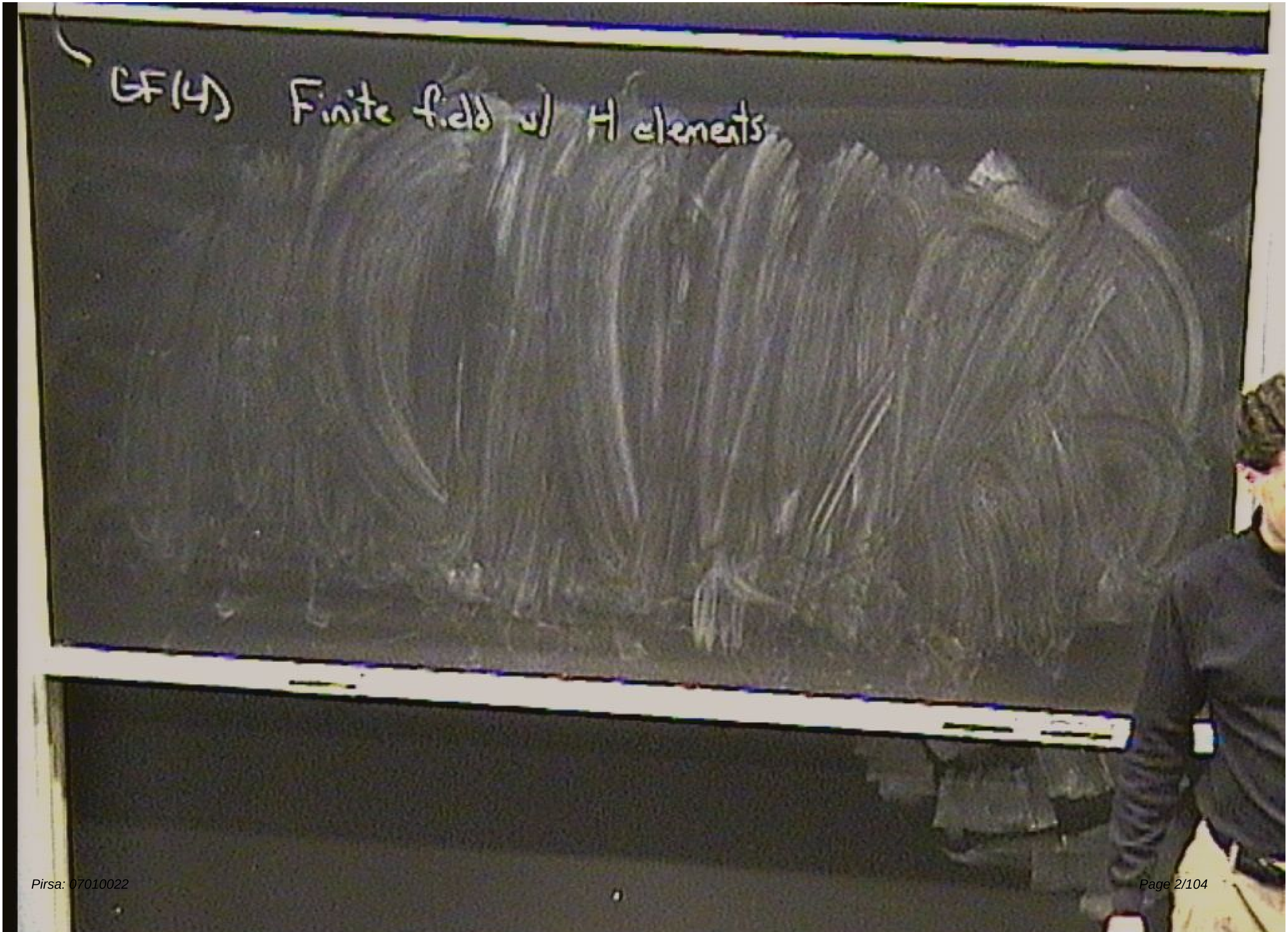
Title: Quantum Error Correction 3A

Date: Jan 23, 2007 03:30 PM

URL: http://pirsa.org/07010022

Abstract: Finite field GF(4), stabilizer codes as GF(4) codes, perfect quantum codes, definition of Clifford group, sample elements of Clifford group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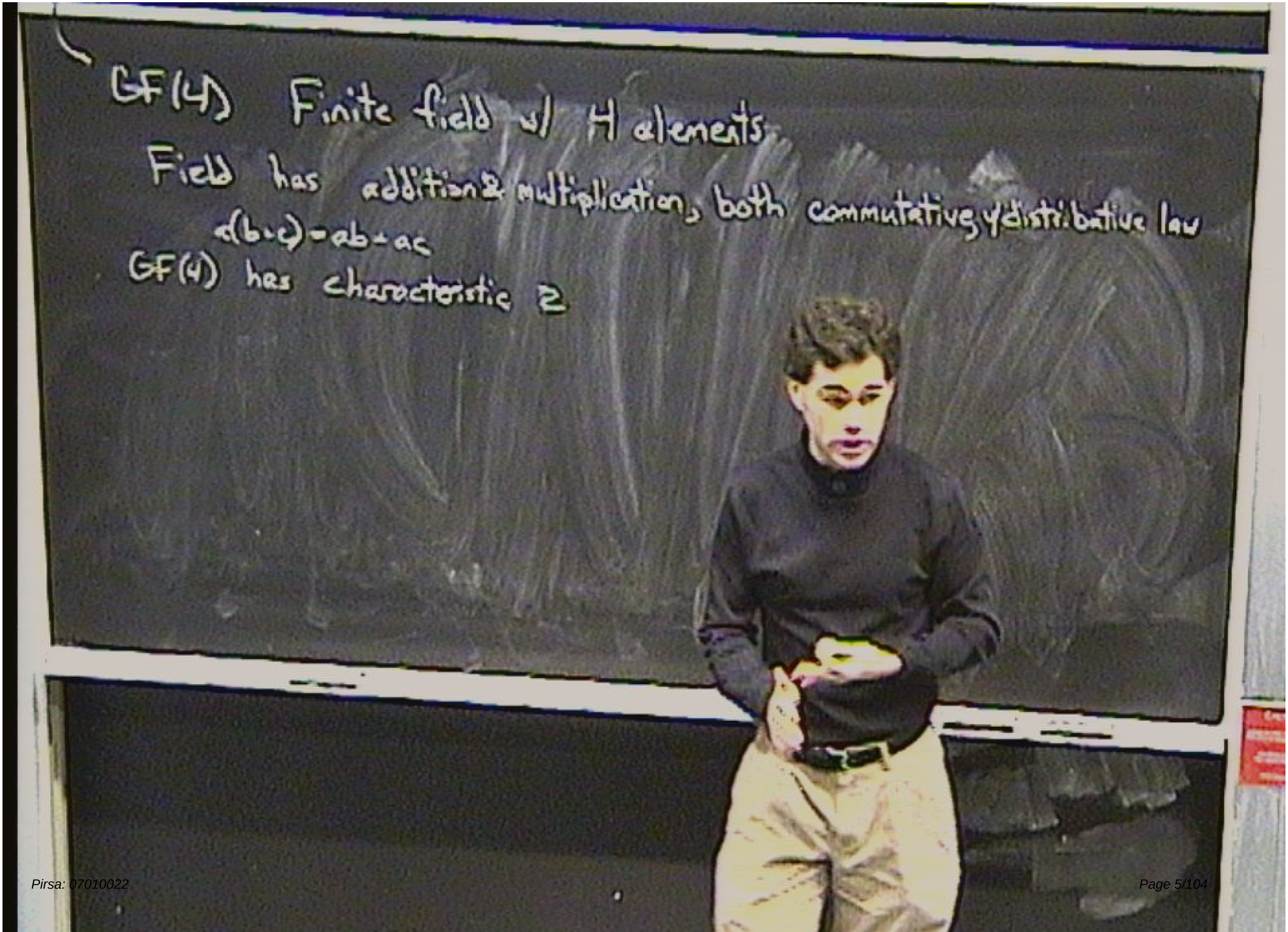
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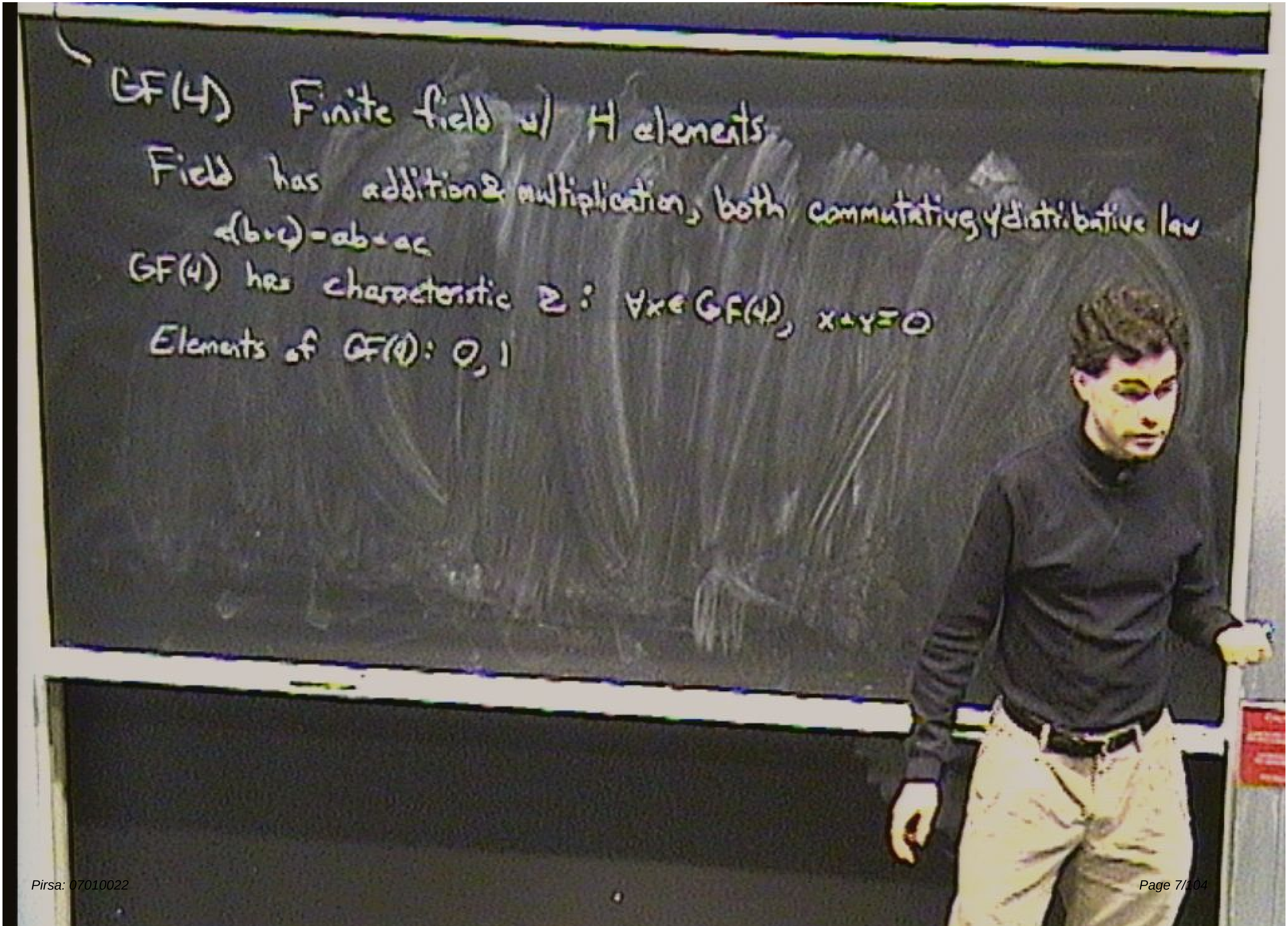
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Elements of GF(4): 0, 1

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Elements of GF(4): $0, 1, \omega, \omega^2$ $\Longleftrightarrow$ GF(4) is a 2-dim vector space over $\mathbb{Z}_2$

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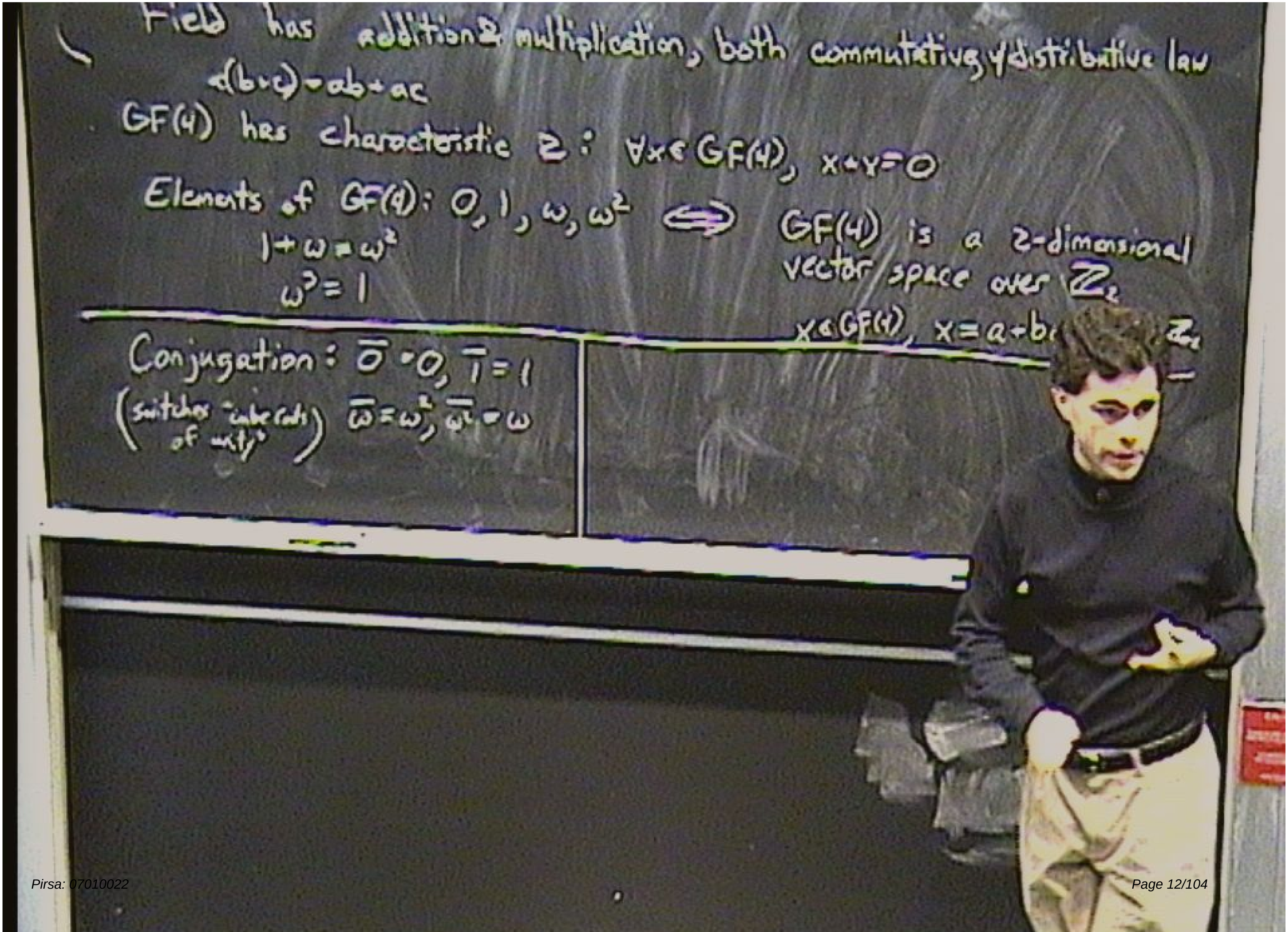
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GF(4) has characteristic 2: ∀x∈GF(4), x+x=0
Elements of GF(4): 0, 1, ω, ω² ⟺ GF(4) is a 2-dimensional vector space over ℤ₂
1+ω=ω²
ω³=1
x∈GF(4), x=a+b
Conjugation: 0̄=0, 1̄=1
(switches "cube roots of unity") ω̄=ω², ω̄²=ω

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<u>Trace</u>: $\mathrm{tr}\,x=\mathrm{tr}\,M_x$

