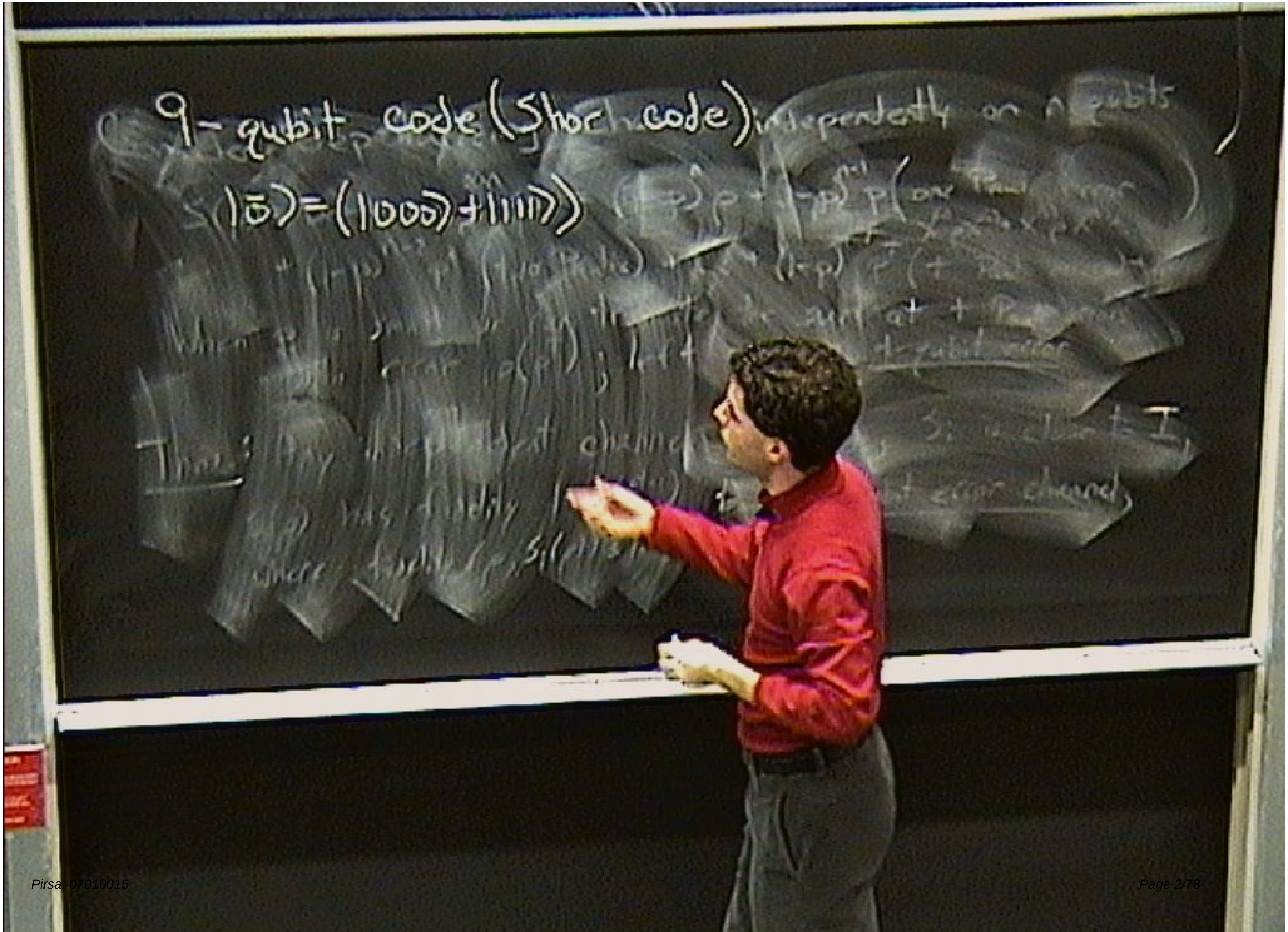
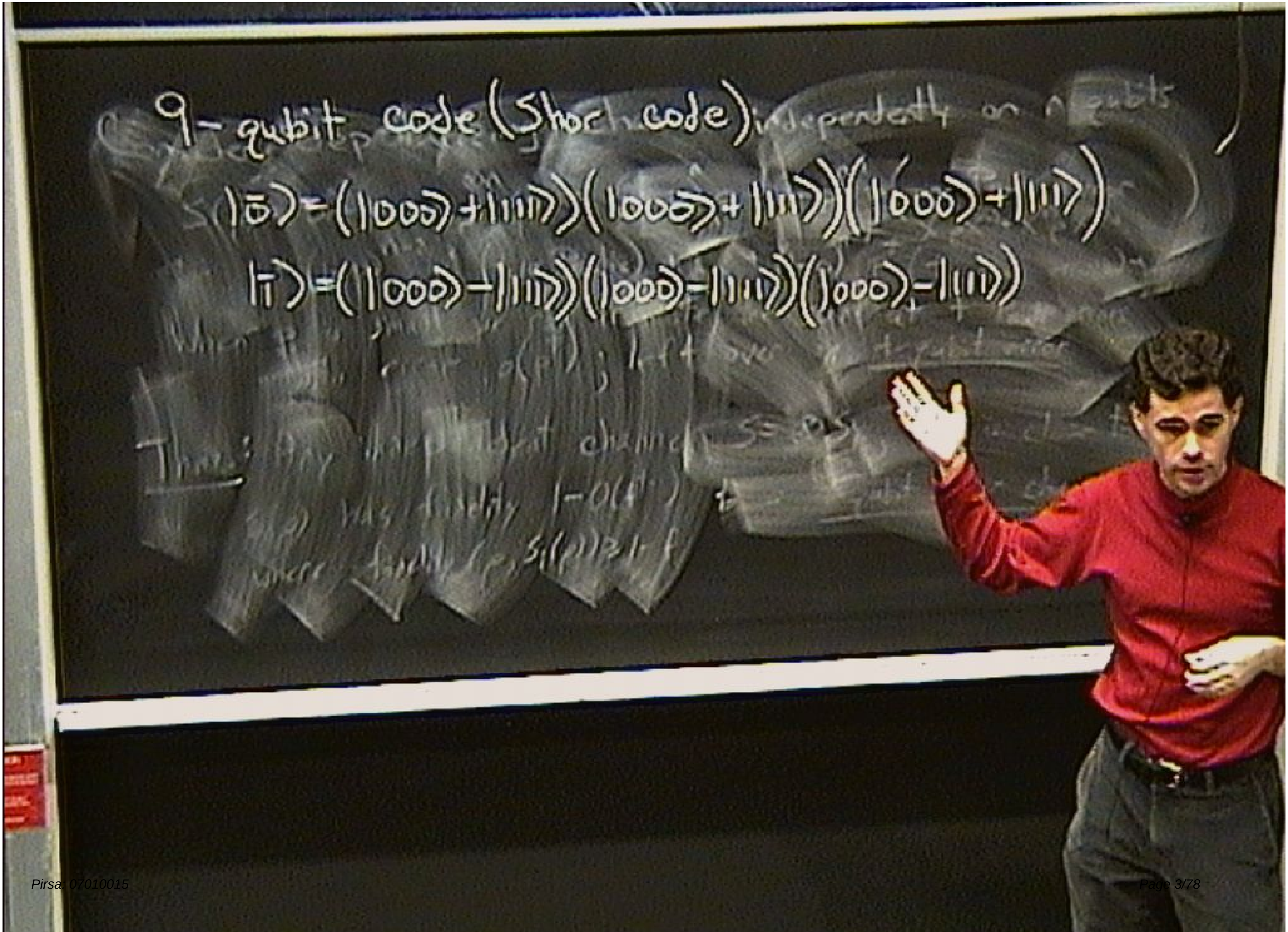
Title: Quantum Error Correction 1B

Date: Jan 09, 2007 05:00 PM

URL: http://pirsa.org/07010015

Abstract: 9-qubit Shor code, definition of a quantum error-correcting code, correcting linear combinations of errors, quantum error correction conditions, definition of distance


9- qubit code (Shor code)
S(|0̄⟩)=(|000⟩+|111⟩)

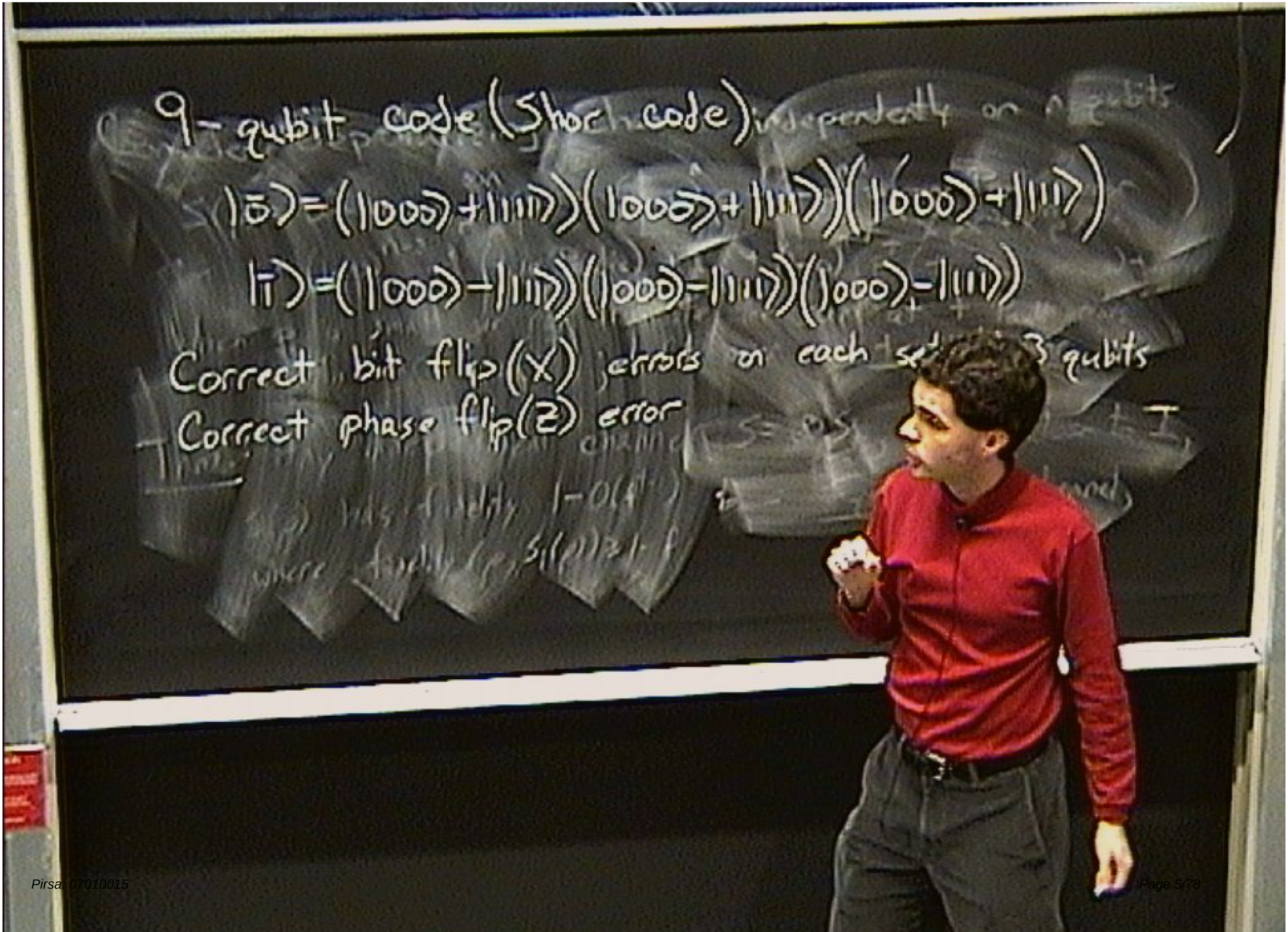

9- qubit code (Shor code)
|0̄⟩ = (|000⟩ + |111⟩)(|000⟩ + |111⟩)(|000⟩ + |111⟩)
|1̄⟩ = (|000⟩ − |111⟩)(|000⟩ − |111⟩)(|000⟩ − |111⟩)

9-qubit code (Shor code)

$$|\bar{0}\rangle = (|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|\bar{1}\rangle = (|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

Correct bit flip (X) errors on each set of 3 qubits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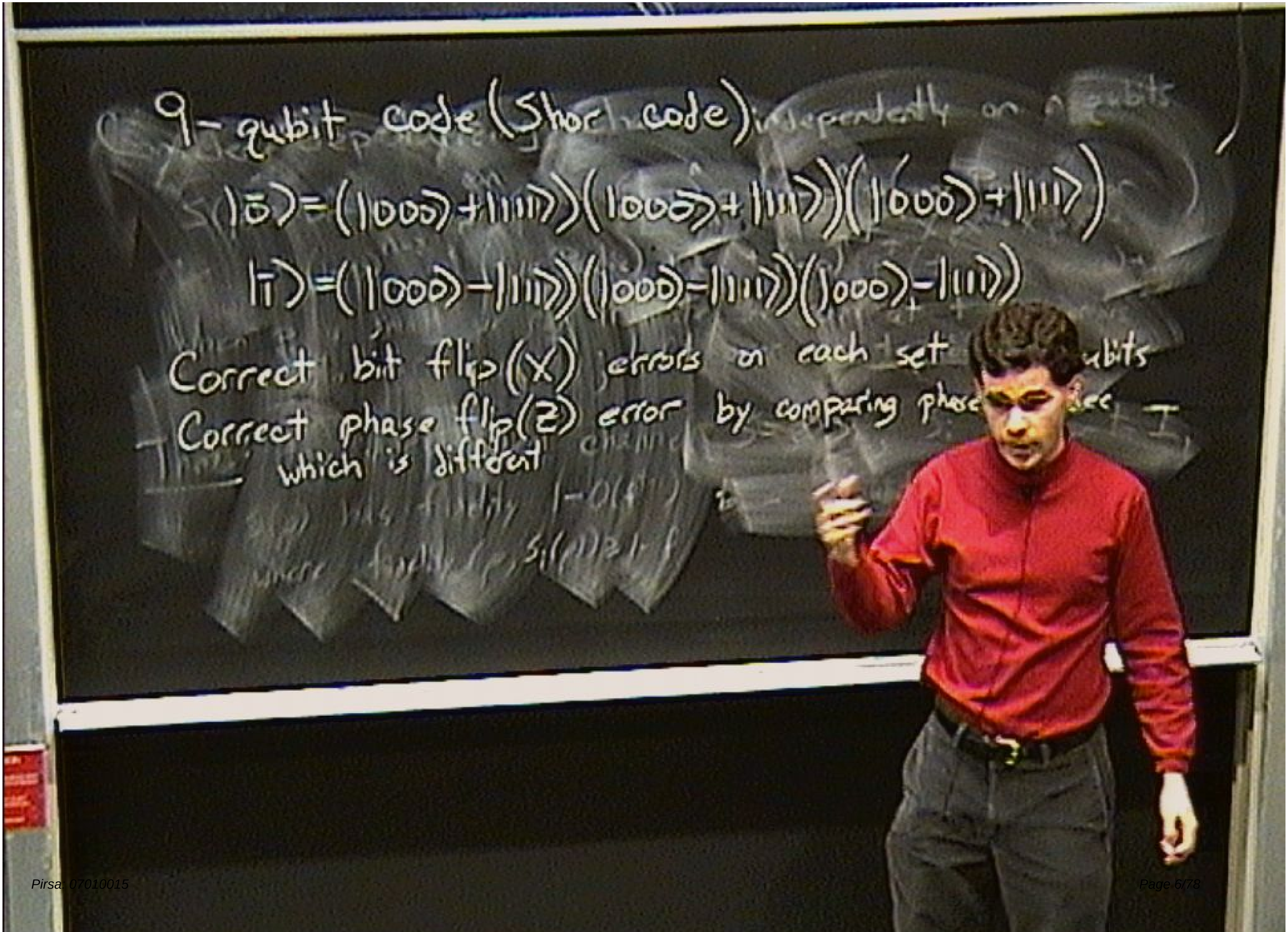

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Correct phase flip (Z) error by comparing
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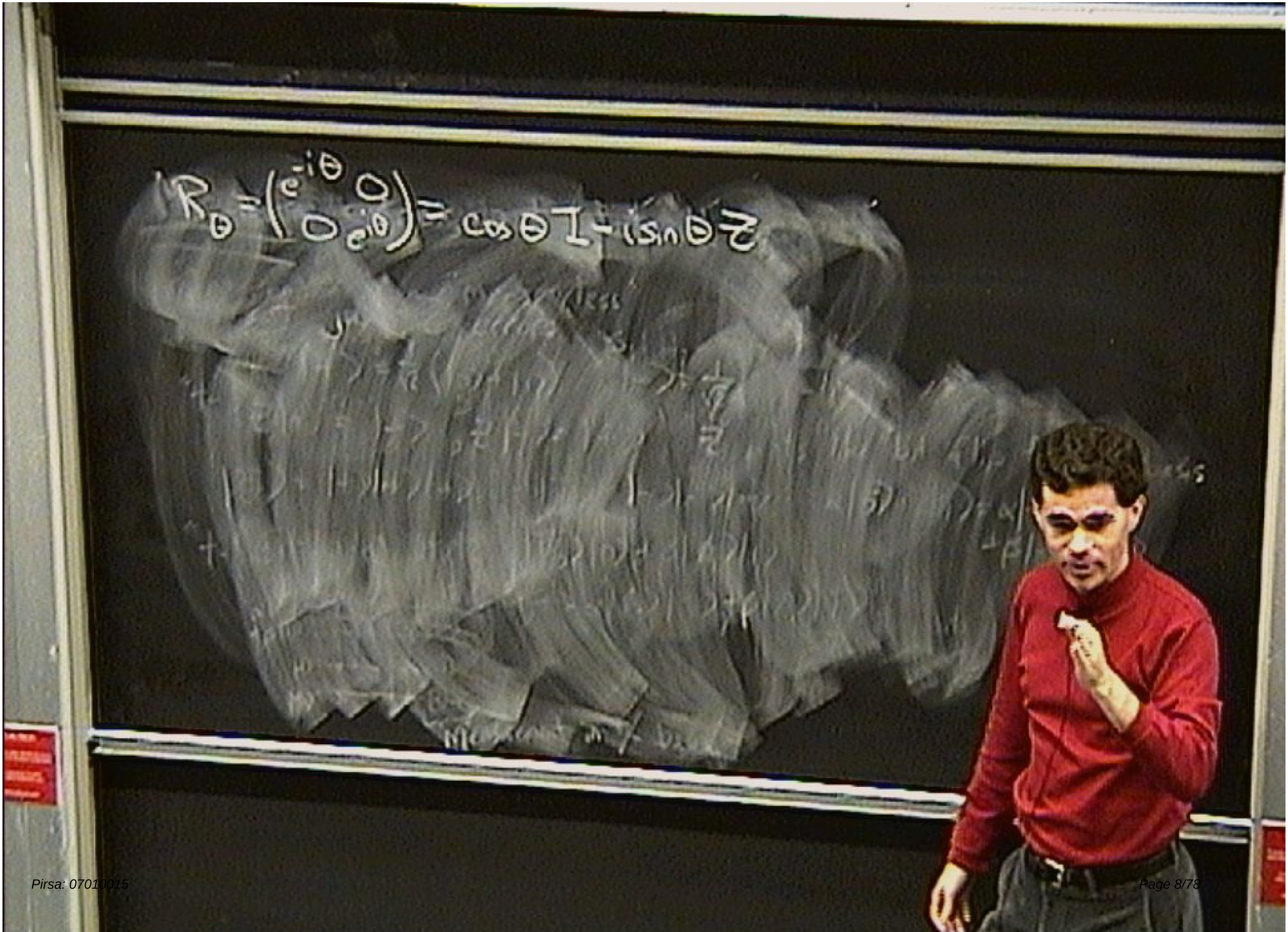
Correct phase flip (Z) error by comparing phases to see which is different

