Title: Geometric measure of entanglement and its applications to multi-partite states and quantum phase transitions

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Abstract: A multi-partite entanglement measure is constructed via the distance or angle of the pure state to its nearest unentangled state. The extention to mixed states is made via the convex-hull construction, as is done in the case of entanglement of formation. This geometric measure is shown to be a monotone. It can be calculated for various states, including arbitrary two-qubit states, generalized Werner and isotropic states in bi-partite systems. It is also calculated for various multi-partite pure and mixed states, including ground states of some physical models and states generated from quantum alogrithms, such as Grover's. A specific application to a spin model with quantum phase transistions will be presented in detail.The connection of the geometric measure to other entanglement properties will also be discussed.

Geometric measure of entanglement & its applications to multipartite states and quantum phase transitions

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Outline

I. Introduction: what is entanglement?

II. Review of standard entanglement measures

1. Entanglement of distillation
2. Entanglement of formation/cost
3. Relative entropy of entanglement

III. Geometric measure of entanglement

1. Definition
2. Benchmark examples
3. Connections of GME to other measures

IV. Application to a many-body model

What is Entanglement?

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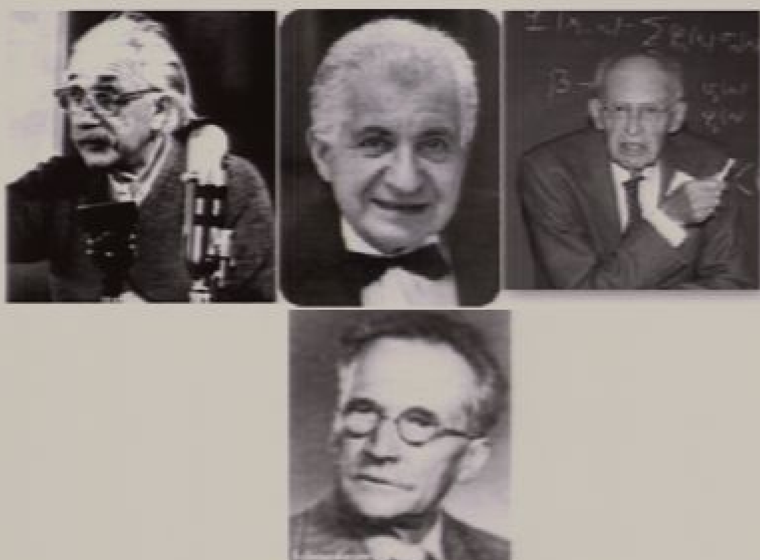
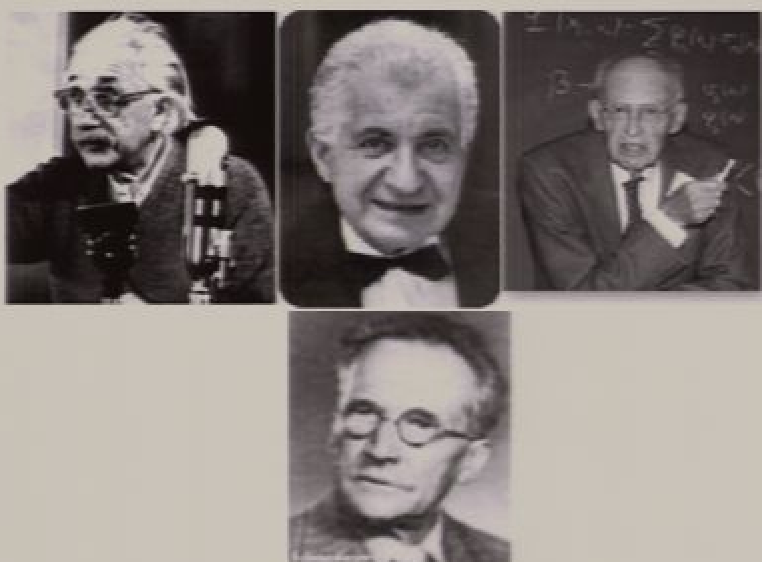


{EPR, Schrodinger, …}

Philosophical

What is Entanglement?



{EPR, Schrodinger, …} **{Bell, Bohm, GHZ,…}**

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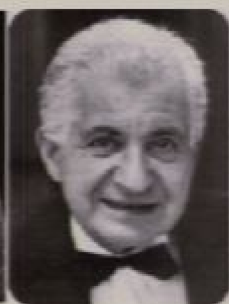
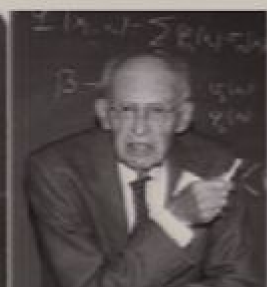




{EPR, Schrodinger, …} {Bell, Bohm, GHZ,…} {Akert, Bennett, Shor,…}

Philosophical → Useful

Entangled vs separable

Entangled vs separable

- A pure state is separable (unentangled) if it can be written as a product state $|\phi\rangle = |\varphi^A\rangle \otimes |\varphi^B\rangle \otimes |\varphi^C\rangle \cdots$

E.g. $|HV\rangle \equiv |H\rangle \otimes |V\rangle$

$$|\uparrow\uparrow\rangle + |\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle + |\downarrow\downarrow\rangle = \left(|\uparrow\rangle + |\downarrow\rangle\right)\left(|\uparrow\rangle + |\downarrow\rangle\right)$$

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- A state is entangled iff it is not separable

E.g. $|\Phi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right)$ $\quad |\Psi^{\pm}\rangle \equiv \frac{1}{\sqrt{2}}\left(|01\rangle \pm |01\rangle\right)$

$$|GHZ\rangle \equiv \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right), \qquad |W\rangle \equiv \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right)$$

Mixed entangled states

Mixed entangled states

- A mixed state is separable if it can be written as a mixture of separable pure states

$$\rho = \sum_i p_i \left|\varphi_i^A\right\rangle\left\langle\varphi_i^A\right| \otimes \left|\varphi_i^B\right\rangle\left\langle\varphi_i^B\right| \otimes \cdots$$

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- A mixed entangled state has no such decomposition

$$\rho_{\text{entangled}} \neq \sum_i p_i \left|\varphi_i^A\right\rangle\left\langle\varphi_i^A\right| \otimes \left|\varphi_i^B\right\rangle\left\langle\varphi_i^B\right| \otimes \left|\varphi_i^C\right\rangle\left\langle\varphi_i^C\right| \otimes \cdots$$

E.g. $\rho = r\left|\Psi^-\right\rangle\left\langle\Psi^-\right| + (1-r)|00\rangle\langle 00|$

$$\left|\Psi^\pm\right\rangle \equiv \frac{1}{\sqrt{2}}\left(|01\rangle \pm |01\rangle\right)$$

A toy model

- Two spins with antiferromagnetic interaction:

$$H = J\vec{\sigma}^1 \cdot \vec{\sigma}^2, \text{ with } J > 0$$

[Nielsen '98]

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- Eigenstates: $|\uparrow\uparrow\rangle,\ |\downarrow\downarrow\rangle,\ (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2},\ (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$
- Eigenvalues: $J, J, J, -3J$

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- The density matrix is a Werner state

$$\rho = \frac{1}{Z}\sum_n e^{-\beta E_n} |n\rangle\langle n| = \frac{1-r}{4} I_{4\times4} + r|\Psi^-\rangle\langle\Psi^-|$$

$$r = (e^{3\beta J} - e^{-\beta J})/(e^{3\beta J} + 3e^{-\beta J})$$

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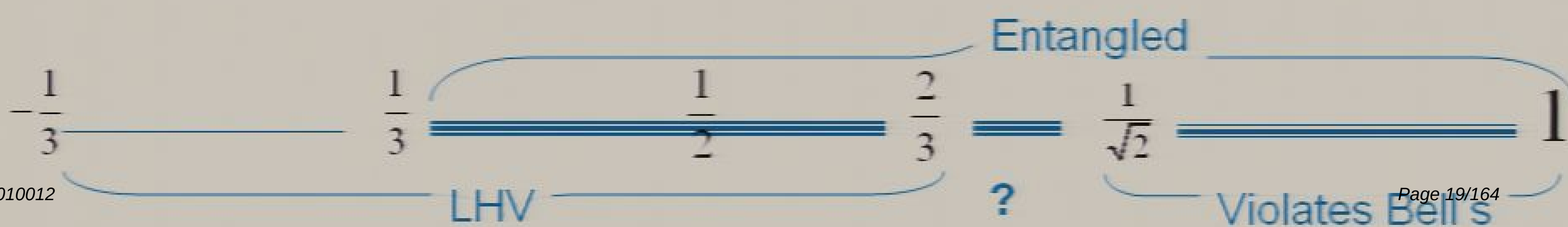
➔ When is this entangled?

Properties of Werner state

$$\rho_{Werner}(r) \equiv r\left|\Psi^-\right\rangle\left\langle\Psi^-\right| + (1-r)I/4$$

- Peres' positive partial transpose (PPT) criterion:
  Werner state is entangled when r >1/3
- Via Horodeckis' results:
  Werner state violates Bell-CHSH if $r > 1/\sqrt{2} \approx 0.707$
- Werner '89: if $r \leq 1/2$, the state can be described by a LHV theory;
  Doherty and Terhal '02 further pushed to $r \leq 2/3$



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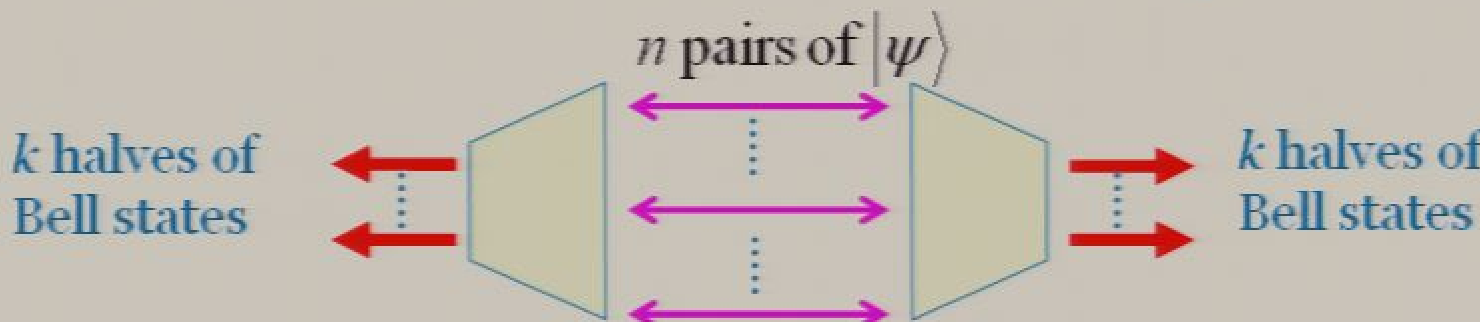
Entanglement of distillation

[Bennett et al. '96]



[Only local operations and classical communication are allowed]

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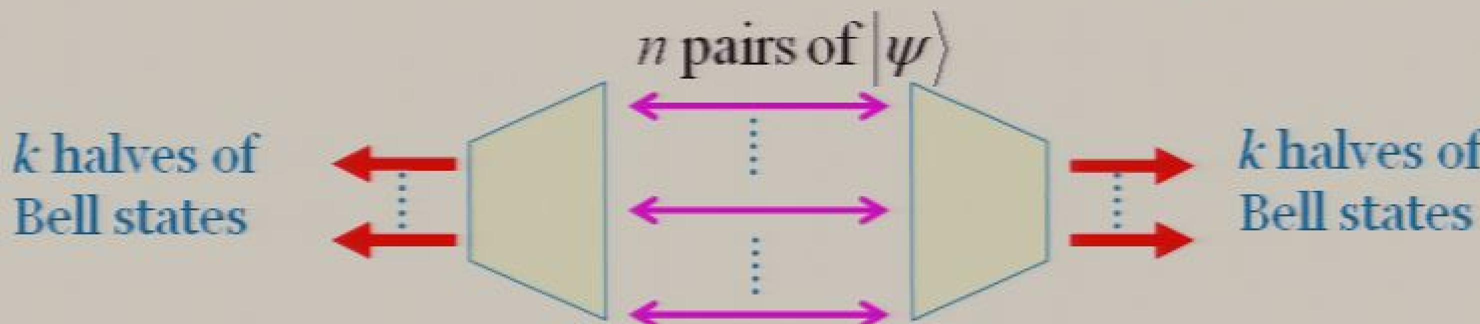

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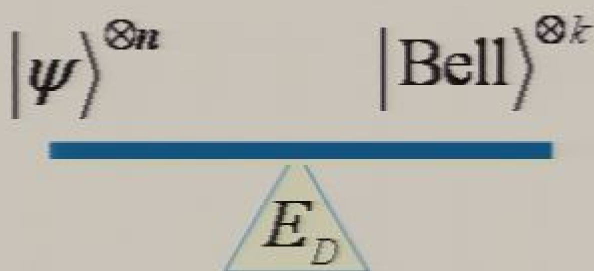
- $E_D(\psi) \equiv \lim_{n\to\infty} (k/n)$ (idea also applies to mixed states)



For pure state ψ , $E_D(\psi) = S_V\left(Tr_B |\psi\rangle\langle\psi|\right)$

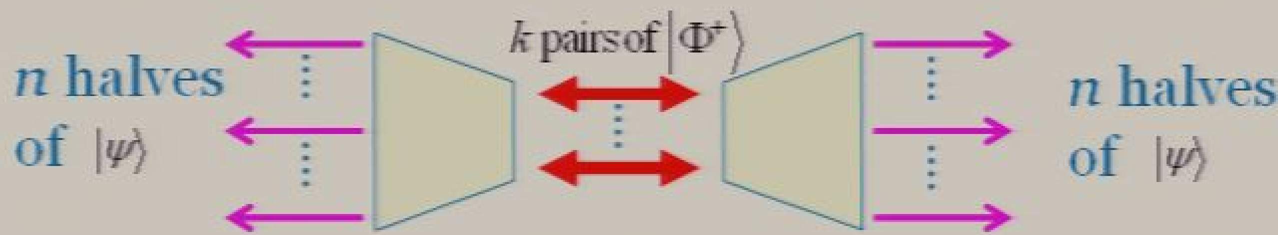
where S_V is the von Neumann entropy

$$S_V(\rho) \equiv -Tr\rho\log\rho$$

Entanglement cost

Dilution process (reverse of distillation):

[Bennett et al. '97]



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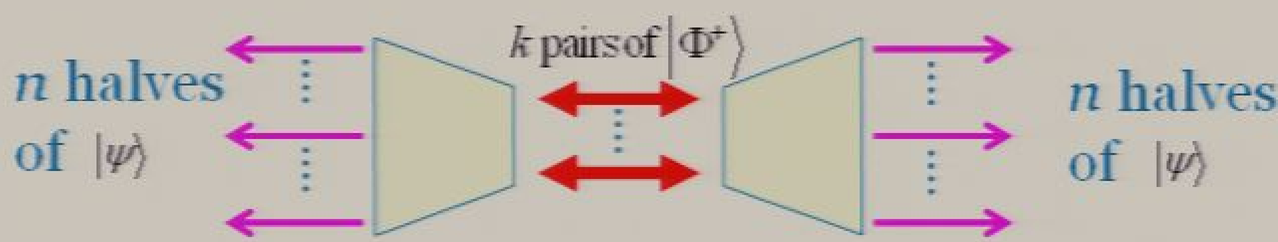

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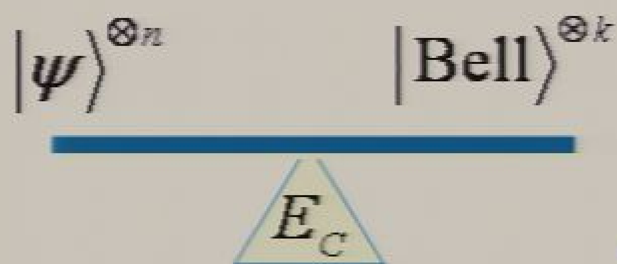

For bipartite pure states,

$$E_C = E_D = S_V\left(Tr_B|\psi\rangle\langle\psi|\right)$$

➔ dilution and distillation are reversible!

