_i = (0 \ldots 0 \overset{i}{1} 0 \ldots 0 \,|\, 0) \to (\vec{a}_i | \vec{c}_i)$ (ith column of $A|C$)


Thm.: $Sp(2n, \mathbb{Z}_2)$ is generated by R, H, CNOT (acting on arbitrary qubits or pairs of qubits)
Proof: $U \in Sp(2n, \mathbb{Z}_2)$ - write as $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$
$X_i = (0 \ldots 0 \ldots 0 \mid 0) \to (\vec{a}_i \mid \vec{c}_i)$
$Z_i \to (\vec{b}_i \mid \vec{d}_i)$
$(\vec{a}_i \mid \vec{c}_i) \cdot (\vec{a}_j \mid \vec{c}_j) = 0$
$(\vec{b}_i \mid \vec{d}_i) \cdot (\vec{b}_j \mid \vec{d}_j) = 0$
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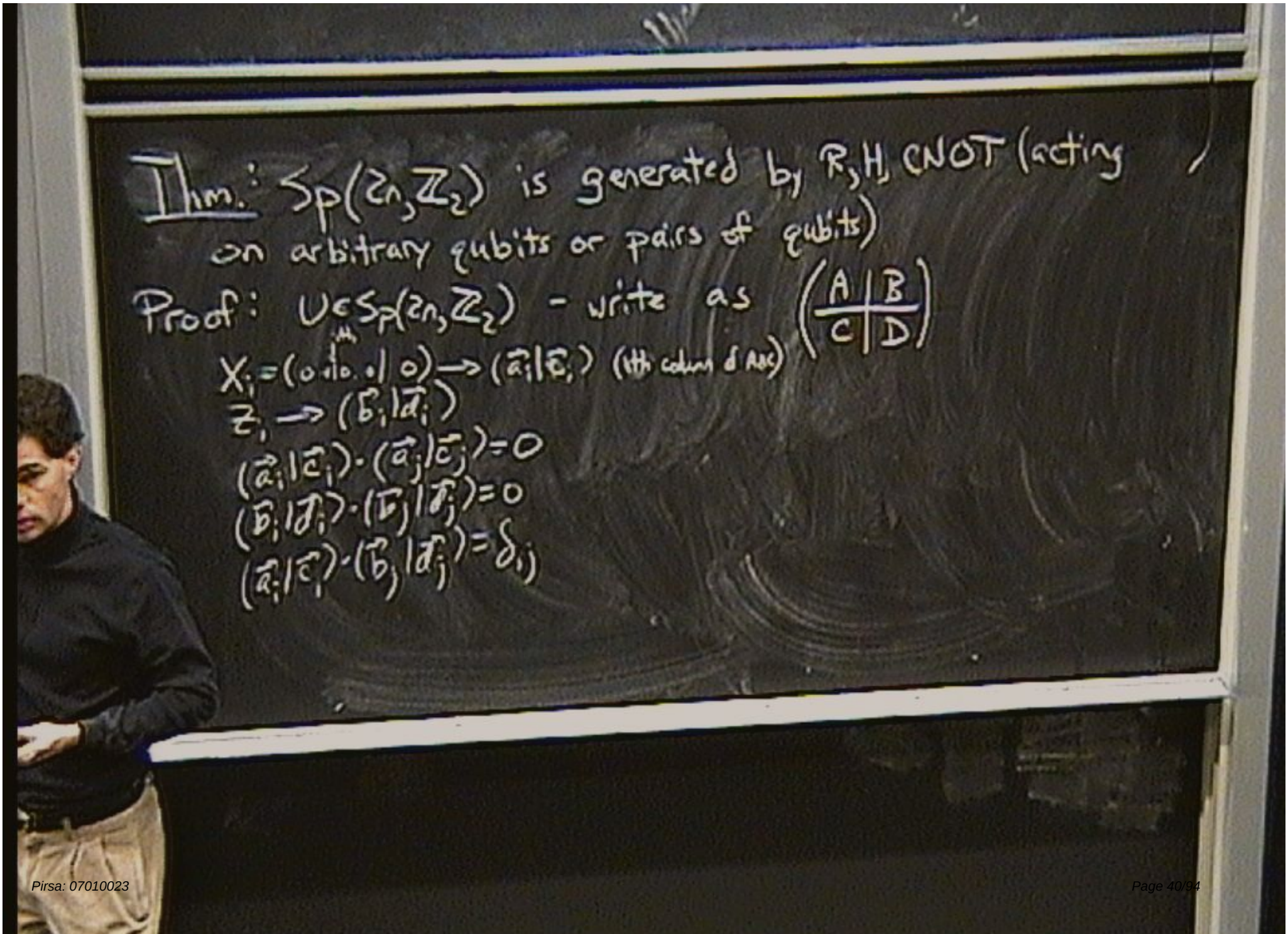
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$(\vec{a}_i \mid \vec{c}_i) \cdot (\vec{b}_j \mid \vec{d}_j) = \delta_{ij}$

Def: The Clifford group (or normalizer group) C_n on n qubits is $\{U \in U(2^n) \mid UPU^\dagger \in P_n \ \forall P \in P_n\}$

E.g.: Paulis $P, Q \in P_n$ $PQP^\dagger = (-1)^{P,Q} QPP^\dagger$ or $C_n / \{e^{i\theta} I\}$

Change Phases

Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

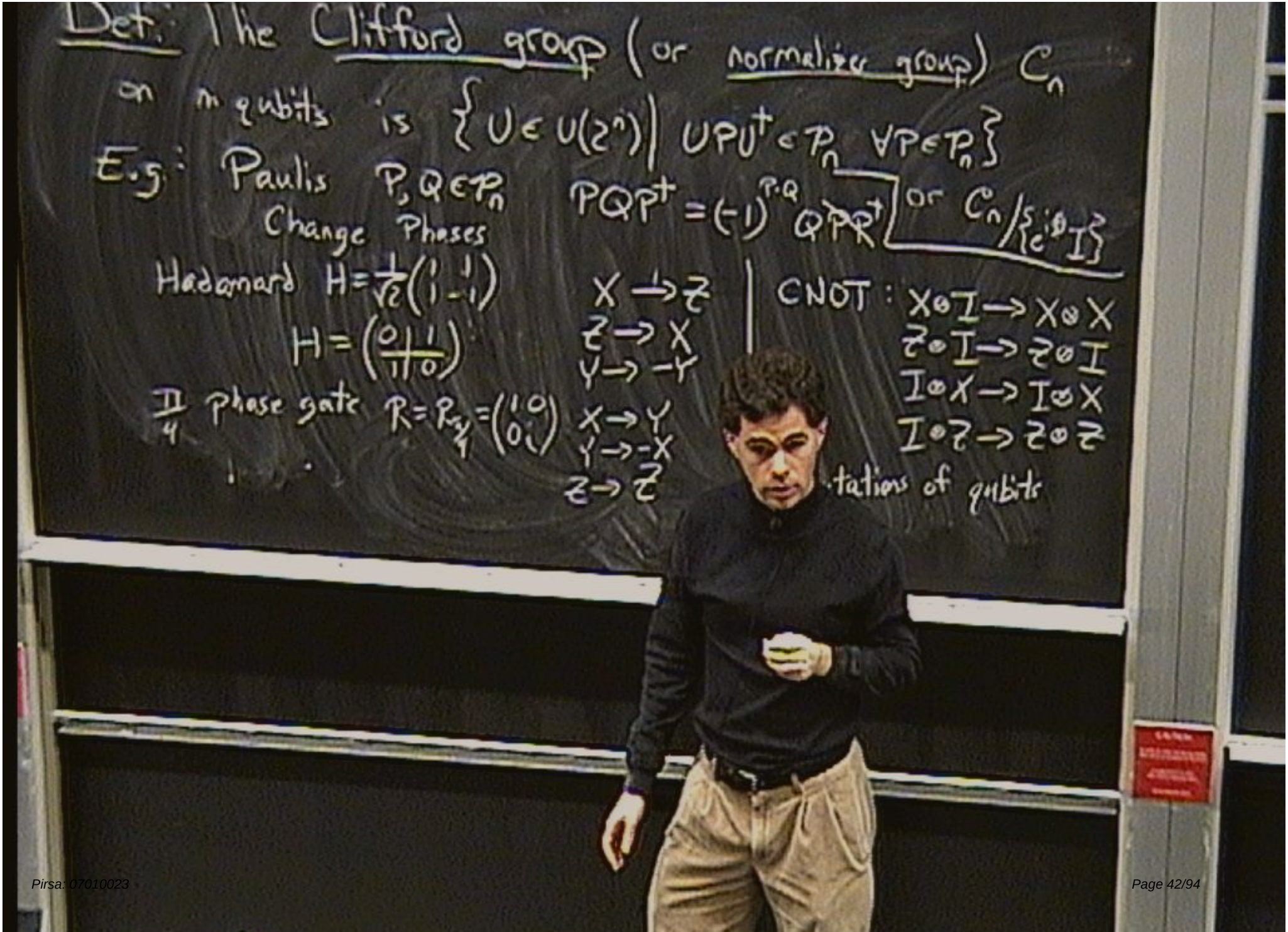
$X \to Z$
$Z \to X$
$Y \to -Y$

$\frac{\pi}{4}$ phase gate $R = R_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$X \to Y$
$Y \to -X$
$Z \to Z$

CNOT: $X \otimes I \to X \otimes X$
$Z \otimes I \to Z \otimes I$
$I \otimes X \to I \otimes X$
$I \otimes Z \to Z \otimes Z$