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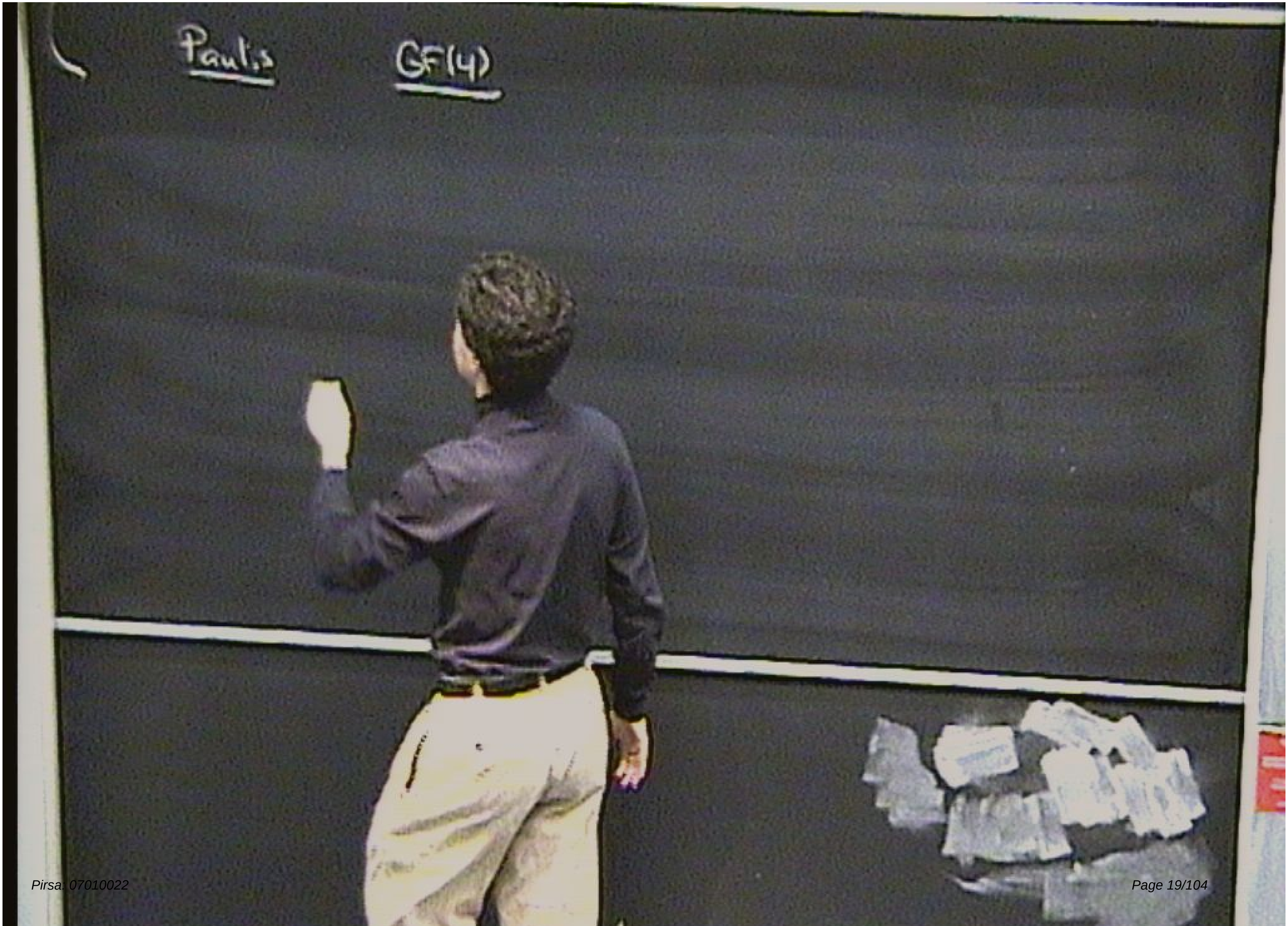
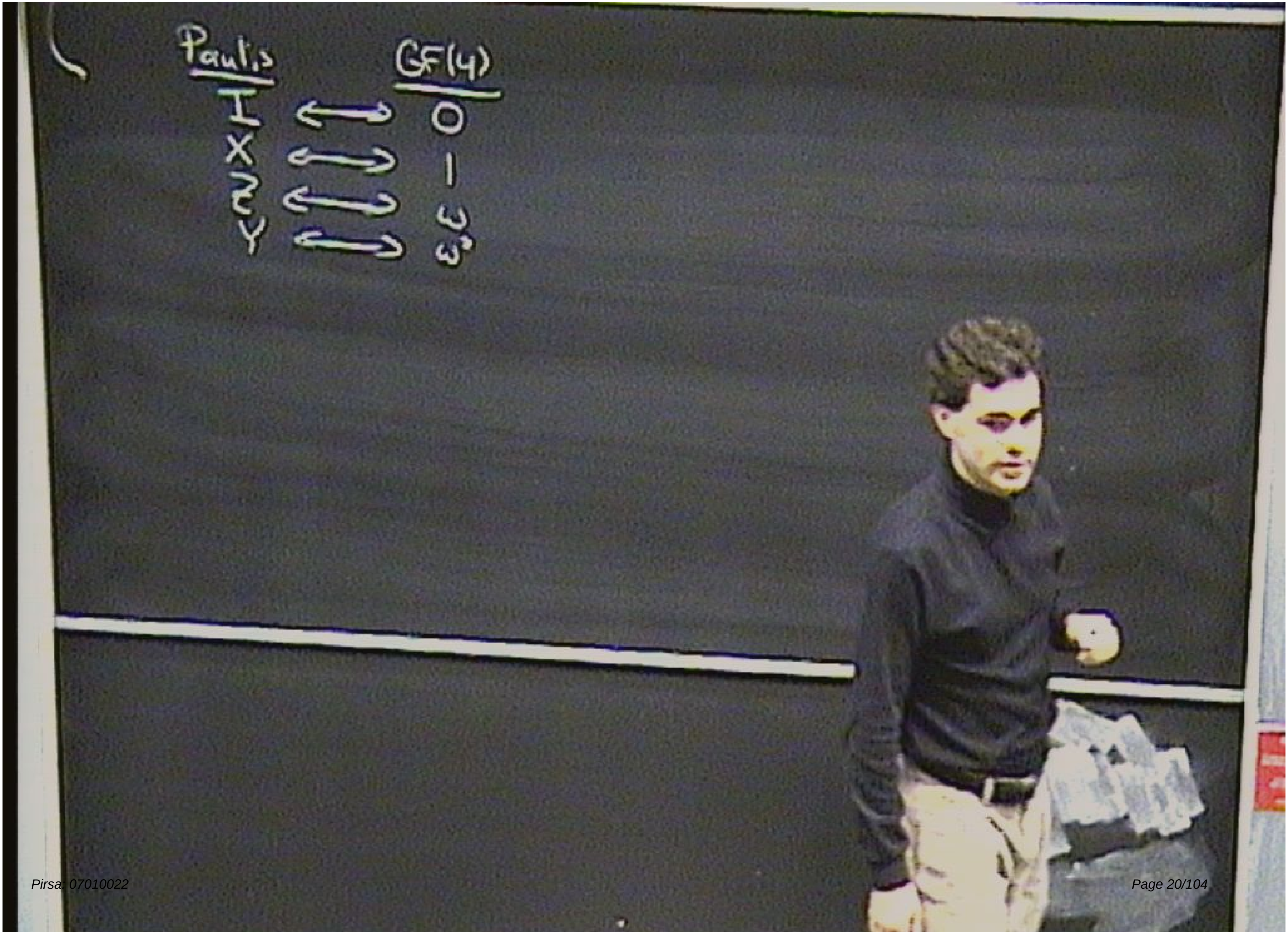
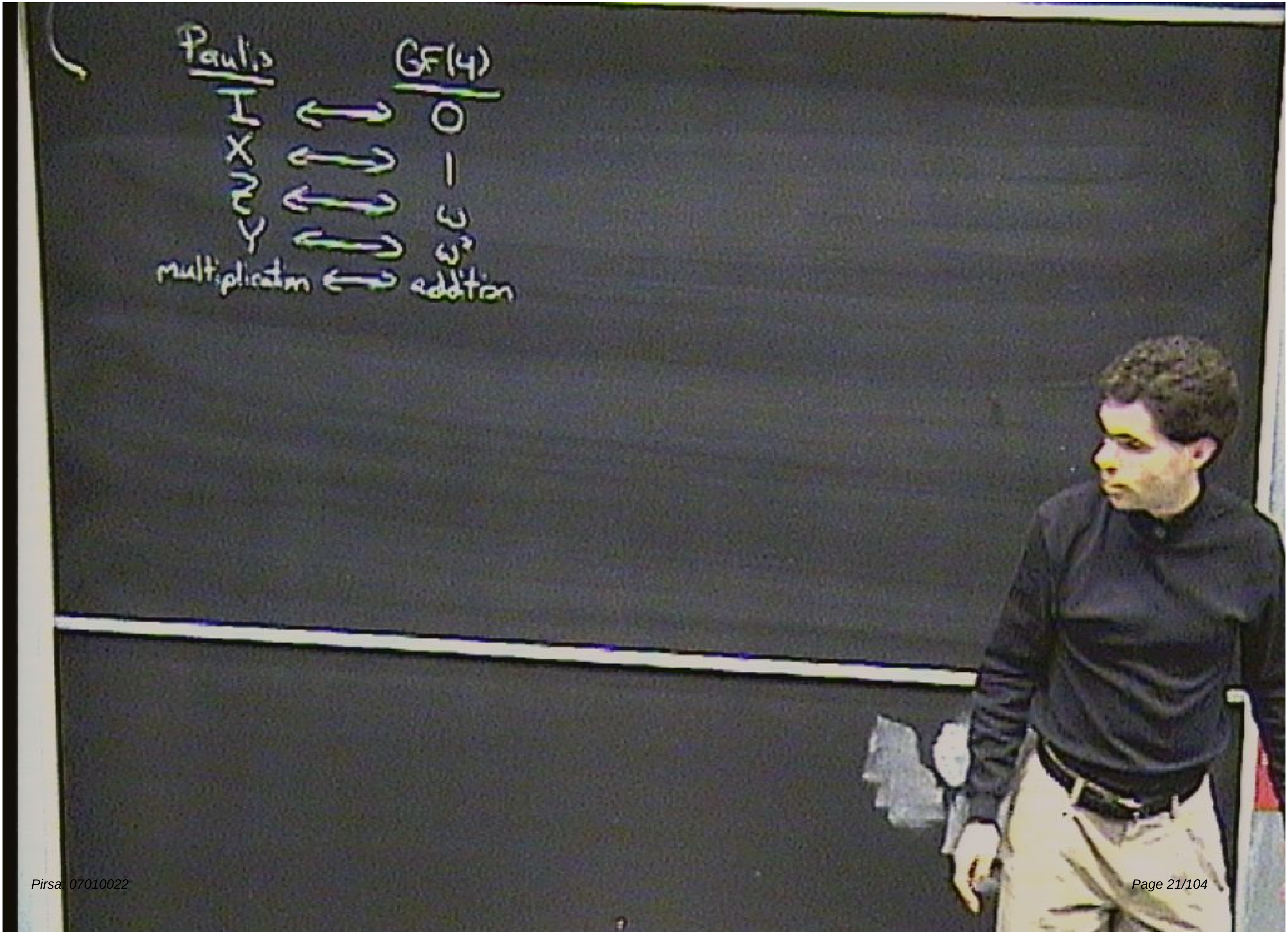
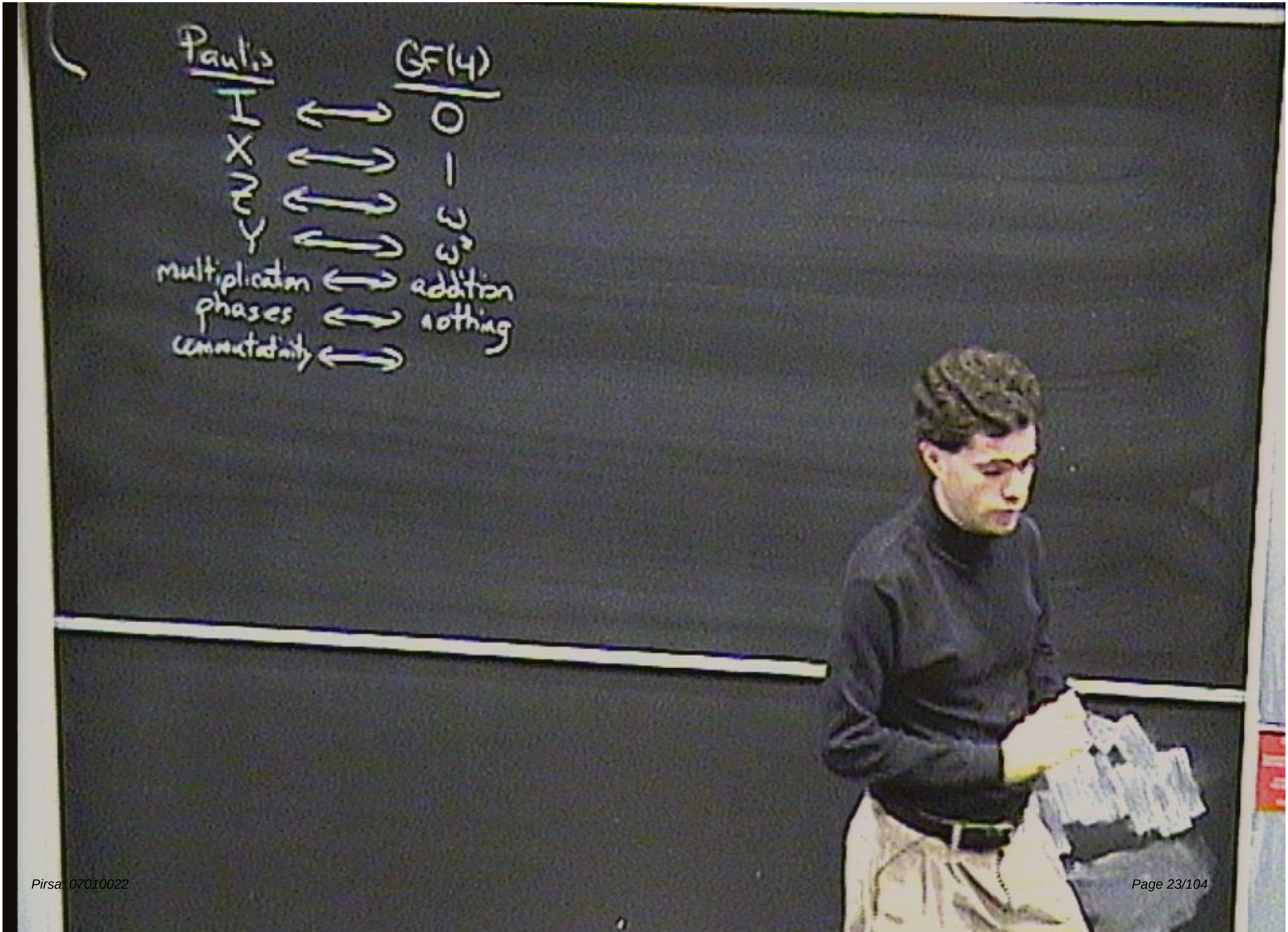

Paulis
GF(4)


Paulis GF(4)
I ⟷ 0
X ⟷ 1
Z ⟷ ω
Y ⟷ ω²


Paulis
GF(4)
I ⟷ 0
X ⟷ 1
Z ⟷ ω
Y ⟷ ω²
multiplication ⟷ addition

Paulis		GF(4)
I	$\longleftrightarrow$	0
X	$\longleftrightarrow$	1
Z	$\longleftrightarrow$	ω
Y	$\longleftrightarrow$	ω^2
multiplication	$\longleftrightarrow$	addition
phases	$\longleftrightarrow$	nothing


Paulis
GF(4)
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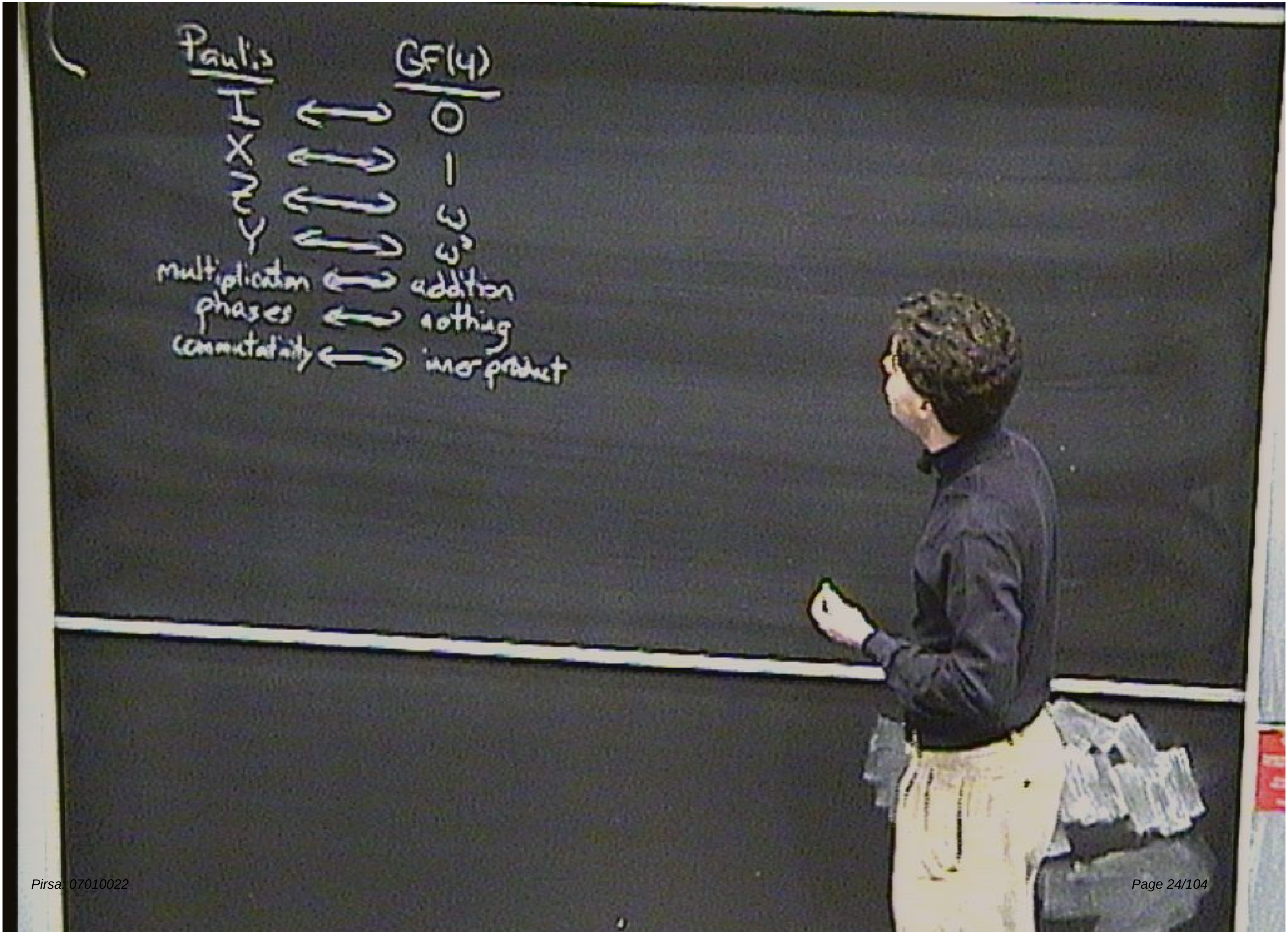
X ⟷ 1
Z ⟷ ω
Y ⟷ ω²
multiplication ⟷ addition
phases ⟷ nothing
commutativity ⟷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$$P, Q \in \mathcal{P}_n \longleftrightarrow \vec{u}, \vec{v} \in GF(4)^n$$

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$$P, Q \in \mathcal{P}_n \longleftrightarrow \vec{u}, \vec{v} \in GF(4)^n$$

$$[P,Q]=0 \Leftrightarrow tr(\vec{u} \cdot \overline{\vec{v}}) = 0$$

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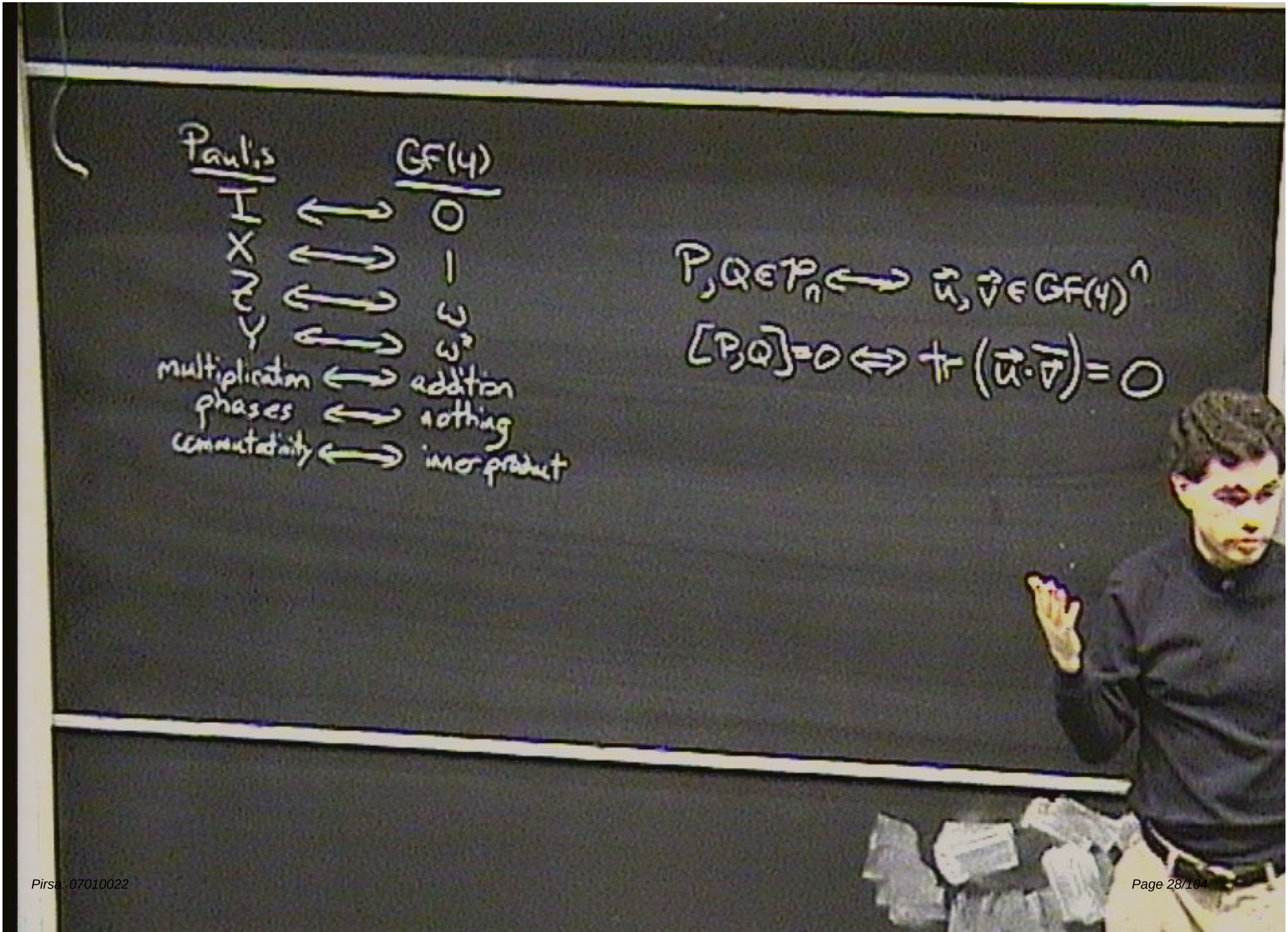
Paulis
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$Y \longleftrightarrow \omega^2$
multiplication $\longleftrightarrow$ addition
phases $\longleftrightarrow$ nothing
commutativity $\longleftrightarrow$ inner product
$P, Q \in \mathcal{P}_n \longleftrightarrow \vec{u}, \vec{v} \in GF(4)^n$
$[P, Q] = 0 \Leftrightarrow \mathrm{tr}(\vec{u} \cdot \bar{\vec{v}}) = 0$

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X	$\longleftrightarrow$	1
Z	$\longleftrightarrow$	ω
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multiplication	$\longleftrightarrow$	addition
phases	$\longleftrightarrow$	nothing
commutativity	$\longleftrightarrow$	inner product

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Y	$\longleftrightarrow$	ω^2
multiplication	$\longleftrightarrow$	addition
phases	$\longleftrightarrow$	nothing
commutativity	$\longleftrightarrow$	inner product multiplication