Entanglement of formation

Entanglement of formation

- Bennett et al. ['96] constructed an average quantity for mixed states:

$$E_F(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_C(\psi_i), \quad \text{with } \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

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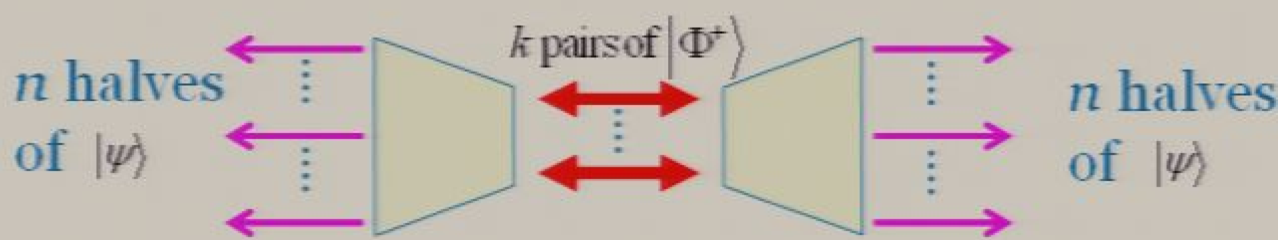

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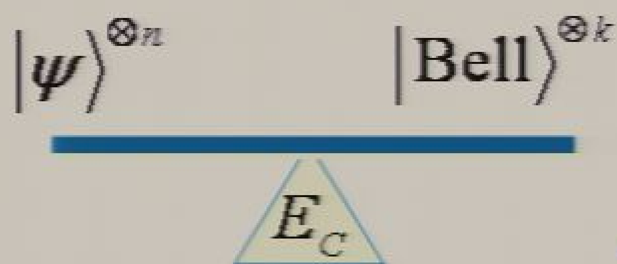

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- Wootters ['98]: an analytical formula of E_F for arbitrary two-qubit states

- Other bi-partite states in higher dimensions (dxd) that allow analytical formulas:

  ① Generalized Werner states: [Vollbrecht & Werner '01]

  ② Isotropic states: [Terhal & Vollbrecht '00]

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- No asymptotic reversible inter-conversion for multi-partite pure entangled states

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- No explicit generalizations of E_D, E_C and E_F yet

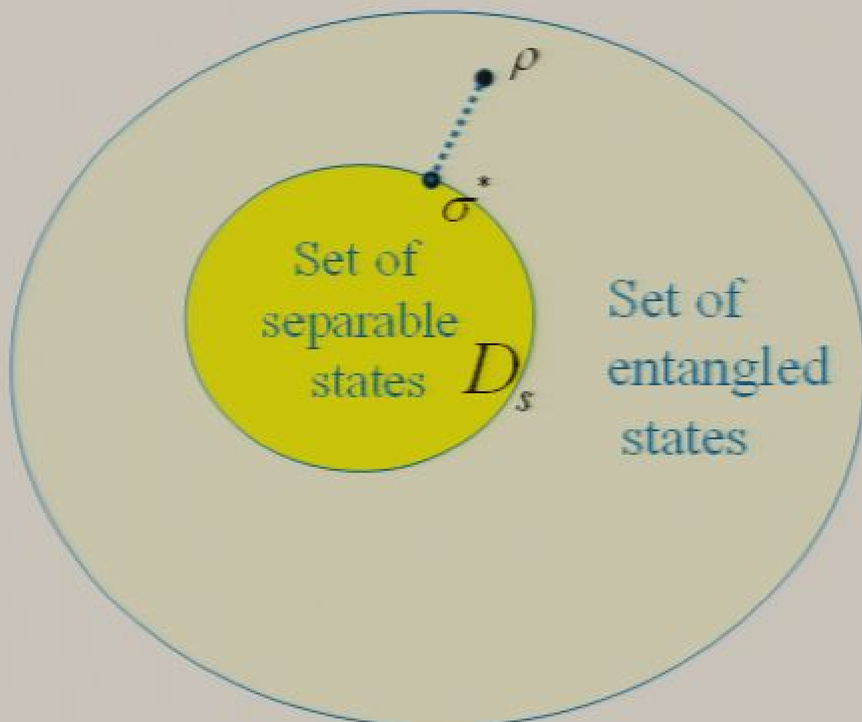
A general multipartite measure: Relative entropy of entanglement

[Vedral et al. '97]

- Define entanglement via relative entropy:

$$E_R(\rho) \equiv \min_{\sigma \in D_s} D\left(\rho \| \sigma\right)$$

$$D(\rho \| \sigma) \equiv \mathrm{Tr}\left[\rho \log_2(\rho / \sigma)\right]$$



$$D_S = \left\{ \sum_i p_i \left|\varphi_i^A\right\rangle\left\langle\varphi_i^A\right| \otimes \left|\varphi_i^B\right\rangle\left\langle\varphi_i^B\right| \otimes \cdots \right\}$$

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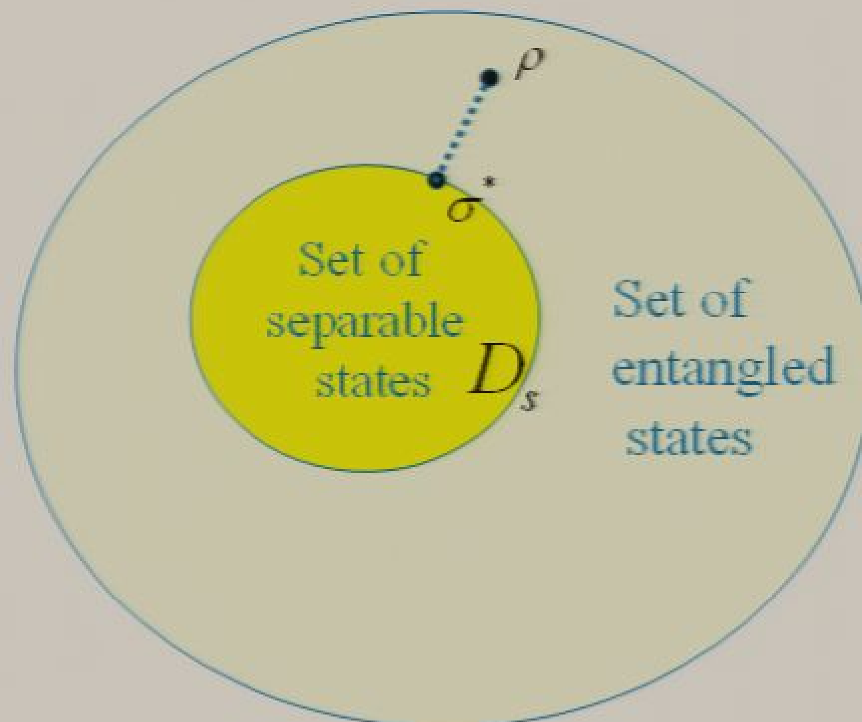

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- Still difficult to calculate even for pure states

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \quad E_R = ?$$

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Why need new measures?

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Why need new measures?

- Some intuitive measures fail for more than two parties

- Existing multi-partite measures (e.g. E_R) are difficult to calculate (even for pure states)

- Want a measure: simple and manageable, at least for pure states

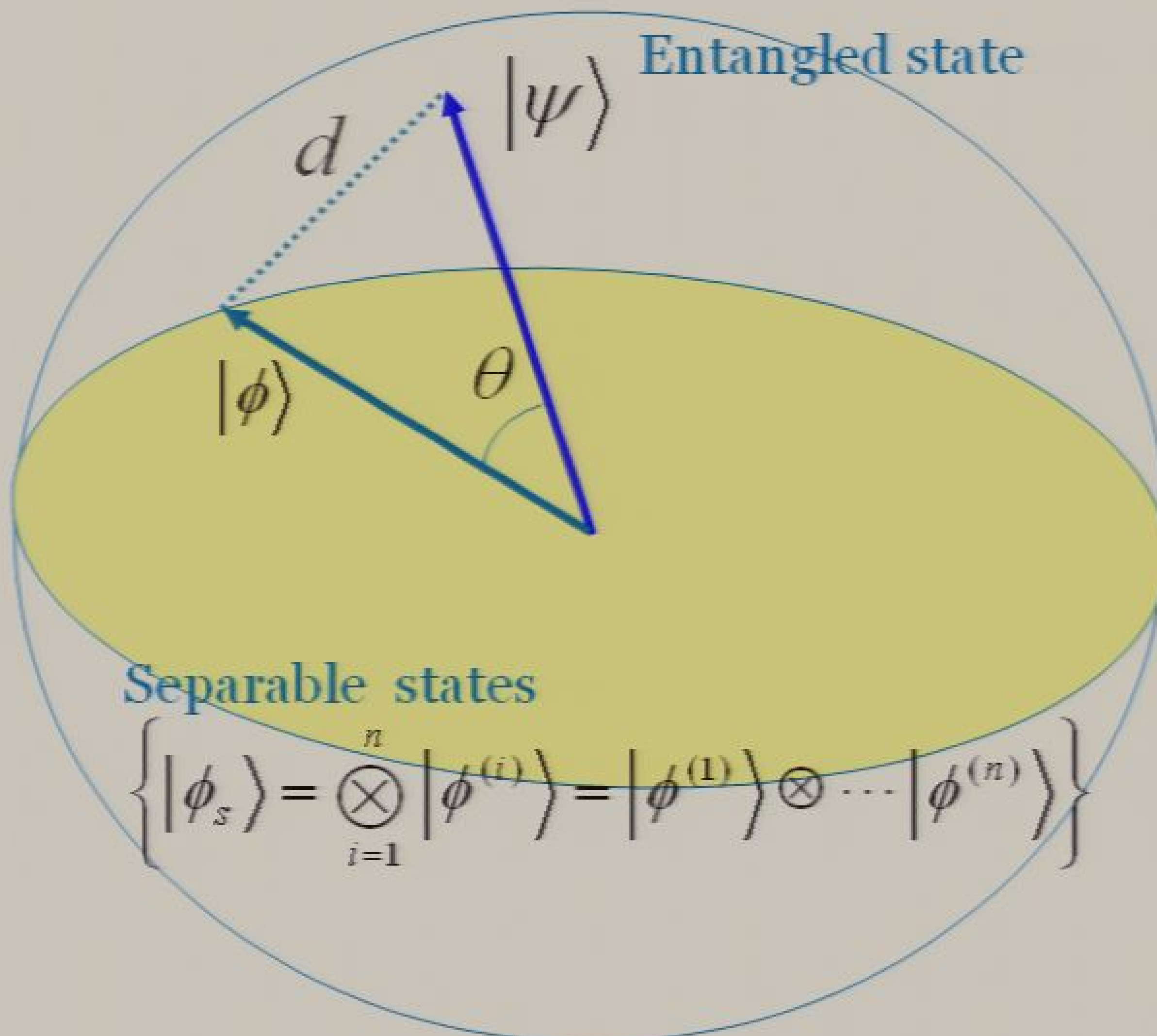
- Study of new measures may help understand existing measures better

Picture for the geometric measure

Pure states

$$\Lambda_{\max}(\psi) \equiv \max_{\phi_s} \left| \langle \phi_s | \psi \rangle \right|$$

$$= \cos\theta_{\min}$$



$$\left\{ |\phi_s\rangle = \bigotimes_{i=1}^{n} \left|\phi^{(i)}\right\rangle = \left|\phi^{(1)}\right\rangle \otimes \cdots \left|\phi^{(n)}\right\rangle \right\}$$

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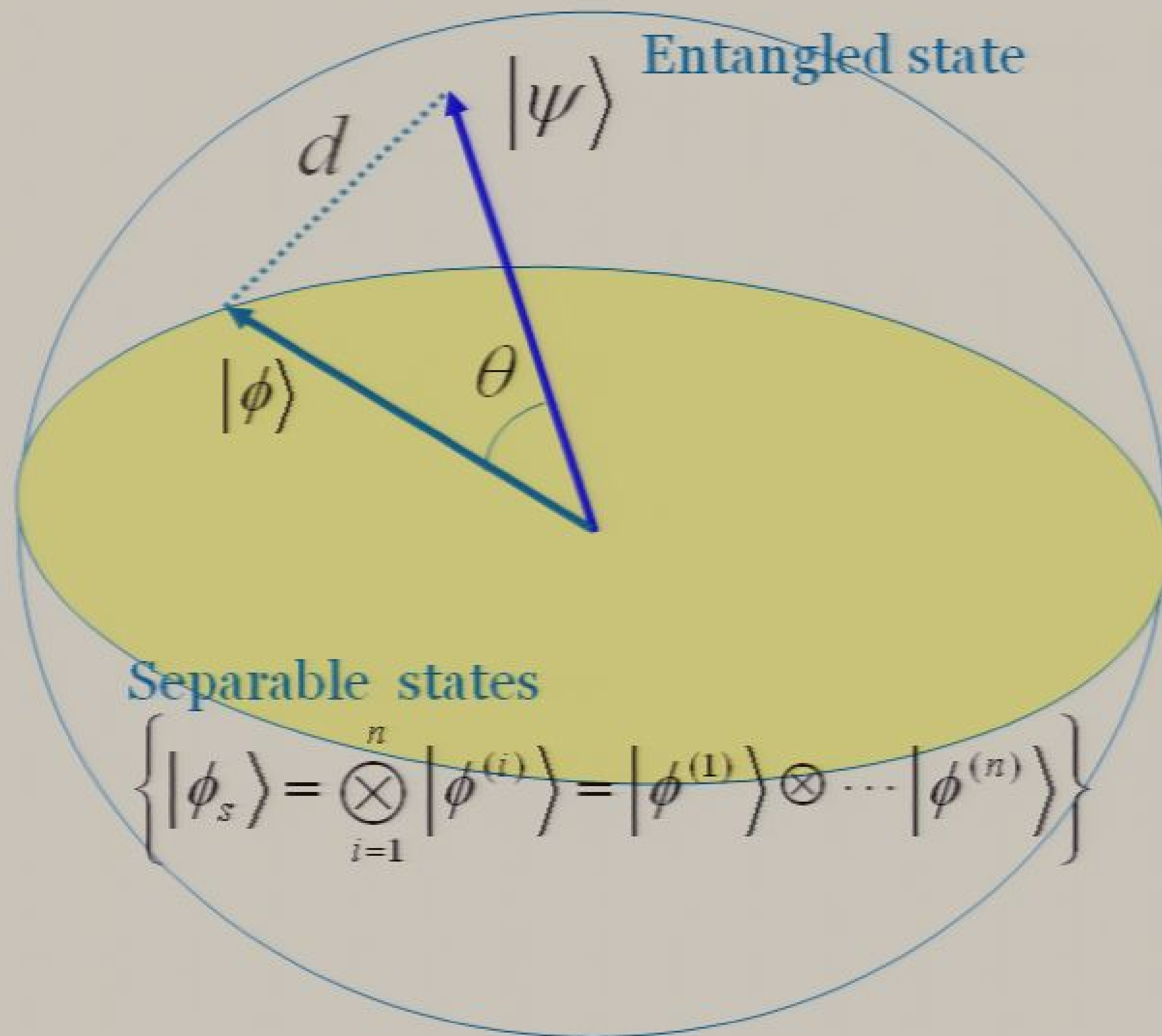