Permutations of qubits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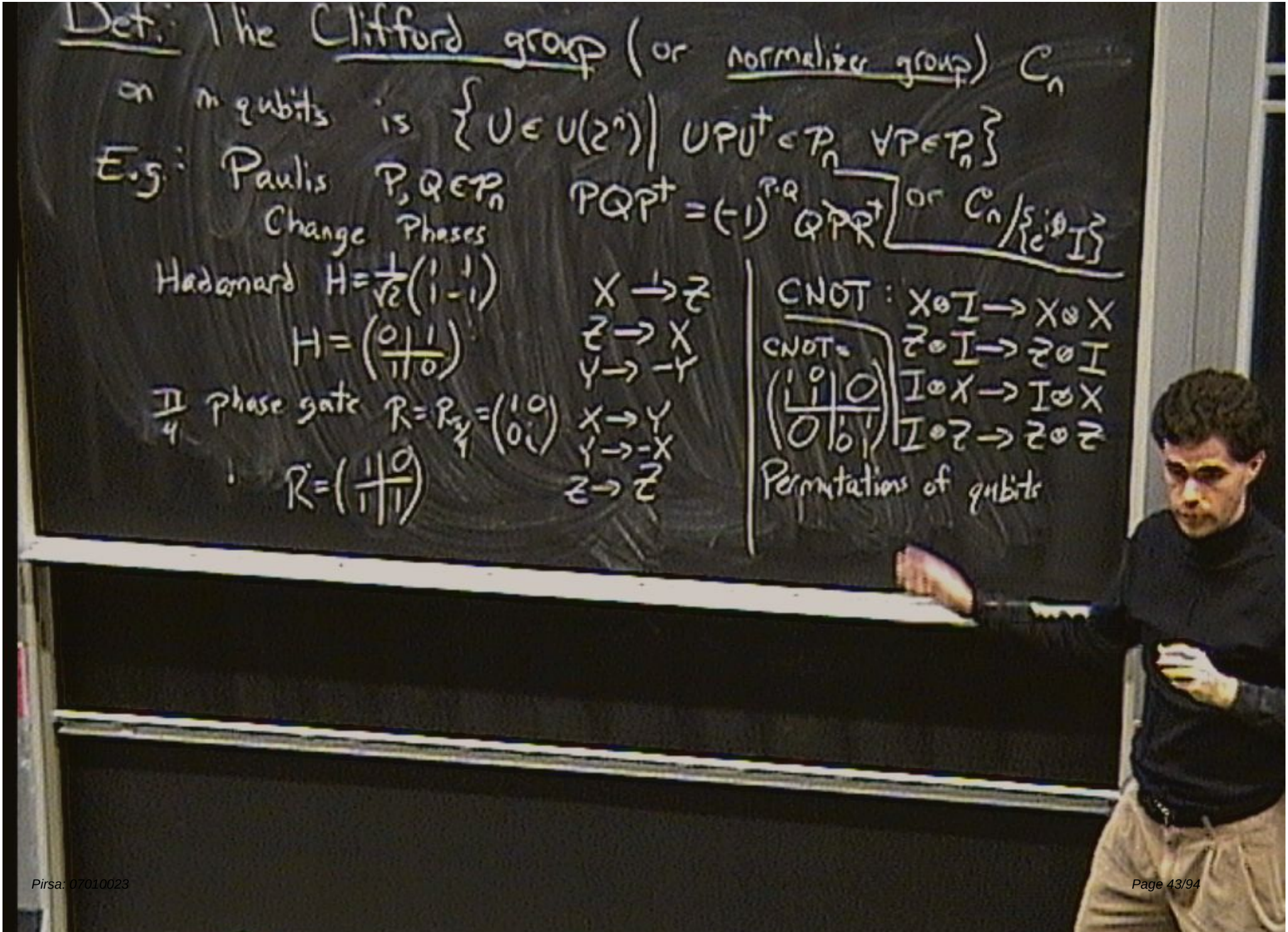
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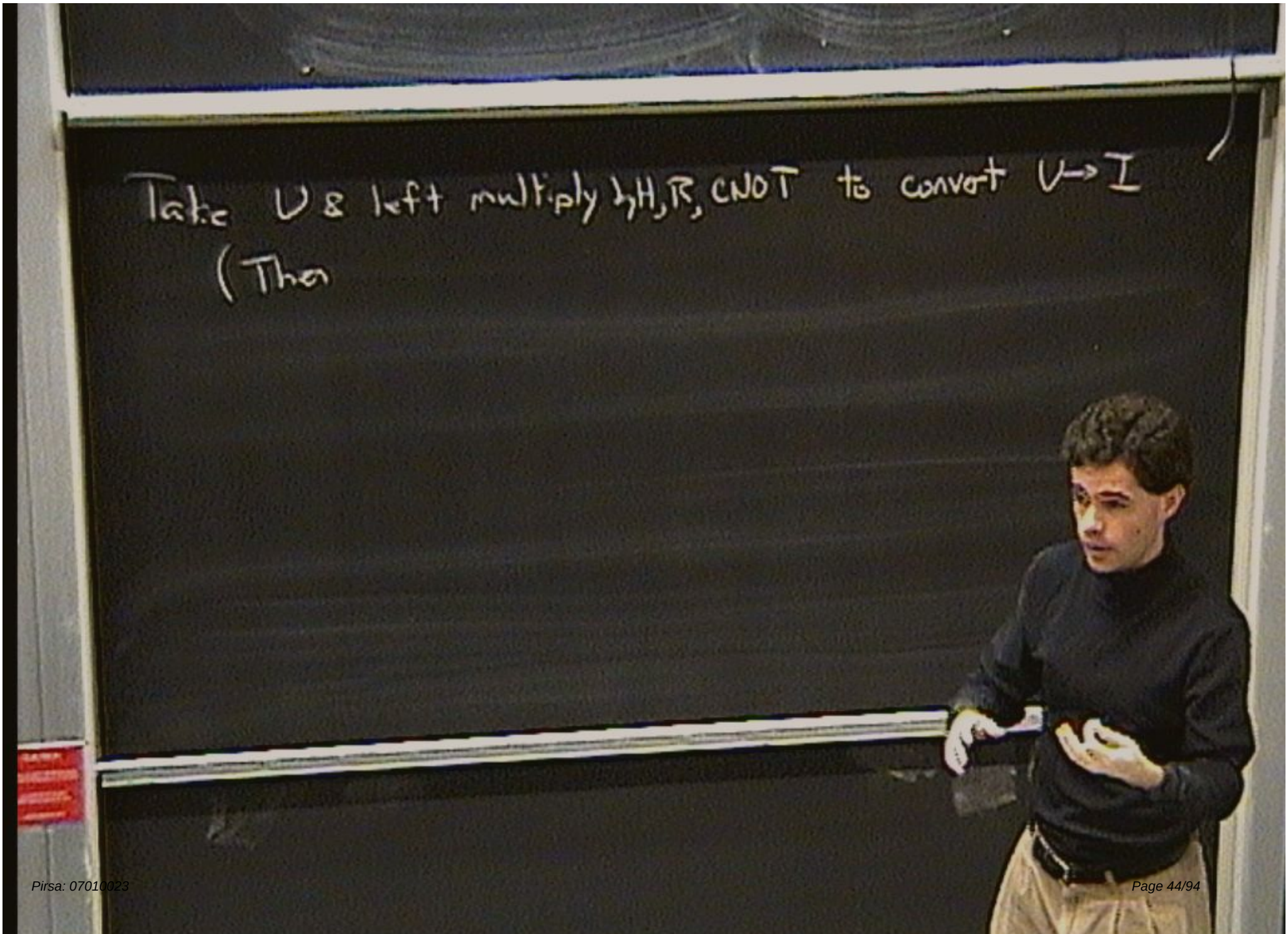
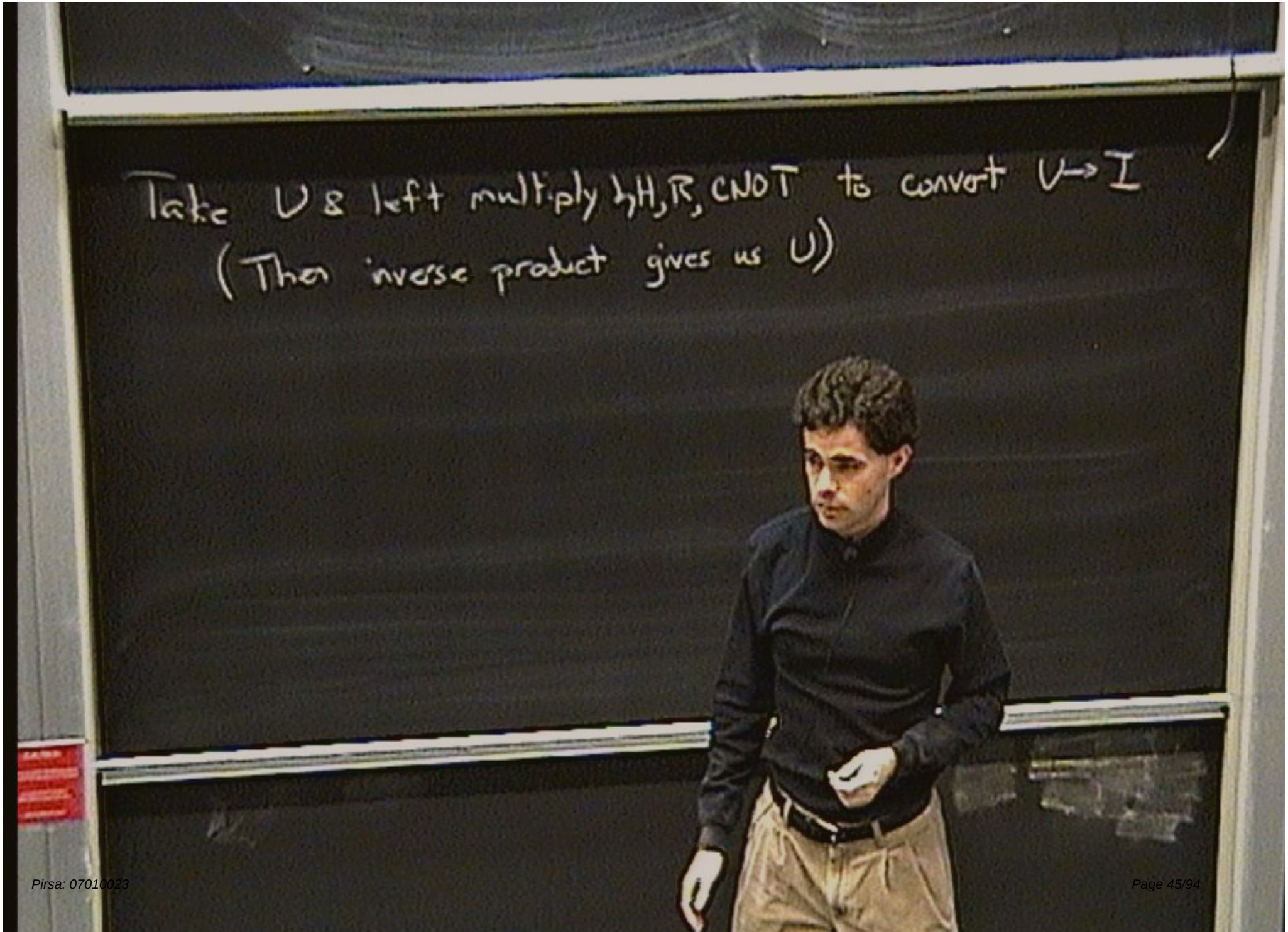
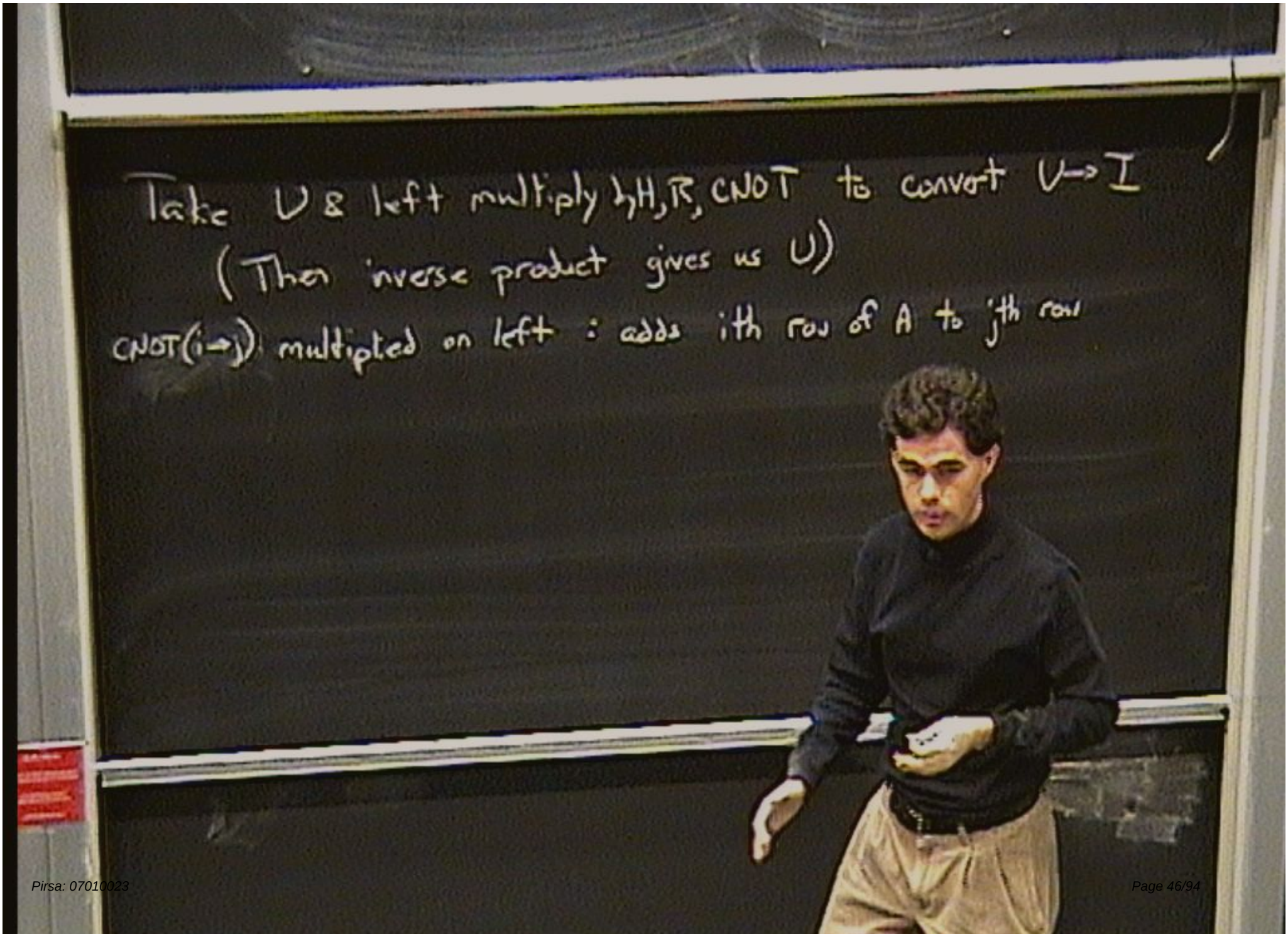

Take U & left multiply by H, R, CNOT to convert U → I
(Then


Take U & left multiply by H, R, CNOT to convert U → I
(Then inverse product gives us U)


Take U & left multiply by H, T, CNOT to convert U -> I
(Then inverse product gives us U)
CNOT(i->j) multiplied on left : adds ith row of A to jth row

Take U & left multiply by H, T, CNOT to convert $U \to I$

(Then inverse product gives us U)

$\text{CNOT}(i \to j)$ multiplied on left : adds ith row of A to j^{th} row

Using CNOTs & row reduction, convert $A = \left(\begin{array}{c|c} I & A' \\ \hline 0 & 0 \end{array}\right)$

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Permutations of qubits

Take U & left multiply by H, R, CNOT to convert U → I

(Then inverse product gives us U)

CNOT(i→j): multiplied on left : adds [illegible] of A to jth row

Using CNOTs & row reduction, co[illegible] $= \left(\begin{array}{c|c} I & A' \\ \hline 0 & 0 \end{array}\right)$

R adds ith row of A to [illegible]

Take U & left multiply by H, R, CNOT to convert $U \to I$

(Then inverse product gives us U)

$\text{CNOT}(i \to j)$: multiplied on left : adds ith row of A to j^{th} row

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R adds ith row of A to ith row of C

Take U & left multiply by H, R, CNOT to convert $U \to I$

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$\text{CNOT}(i \to j)$: multiplied on left : adds ith row of A to j^{th} row

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H switches A/B with C/D

Thm: $Sp(2n, \mathbb{Z}_2)$ is generated by R, H, CNOT (acting on arbitrary qubits or pairs of qubits)