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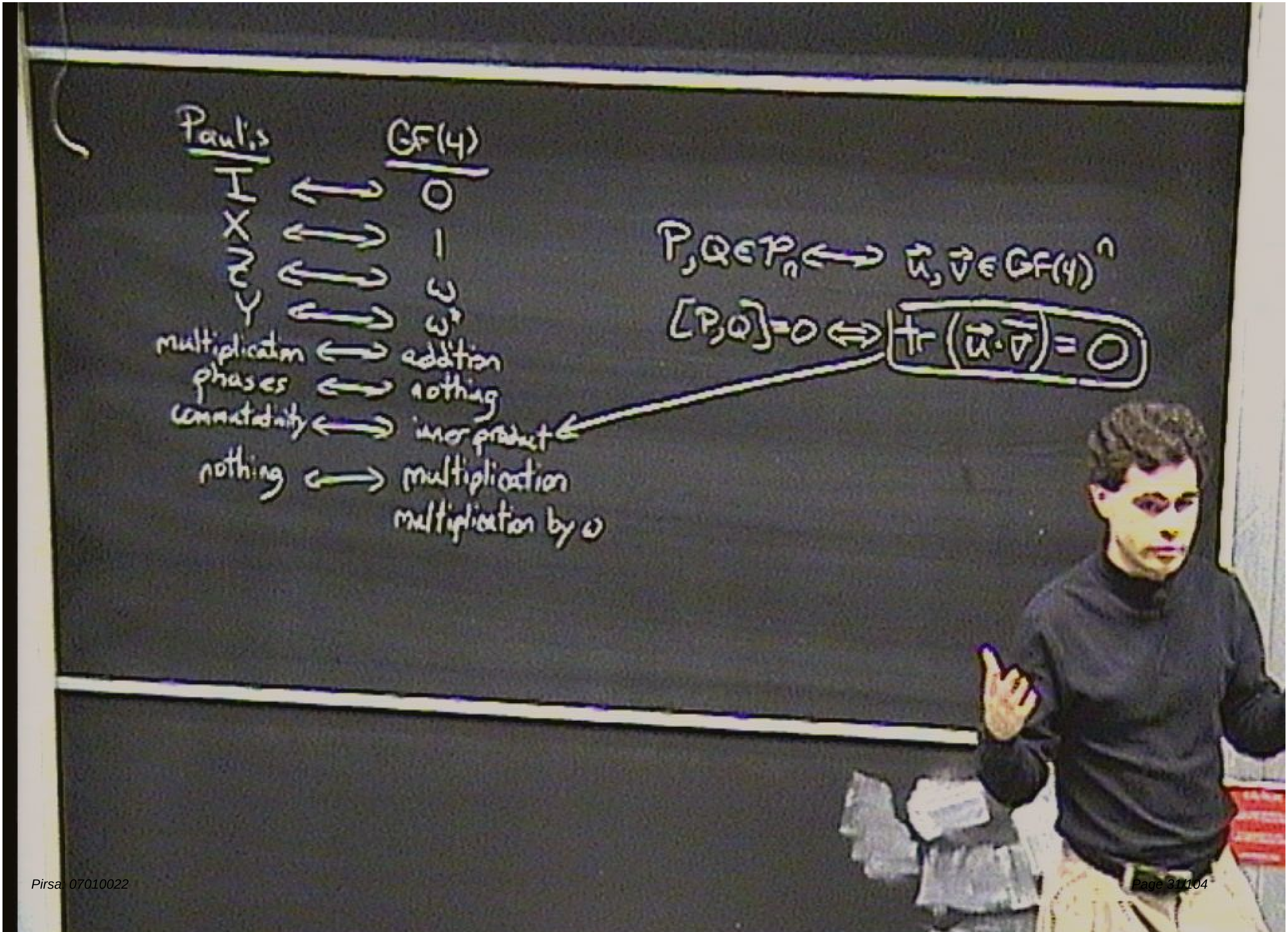
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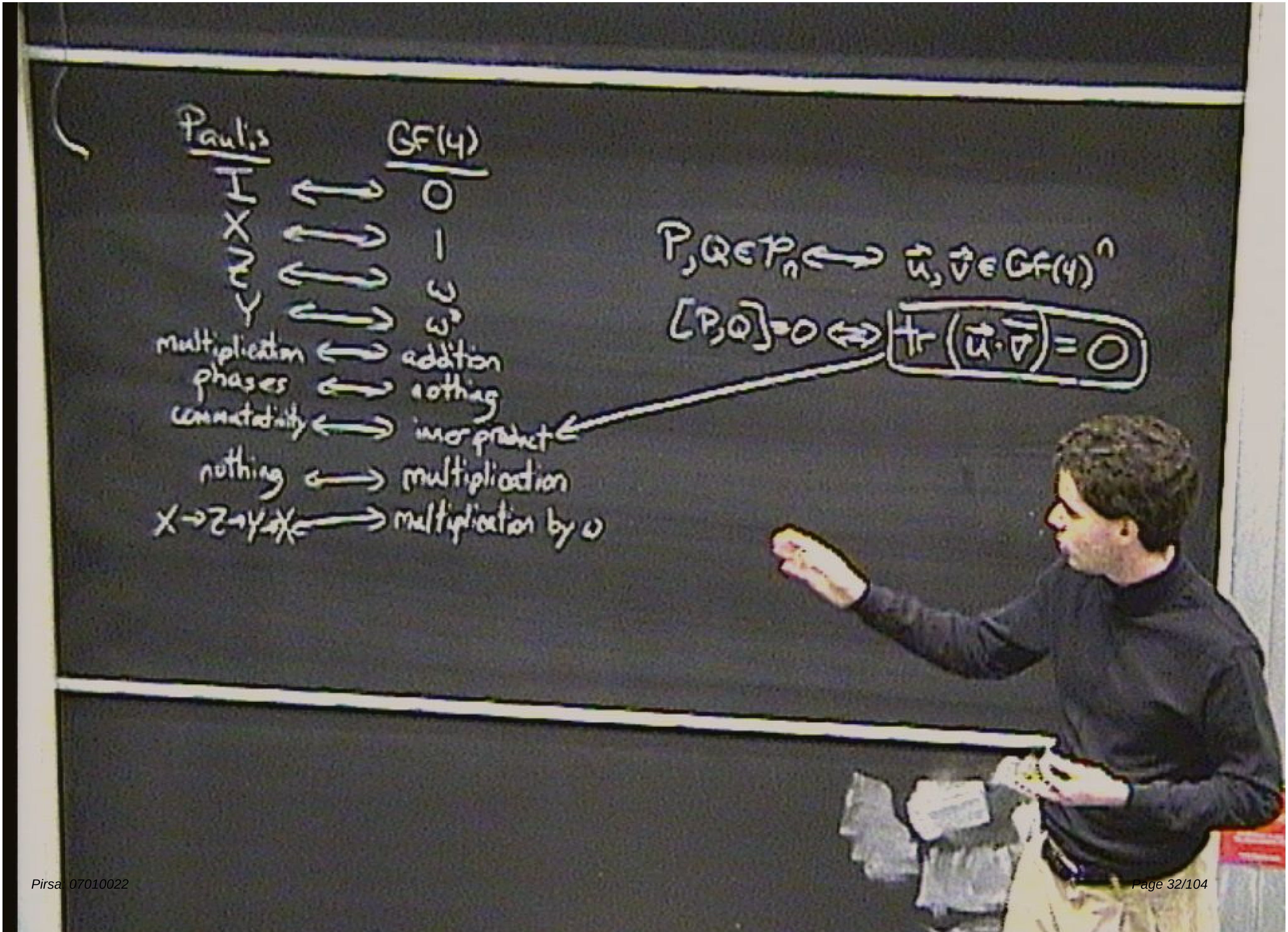
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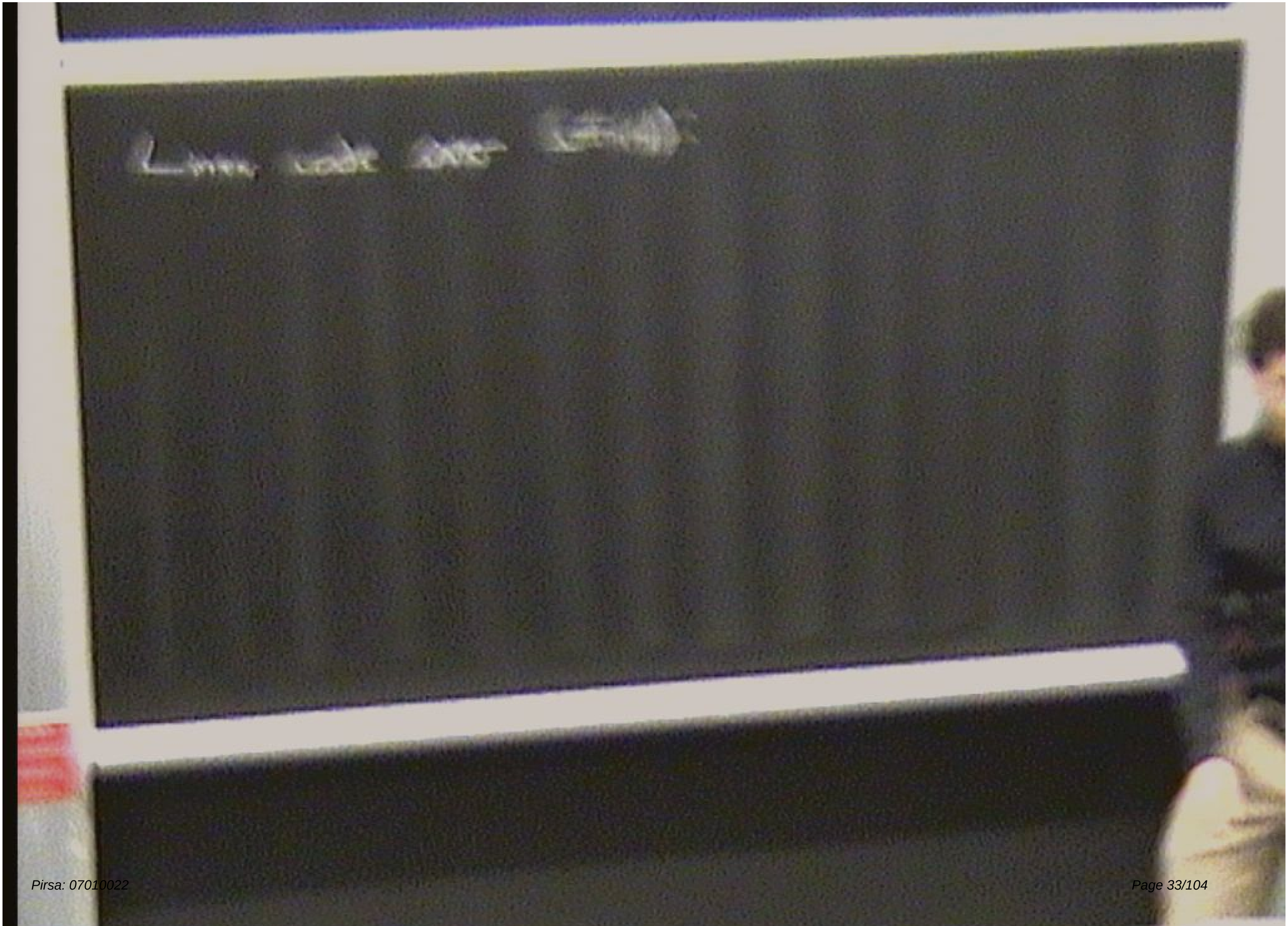
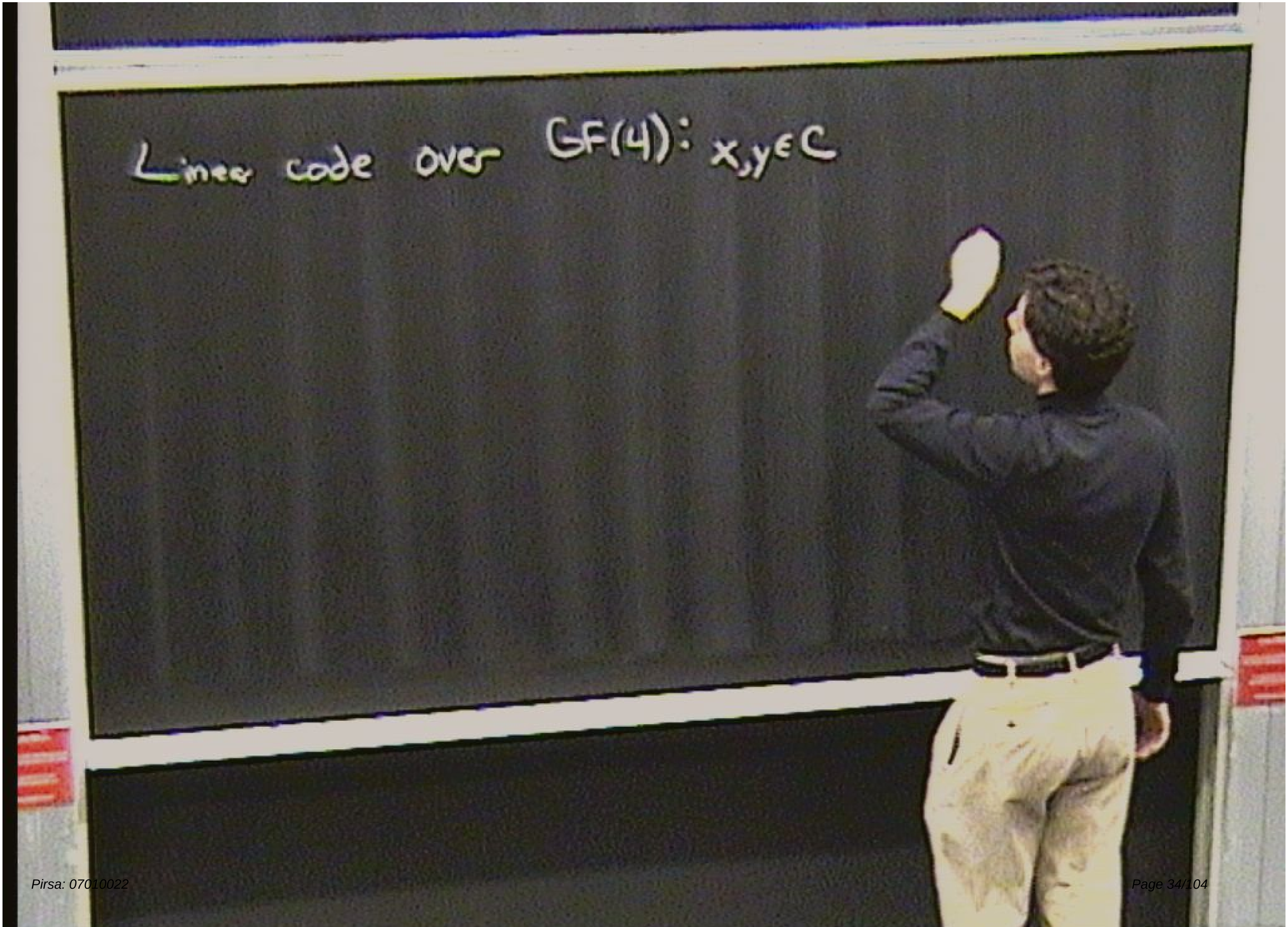
X → Z → Y → X ⟷ multiplication by ω
P, Q ∈ 𝒫ₙ ⟷ u⃗, v⃗ ∈ GF(4)ⁿ
[P, Q] = 0 ⟷ tr(u⃗ · v⃗) = 0




Linear code over GF(4): x,y∈C

Linear code over $GF(4)$: $\vec{x}, \vec{y} \in C \Rightarrow \vec{x} + \alpha \vec{y} \in C$, $\alpha \in GF(4)$

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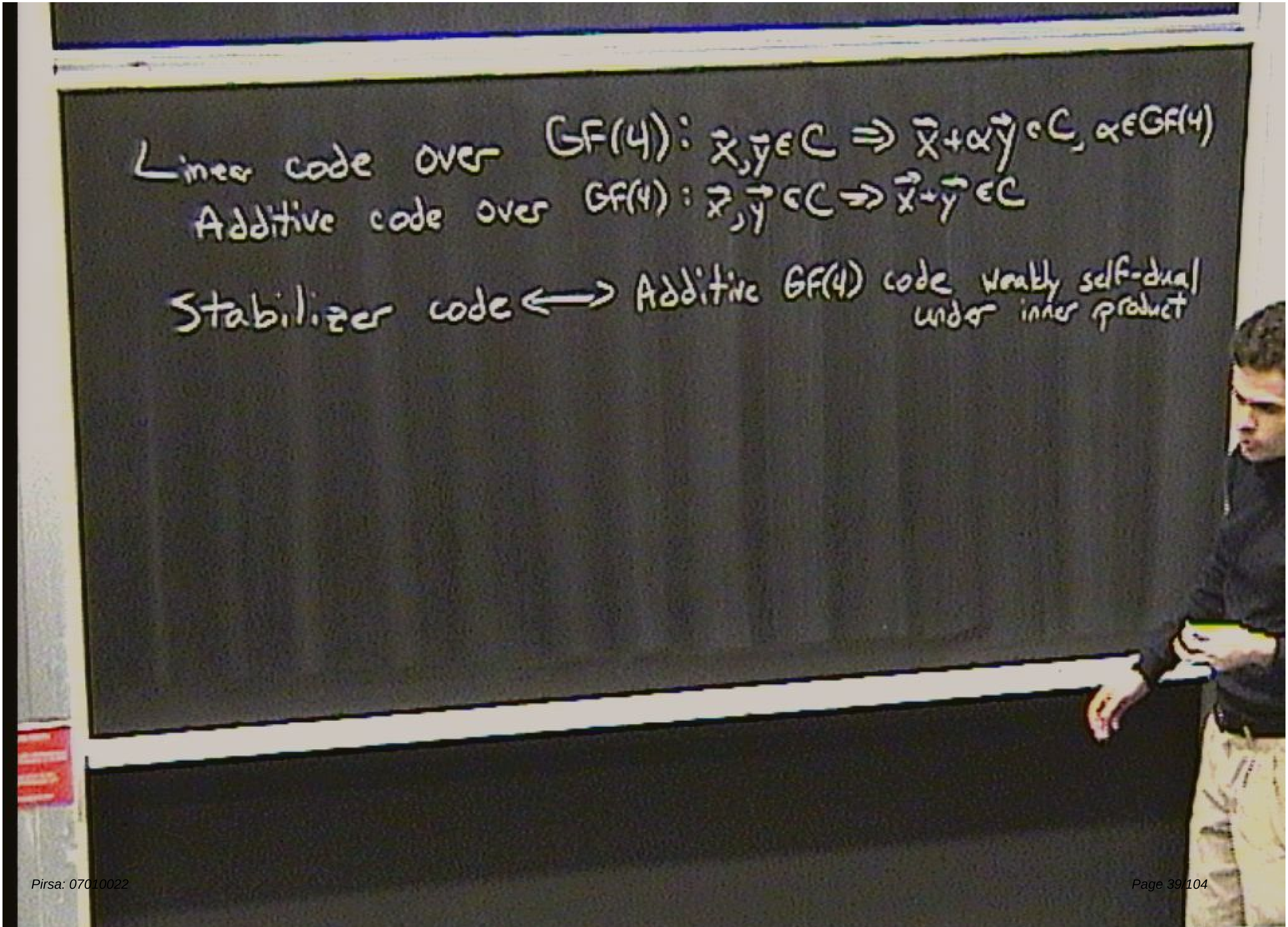
Stabilizer code $\longleftrightarrow$

Linear code over GF(4): $\vec{x},\vec{y}\in C \Rightarrow \vec{x}+\alpha\vec{y}\in C,\ \alpha\in GF(4)$

Additive code over GF(4): $\vec{x},\vec{y}\in C \Rightarrow \vec{x}+\vec{y}\in C$

Stabilizer code $\longleftrightarrow$ Additive GF(4) code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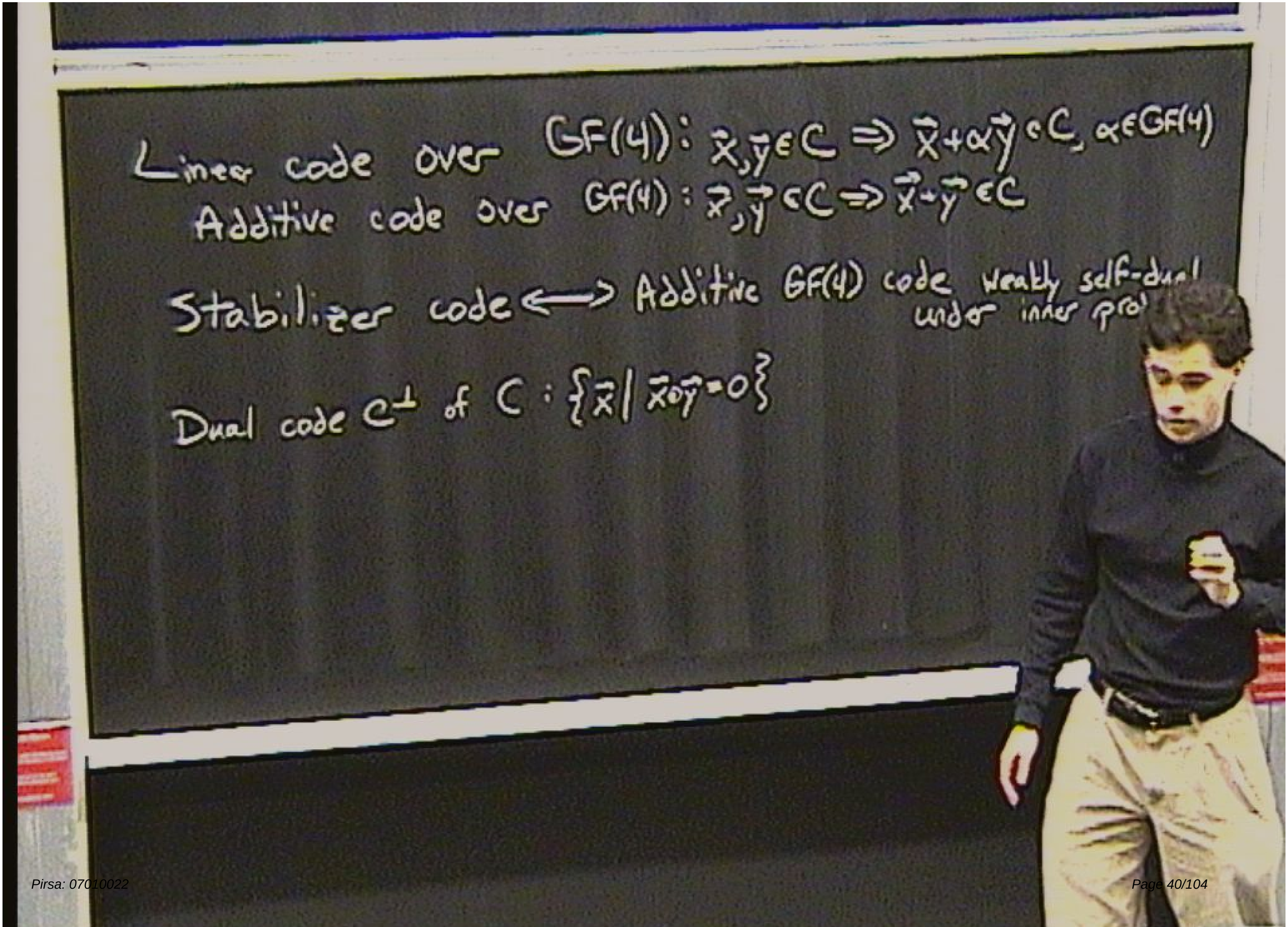
Stabilizer code $\longleftrightarrow$ Additive GF(4) code weakly self-dual under inner product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Dual code $C^{\perp}$ of C: $\{\vec{x} \mid \vec{x} \circ \vec{y} = 0\}$

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Dual code $C^{\perp}$ of C: $\{\vec{x} \mid \vec{x}\odot\vec{y}=0 \ \forall \vec{y}\in C\}$, $\vec{x}\odot\vec{y} = tr(\vec{x}\cdot\bar{\vec{y}})$

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Weakly self-dual: $C \subseteq C^{\perp}$

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Weakly self-dual: $C\subseteq C^{\perp}$ $\Leftrightarrow$ stabilizer Abelian

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$N(S) \Leftrightarrow C^{\perp}$ Distance = min wt. $x \mid x \in C^{\perp} \setminus C$

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Note:

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Stabilizer code $\Longleftrightarrow$ Additive $GF(4)$ code weakly self-dual under inner product

Dual code $C^{\perp}$ of C: $\{\vec{x}\,|\,\vec{x}\odot\vec{y}=0\ \forall\vec{y}\in C\}$, $\vec{x}\odot\vec{y}=\mathrm{tr}(\vec{x}\cdot\vec{\bar{y}})$

Weakly self-dual: $C\subseteq C^{\perp}$ $\Longleftrightarrow$ Stabilizer Abelian

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Note: Distance can be larger than distance of $C^{\perp}$ because of degeneracy

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Hamming codes over GF(4):

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Hamming codes over GF(4): Linear codes

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Hamming codes over GF(4): Linear codes