X & Z correction are independent ⇒ can correct 1 of each, such as Y



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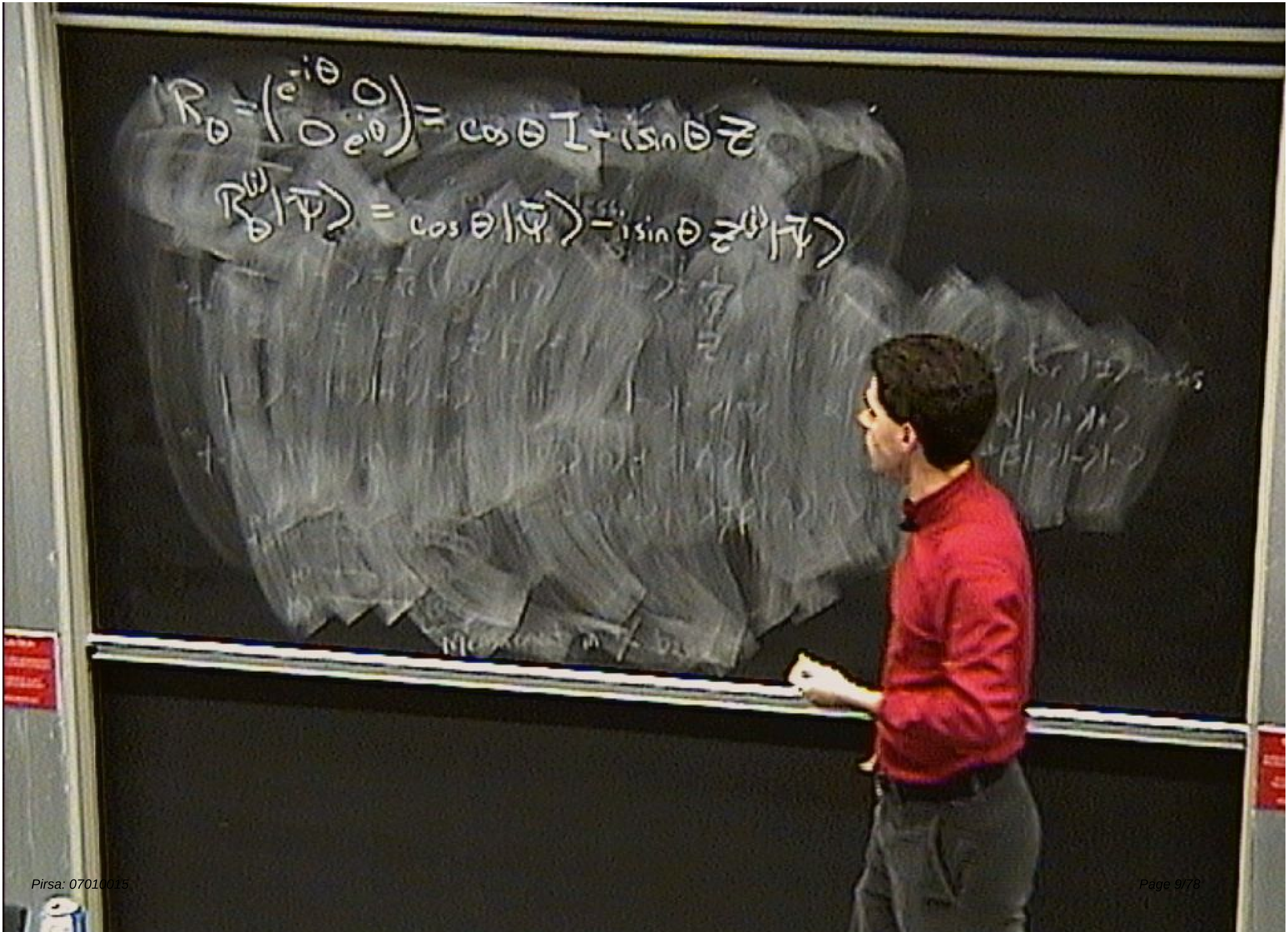
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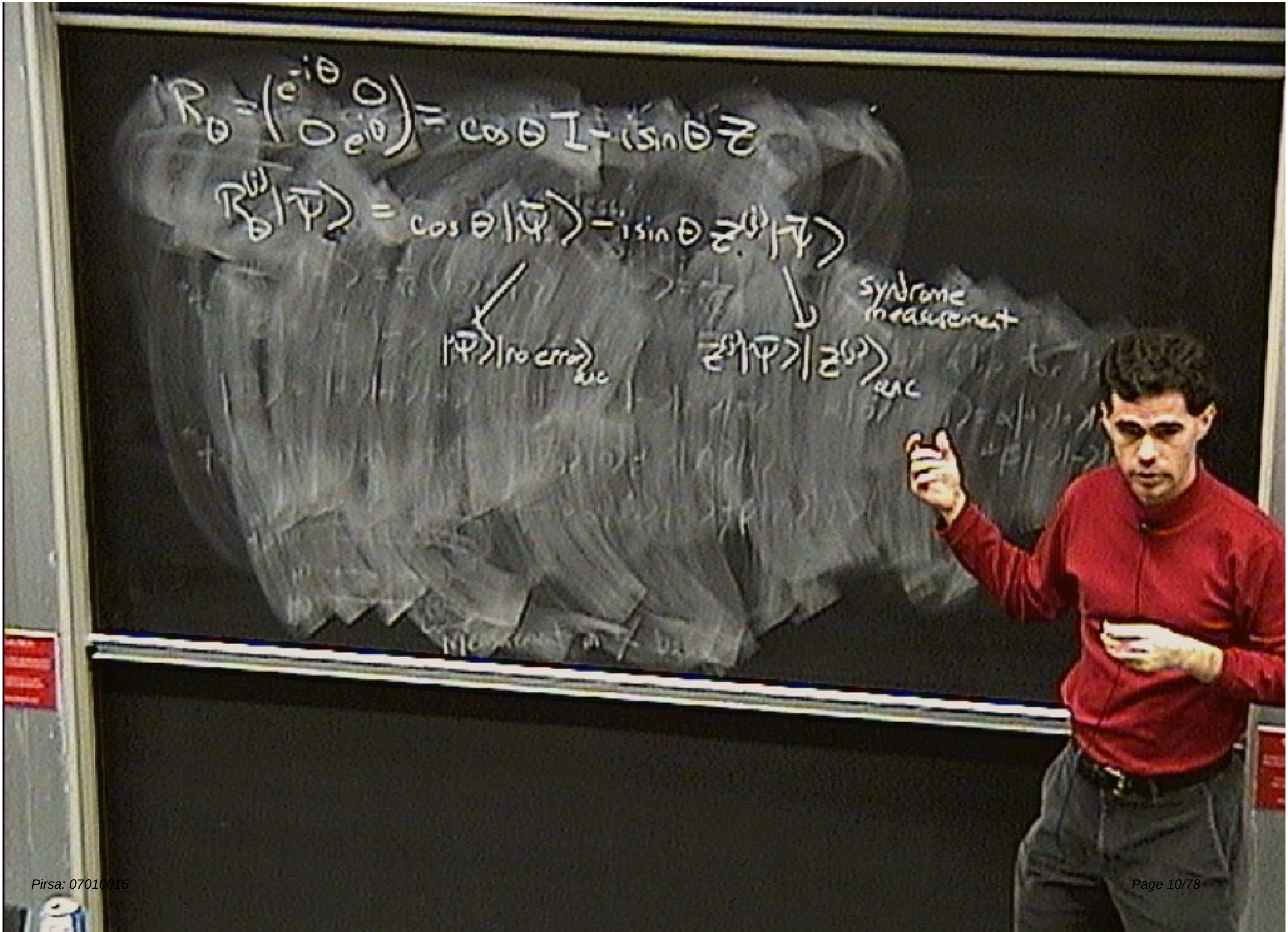
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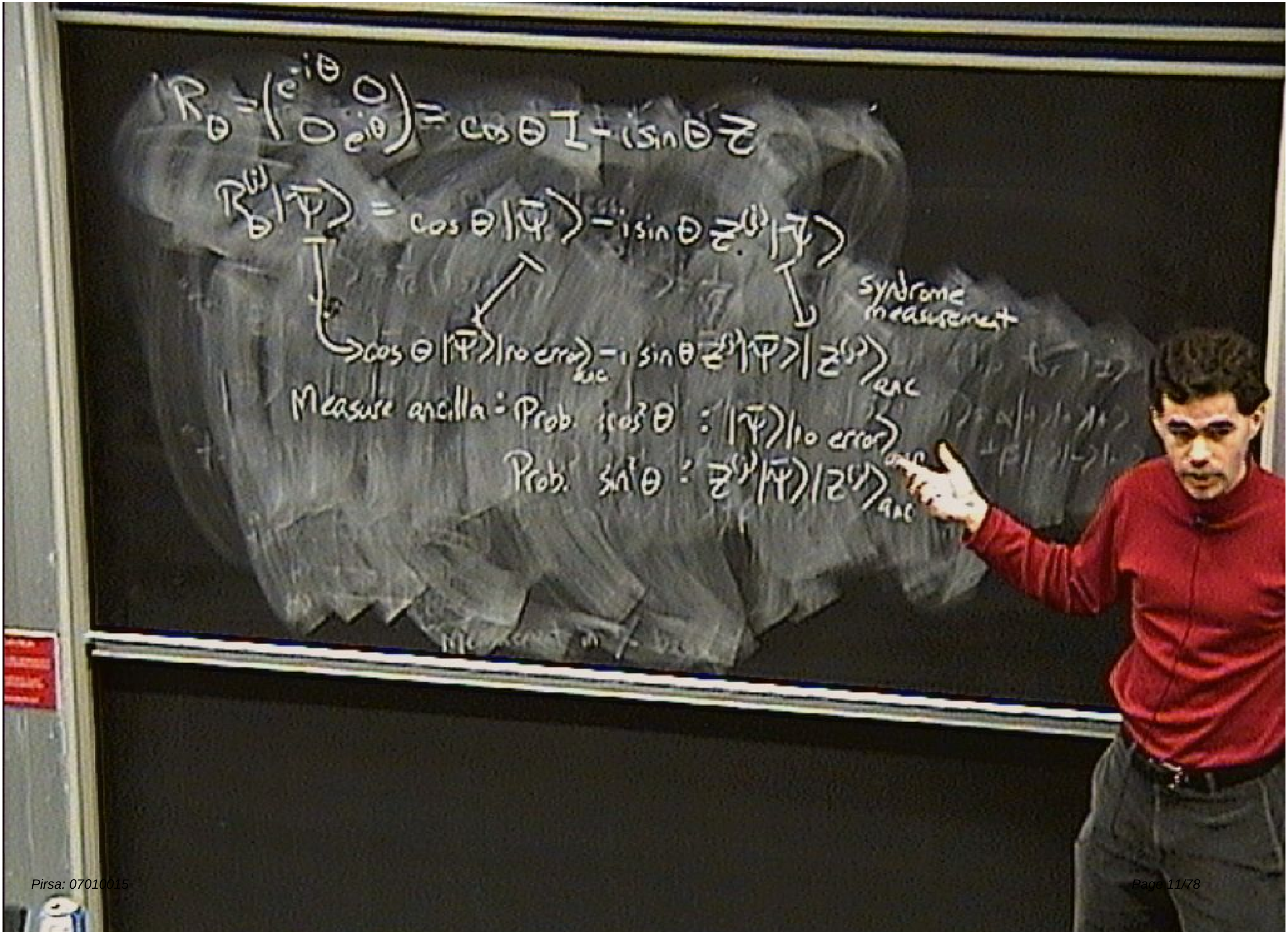
syndrome measurement
$|\bar{\Psi}\rangle|\text{no error}\rangle_{anc}$
$Z^{(j)}|\bar{\Psi}\rangle|Z^{(j)}\rangle_{anc}$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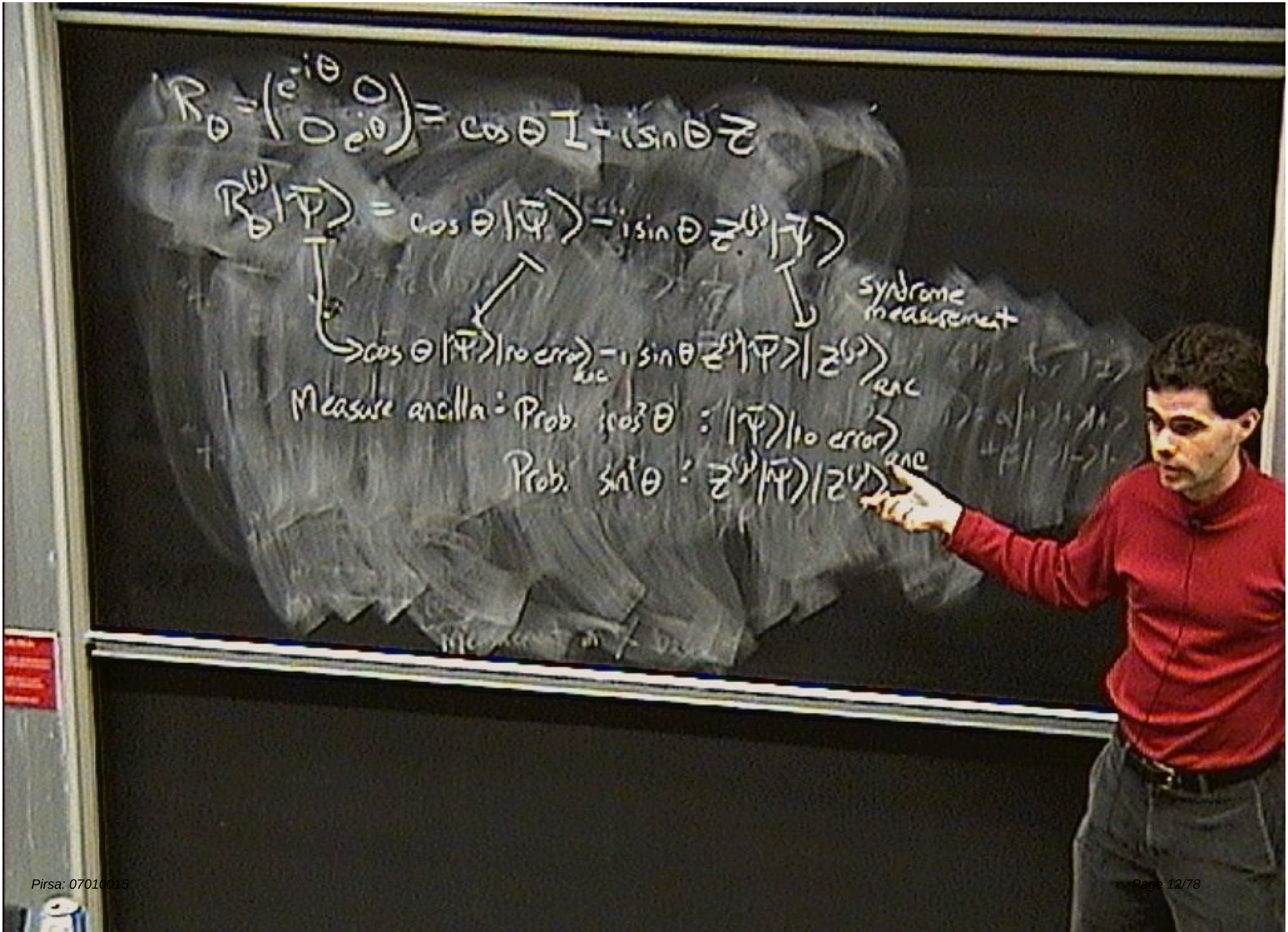