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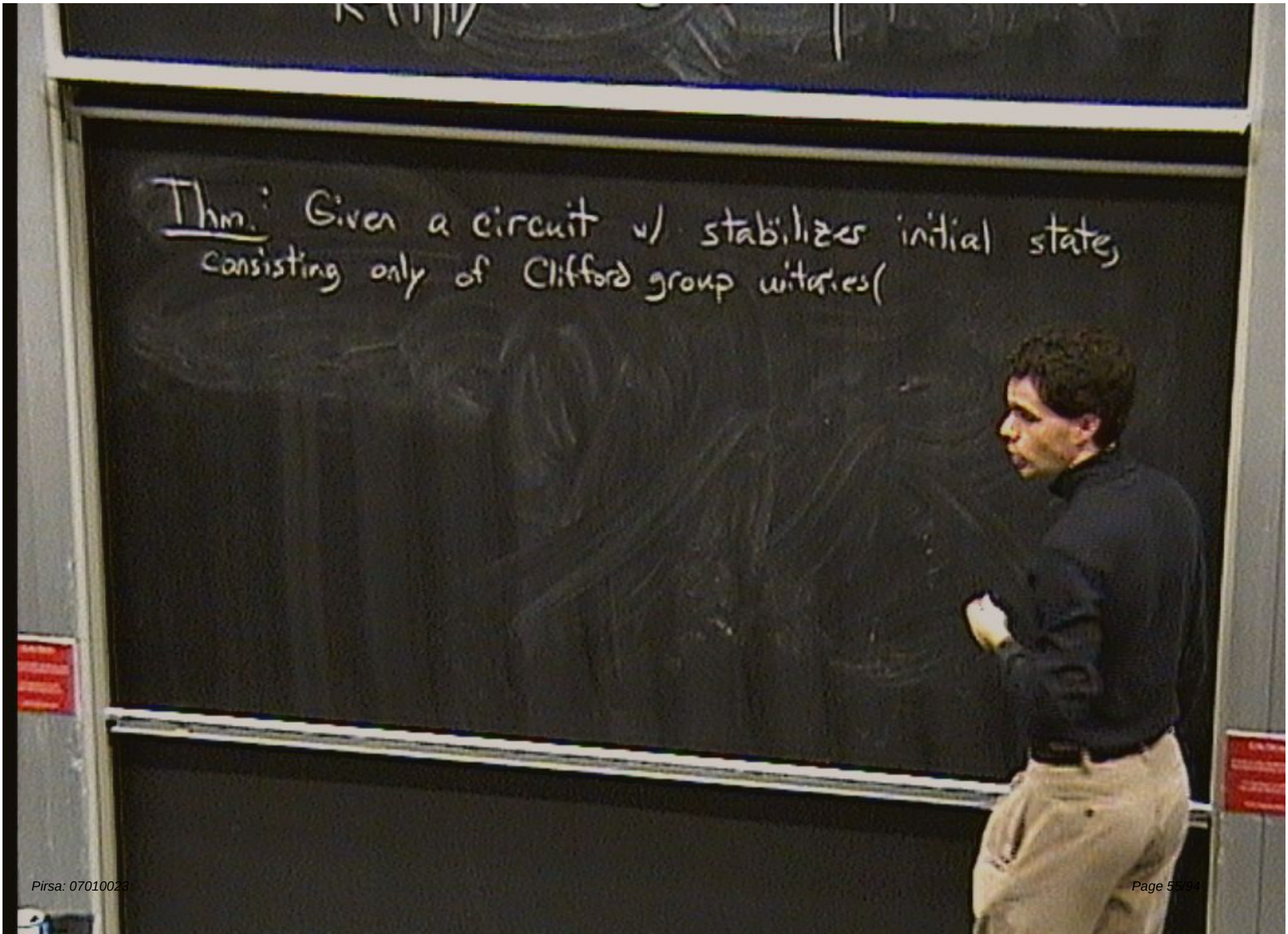
Using CNOTs & row reduction, convert $A = \left(\begin{array}{c|c} I & A' \\ \hline 0 & 0 \end{array}\right)$

R adds i^{th} row of A to i^{th} row of C

H switches A/B with C/D


Thm: Given a circuit w/ stabilizer initial state,
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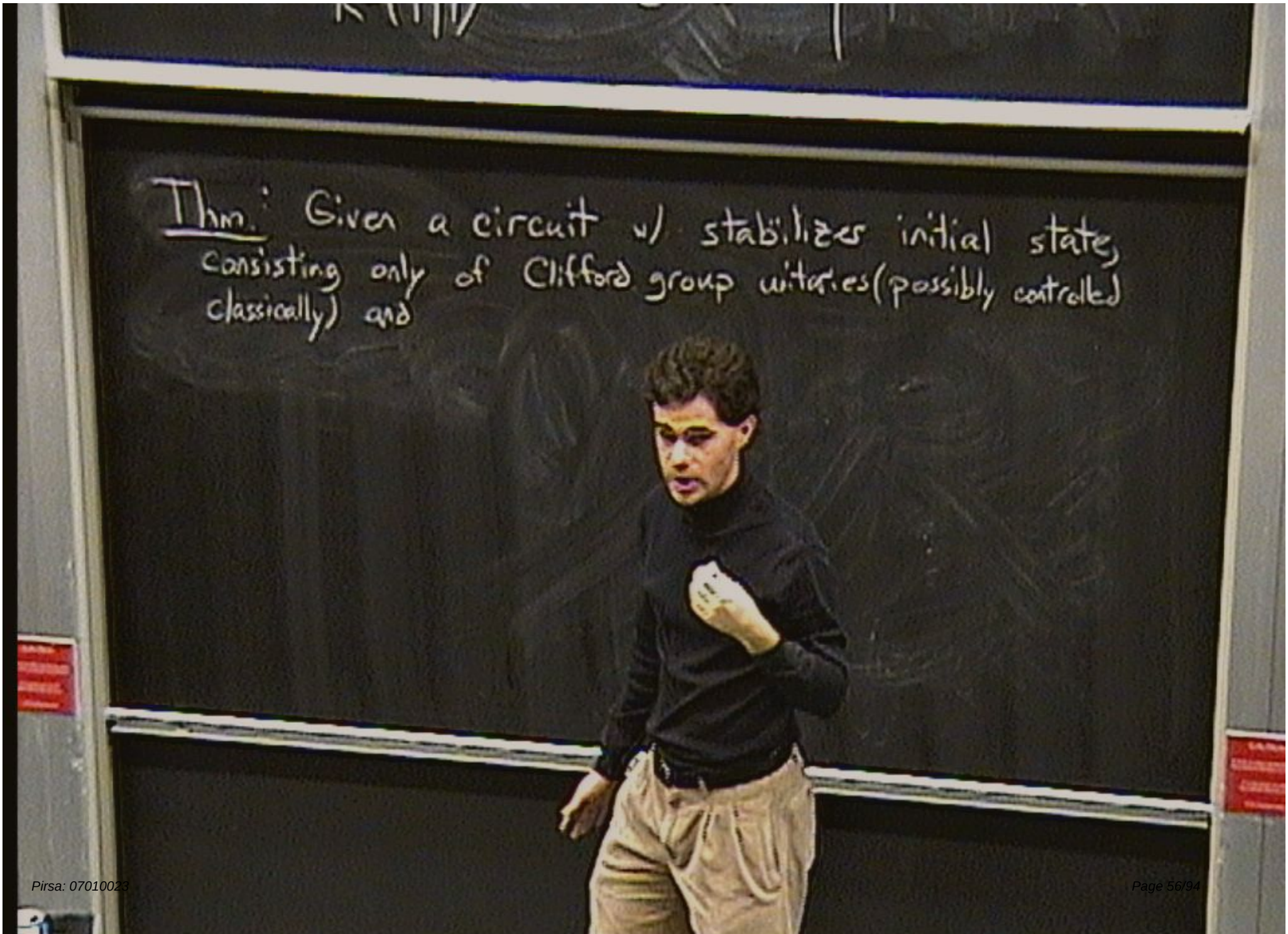
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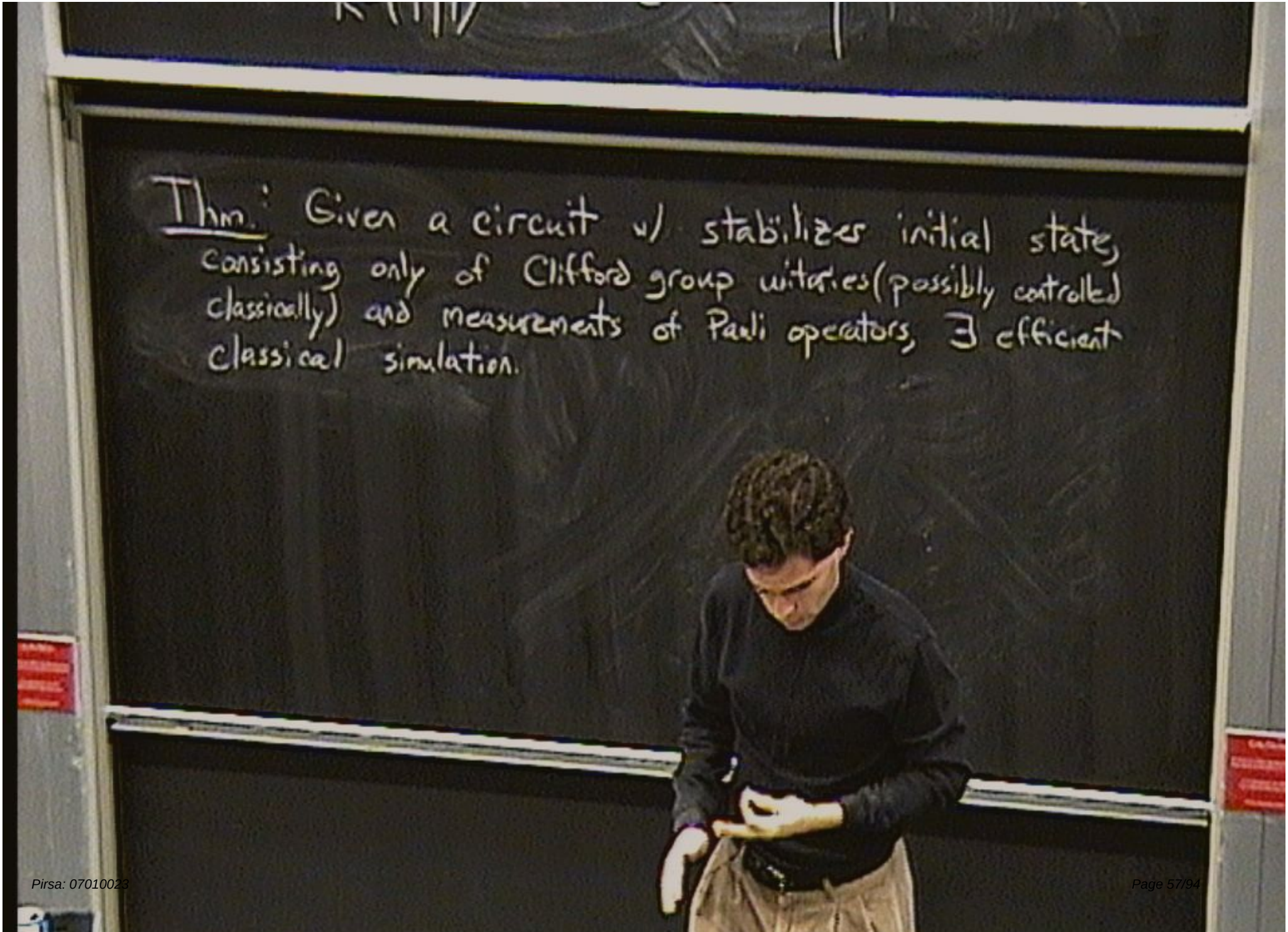
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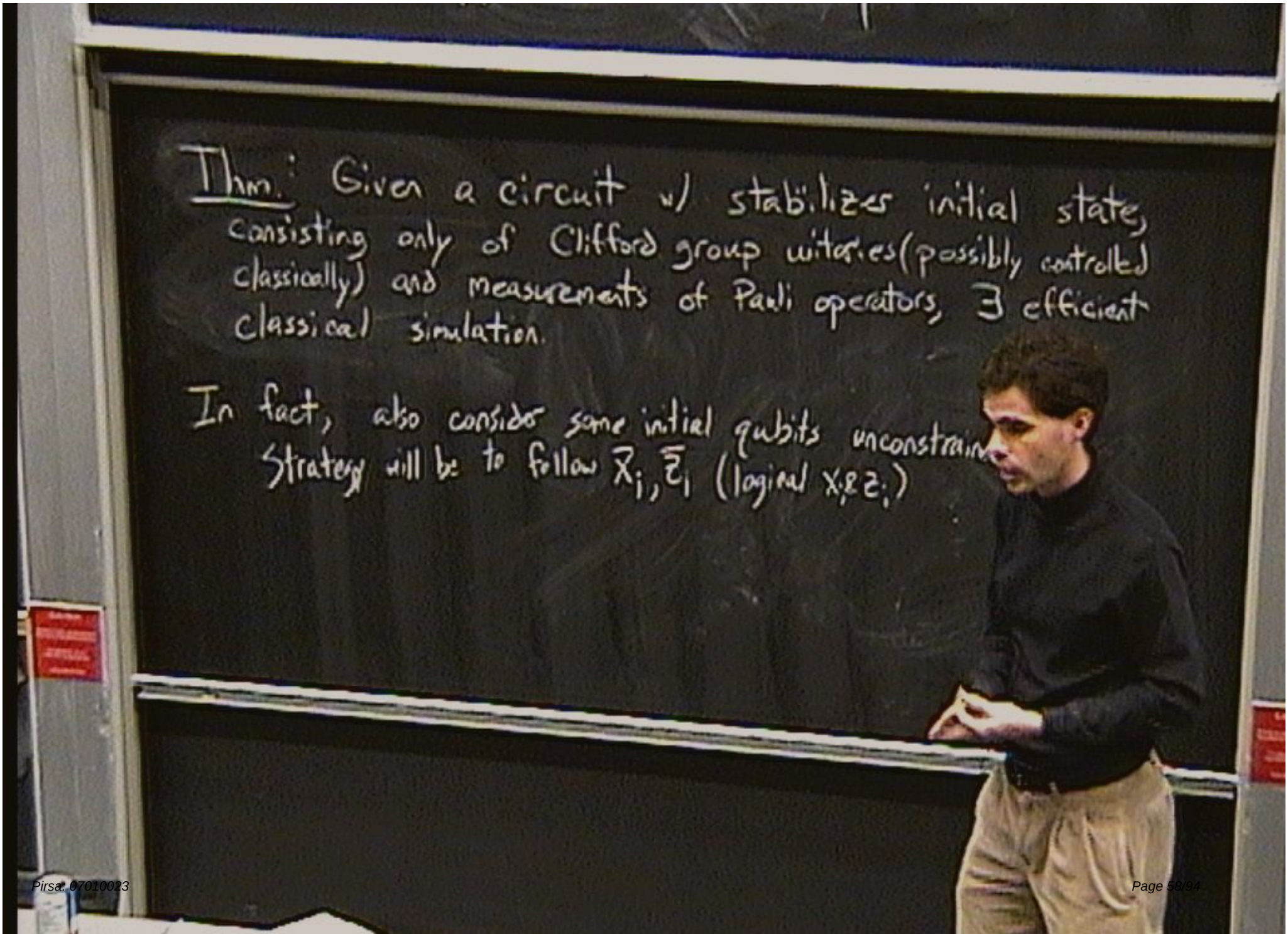
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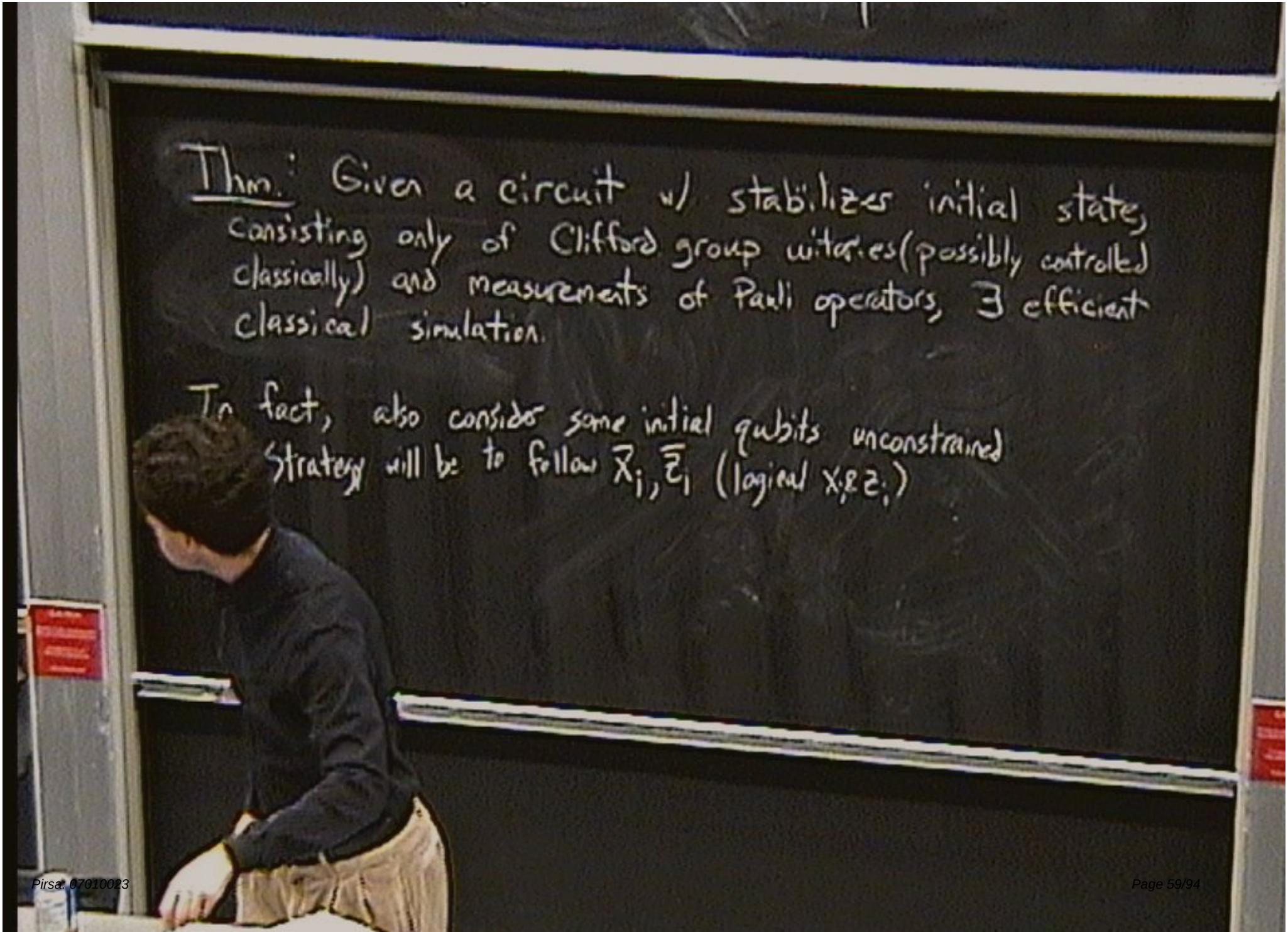
In fact, also consider some initial qubits unconstrain
Strategy will be to follow $\bar{X}_i, \bar{Z}_i$ (logical X_i & Z_i)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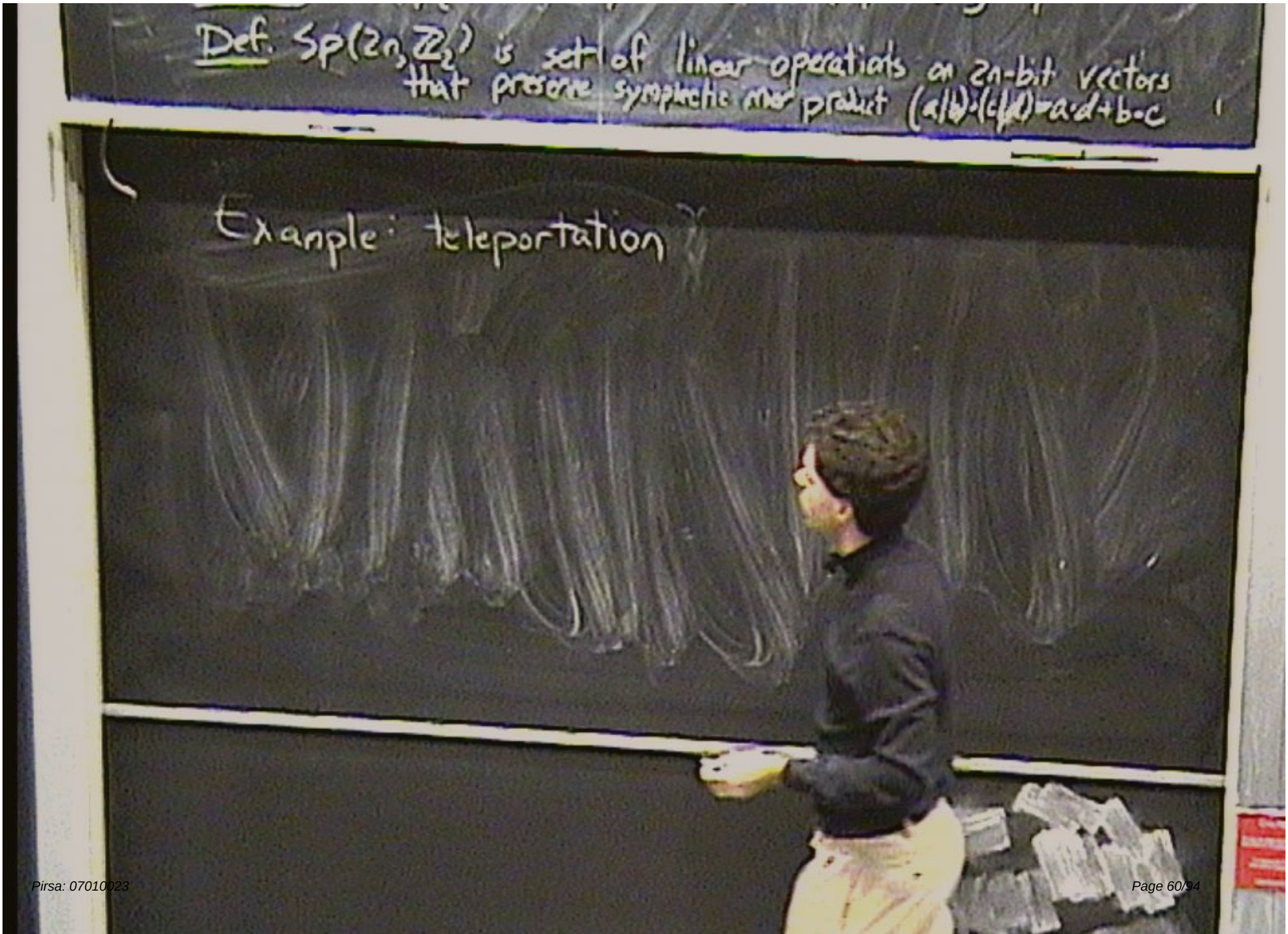
Def. $Sp(2n, \mathbb{Z}_2)$ is set of linear operations on 2n-bit vectors that preserve symplectic inner product $(a|b)\cdot(c|d) = a\cdot d + b\cdot c$
Example: teleportation

