Title: Quantum Error Correction 2A

Date: Jan 16, 2007 03:30 PM

URL: http://pirsa.org/07010020

Abstract: Stabilizer codes (definition of stabilizer, basic properties of stabilizer, binary vector representation of stabilizer)

Tensor produc[illegible] Z w/ overall phase $\pm 1, \pm i$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} [illegible] \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad Y = iXZ$

$XY = -YX$, $XZ = -ZX$, $YZ = -ZY$, $X^2 = Y^2 = Z^2 = I$

$\forall P, Q \in \mathcal{P}_n$, $PQ = QP \; (\Leftrightarrow [P,Q] = 0)$ or $PQ = -QP \; (\Leftrightarrow \{P,Q\} = 0)$

$P^2 = \pm I$

wt P = # of non-identity tensor factors

Pauli group n qubits:

Tensor products of I, X, Y, Z w/ overall phase $\pm 1, \pm i$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad Y = iXZ$

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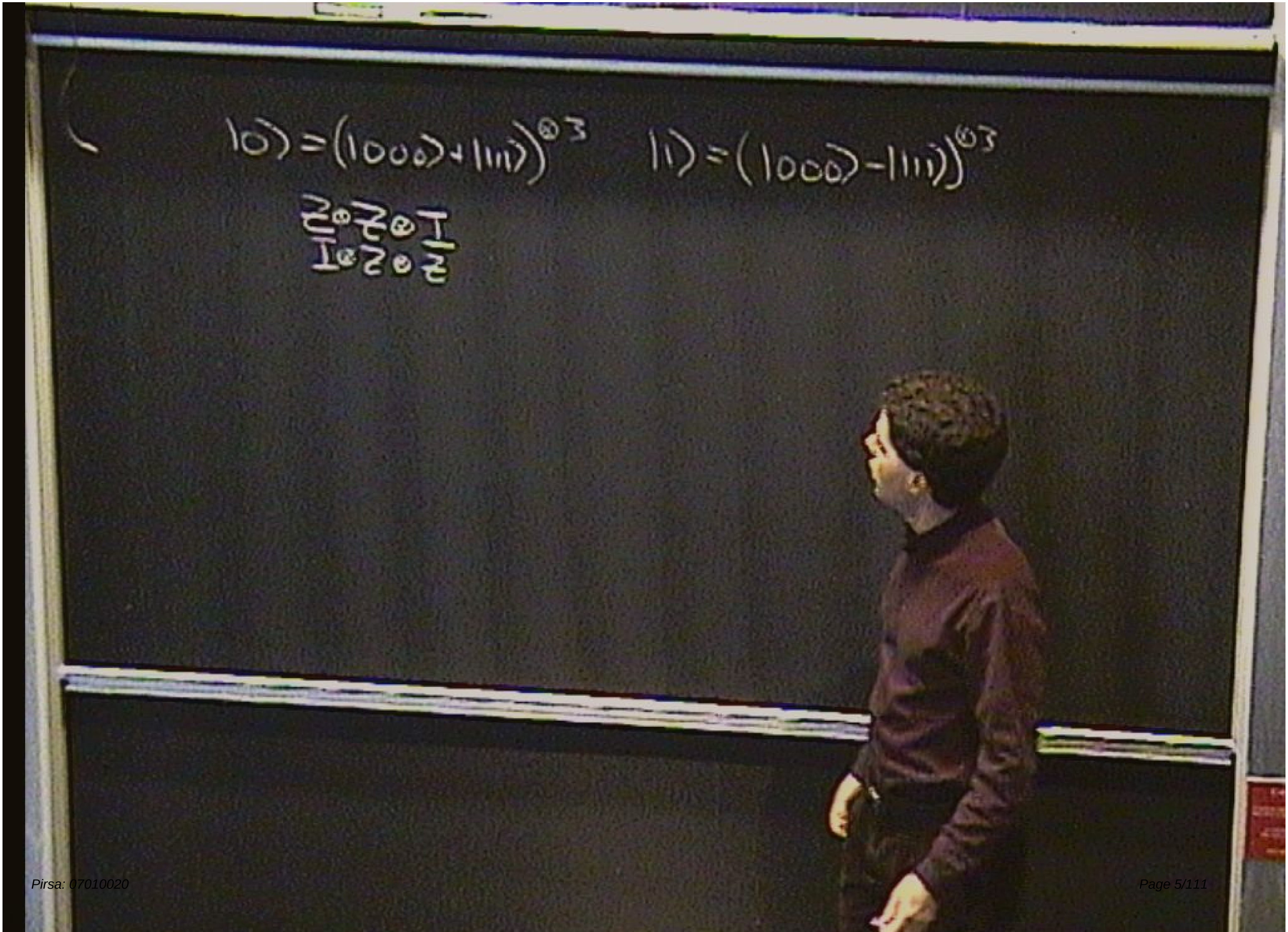