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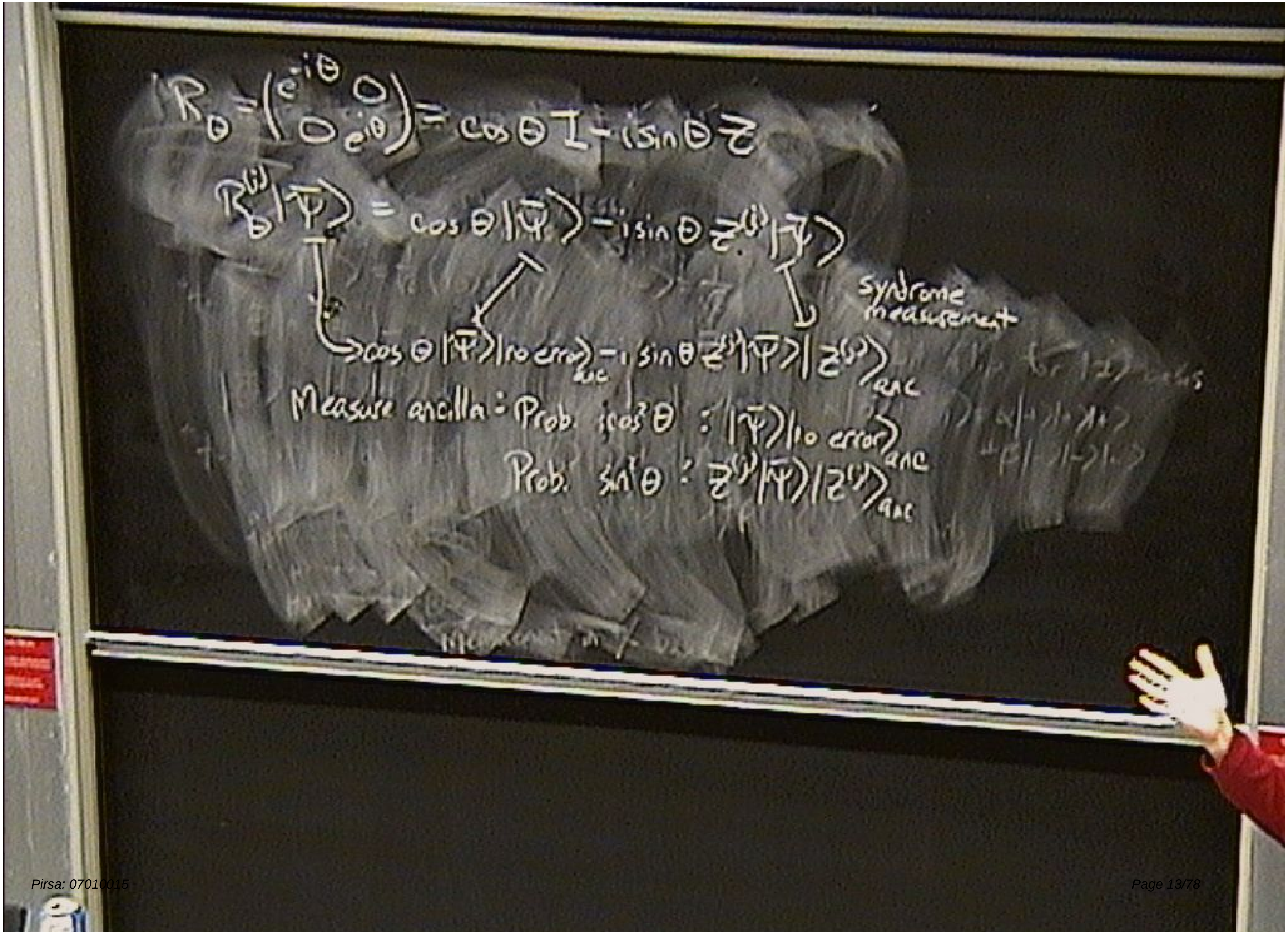
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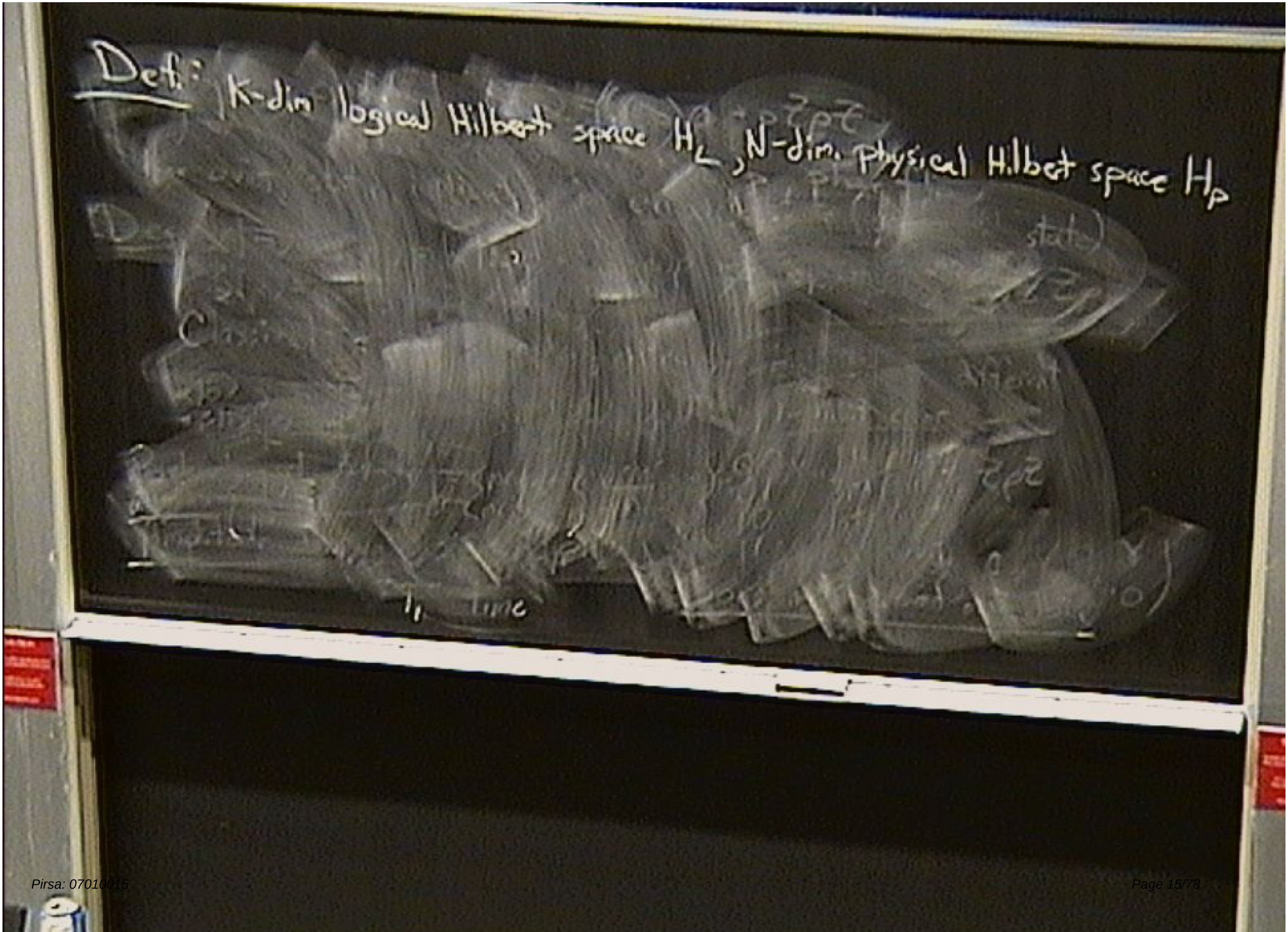
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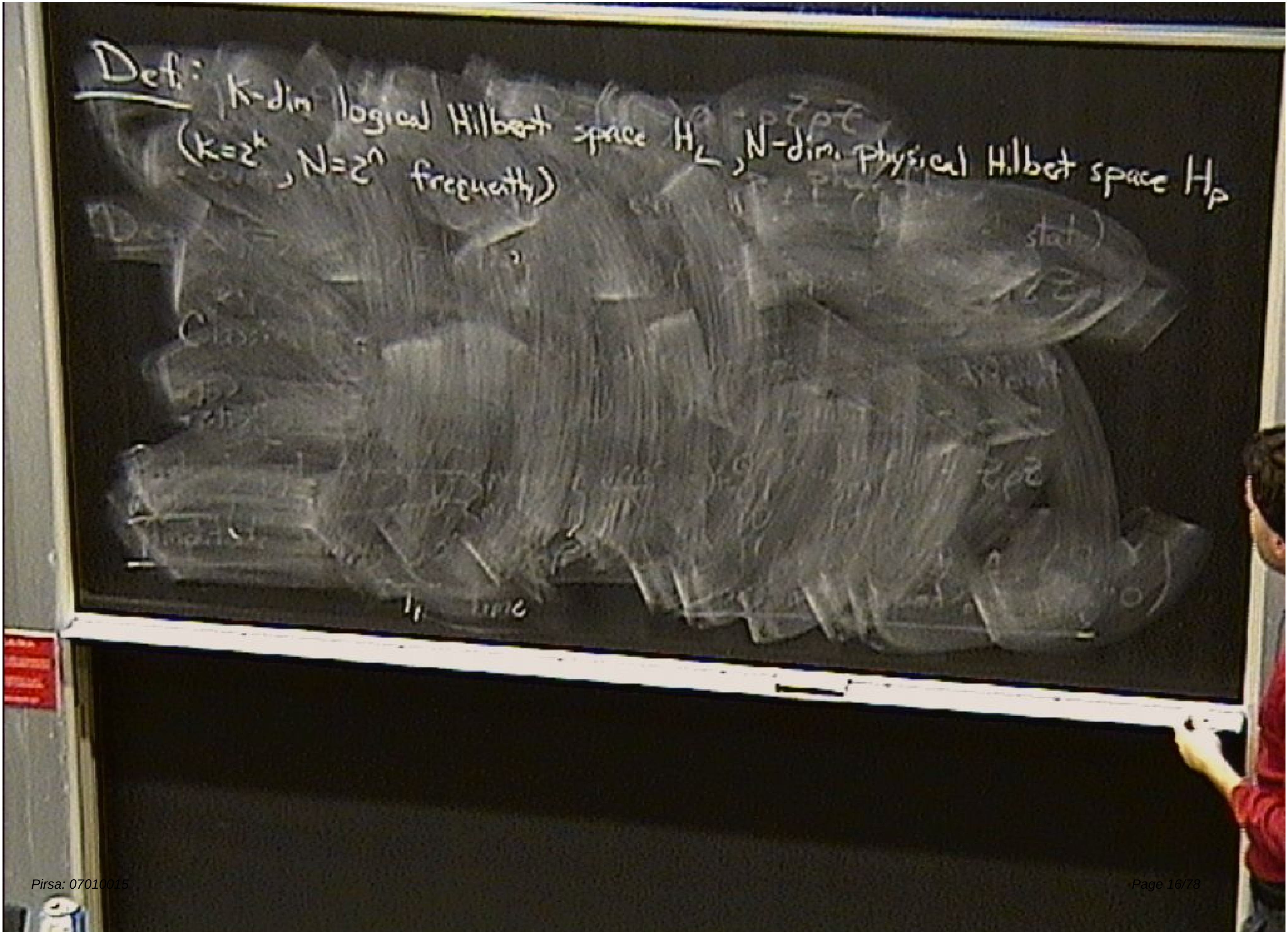
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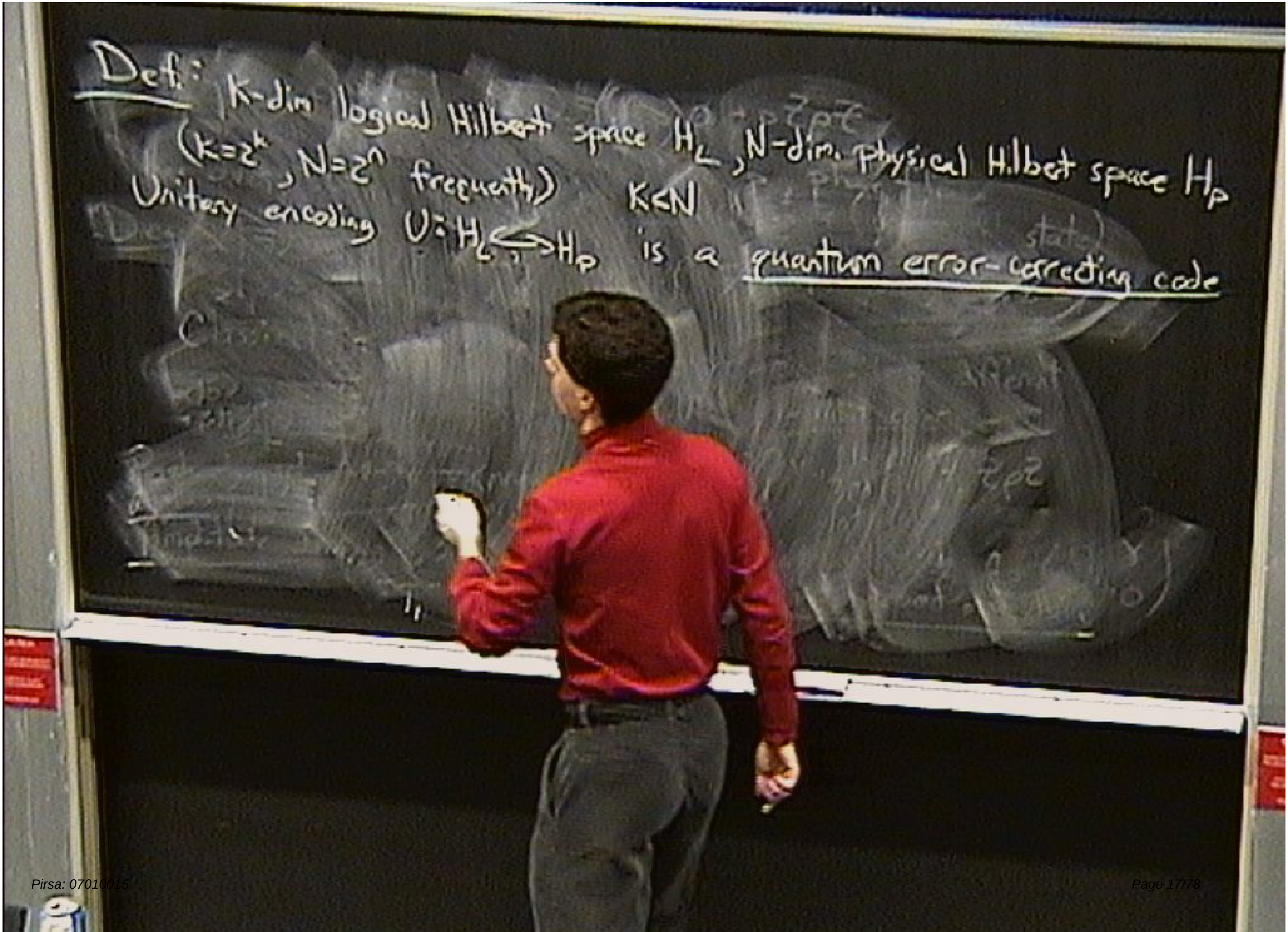
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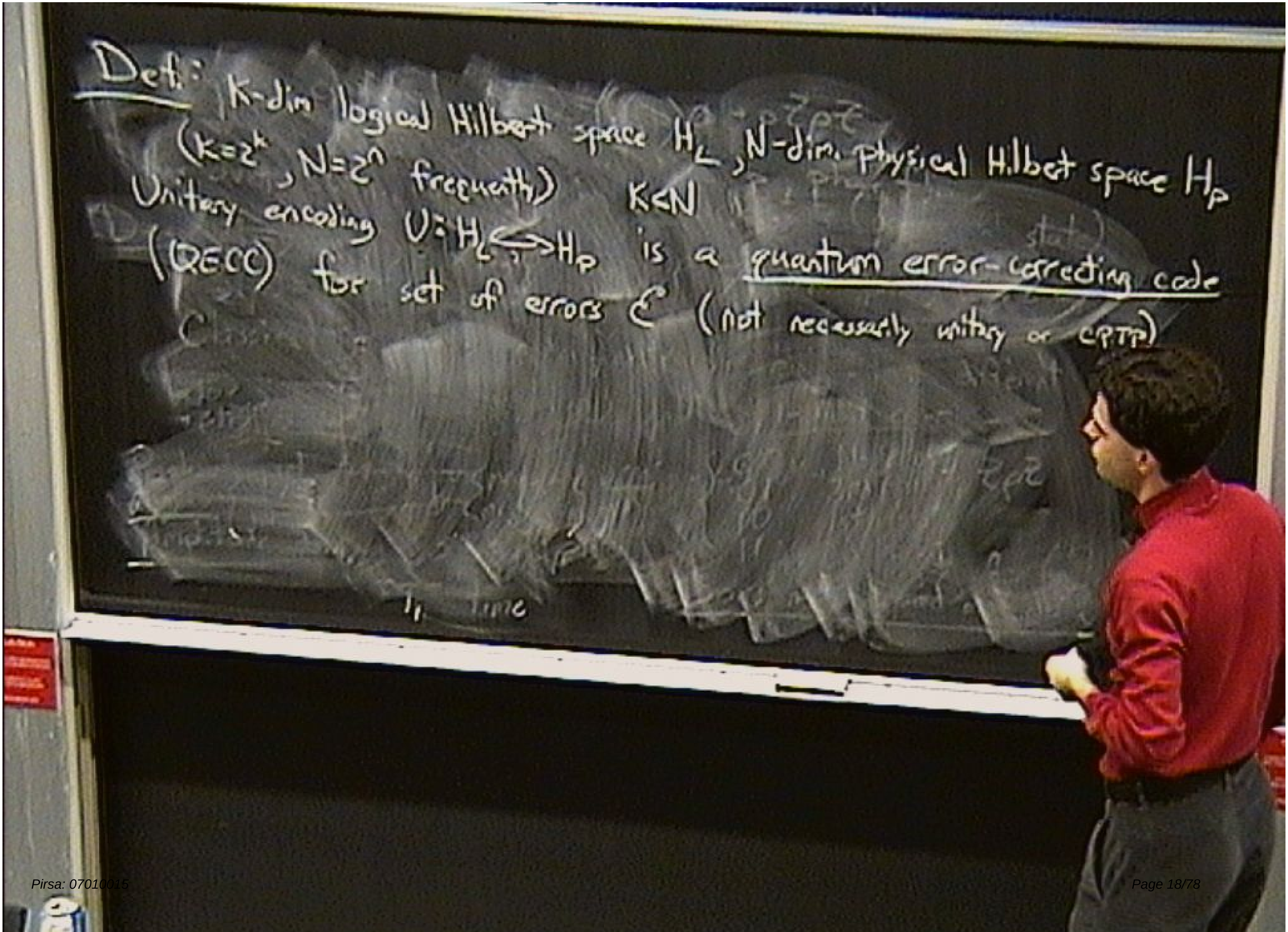
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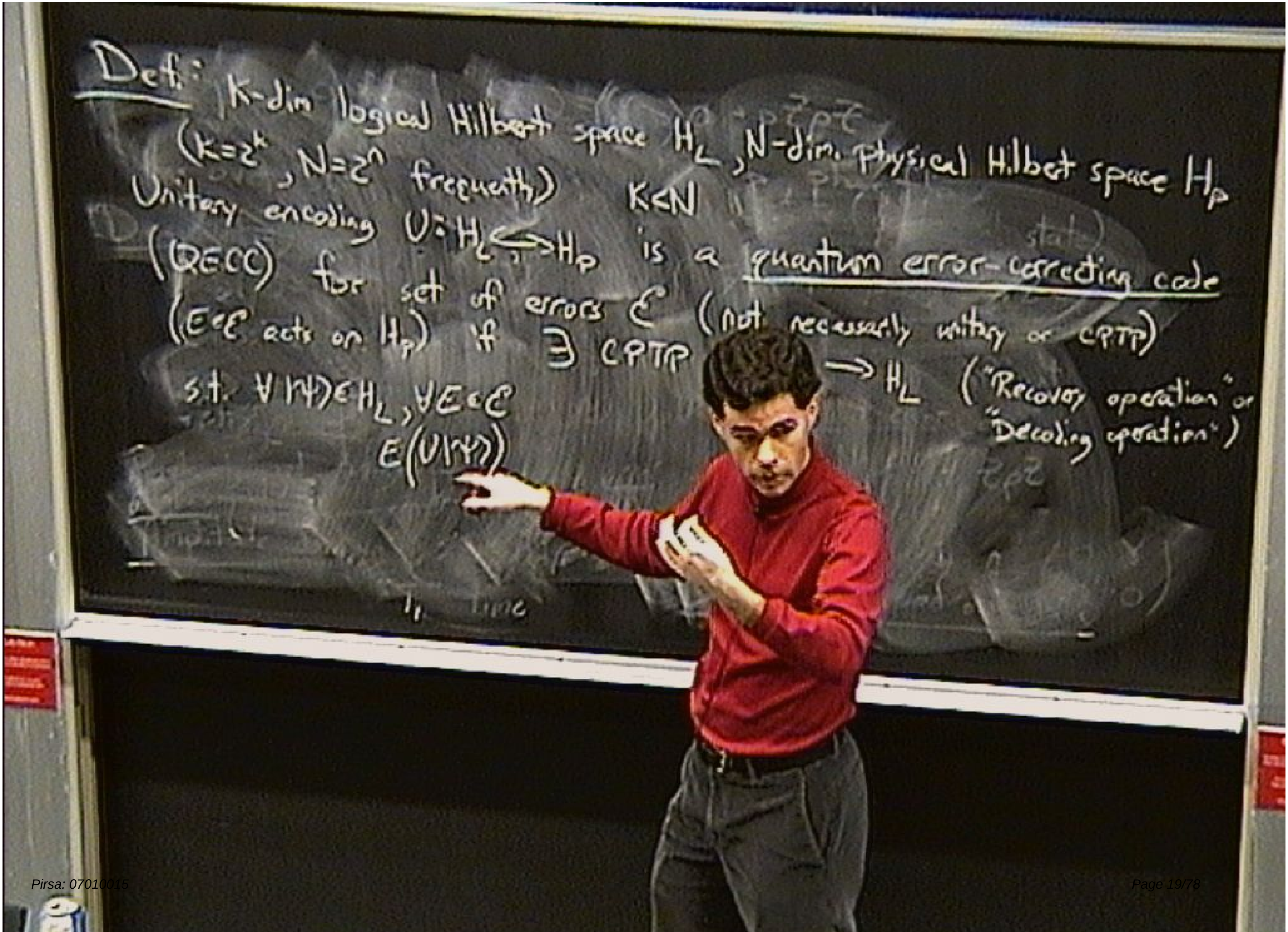
(QECC) for set of errors $\mathcal{E}$ (not necessarily unitary or CPTP)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(E∈ℰ acts on H_P) if ∃ CPTP
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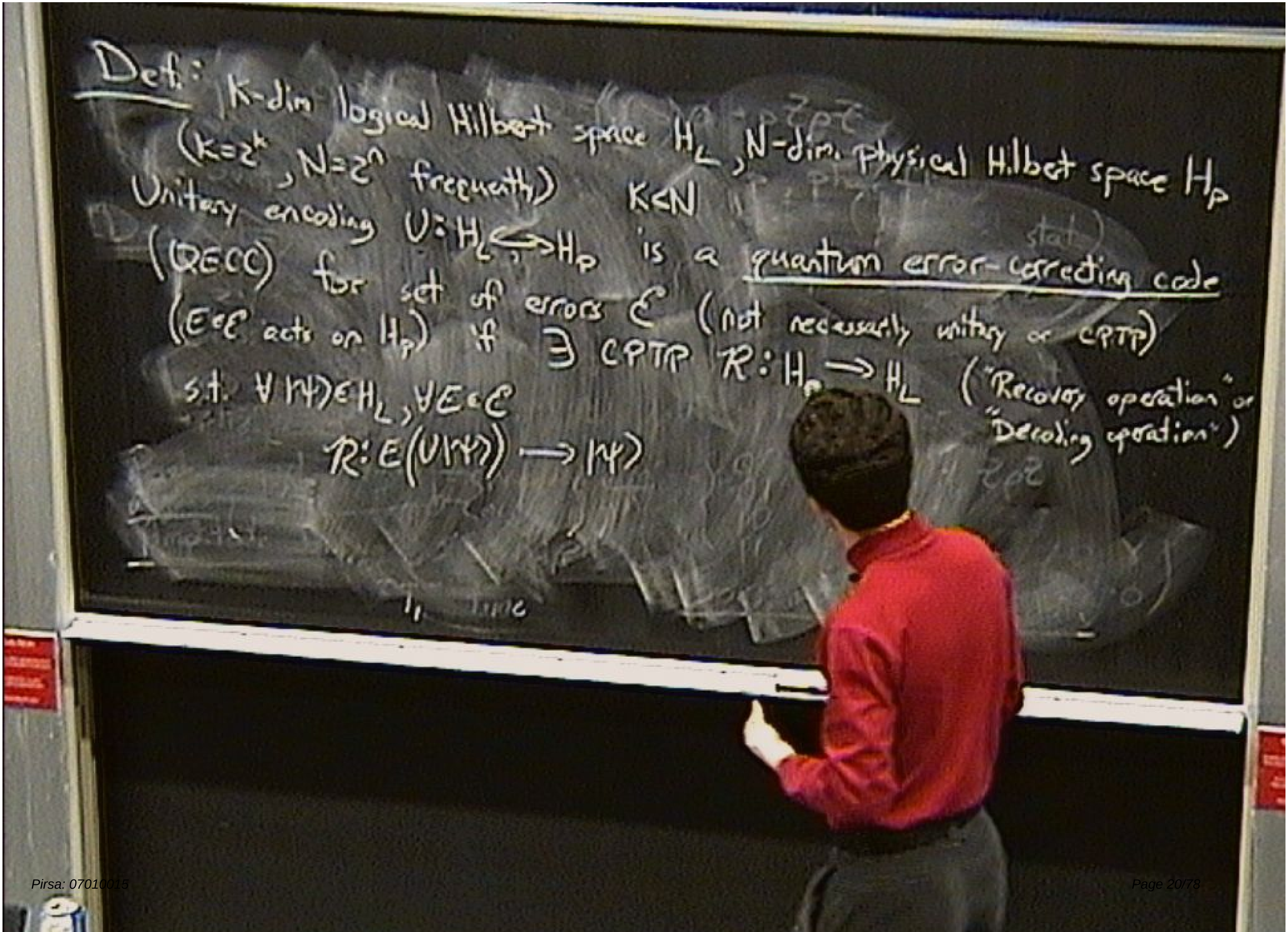
s.t. ∀|ψ⟩∈H_L, ∀E∈ℰ
E(U|ψ⟩)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("Recovery operation" or "Decoding operation")
s.t. $\forall |\psi\rangle \in H_L$, $\forall E \in \mathcal{E}$
$\mathcal{R}: E(U|\psi\rangle) \longrightarrow |\psi\rangle$

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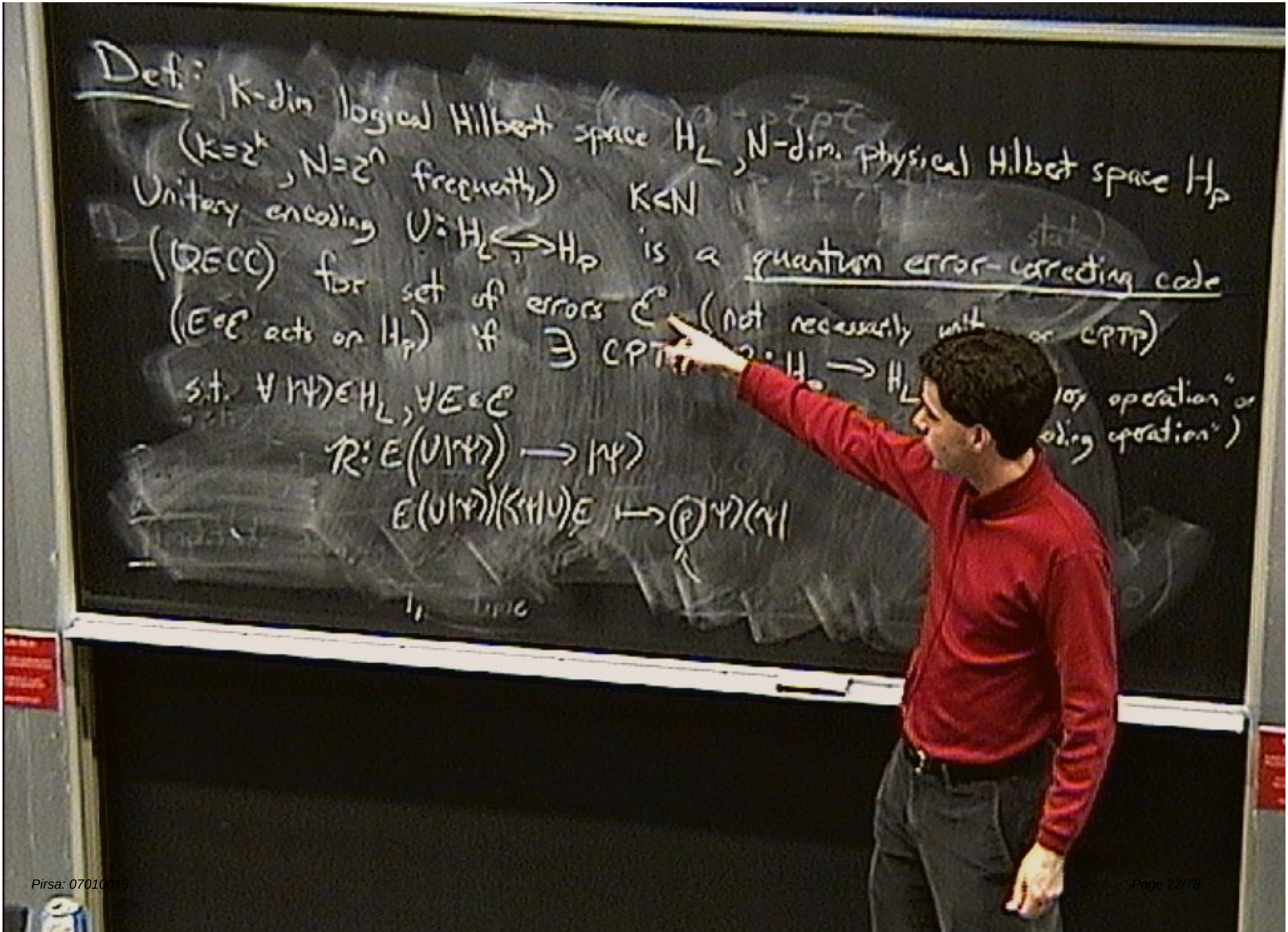
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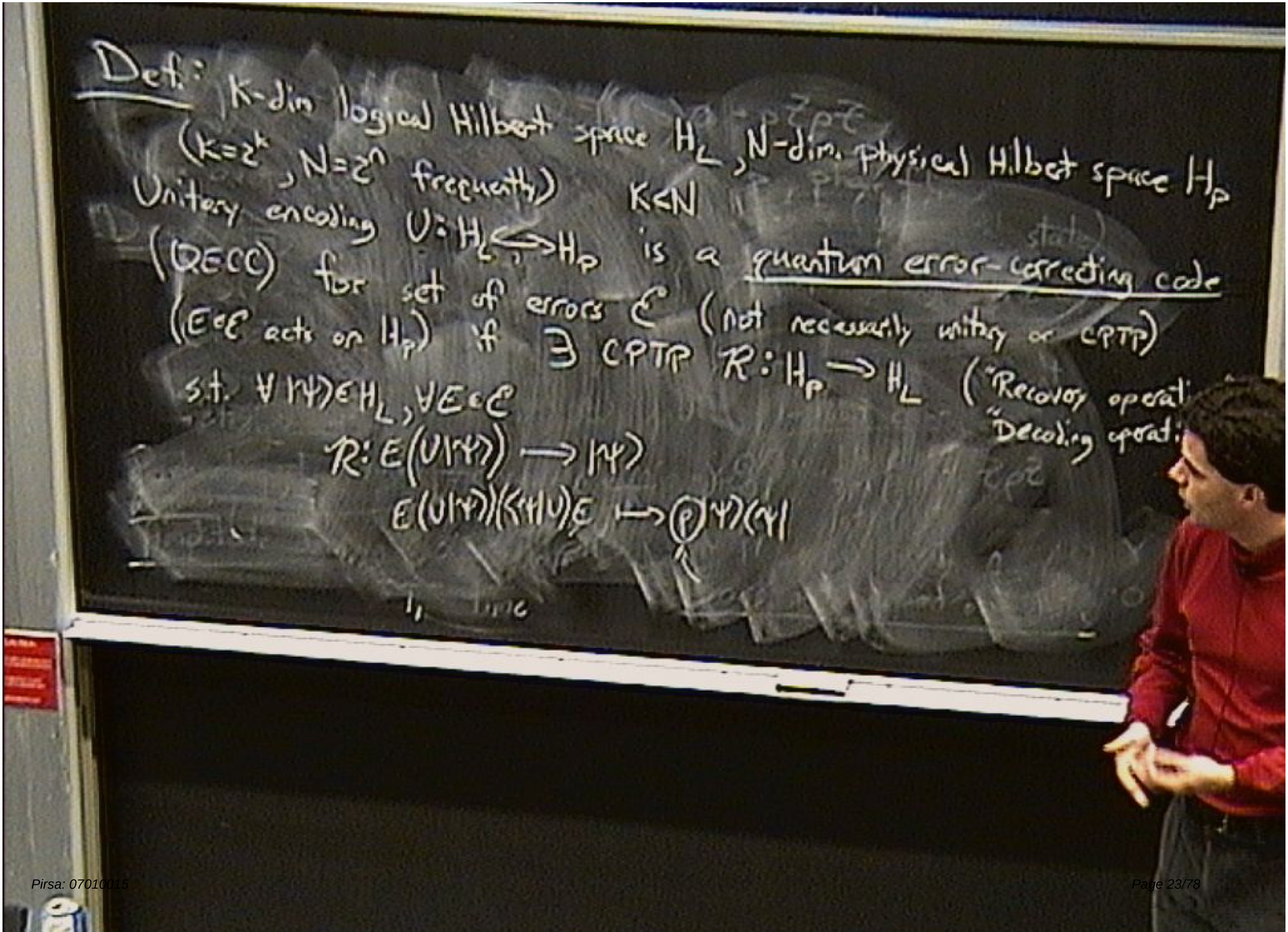
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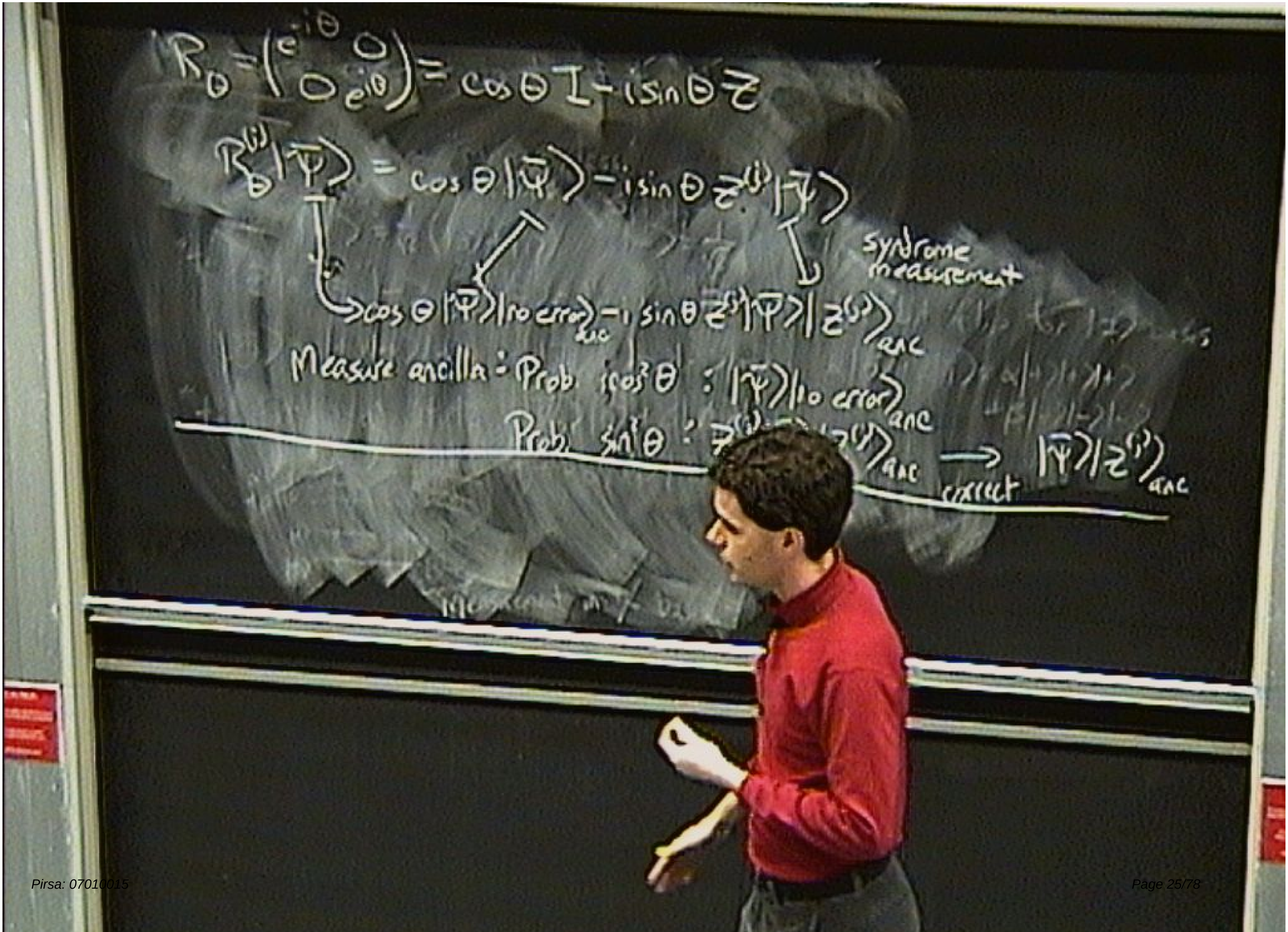
$$E(U|\psi\rangle)(\langle\psi|U^\dagger)E^\dagger \longmapsto p\,|\psi\rangle\langle\psi|$$