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Example: teleportation

Stabilizer $\overline{1}$ $\overline{X}_2$ $\overline{X}_2$

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Example: teleportation

Stabilizer

Z_a	Z	I	I
	I	X	X
	I	Z	Z

Def. $Sp(2n, \mathbb{Z}_2)$ is set of linear operations on $2n$-bit vectors that preserve symplectic inner product $(a|b)\cdot(c|d) = a\cdot d + b\cdot c$

Example: teleportation

Stabilizer

Z_3	I	I	I
Z_2	I	I	I
X_3	X	I	I
X_2	I	X	I
I	I	I	I

CNOT

Def. $Sp(2n, \mathbb{Z}_2)$ is set of linear operations on $2n$-bit vectors that preserve symplectic inner product $(a|b)\cdot(c|d) = a\cdot d + b\cdot c$

Example: teleportation

Stabilizer

	II	XX	XX
	XI	I	I
	ZI	I	I

CNOT →

IXX

Def. $Sp(2n, \mathbb{Z}_2)$ is set of linear operations on $2n$-bit vectors that preserve symplectic inner product $(a|b)\cdot(c|d) = a\cdot d + b\cdot c$

Example: teleportation

Stabilizers

I	X	X
I	Z	Z
X	I	I
Z	I	I

CNOT →

I	X	X
I	Z	Z
X	X	I
Z	Z	I

$\frac{\pi}{4}$ phase gate $R = R_z(\pi/2)$ $\quad X \to -X$, $Y \to -Y$, $Z \to Z$

$R = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

Permutations of qubits

Thm: Given a circuit w/ stabilizer initial state, consisting only of Clifford group unitaries (possibly controlled classically) and measurements of Pauli operators, $\exists$ efficient classical simulation.

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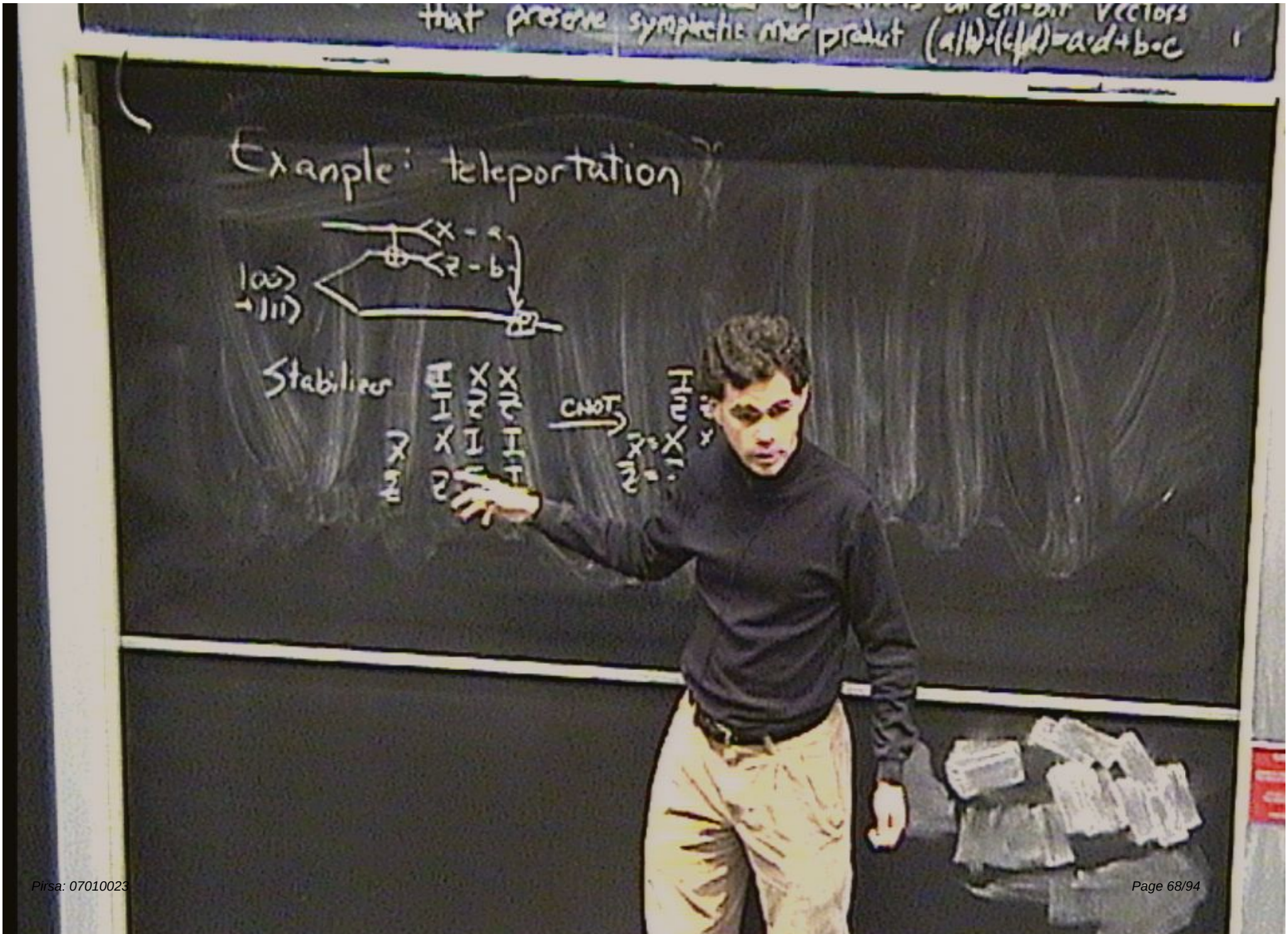
Simulate Cliffords by following transformations of stabilizer, $\bar{X}_i, \bar{Z}_i$ ✓