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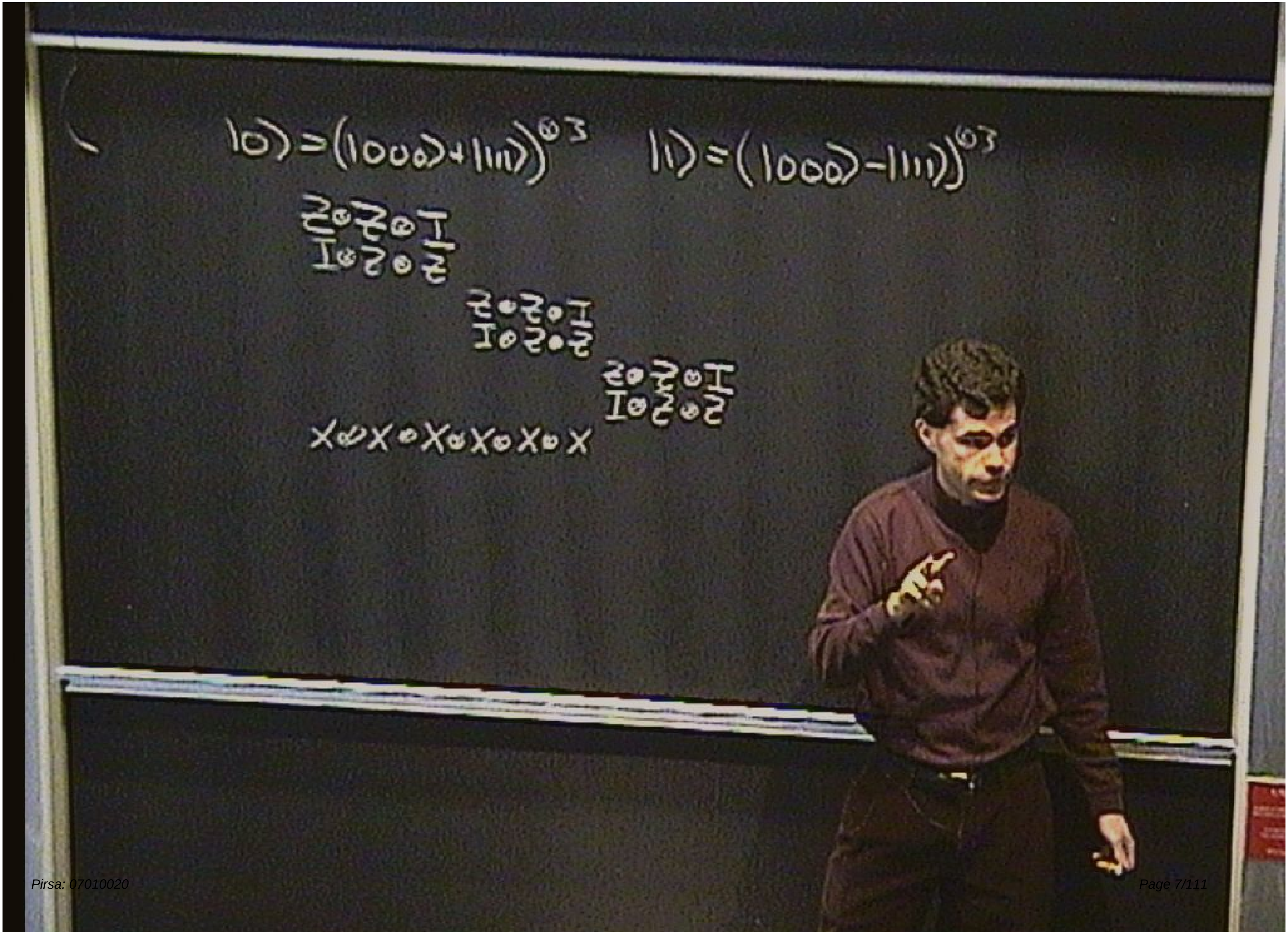
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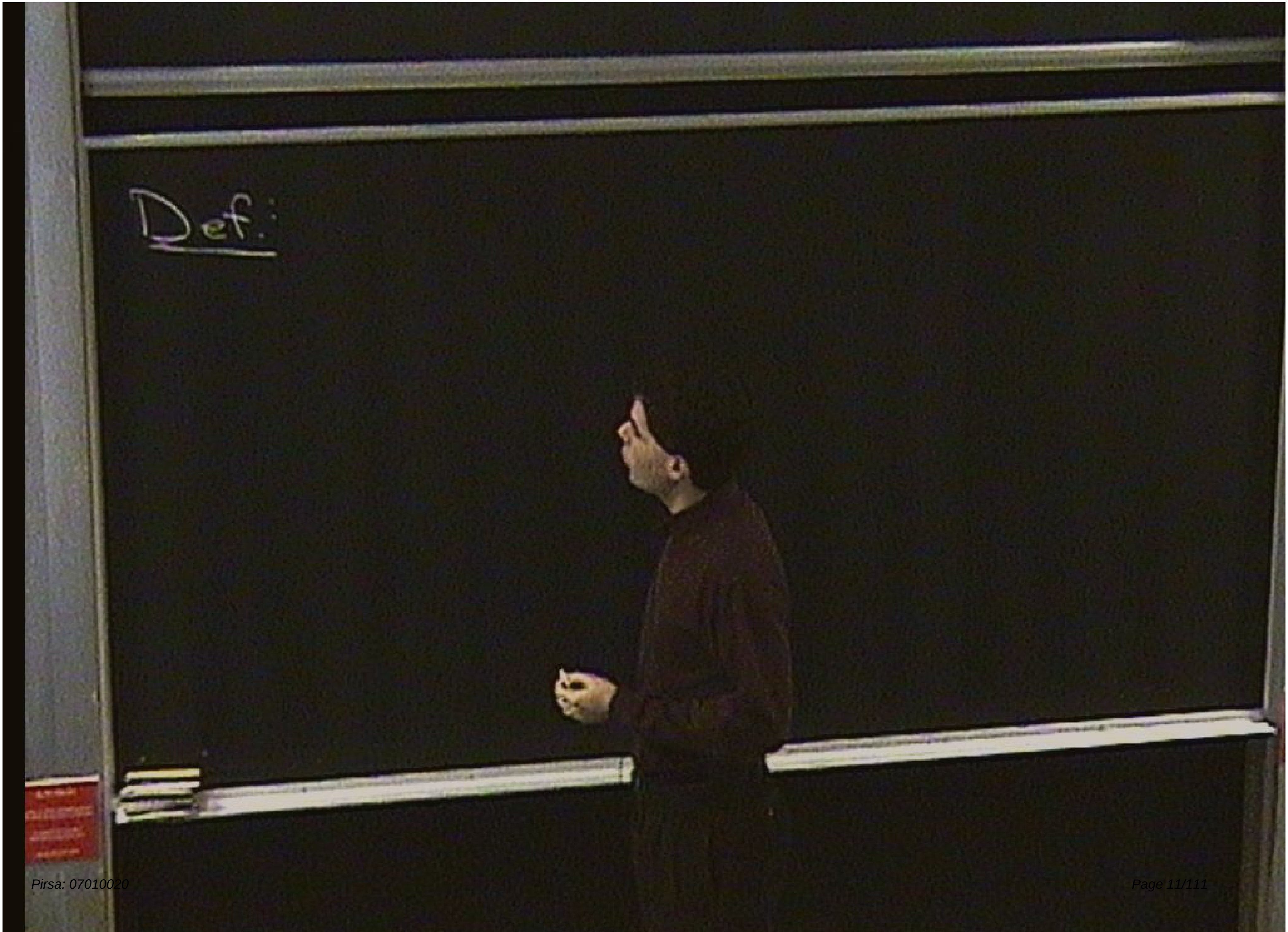
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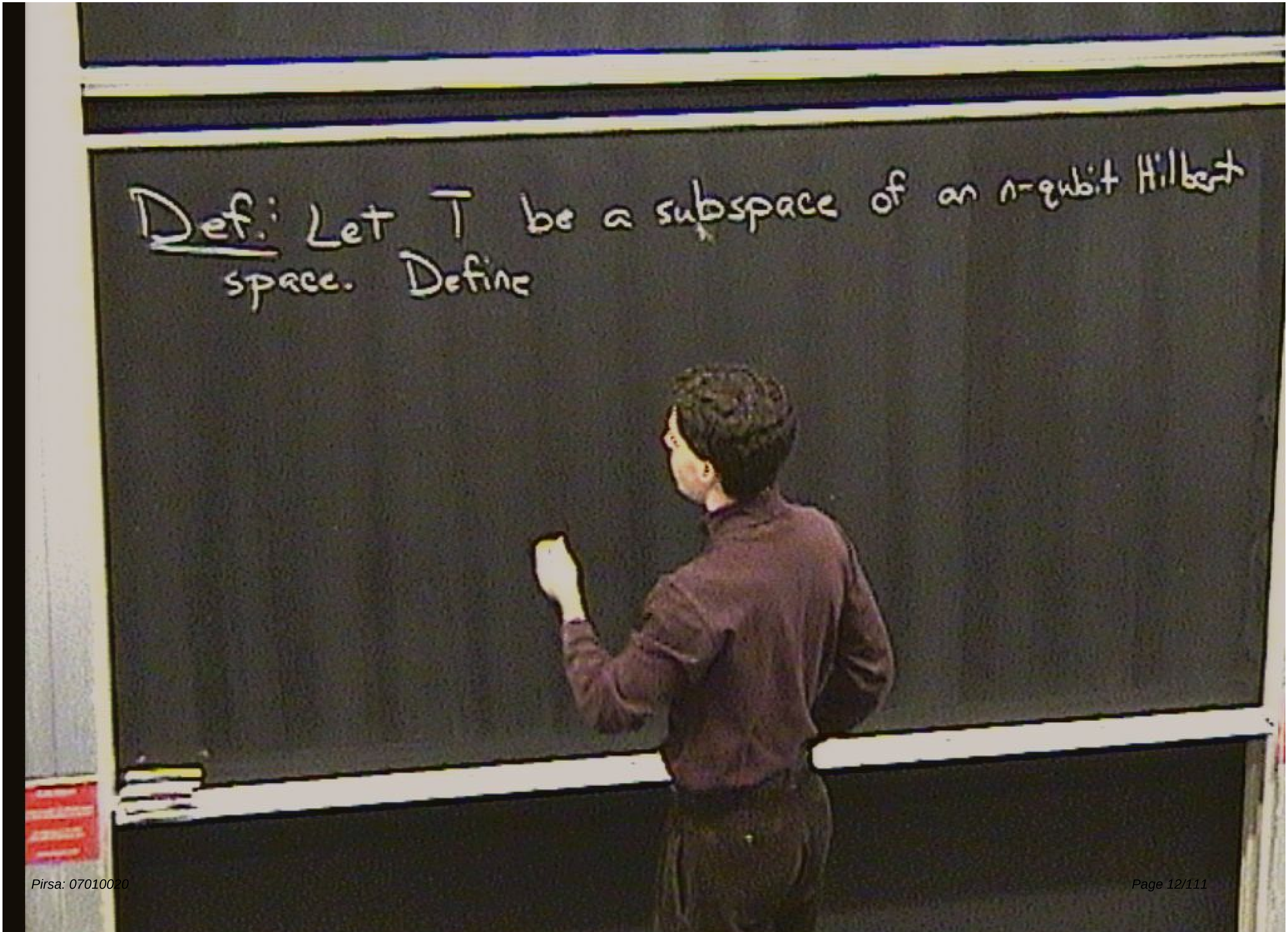
Def.: Let T be a subspace of an n-qubit Hilbert
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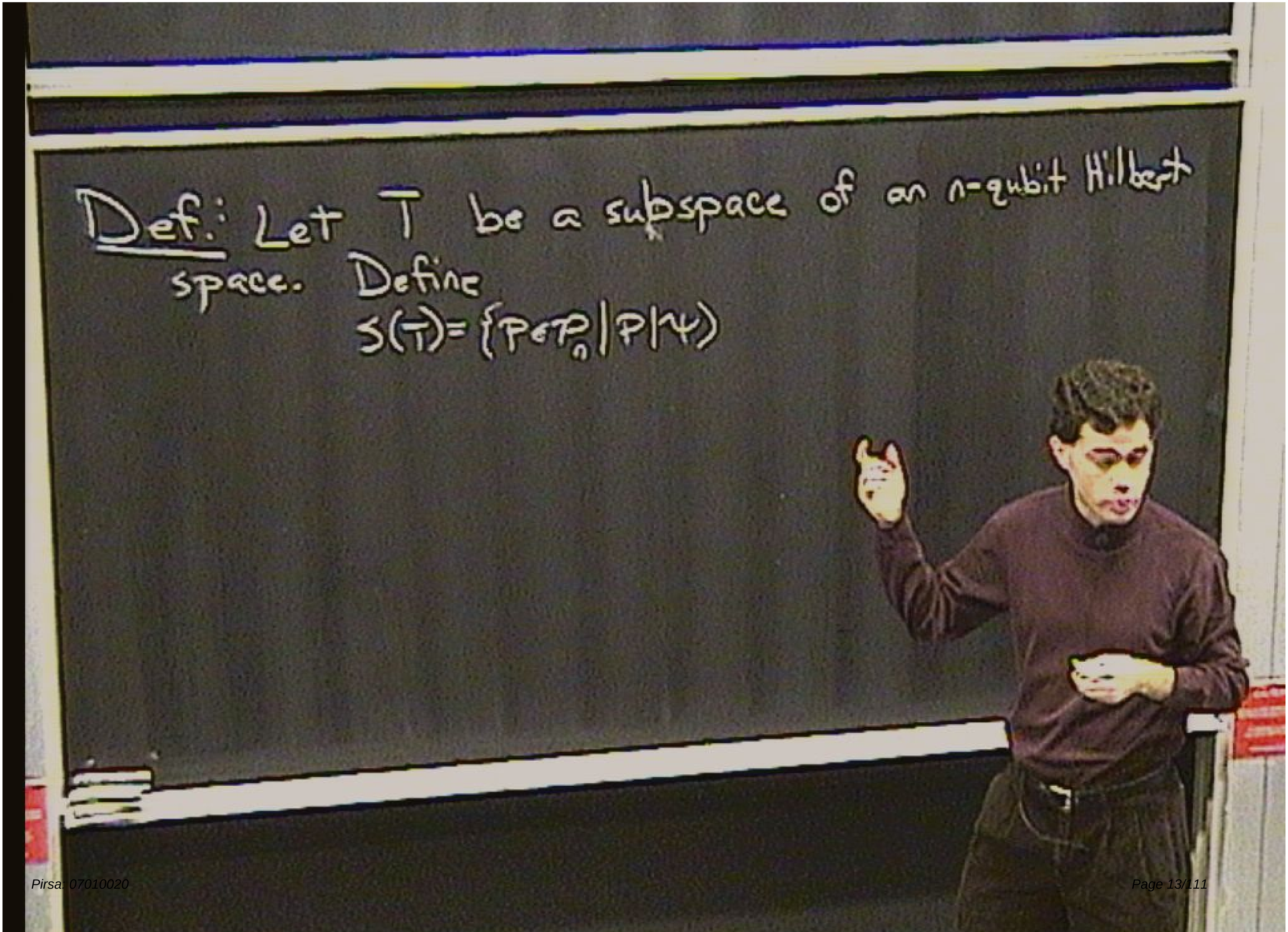
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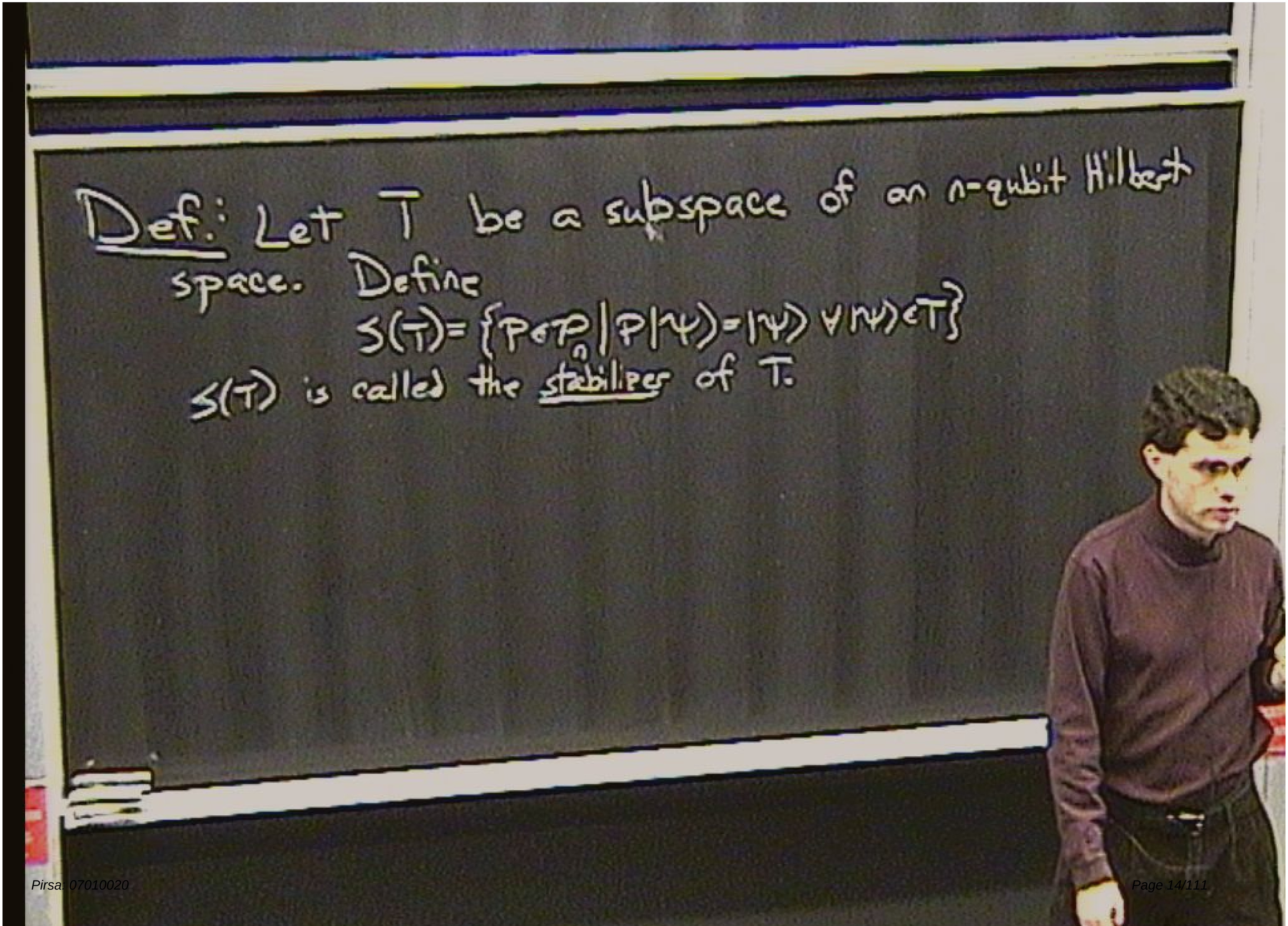
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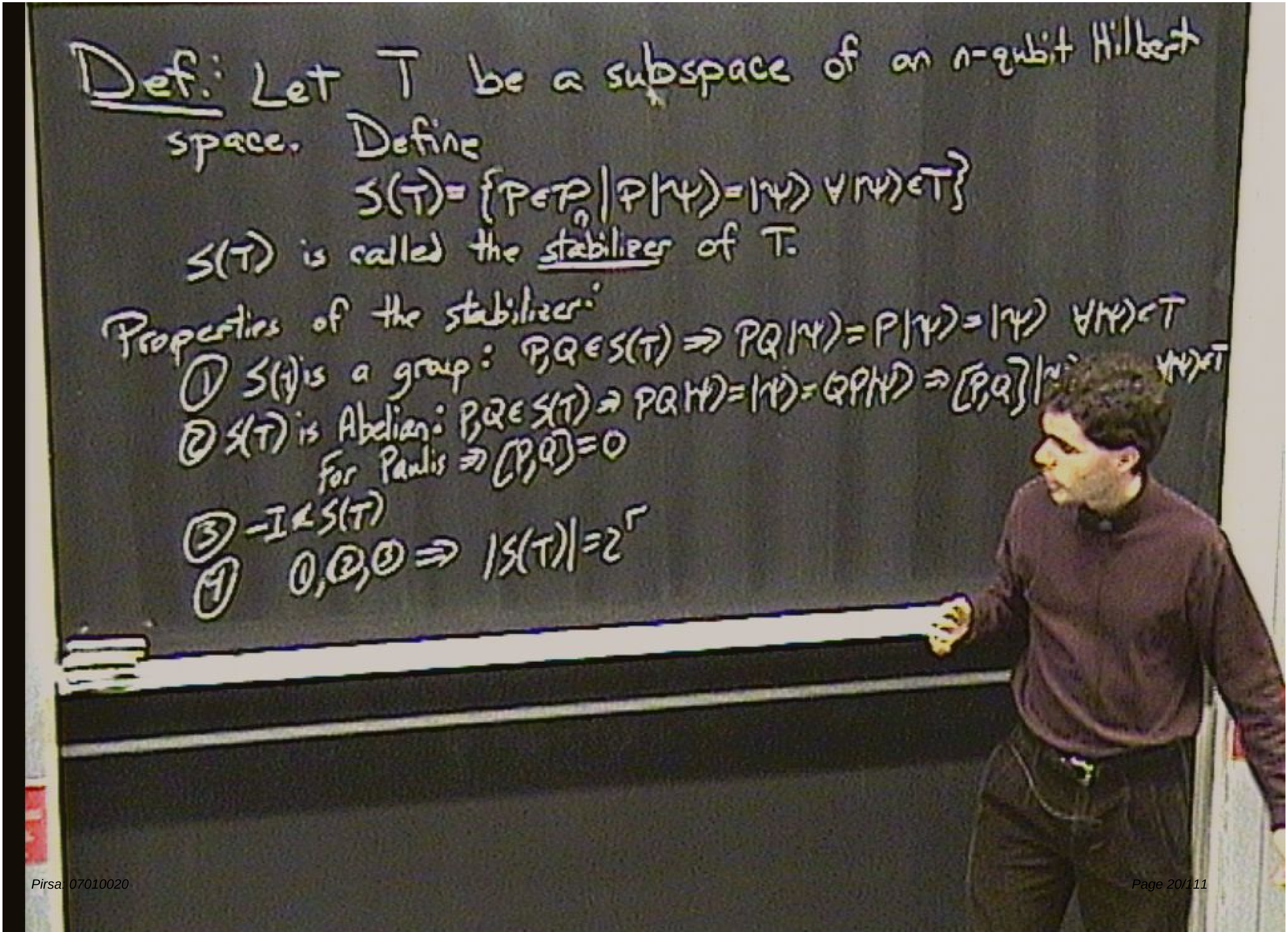
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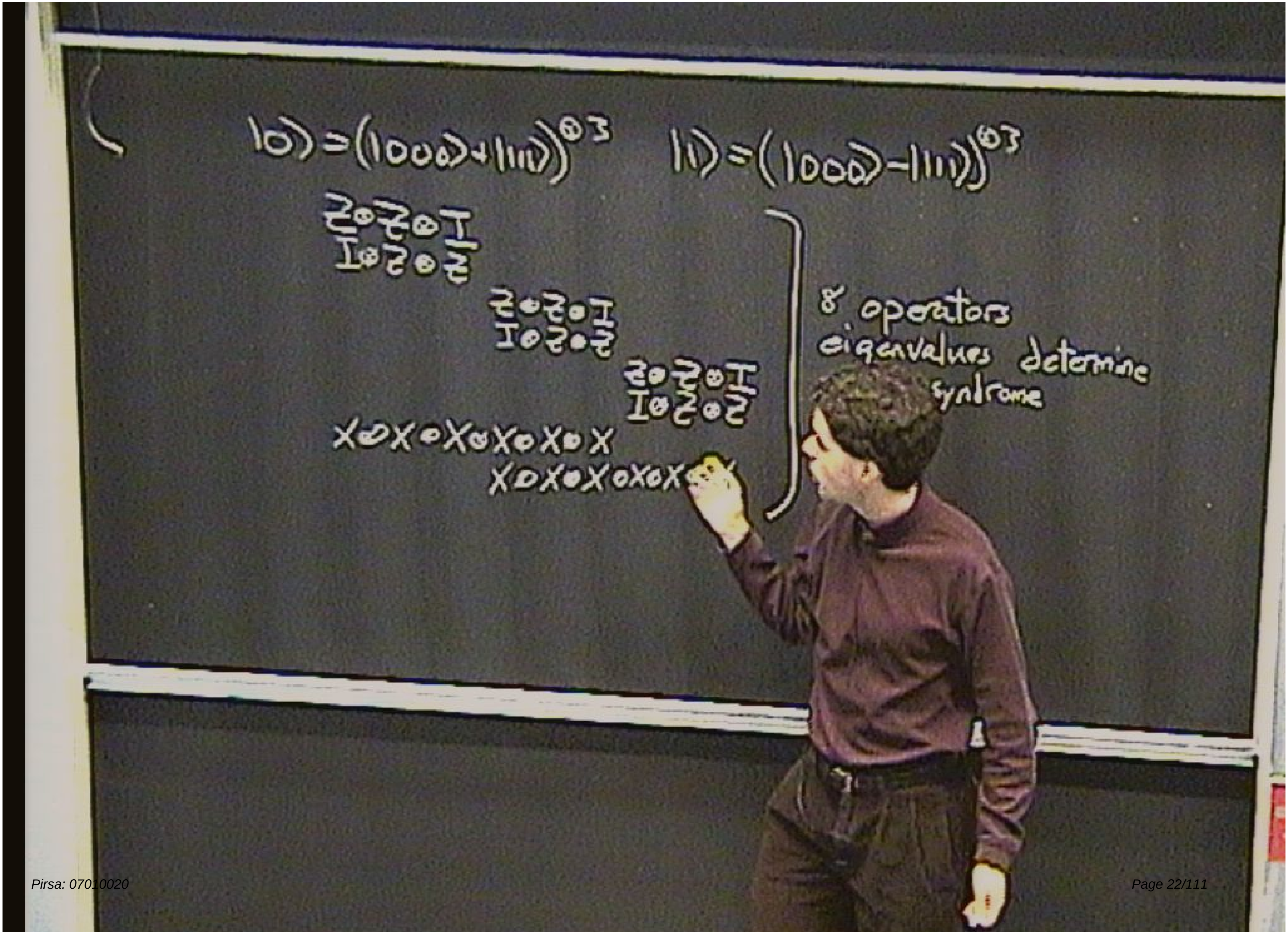
$M_1^{a_1} M_2^{a_2} \cdots M_r^{a_r} \quad (a_i \in \{0,1\})$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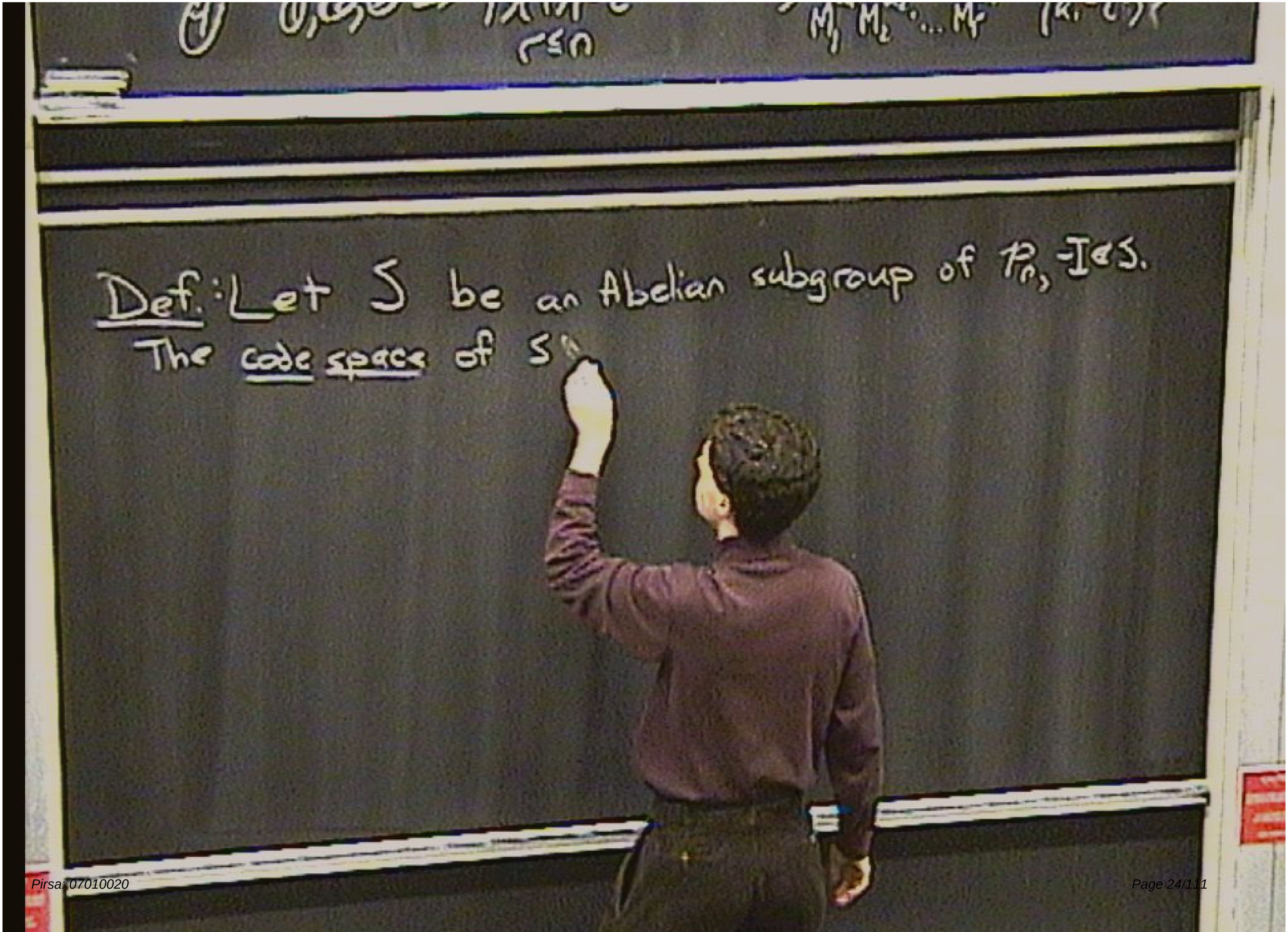
   $M_1^{a_1} M_2^{a_2} \cdots M_r^{a_r}$ $(a_i \in \{0,1\})$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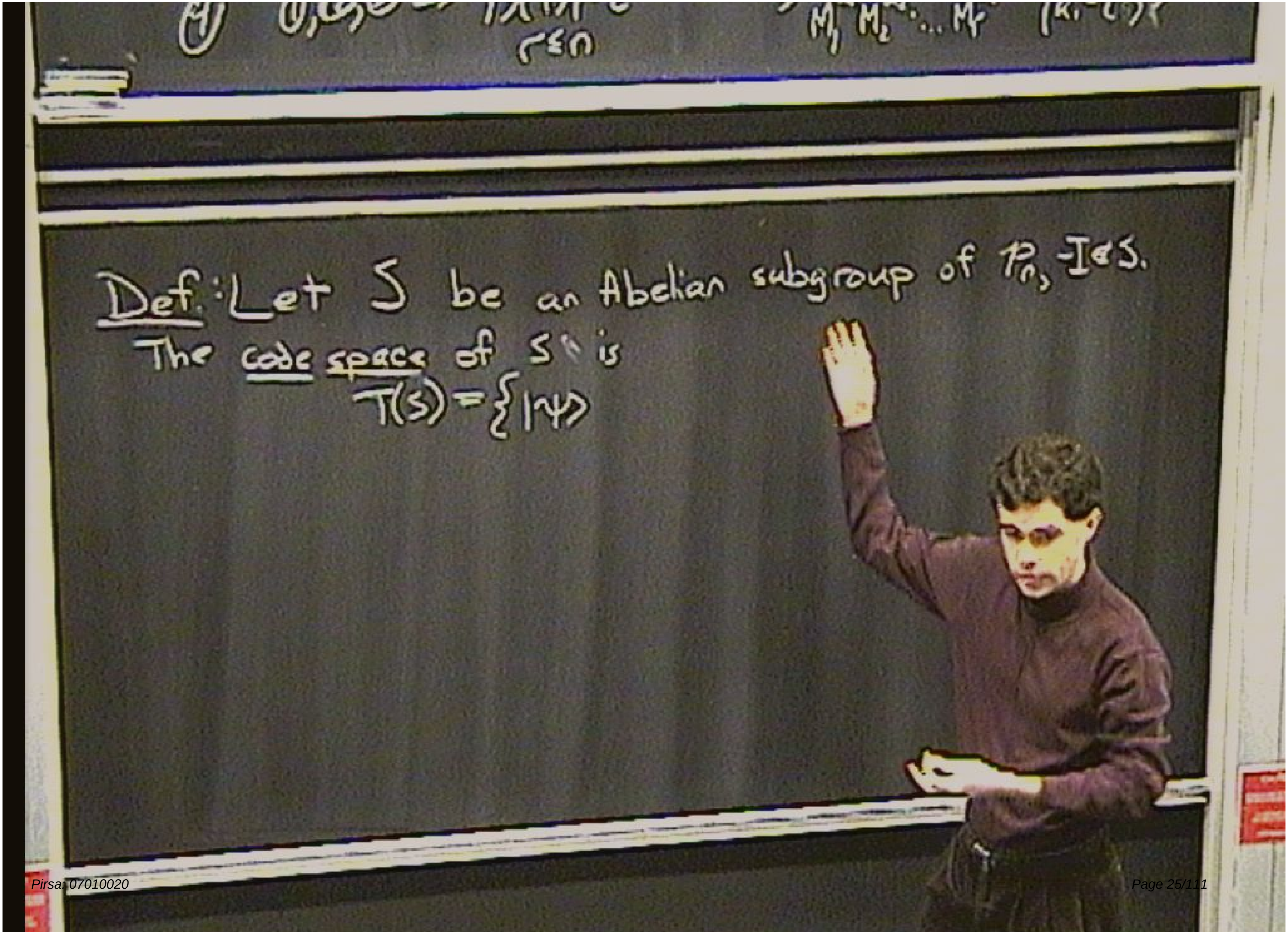