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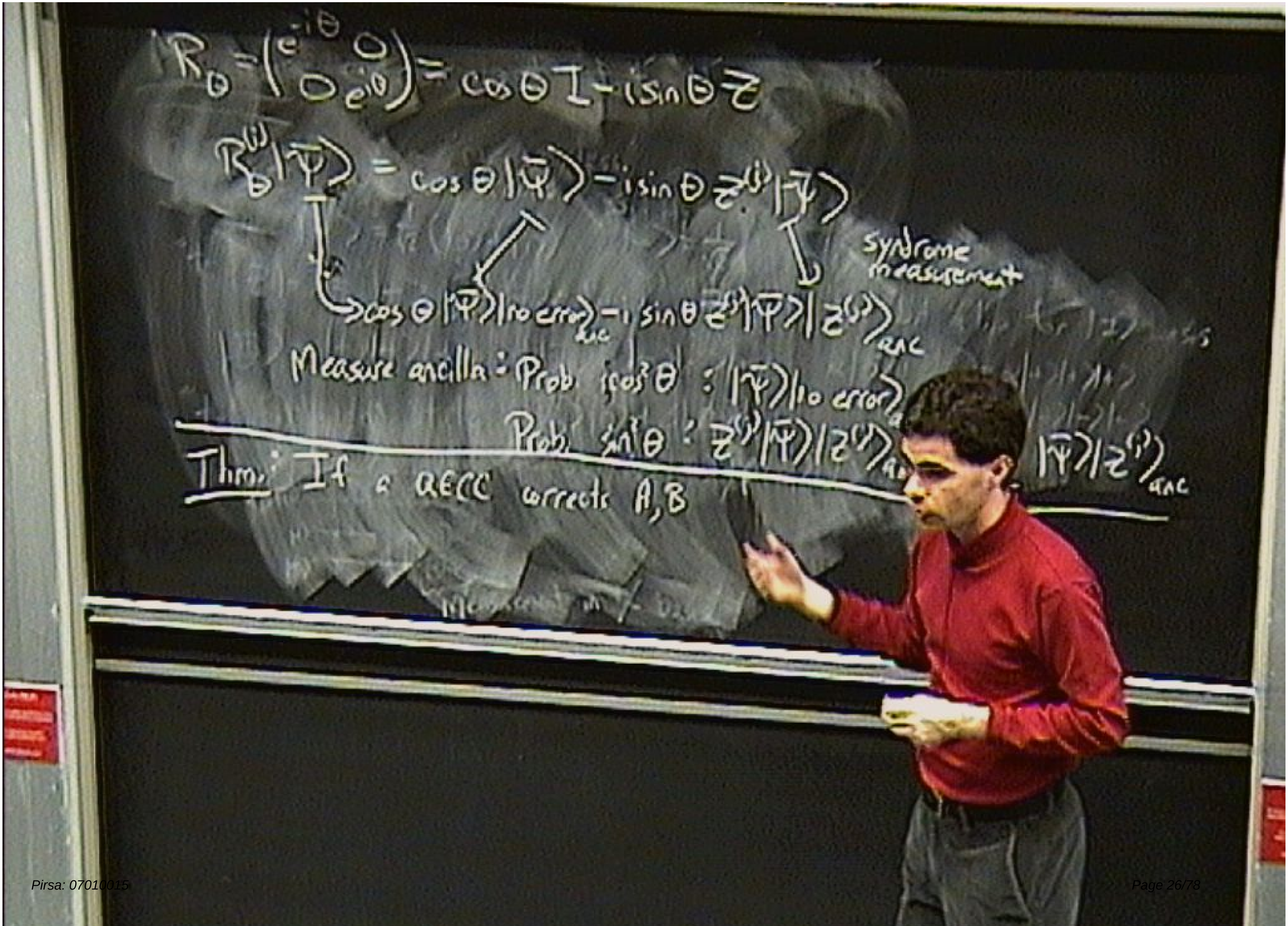
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Thm: If a QECC corrects A, B

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Thm: If a QECC corrects A, B, then it corrects $\alpha A + \beta B$

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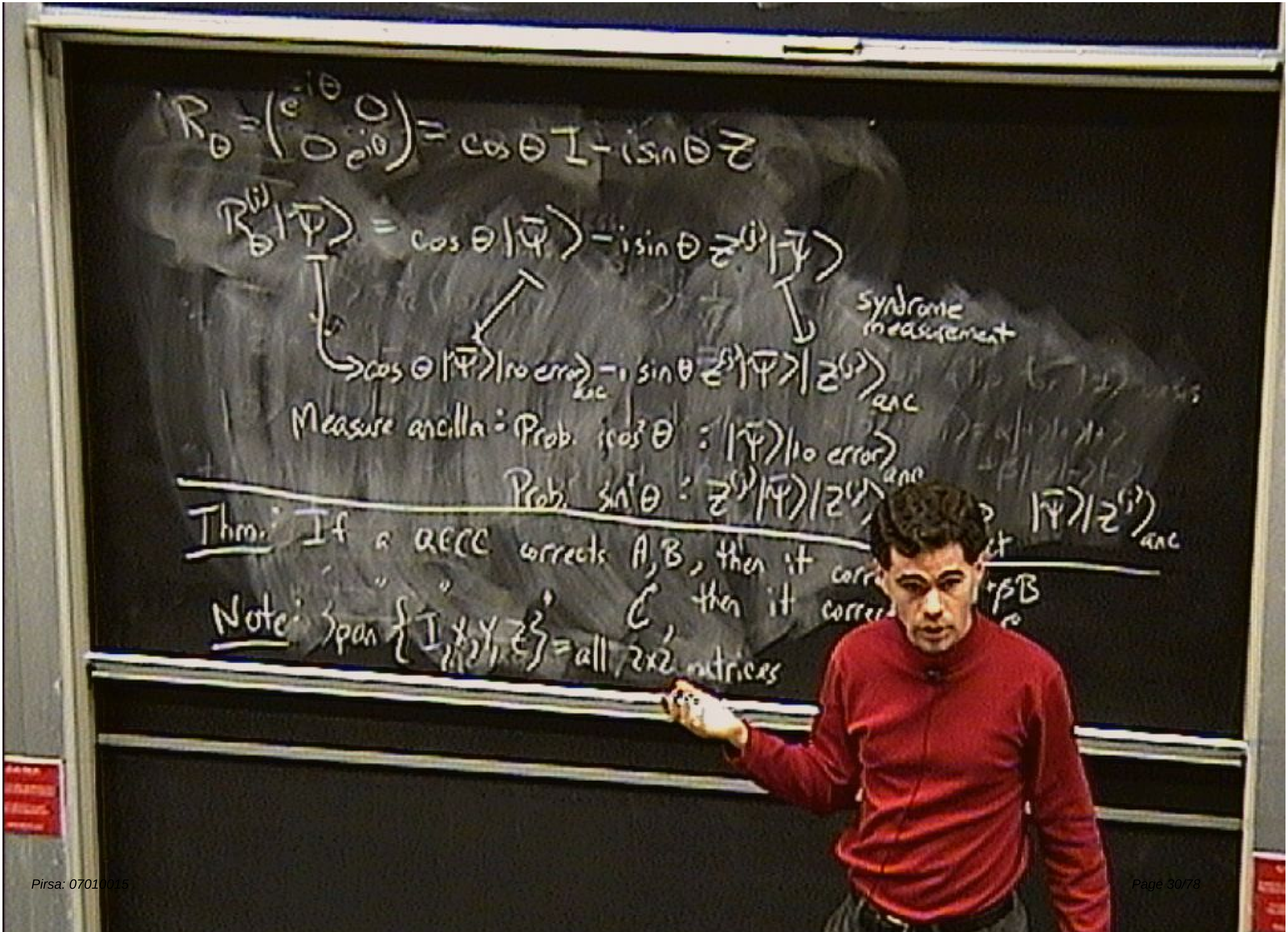
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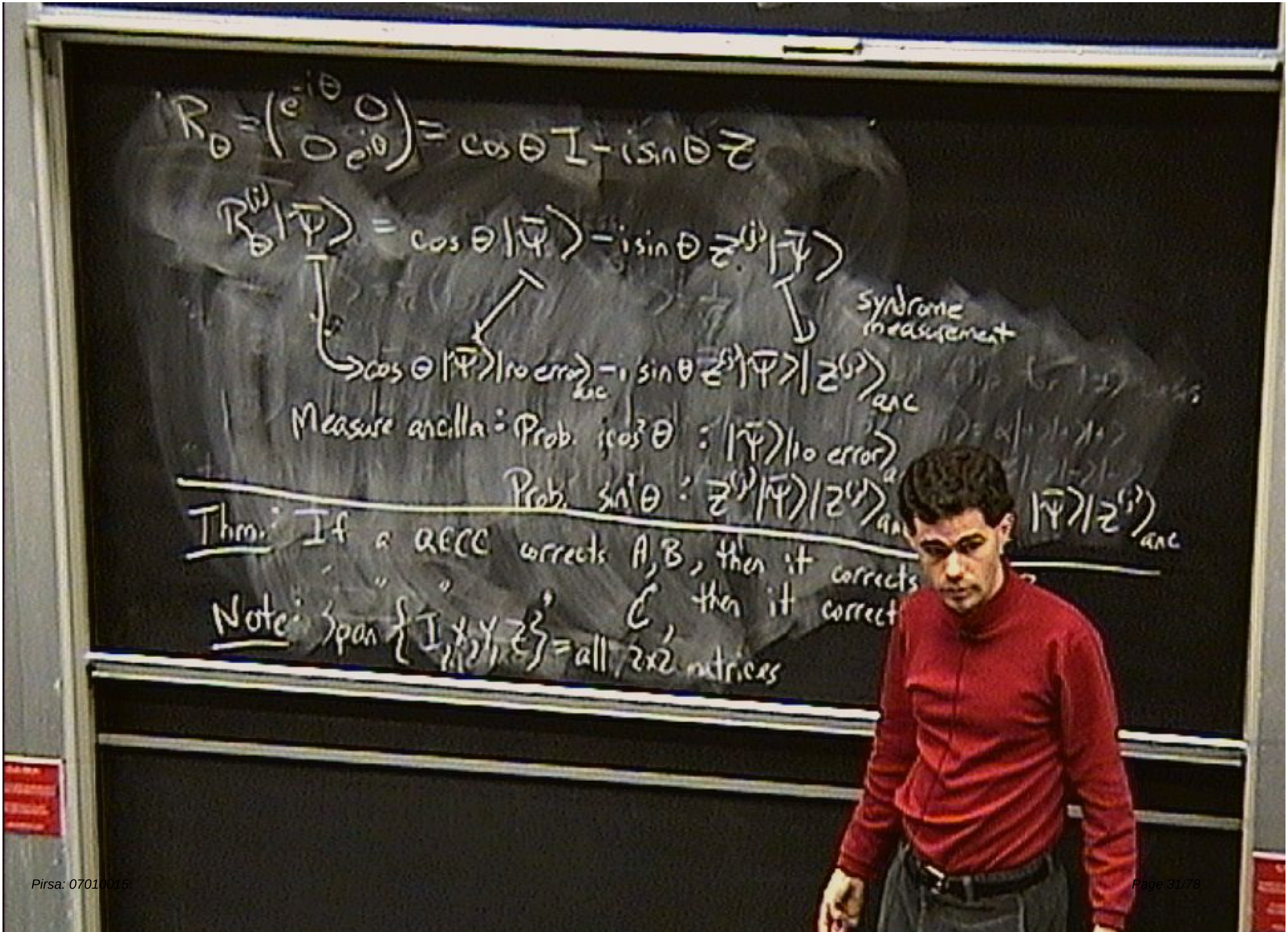
Note: Span{I, X, Y, Z} = all 2x2 matrices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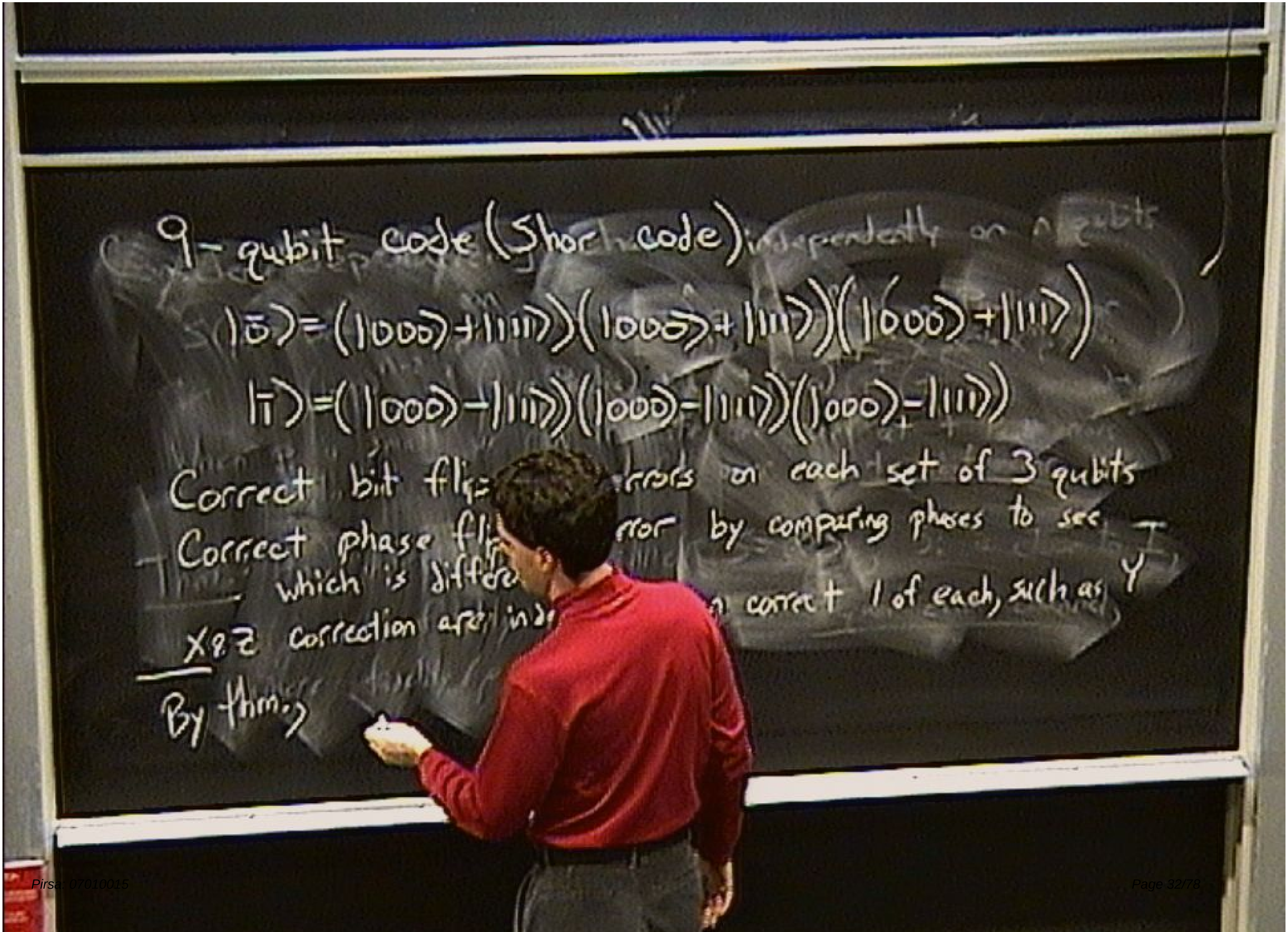
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9-qubit code (Shor

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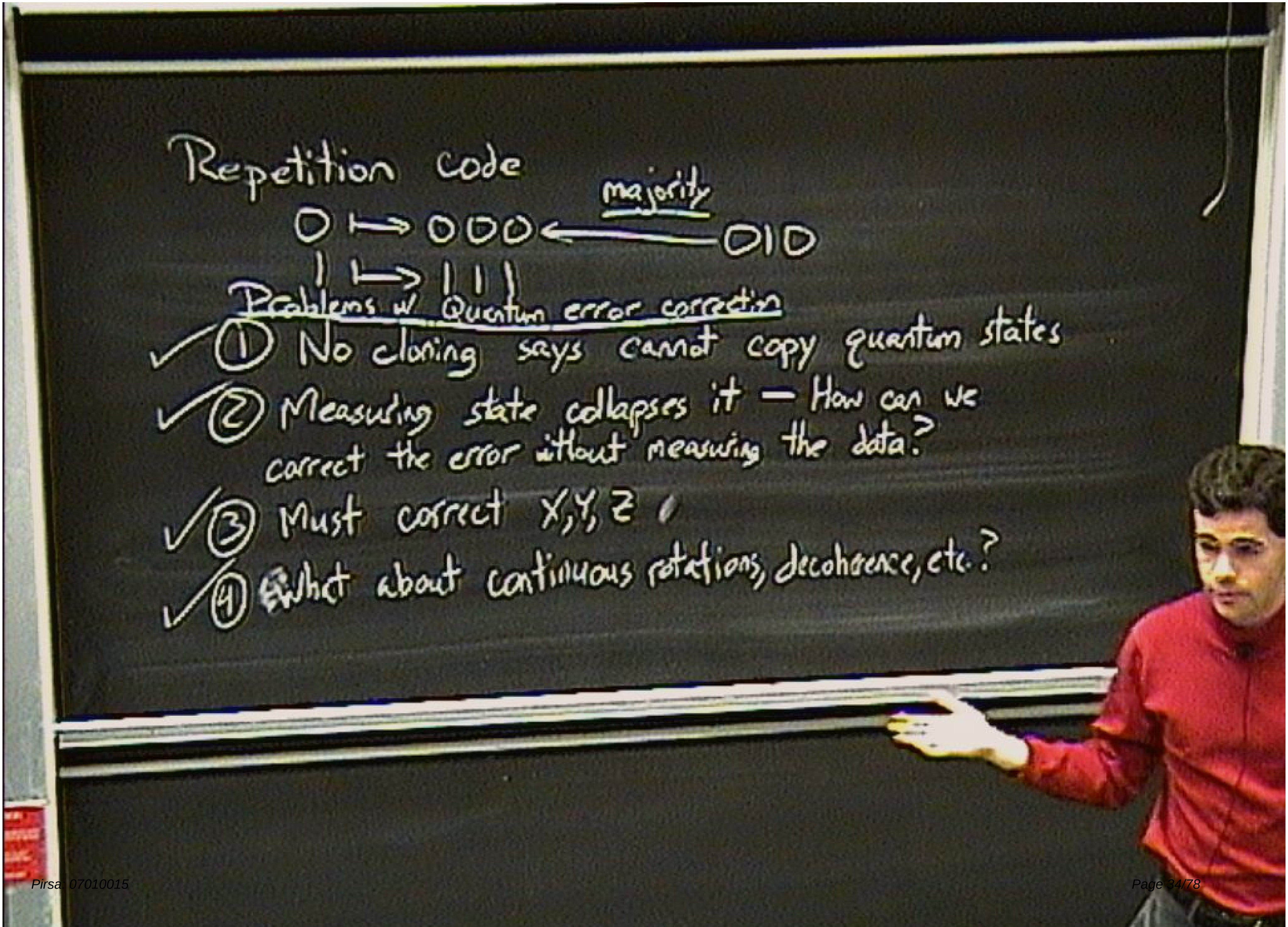
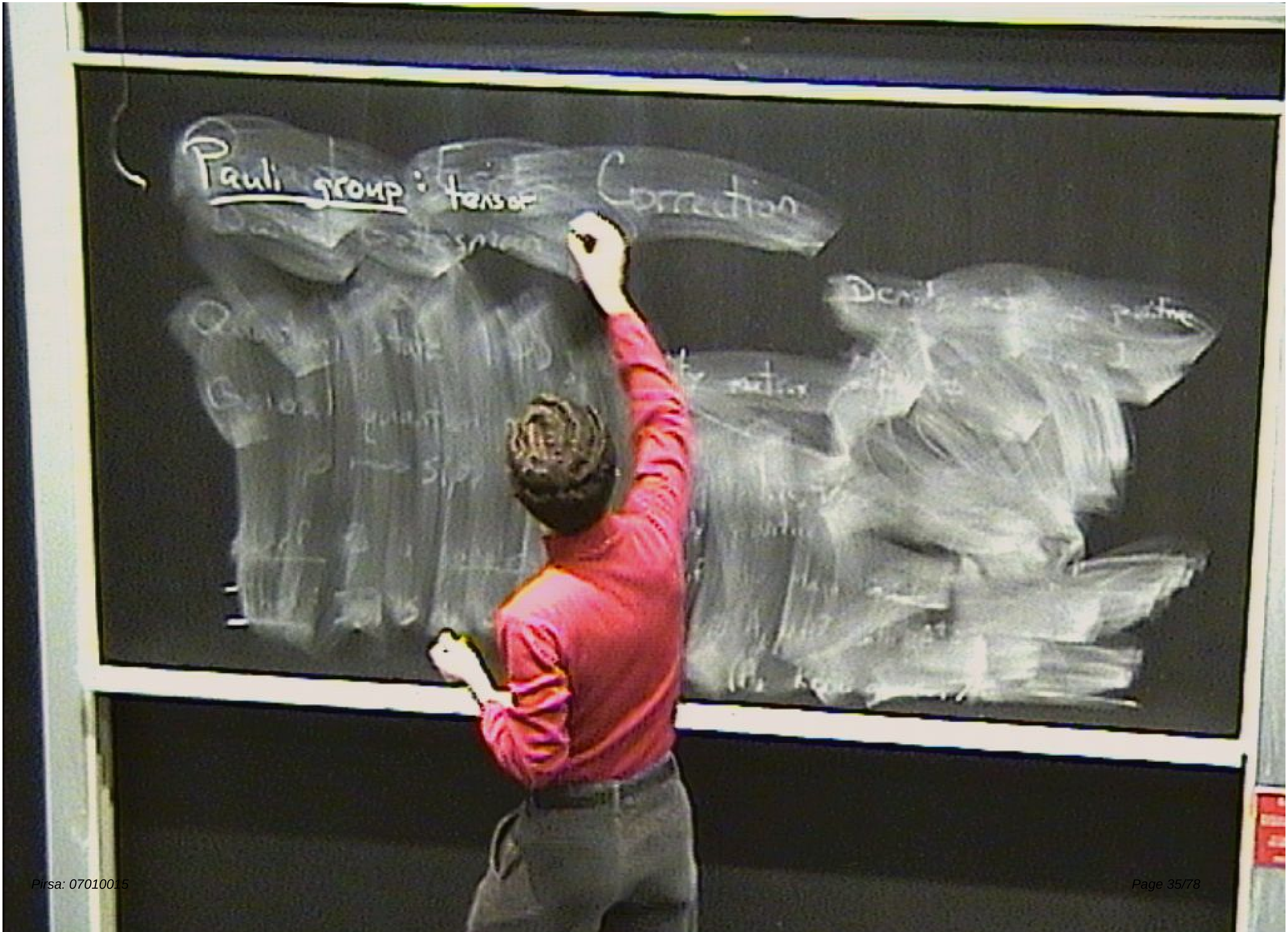
Correct bit flip (X) errors on each set of 3 qubits