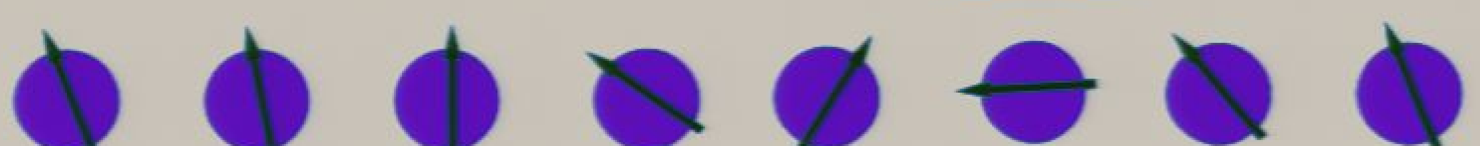

Geometric measure of entanglement

<u>Pure states</u>

[Shimony '95, Barnum & Linden '01, Wei & Goldbart '03]

- A n-partite pure state described by

$$|\psi\rangle = \sum_{p_1 p_2 \cdots p_n} \chi_{p_1 p_2 \cdots p_n} \left|e^{(1)}_{p_1}\right\rangle \otimes \left|e^{(2)}_{p_2}\right\rangle \otimes \cdots \otimes \left|e^{(n)}_{p_n}\right\rangle$$

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- The larger $\Lambda_{\max}(\psi)$ is, the less entangled $|\psi\rangle$ is

GME and Groverian measure

- Groverian measure [Biham, Nielsen & Osborne '02]

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Derived from a modified Grover search
with a general initial state Φ and general local unitaries

$$P_{\max} = \max_{|e_1,\ldots,e_n\rangle} |\langle e_1,\ldots,e_n|\phi\rangle|^2 + O\left(\frac{1}{\sqrt{N}}\right)$$

$$G(\phi) \equiv \sqrt{1 - P_{\max}}$$

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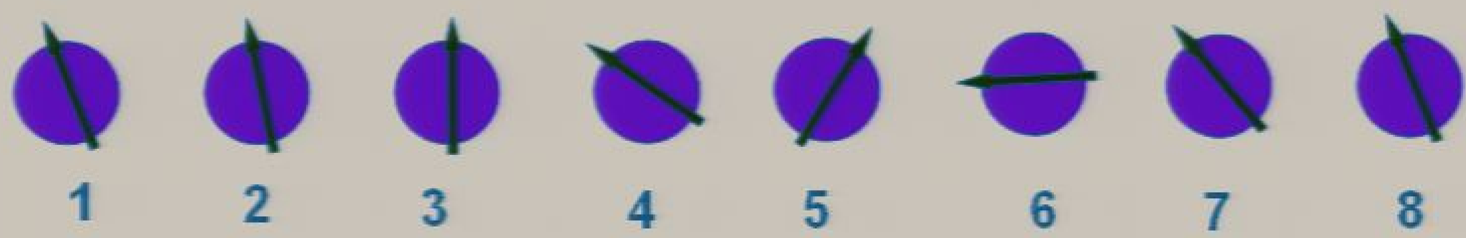
- Equivalent to GME if ignore $O\left(1/\sqrt{N}\right)$

Entanglement among partitions



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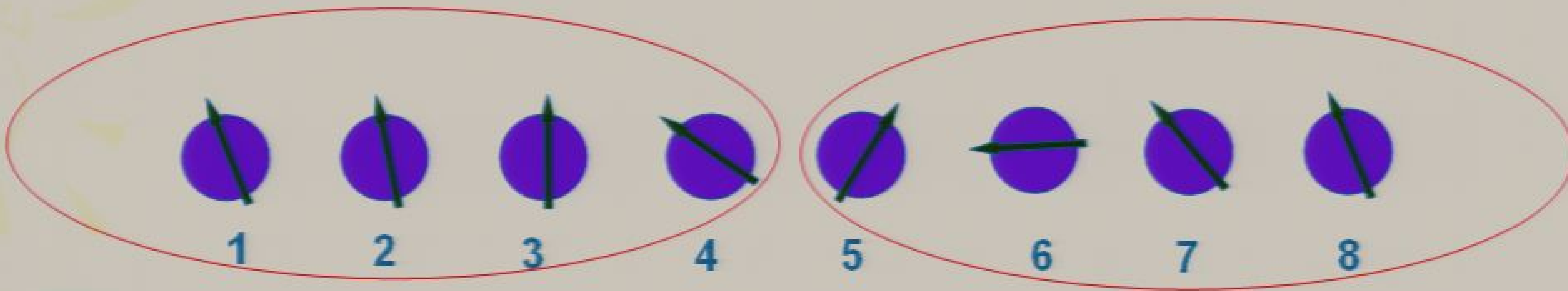
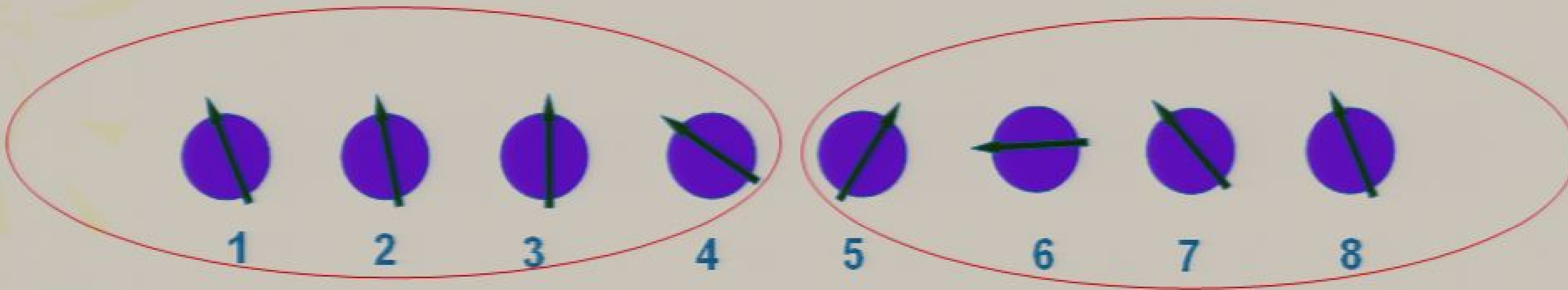


- To study entanglement between two groups: {1,2,3,4} and {5,6,7,8}, take the separable state

$$\left|\phi_s\right\rangle = \left|\phi^{(1,2,3,4)}\right\rangle \otimes \left|\phi^{(5,6,7,8)}\right\rangle$$

Entanglement among partitions



- To study entanglement between two groups: {1,2,3,4} and {5,6,7,8}, take the separable state

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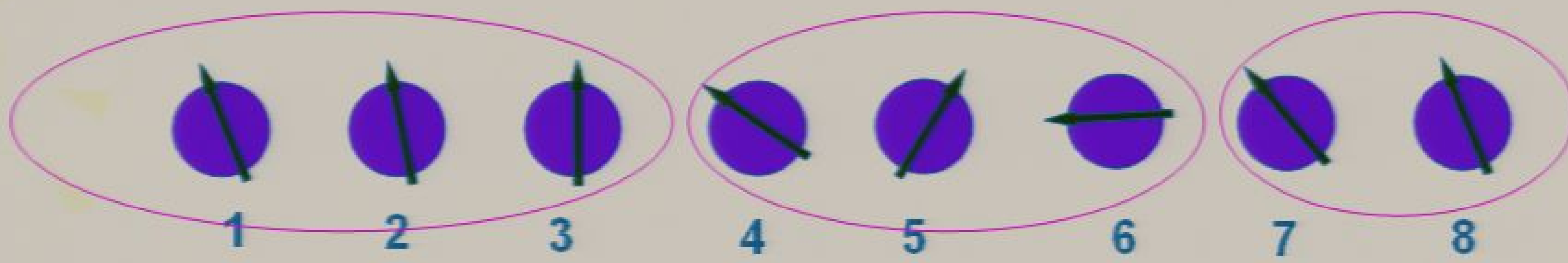

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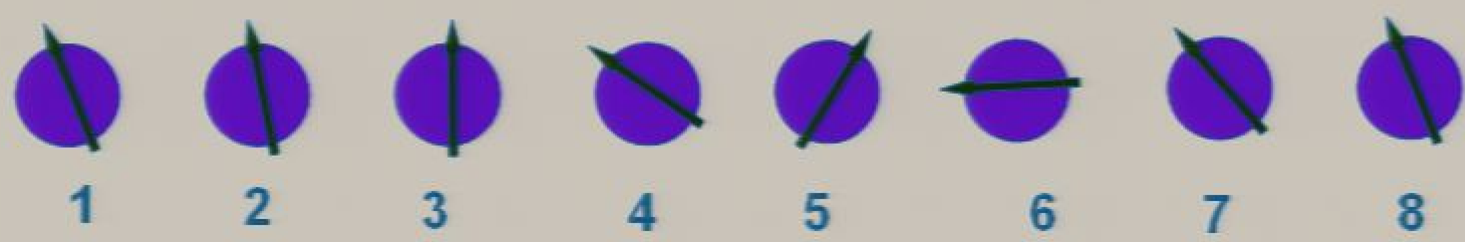
- To study entanglement among {1,2,3}, {4,5,6}, and {7,8}, take the separable state

$$\left|\phi_s\right\rangle = \left|\phi^{(1,2,3)}\right\rangle \otimes \left|\phi^{(4,5,6)}\right\rangle \otimes \left|\phi^{(7,8)}\right\rangle$$

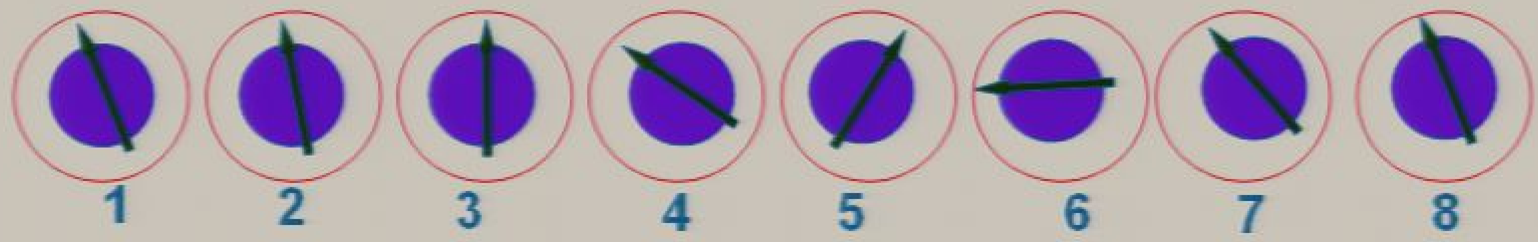
Global entanglement



- Making the finest partition in the separable state

$$\left|\phi_s\right\rangle = \left|\phi^{(1)}\right\rangle \otimes \left|\phi^{(2)}\right\rangle \otimes \cdots \otimes \left|\phi^{(8)}\right\rangle$$

Global entanglement



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we are studying global entanglement of the system

GME: Two specific forms

[Wei & Goldbart '03]

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$$\mathcal{E}(\psi) \equiv \lim_{N\to\infty} \frac{1}{N} E_{\log_2}(\psi)$$

GME: Mixed states via convex hull

$$E_{mixed}(\rho) \equiv \min_{\{p_i, \psi_i\}} \sum_i p_i E_{pure}(\psi_i), \quad \text{with } \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

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- It complicates the calculation for mixed-state entanglement

GME is a monotone

- We can show $E_{\sin^2}(\rho)$ satisfies the following

[Wei & Goldbart '03]

Criteria for good entanglement measures

[Vedral et al. '97, Vidal '00, Horodecki et al. '00]

1. (a) $E(\rho) \geq 0$; (b) $E(\rho) = 0$, if ρ is not entangled
2. Local unitary transformations should not change the amount of entanglement
3. Local operations and classical communication should not increase the expectation value of entanglement
4. Entanglement cannot increase under discarding information

$$\sum_i p_i E(\rho_i) \geq E\left(\sum_i p_i \rho_i\right)$$

Benchmark states

- Bipartite states:
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GME for benchmark states

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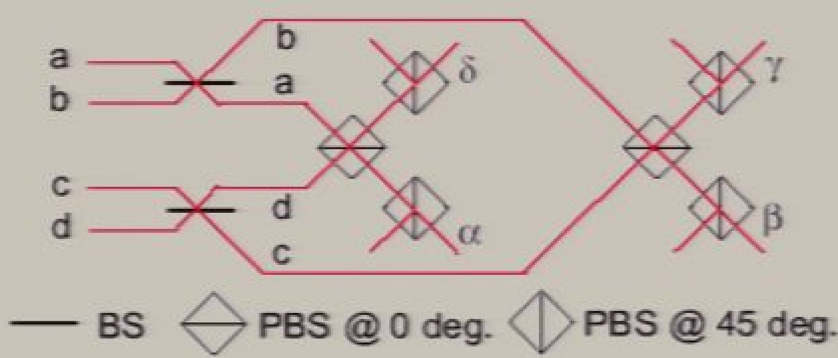
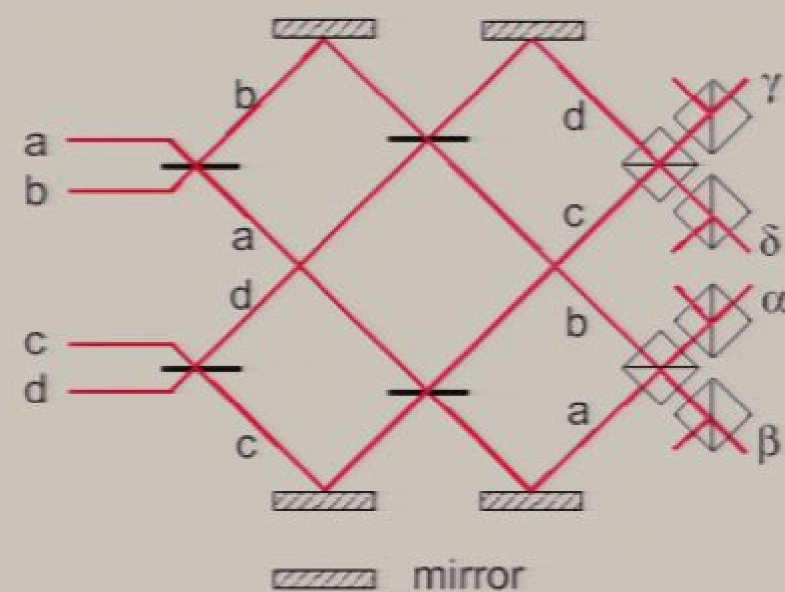
$$\rho_{Werner}(f) \equiv \frac{d^2 - fd}{d^4 - d^2} I \otimes I + \frac{fd^2 - d}{d^4 - d^2} F, \quad \text{where } F \equiv \sum_{ij} |ij\rangle\langle ji|$$