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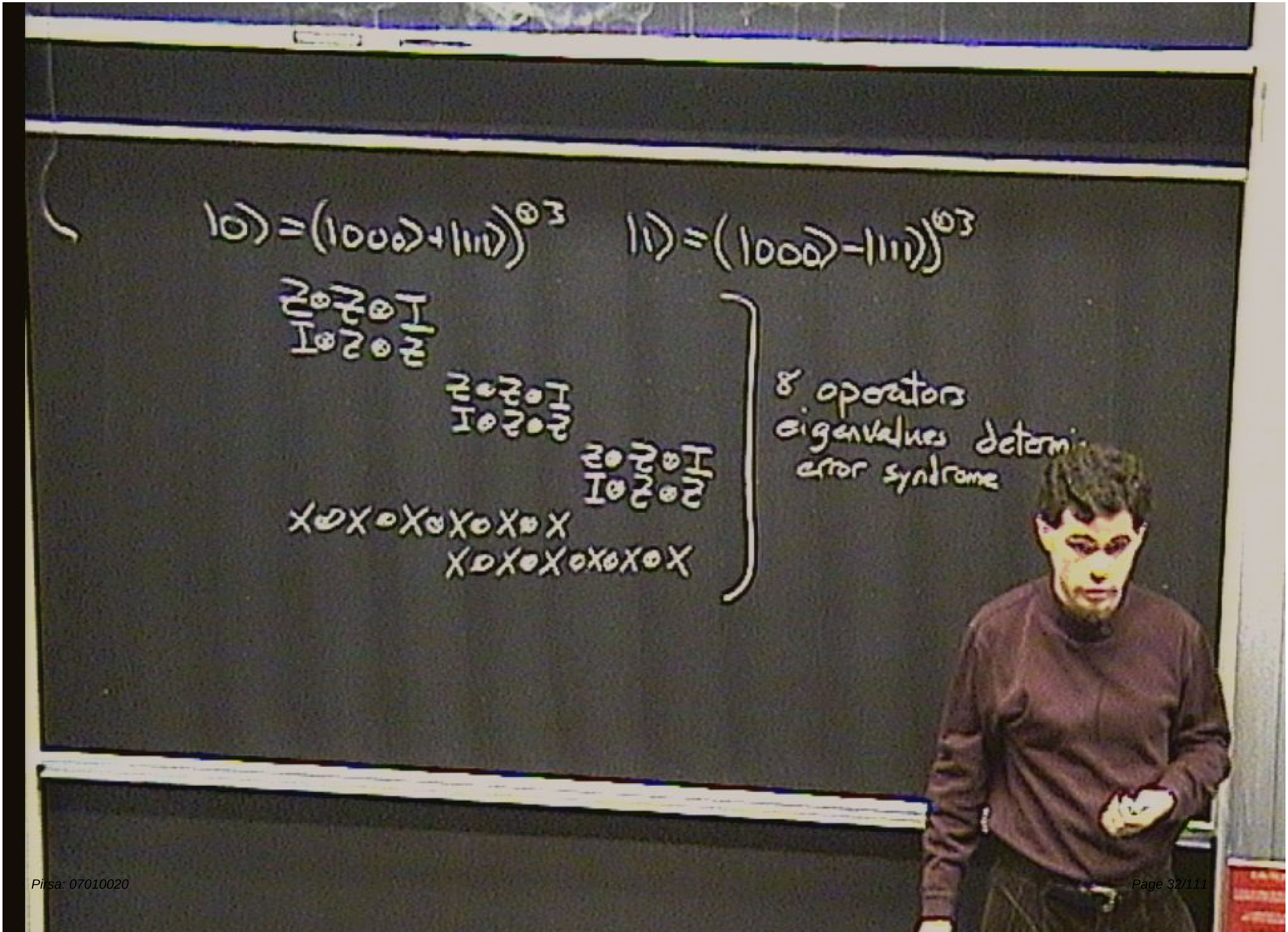
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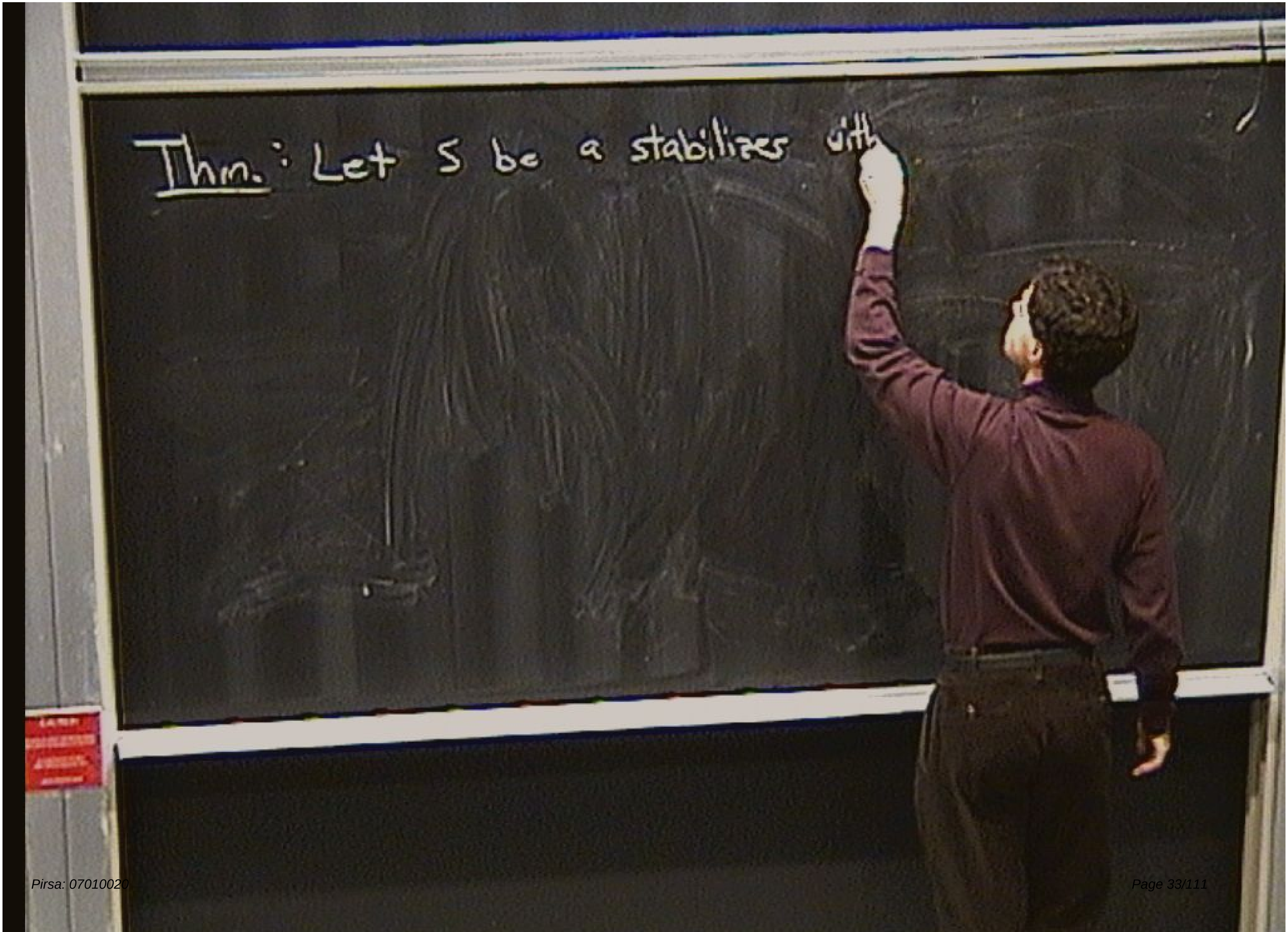
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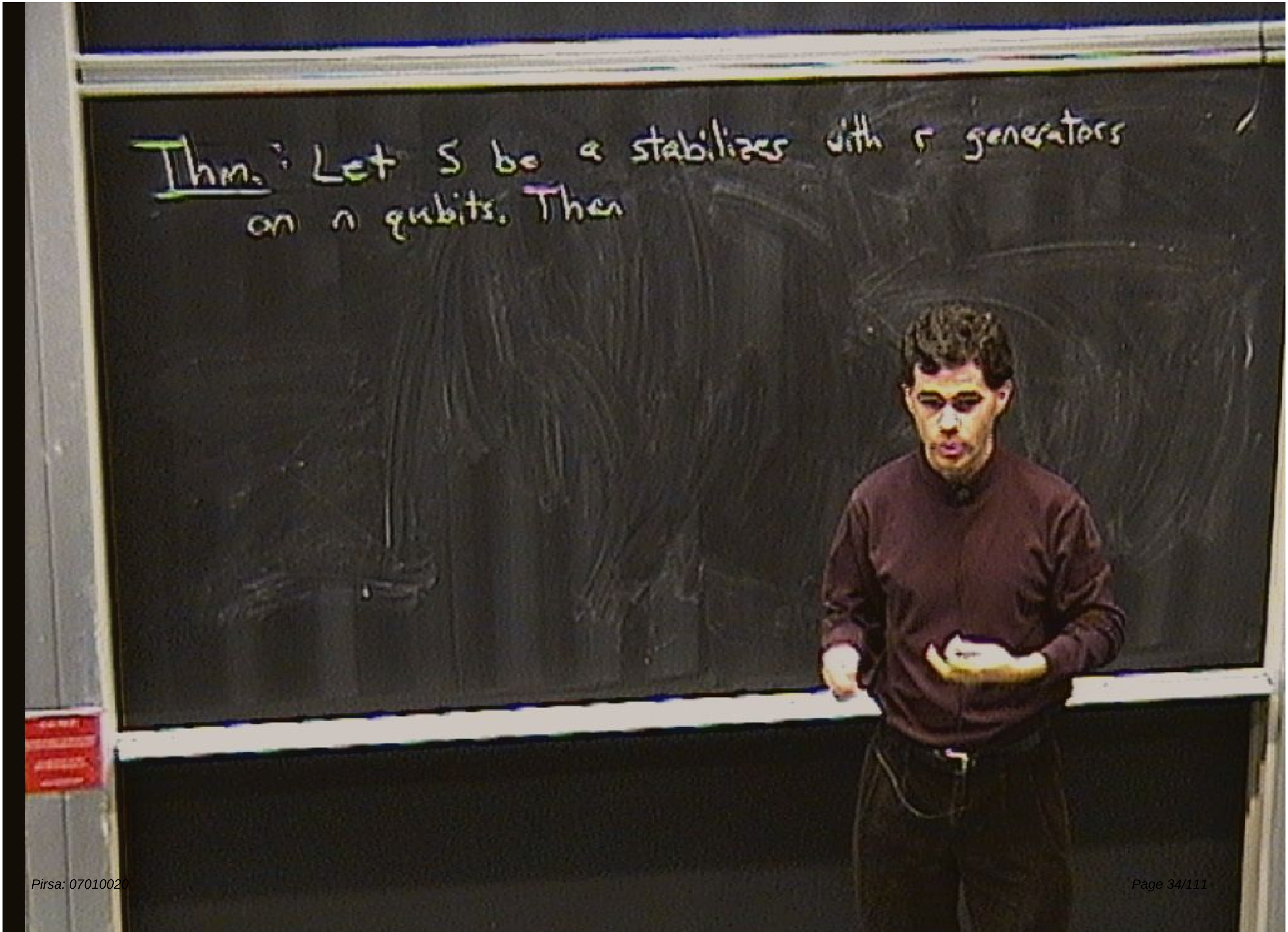
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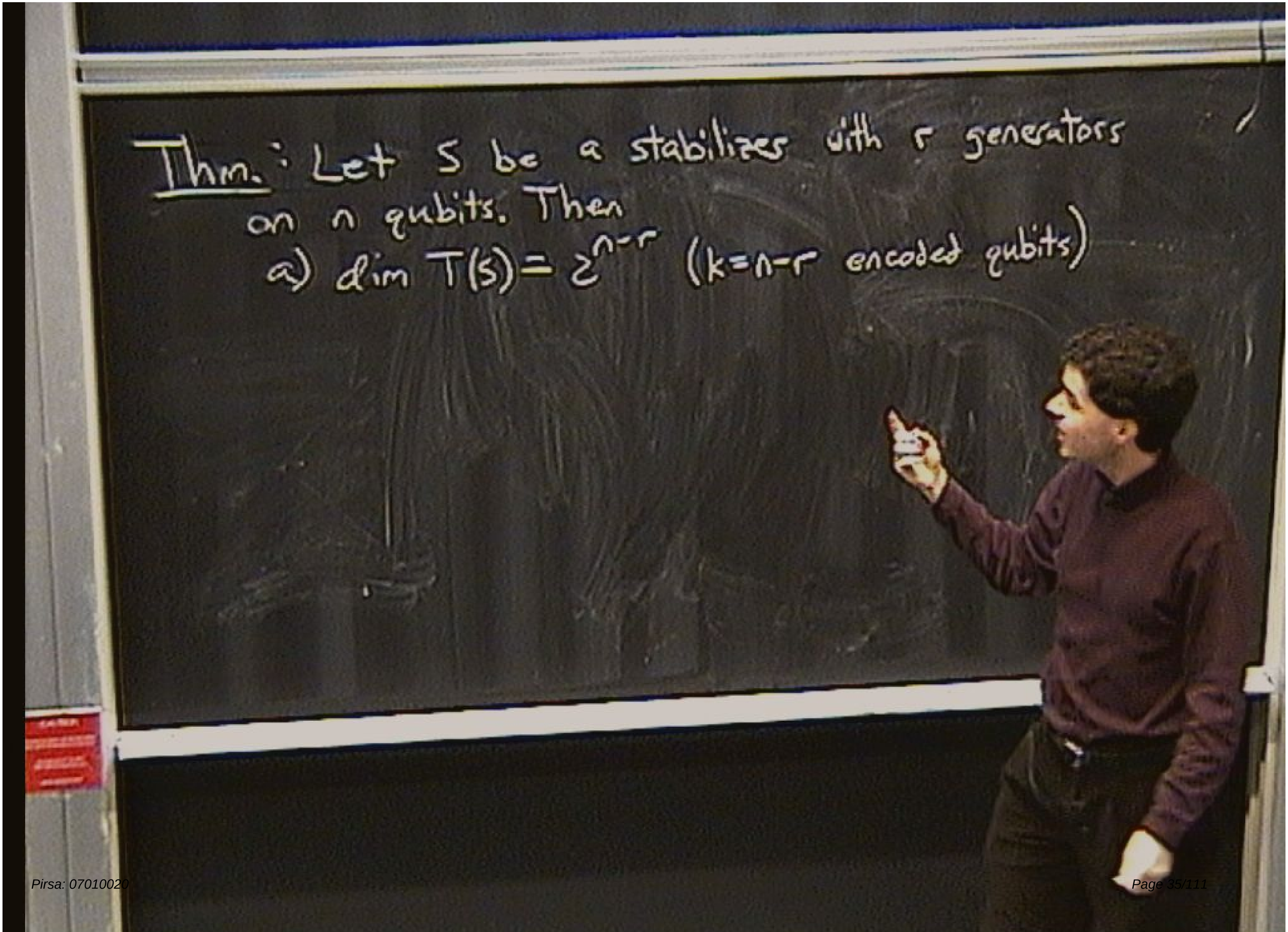
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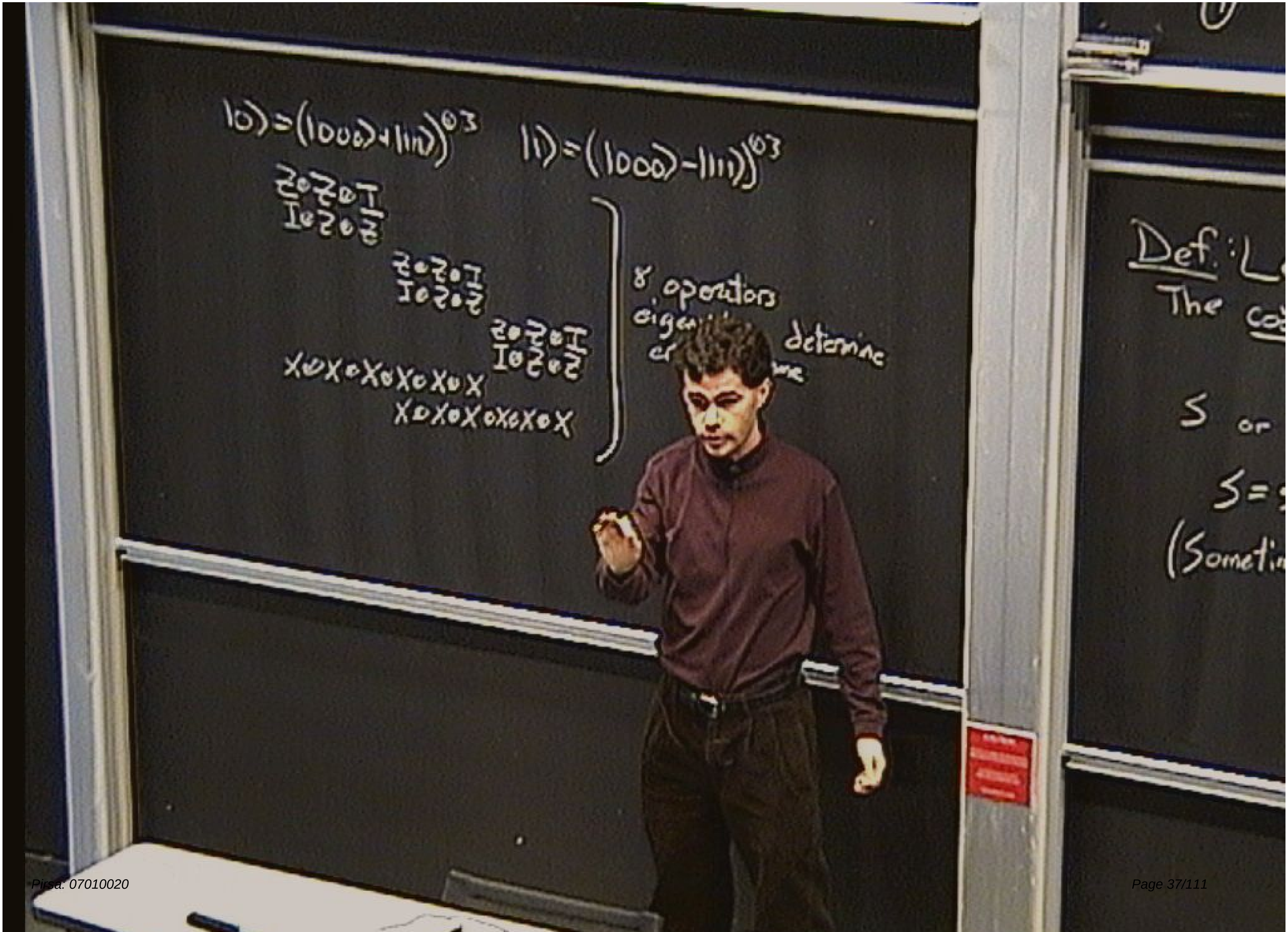
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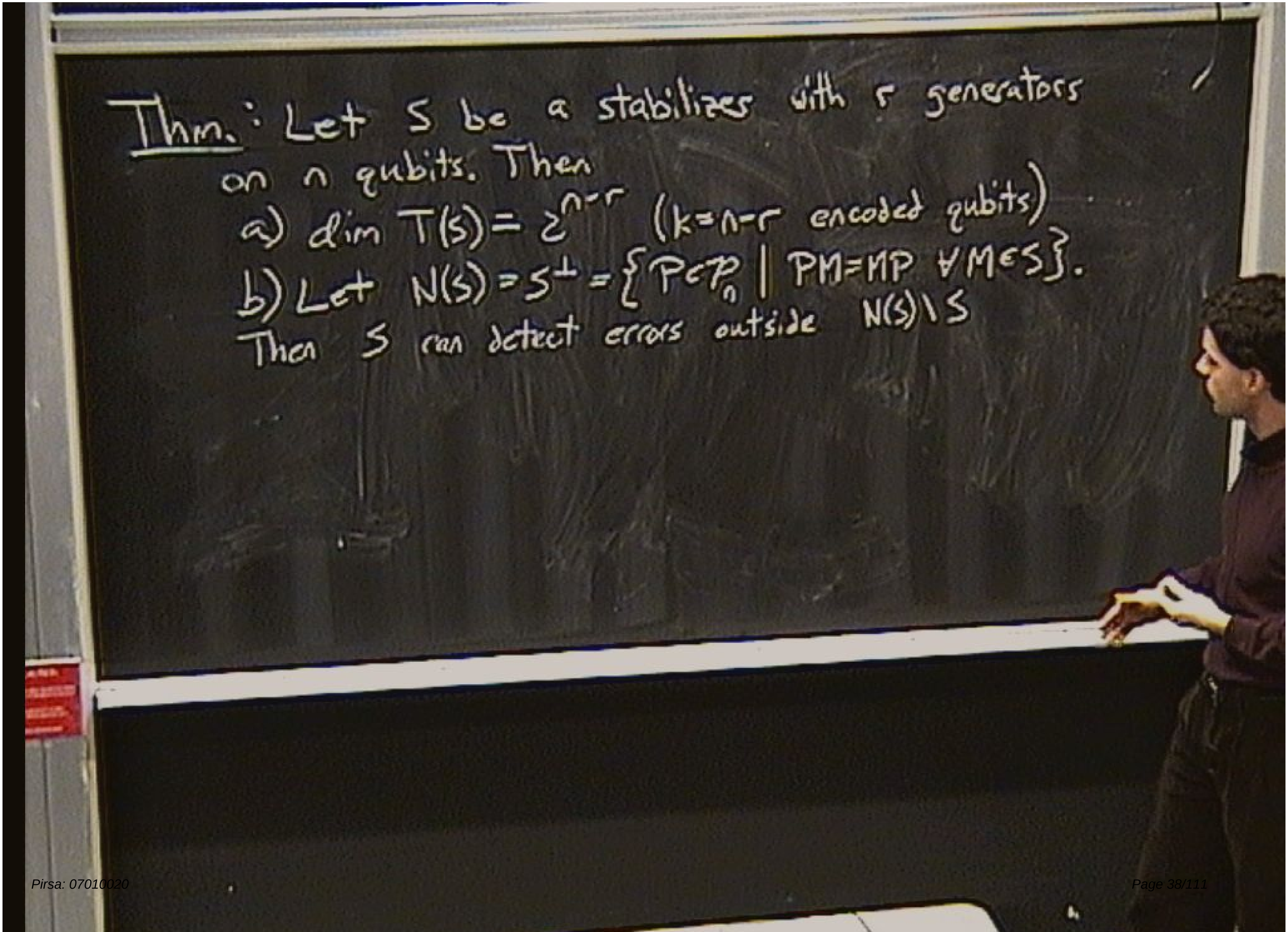
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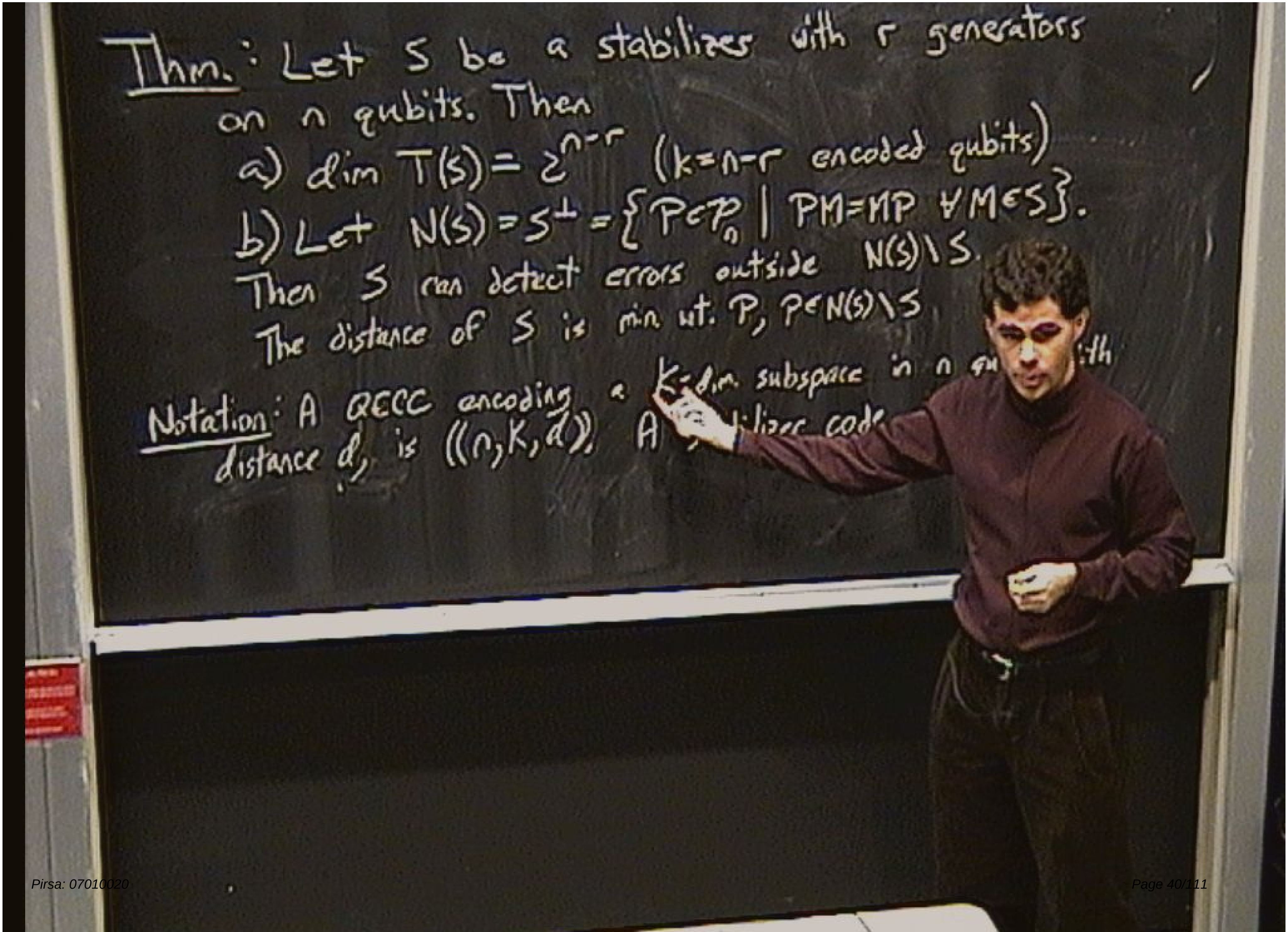
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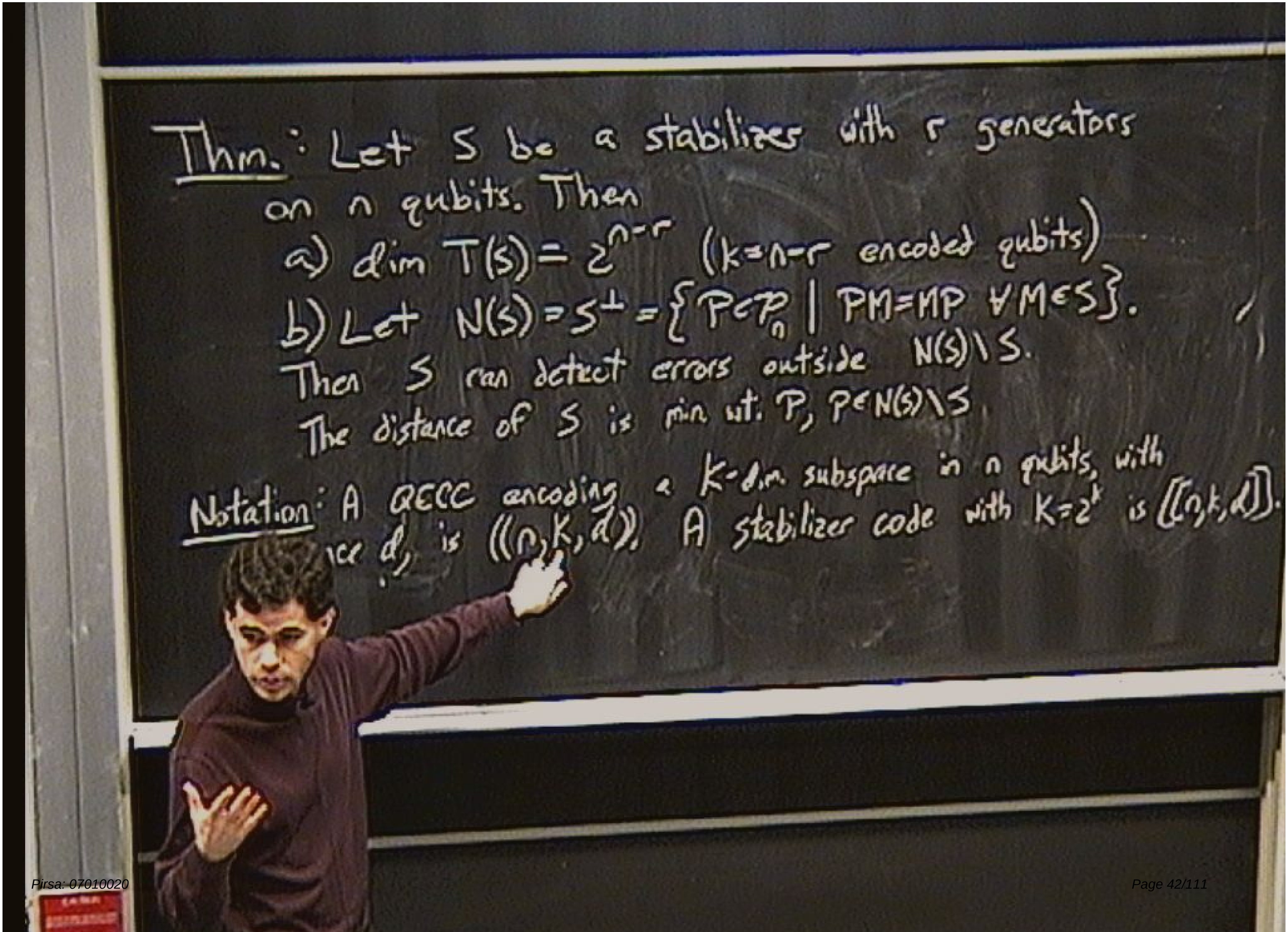
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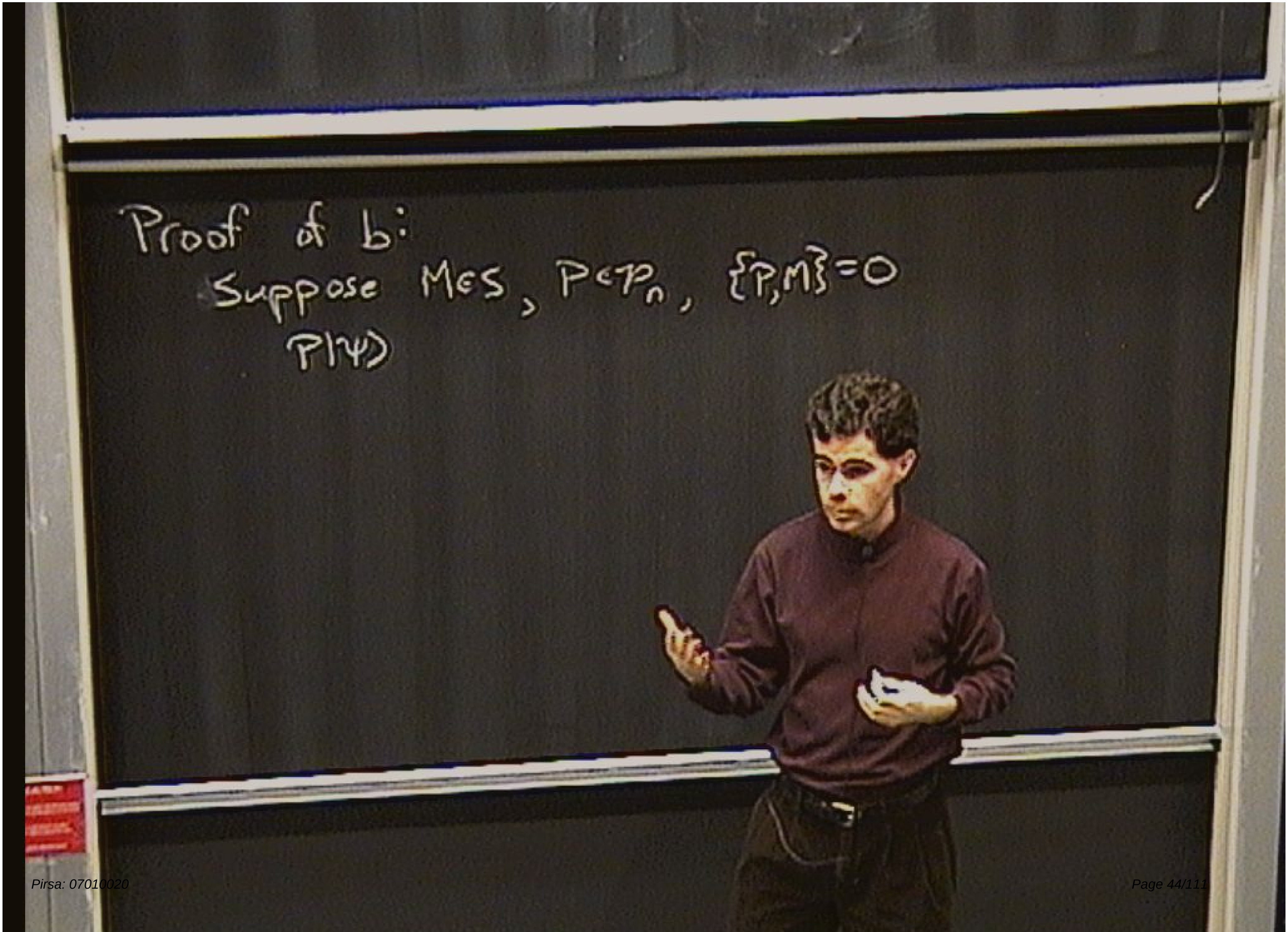
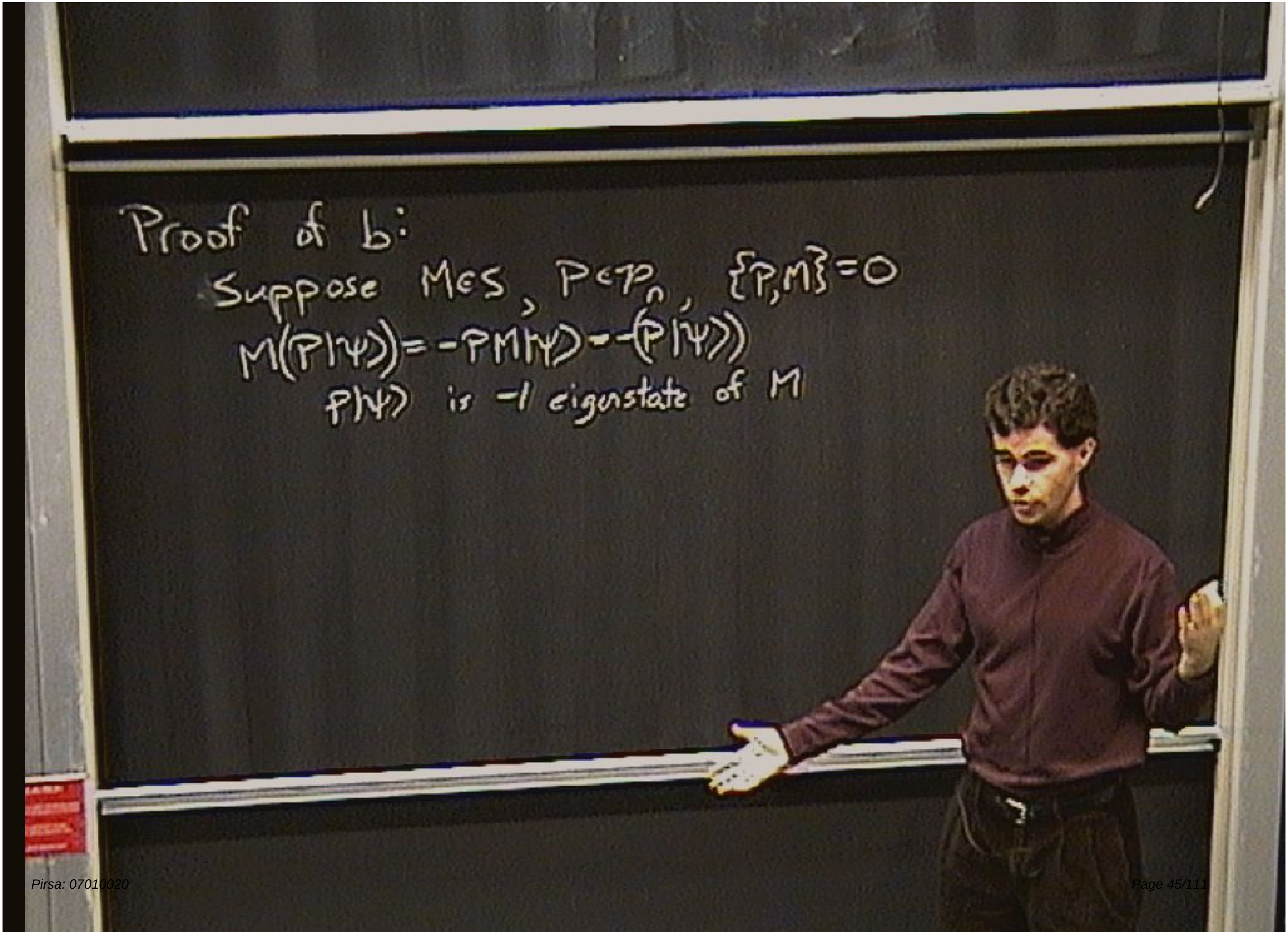