Correct phase flip (Z) error by comparing phases to see which is different

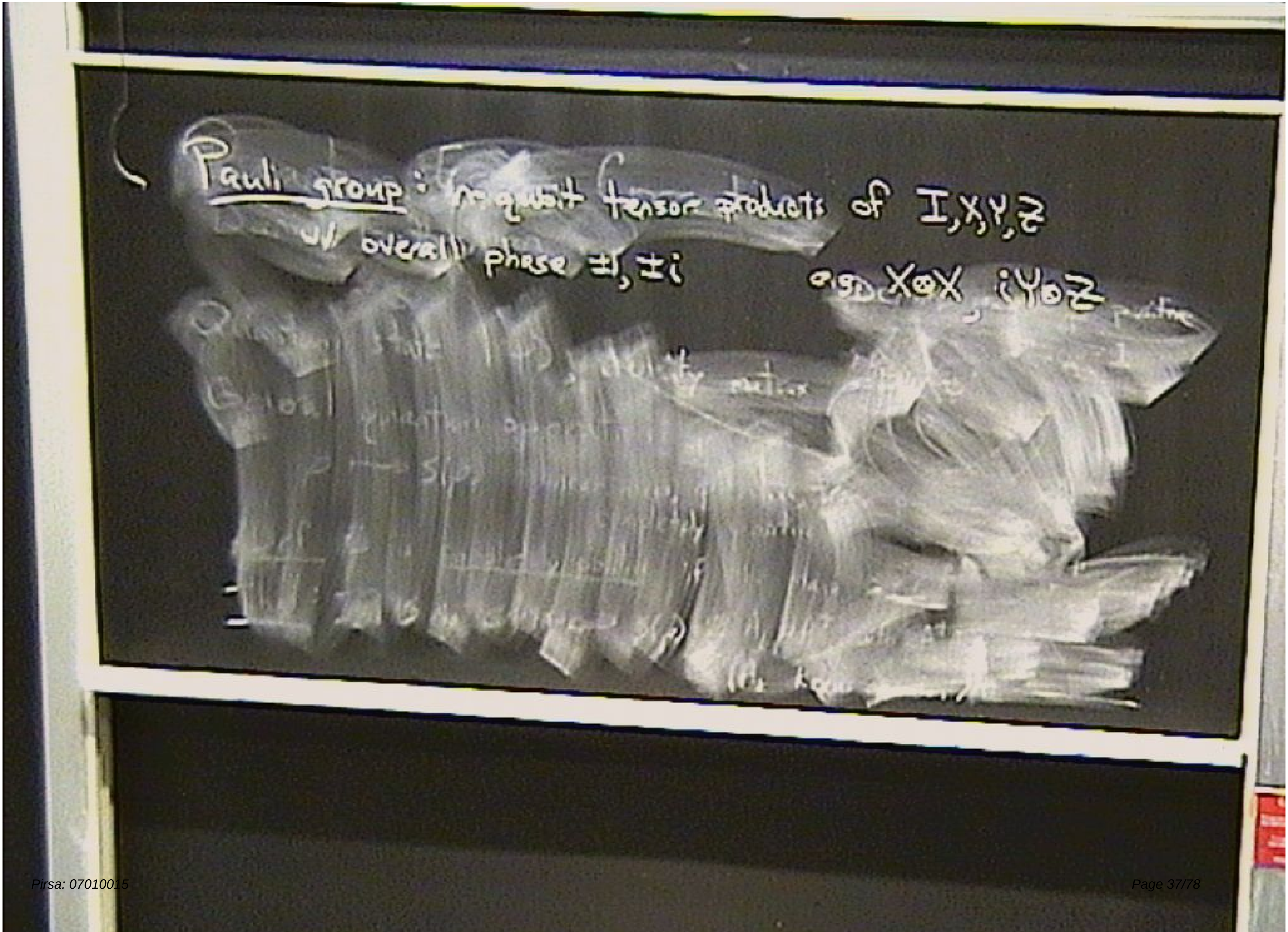
X & Z correction are independent → can correct 1 of each, such as Y

By thm., 9-qubit code can correct *any* single-qubit error.


Repetition code
majority
0 ↦ 000 ← 010
1 ↦ 111
Problems w. Quantum error correction
① No cloning says cannot copy quantum states
② Measuring state collapses it — How can we correct the error without measuring the data?
③ Must correct X, Y, Z
④ What about continuous rotations, decoherence, etc.?


Pauli group: tensor
Correction

Pauli group: n-qubit tensor products of I, X, Y, Z
w/ overall phase $\pm 1, \pm i$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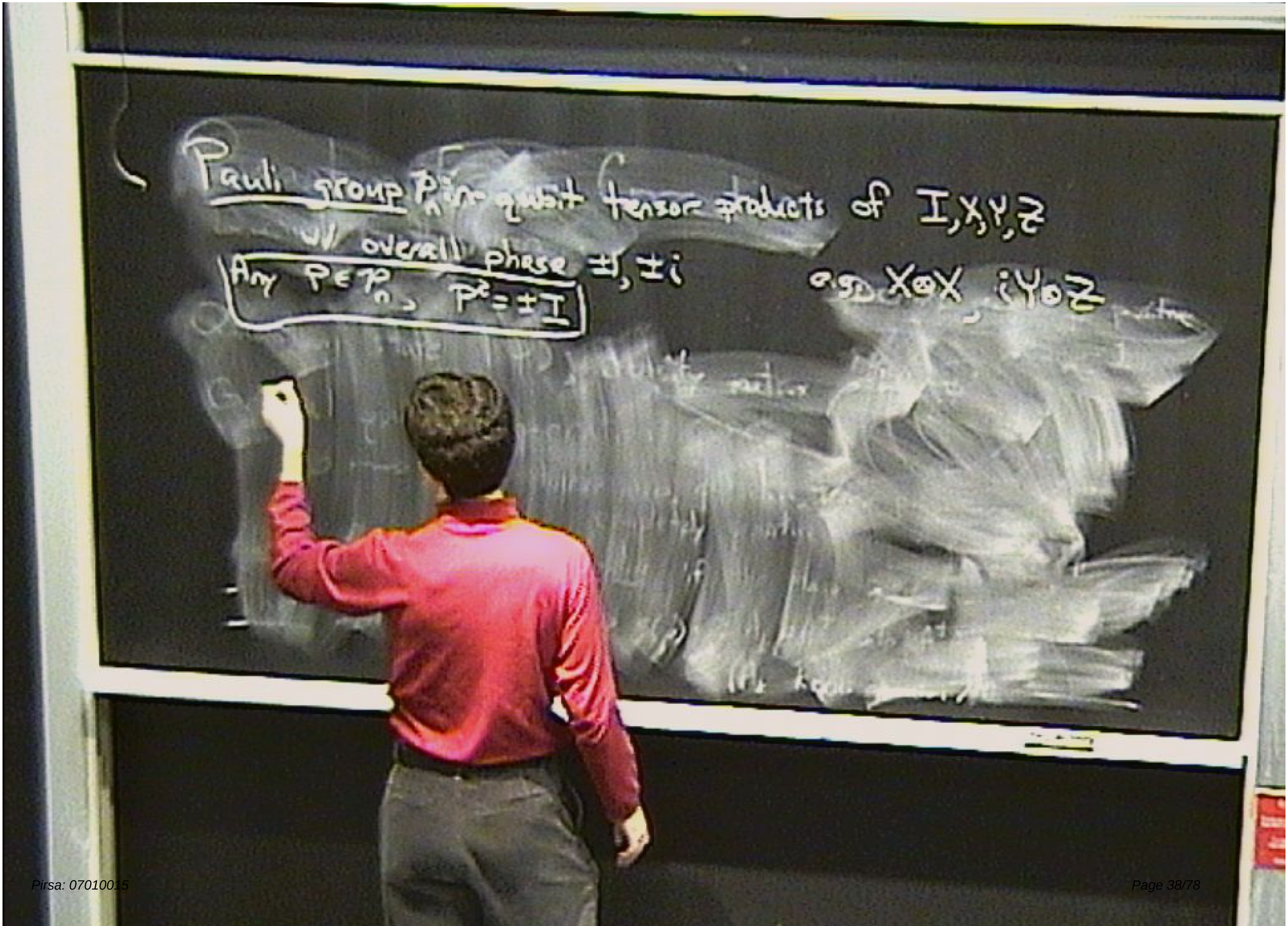
Pauli group P_n = n-qubit tensor products of I, X, Y, Z
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Any P ∈ P_n, P² = ±I
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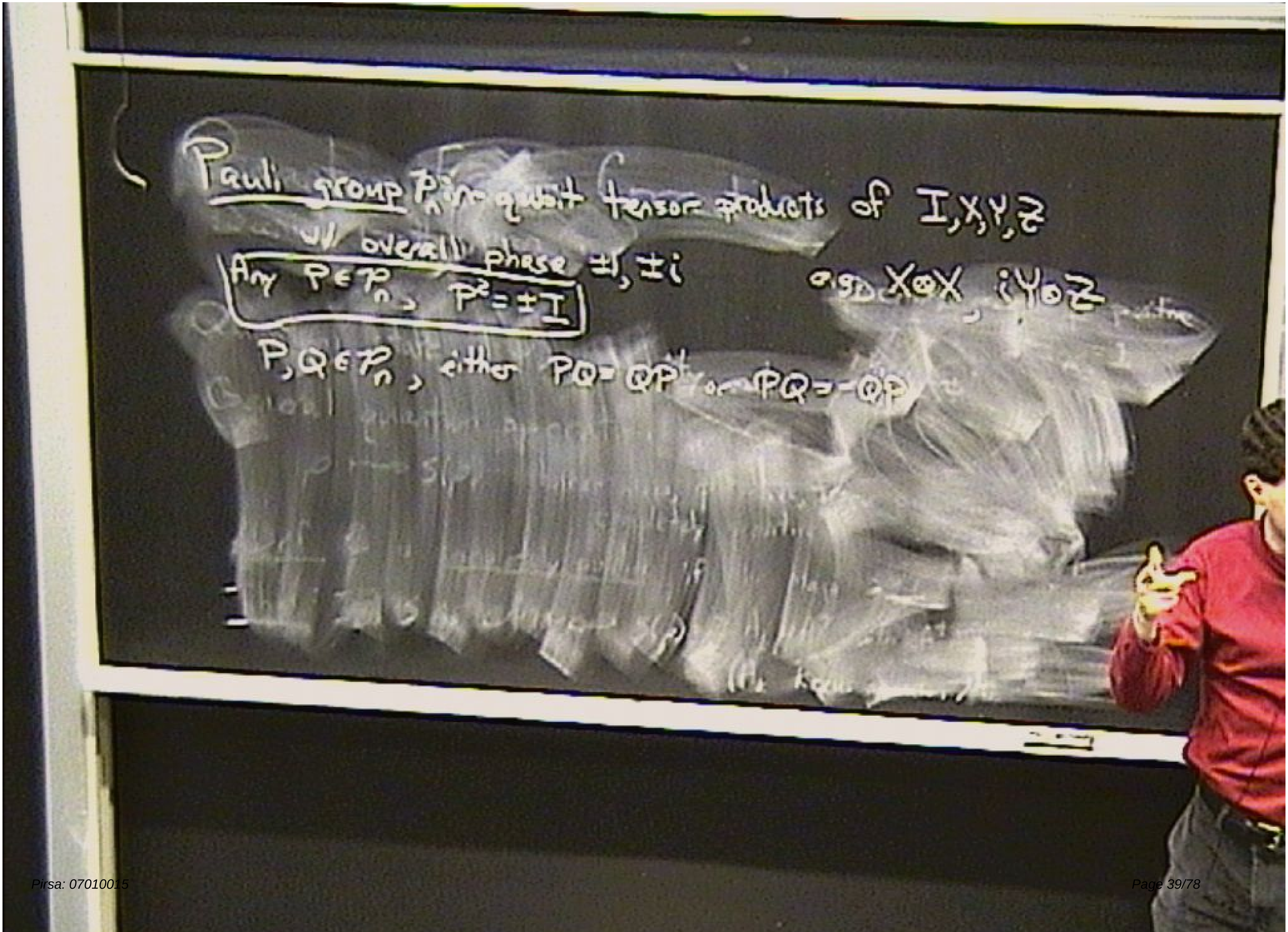
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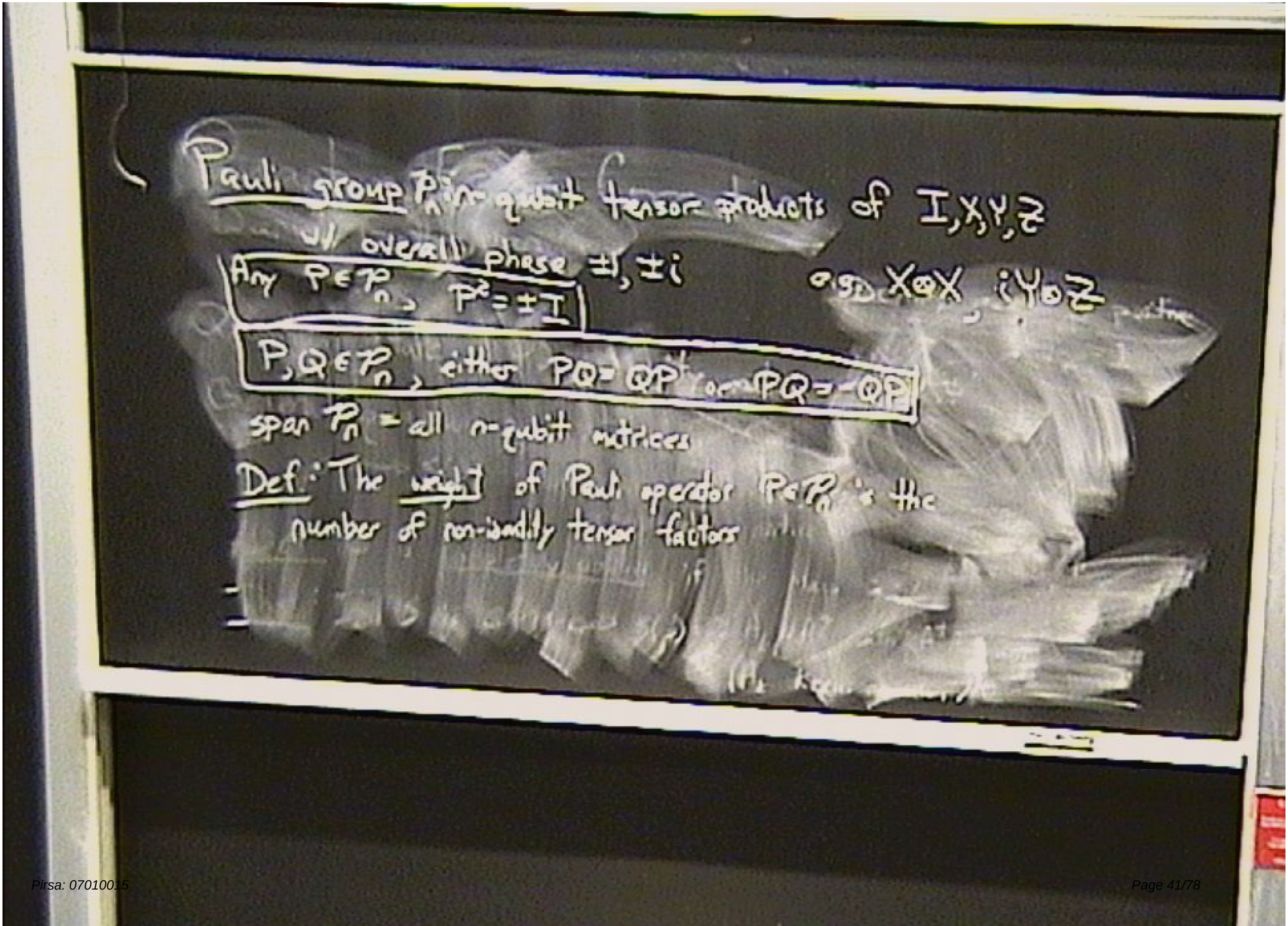
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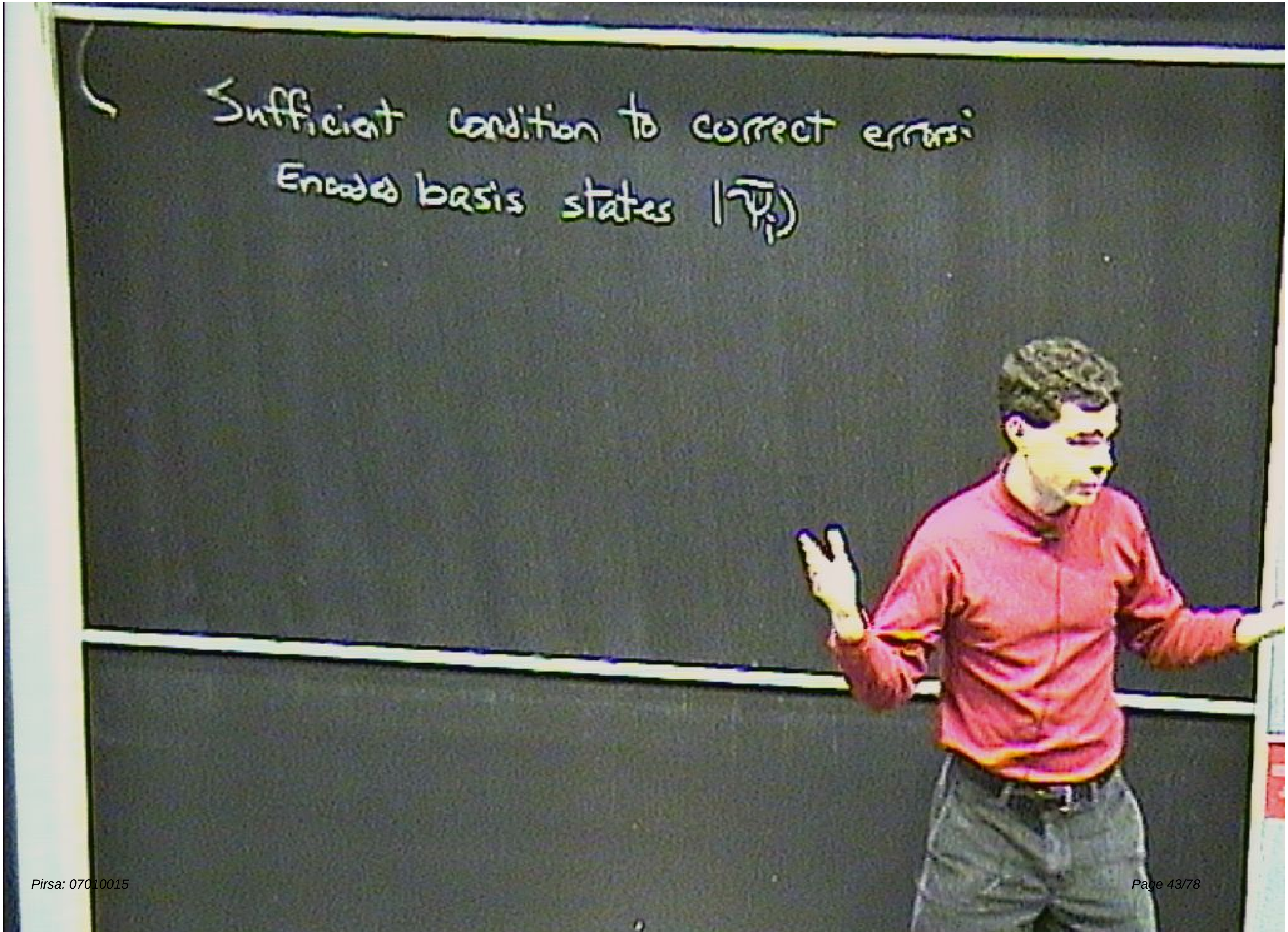
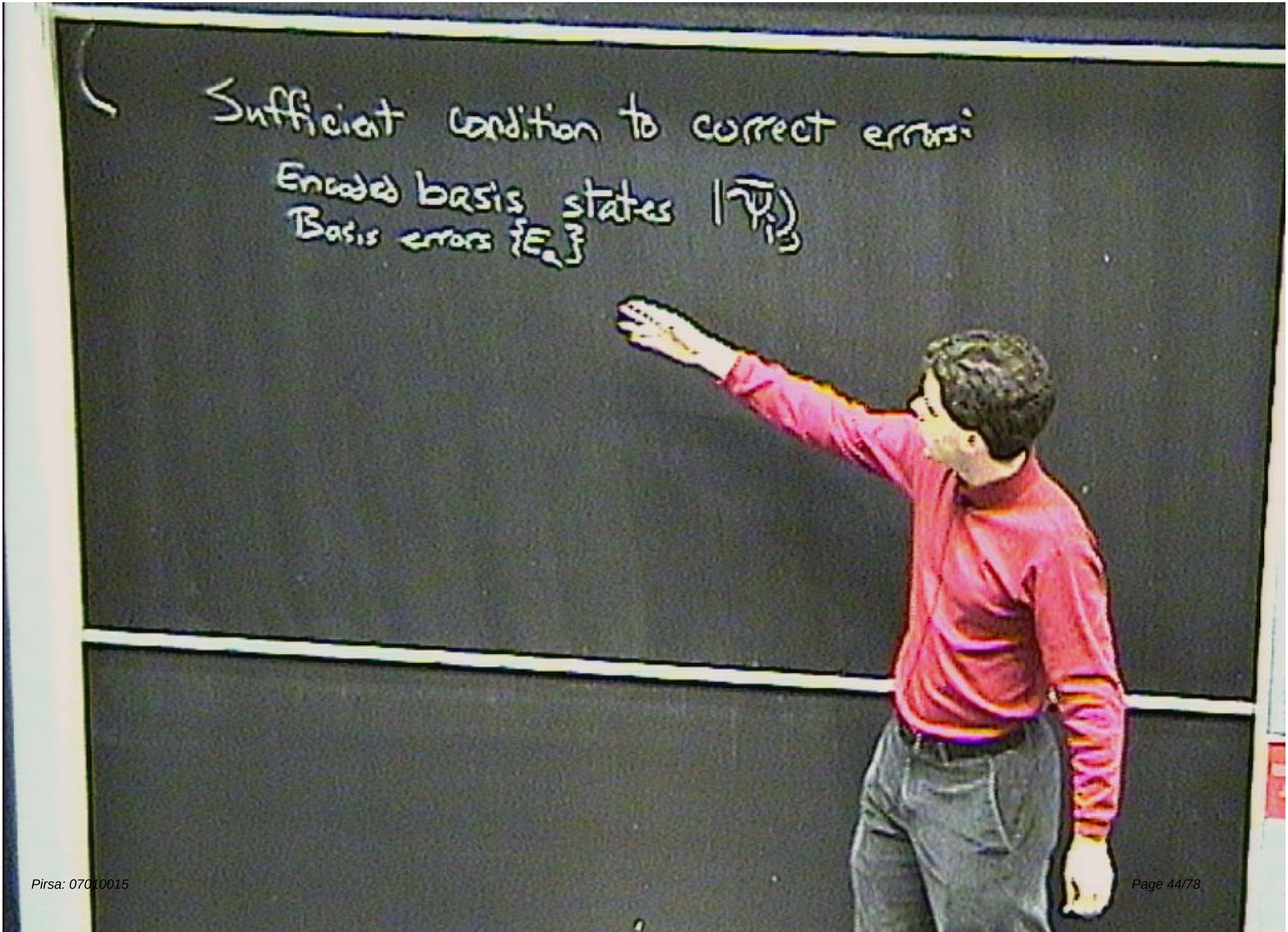
<u>Def</u>: The <u>weight</u> of Pauli operator $P \in \mathcal{P}_n$ is the number of non-identity tensor factors