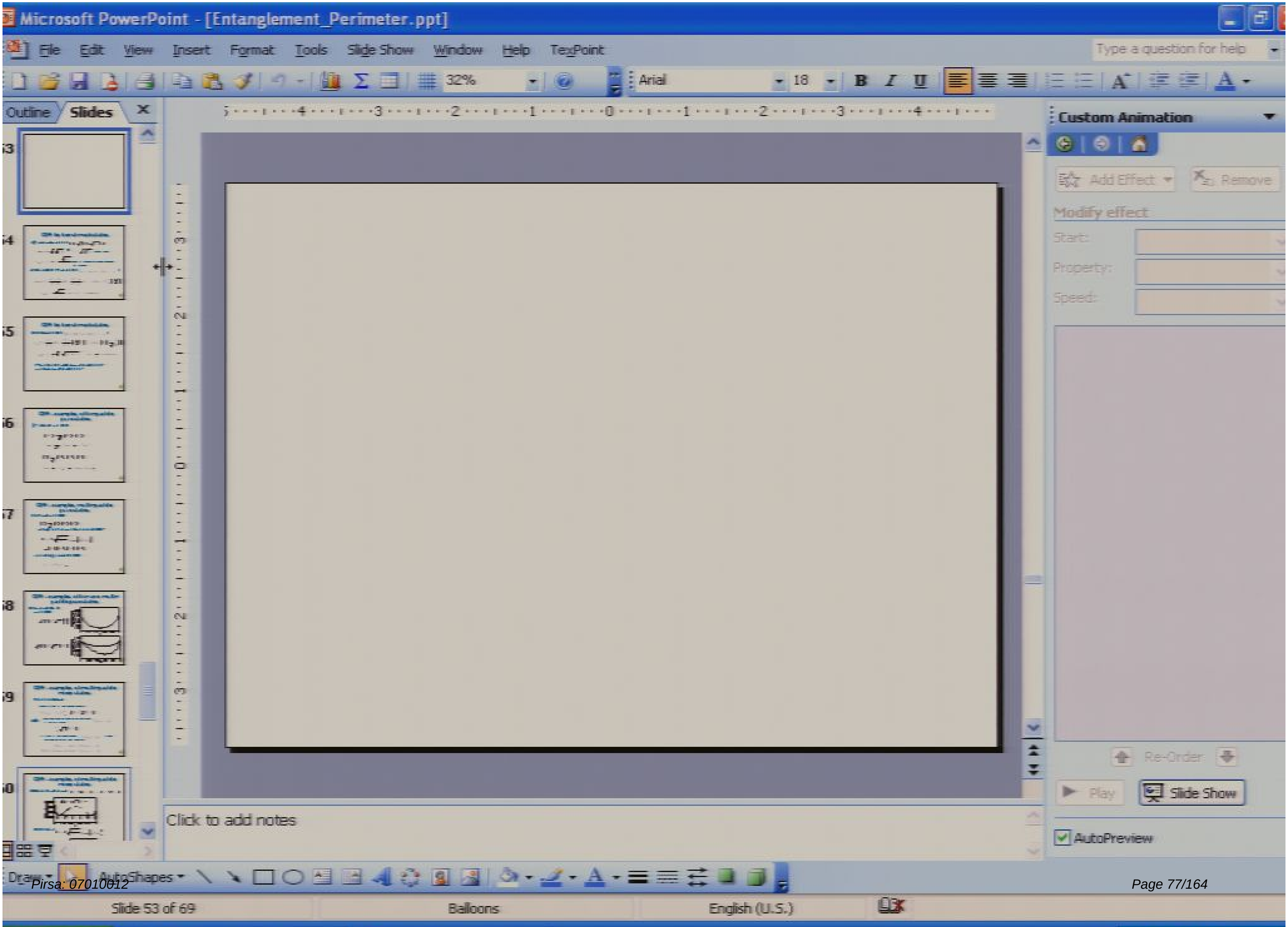
$$E_{\sin^2}(f) = \frac{1-\sqrt{1-f^2}}{2}, \quad \text{for } f \le 0; \quad 0 \text{ otherwise}$$

Unambiguous discrimination

With two copies of any one of the 16 Bell states, we can unambiguously determine the state



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Slide 53 of 69
Balloons
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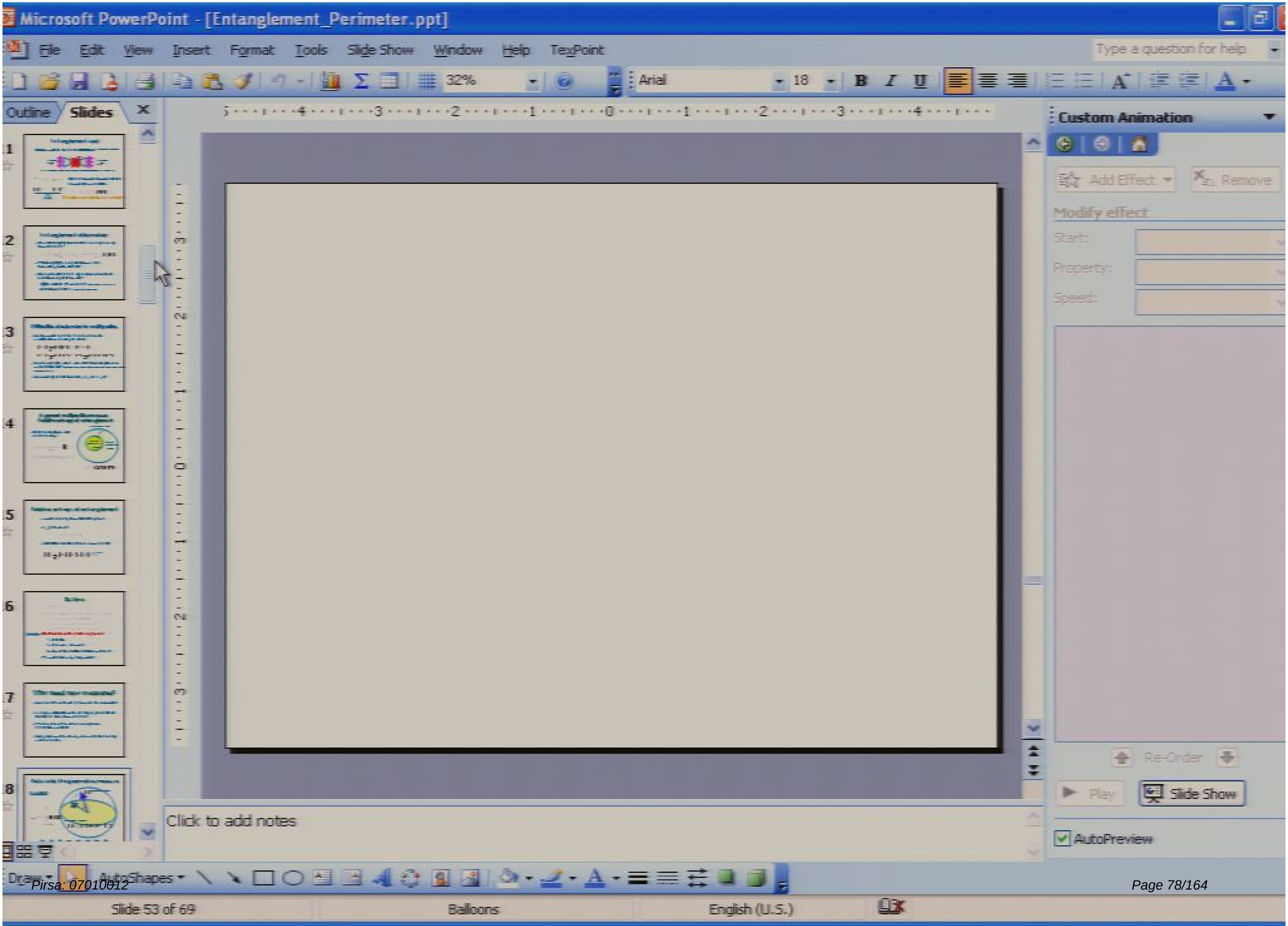
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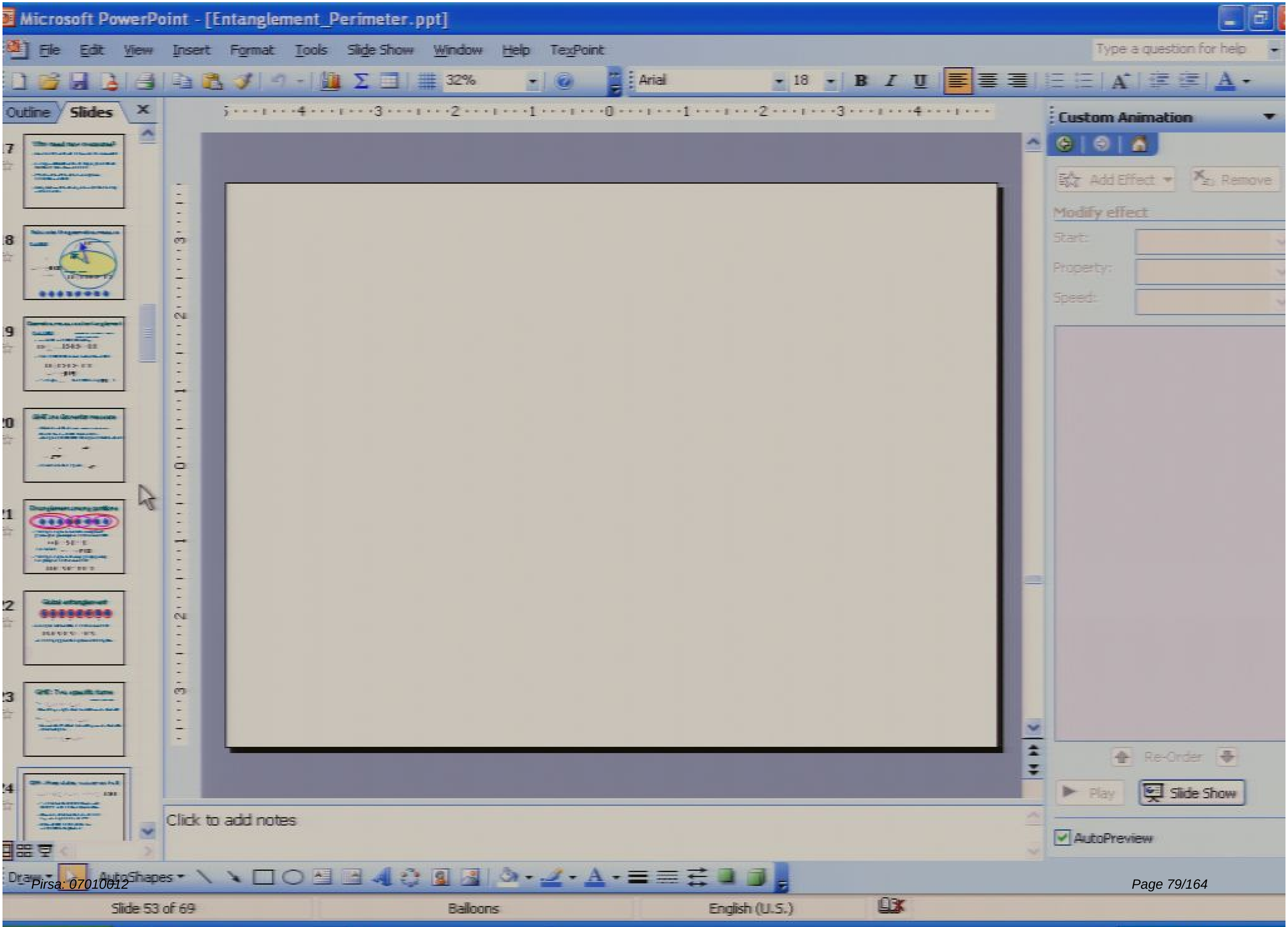
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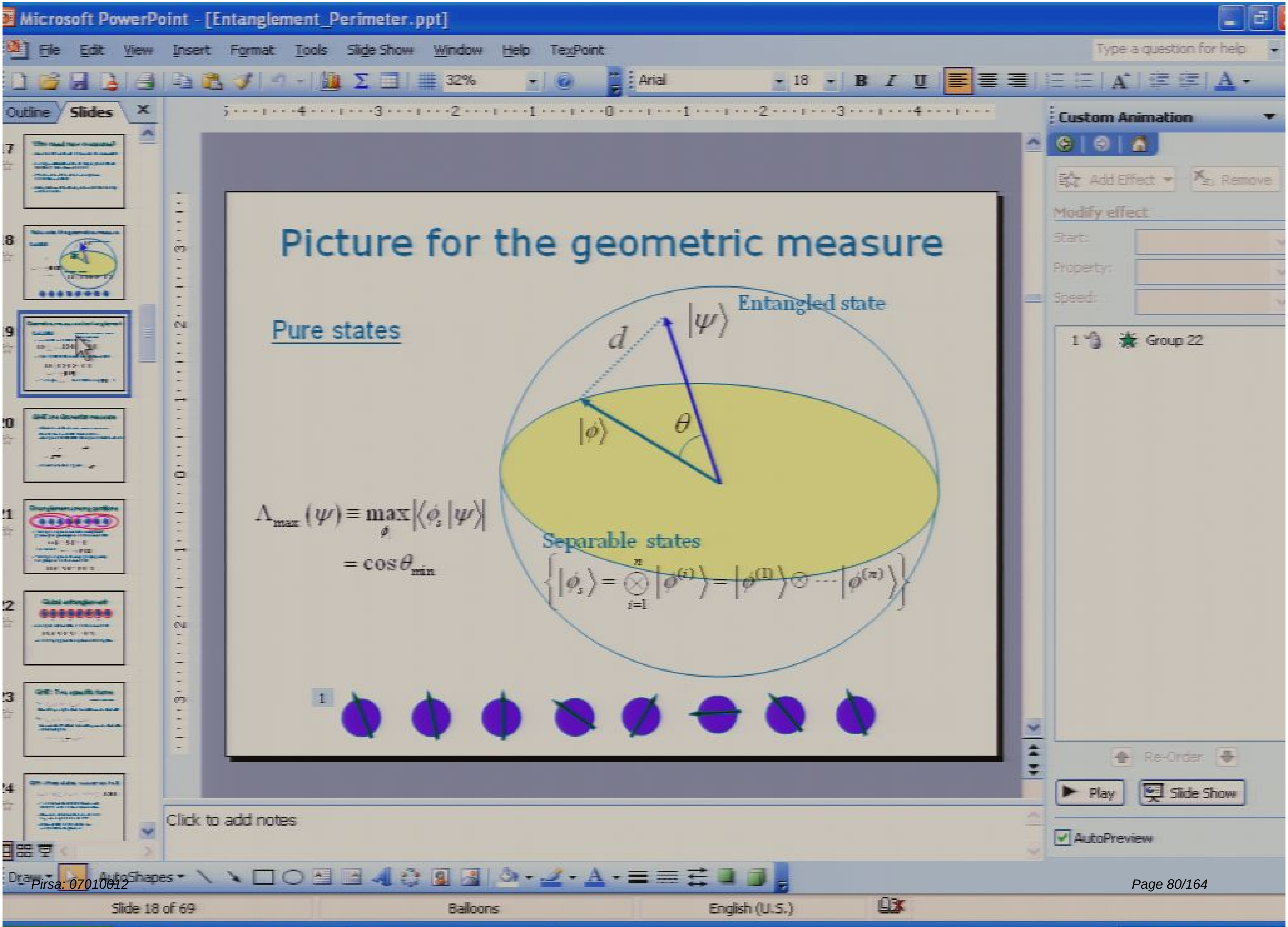
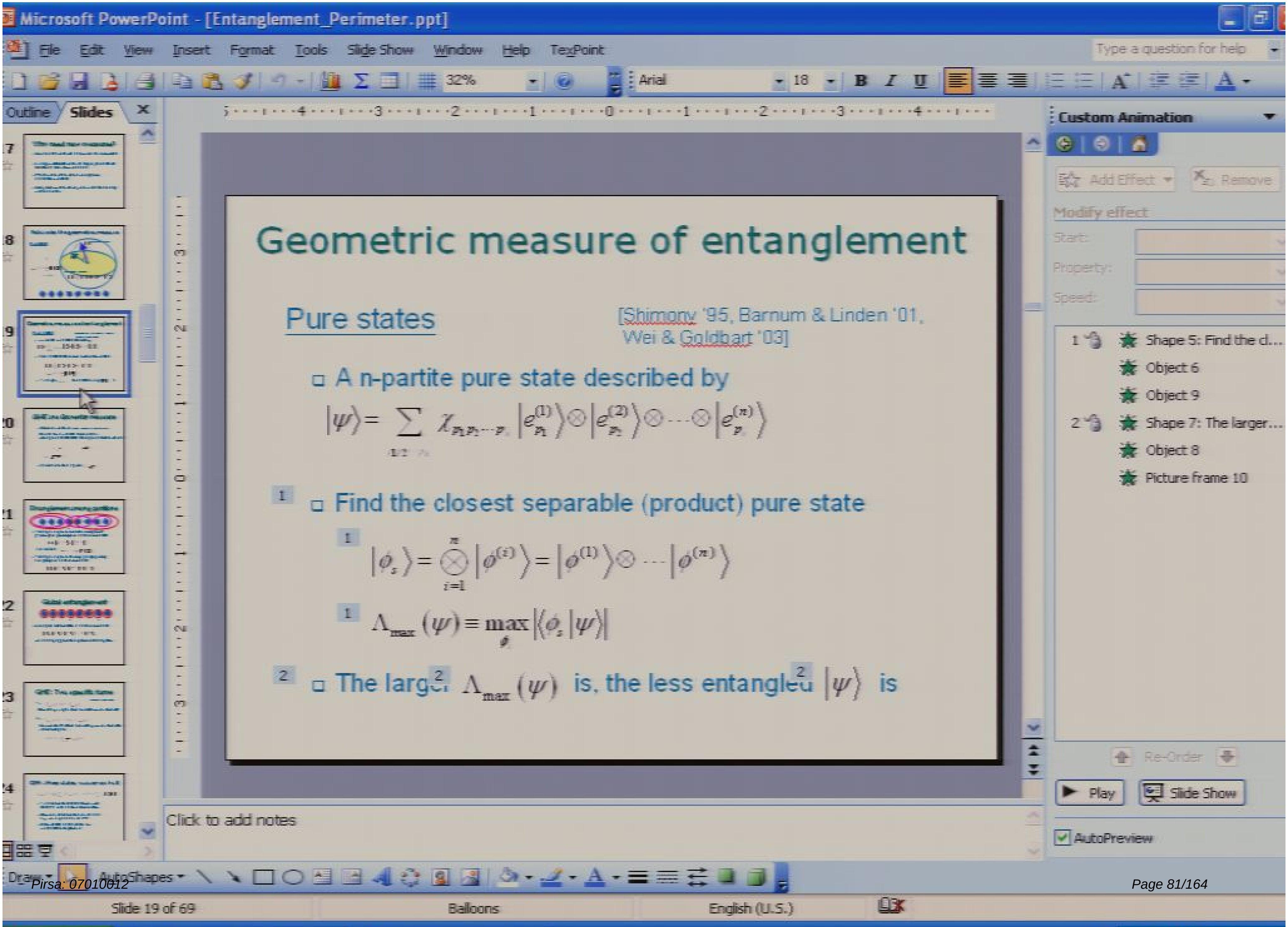