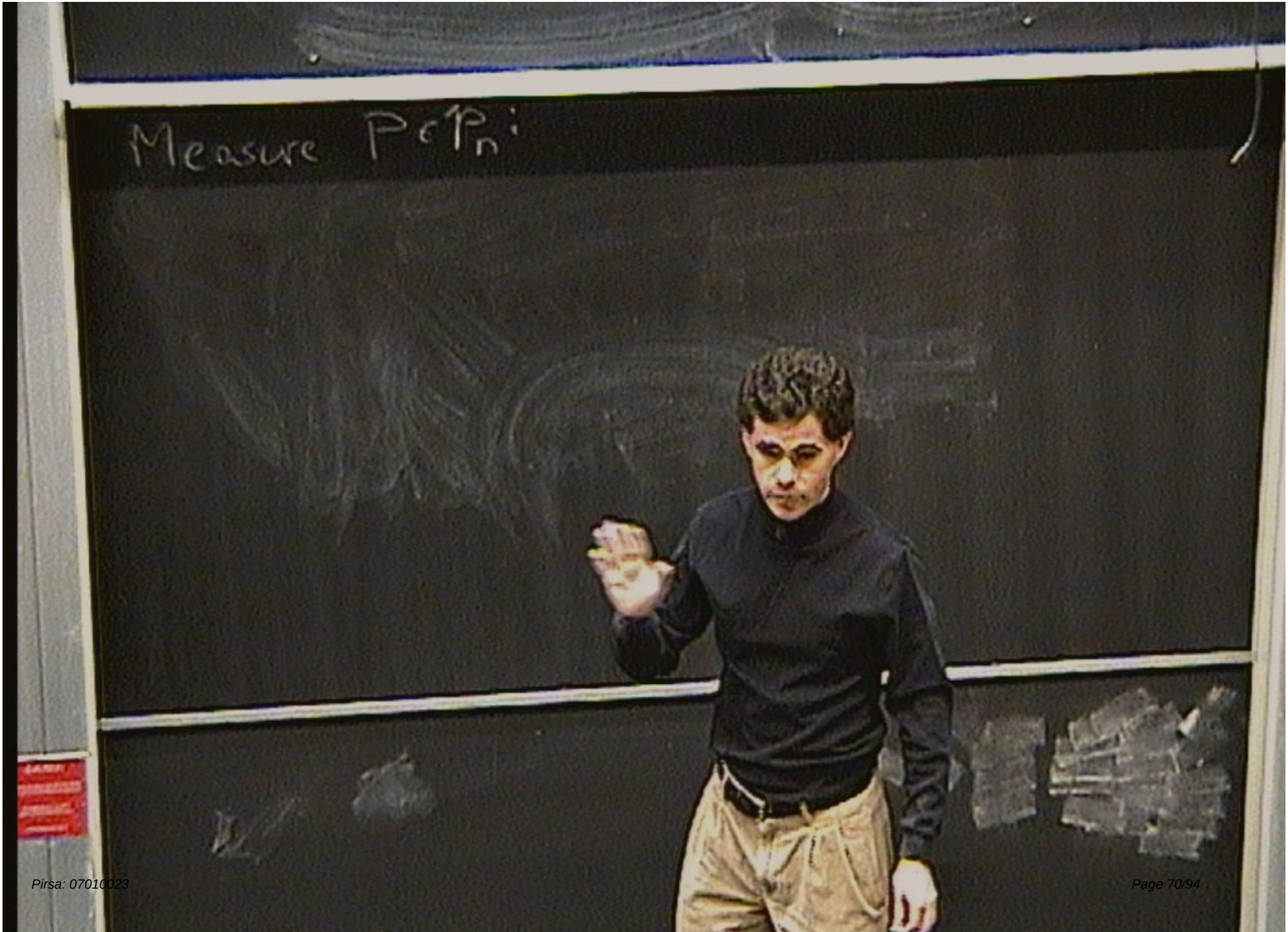

that preserve symplectic inner product
Example: teleportation
Stabilizer
CNOT

that preserve symplectic inner product $(a|b)\cdot(c|d) = a\cdot d + b\cdot c$

Example: teleportation

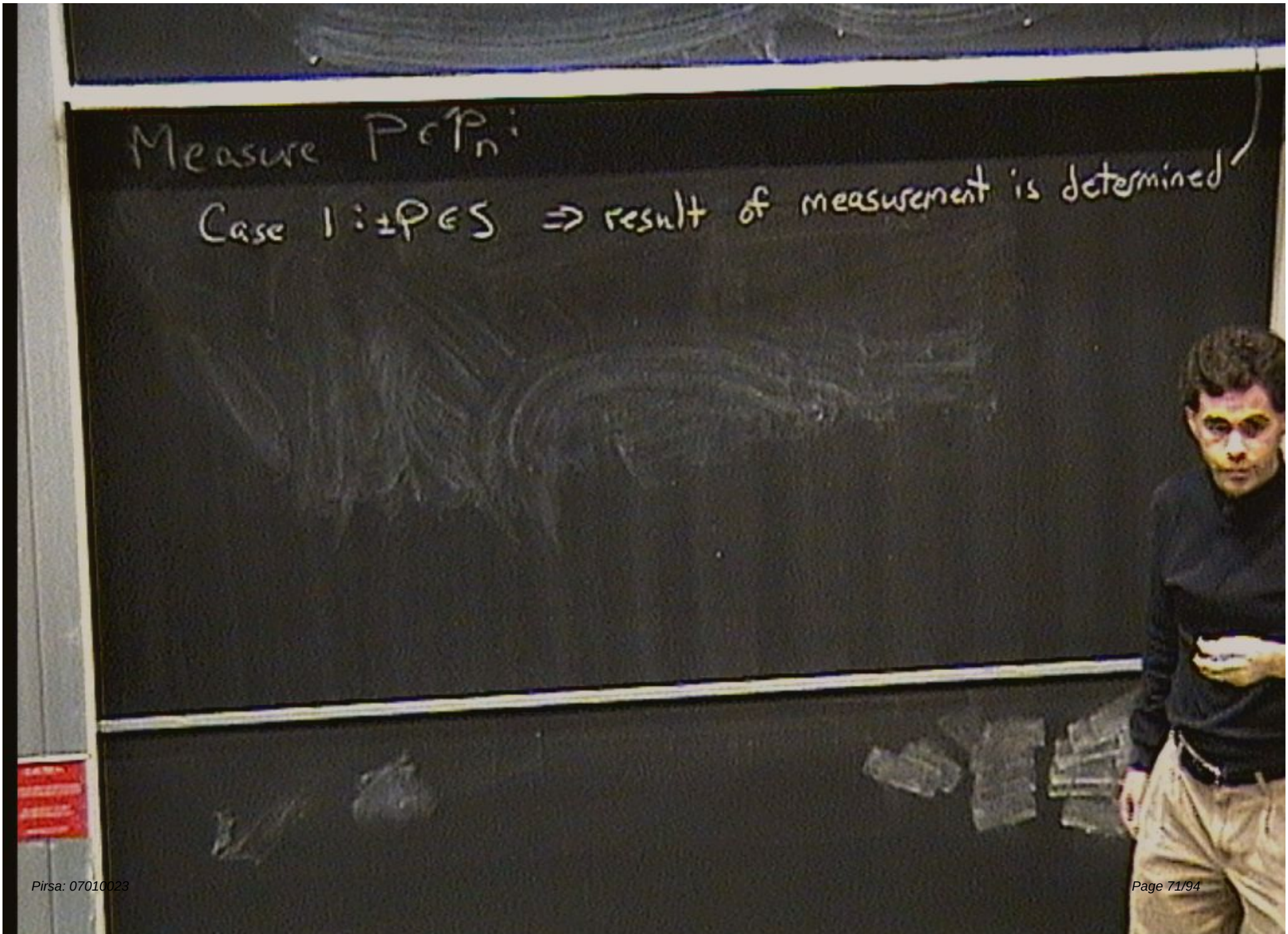
Stabilizer

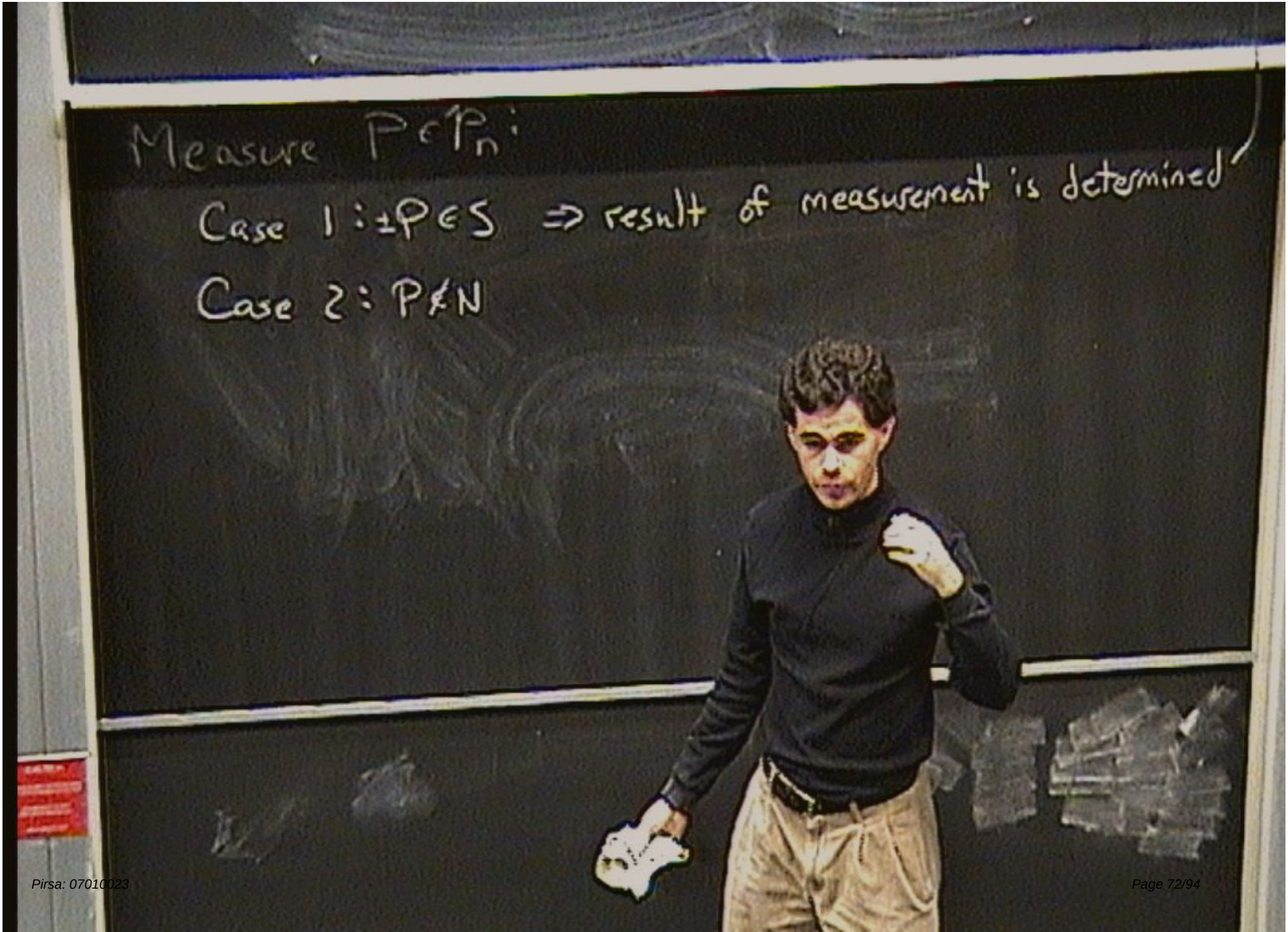
$$\begin{matrix} I & X & X \\ Z & I & I \\ I & Z & I \end{matrix} \xrightarrow{\text{CNOT}} \begin{matrix} I & X & X \\ Z & I & I \\ I & Z & Z \end{matrix}$$