Weight t Pauli errors form a basis for all t-qubit errors


Sufficient condition to correct errors:
Encoded basis states $|\overline{\Psi}_i\rangle$


Sufficient condition to correct errors:
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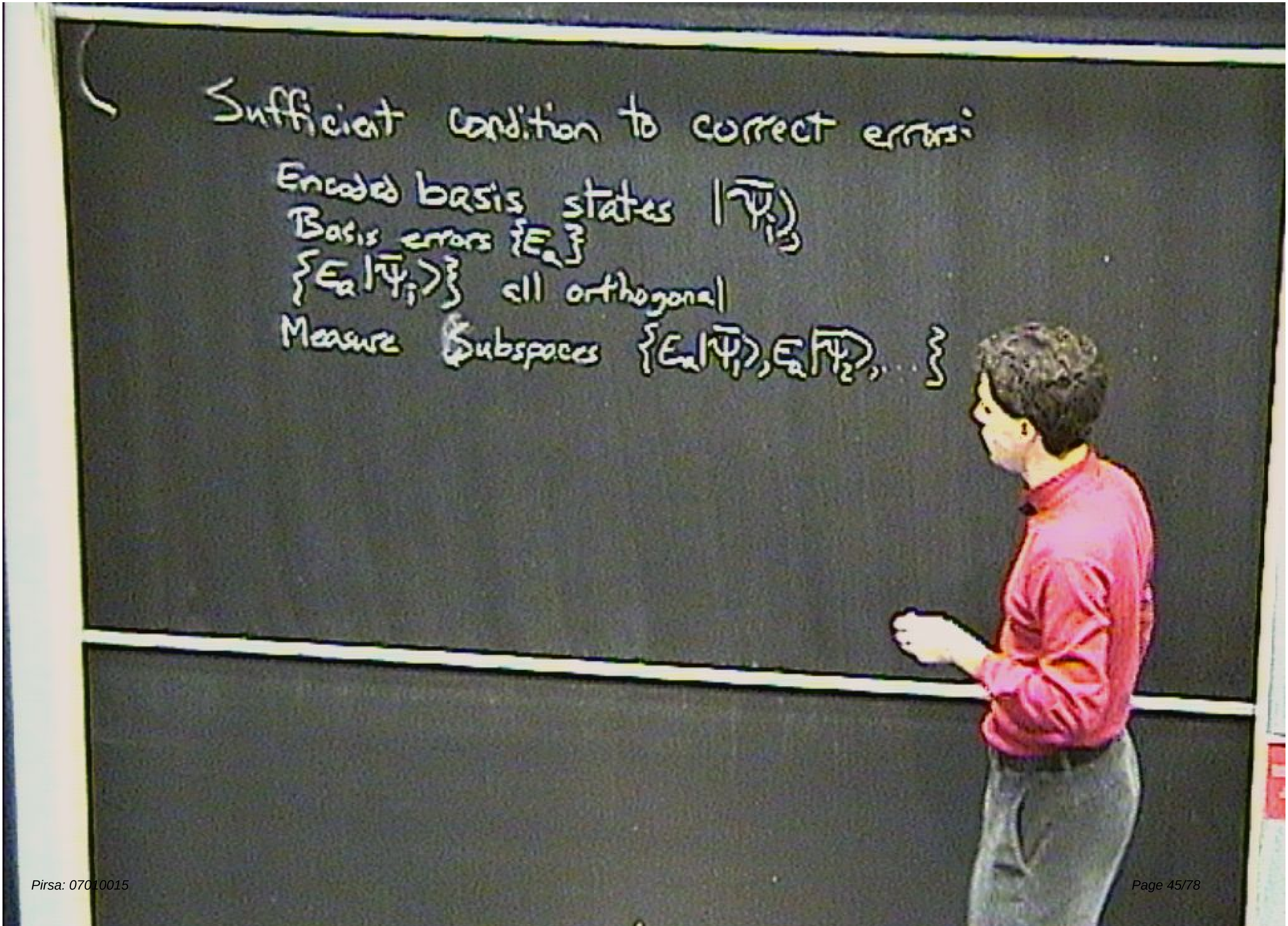
Basis errors $\{E_a\}$


Sufficient condition to correct errors:
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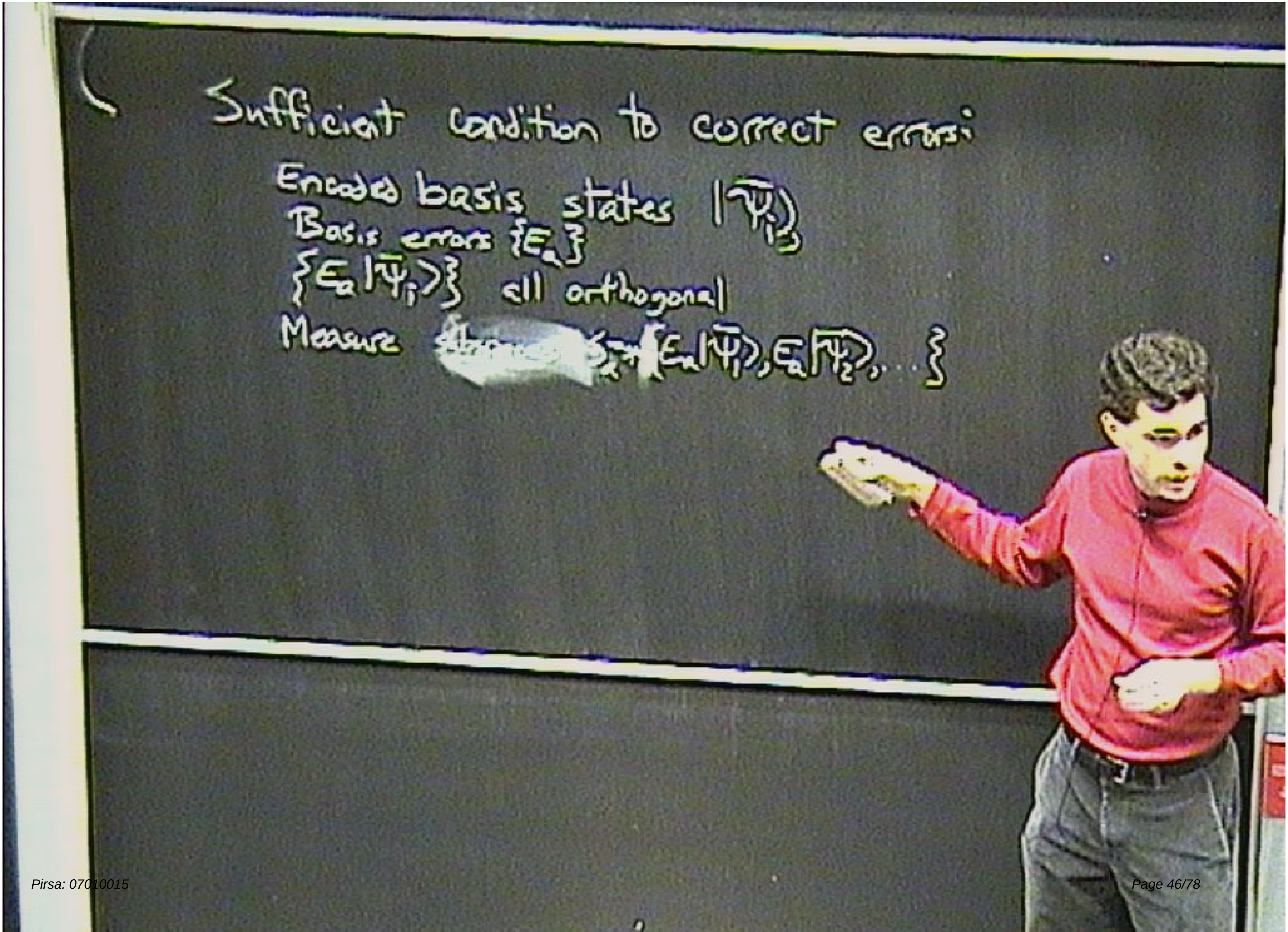
$\{E_a|\bar{\Psi}_i\rangle\}$ all orthogonal
Measure subspaces $\{E_a|\bar{\Psi}_1\rangle, E_a|\bar{\Psi}_2\rangle, \dots\}$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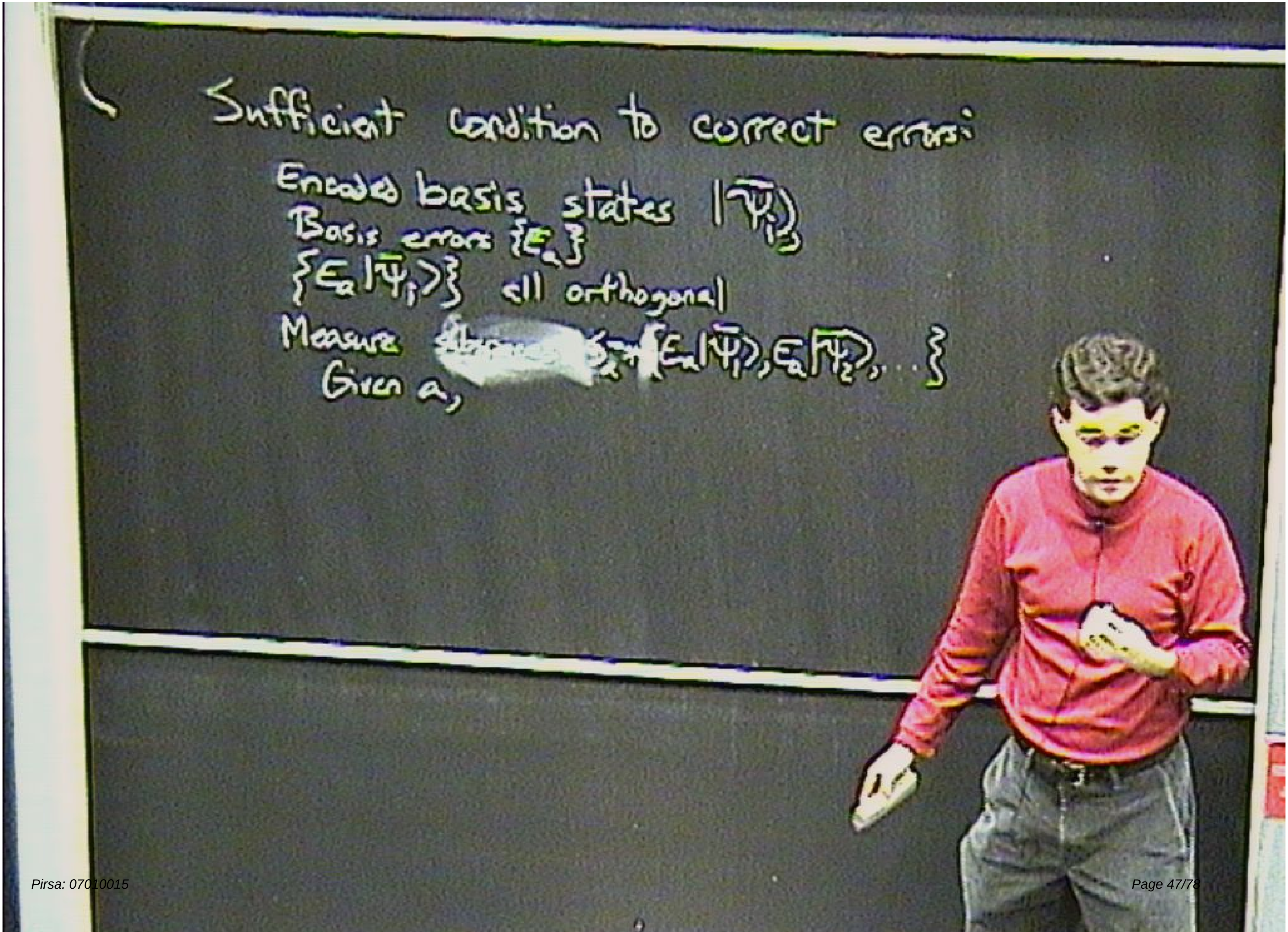
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Given a, map $S_a \rightarrow H_L$ $\quad E_a|\bar{\Psi}_i\rangle \mapsto |\bar{\Psi}_i\rangle$

$\langle \bar{\Psi}_i | E_a^{\dagger} E_b | \bar{\Psi}_j \rangle = \delta_{ab}\delta_{ij}$

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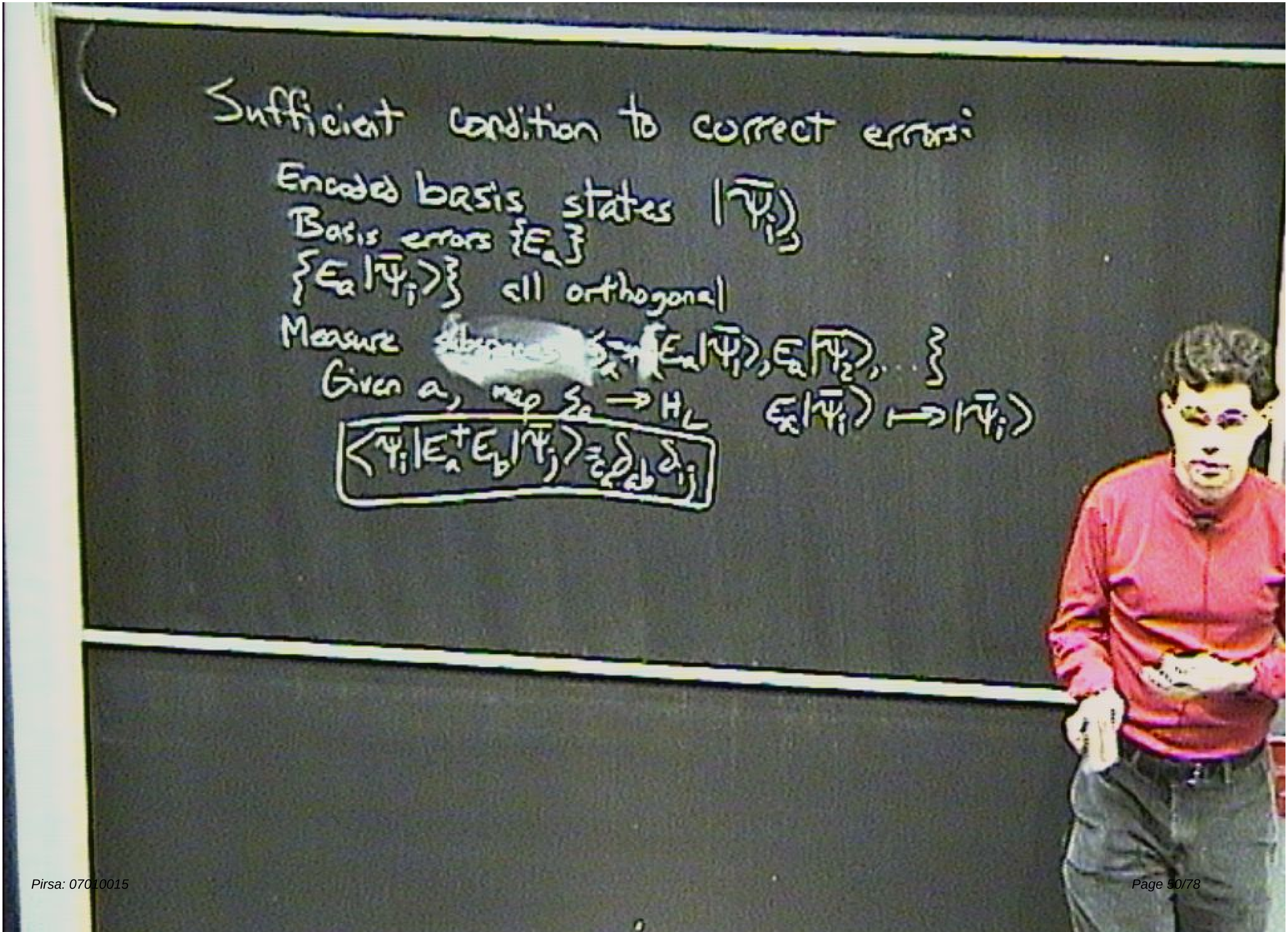
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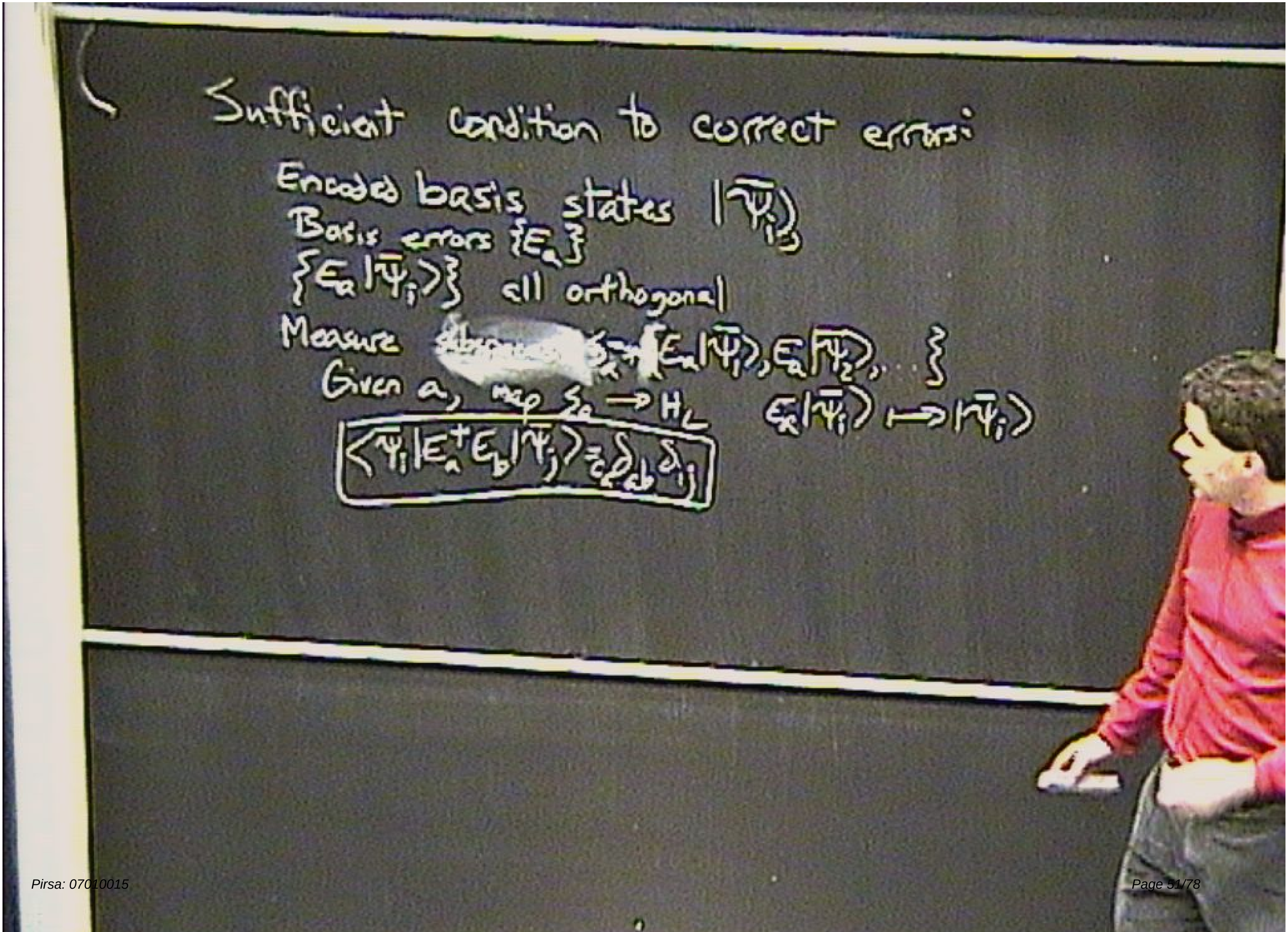
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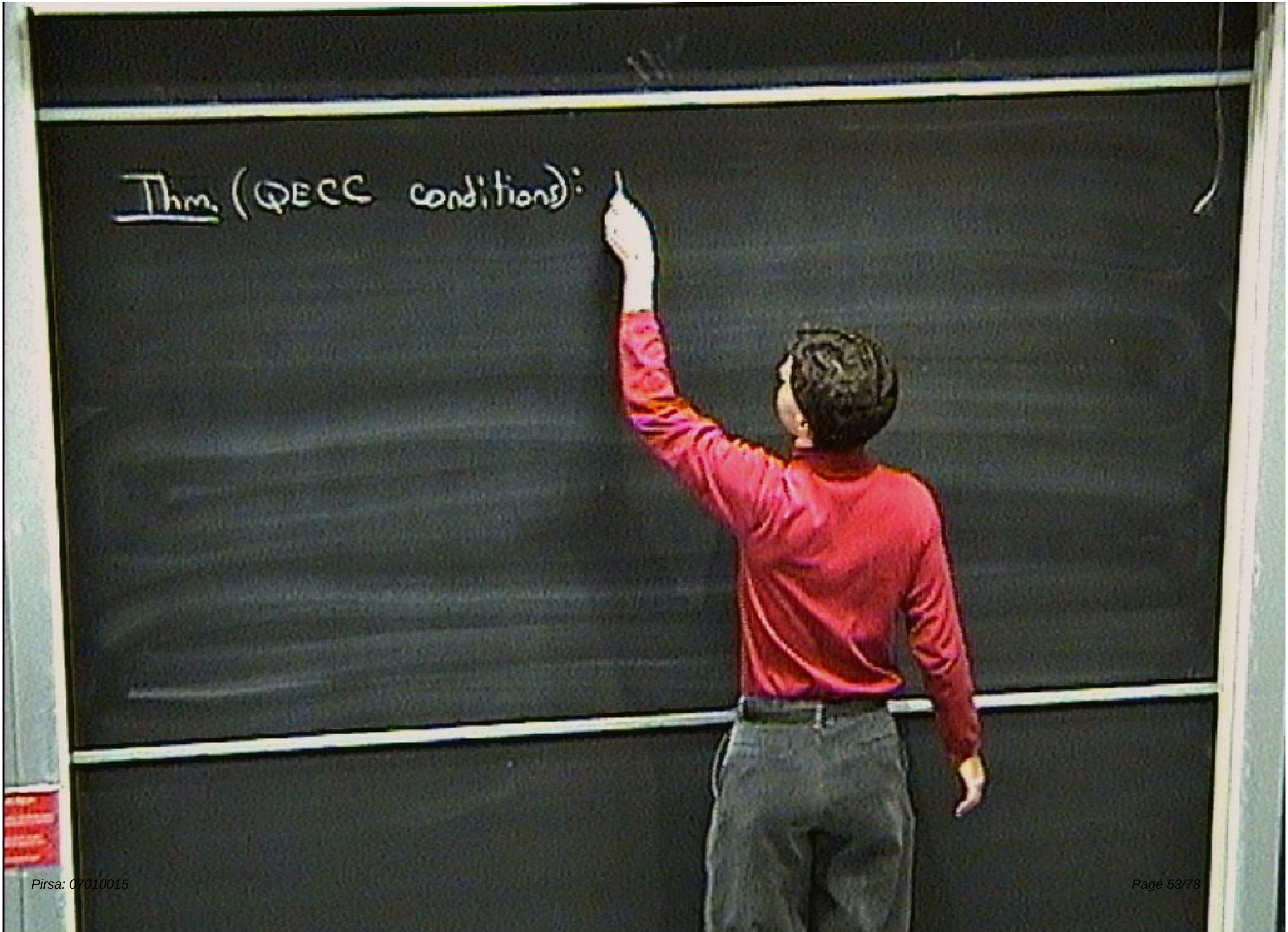
True for some basis E_a of $\mathcal{E}$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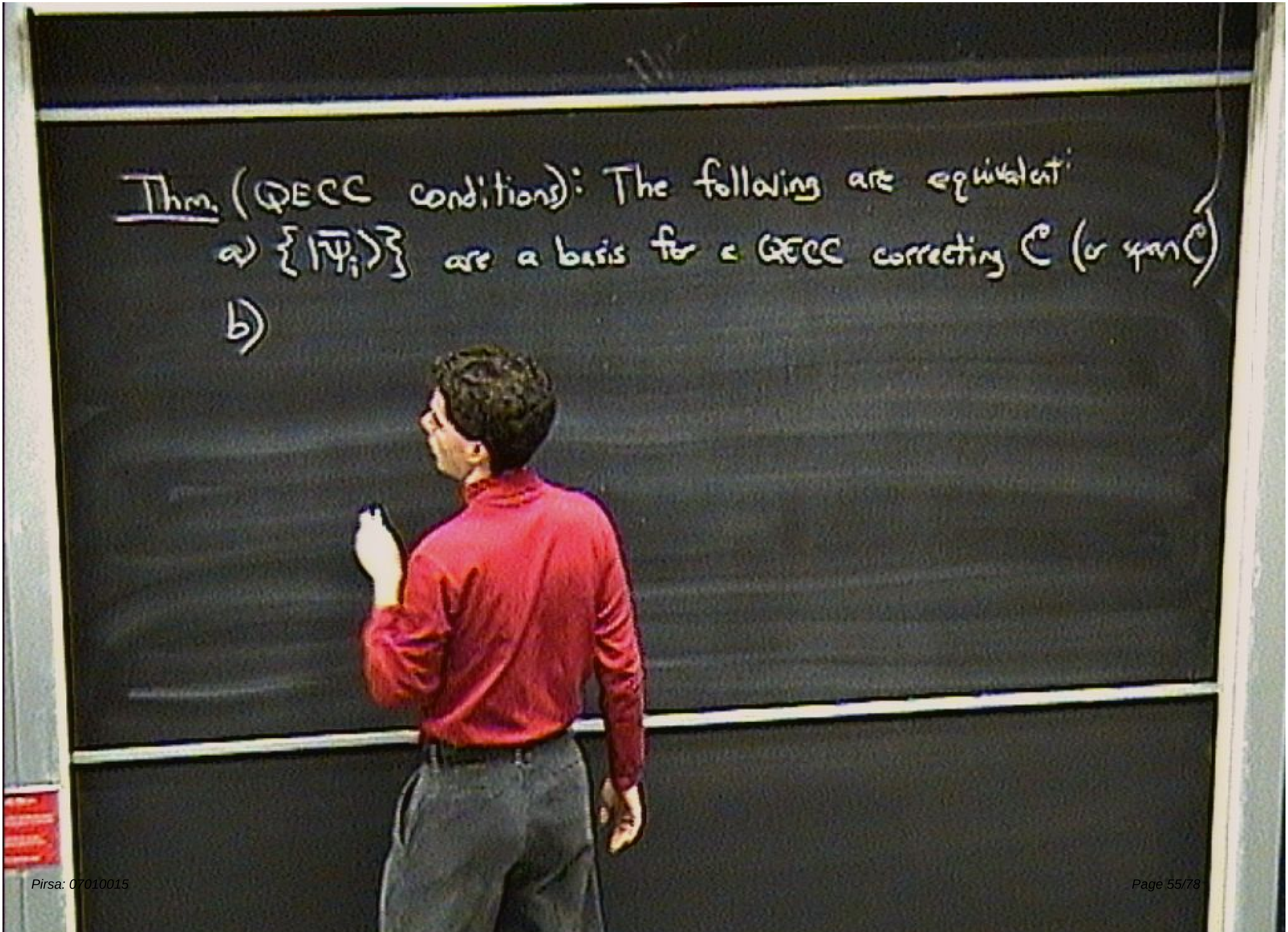
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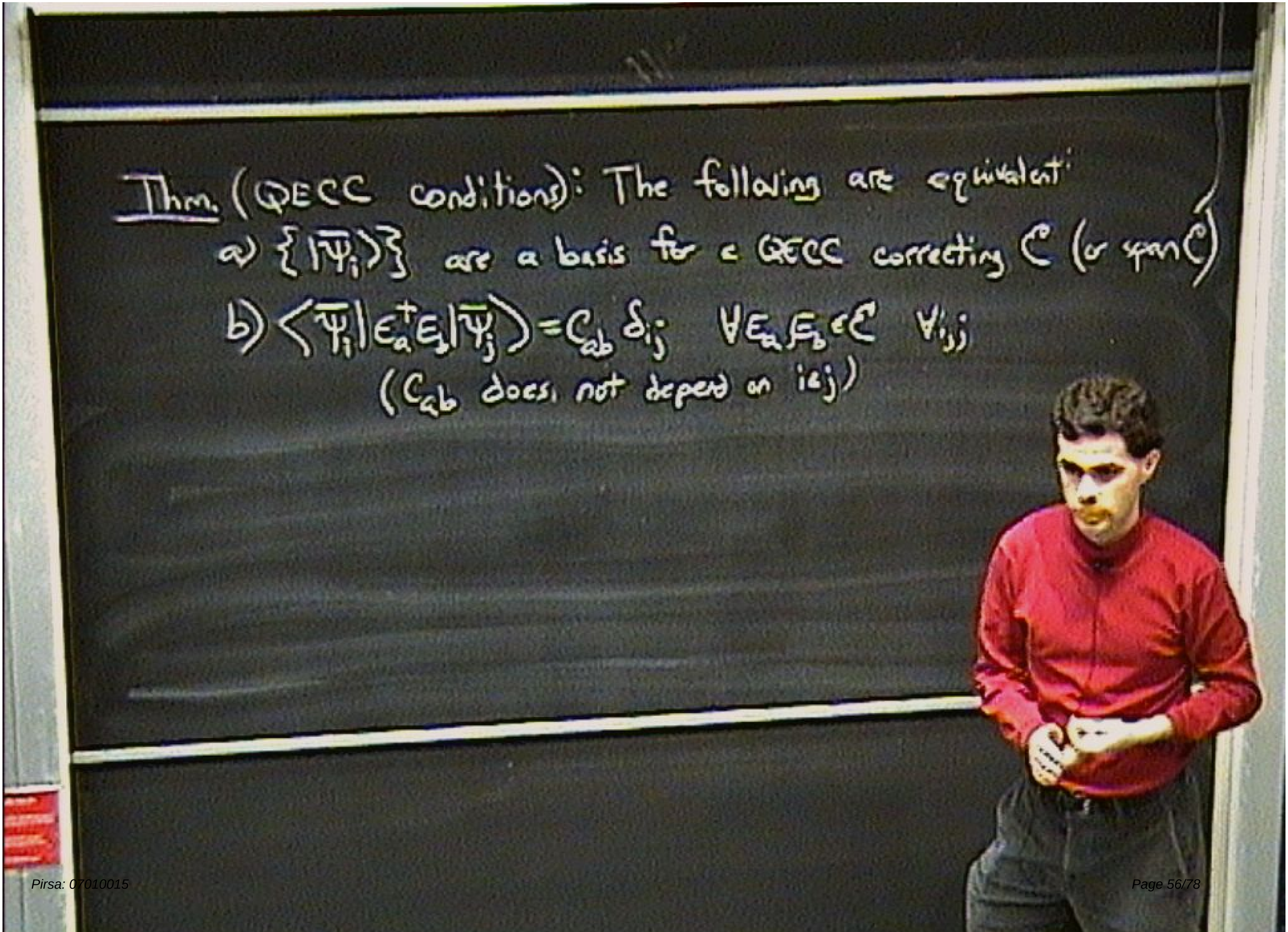
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Proof: b) $\Rightarrow$ a) Diagonalize C_{ab} by choosing basis E_c for $\mathcal{E}$, get sufficient condition