Microsoft PowerPoint - [Entanglement_Perimeter.ppt]
Picture for the geometric measure
Pure states
Entangled state
$|\psi\rangle$
d
θ
$|\phi\rangle$
$\Lambda_{\max}(\psi) \equiv \max_{\phi} |\langle \phi_s | \psi \rangle|$
$= \cos\theta_{\min}$
Separable states
$\left\{ |\phi_s\rangle = \bigotimes_{i=1}^{n} |\phi^{(i)}\rangle = |\phi^{(1)}\rangle \otimes \cdots |\phi^{(n)}\rangle \right\}$
Click to add notes
Custom Animation
Group 22
Slide 18 of 69
Balloons
English (U.S.)


Microsoft PowerPoint - [Entanglement_Perimeter.ppt]
Geometric measure of entanglement
Pure states
[Shimony '95, Barnum & Linden '01, Wei & Goldbart '03]
A n-partite pure state described by
$|\psi\rangle = \sum_{p_1 p_2 \cdots p_n} \chi_{p_1 p_2 \cdots p_n} |e^{(1)}_{p_1}\rangle \otimes |e^{(2)}_{p_2}\rangle \otimes \cdots \otimes |e^{(n)}_{p_n}\rangle$
Find the closest separable (product) pure state
$|\phi_s\rangle = \bigotimes_{i=1}^{n} |\phi^{(i)}\rangle = |\phi^{(1)}\rangle \otimes \cdots |\phi^{(n)}\rangle$
$\Lambda_{\max}(\psi) \equiv \max_{\phi} |\langle \phi_s | \psi \rangle|$
The larger $\Lambda_{\max}(\psi)$ is, the less entangled $|\psi\rangle$ is
Custom Animation
Shape 5: Find the cl...
Object 6
Object 9
Shape 7: The larger...
Object 8
Picture frame 10
Click to add notes
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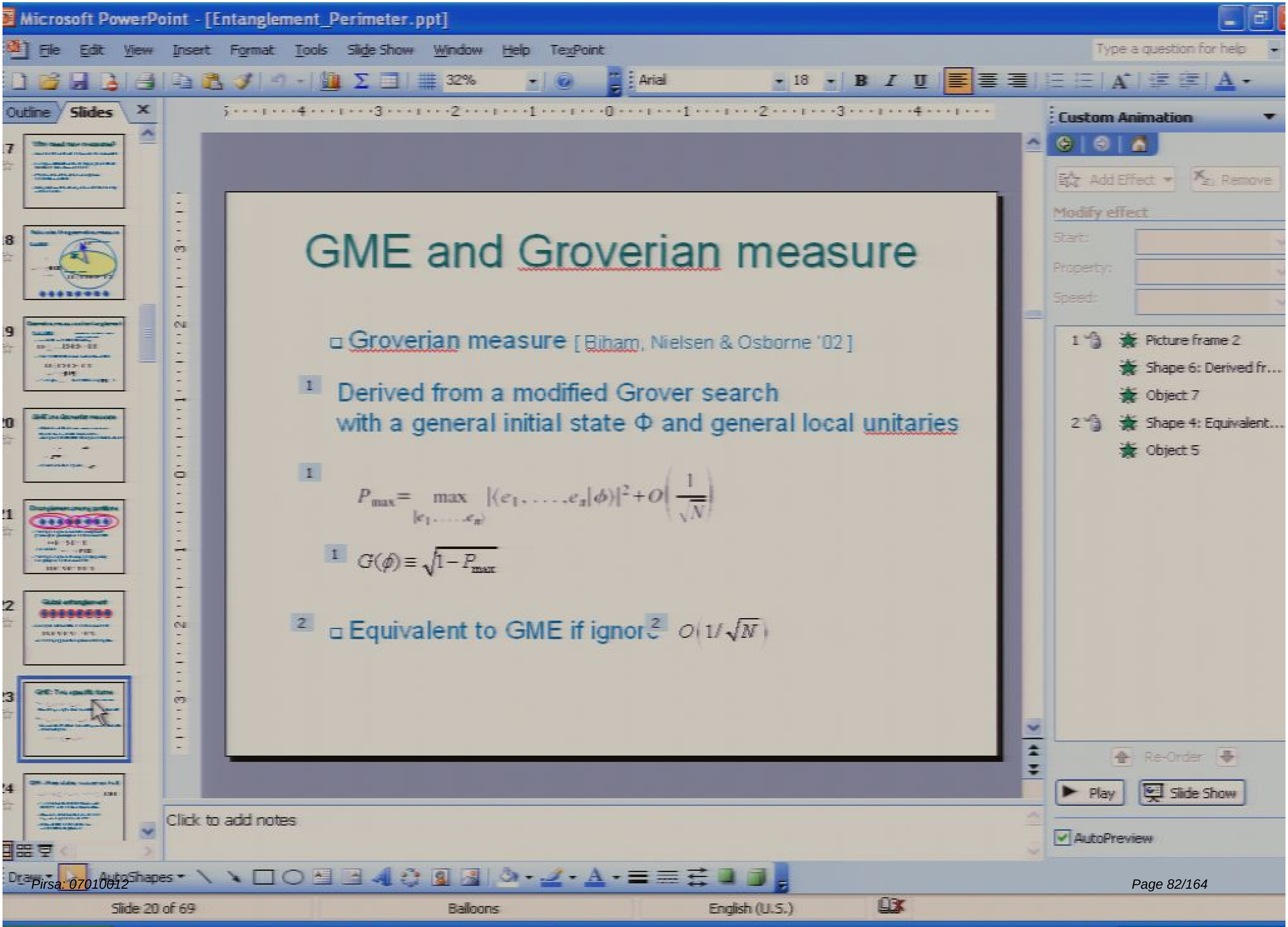
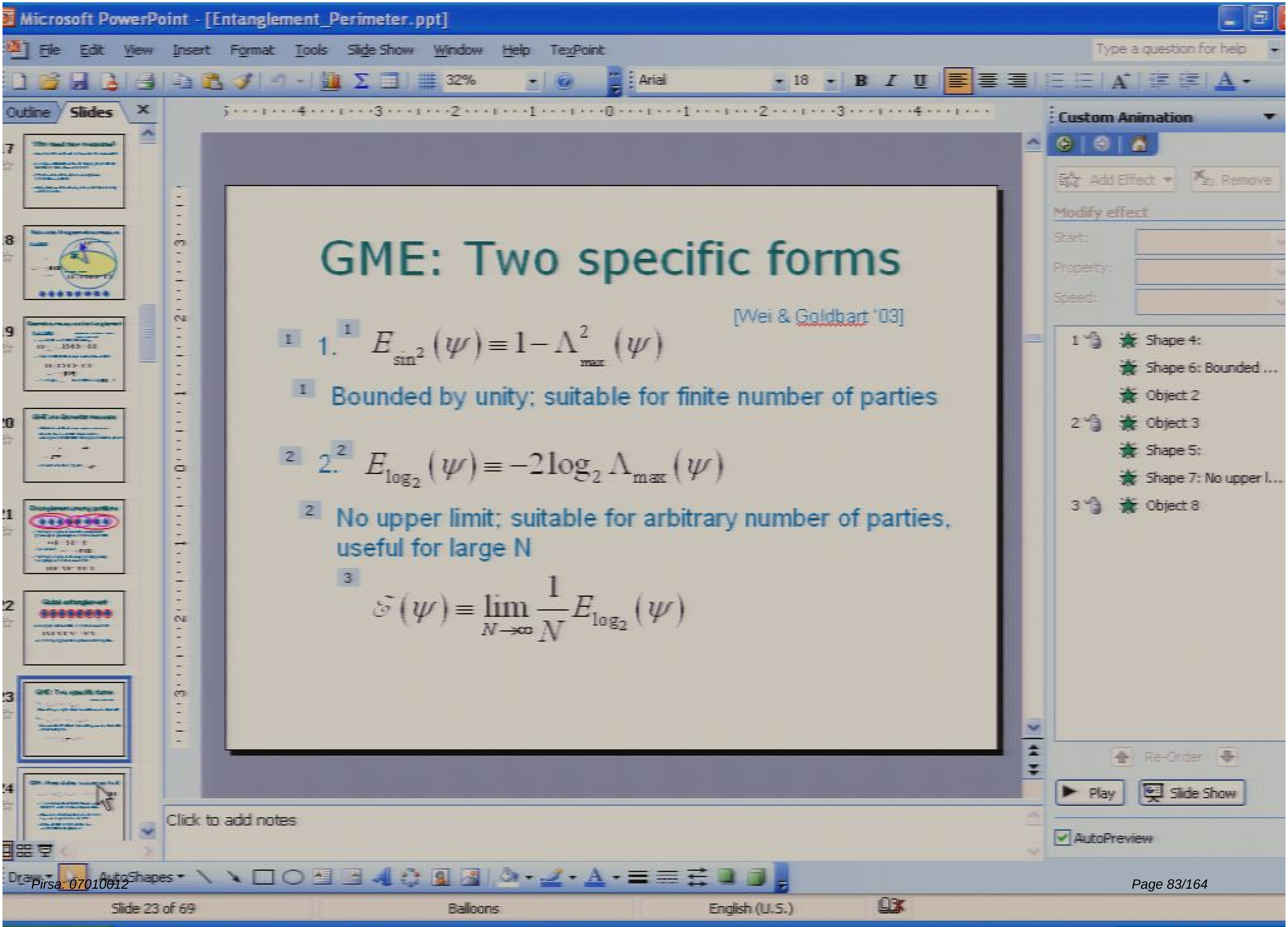
Balloons
English (U.S.)