2-row parity check matrix

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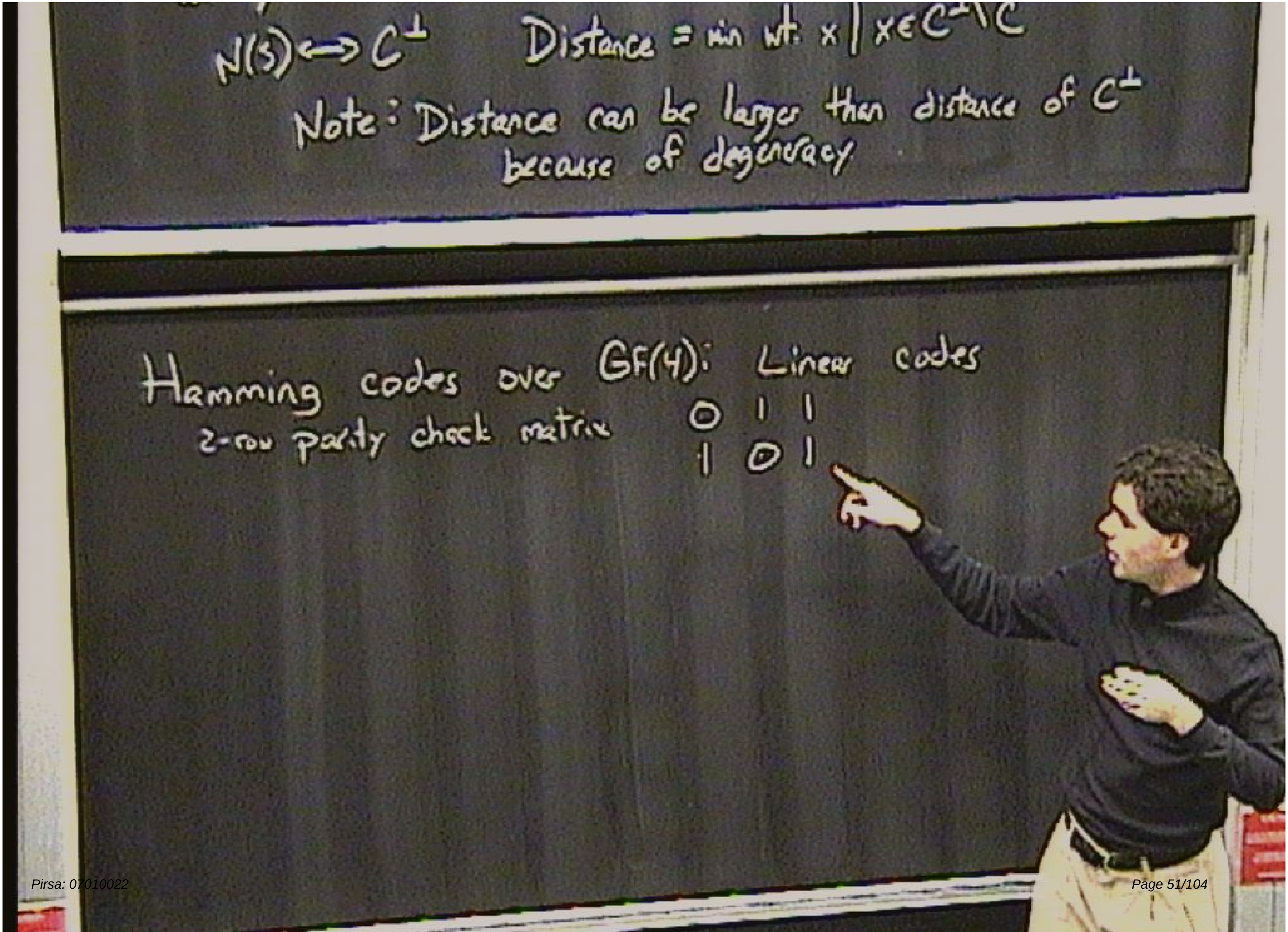
Hamming codes over GF(4): Linear codes

2-row parity check matrix
$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$$


$N(S) \Longleftrightarrow C^{\perp}$
Distance = min wt. $x \mid x \in C^{\perp} \setminus C$
Note: Distance can be larger than distance of $C^{\perp}$
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Hamming codes over GF(4): Linear codes
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Convert to stabilizer code linear $\rightarrow$ additive

Linear code over GF(4): $\vec{x},\vec{y}\in C \Rightarrow \vec{x}+\alpha\vec{y}\in C,\ \alpha\in GF(4)$

Additive code over GF(4): $\vec{x},\vec{y}\in C \Rightarrow \vec{x}+\vec{y}\in C$

Stabilizer code $\Longleftrightarrow$ Additive GF(4) code weakly self-dual under inner product

Dual code $C^\perp$ of C: $\{\vec{x} \mid \vec{x}\circ\vec{y}=0\ \forall \vec{y}\in C\}$, $\vec{x}\circ\vec{y}=\mathrm{tr}(\vec{x}\cdot\bar{\vec{y}})$

Weakly self-dual: $C\subseteq C^\perp \Longleftrightarrow$ Stabilizer Abelian

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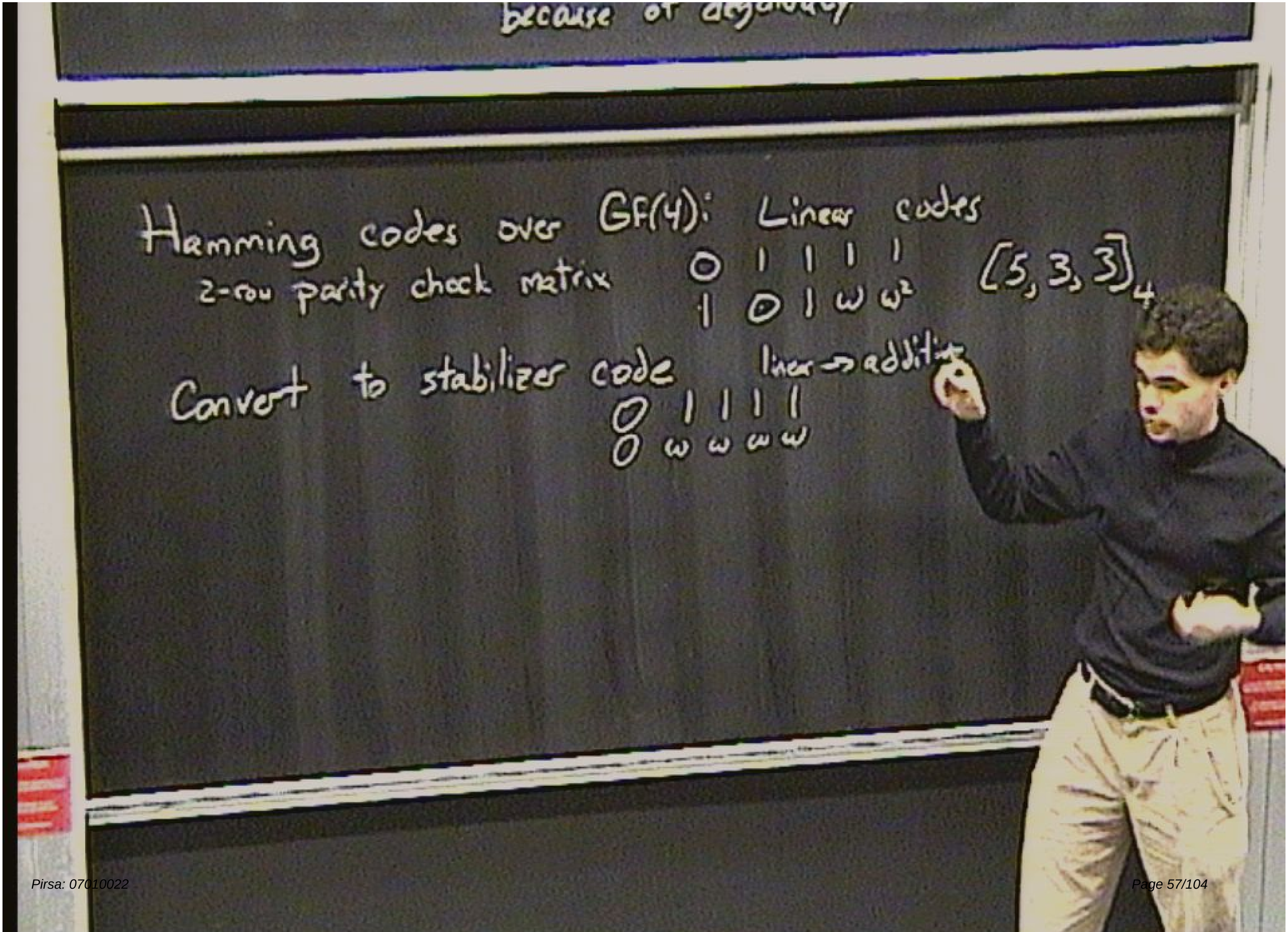
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Hamming codes over GF(4): Linear codes
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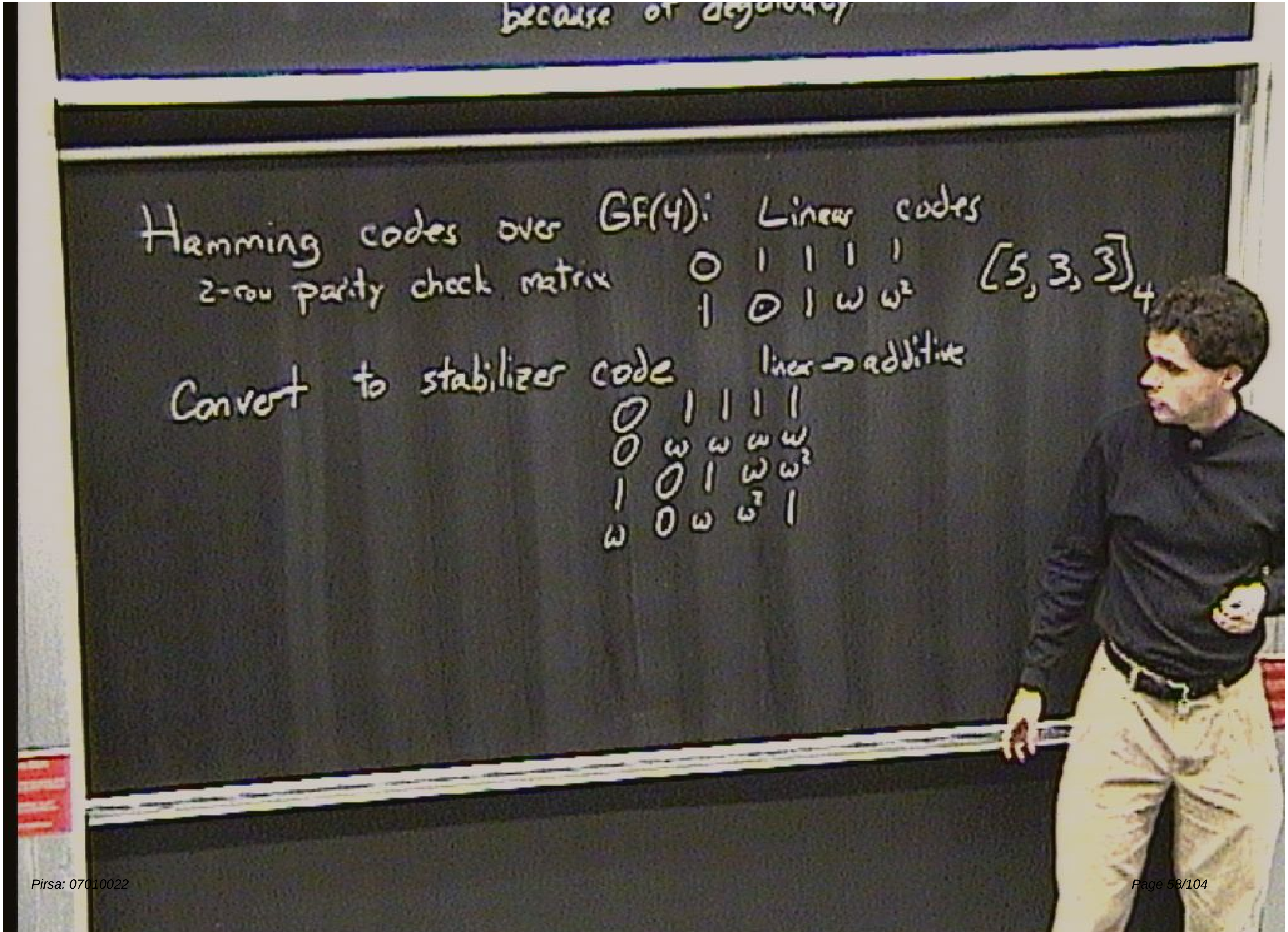
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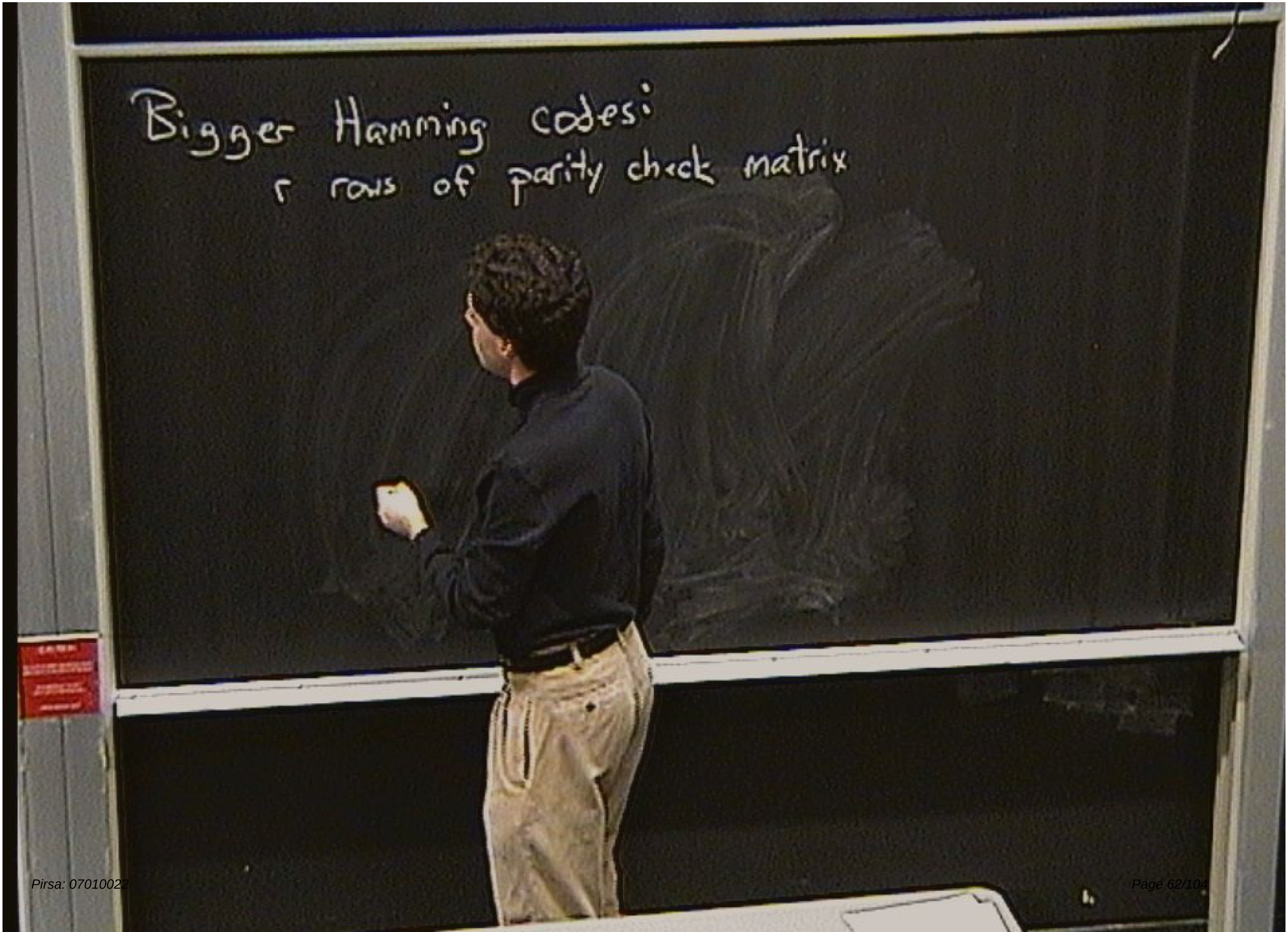
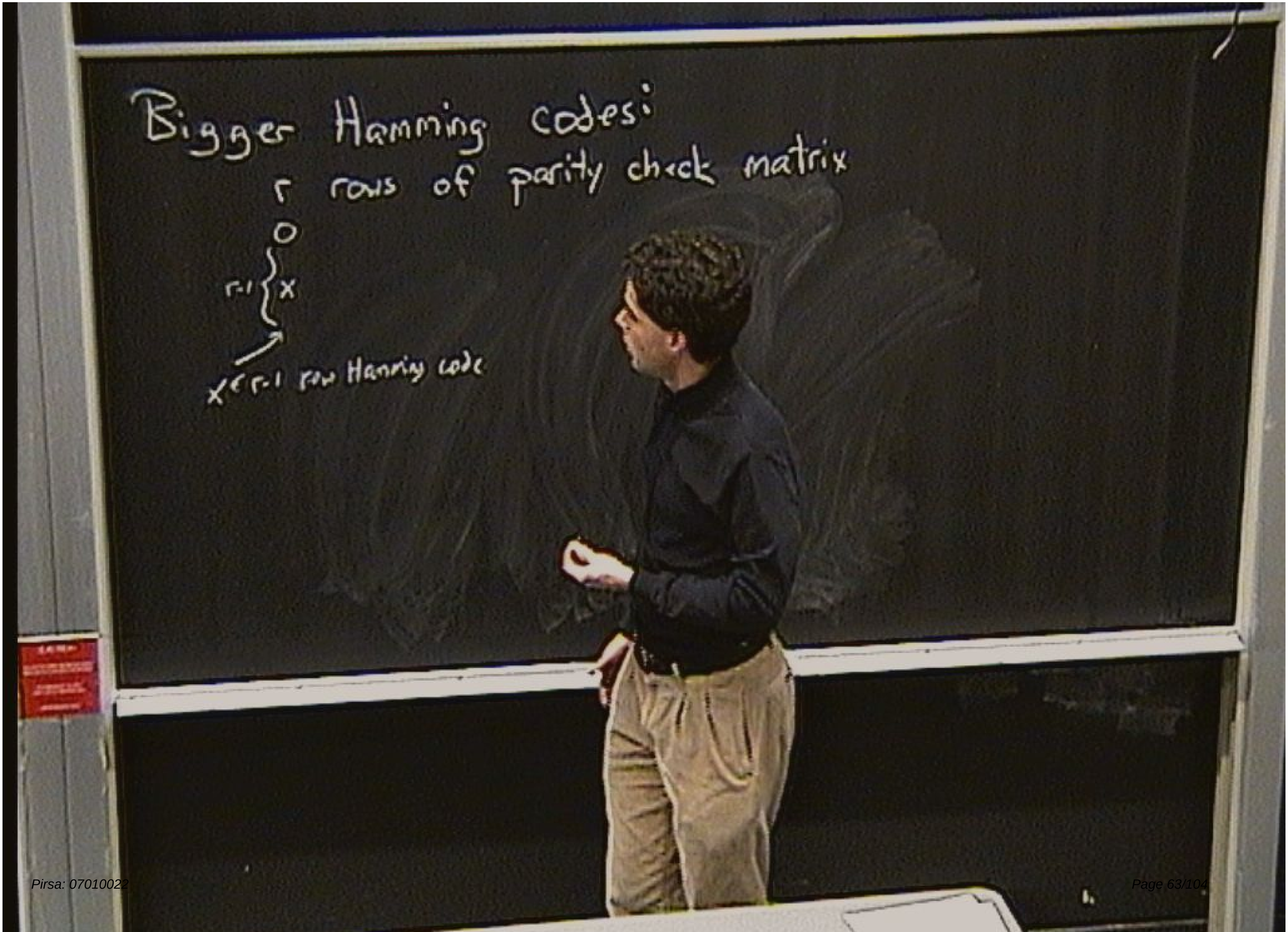
$[[5,1,3]]$