a) $\Rightarrow$ c) U recovery: $UE|\overline{\Psi}\rangle = a|\overline{\Psi}\rangle|anc_1\rangle$, $UE|\overline{\Phi}\rangle = b|\overline{\Phi}\rangle|anc_2\rangle$

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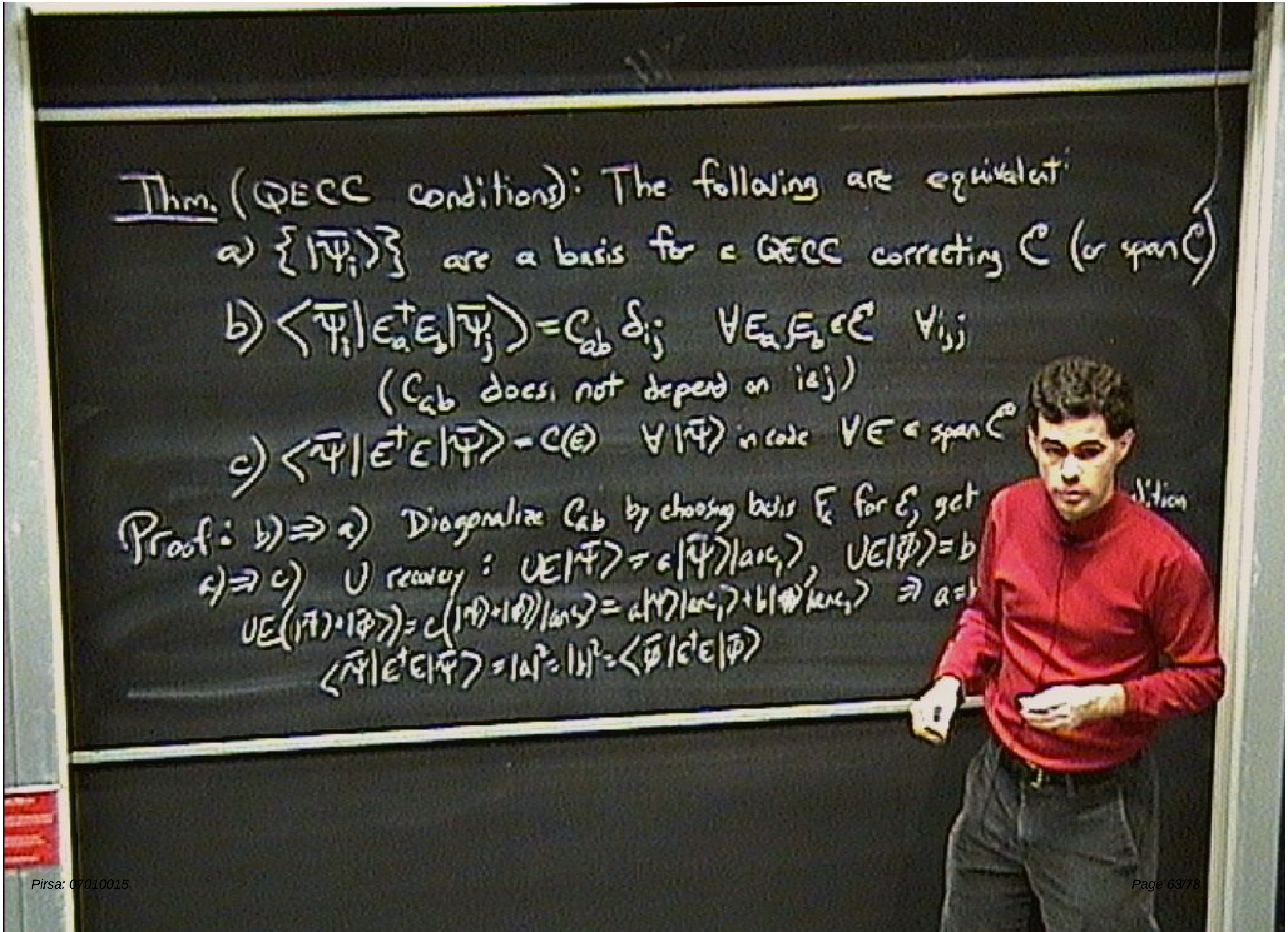
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Proof: b) $\Rightarrow$ a) Diagonalize C_{ab} by choosing basis E_a for $\mathcal{E}$, get sufficient condition

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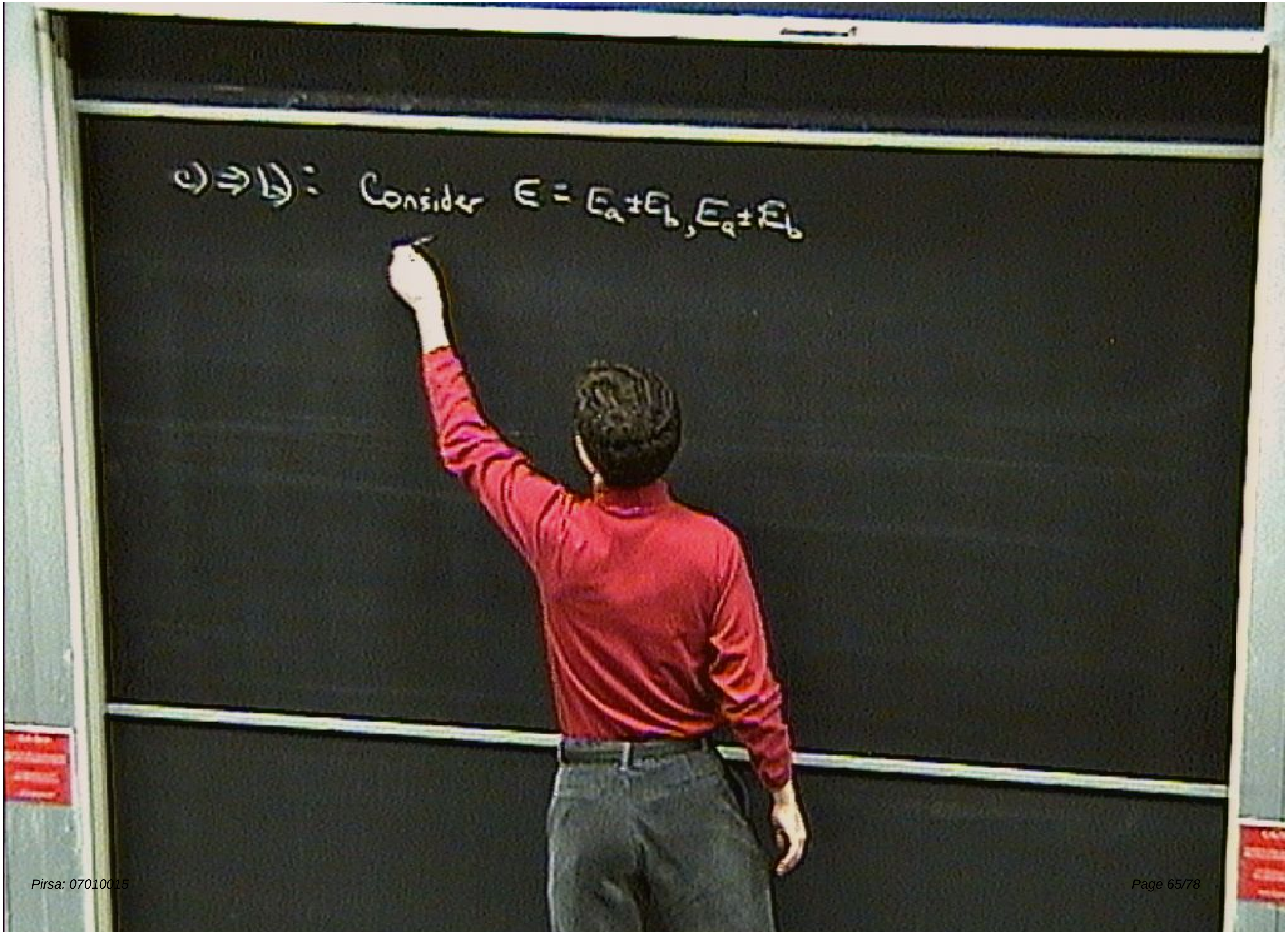
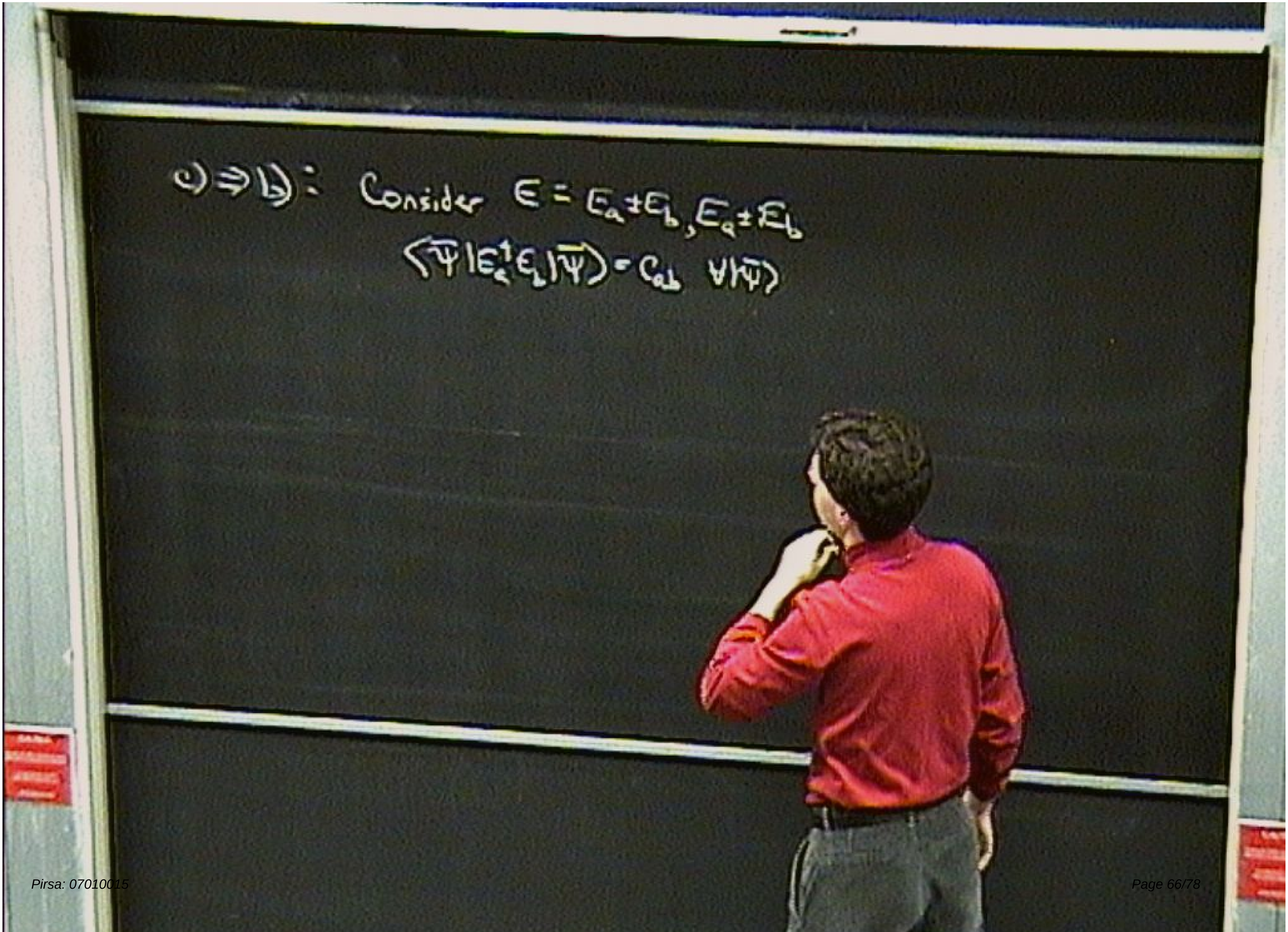

c) ⇒ b): Consider $\mathcal{E} = E_a \pm E_b, E_a \pm E_b$


c) ⇒ b): Consider E = E_a ± E_b, E_a ± iE_b
⟨Ψ|E_a^† E_b|Ψ⟩ = c_ab ∀|Ψ⟩

c) $\Rightarrow$ b): Consider $E = E_a \pm E_b, E_a \pm E_b$

$\langle \bar{\Psi} | E_a^\dagger E_b | \bar{\Psi} \rangle = C_{ab} \ \forall |\bar{\Psi}\rangle$

Consider $|\bar{\Psi}\rangle = |\bar{\Psi}_i\rangle \pm |\bar{\Psi}_j\rangle, |\bar{\Psi}_i\rangle \pm i|\bar{\Psi}_j\rangle$, get b)

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Def.: <u>Distance</u> of a QECC as minimum weight Pauli operator P for which $\langle\bar{\Psi}_i|P|\bar{\Psi}_j\rangle \neq c(P)\delta_{ij}$

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Cor: A QECC of distance $2t+1$ will correct t-qubit errors

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" " " " d will correct $(d-1)$-qubit erasure errors

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Def.: Distance of a QECC as minimum weight Pauli operator P for which $\langle \bar{\Psi}_i | P | \bar{\Psi}_j \rangle \neq c(P)\,\delta_{ij}$