Microsoft PowerPoint - [Entanglement_Perimeter.ppt]
GME and Groverian measure
Groverian measure [Biham, Nielsen & Osborne '02]
Derived from a modified Grover search
with a general initial state Φ and general local unitaries
$P_{max} = \max_{|e_1,\ldots,e_n\rangle} |\langle e_1,\ldots,e_n|\phi\rangle|^2 + O\left(\frac{1}{\sqrt{N}}\right)$
$G(\phi) \equiv \sqrt{1 - P_{max}}$
Equivalent to GME if ignore $O(1/\sqrt{N})$
Custom Animation
Picture frame 2
Shape 6: Derived fr...
Object 7
Shape 4: Equivalent...
Object 5
Click to add notes
Slide 20 of 69
Balloons
English (U.S.)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No upper limit; suitable for arbitrary number of parties, useful for large N
$\mathcal{E}(\psi) \equiv \lim_{N\to\infty} \frac{1}{N} E_{\log_2}(\psi)$
Click to add notes
Slide 23 of 69

GME: Mixed states via convex hull

$$E_{mixed}(\rho) \equiv \min_{\{p_i,\psi_i\}} \sum_i p_i E_{pure}(\psi_i), \quad \text{with } \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

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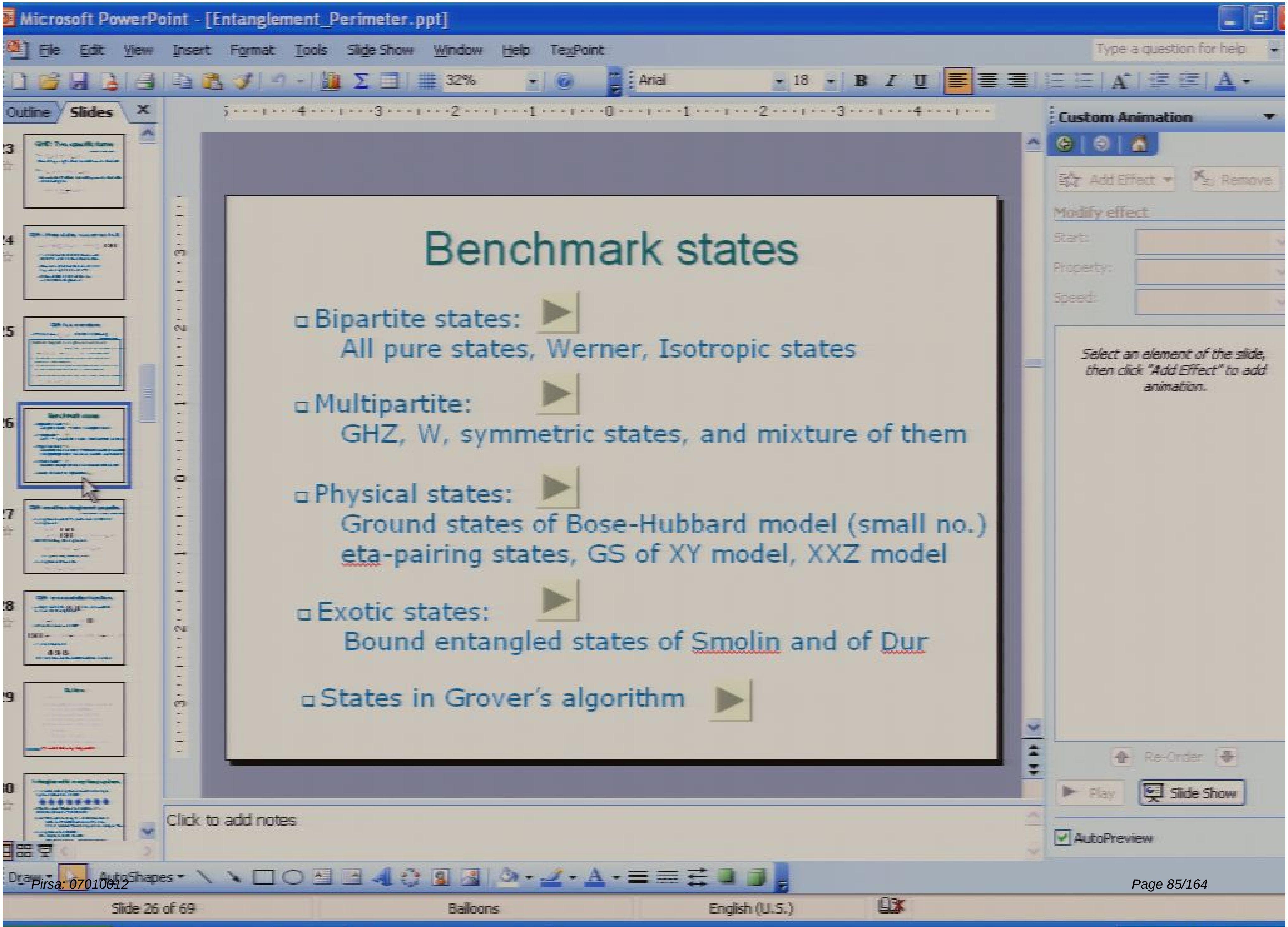
Microsoft PowerPoint - [Entanglement_Perimeter.ppt]
Benchmark states
Bipartite states:
All pure states, Werner, Isotropic states
Multipartite:
GHZ, W, symmetric states, and mixture of them
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eta-pairing states, GS of XY model, XXZ model
Exotic states:
Bound entangled states of Smolin and of Dur
States in Grover's algorithm
Custom Animation
Select an element of the slide, then click "Add Effect" to add animation.
Click to add notes
Slide 26 of 69
Balloons
English (U.S.)

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[cf Vidal '02]

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GME for benchmark states

③ Isotropic states ($\int dU\, U \otimes U^{*} \rho U^{+} \otimes (U^{+})^{*} = \rho$)

$$\rho_{iso}(F) \equiv \frac{1-F}{d^2-1} I \otimes I + \frac{Fd^2-1}{d^2-1} |\Phi^+\rangle\langle\Phi^+|, \qquad \text{where } |\Phi^+\rangle \equiv \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$$

$$E_{\sin^2}(F) = 1 - \frac{1}{d}\left(\sqrt{F} + \sqrt{(1-F)(d-1)}\right)^2, \quad \text{for } F \geq \frac{1}{d}; \;\; 0 \text{ otherwise}$$

Works for these well-known bi-partite states;
what about multi-partite states?

GME: examples of tri-partite pure states

④ Tri-partite pure states

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right),$$

$$\Lambda_{max} = \frac{1}{\sqrt{2}}, \quad E_{\sin^2} = \frac{1}{2}, \quad E_{\log_2} = 1$$

$$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right),$$

$$\Lambda_{max} = \frac{2}{3}, \quad E_{\sin^2} = \frac{5}{9}, \quad E_{\log_2} = \log_2\left(\frac{9}{4}\right)$$

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GME and other entanglement properties

[Wei, Ericsson, Goldbart & Munro '04]

- Entanglement witness: an observable that detects entanglement

$$W = \lambda^2 I - |\psi\rangle\langle\psi|$$

$$\min_{W} Tr\left(W|\psi\rangle\langle\psi|\right) = -1 + \Lambda^2(\psi) = -E_{\sin^2}(\psi)$$

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- Relative entropy of entanglement

$$E_R(\psi) \geq -2\log_2 \Lambda_{\max}(\psi) \equiv E_{\log_2}(\psi)$$

e.g. $E_R(W) = -2 \log_2 \Lambda(W) = \log_2 (9/4)$

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- Entanglement of formation

$$(\log_2 e) E_{\sin^2} \leq E_{\log_2} \leq E_F$$

GME and correlation functions

- A single spin state $\alpha|\uparrow\rangle+\beta|\downarrow\rangle$ can be expressed in terms of a density matrix

$$\rho=\frac{1}{2}\left(I+\vec{r}\cdot\vec{\sigma}\right), \qquad \text{with } |r|=1$$

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- N-spin separable pure state:

$$|\phi_S\rangle\langle\phi_S|=\frac{1}{2}\left(I+\vec{r}_1\cdot\vec{\sigma}\right)\otimes\frac{1}{2}\left(I+\vec{r}_2\cdot\vec{\sigma}\right)\otimes\cdots\otimes\frac{1}{2}\left(I+\vec{r}_N\cdot\vec{\sigma}\right)$$

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- The overlap square

$$\Lambda^2 = \left\langle\left(|\phi_S\rangle\langle\phi_S|\right)\right\rangle_\psi$$

Is a linear combination of all correlation functions

Outline

I. Introduction

II. Review of standard entanglement measures

1. Entanglement of distillation
2. Entanglement of formation/cost
3. Relative entropy of entanglement

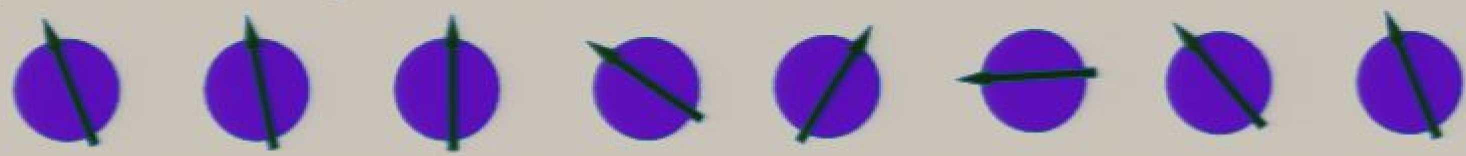
III. Geometric measure of entanglement

1. Definition
2. Benchmark examples
3. Connections of GME to other measures

IV. Applied to a many-body model

Entanglement in many-body systems

- The notion of entanglement applies naturally to systems of spins on a lattice



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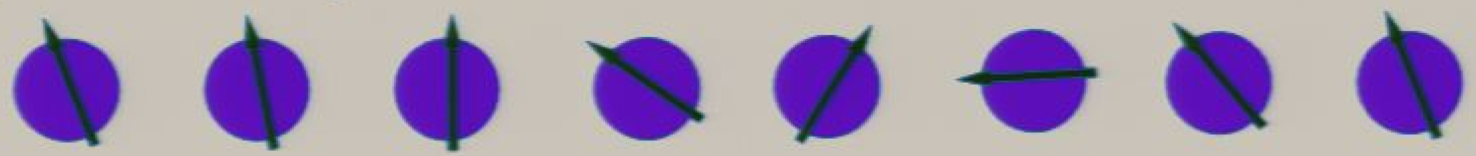

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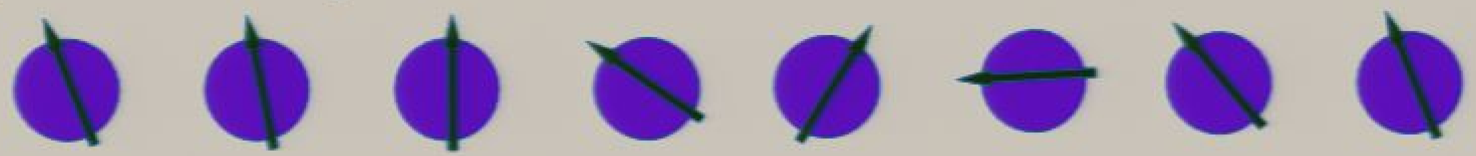

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More: Calabrese & Cardy '04, Refael & Moore '04
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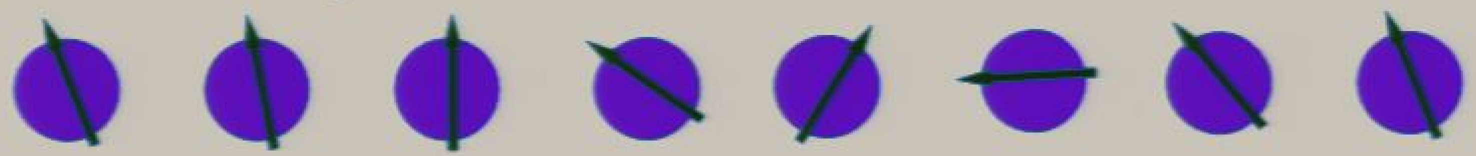

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- Entanglement and DMRG:

QI helps to revitalize DMRG

[Verstraete et al. '04, Vidal et al. '03-'04]

Multipartite Entanglement?

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- Concurrence and entropy approaches are essentially bipartite

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Concurrence:
ent. of two spins

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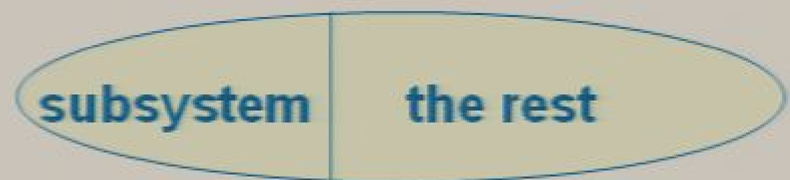

Entropy

Multipartite Entanglement?