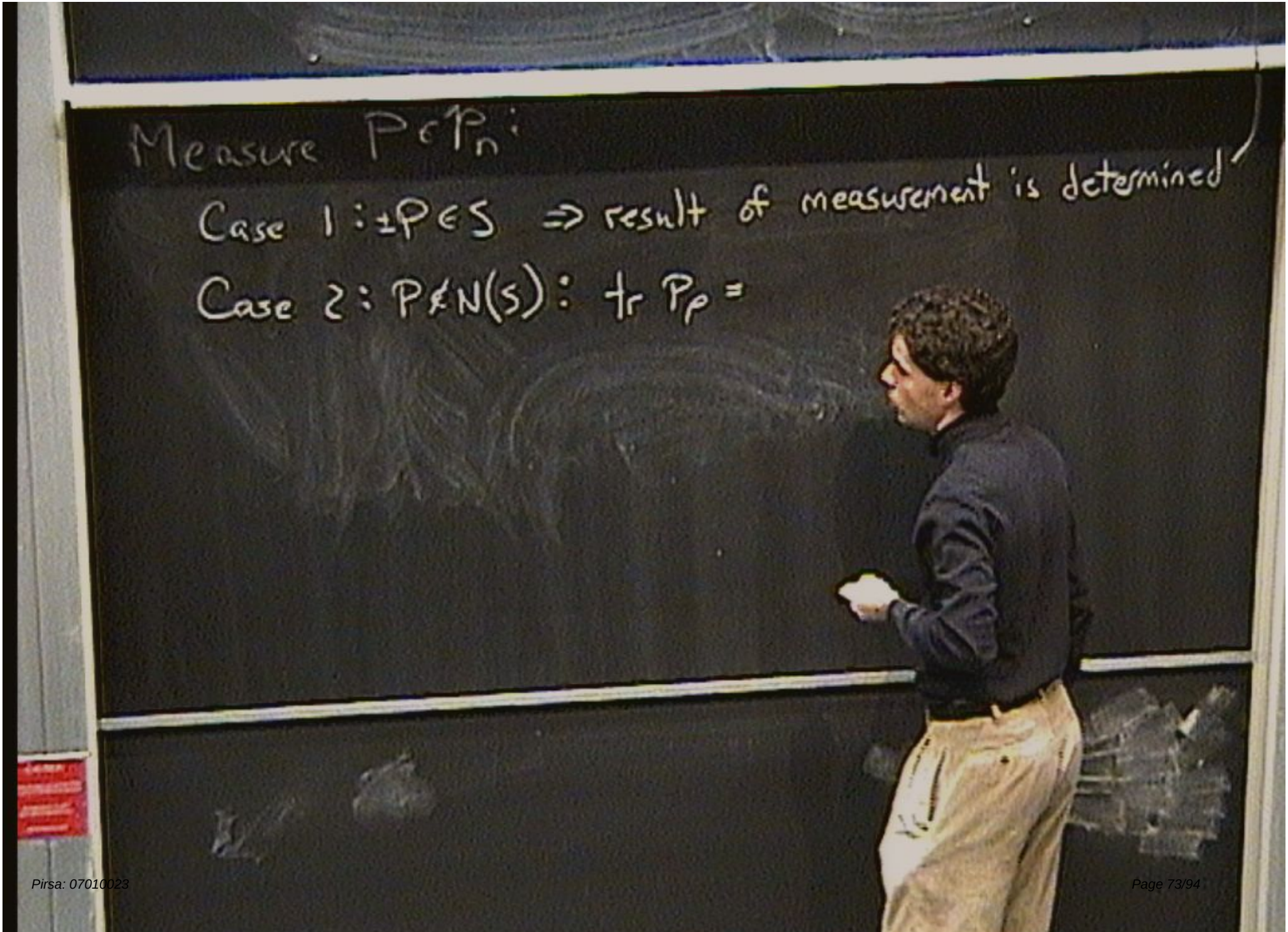

Measure PcPn

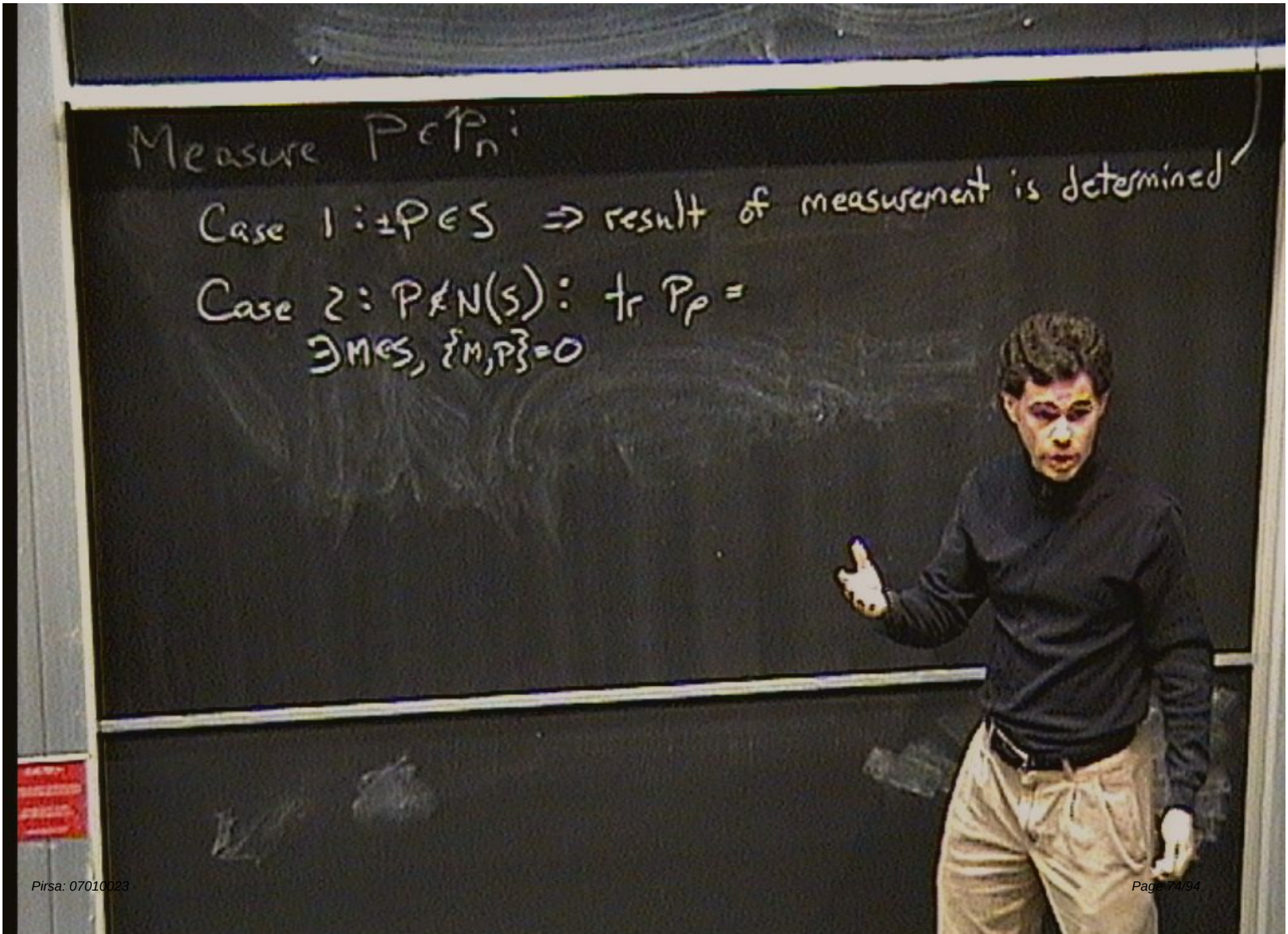
Measure $P \in P_n$:

Case 1: $\pm P \in S \Rightarrow$ result of measurement is determined




Measure P∈Pn:
Case 1: ±P∈S ⇒ result of measurement is determined
Case 2: P∉N


Measure $P \in P_n$:
Case 1: $\pm P \in S \Rightarrow$ result of measurement is determined
Case 2: $P \notin N(S)$: $\pm \operatorname{tr} P_P =$

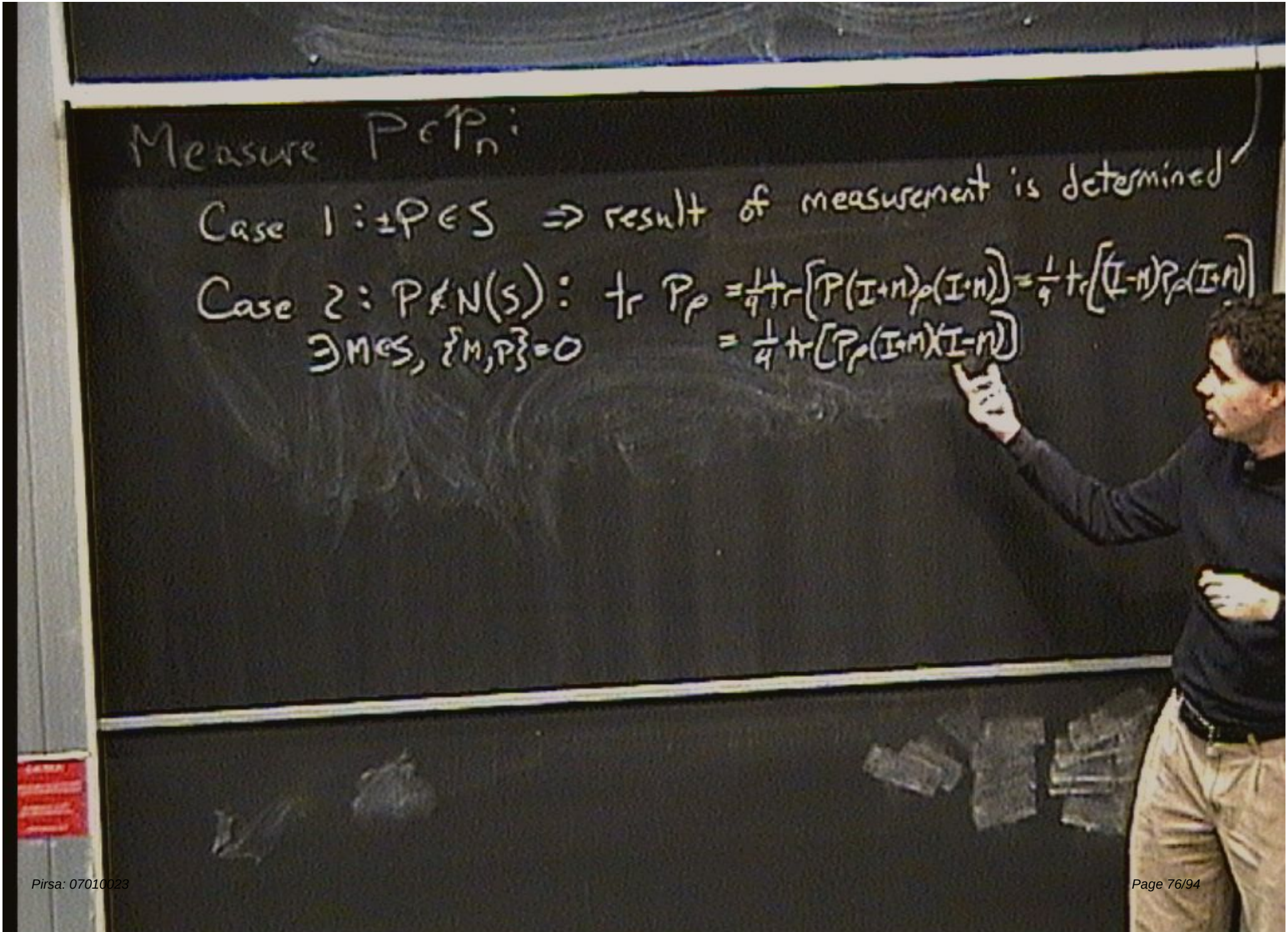

Measure $P \in \mathcal{P}_n$:
Case 1: $\pm P \in S \Rightarrow$ result of measurement is determined
Case 2: $P \notin N(S)$: $\pm \mathrm{tr}\, P_P =$
$\exists M \in S, \{M, P\} = 0$

Measure $P \in \mathcal{P}_n$:

Case 1: $\pm P \in S \Rightarrow$ result of measurement is determined

Case 2: $P \notin N(S)$: $\mathrm{tr}\, P\rho = \frac{1}{q}\mathrm{tr}\left[P(I+n)\rho(I+n)\right]$

$\exists M \in S,\ \{M, P\} = 0$


Measure $P \in \mathcal{P}_n$:
Case 1: $\pm P \in S \Rightarrow$ result of measurement is determined
Case 2: $P \notin N(S)$: $\exists M \in S, \{M, P\} = 0$
$\mathrm{tr}\, P\rho = \frac{1}{4}\mathrm{tr}[P(I+M)\rho(I+M)] = \frac{1}{4}\mathrm{tr}[(I-M)P\rho(I+M)]$
$= \frac{1}{4}\mathrm{tr}[P\rho(I+M)(I-M)]$

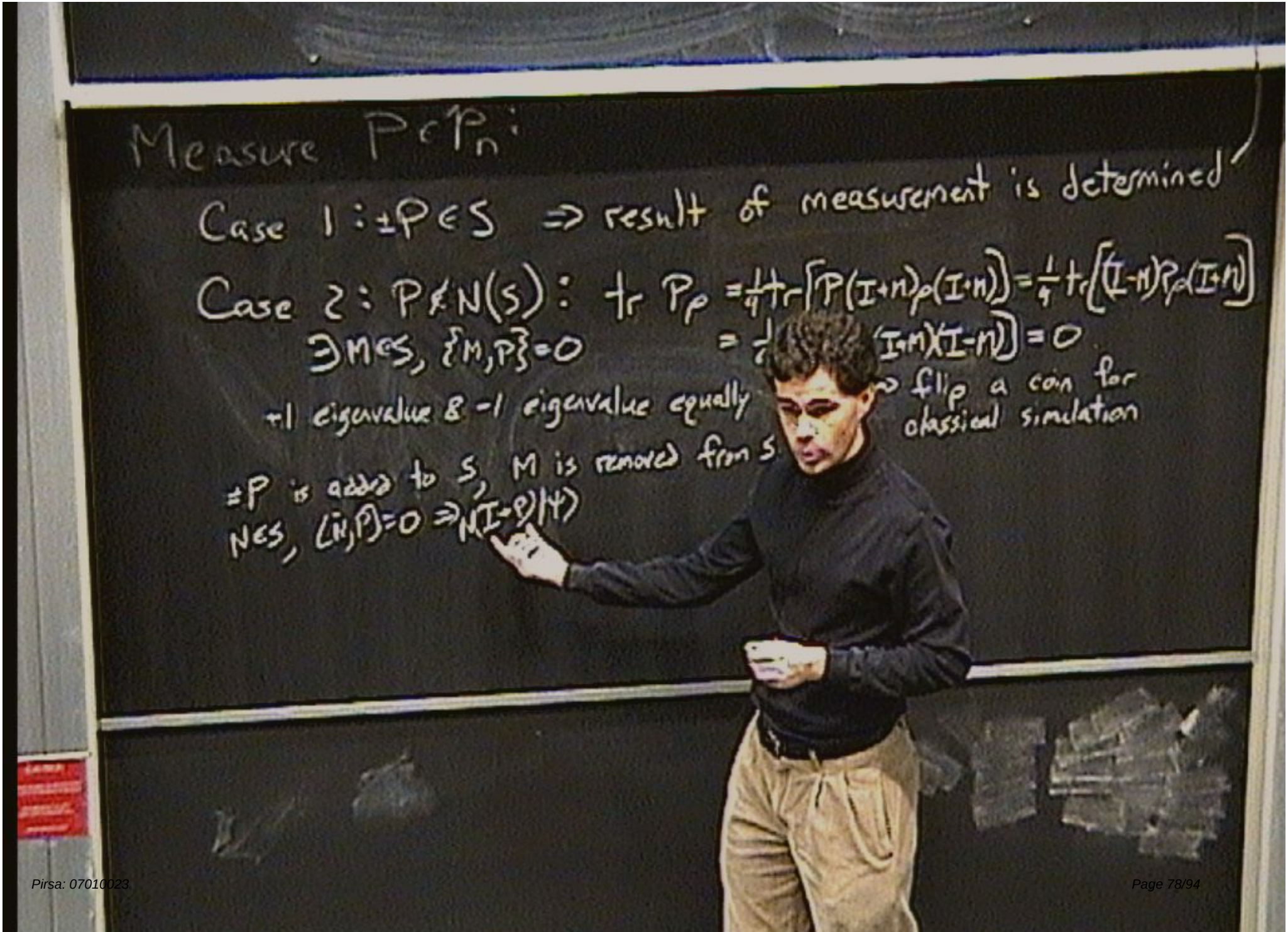
Measure $P \in \mathcal{P}_n$:

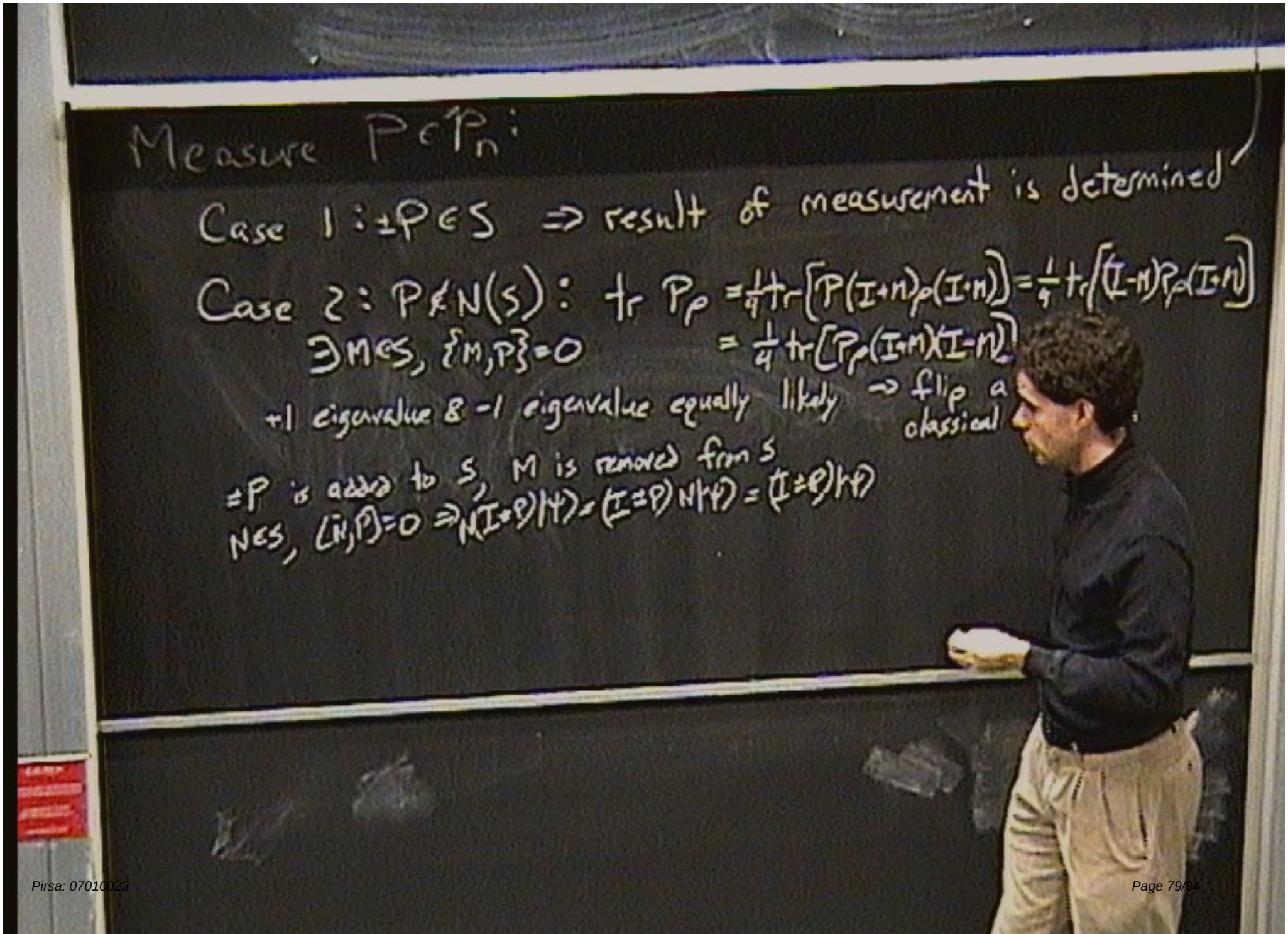
Case 1: $\pm P \in S \Rightarrow$ result of measurement is determined

Case 2: $P \notin N(S)$: $\exists M \in S,\ \{M, P\} = 0$

$$\mathrm{tr}\, P\rho = \frac{1}{4}\mathrm{tr}\left[P(I+M)\rho(I+M)\right] = \frac{1}{4}\mathrm{tr}\left[(I-M)P\rho(I+M)\right]$$
$$= \frac{1}{4}\mathrm{tr}\left[P\rho(I+M)(I-M)\right] = 0$$

$+1$ eigenvalue & -1 eigenvalue equally likely $\Rightarrow$ flip a coin for classical simulation


Measure $P \in P_n$:
Case 1: $\pm P \in S$ ⇒ result of measurement is determined
Case 2: $P \notin N(S)$: $\exists M \in S, \{M,P\}=0$
± 1 eigenvalue & -1 eigenvalue equally
$\pm P$ is added to S, M is removed from S
$N \in S, [N,P]=0 \Rightarrow N(I-P)|\psi\rangle$
$\mathrm{tr}\, P\rho = \frac{1}{4}\mathrm{tr}[P(I+n)\rho(I+n)] = \frac{1}{4}\mathrm{tr}[(I-n)P\rho(I+n)]$
$= \frac{1}{2}$ (I+n)(I-n)] = 0
⇒ flip a coin for classical simulation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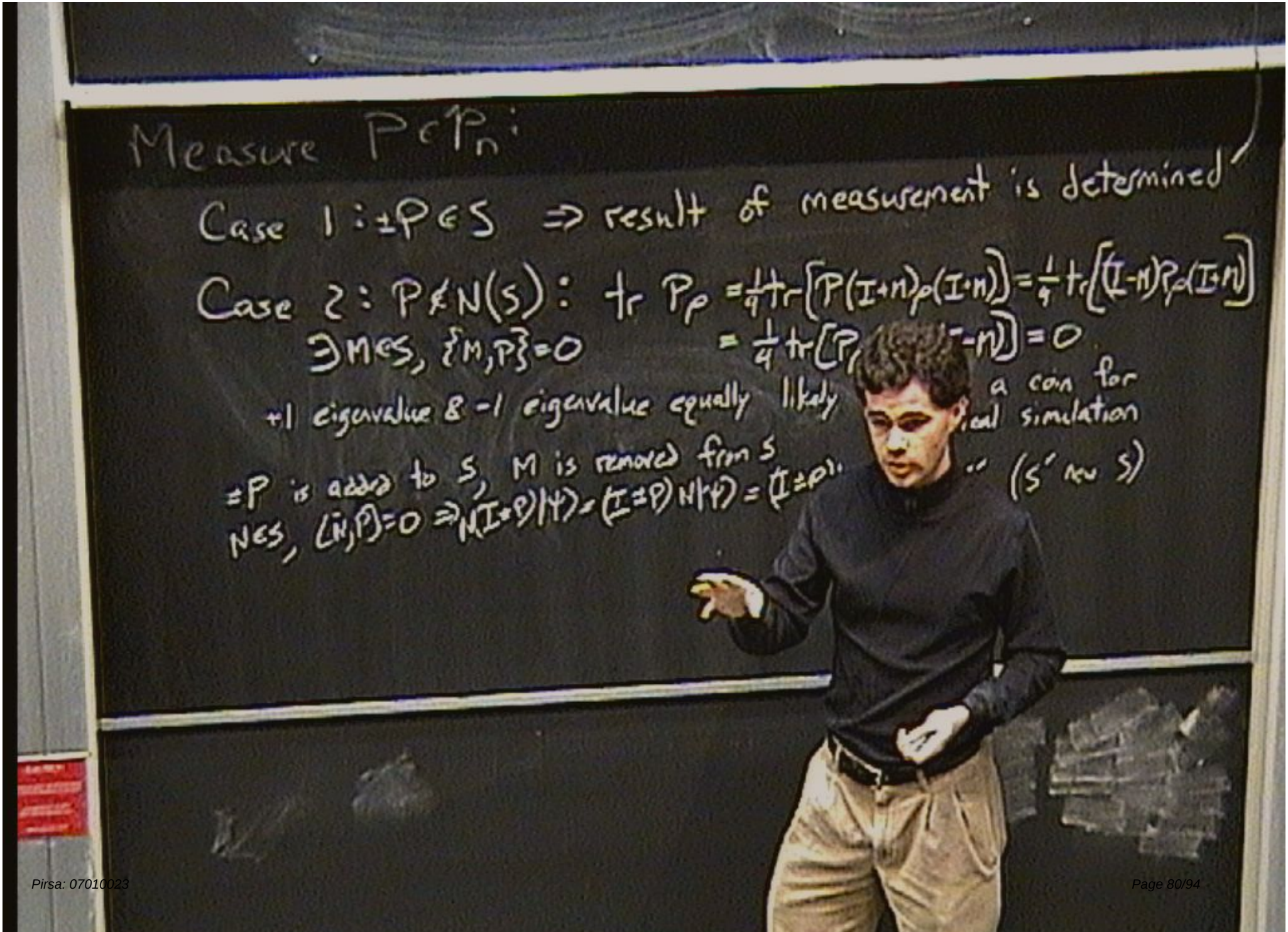
Case 2: $P \notin N(S)$: $\exists M \in S, \{M,P\}=0$
$\mathrm{tr}\, P\rho = \frac{1}{4}\mathrm{tr}[P(I+n)\rho(I+n)] = \frac{1}{4}\mathrm{tr}[(I-n)P\rho(I+n)]$
$= \frac{1}{4}\mathrm{tr}[P\rho(I+n)(I-n)]$
+1 eigenvalue & -1 eigenvalue equally likely $\rightarrow$ flip a classical
$\pm P$ is added to S, M is removed from S
$N \in S, [N,P]=0 \Rightarrow N(I \pm P)|\psi\rangle = (I \pm P)N|\psi\rangle = (I \pm P)|\psi\rangle$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$+1$ eigenvalue & -1 eigenvalue equally likely
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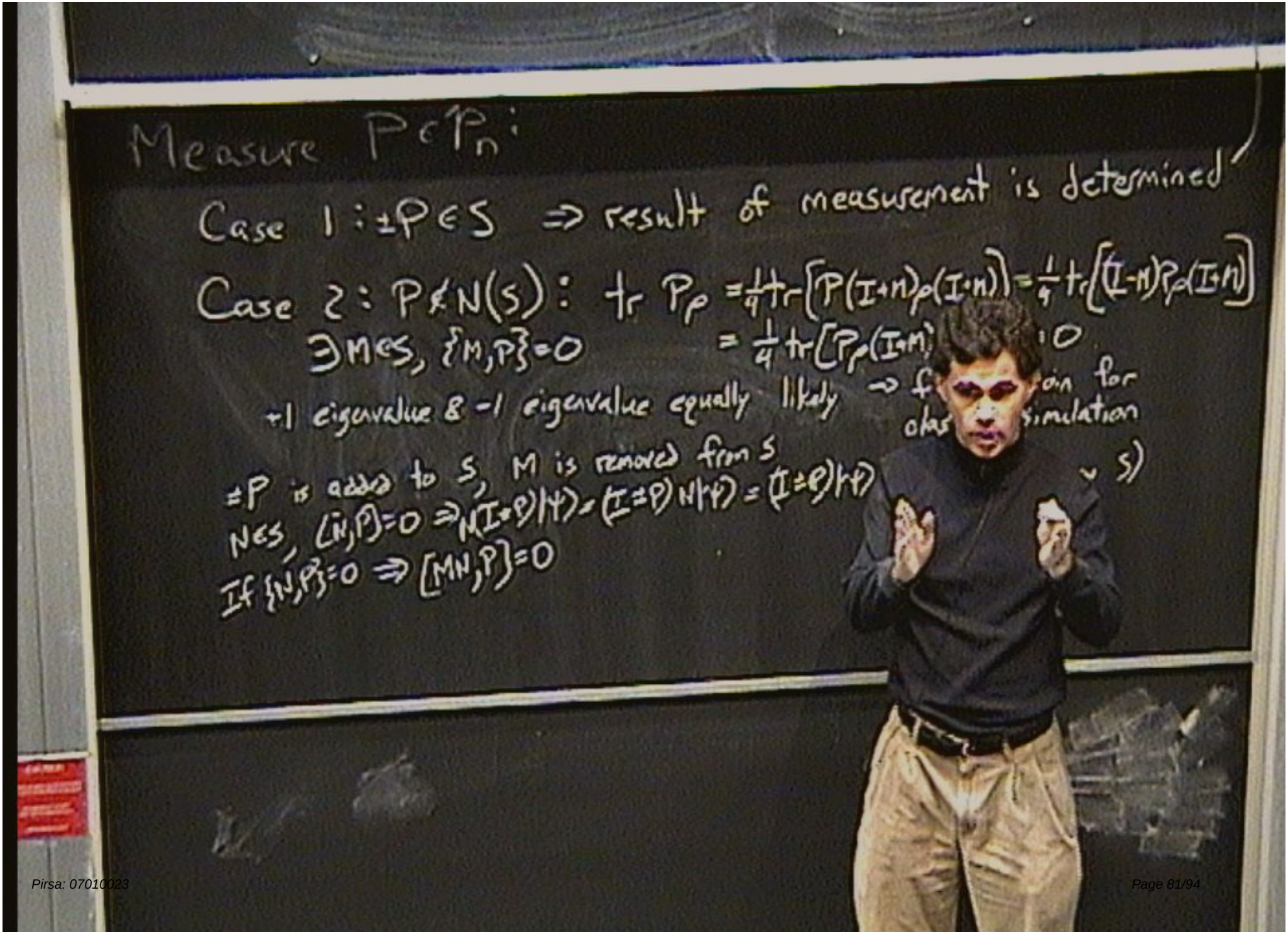
$\therefore$ (S' new S)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$\pm P$ is added to S, M is removed from S