Bigger Hamming codes:
r rows of parity check matrix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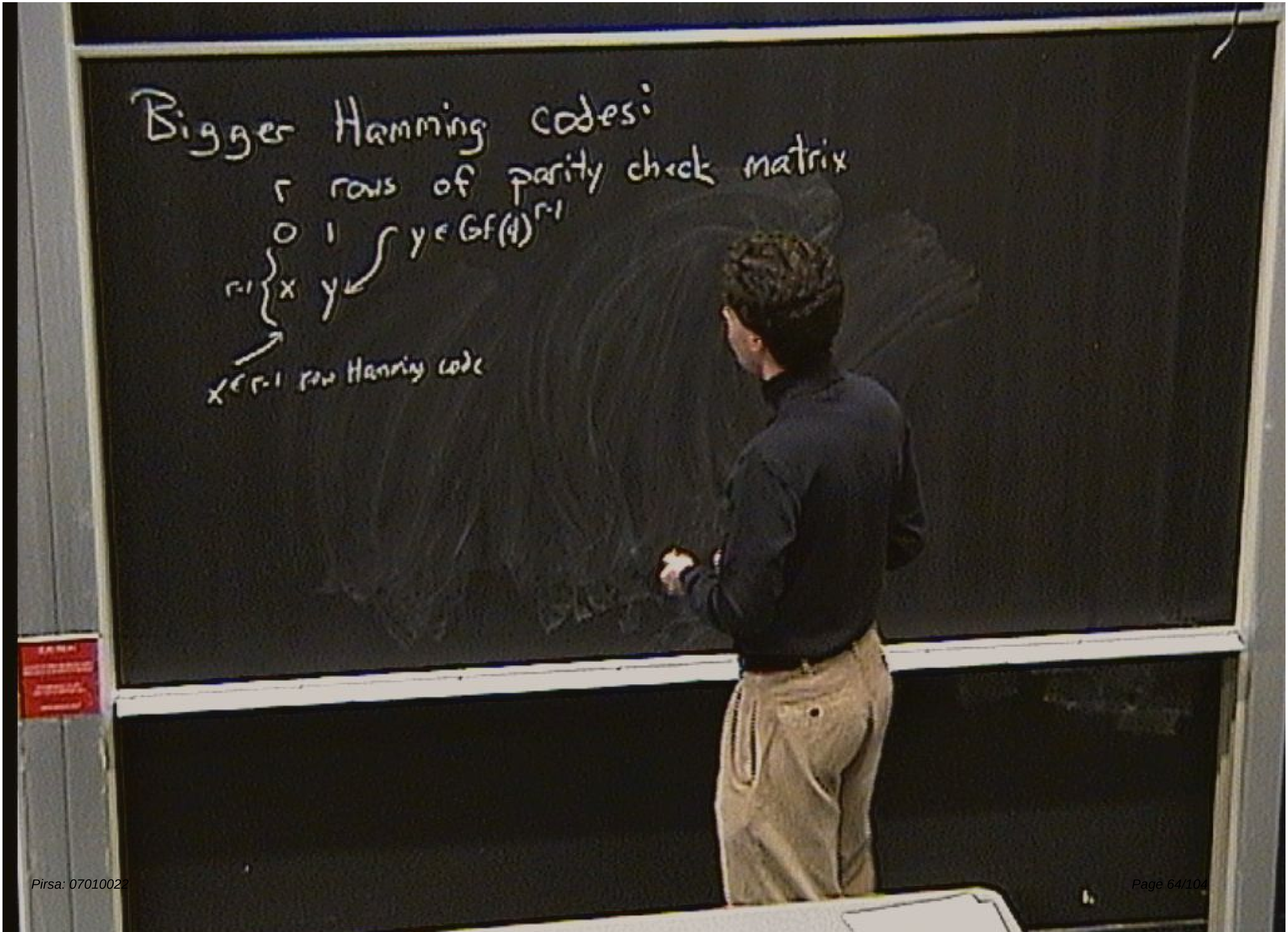
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$x \in r-1$ row Hamming code

$$n_r = n_{r-1} + 4^{r-1} = 1 + 4 + 4^2 \ldots + 4^{r-1}$$

$$= \frac{4^r - 1}{3}$$

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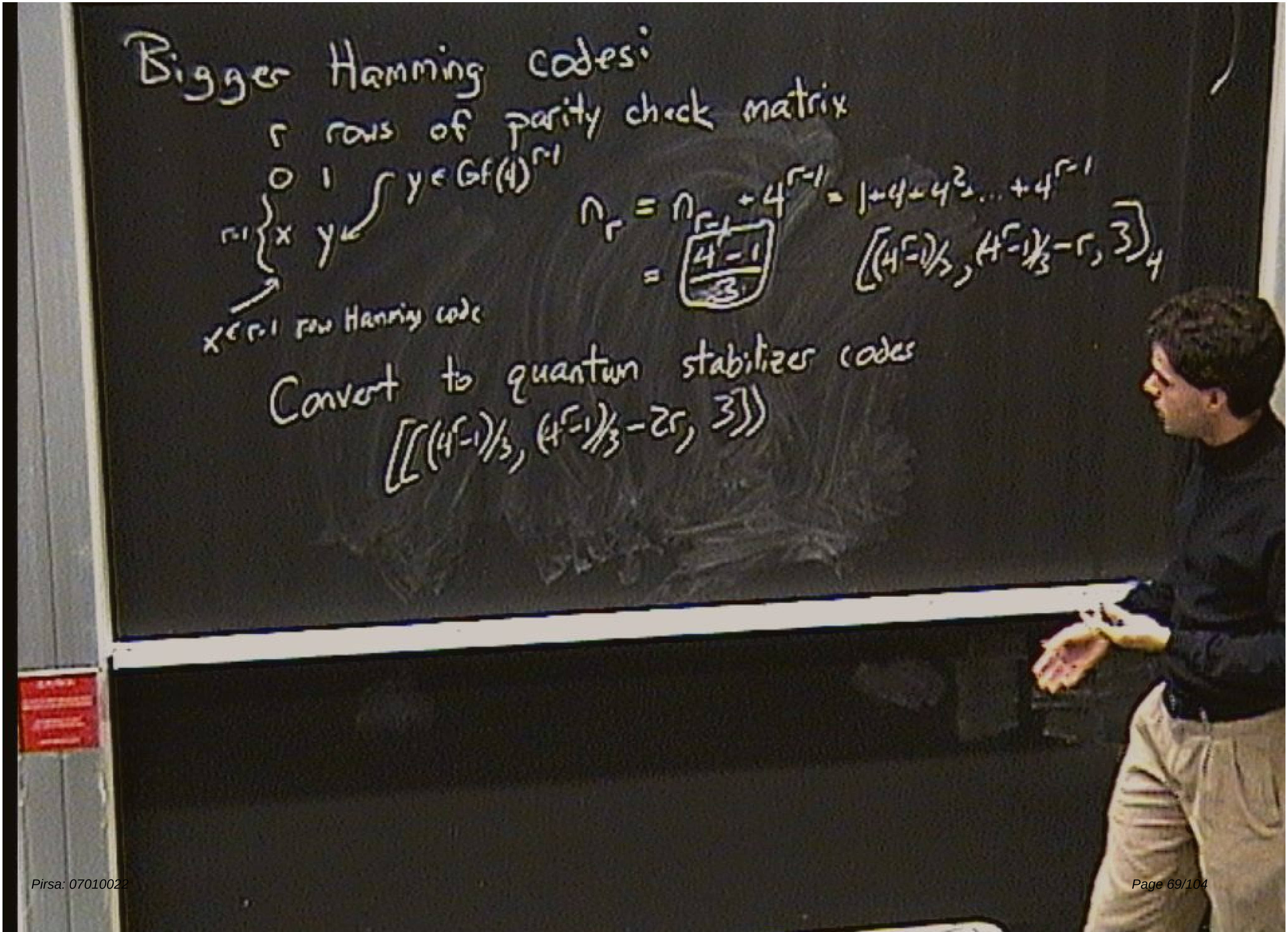
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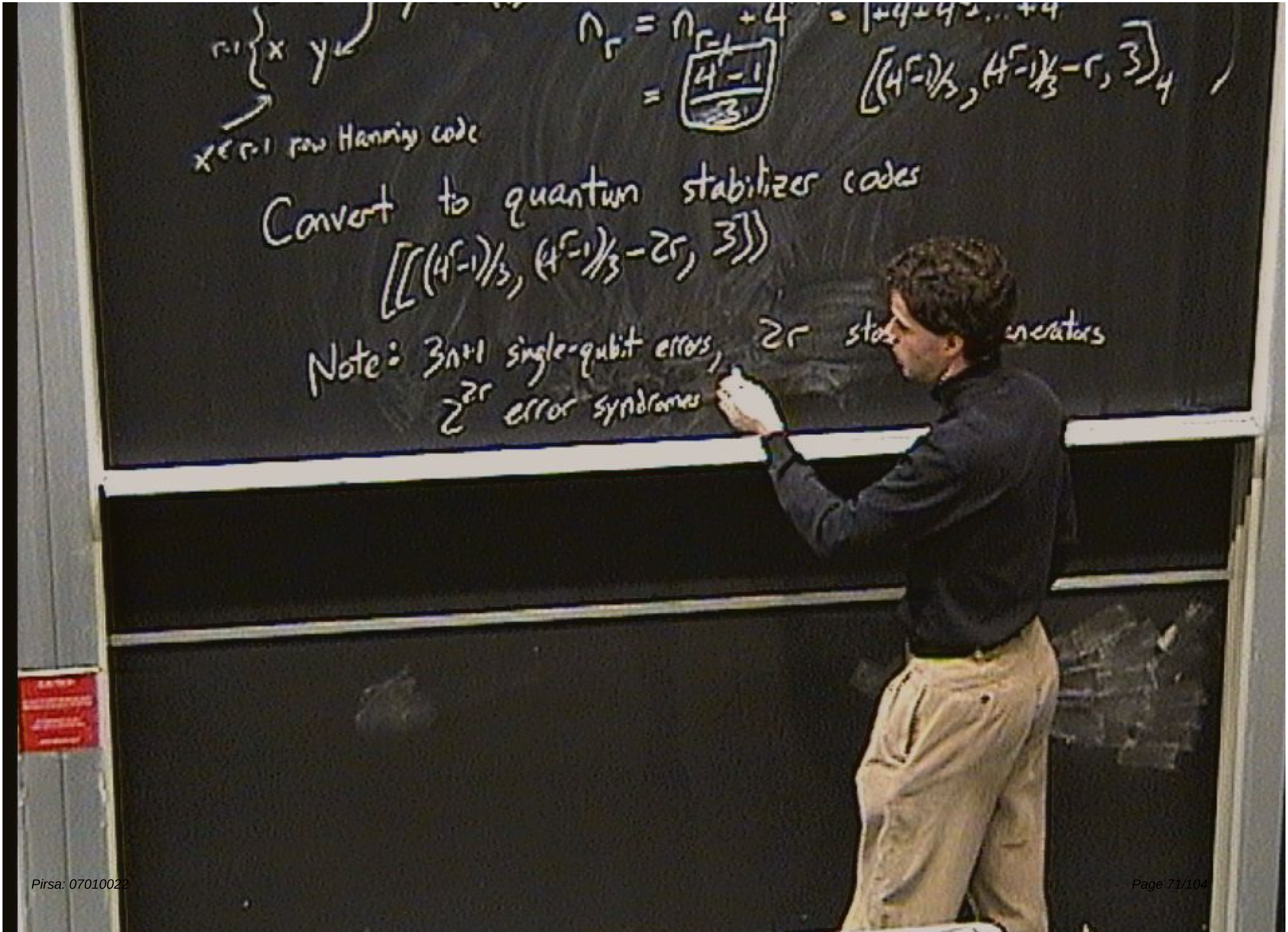
Note: $3n+1$ single-qubit errors, 2^k encoded states



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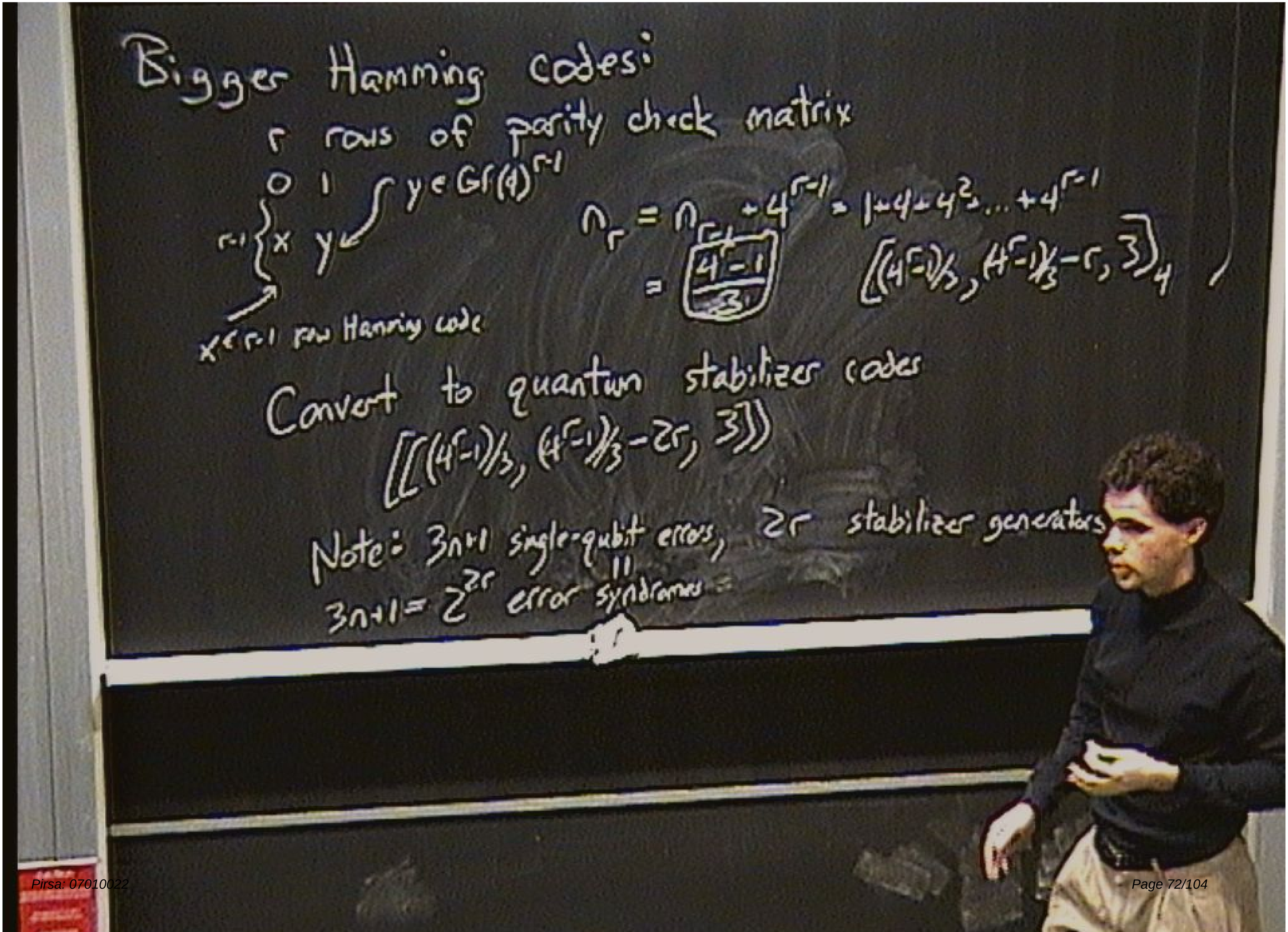
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\\
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$r=2$ $[[5,1,3]]$

$r=3$ $[[21,15,3]]$

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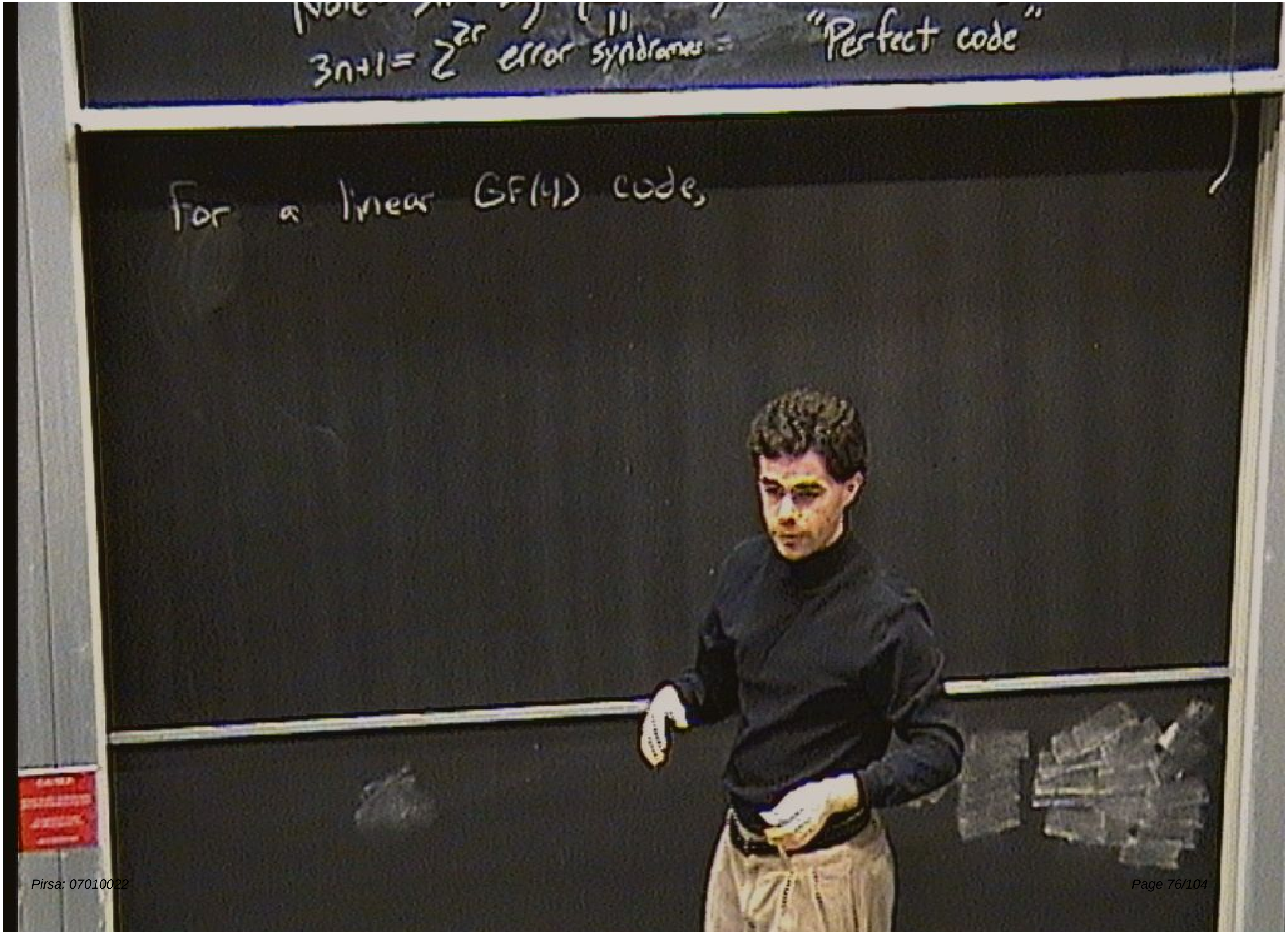
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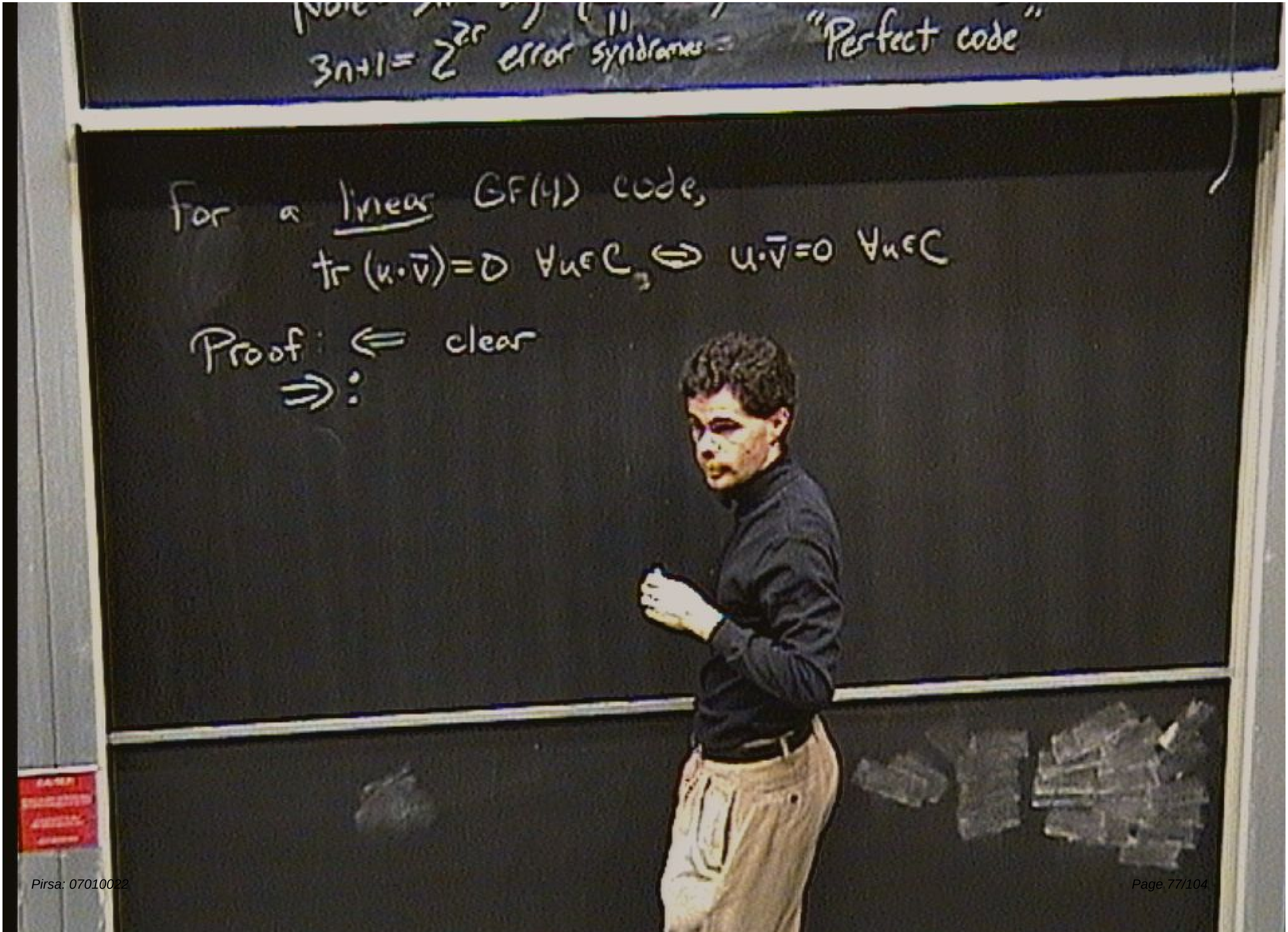
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tr (u·v̄)=0 ∀u∈C_2 ⟺ u·v̄=0 ∀u∈C
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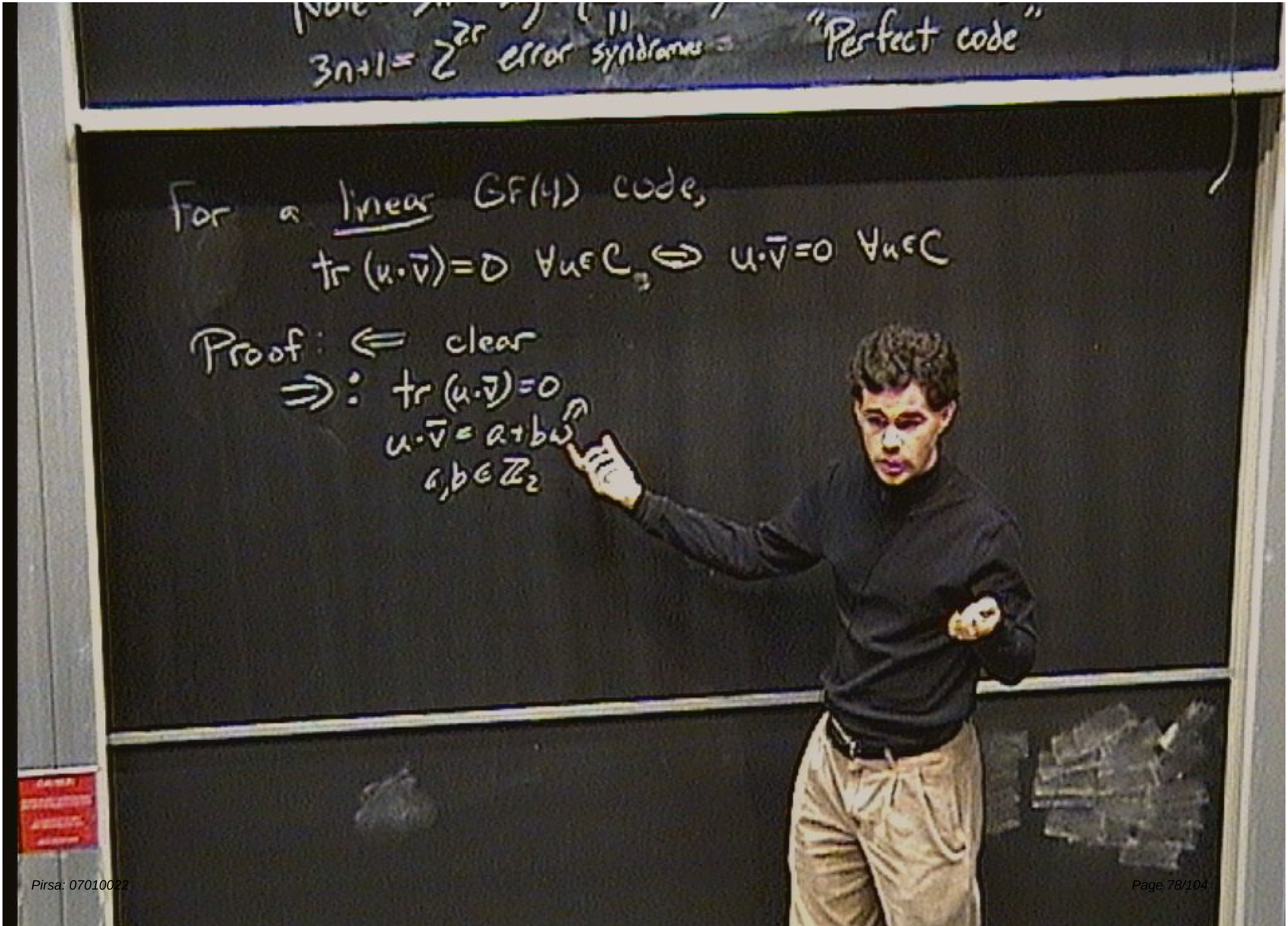
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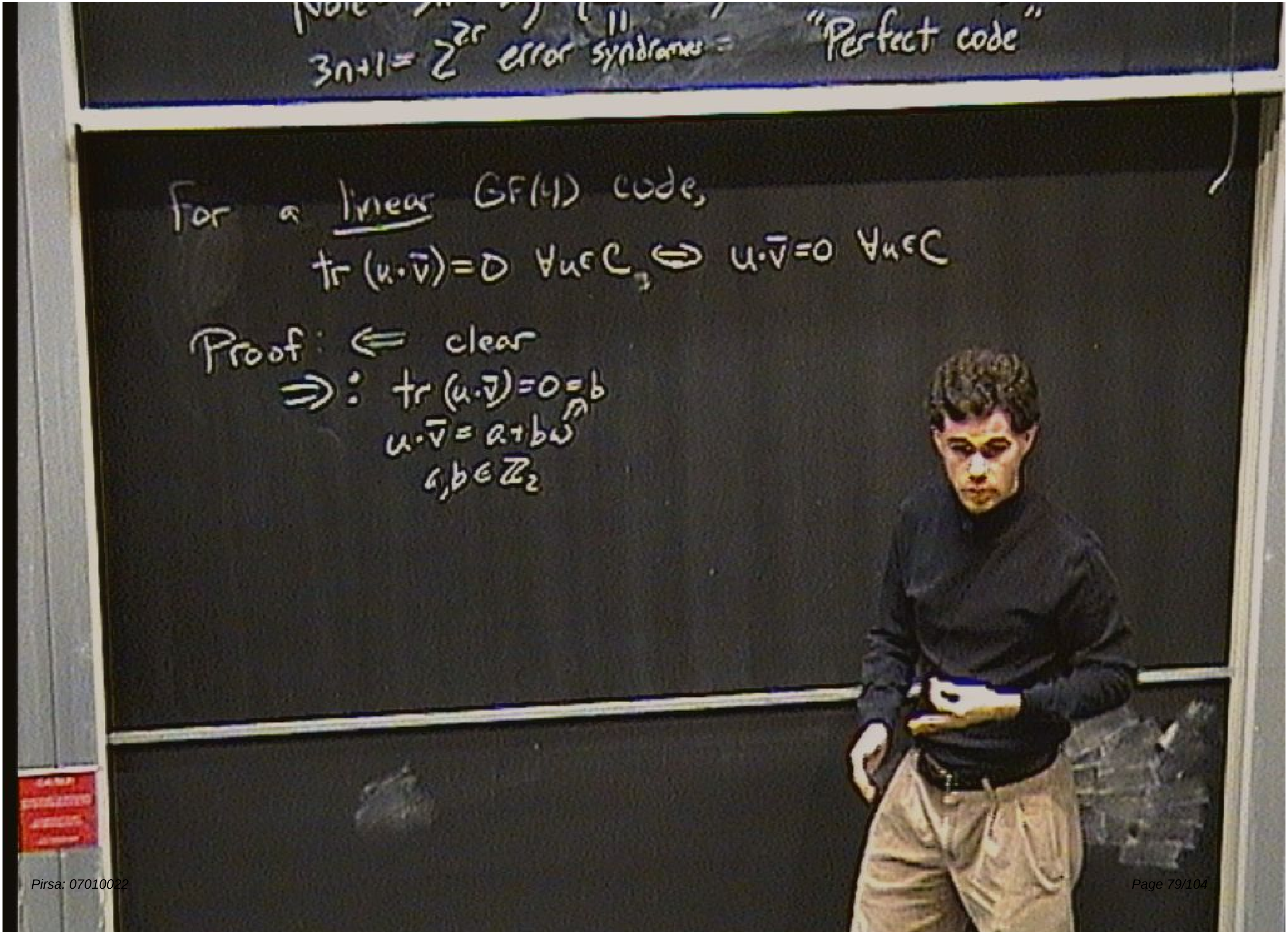
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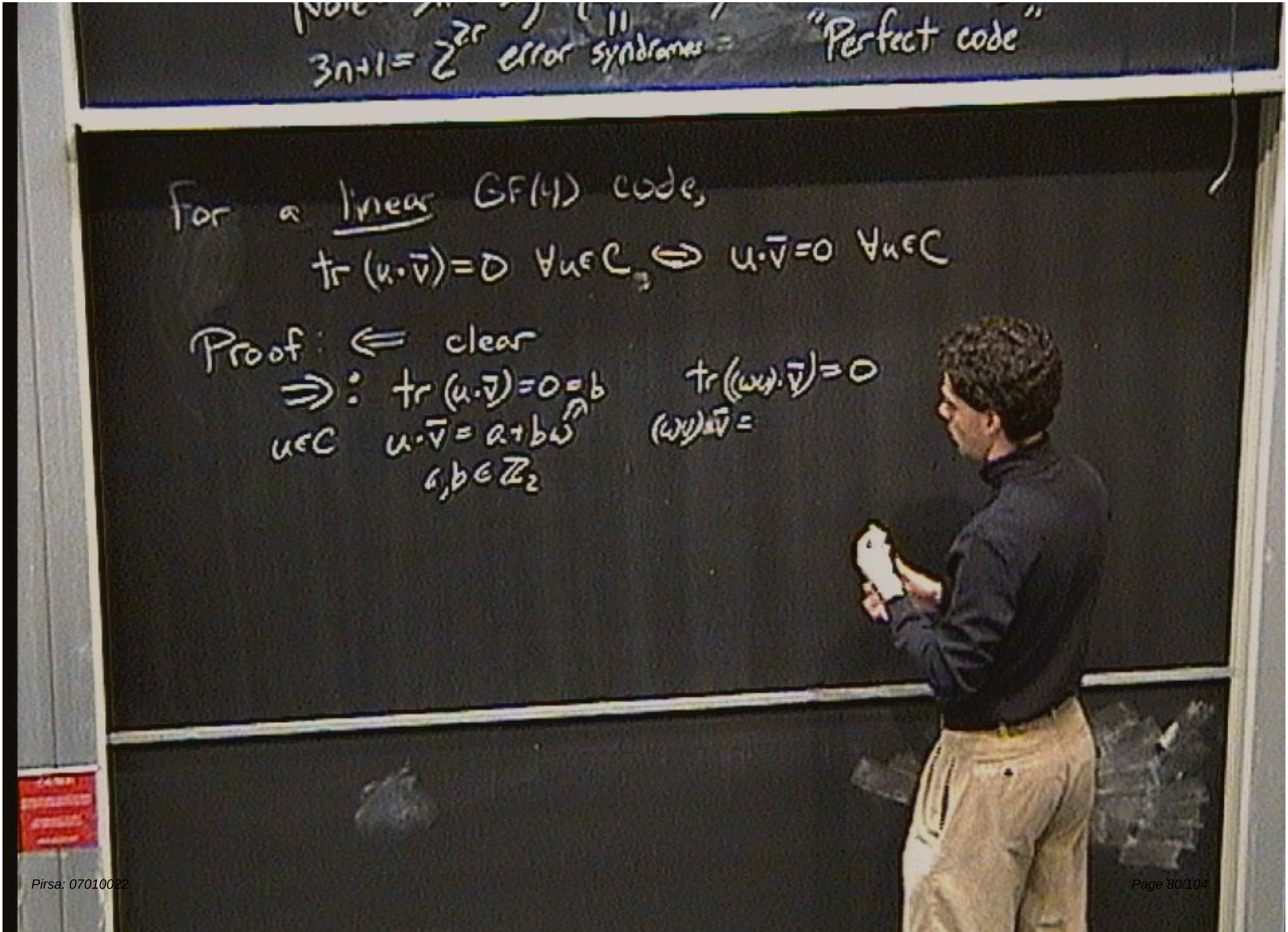