Proof of b:
Suppose $M \in S$, $P \in \mathcal{P}_n$, $\{P, M\} = 0$
$P|\psi\rangle$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M(P|ψ⟩) = −PM|ψ⟩ = −(P|ψ⟩)
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$P|\psi\rangle \in T(S)$

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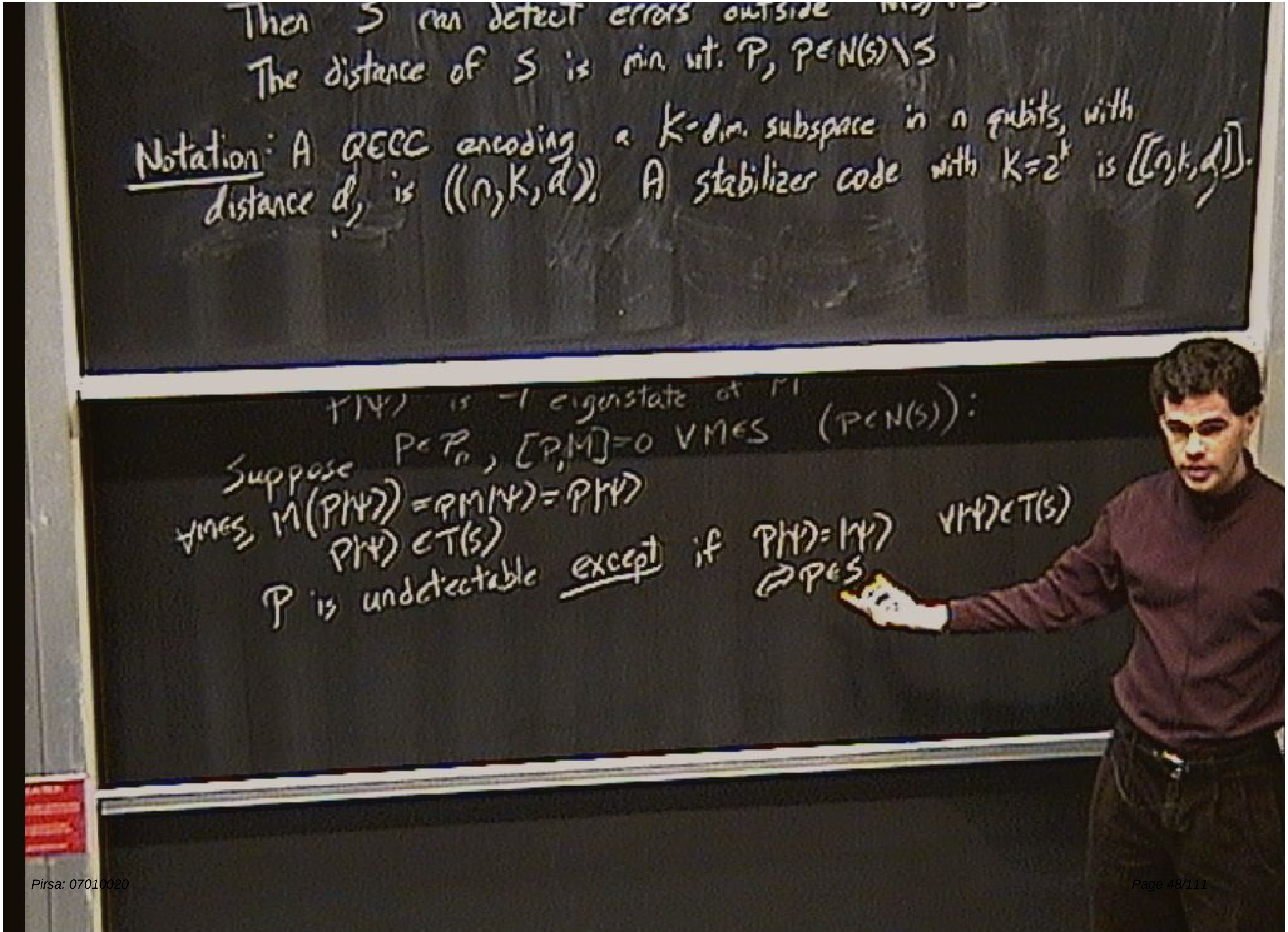
P is undetectable <u>except</u> if $P|\psi\rangle = |\psi\rangle$ $\forall |\psi\rangle \in T(S)$

$\Leftrightarrow P \in S$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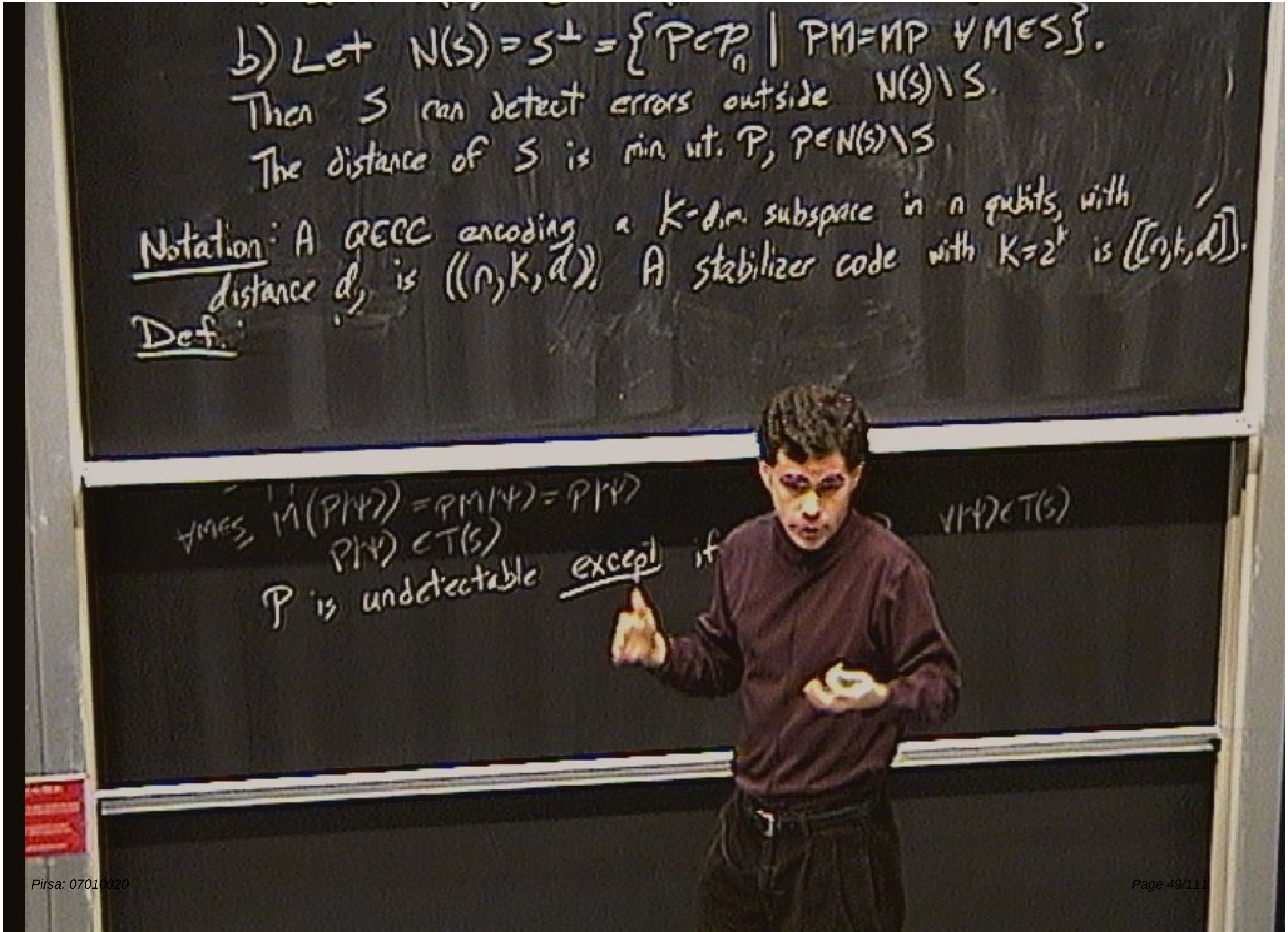
Notation: A QECC encoding a K-dim. subspace in n qubits, with distance d, is $((n,K,d))$. A stabilizer code with $K=2^k$ is $[[n,k,d]]$.
$P|\psi\rangle$ is -1 eigenstate of M
Suppose $P \in \mathcal{P}_n$, $[P,M]=0$ $\forall M \in S$ $(P \in N(S))$:
$\forall M \in S$, $M(P|\psi\rangle) = PM|\psi\rangle = P|\psi\rangle$
$P|\psi\rangle \in T(S)$
P is undetectable except if $P|\psi\rangle = |\psi\rangle$ $\forall |\psi\rangle \in T(S)$
$\Leftrightarrow P \in S$


b) Let N(S) = S⊥ = {P ∈ 𝒫n | PM = MP ∀M ∈ S}.
Then S can detect errors outside N(S)\S.
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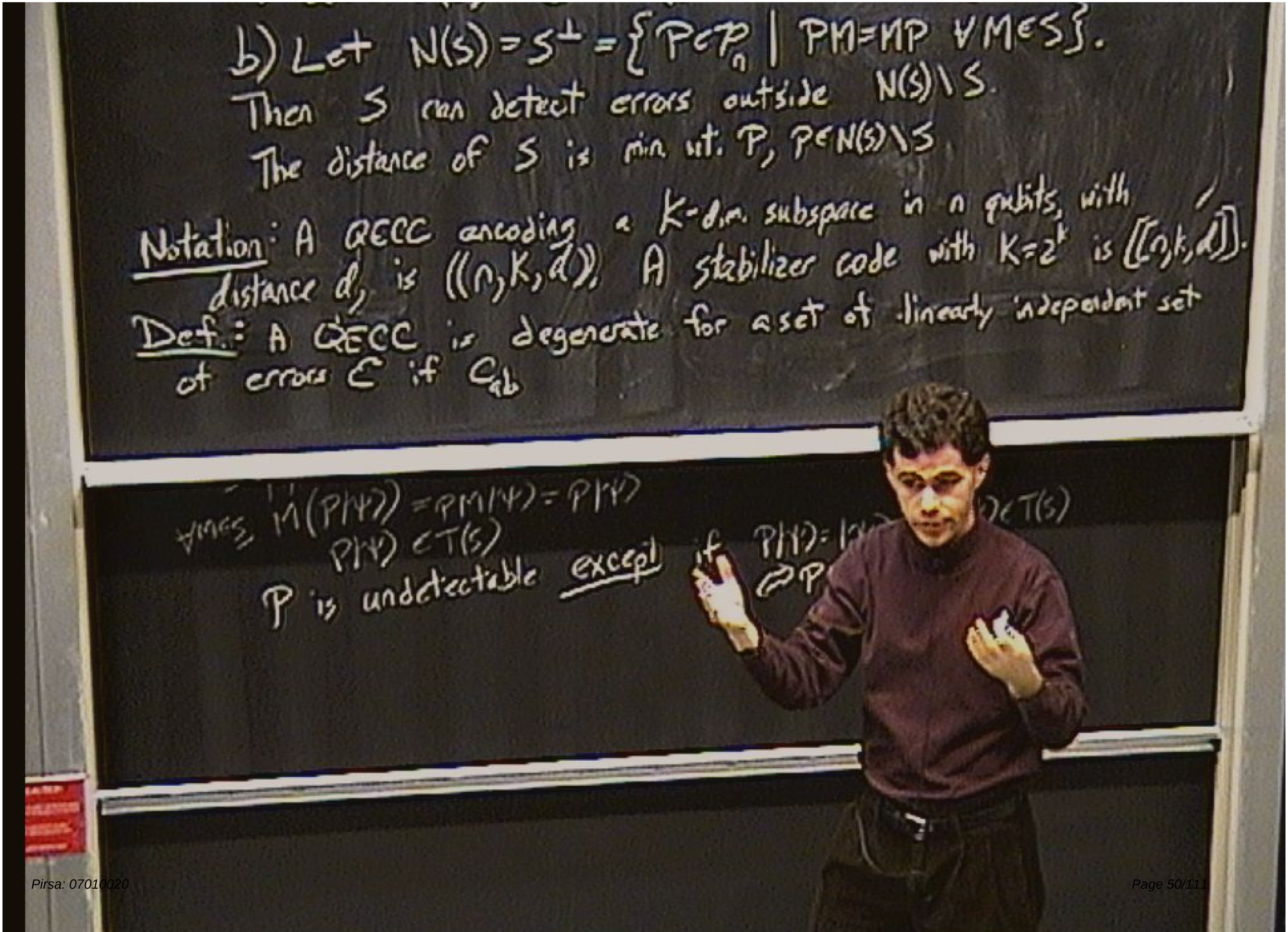
P is undetectable except if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Notation: A QECC encoding a K-dim. subspace in n qubits, with distance d, is ((n, K, d)). A stabilizer code with K = 2^k is [[n, k, d]].
Def.: A QECC is degenerate for a set of linearly independent set of errors E if C_ab
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P is undetectable except if P|ψ⟩ = |

b) Let $N(S) = S^{\perp} = \{ P \in \mathcal{P}_n \mid PM = MP \ \forall M \in S \}$.

Then S can detect errors outside $N(S) \setminus S$.

The distance of S is min wt. P, $P \in N(S) \setminus S$

<u>Notation</u>: A QECC encoding a K-dim. subspace in n qubits, with distance d, is $((n, K, d))$. A stabilizer code with $K = 2^k$ is $[[n, k, d]]$.

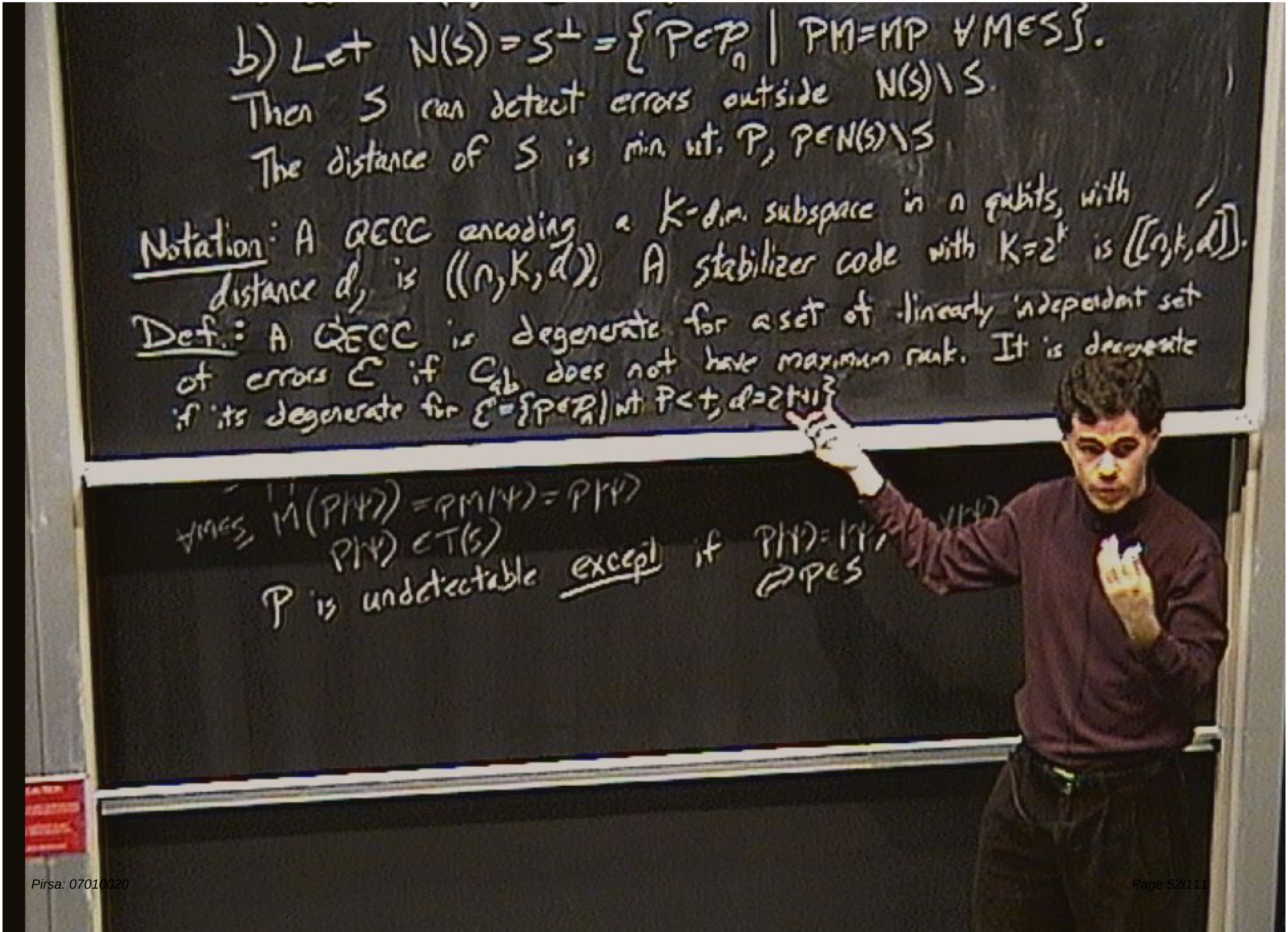
<u>Def.</u>: A QECC is degenerate for a set of linearly independent set of errors $\mathcal{E}$ if C_{ab} does not have maximum rank. It is degenerate

$\forall M \in S$, $M(P|\psi\rangle) = PM|\psi\rangle = P|\psi\rangle$ $\quad \forall |\psi\rangle \in T(S)$

$P|\psi\rangle \in T(S)$

P is undetectable <u>except</u> if $P|\psi\rangle = |\psi\rangle$

$\Leftrightarrow P \in S$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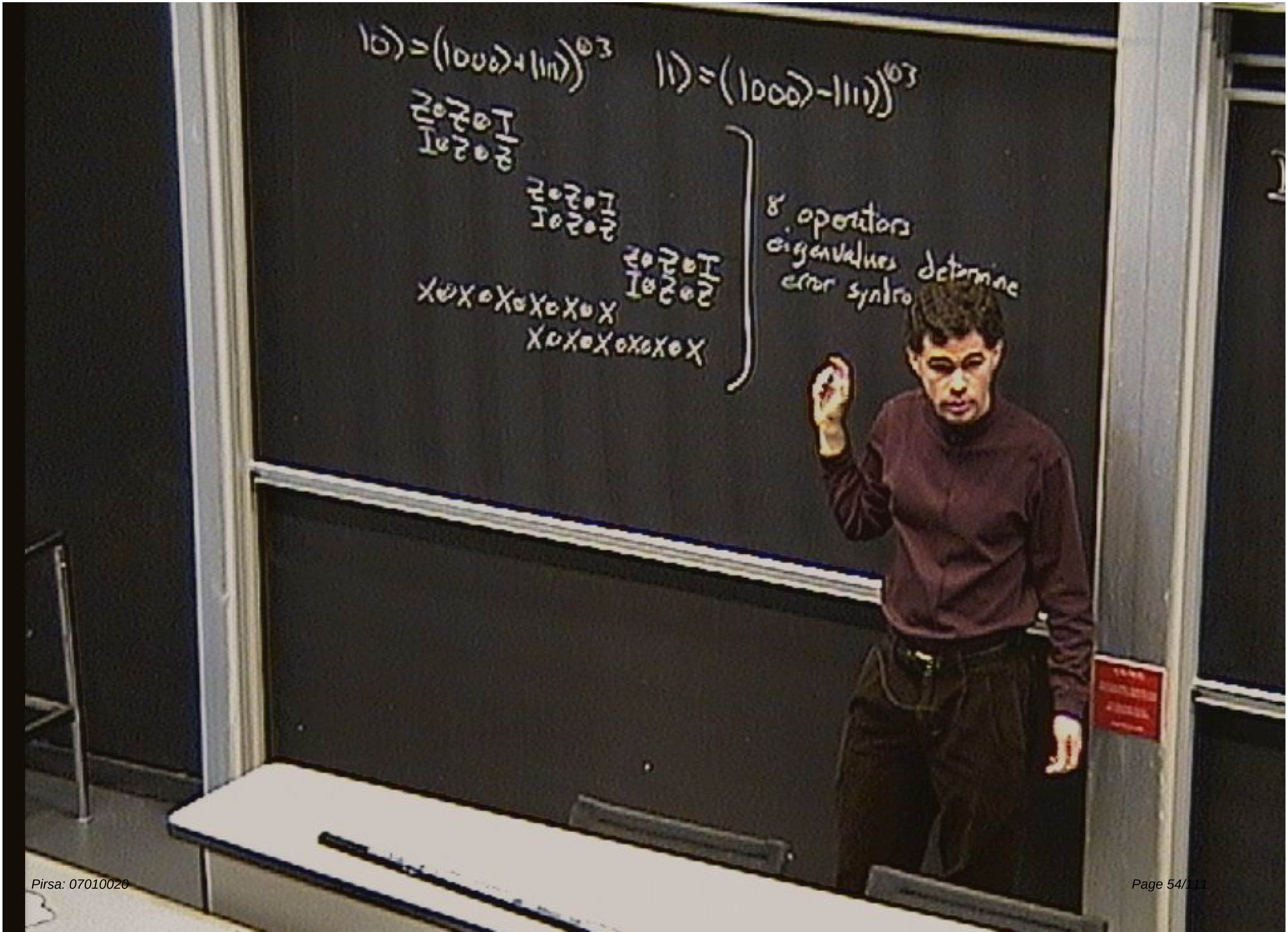
P is undetectable <u>except</u> if $P|\psi\rangle = |\psi\rangle \quad \forall |\psi\rangle \in T(S)$

$\Leftrightarrow P \in S$


|0⟩=(|000⟩+|111⟩)⊗3
|1⟩=(|000⟩-|111⟩)⊗3
Z⊗Z⊗I
I⊗Z⊗Z
Z⊗Z⊗I
I⊗Z⊗Z
Z⊗Z⊗I
I⊗Z⊗Z
X⊗X⊗X⊗X⊗X⊗X
X⊗X⊗X⊗X⊗X⊗X
8 operators
eigenvalues determine
error syndro

of errors $\mathcal{E}$ if C_{ab} does not have maximum rank. It is degenerate if its degenerate for $\mathcal{E}=\{P\in\mathcal{P}_n \mid wt\,P<t,\ d=2t+1\}$

Proof of b:

Suppose $M\in S$, $P\in\mathcal{P}_n$, $\{P,M\}=0$ $\quad P\notin N(S)$

$M(P|\psi\rangle)=-PM|\psi\rangle=-(P|\psi\rangle)$

$P|\psi\rangle$ is -1 eigenstate of M

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$\Leftrightarrow P\in S$

(A stabilizer code is degenerate if $C_{ab}E_b$

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P is undetectable <u>except</u> if $P|\psi\rangle=|\psi\rangle$ $\quad \forall|\psi\rangle\in T(S)$

$\Leftrightarrow P\in S$

(A stabilizer code is degenerate if $E_a^\dagger E_b\in S$ for E_a, E_b distinct correctable errors.

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(A <u>stabilizer</u> code is degenerate if $E_a^\dagger E_b \in S$ for E_a, E_b distinct correctable errors.)