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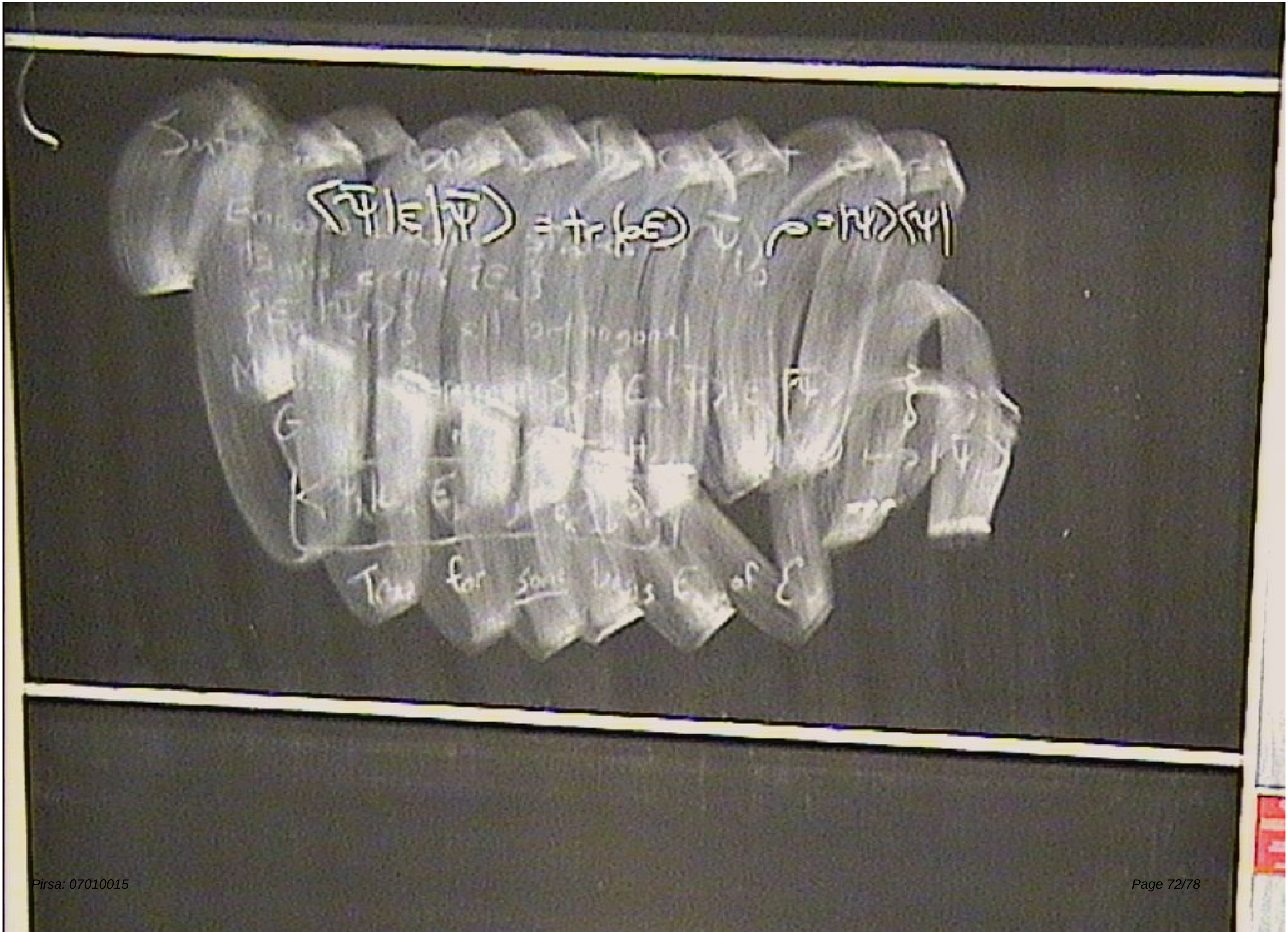
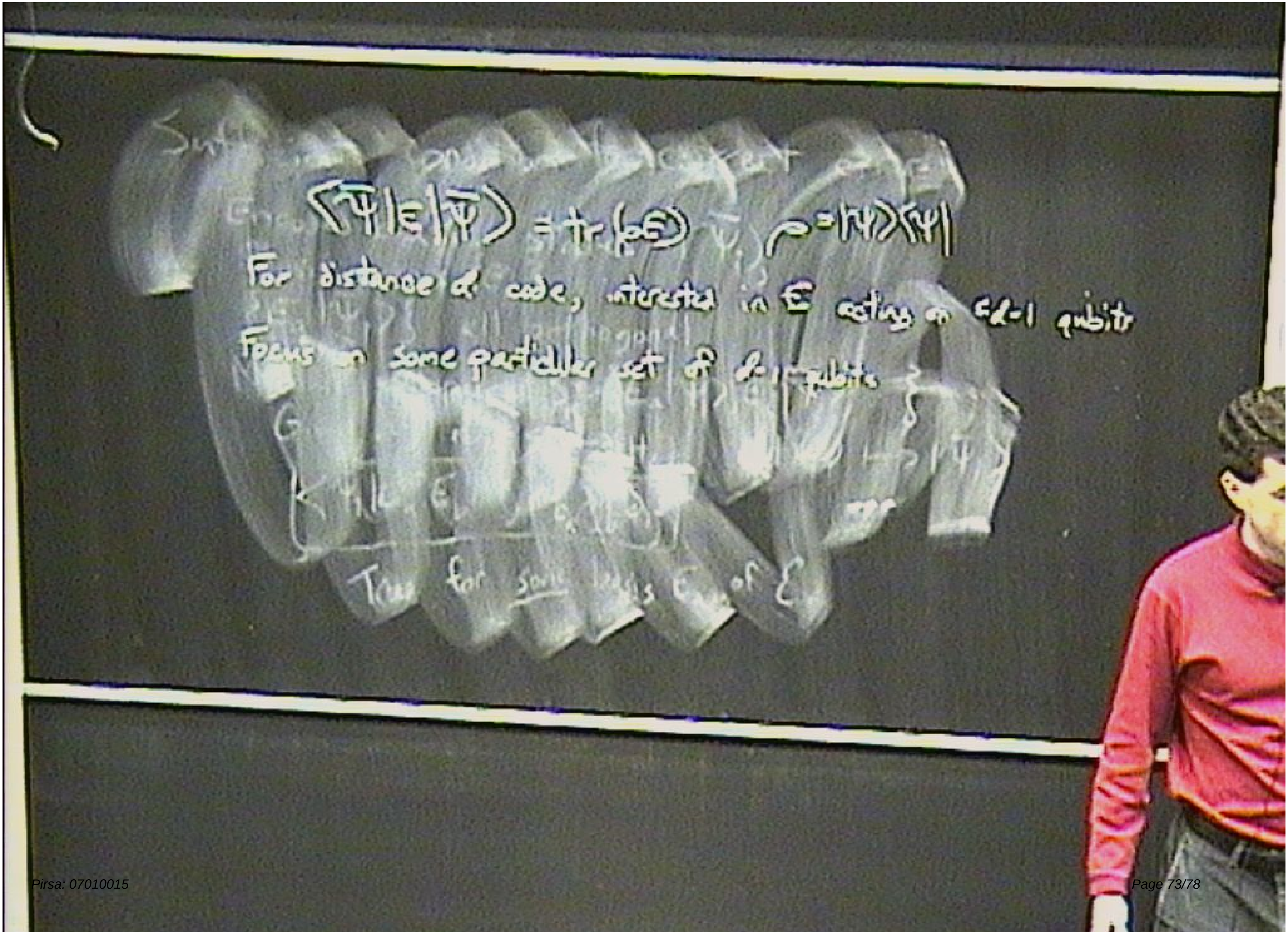
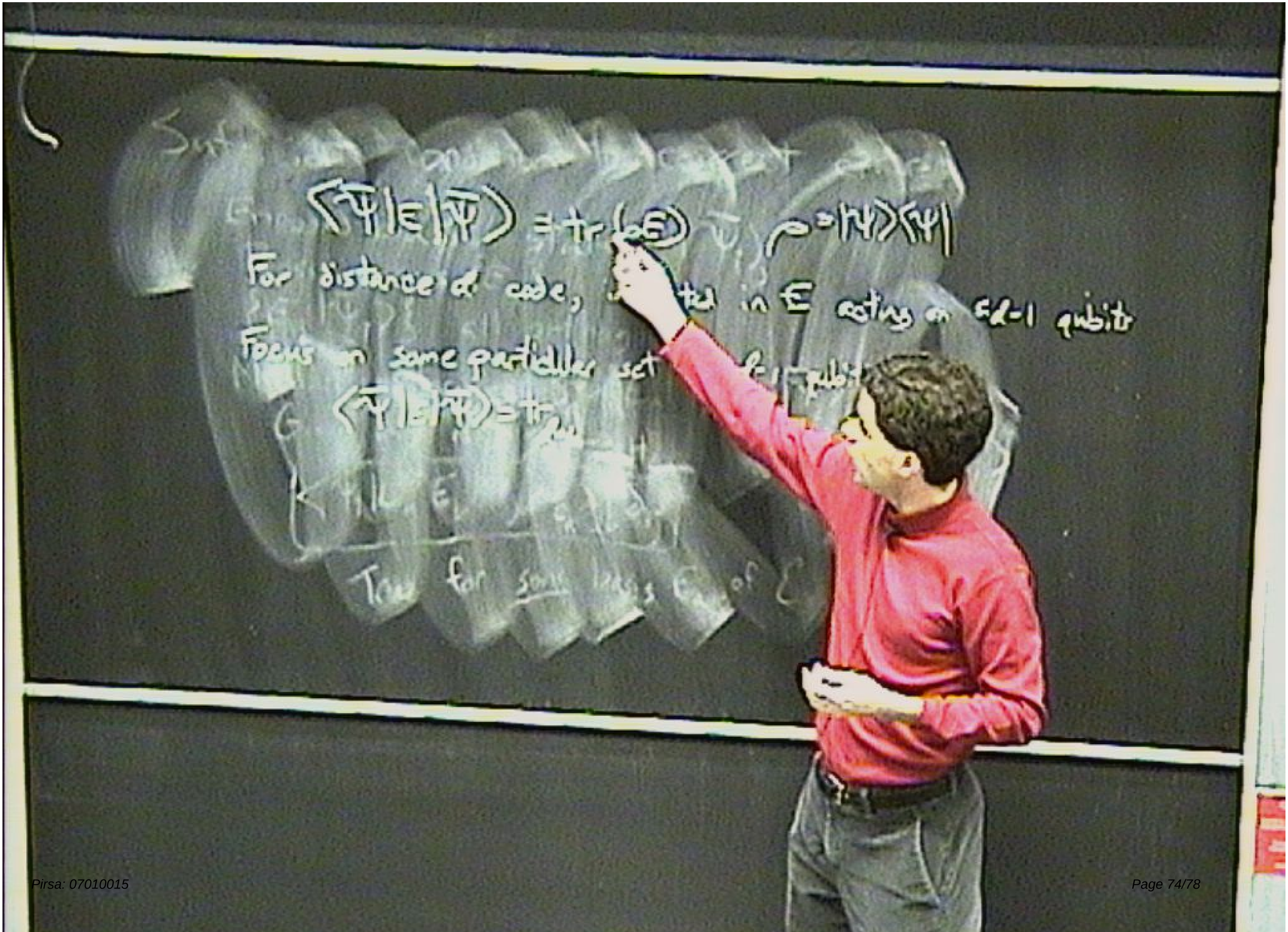
Prop.: A QECC of distance d will detect $d-1$ errors


⟨ψ|E|ψ⟩ = tr(ρE) ∀ψ ρ=|ψ⟩⟨ψ|
True for some basis C of E


$\langle\Psi|E|\Psi\rangle = tr(\rho E)$ $\rho=|\psi\rangle\langle\psi|$
For distance d code, interested in E acting on ≤d-1 qubits
Focus on some particular set of d-1 qubits


⟨Ψ|E|Ψ⟩ = tr(ρE)   ρ=|Ψ⟩⟨Ψ|
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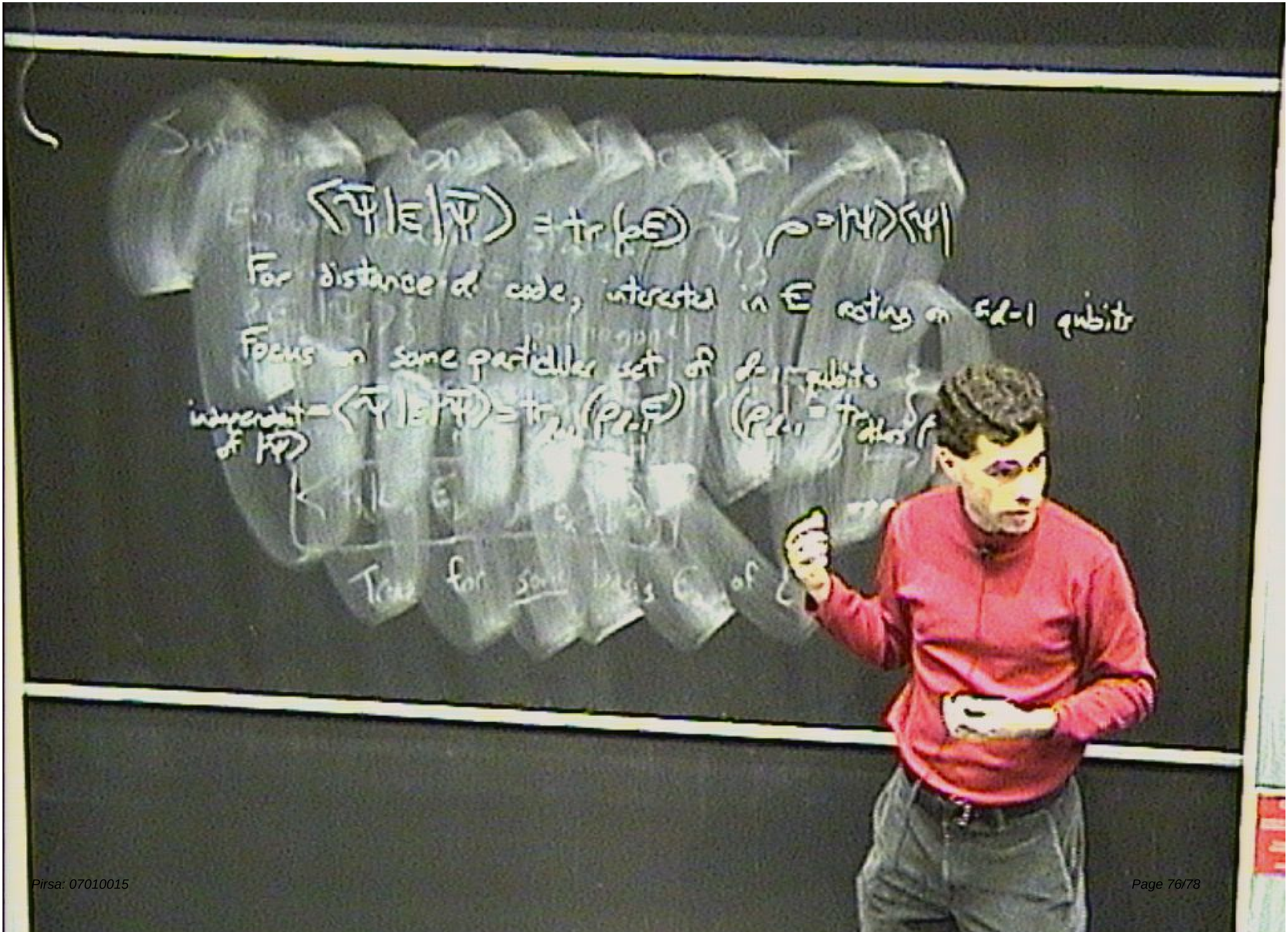
Focus on some particular set of $d-1$ qubits

$\langle\Psi|E|\Psi\rangle = \mathrm{tr}_{d-1}(\rho_{d-1} E)$ $\left(\rho_{d-1} = \mathrm{tr}_{\text{others}}\,\rho\right)$


⟨Ψ|E|Ψ⟩ = tr(ρE)   ρ=|ψ⟩⟨ψ|
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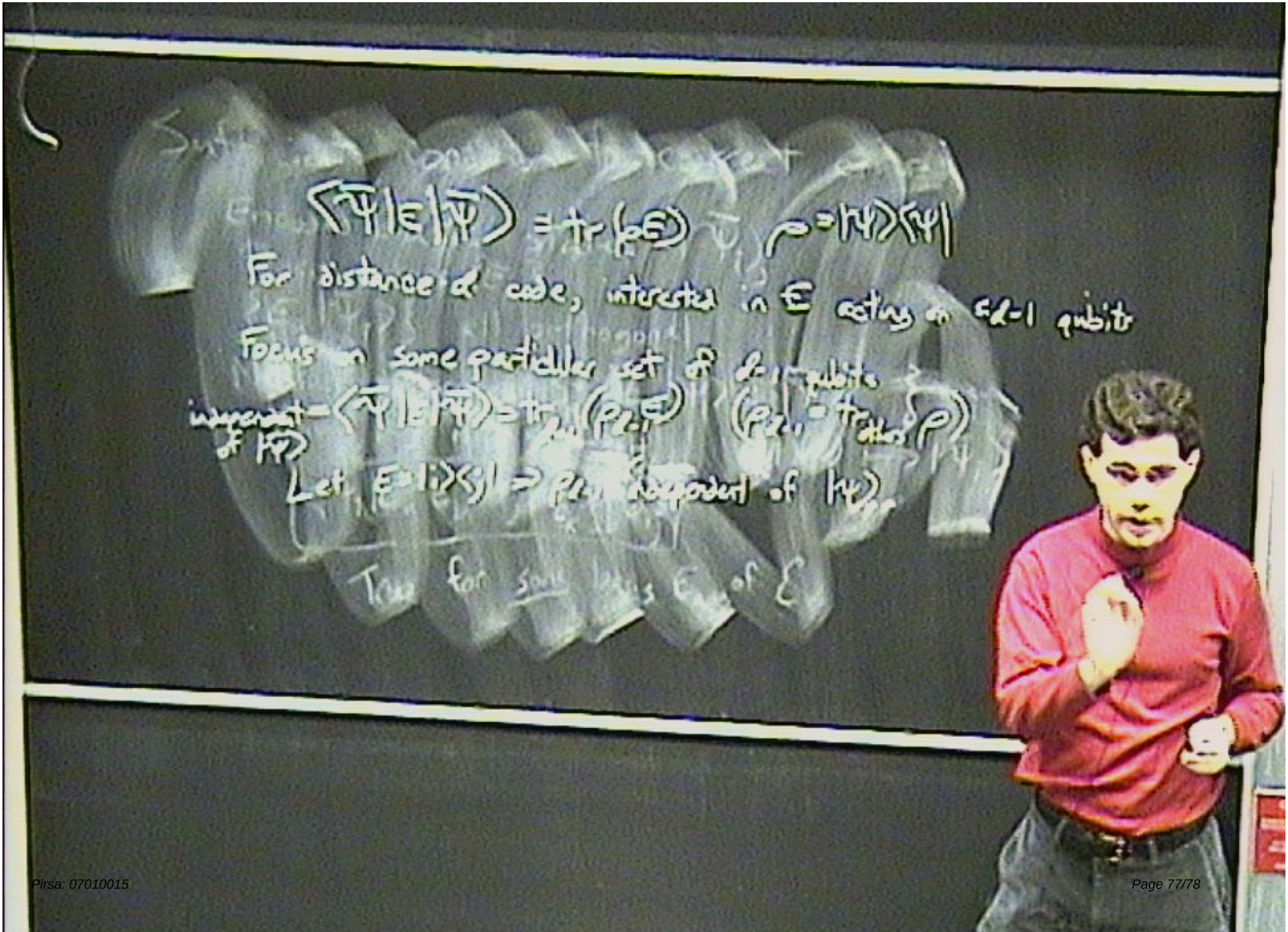
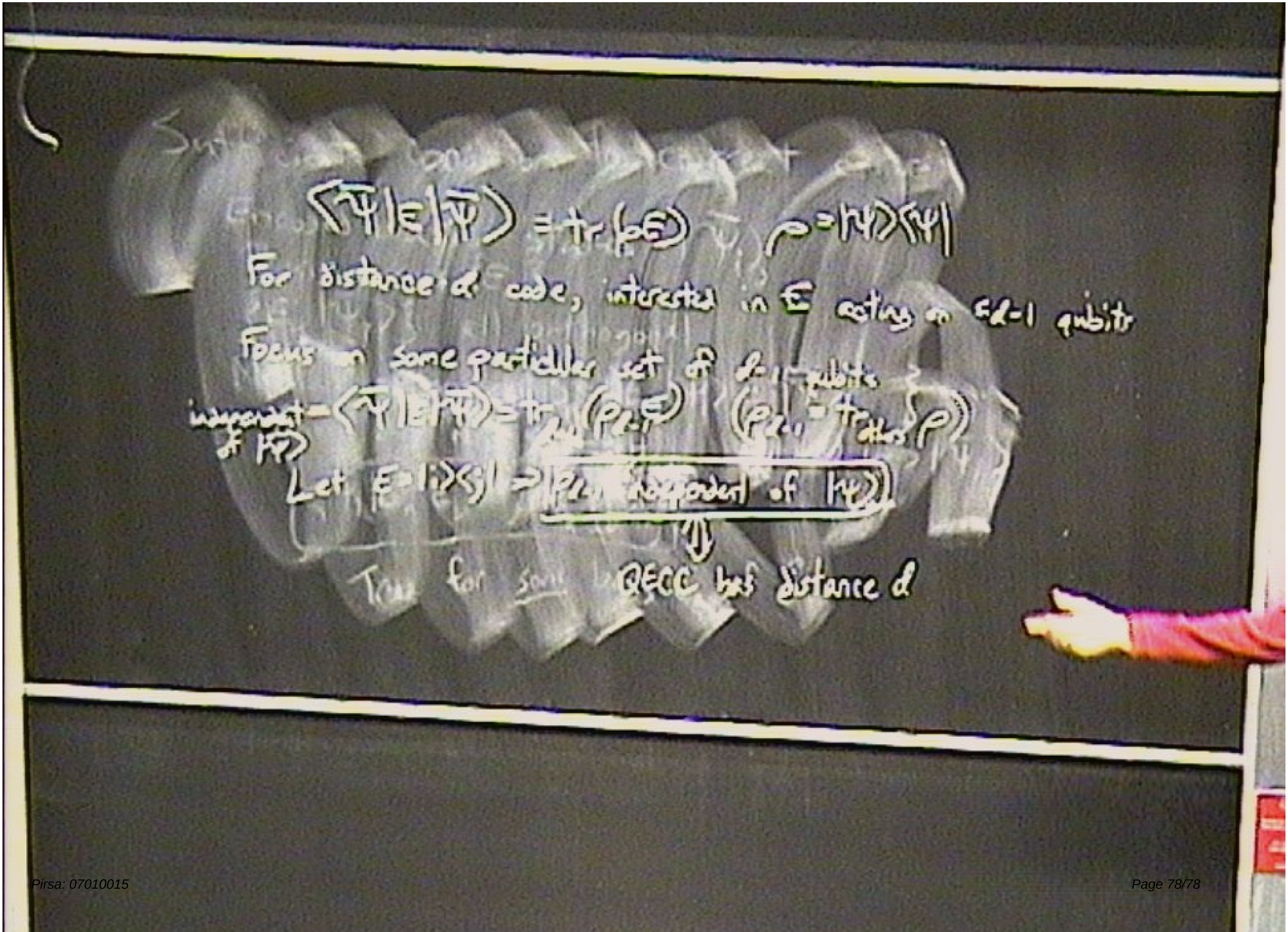
Focus on some particular set of d-1 qubits
independent of |Ψ⟩ = ⟨Ψ|E|Ψ⟩ = tr_{d-1}(ρ_{d-1}E)   (ρ_{d-1} = tr_{others}(ρ
Trace for some basis


$\langle\Psi|E|\Psi\rangle = tr(\rho E)$ $\rho = |\psi\rangle\langle\psi|$
For distance d code, interested in E acting on ≤d-1 qubits
Focus on some particular set of d-1 qubits
independent of $|\psi\rangle$ — $\langle\Psi|E|\Psi\rangle = tr_{d-1}(\rho_{d-1}E)$ $(\rho_{d-1} = tr_{others}\,\rho)$
Let $E = |i\rangle\langle j| \Rightarrow \rho_{d-1}$ independent of $|\psi\rangle$


$\langle \Psi | E | \Psi \rangle = tr(\rho E)$ $\rho = |\psi\rangle\langle\psi|$
For distance d code, interested in E acting on $\leq d-1$ qubits
Focus on some particular set of $d-1$ qubits
independent of $|\psi\rangle$ $\langle \Psi | E | \Psi \rangle = tr_{d-1}(\rho_{d-1} E)$ $(\rho_{d-1} = tr_{others}\, \rho)$
Let $E = |i\rangle\langle j| \Rightarrow$ $\boxed{\rho_{d-1} \text{ independent of } |\psi\rangle}$
$\Updownarrow$
QECC has distance d