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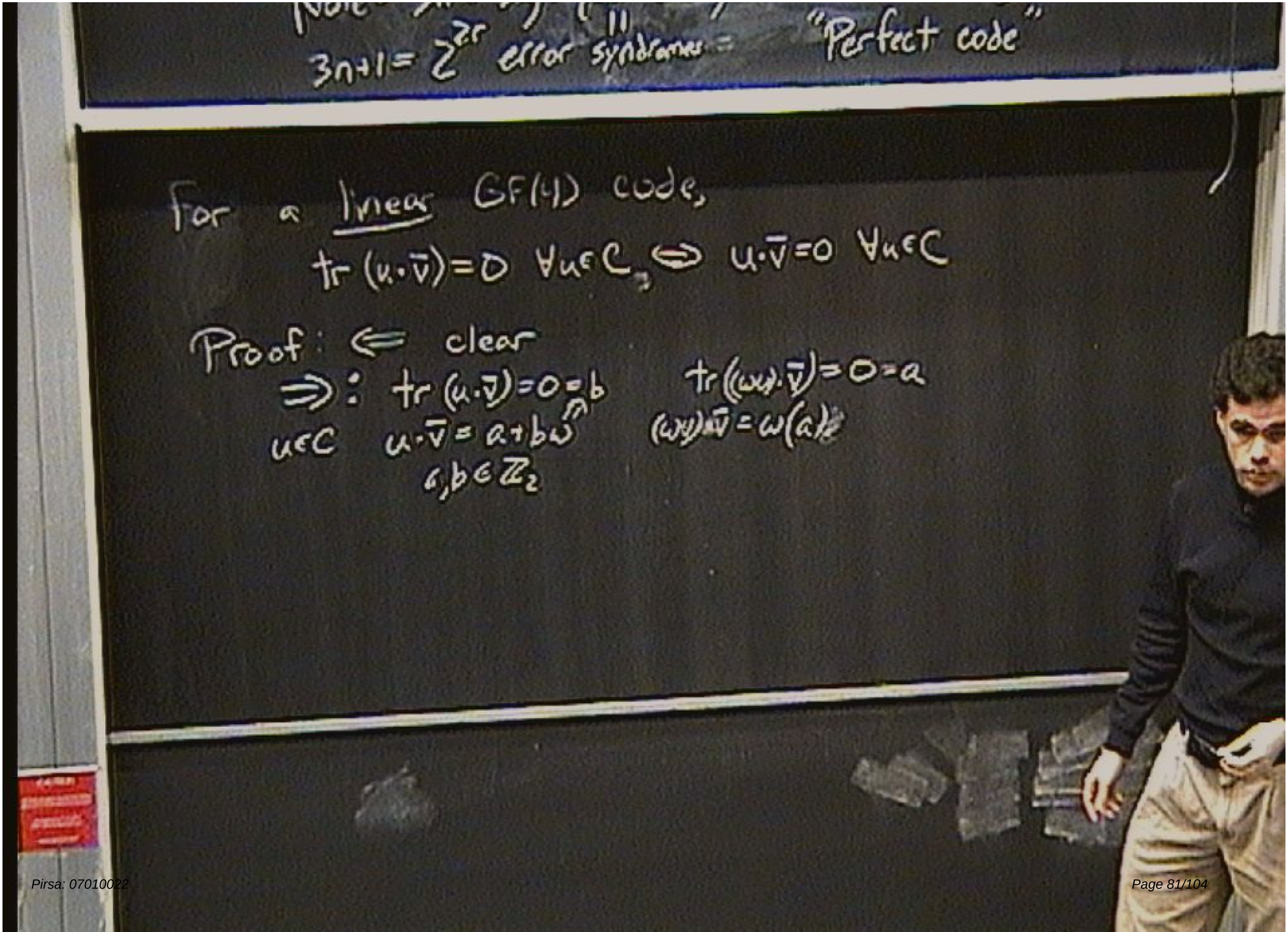
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u∈C u·v̄ = a+bω̂ (ωu)·v̄ = ω(a)
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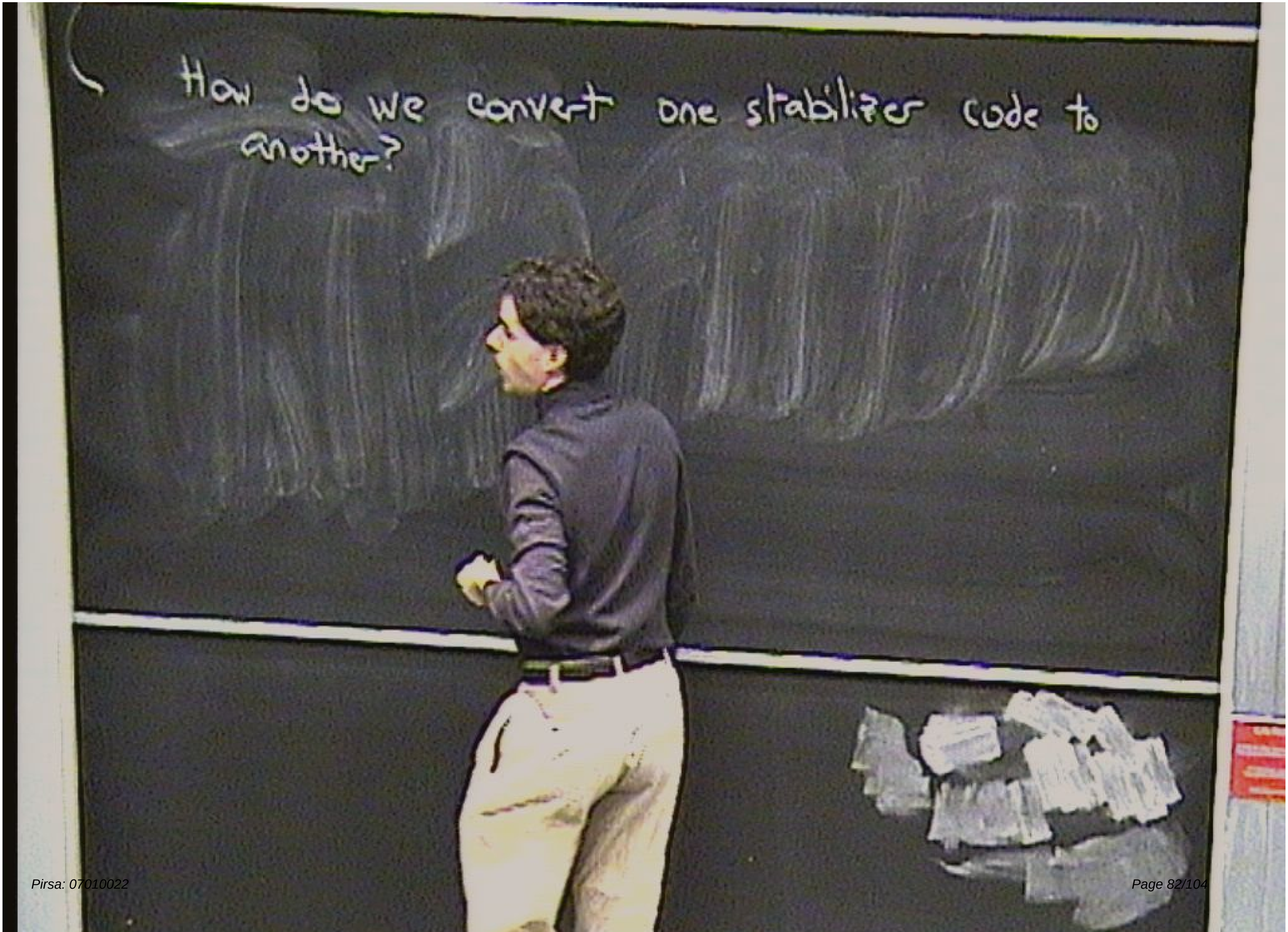
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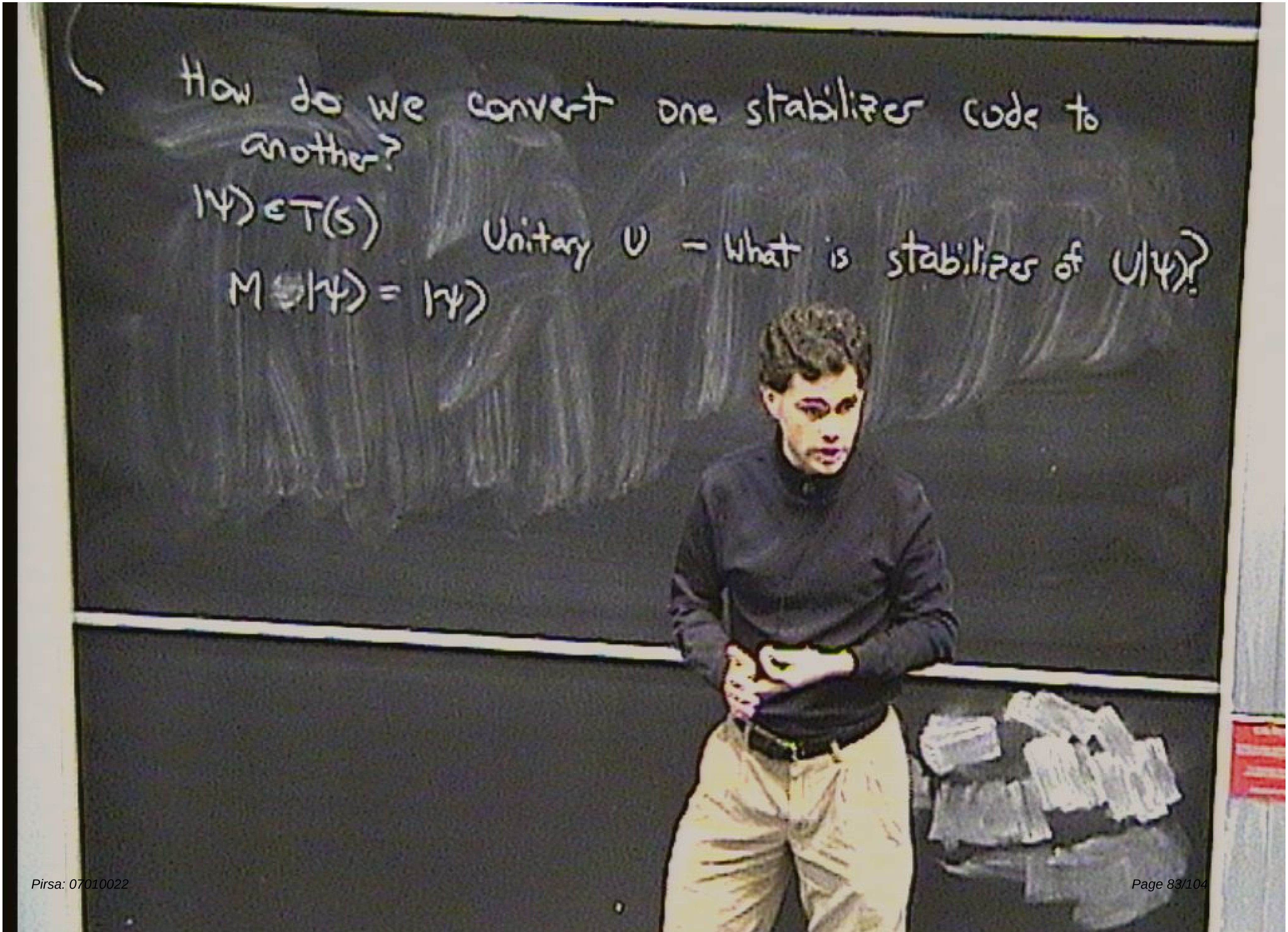
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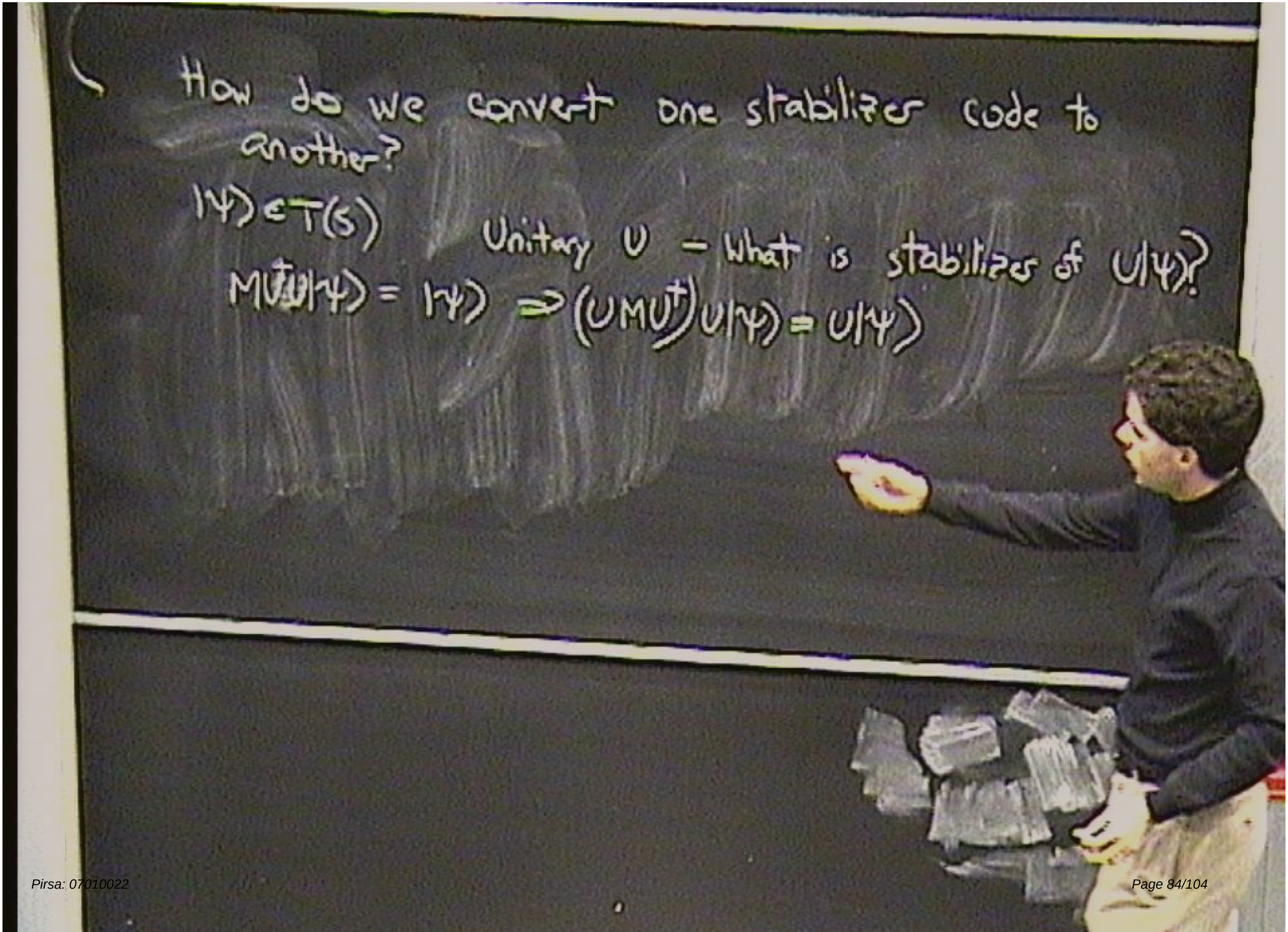
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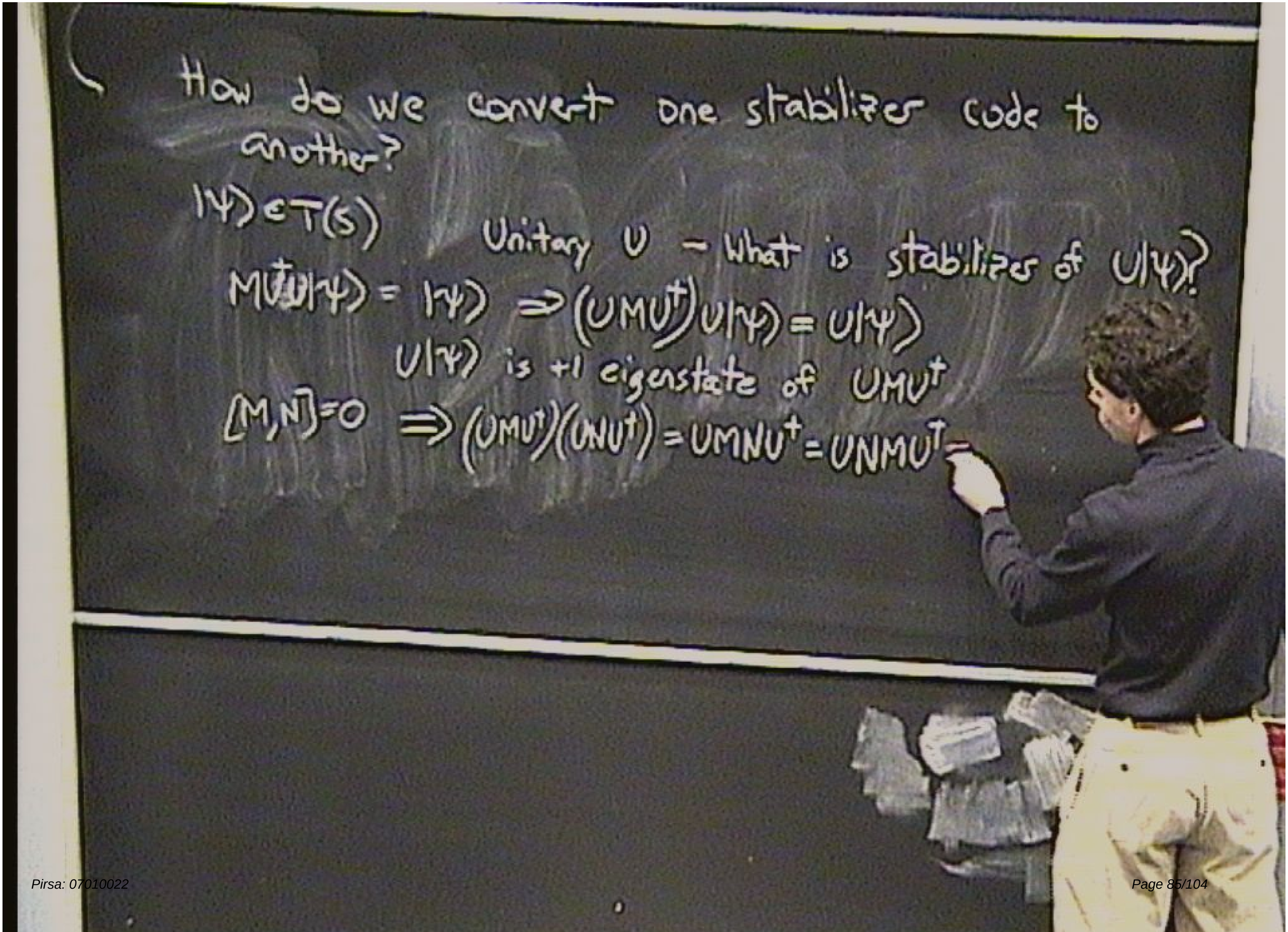
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$$[UMU^\dagger, UNU^\dagger]=0$$