In a non-degenerate stabilizer code, dist = min wt $N(S)$

of errors $\mathcal{E}$ if C_{ab} does not have maximum rank. It is degenerate if its degenerate for $\mathcal{E} = \{P \in \mathcal{P}_n \mid \text{wt } P < t, d = 2t+1\}$

Proof of b:

Suppose $M \in S$, $P \in \mathcal{P}_n$, $\{P, M\} = 0$ $\quad P \notin N(S)$

$M(P|\psi\rangle) = -PM|\psi\rangle = -(P|\psi\rangle)$

$P|\psi\rangle$ is -1 eigenstate of M

Suppose $P \in \mathcal{P}_n$, $[P, M] = 0$ $\forall M \in S$ $\quad (P \in N(S))$:

$\forall M \in S$, $M(P|\psi\rangle) = PM|\psi\rangle = P|\psi\rangle$

$P|\psi\rangle \in T(S)$

P is undetectable except if $P|\psi\rangle = |\psi\rangle$ $\forall |\psi\rangle \in T(S)$
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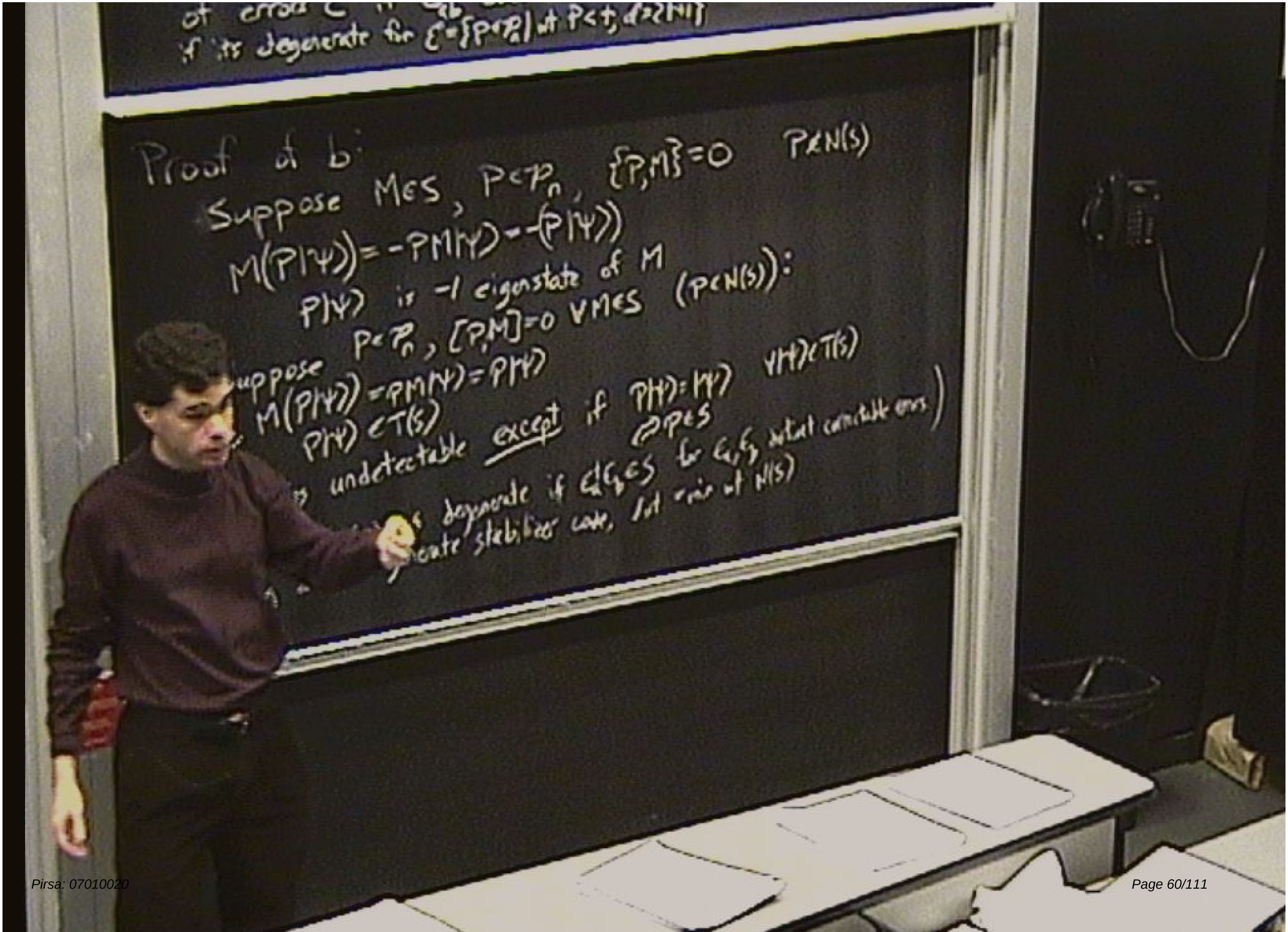
(A stabilizer code is degenerate if $E_a^\dagger E_b \in S$ for E_a, E_b distinct correctable errors.)

In a non-degenerate stabilizer code, $d = \min \text{wt } N(S)$

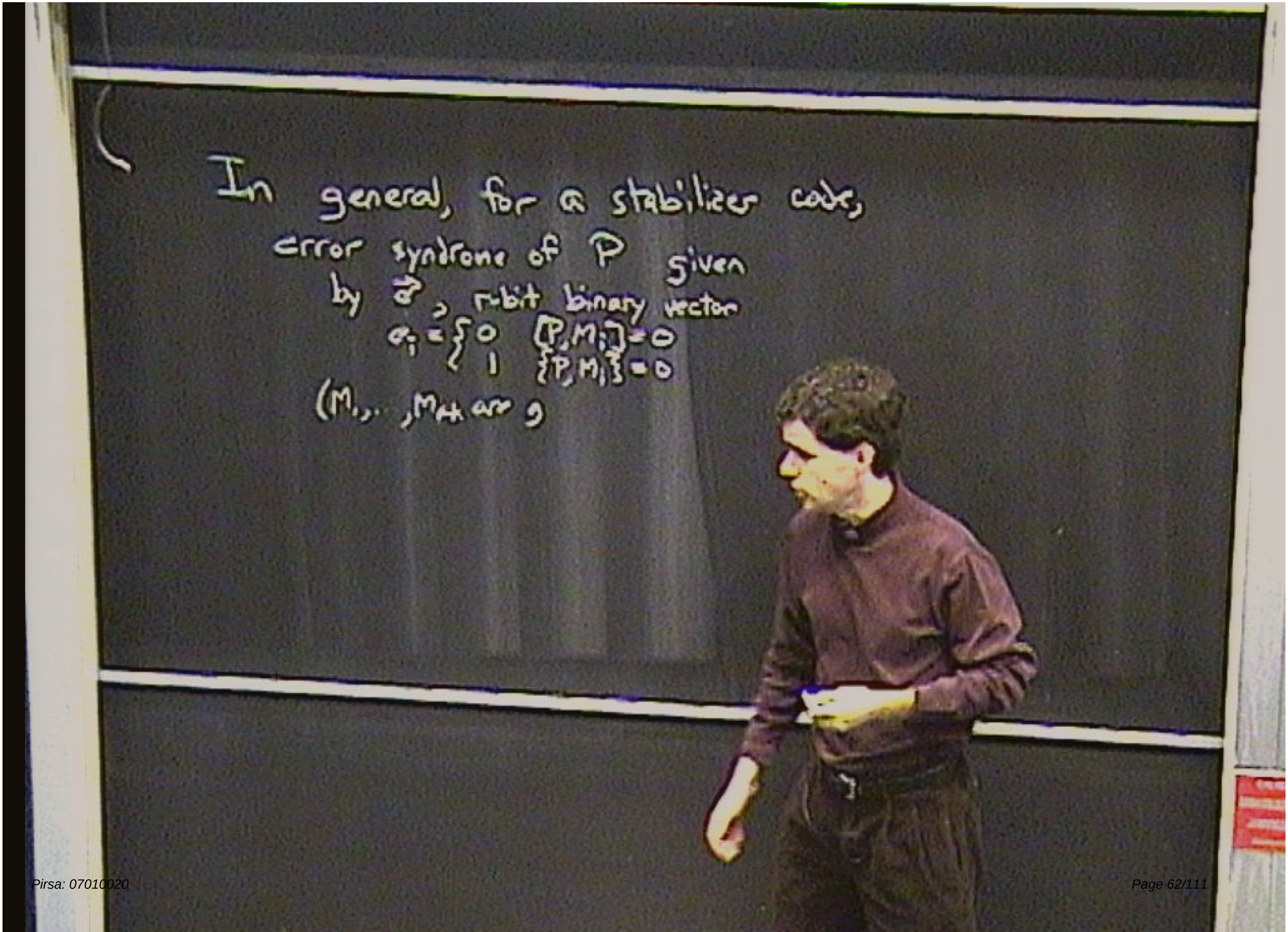
$$|0\rangle = (|000\rangle + |111\rangle)^{\otimes 3} \qquad |1\rangle = (|000\rangle - |111\rangle)^{\otimes 3}$$

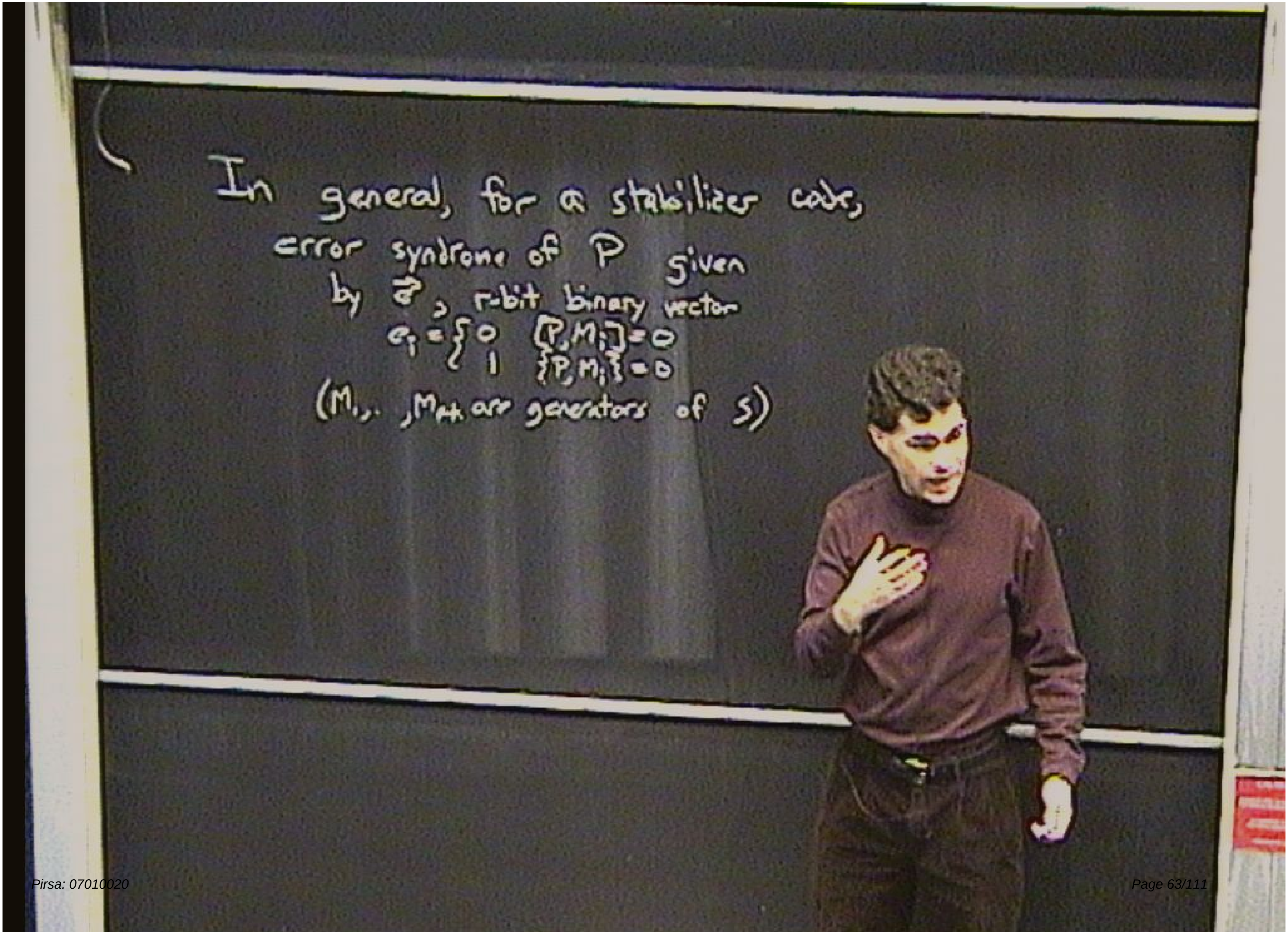
$$\left.\begin{array}{lll}
Z\otimes Z\otimes I & & \\
I\otimes Z\otimes Z & & \\
 & Z\otimes Z\otimes I & \\
 & I\otimes Z\otimes Z & \\
 & & Z\otimes Z\otimes I \\
 & & I\otimes Z\otimes Z \\
X\otimes X\otimes X\otimes X\otimes X\otimes X & & \\
 & X\otimes X\otimes X\otimes X\otimes X\otimes X &
\end{array}\right\}$$

8 operators
eigenvalues determine error syndrome


Proof of b:
Suppose M∈S, P∈P_n, {P,M}=0 P∉N(S)
M(P|ψ⟩)=-PM|ψ⟩=-(P|ψ⟩)
P|ψ⟩ is -1 eigenstate of M
P∈P_n, [P,M]=0 ∀M∈S (P∈N(S)):
Suppose
M(P|ψ⟩)=PM|ψ⟩=P|ψ⟩
P|ψ⟩∈T(S)
undetectable except if P|ψ⟩=|ψ⟩ ∀|ψ⟩∈T(S)
P∈S

In general, for a stabilizer code,
error syndrome of P


In general, for a stabilizer code,
error syndrome of P given
by $\vec{a}$, r-bit binary vector
$a_i = \begin{cases} 0 & [P, M_i] = 0 \\ 1 & \{P, M_i\} = 0 \end{cases}$
($M_1, \ldots, M_{n-k}$ are g


In general, for a stabilizer code,
error syndrome of P given
by $\vec{e}$, r-bit binary vector
$e_i = \begin{cases} 0 & [P,M_i]=0 \\ 1 & \{P,M_i\}=0 \end{cases}$
($M_1, \ldots, M_{n-k}$ are generators of S)

In general, for a stabilizer code,

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$$e_i = \begin{cases} 0 & [P, M_i] = 0 \\ 1 & \{P, M_i\} = 0 \end{cases}$$

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$$\text{Syndrome}(PQ) = \text{Syndrome}(P) + \text{syndrome}(Q)$$

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Syndrome (PQ) = Syndrome (P) + syndrome (Q) ((binary))

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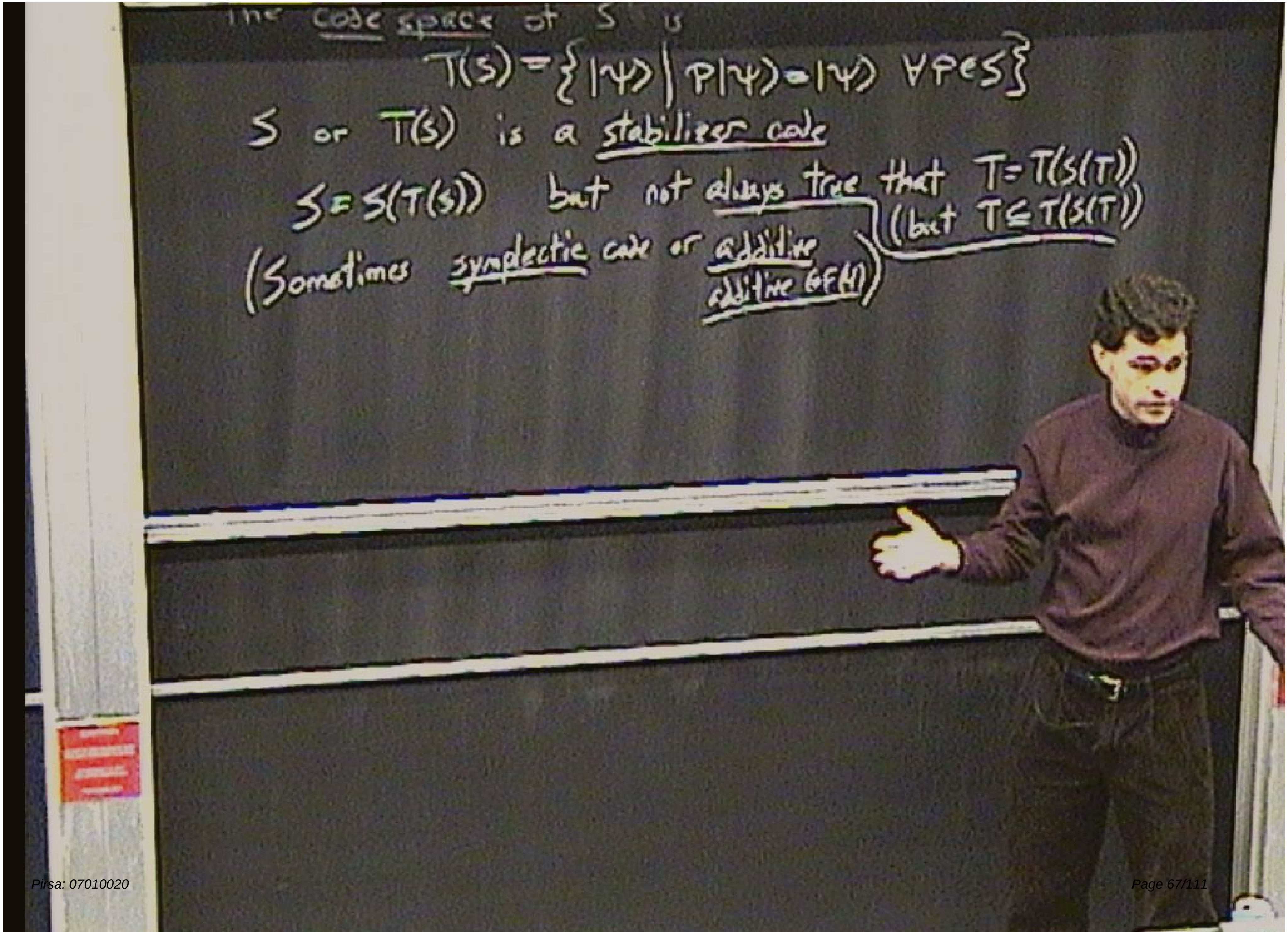
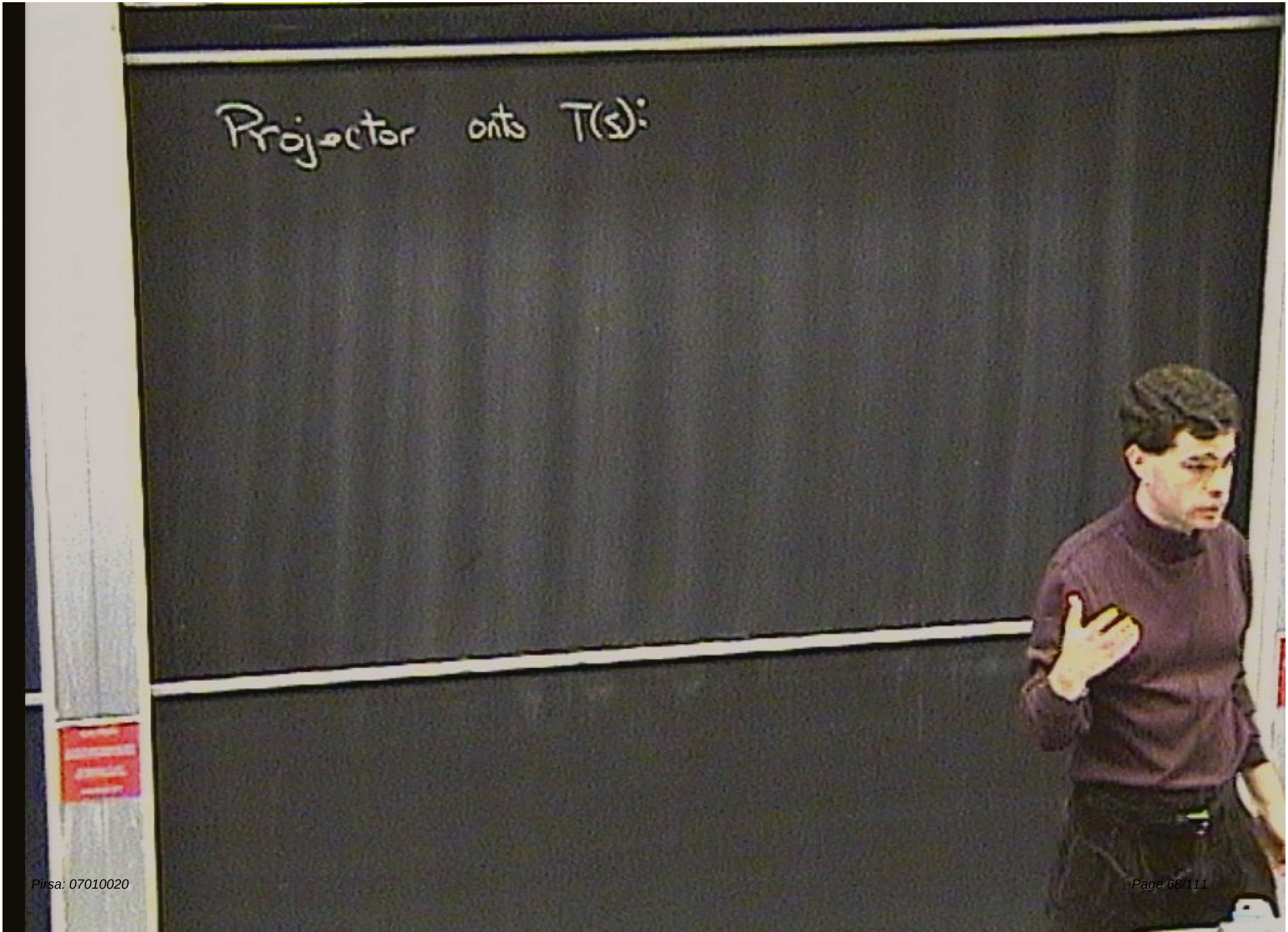
$$\text{Syndrome}(PQ) = \text{Syndrome}(P) + \text{syndrome}(Q) \quad (\text{(binary)})$$

$$\text{syndrome } P = \text{syndrome } Q \Leftrightarrow \text{syndrome}(PQ) = 0 \Leftrightarrow PQ \in N(S)$$


The code space of S is
$T(S) = \{ |\psi\rangle \mid P|\psi\rangle = |\psi\rangle \ \forall P \in S \}$
S or T(S) is a stabilizer code
$S = S(T(S))$ but not always true that $T = T(S(T))$
(but $T \subseteq T(S(T))$)
(Sometimes symplectic code or additive
additive GF(4))


Projector onto T(s):

Projector onto $T(S)^\dagger \Pi_S$:

Projector onto +1 eigenspace of $M \in \mathcal{P}_n$:

Projector onto $T(S)$, Π_S:

Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $I+M$

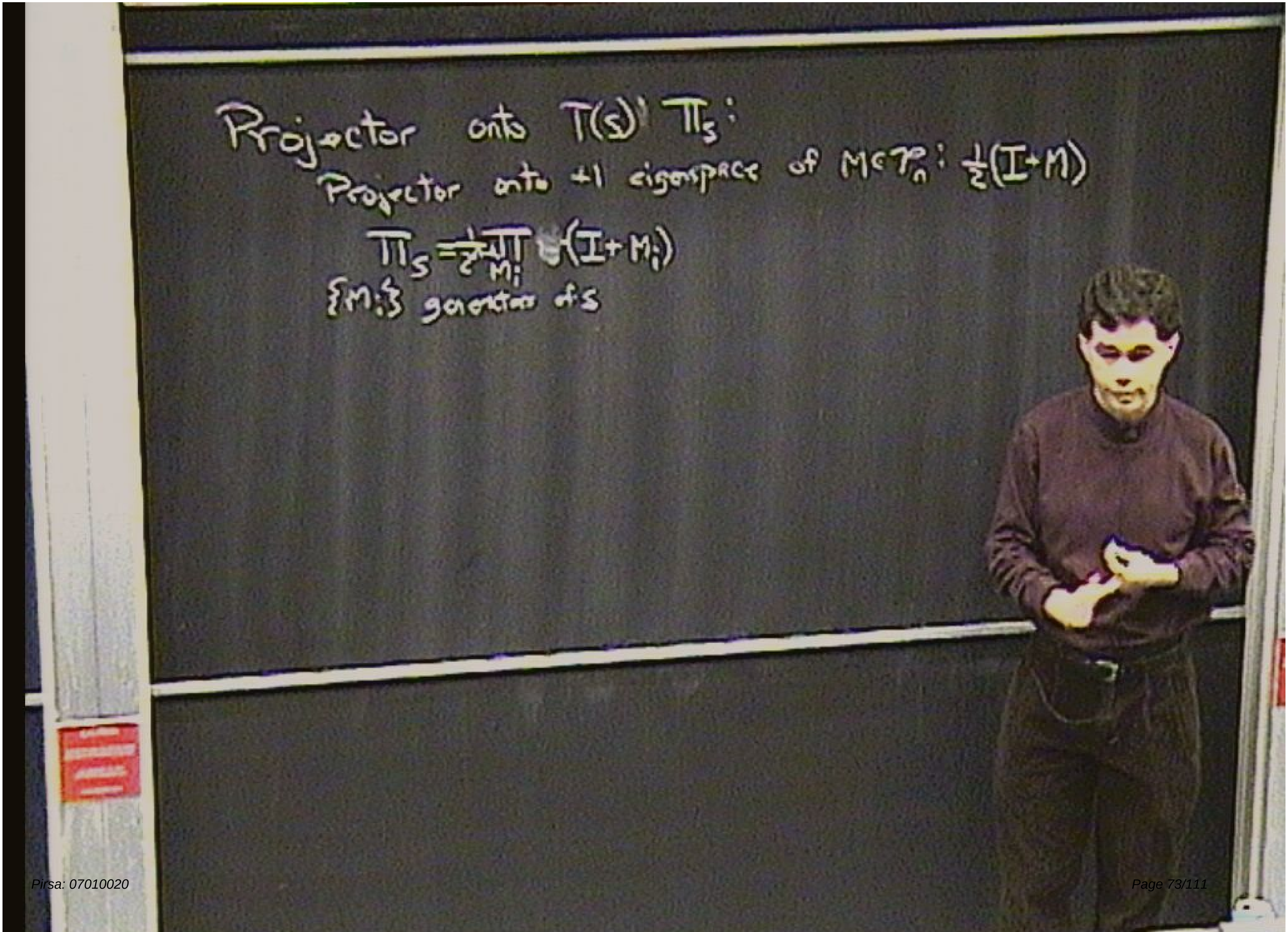
Projector onto $T(S)$ Π_S:

Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$

Projector onto $T(S)$: Π_S:

Projector onto $+1$ eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$

$\Pi_S =$

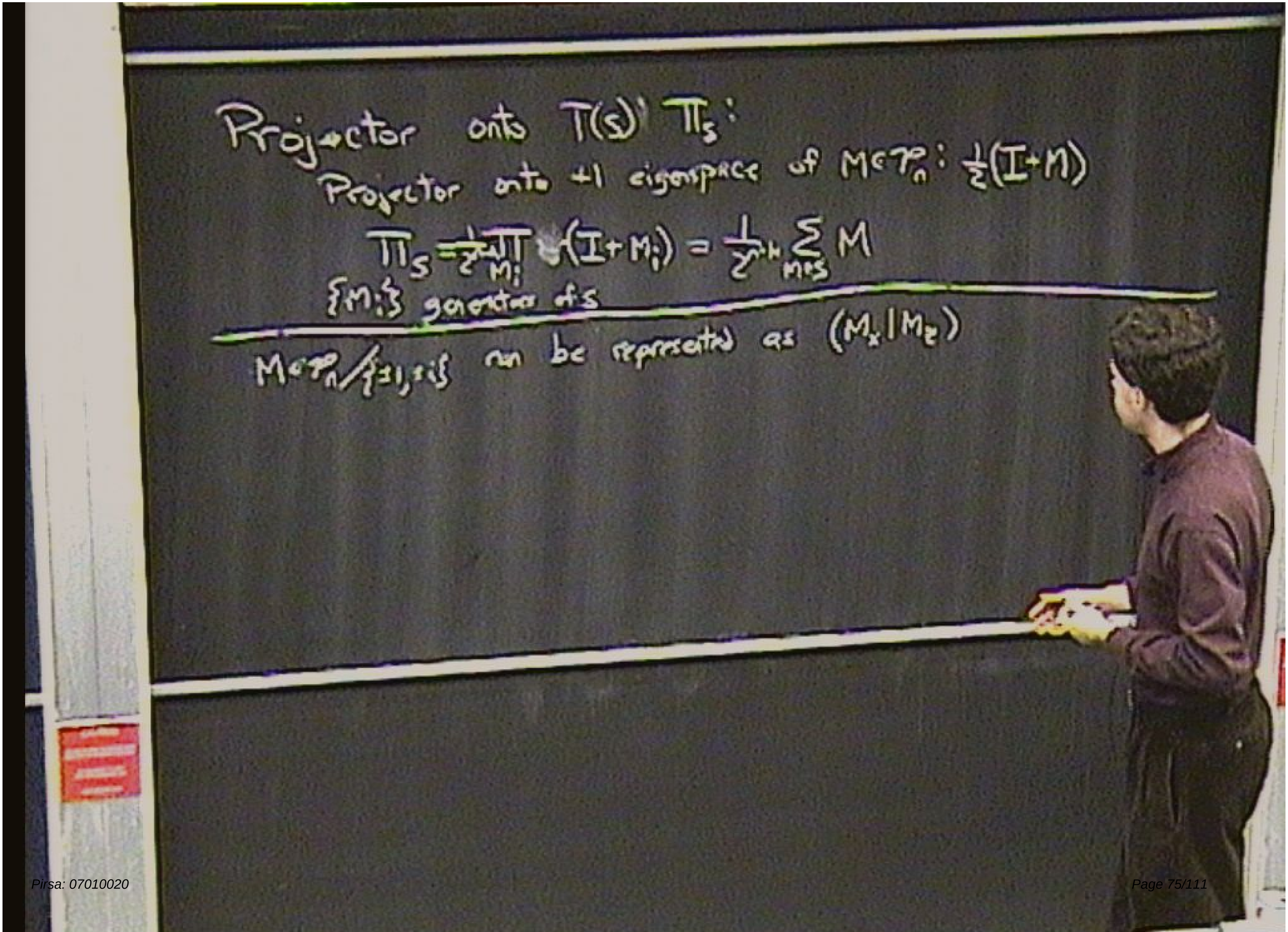

Projector onto $T(S)$: Π_S:
Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$
$\Pi_S = \prod_{M_i} \frac{1}{2}(I+M_i)$
$\{M_i\}$ generators of S

Projector onto $T(S)$: Π_S:

Projector onto $+1$ eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$

$$\Pi_S = \frac{1}{2^{n-k}} \prod_{M_i} (I+M_i) = \frac{1}{2^{n-k}} \sum_{M \in S} M$$

$\{M_i\}$ generators of S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$M \in \mathcal{P}_n / \{\pm 1, \pm i\}$ can be represented as $(M_x | M_z)$

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Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$

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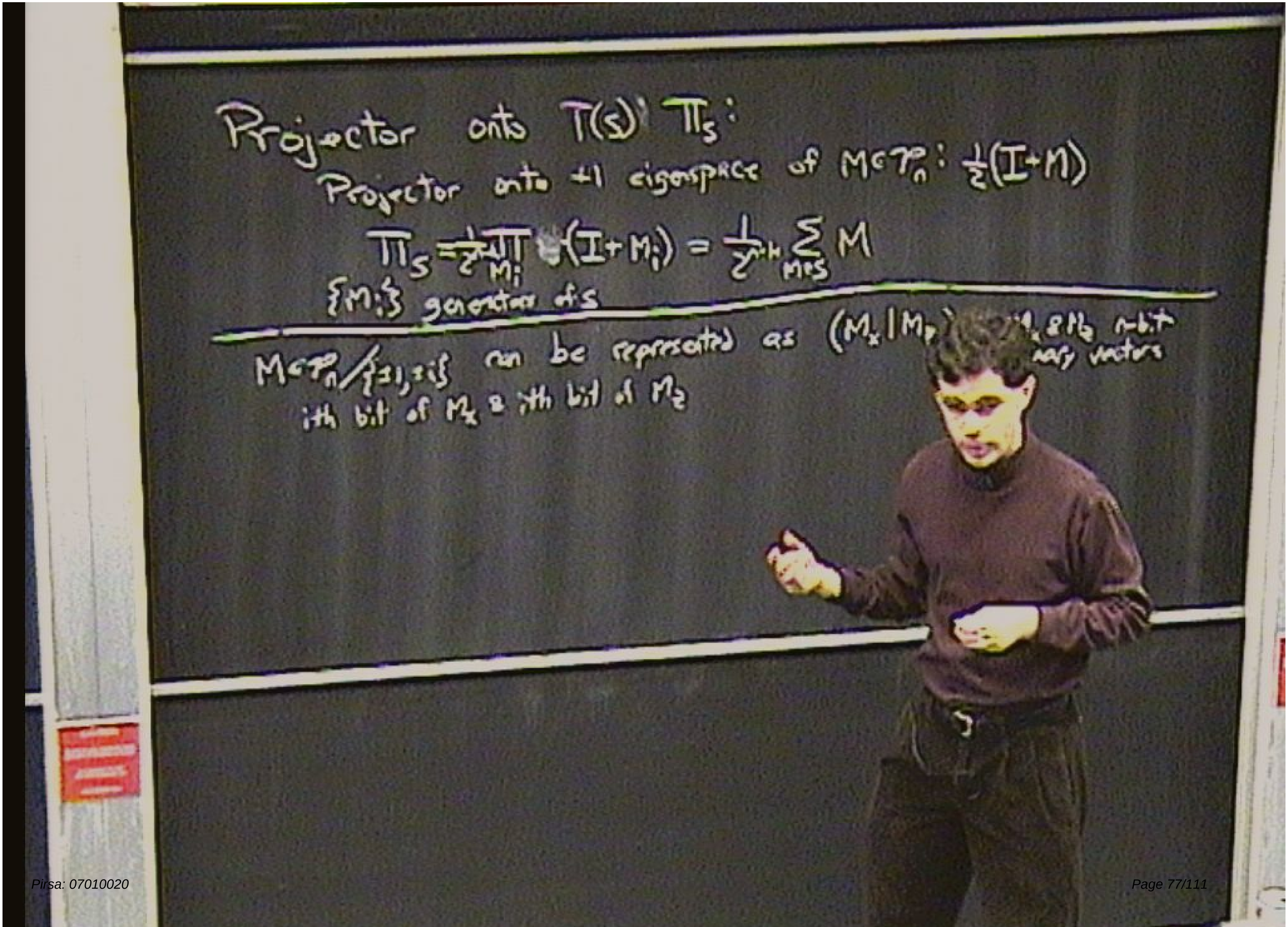
$\{M_i\}$ generators of S

$M \in \mathcal{P}_n / \{\pm 1, \pm i\}$ can be represented as $(M_x | M_z)$, M_x & M_z n-bit binary vectors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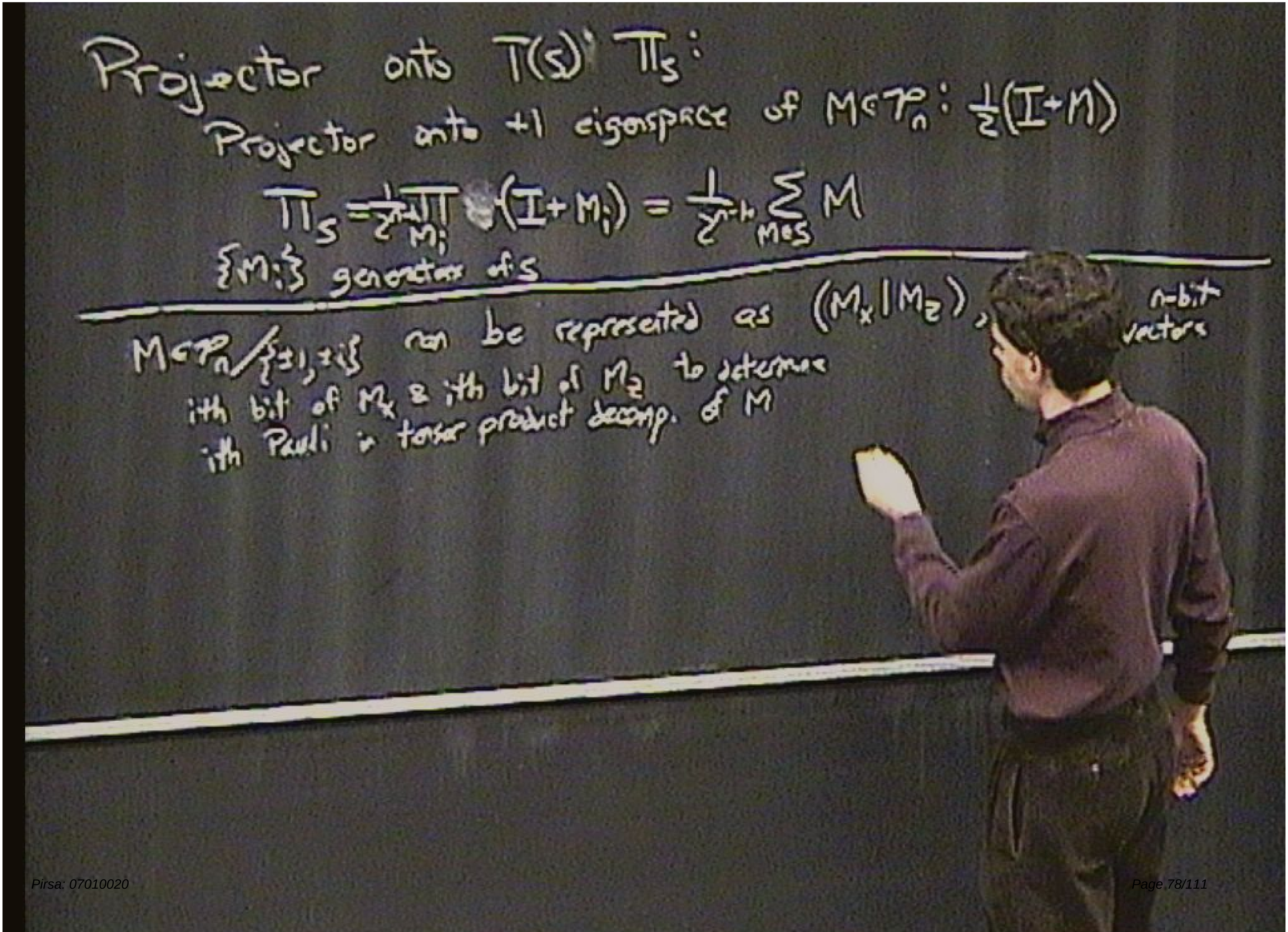
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$M \in \mathcal{P}_n / \{\pm 1, \pm i\}$ can be represented as $(M_x | M_z)$, n-bit vectors
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Projector onto $T(S)$: Π_S:

Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$

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$(M_x)_i = 1$, $(M_z)_i = 0$ $\Rightarrow X$ in ith position

$\vdots$

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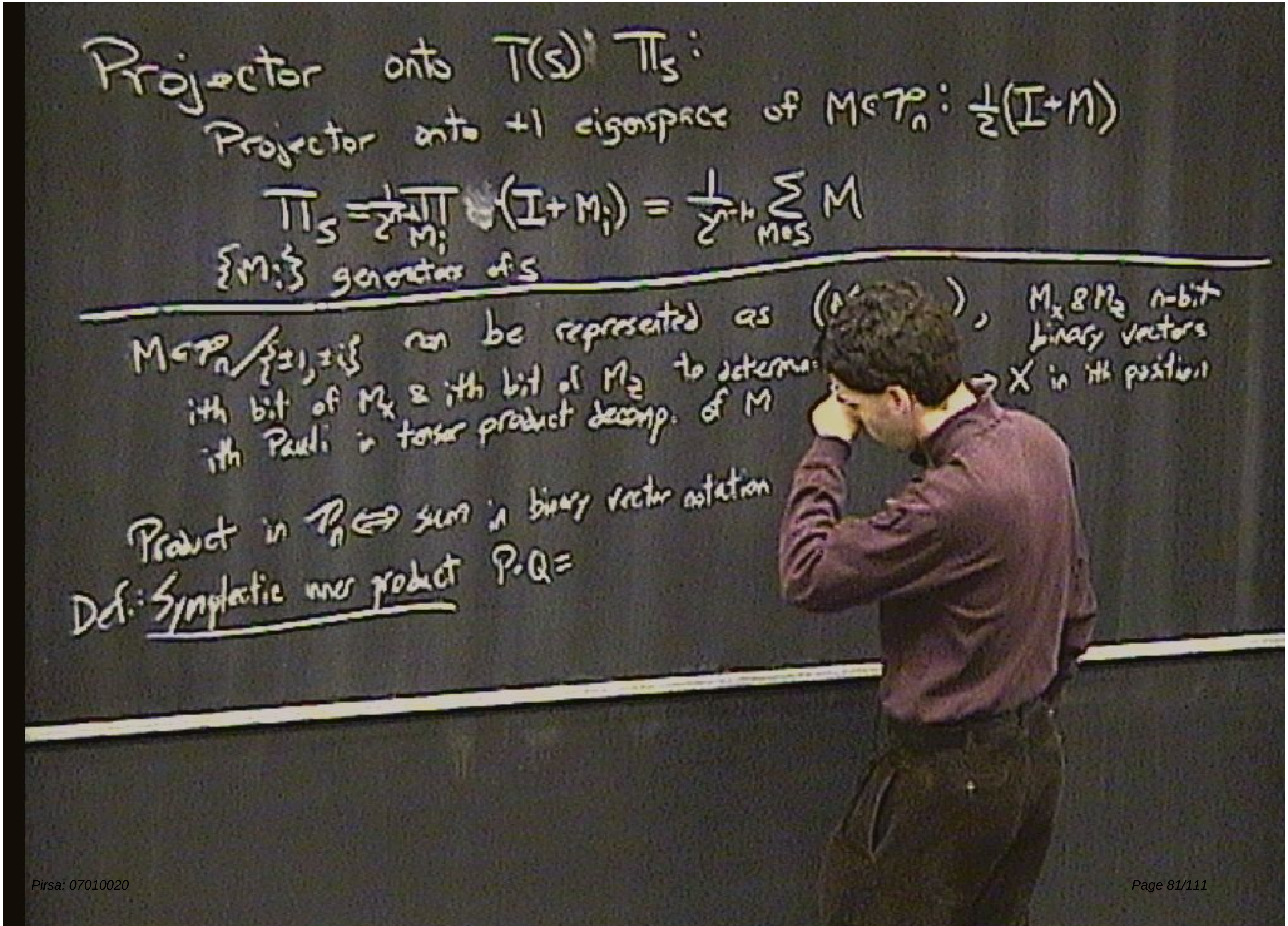
Product in $\mathcal{P}_n \Leftrightarrow$ sum in binary vector notation


Projector onto T(S): Π_S:
Projector onto +1 eigenspace of M∈P_n: ½(I+M)
Π_S = ½ⁿ⁻ᵏ... ∏_{M_i} (I+M_i) = 1/2^{n-k} Σ_{M∈S} M
{M_i} generators of S
M∈P_n/{±1,±i} can be represented as ( ), M_x & M_z n-bit binary vectors
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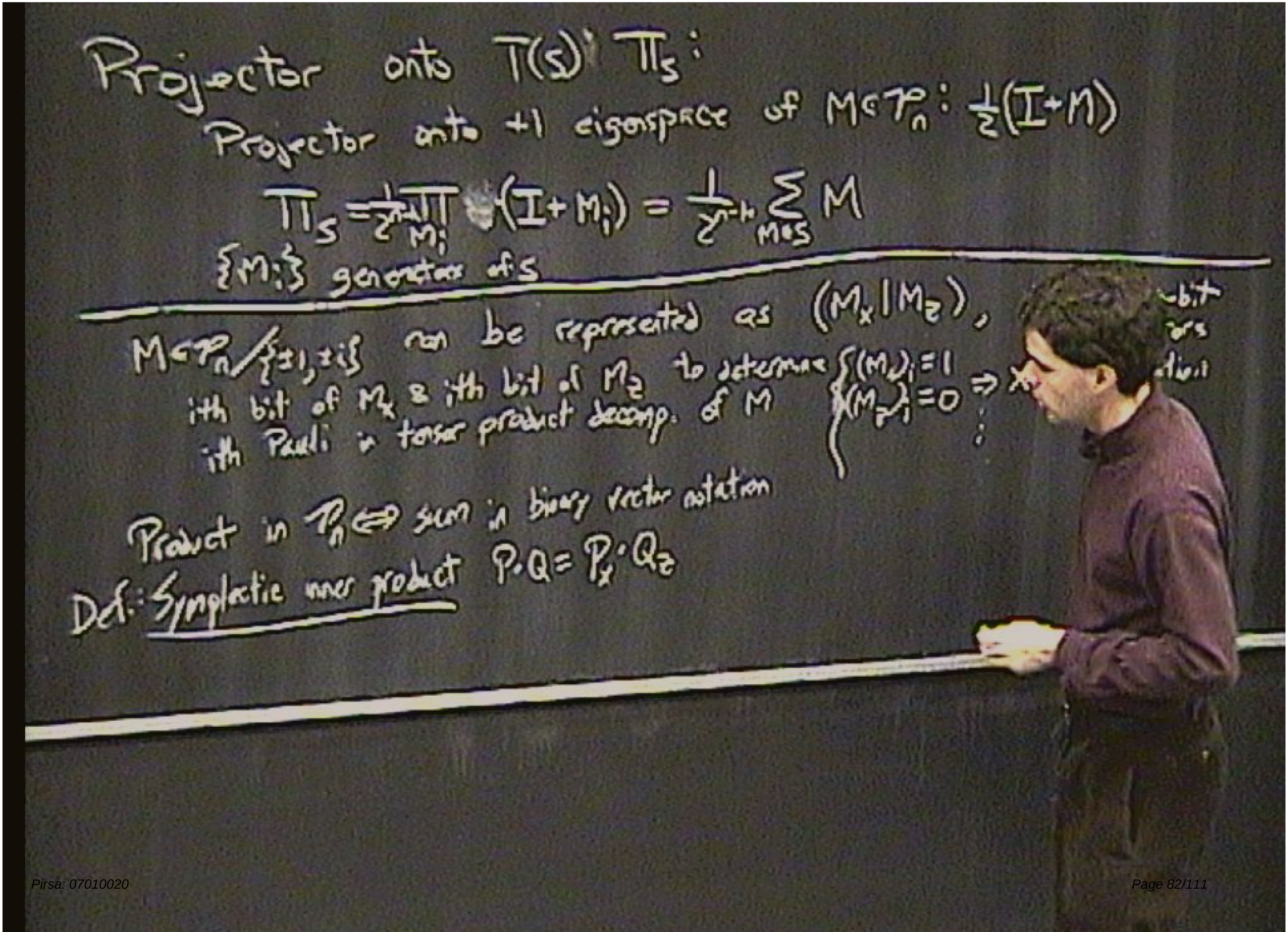
ith Pauli in tensor product decomp. of M
Product in P_n ⟺ sum in binary vector notation
Def.: Symplectic inner product P·Q=


Projector onto $T(S)$: Π_S:
Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$
$\Pi_S = \frac{1}{2^{n-k}} \prod_{M_i} (I+M_i) = \frac{1}{2^{n-k}} \sum_{M \in S} M$
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$(M_x)_i = 1$, $(M_z)_i = 0 \Rightarrow X$
Product in $\mathcal{P}_n \Leftrightarrow$ sum in binary vector notation
Def: Symplectic inner product $P \cdot Q = P_x \cdot Q_z$

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Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$

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$\begin{cases} (M_x)_i = 1 \\ (M_z)_i = 0 \Rightarrow X \text{ in ith position} \\ \vdots \end{cases}$

Product in $\mathcal{P}_n \Leftrightarrow$ sum in binary vector notation

Def.: Symplectic inner product $P \cdot Q = P_x \cdot Q_z + Q_x \cdot P_z$

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$(M_x)_i = 1$, $(M_z)_i = 0$ $\Rightarrow X$ in ith position

⋮

Product in $\mathcal{P}_n \Leftrightarrow$ sum in binary vector notation

$(P_x | P_z)$

$(Q_x | Q_z)$

Def.: Symplectic inner product $P \cdot Q = P_x \cdot Q_z + Q_x \cdot P_z$

Projector onto $T(S)$: Π_S:

Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$

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$\{M_i\}$ generators of S

$M \in \mathcal{P}_n/\{\pm 1, \pm i\}$ can be represented as $(M_x|M_z)$, M_x & M_z n-bit binary vectors

ith bit of M_x & ith bit of M_z to determine ith Pauli in tensor product decomp. of M

$$\begin{cases} (M_x)_i = 1 \\ (M_z)_i = 0 \\ \vdots \end{cases} \Rightarrow X \text{ in ith position}$$

Product in $\mathcal{P}_n \Leftrightarrow$ sum in binary vector notation

$(P_x|P_z)$

$(Q_x|Q_z)$

Def: Symplectic inner product $P\cdot Q = P_x\cdot Q_z + Q_x\cdot P_z$ (mod 2 arithmetic)

Projector onto $T(S)$: Π_S:

Projector onto +1 eigenspace of $M \in \mathcal{P}_n$: $\frac{1}{2}(I+M)$

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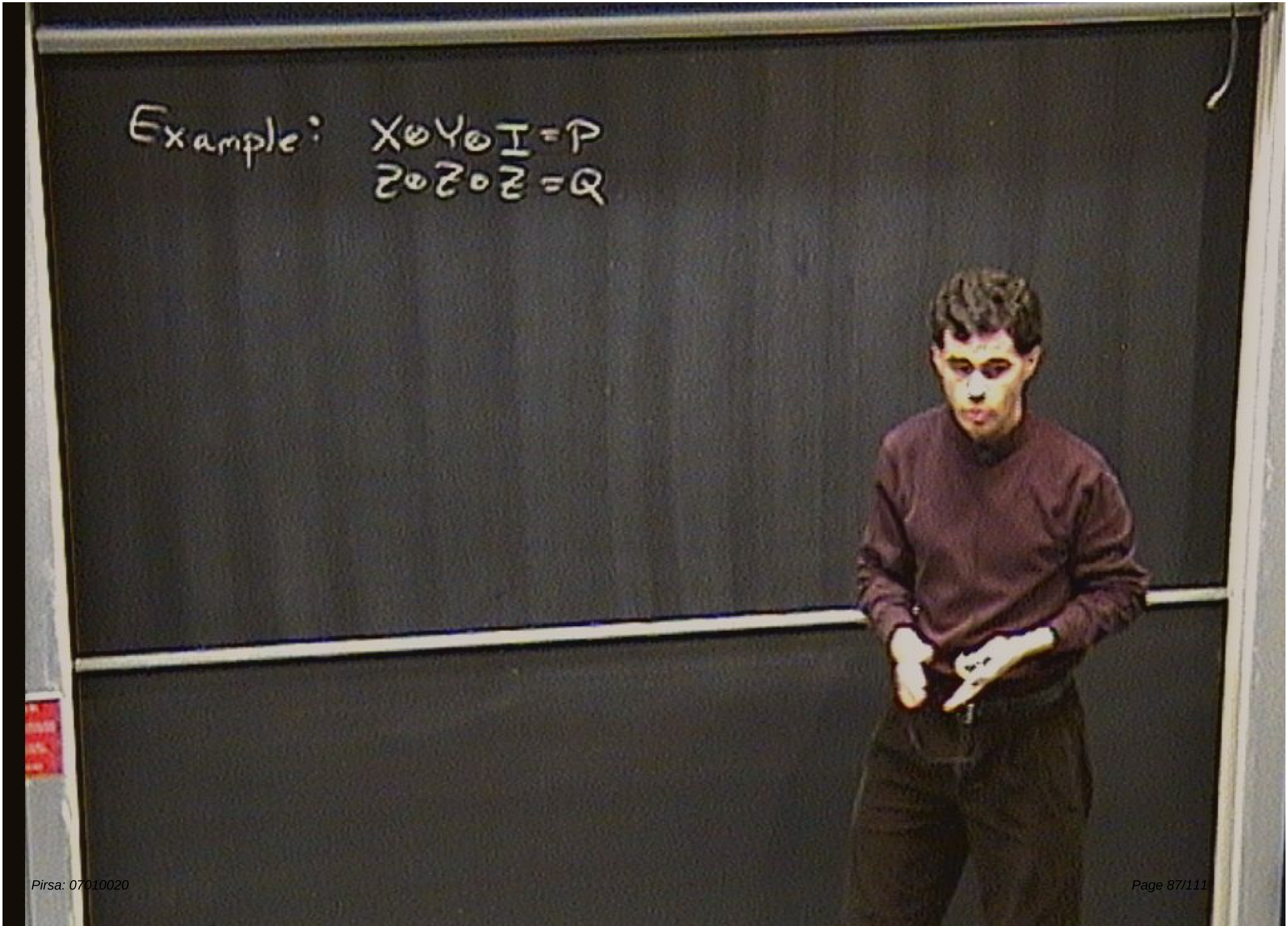
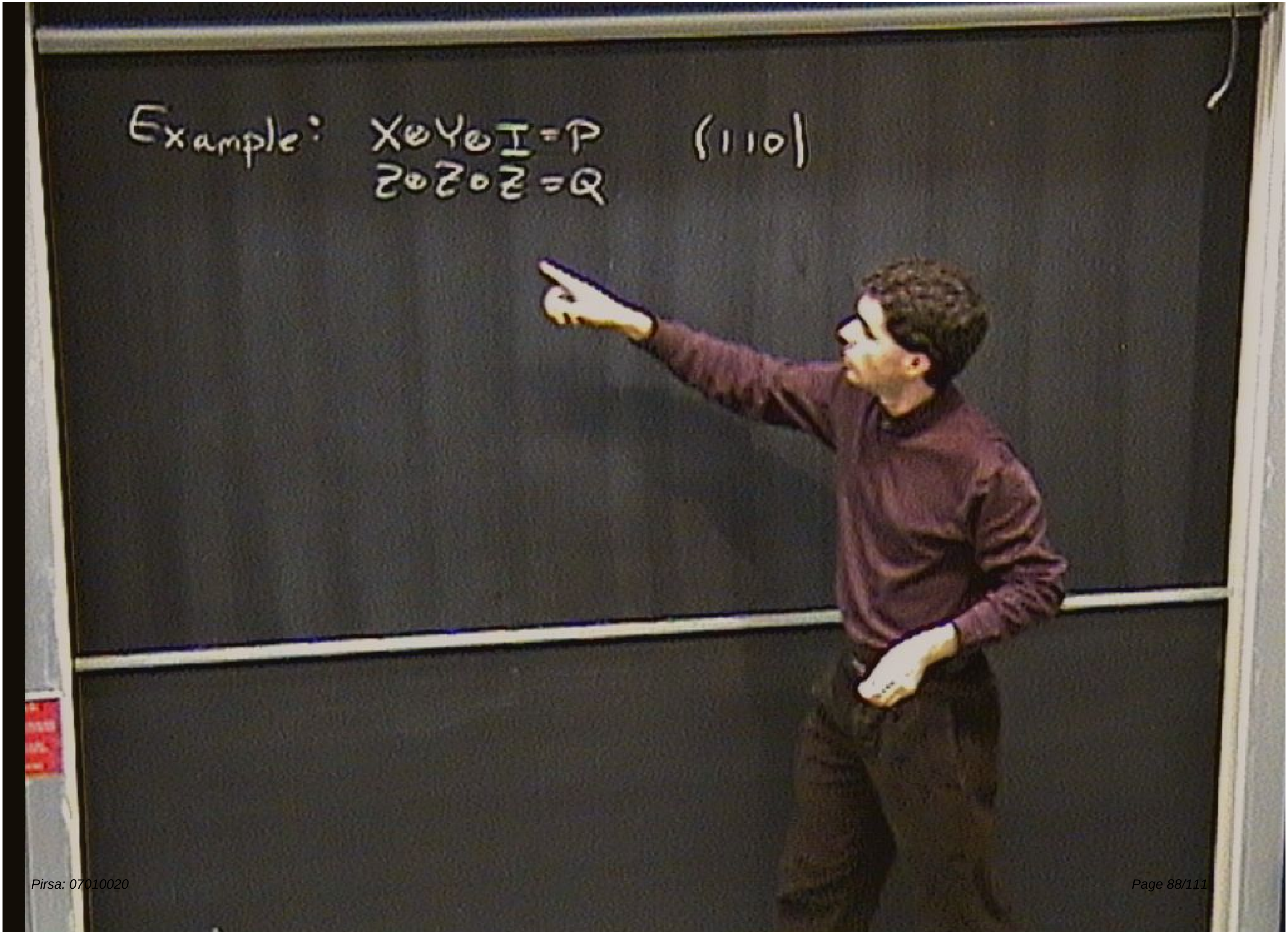

Example: $X \otimes Y \otimes I = P$
$Z \otimes Z \otimes Z = Q$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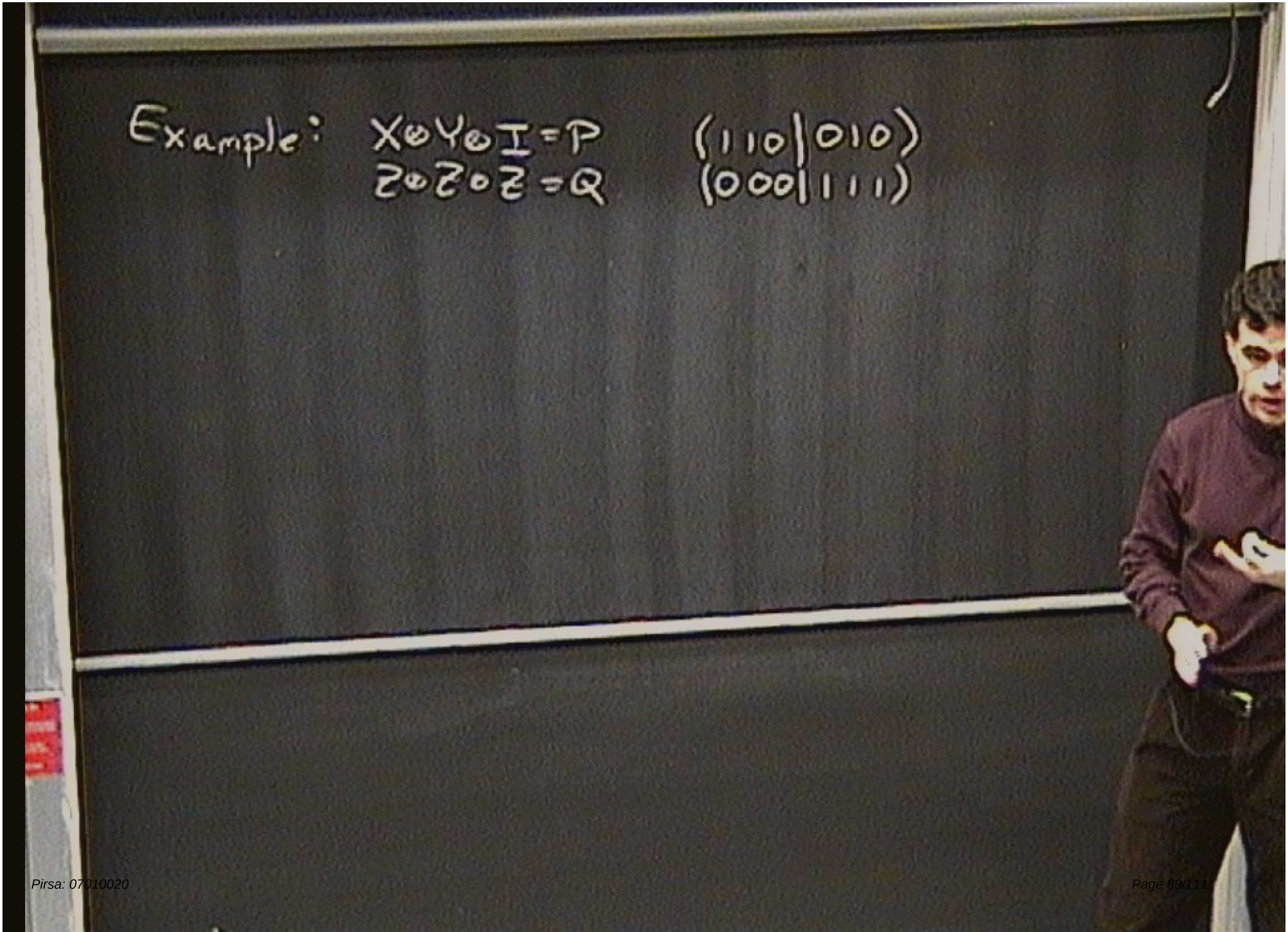
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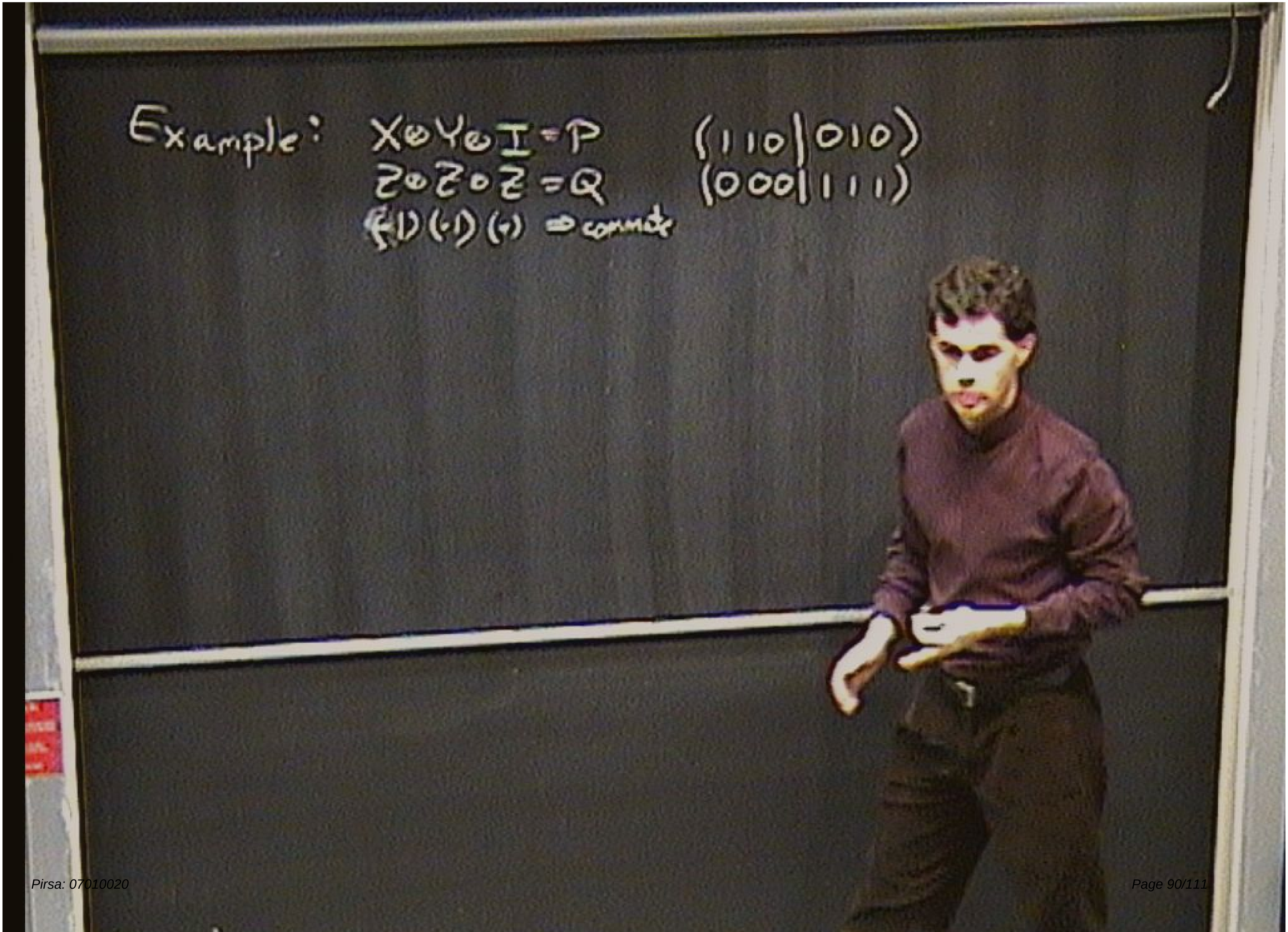
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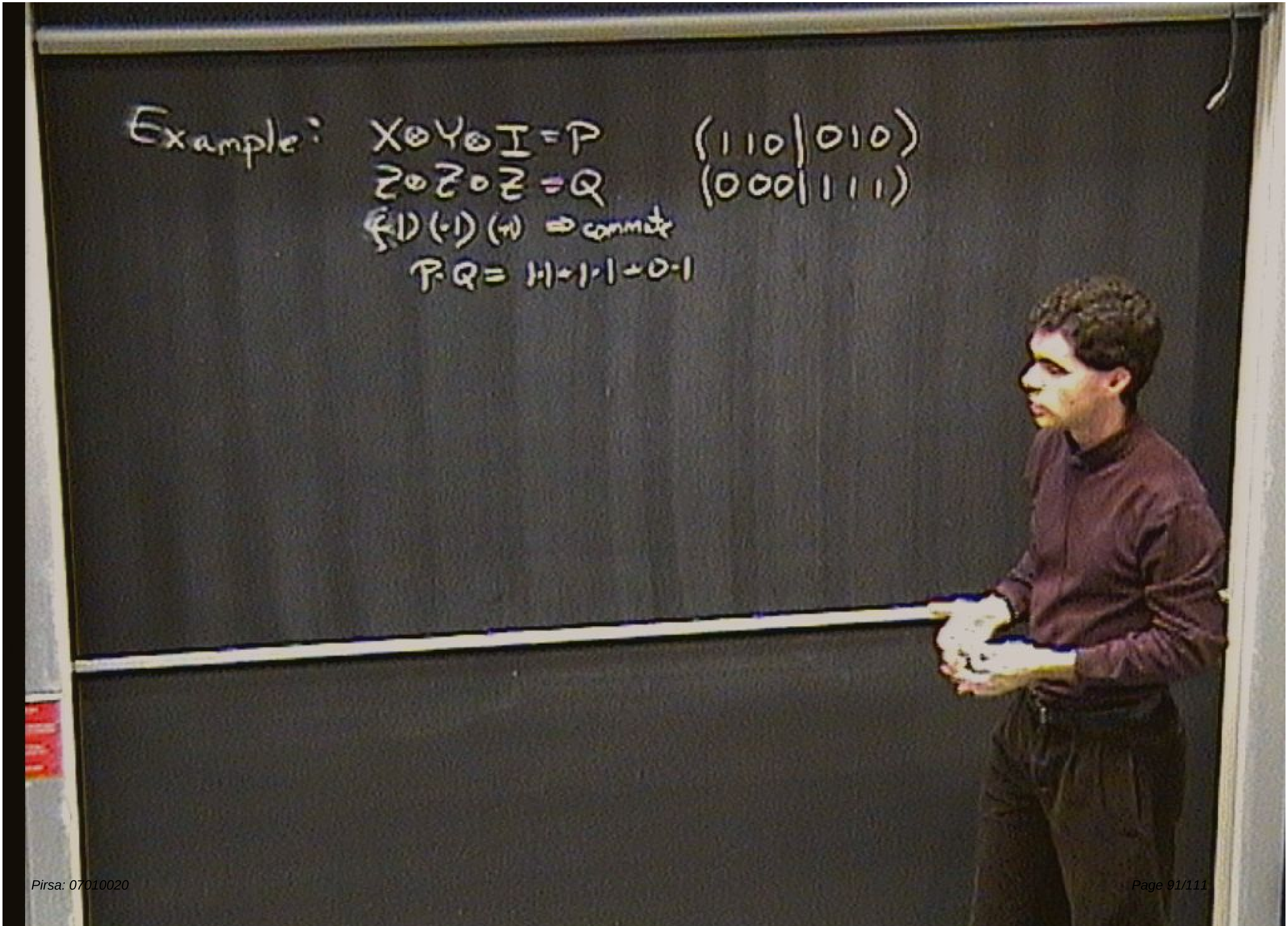
(-1)(-1)(+) ⇒ commute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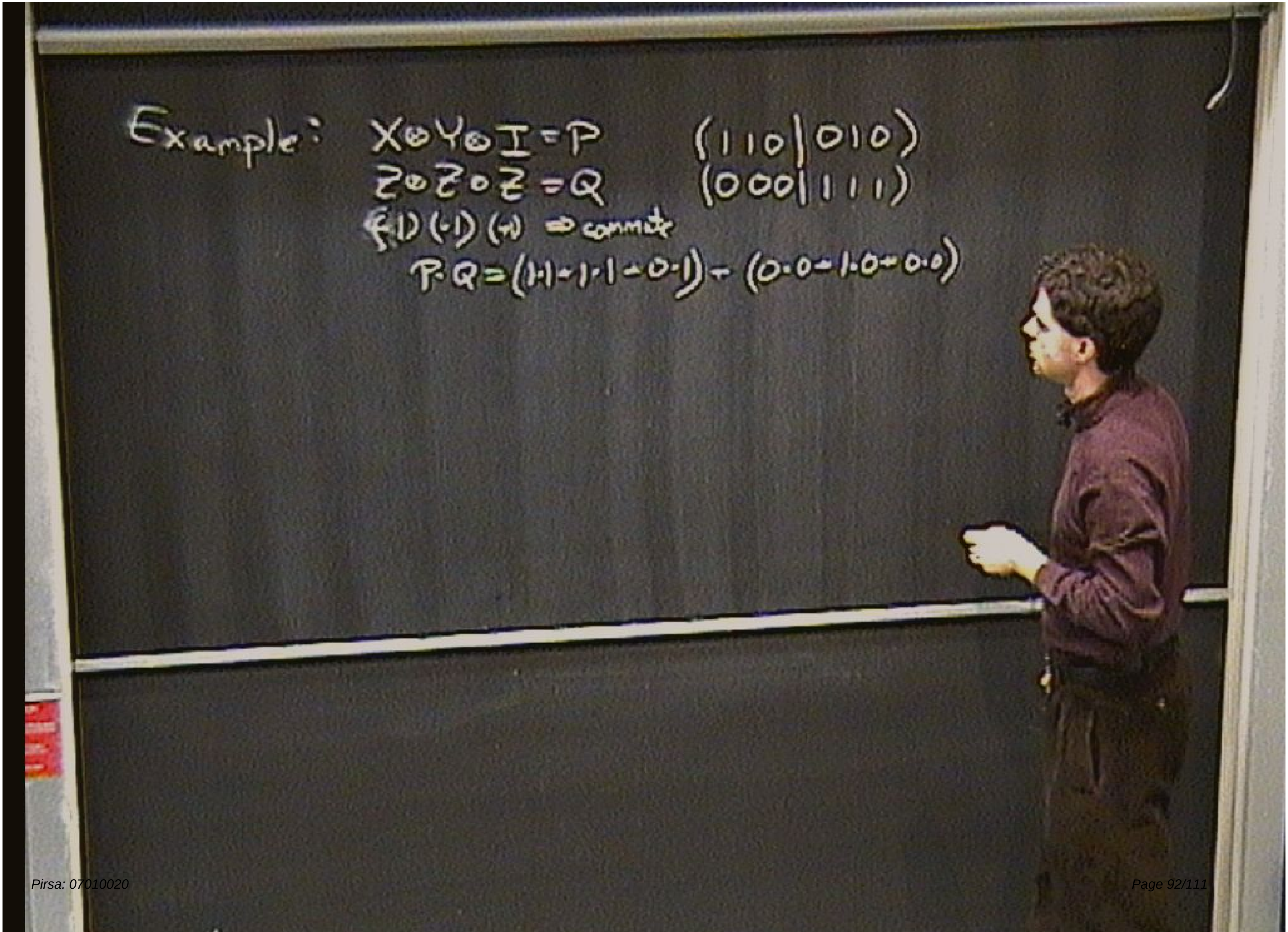
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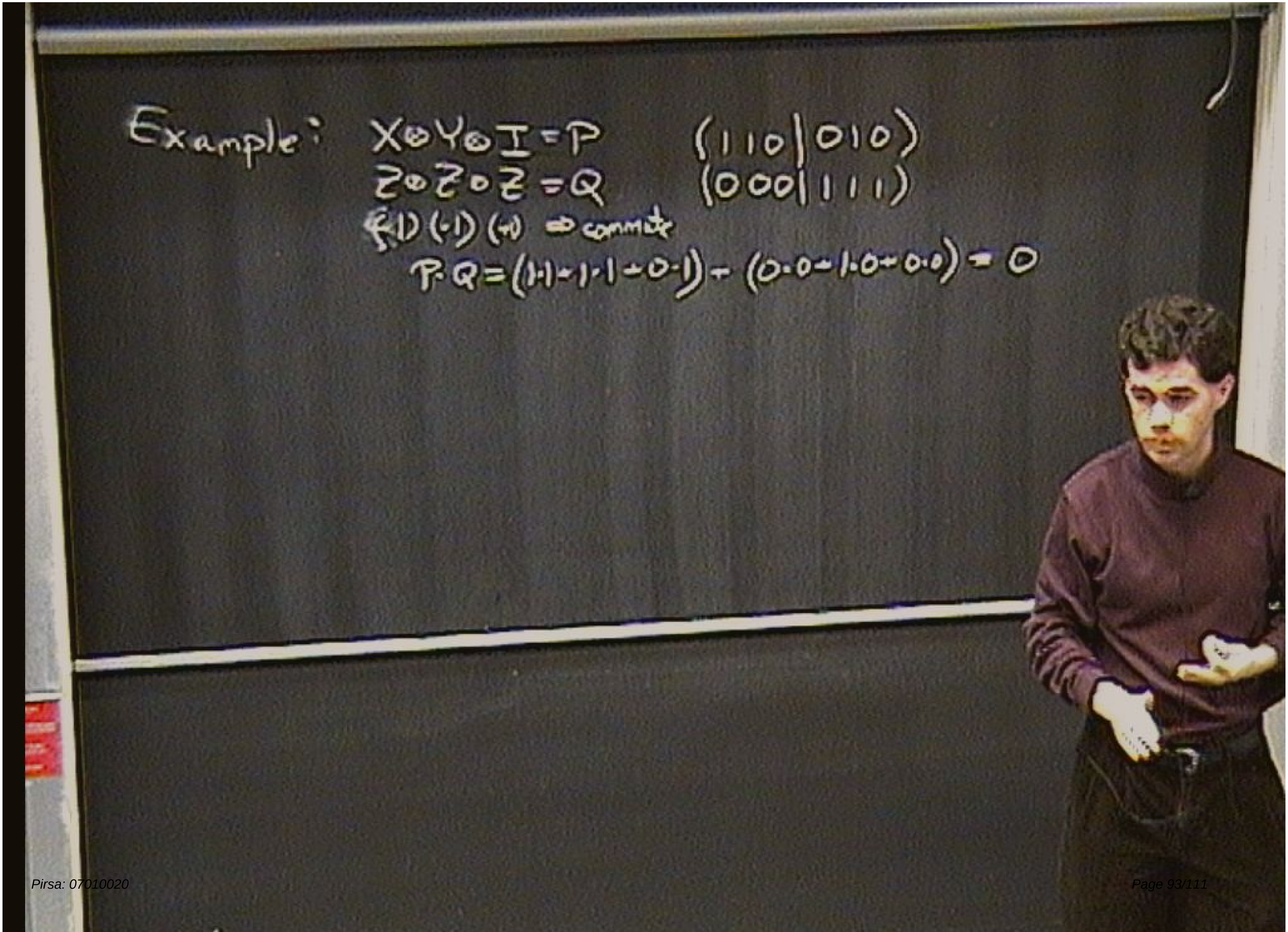
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Proof of b:

<u>Lemma</u>:

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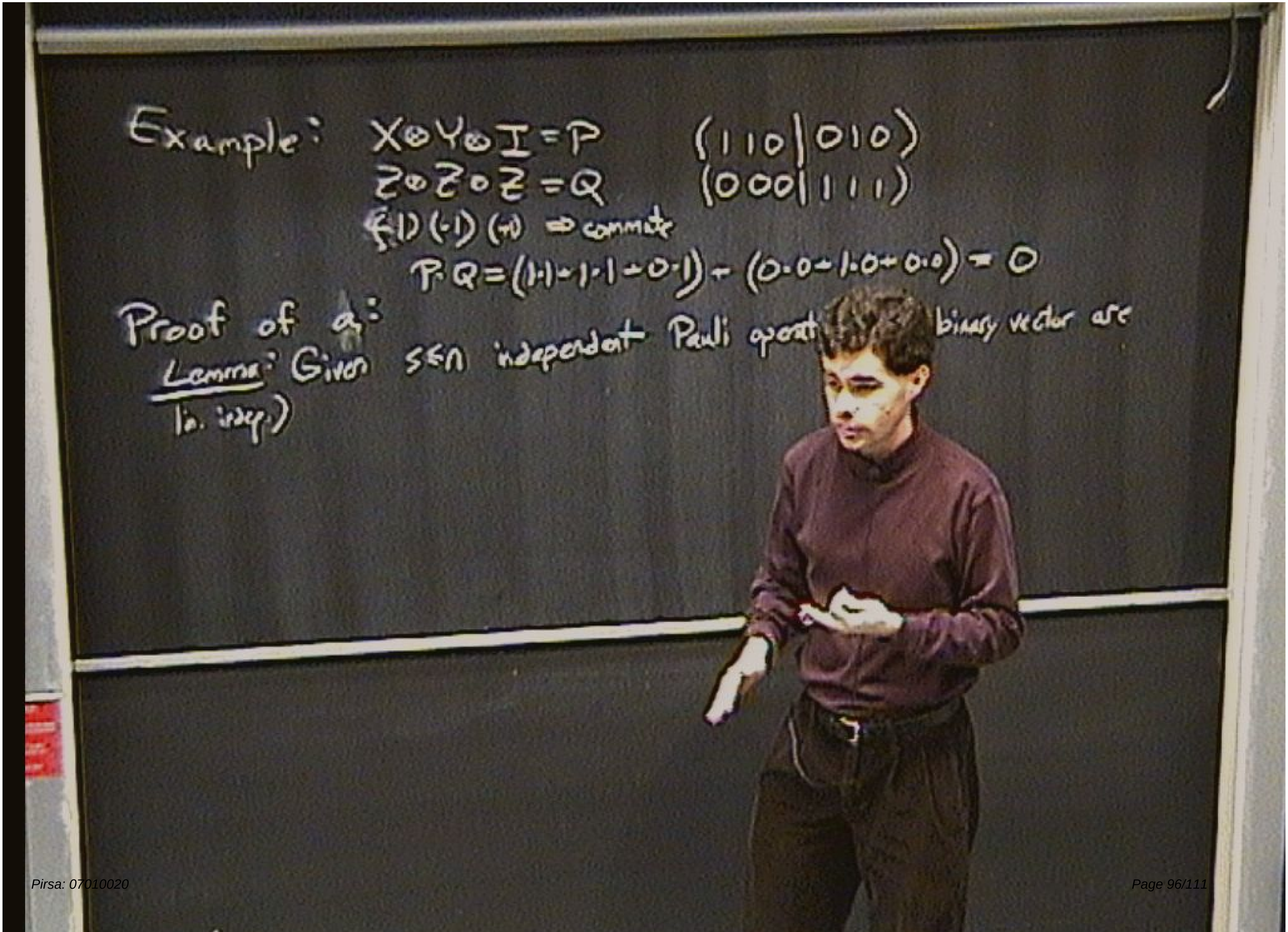
Lemma: Given $s \leq n$ independent Pauli operators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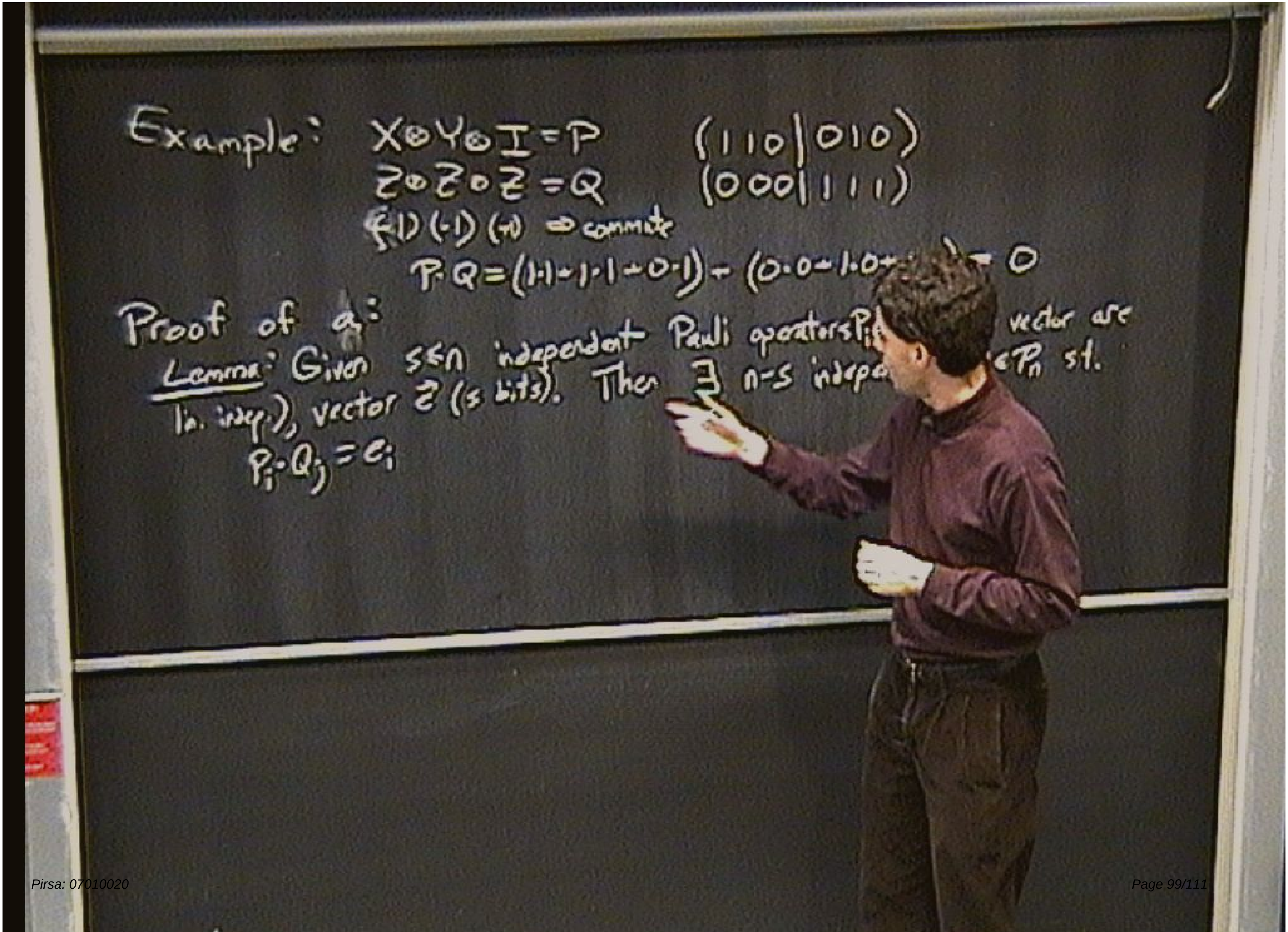
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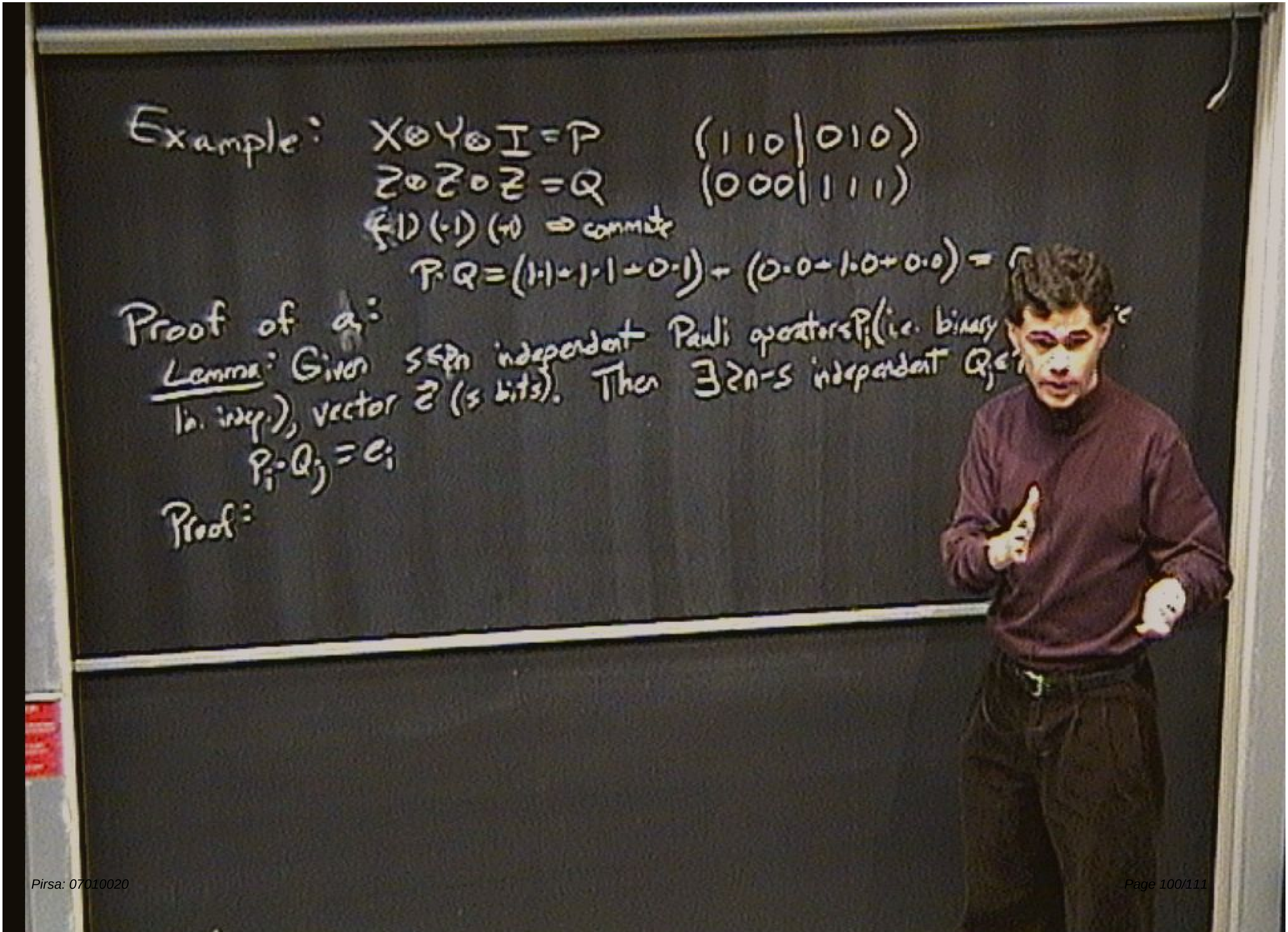
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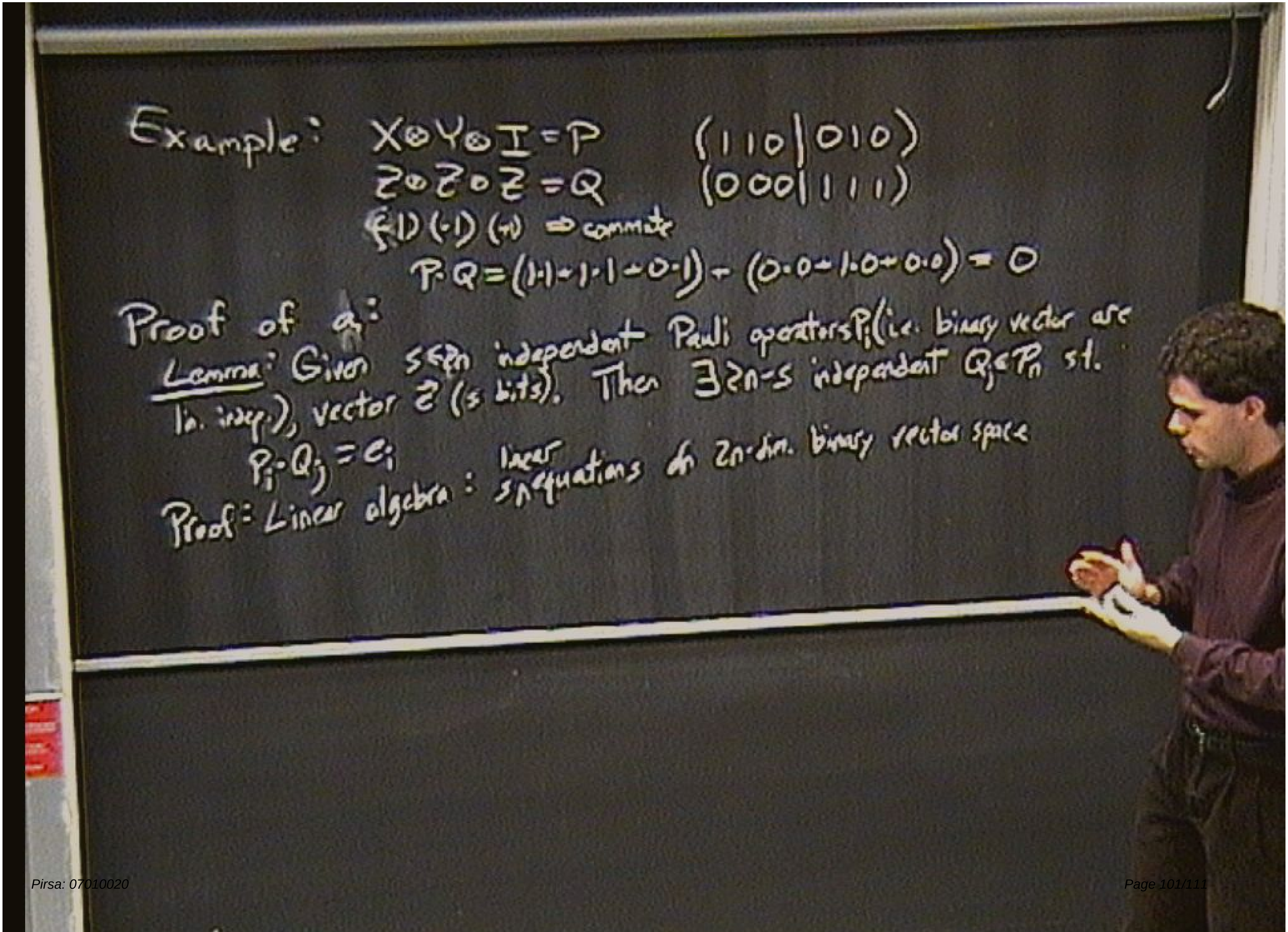
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Proof: Linear algebra: s linear equations on $2n$-dim. binary vector space

r' generators for S are $P_1, \ldots, P_r$

$$a_i = \begin{cases} 0 & [P, M_i] = 0 \\ 1 & \{P, M_i\} = 0 \end{cases}$$

($M_1, \ldots, M_{n-k}$ are generators of S)

$$\text{Syndrome}(PQ) = \text{Syndrome}(P) + \text{Syndrome}(Q) \quad (\text{binary})$$

$$\text{Syndrome } P = \text{Syndrome } Q \Leftrightarrow \text{Syndrome}(PQ) = 0 \Leftrightarrow PQ \in N(S)$$

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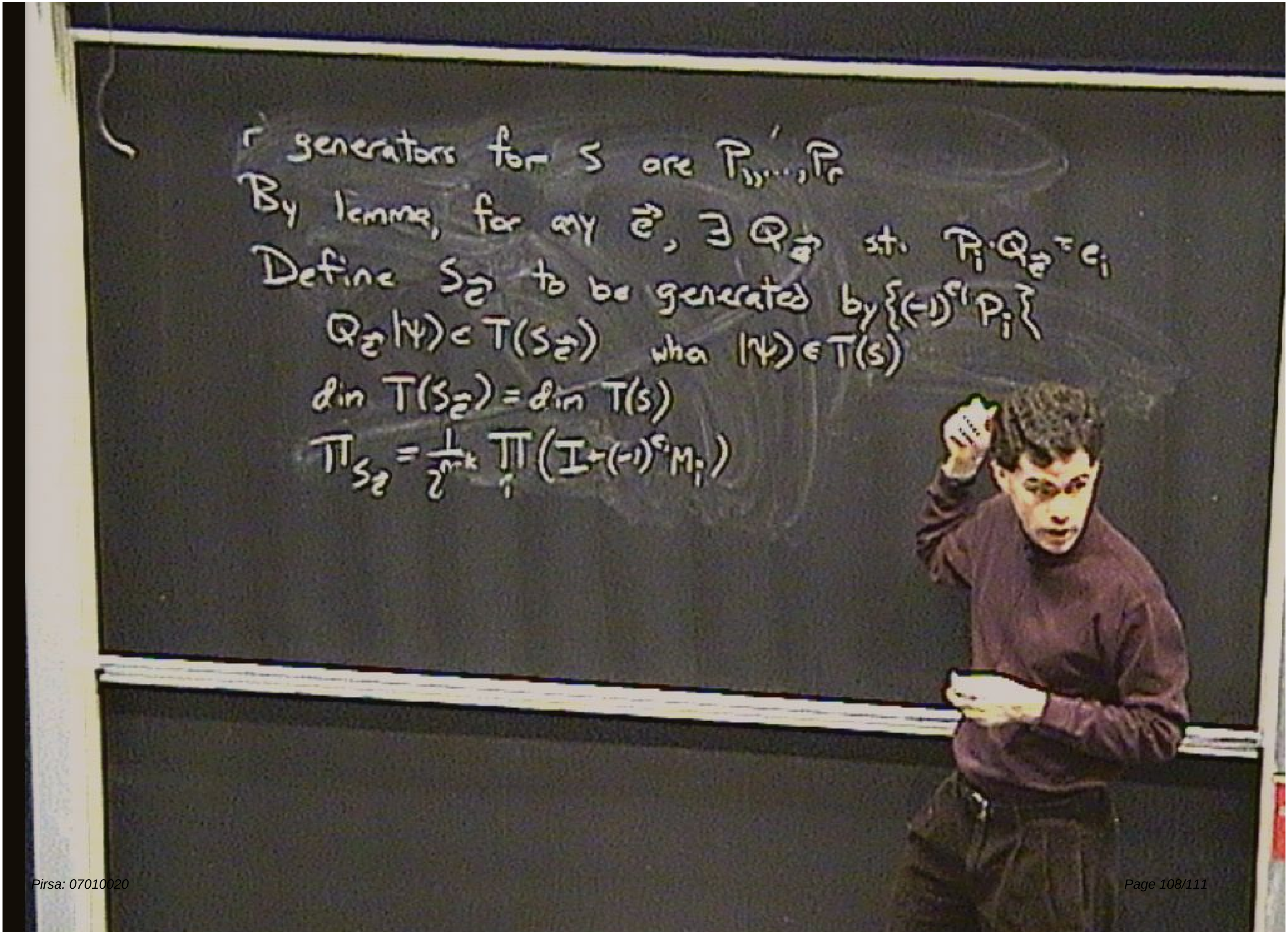
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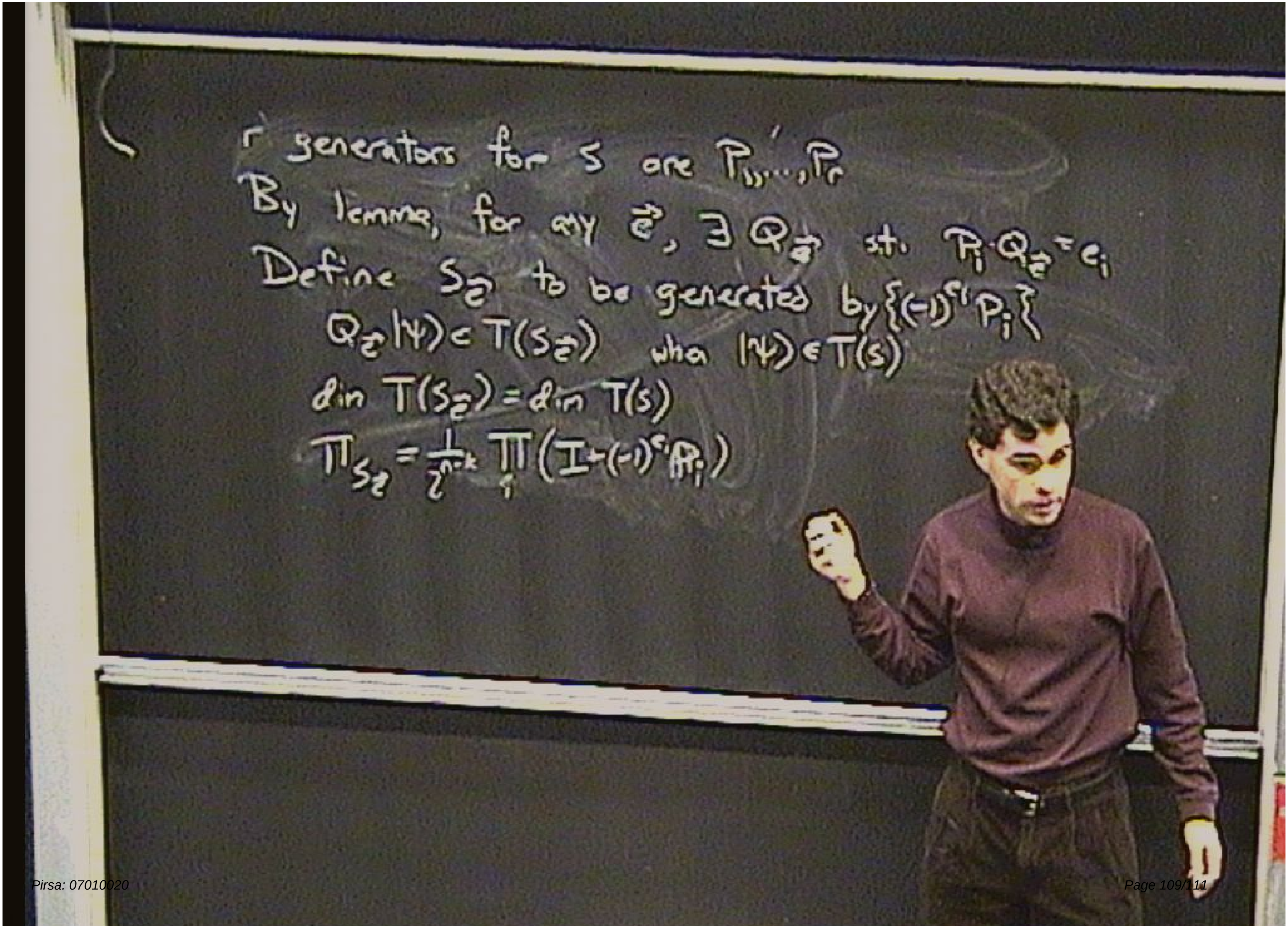
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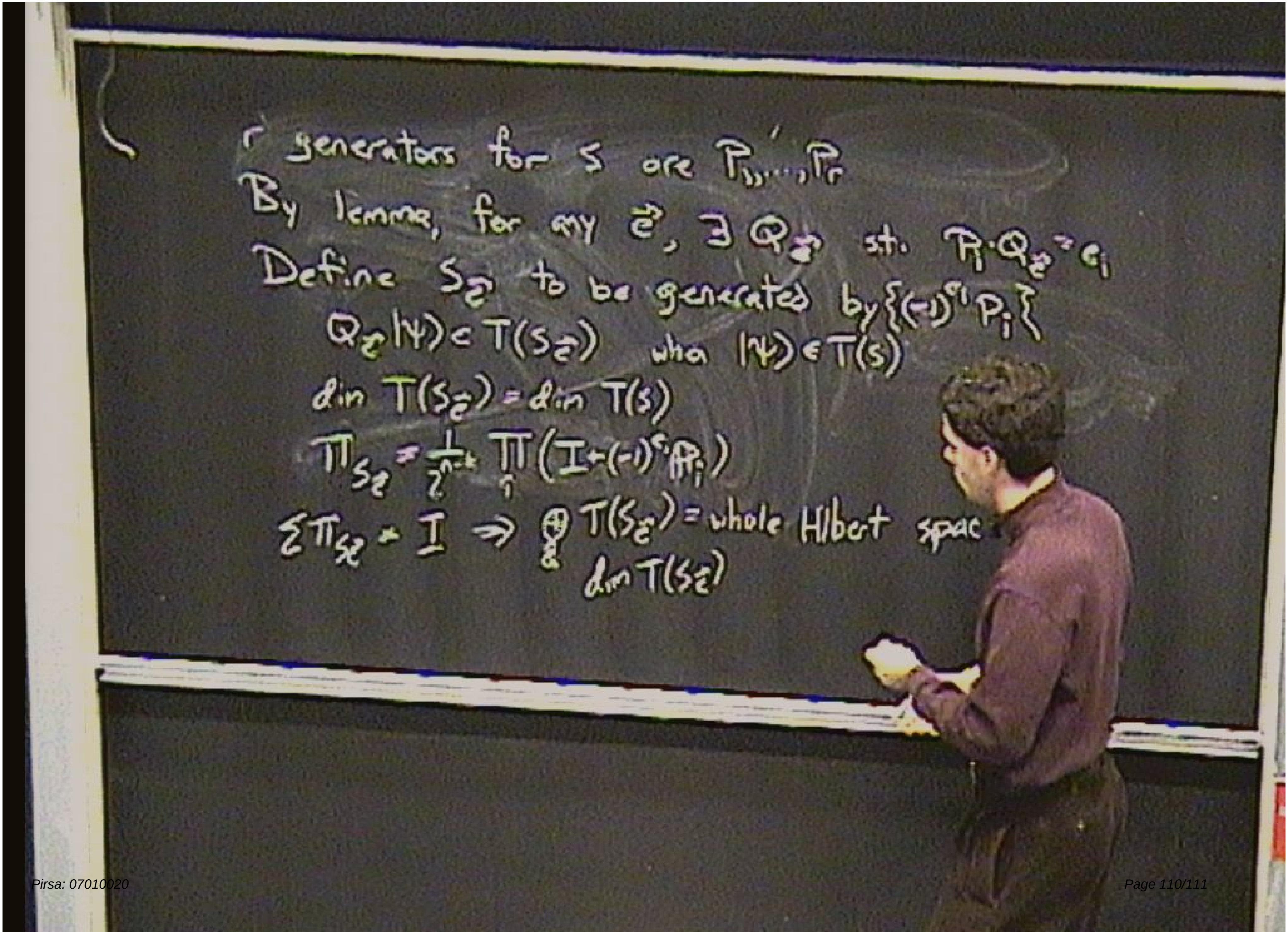
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$\dim T(S_{\vec{e}}) = 2^n / 2^r = 2^{n-r}$