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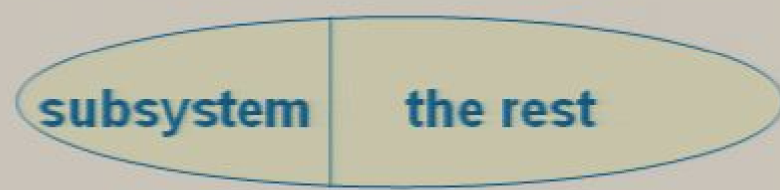

Entropy

- Goal:

  To quantify multipartite entanglement of many-body system near QPT

Model: transverse XY

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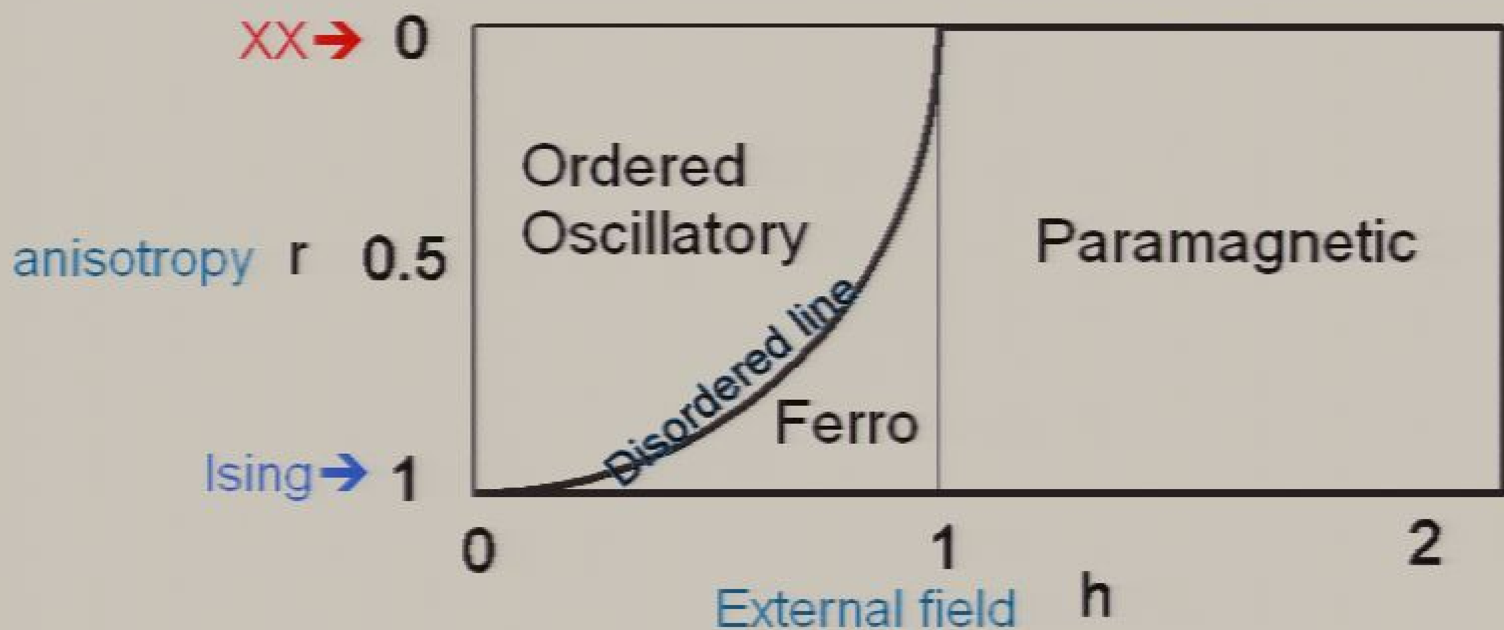

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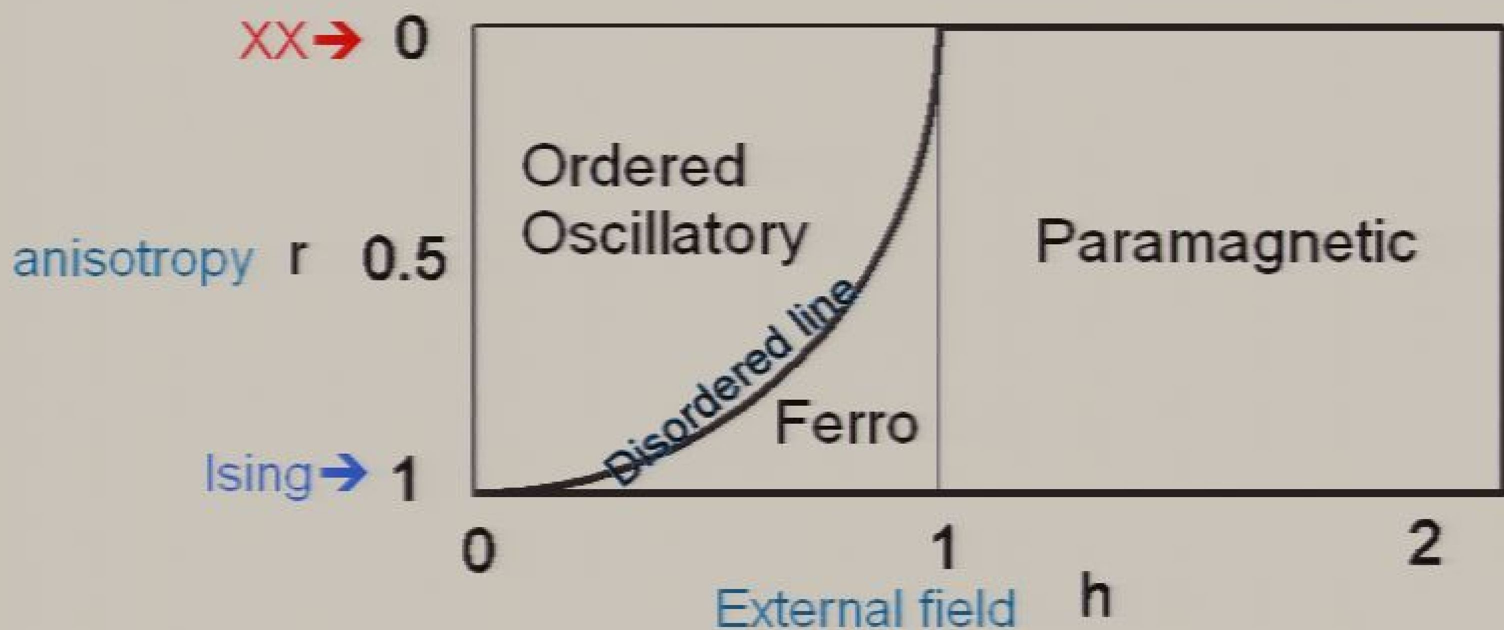

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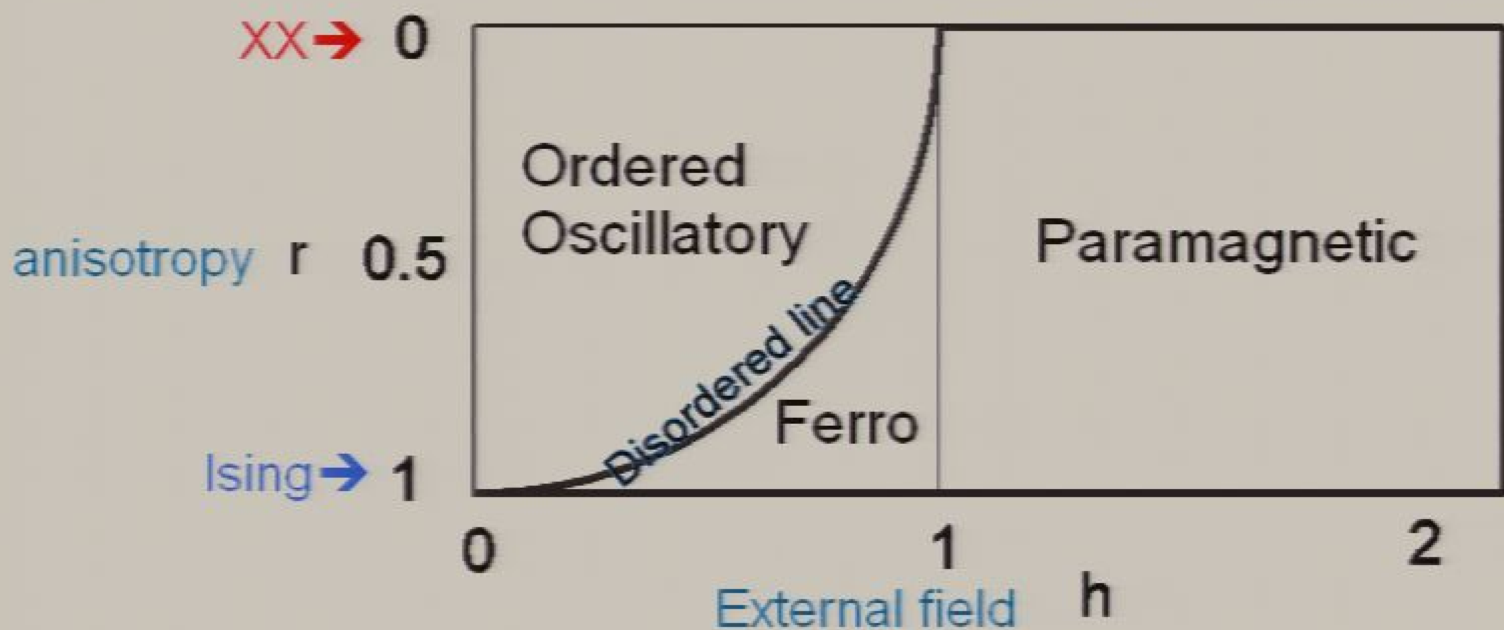

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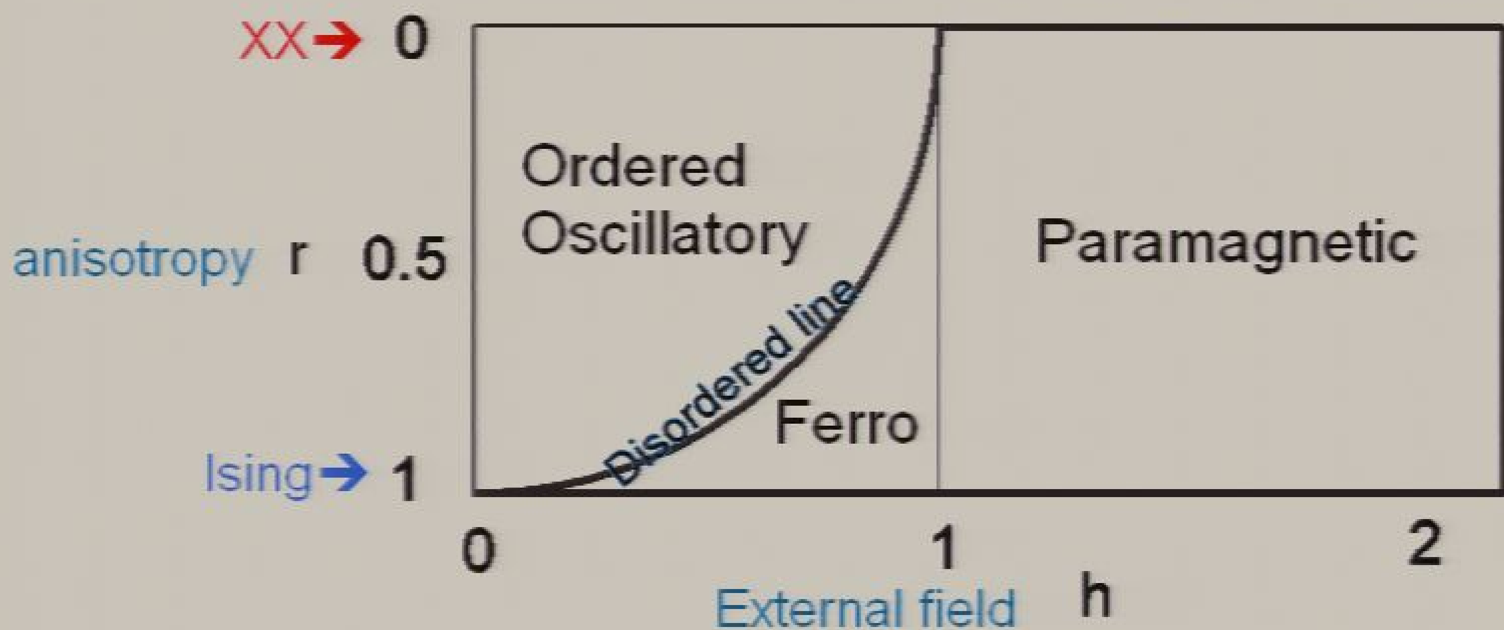

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- Can study ground-state entanglement

J-W and Bogoliubov

- J-W transformation

$$\sigma_i^z = 1 - 2c_i^+ c_i$$

$$\sigma_i^x = \prod_{j=1}^{i-1} (1 - 2c_j^+ c_j)(c_i + c_i^+)$$

$$\sigma_i^y = -i \prod_{j=1}^{i-1} (1 - 2c_j^+ c_j)(c_i - c_i^+)$$

- Bogoliubov transformation

$$\tilde{c}_k = \cos\theta_k \, \gamma_k + i \sin\theta_k \, \gamma_{-k}^+$$

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4. Evaluate the overlap between ground state $\left|\Psi_0(r,h)\right\rangle$ and the Ansatz state

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6. Entanglement density in the thermodynamic limit is

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Global entanglement in XY model

$$H = -\sum_i \left(\frac{1+r}{2} \sigma_i^x \sigma_{i+1}^x + \frac{1-r}{2} \sigma_i^y \sigma_{i+1}^y + h\sigma_i^z \right)$$

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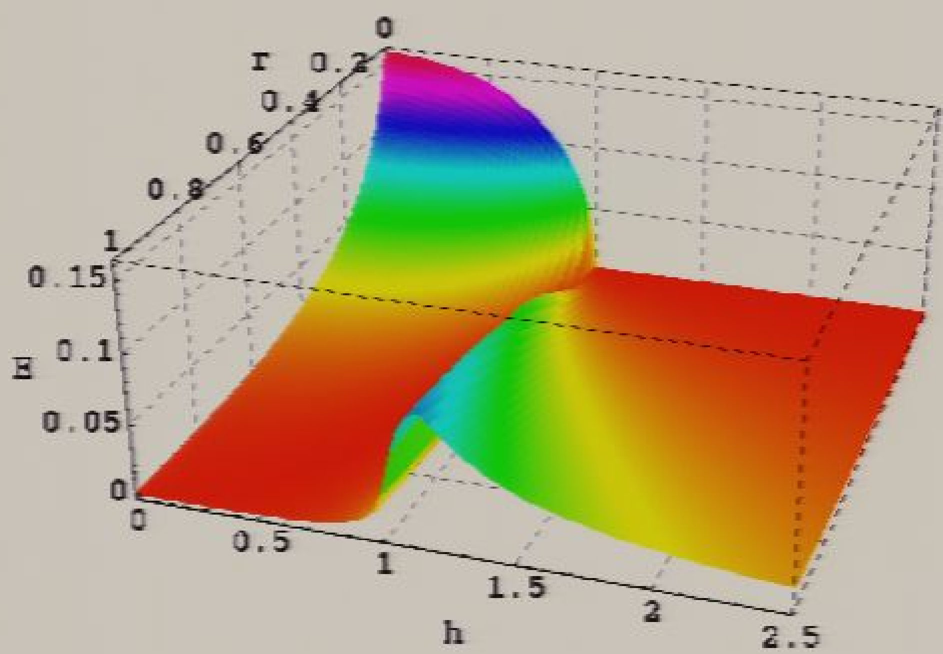