Def.: The **Clifford group** (or **normalizer group**) $\mathcal{C}_n$

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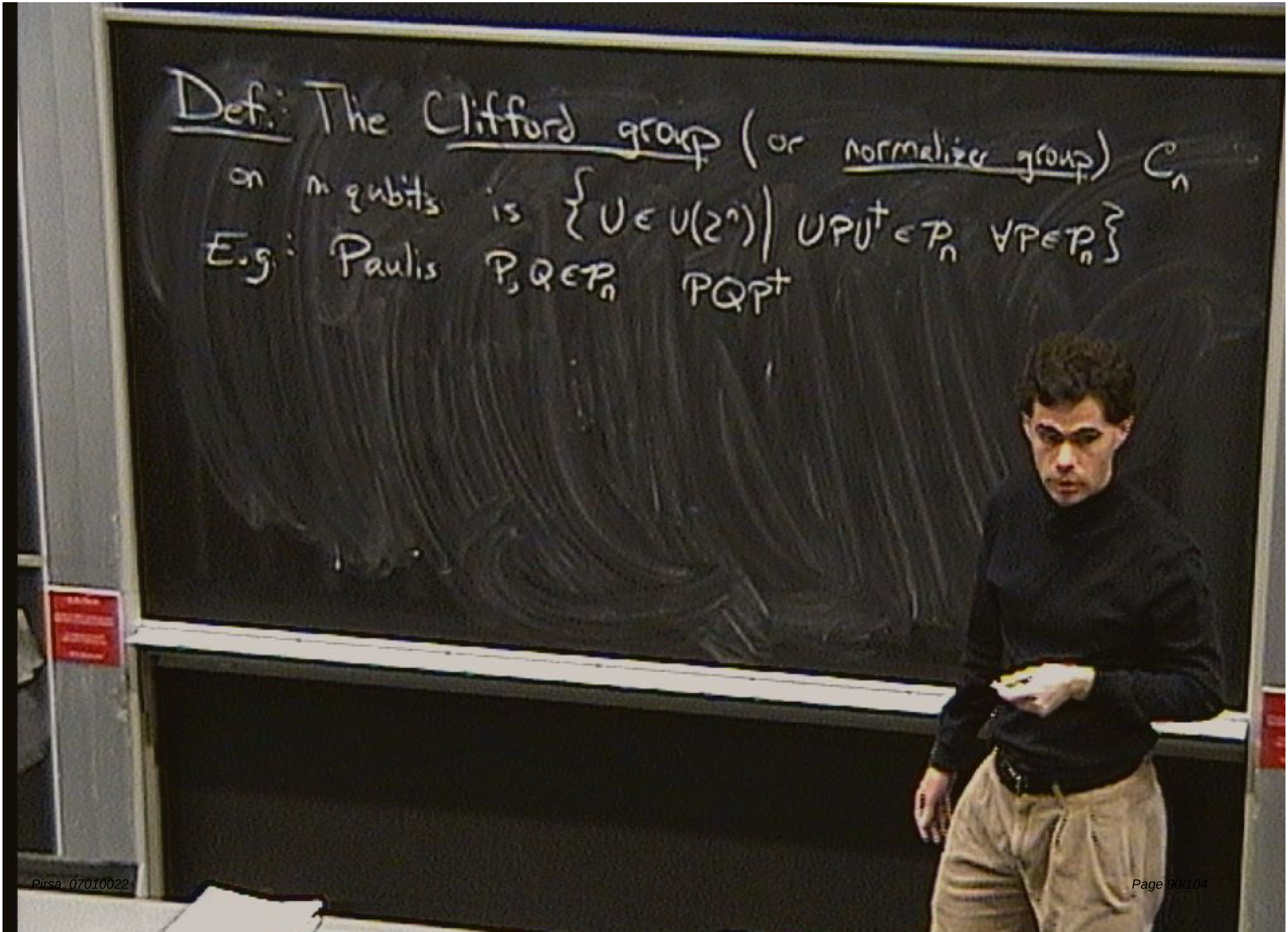
E.g.: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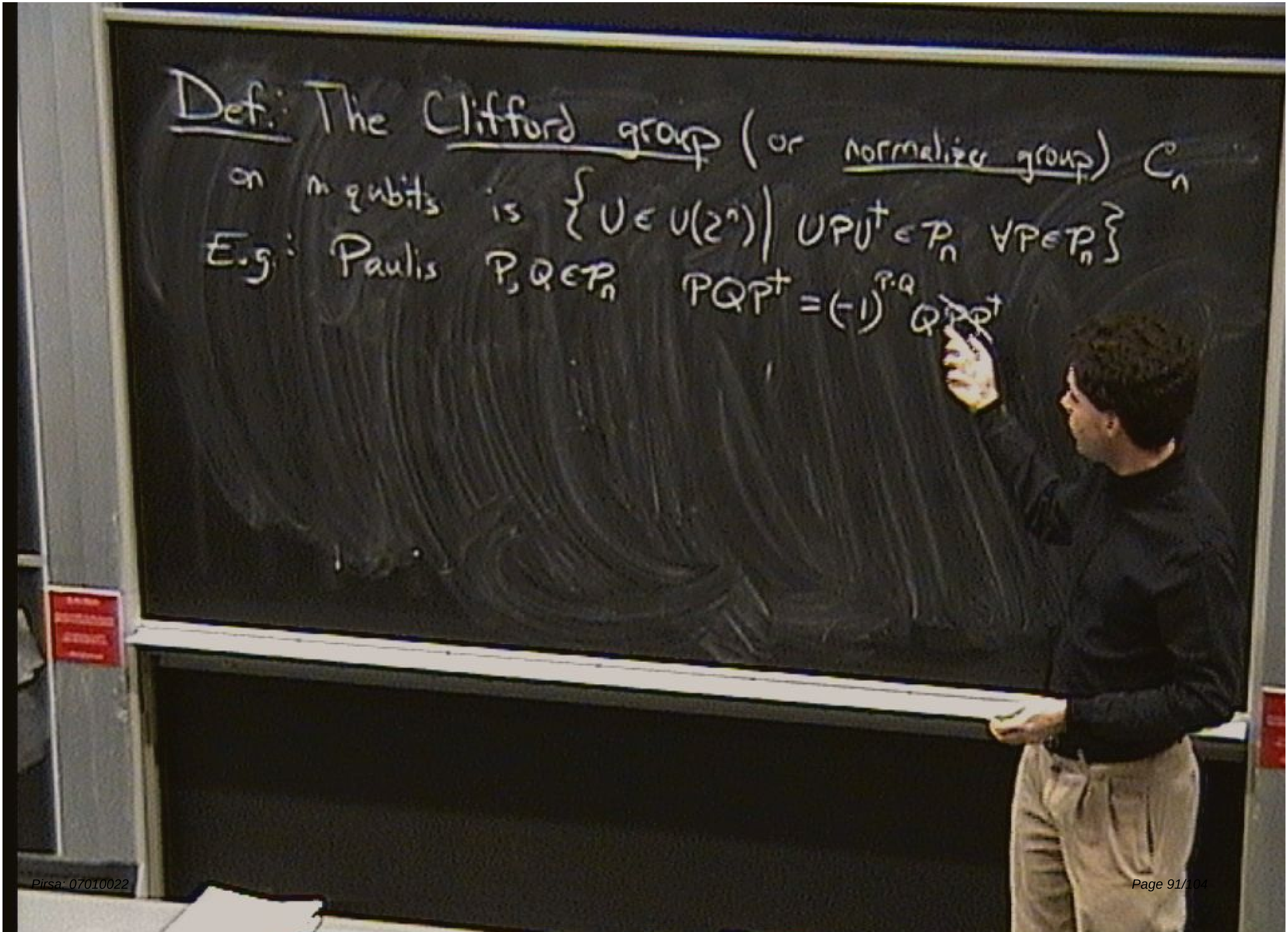
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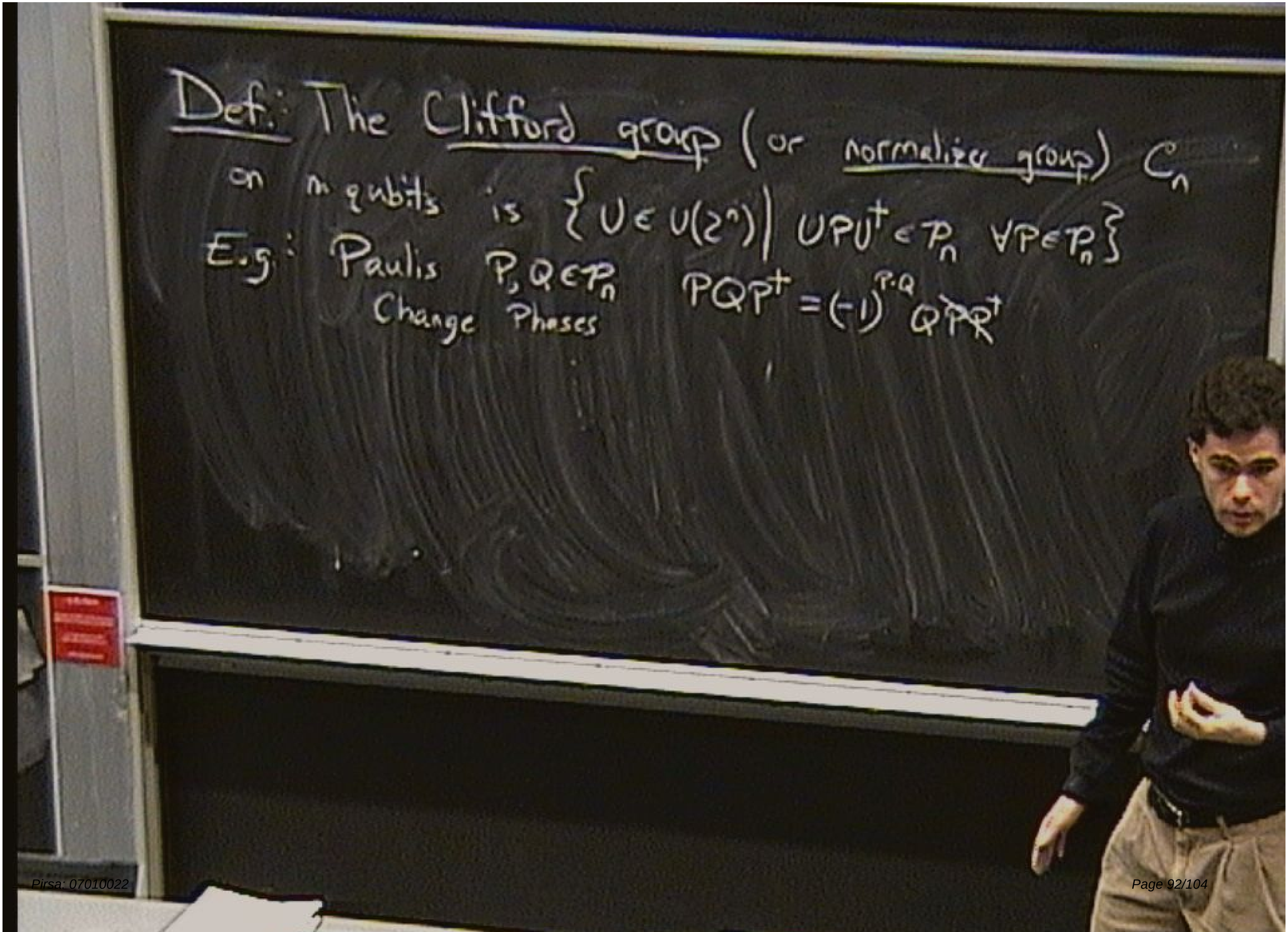
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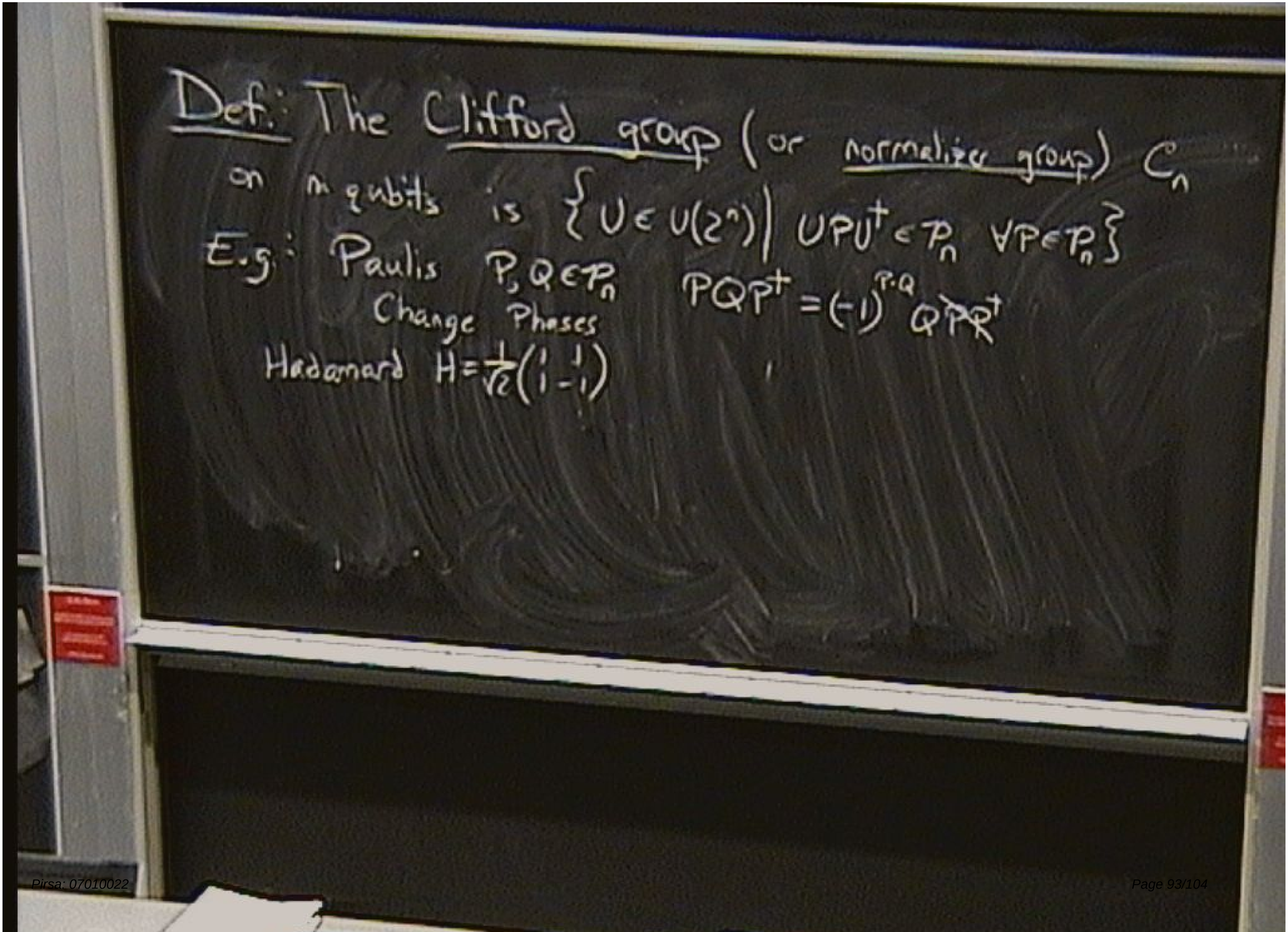
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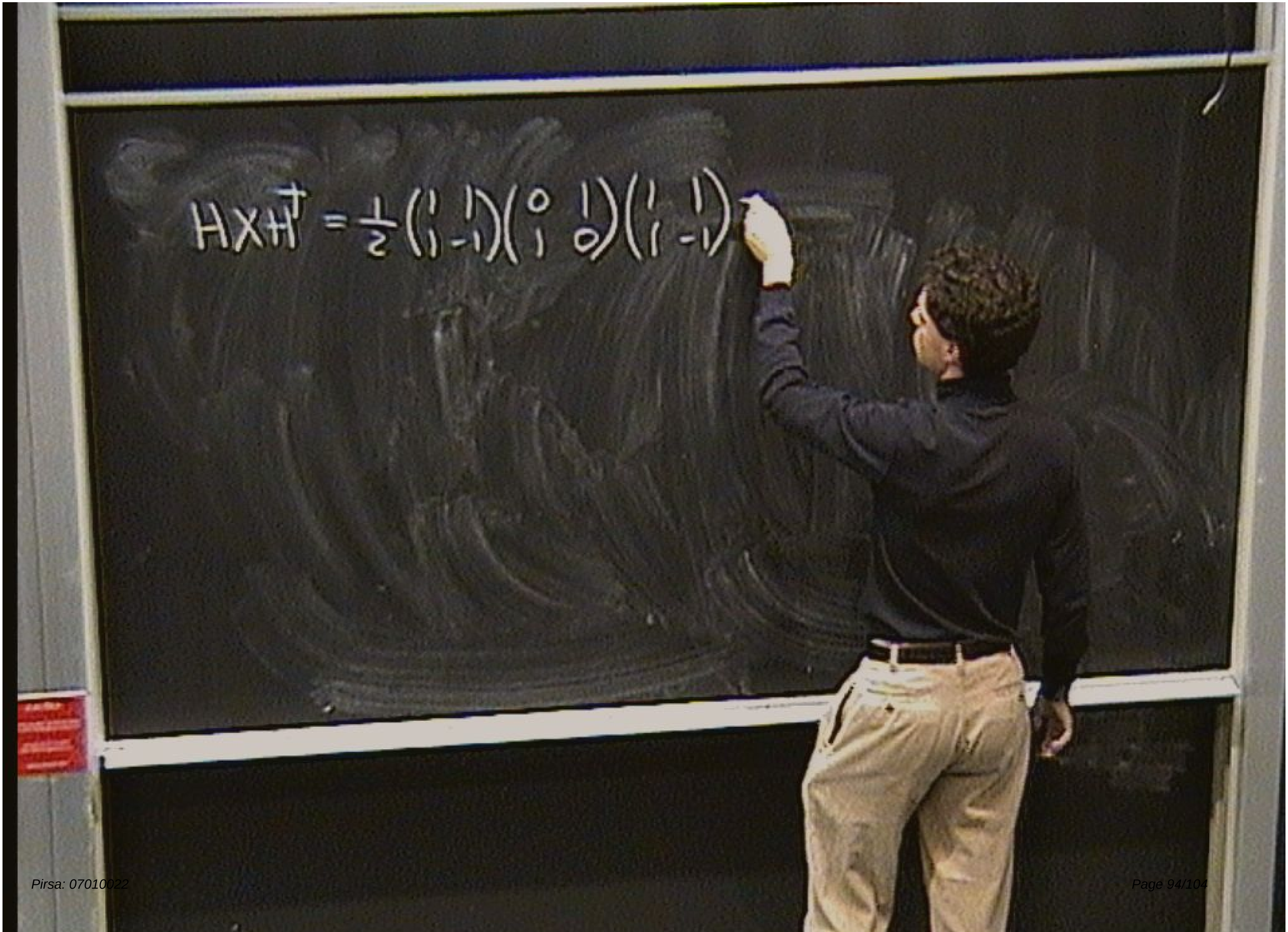
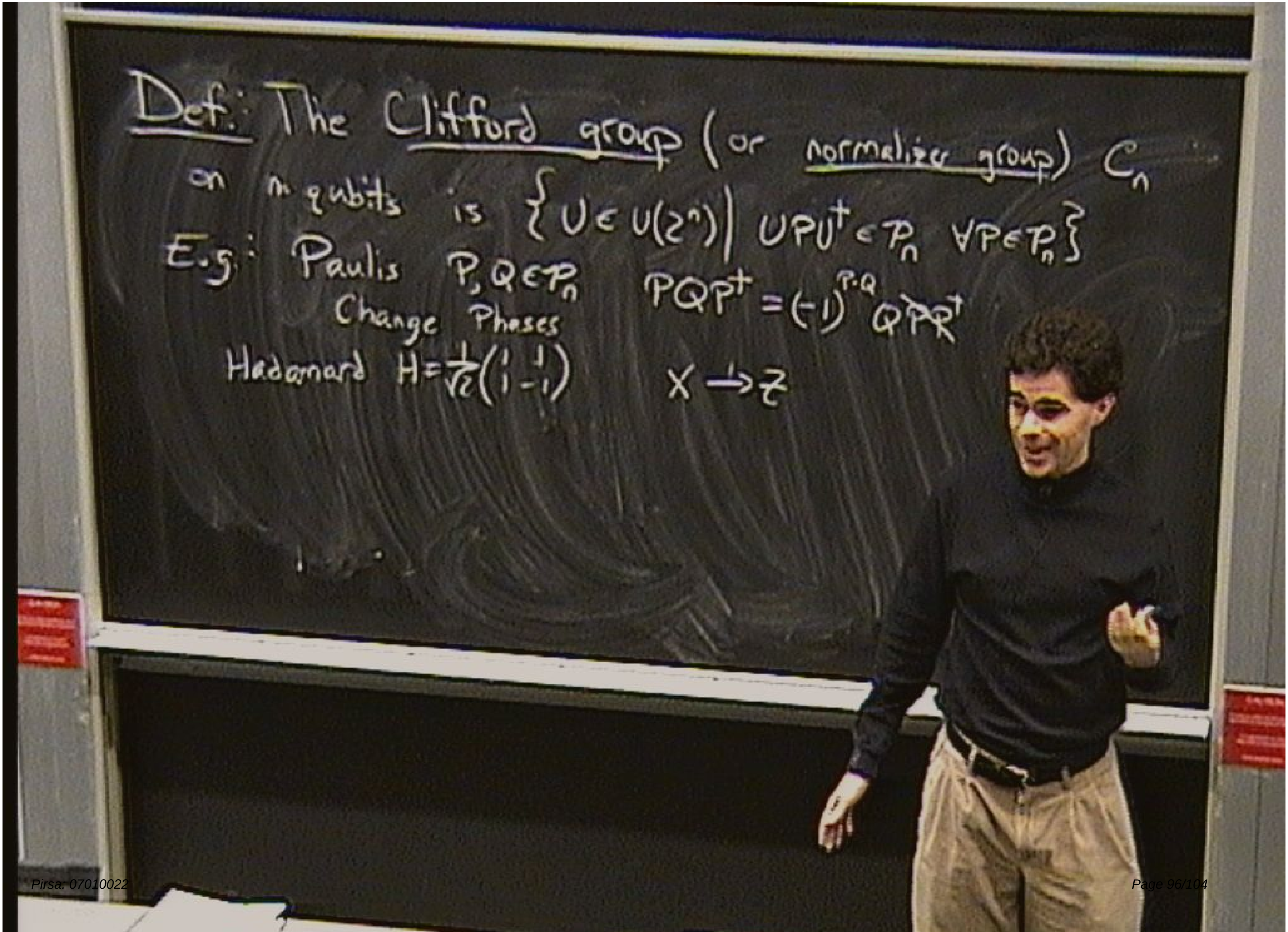
Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$