$N \in S$, $[N,P]=0 \Rightarrow N(I \pm P)|\psi\rangle = (I \pm P)N|\psi\rangle = (I \pm P)|\psi\rangle \Rightarrow N \in S'$ (S' new S)

If $\{N,P\}=0 \Rightarrow [MN,P]=0 \Rightarrow$ Replace generators N_i of S:

If $[N_i,P]=0$ keep N_i

$\{N_i,P\}=0$ $N_i \to MN_i$

Similarly $\bar{X}$ & $\bar{Z}$'s

Def. $Sp(2n, \mathbb{Z}_2)$ is set of linear operations on $2n$-bit vectors that preserve symplectic inner product $(a|b)\cdot(c|d) = a\cdot d + b\cdot c$

Example: teleportation

Stabilizer $\begin{matrix} I & X & X \\ Z & X & I \\ I & Z & I \end{matrix}$ $\xrightarrow{CNOT}$ $\begin{matrix} I & X & X \\ Z & X & I \\ I & Z & I \end{matrix}$ $\xrightarrow{Measure}$

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Example: teleportation

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Example: teleportation

Stabilizer $\begin{matrix} I & X & X \\ I & Z & Z \\ X & I & I \\ Z & I & I \end{matrix}$ $\xrightarrow{\text{CNOT}}$ $\begin{matrix} I & X & X \\ I & Z & Z \\ X & X & I \\ Z & I & I \end{matrix}$ $\xrightarrow{\text{Measure}}$ $\begin{matrix} I & X & X \\ X & I & I \\ Z & Z & I \\ I & I & Z \end{matrix}$

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Example: teleportation

Stabilizer $\begin{matrix} II & XX \\ XI & IX \\ ZI & IZ \end{matrix}$ CNOT $\begin{matrix} II & XX \\ XX & IX \\ ZI & ZZ \end{matrix}$ Measure $\begin{matrix} II & XX \\ XX & IX \\ ZZ & ZI \end{matrix}$ $\begin{matrix} \bar{Z} = XX \\ \bar{X} = IX \end{matrix}$ $\bar{Z} = XX$ $\bar{X} = IX$ $\bar{Z} = IZ$ $\bar{X} = ZI$ $\bar{Z} = ZZ$ $\bar{X} = XI$ $\bar{Z} = IZ$ $\bar{X} = XI$ Measure Z $\bar{Z} = IZ$ $\bar{X} = XI$ $T \times I$

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Example: teleportation

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Stabilizer $\begin{matrix} II & XX & XX \\ XI & XI & II \\ ZI & ZI & II \end{matrix}$ CNOT $\begin{matrix} II & XX & XX \\ XI & XI & II \\ ZI & ZI & II \end{matrix}$ Measure $\begin{matrix} II & XX & XX \\ XI & XI & II \\ ZI & ZI & II \end{matrix}$ $(-1)^{}$

$\begin{matrix} II & XX & XX \\ XI & XI & II \\ ZI & ZI & II \end{matrix}$ $(-1)^{} = (-1)^{} IXI$

Measure $\begin{matrix} II & XX & XX \\ XI & XI & II \\ ZI & ZI & II \end{matrix}$

$ZZI = IZZ$

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Example: teleportation

Stabilizer $\begin{matrix} I & X & X \\ Z & X & I \\ Z & I & I \end{matrix}$ $\xrightarrow{CNOT}$ $\begin{matrix} I & X & X \\ Z & X & I \\ Z & I & I \end{matrix}$ $\xrightarrow{Measure}$ $\begin{matrix} I & X & X \\ (-1)^{x} X & I & I \\ Z & I & I \end{matrix}$

$\xrightarrow{Measure}$ $\begin{matrix} I & X & X \\ (-1)^{x} X & I & I \\ (-1)^{z} Z & I & I \end{matrix}$

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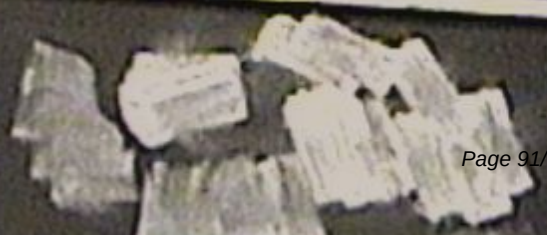
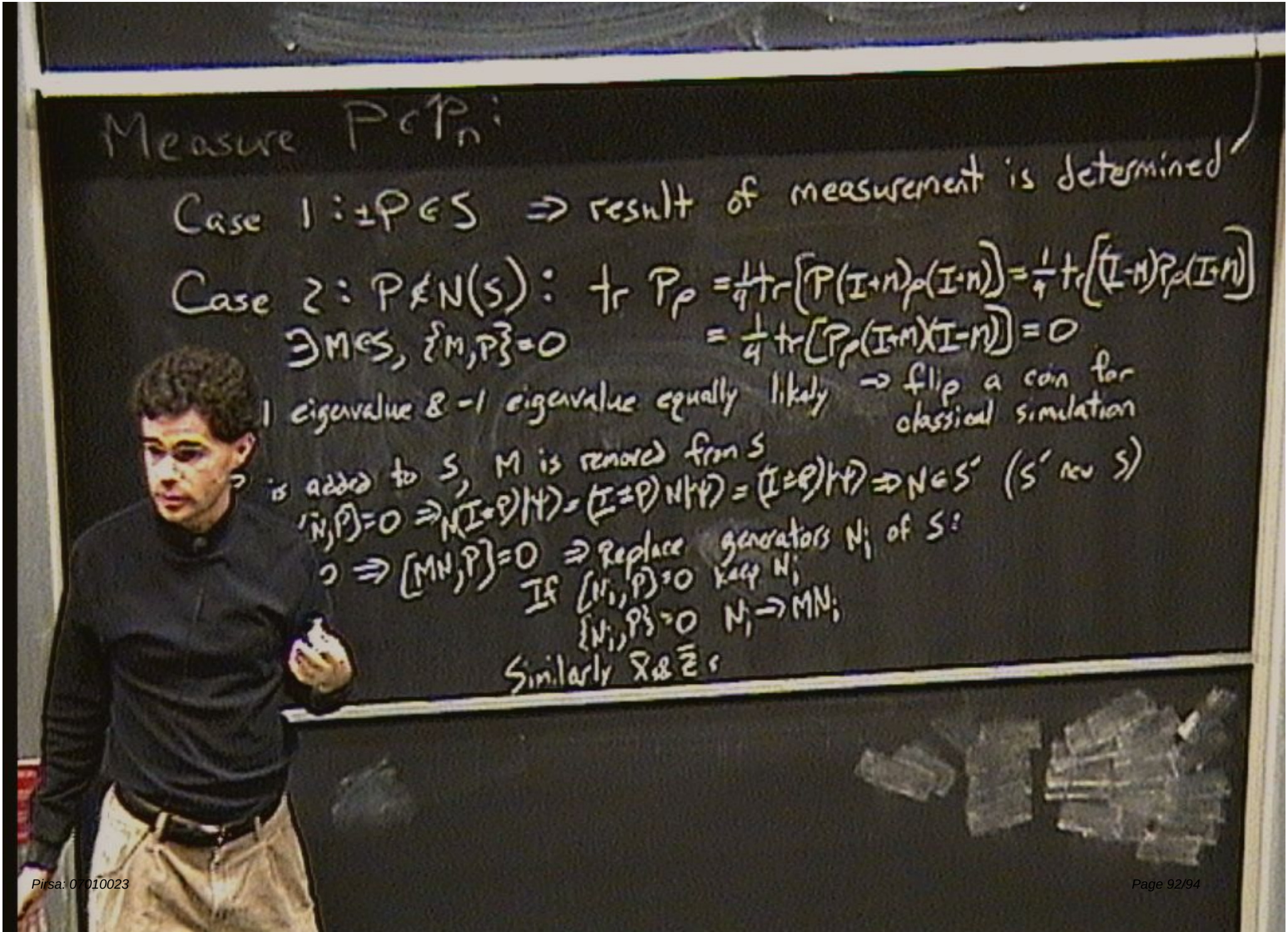
$\xrightarrow{Measure}$ $\begin{matrix} I & I & X \\ X & I & I \\ I & I & I \end{matrix}$ $\xrightarrow{Pauli}$ $\begin{matrix} I & I & X \\ X & I & I \\ I & I & I \end{matrix}$

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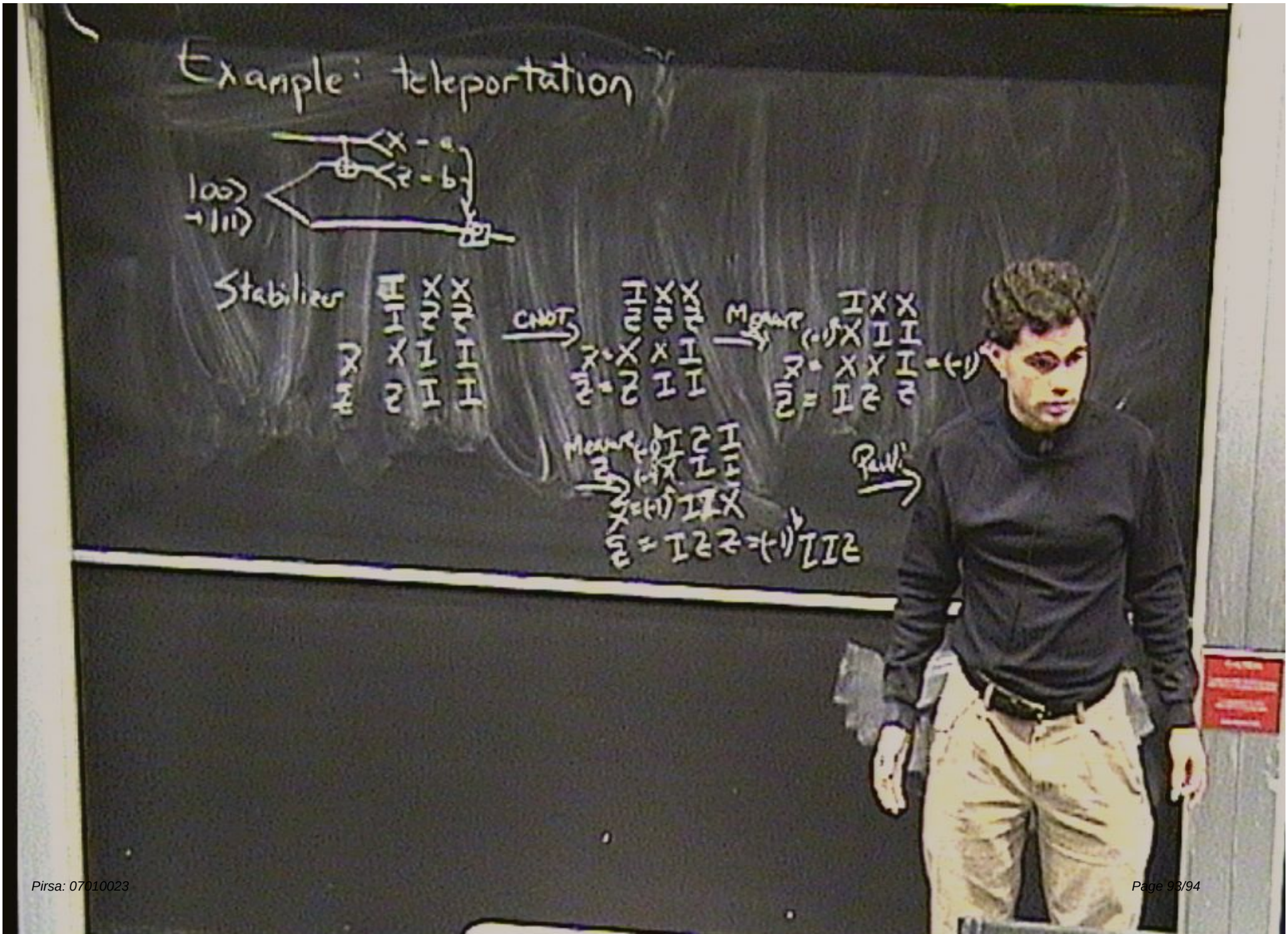
Example: teleportation
|00⟩ + |11⟩
Stabilizer
CNOT
Measure
Measure
Pauli

Example: teleportation

Stabilizer

CNOT

Measure

Measure

Rot