$HXH^\dagger = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$$HXH^\dagger = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = Z$$


Def: The Clifford group (or normalizer group) C_n
on n qubits is $\{U \in U(2^n) \mid UPU^\dagger \in P_n \ \forall P \in P_n\}$
E.g.: Paulis $P, Q \in P_n$ $PQP^\dagger = (-1)^{P\cdot Q} QPP^\dagger$
Change Phases
Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ $X \rightarrow Z$

Def: The Clifford group (or normalizer group) $\mathcal{C}_n$ on n qubits is $\{U \in U(2^n) \mid UPU^\dagger \in \mathcal{P}_n \ \forall P \in \mathcal{P}_n\}$

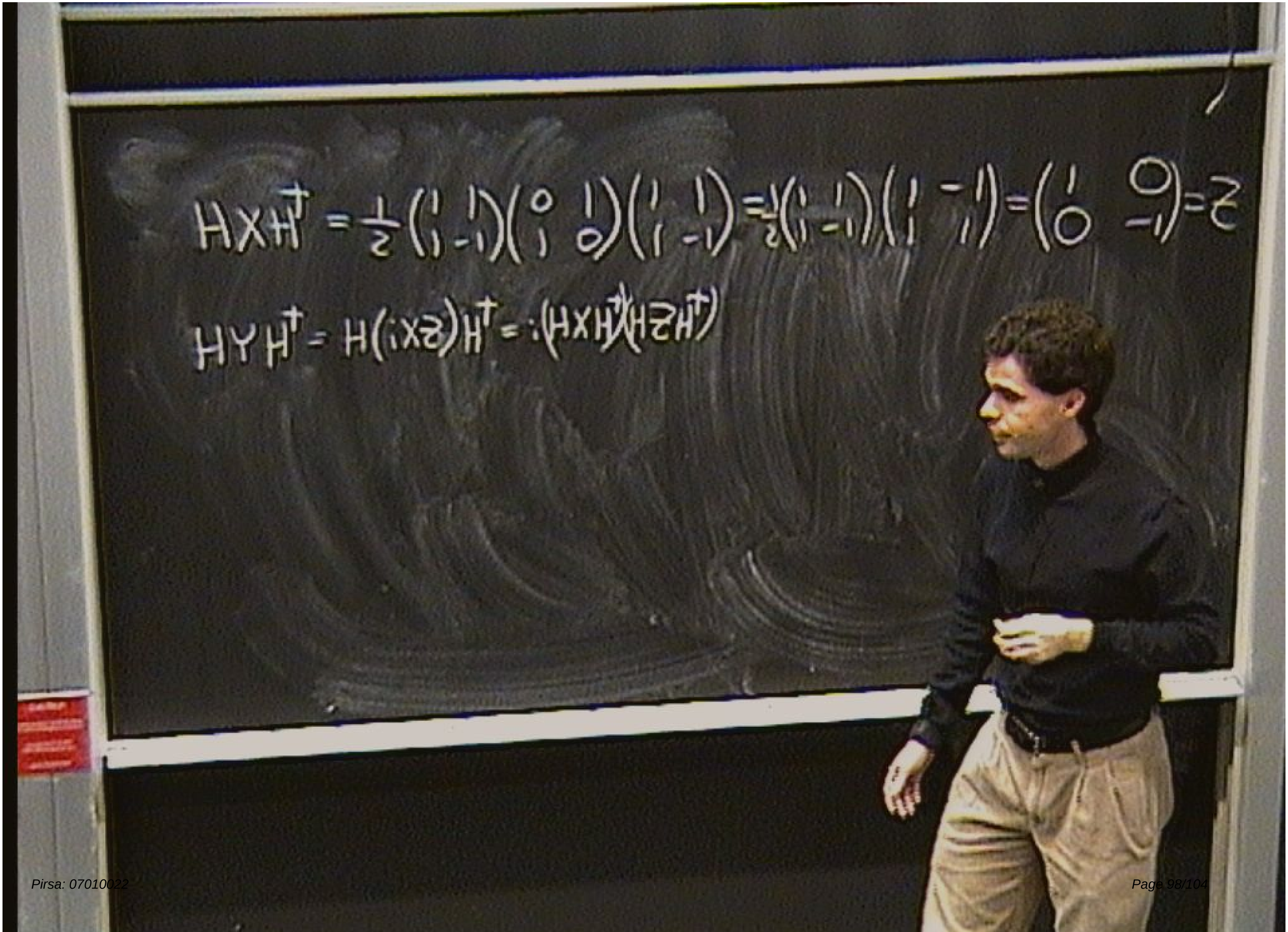
E.g.: Paulis $P, Q \in \mathcal{P}_n$ $\quad PQP^\dagger = (-1)^{P\cdot Q}\, QPP^\dagger$

Change Phases

Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$X \to Z$

$Z \to X$

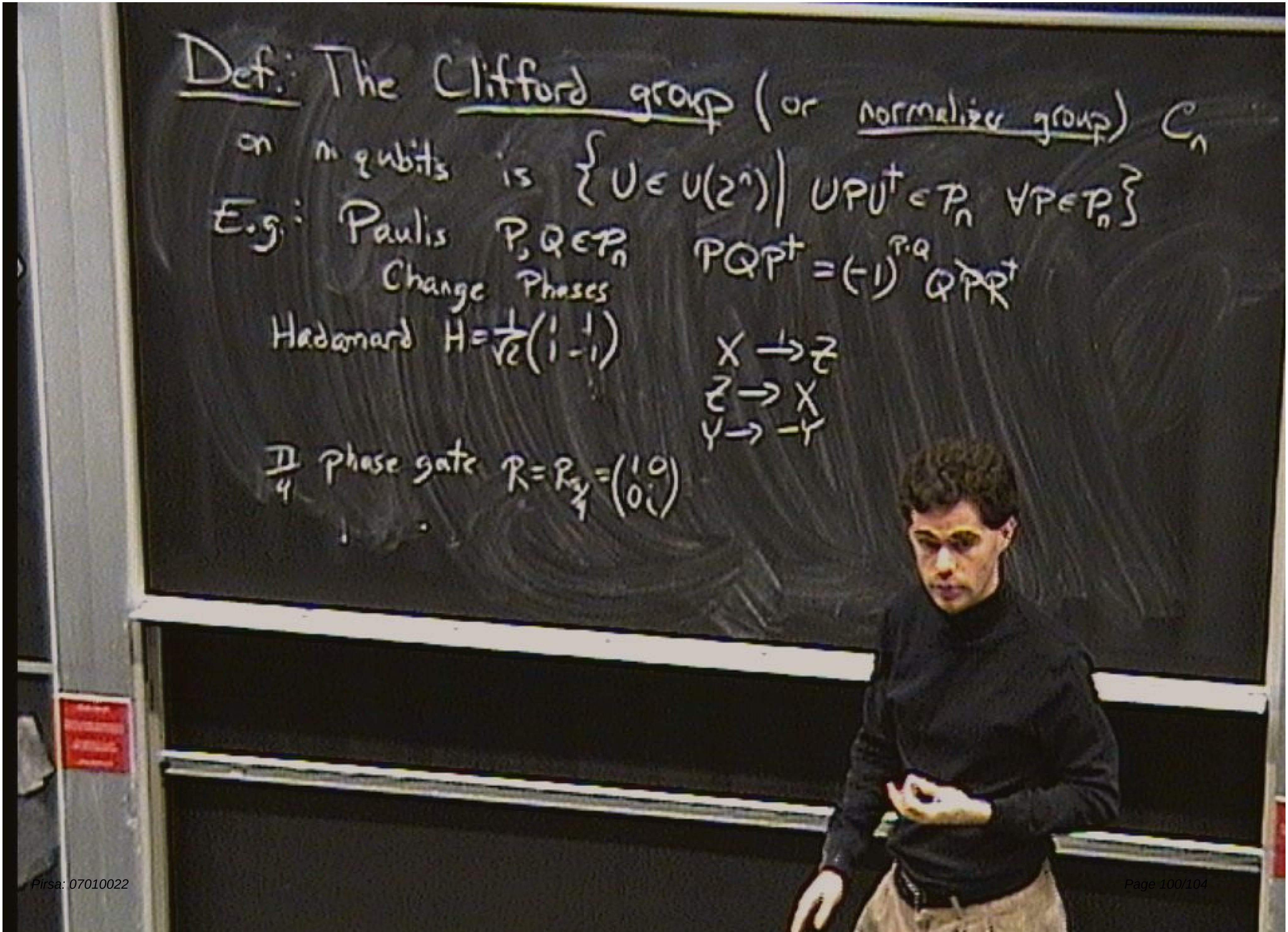


$HXH^\dagger = \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&-1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} = Z$
$HYH^\dagger = H(iXZ)H^\dagger = i(HXH^\dagger)(HZH^\dagger)$

$$HXH^{\dagger} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$$

$$HYH^{\dagger} = H(iXZ)H^{\dagger} = i(HXH^{\dagger})(HZH^{\dagger}) = iZX = -Y$$

Sufficient to check $UPU^{\dagger} \in \mathcal{P}_n$ for generating set of $\mathcal{P}_1$ (for $\mathcal{P}_n$)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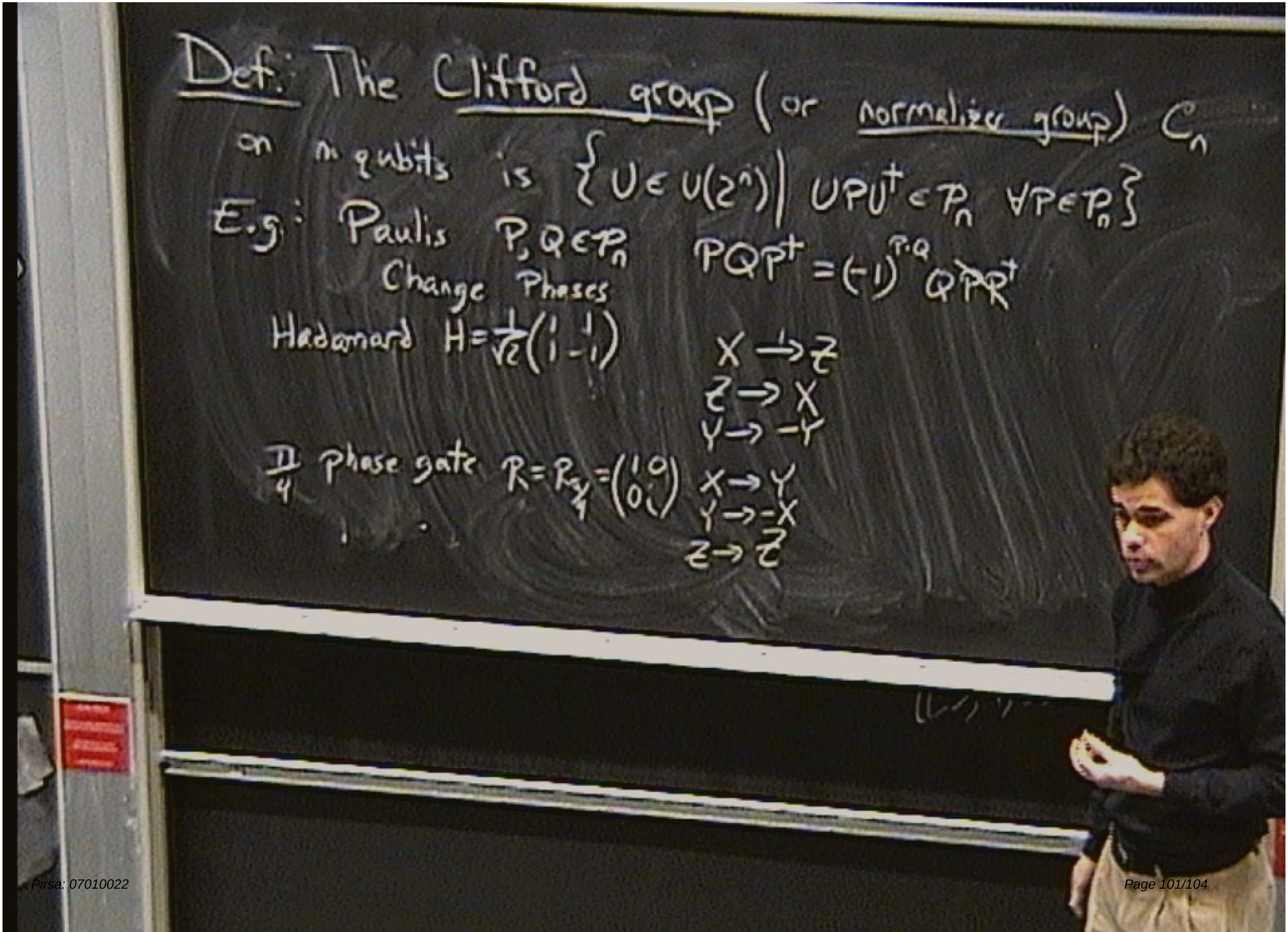
E.g.: Paulis $P, Q \in \mathcal{P}_n$ $PQP^\dagger = (-1)^{P \cdot Q} Q P P^\dagger$
Change Phases
Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$X \to Z$
$Z \to X$
$Y \to -Y$
$\frac{\pi}{4}$ phase gate $R = R_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$


Def: The Clifford group (or normalizer group) $\mathcal{C}_n$
on n qubits is $\{ U \in U(2^n) \mid UPU^\dagger \in \mathcal{P}_n \ \forall P \in \mathcal{P}_n \}$
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Change Phases
Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$
$X \to Z$
$Z \to X$
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Def: The Clifford group (or normalizer group) C_n on n qubits is $\{U \in U(2^n) \mid UPU^\dagger \in \mathcal{P}_n \ \forall P \in \mathcal{P}_n\}$

E.g.: Paulis $P, Q \in \mathcal{P}_n$ $\quad PQP^\dagger = (-1)^{P\cdot Q} QPP^\dagger$

Change Phases

Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$X \rightarrow Z$
$Z \rightarrow X$
$Y \rightarrow -Y$

$\frac{\pi}{4}$ phase gate $R = R_{\frac{\pi}{4}} = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

$X \rightarrow Y$
$Y \rightarrow -X$
$Z \rightarrow Z$

CNOT: $X \otimes I \rightarrow X \otimes X$
$Z \otimes I \rightarrow Z \otimes I$
$I \otimes X \rightarrow I \otimes X$
$I \otimes Z \rightarrow Z \otimes Z$

Def: The Clifford group (or normalizer group) C_n on n qubits is $\{U \in U(2^n) \mid UPU^\dagger \in \mathcal{P}_n \ \forall P \in \mathcal{P}_n\}$

E.g.: Paulis $P, Q \in \mathcal{P}_n$ $\quad PQP^\dagger = (-1)^{P\cdot Q} QPP^\dagger$

Change Phases

Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$X \rightarrow Z$

$Z \rightarrow X$

$Y \rightarrow -Y$

$\frac{\pi}{4}$ phase gate $R = R_{\frac{\pi}{4}} = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$

$X \rightarrow Y$

$Y \rightarrow -X$

$Z \rightarrow Z$

CNOT: $X \otimes I \rightarrow X \otimes X$

$Z \otimes I \rightarrow Z \otimes I$

$I \otimes X \rightarrow I \otimes X$

$I \otimes Z \rightarrow Z \otimes Z$

Permutations of qubits

Def: The Clifford group (or normalizer group) C_n on n qubits is $\{U \in U(2^n) \mid UPU^\dagger \in \mathcal{P}_n \ \forall P \in \mathcal{P}_n\}$ or $C_n / \{e^{i\theta} I\}$

E.g.: Paulis $P, Q \in \mathcal{P}_n$ $\quad PQP^\dagger = (-1)^{P\cdot Q} QPP^\dagger$

Change Phases

Hadamard $H = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$

$X \to Z$
$Z \to X$
$Y \to -Y$

$\frac{\pi}{4}$ phase gate $R = R_{\frac{\pi}{4}} = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$

$X \to Y$
$Y \to -X$
$Z \to Z$

CNOT: $X\otimes I \to X \otimes X$
$Z \otimes I \to Z \otimes I$
$I \otimes X \to I \otimes X$
$I \otimes Z \to Z \otimes Z$

Permutations of qubits