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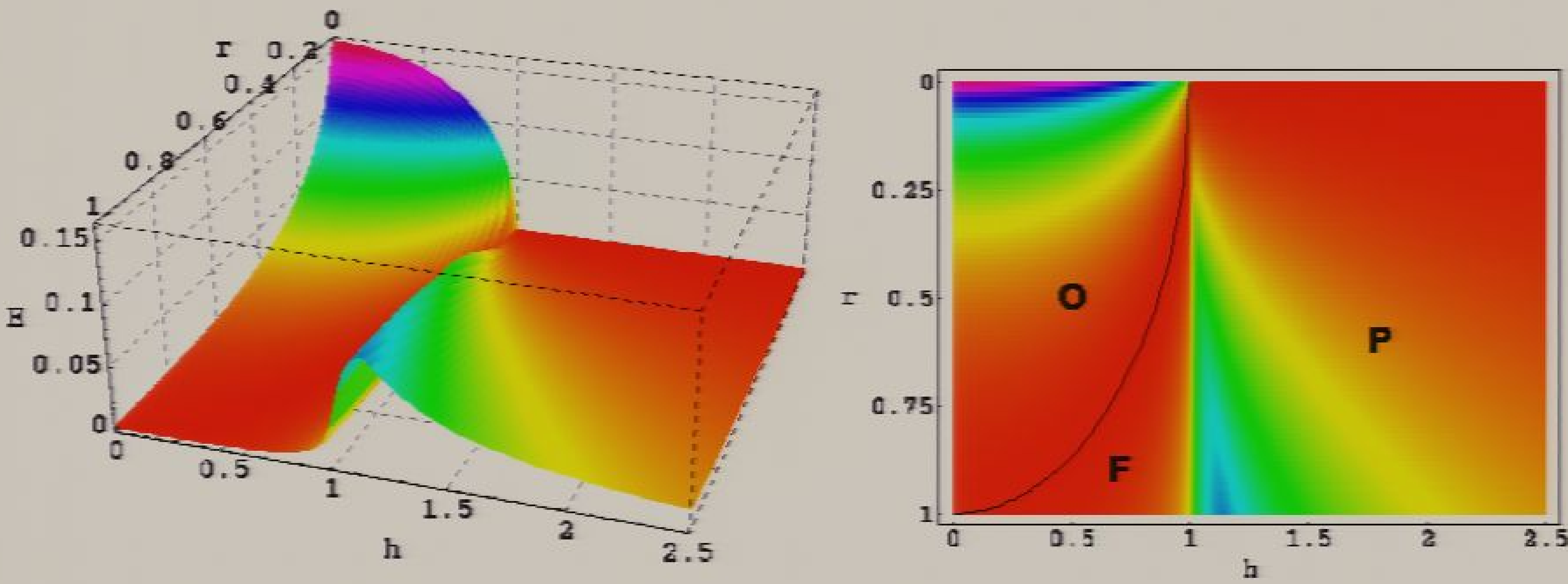
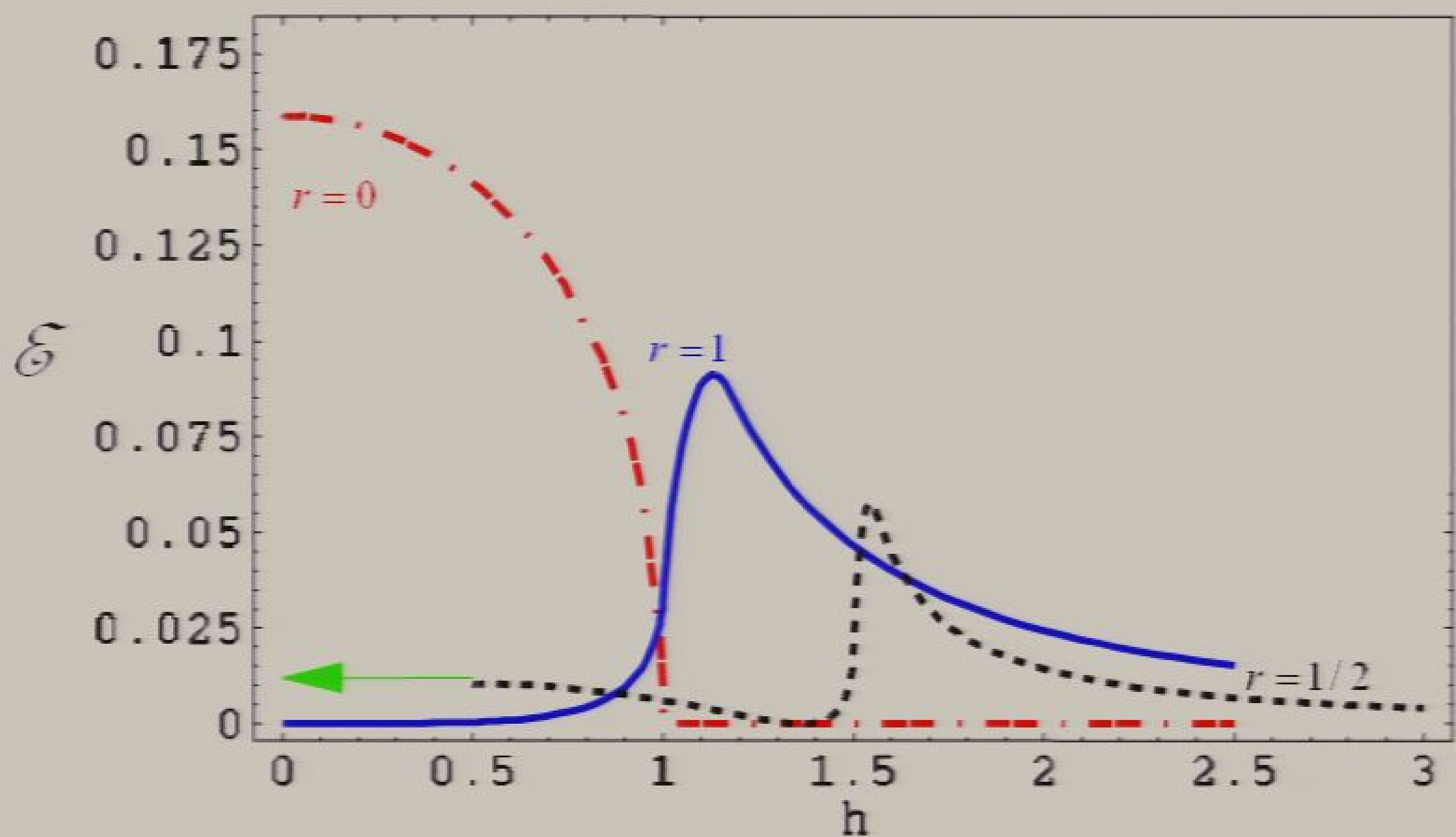


 [Wei, Das, Muhkopadyay, Vishweshwara & Goldbart '05]

Entanglement in different universality classes



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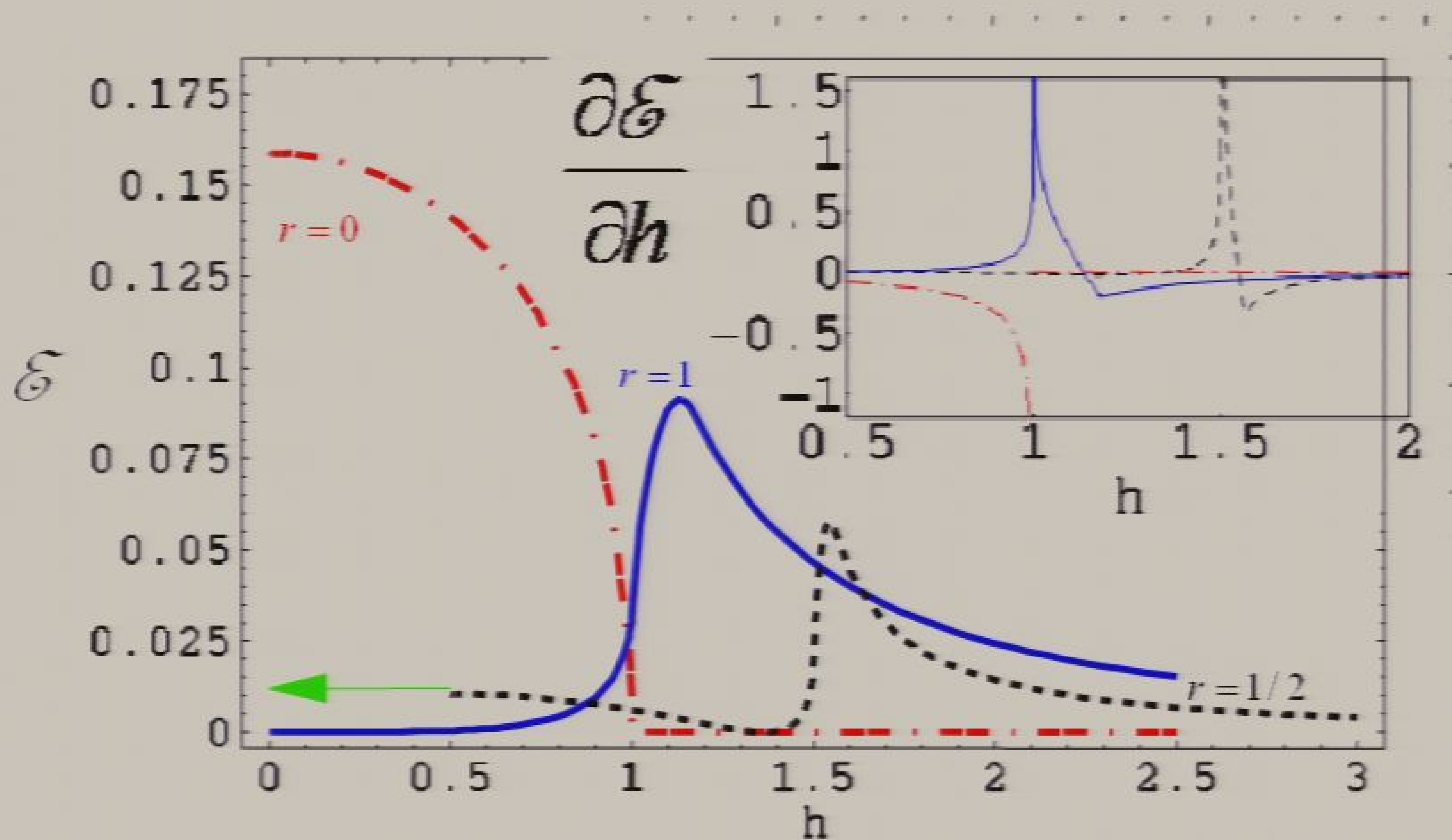

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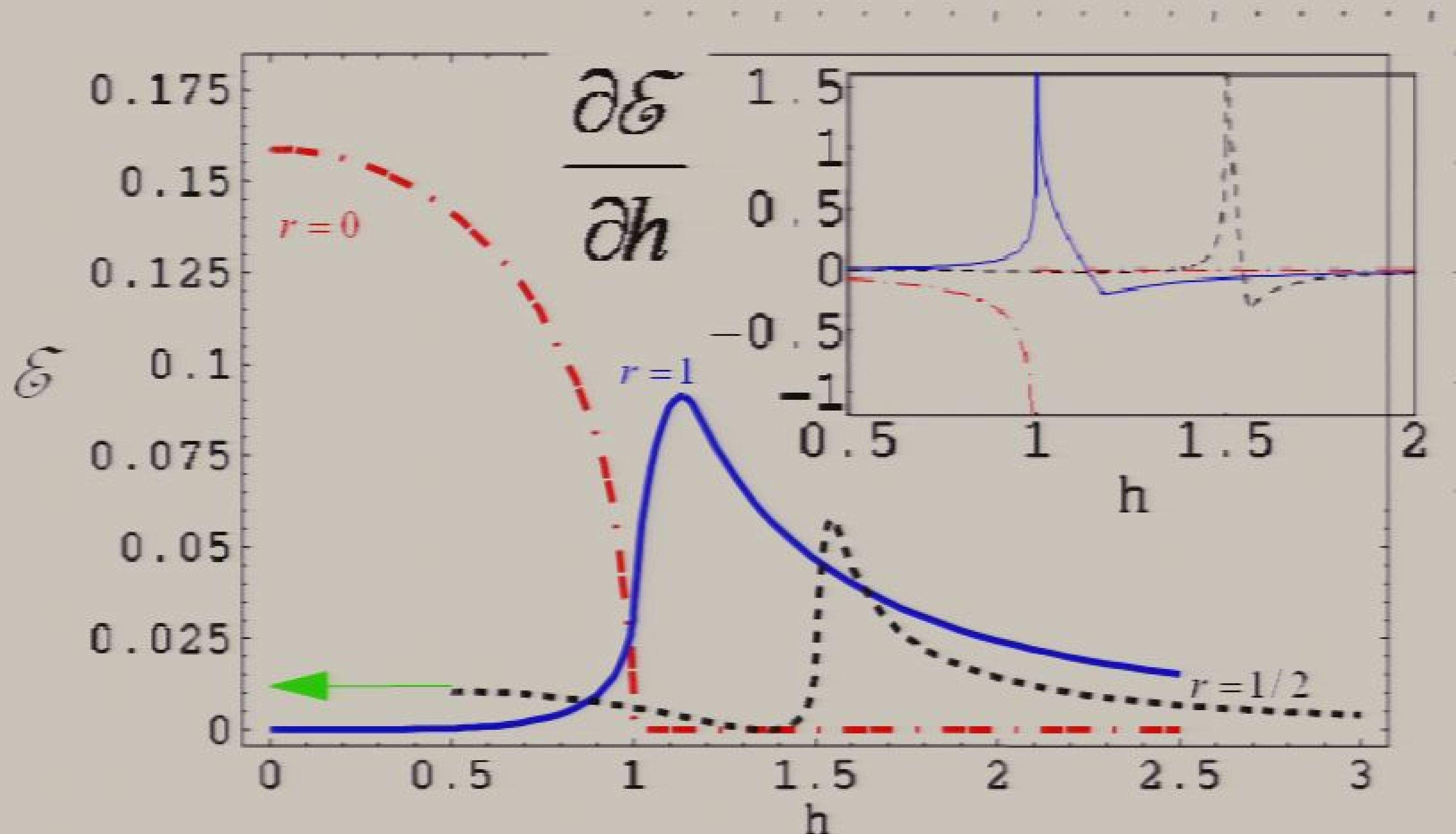
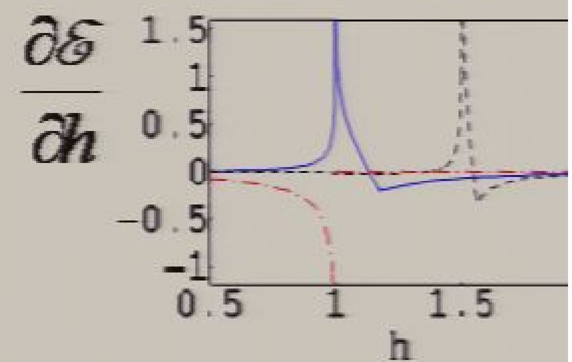


Singular behavior of entanglement near critical points



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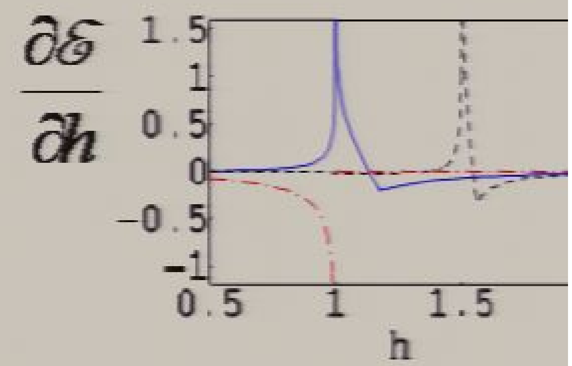
- Across critical line h=1, field-derivative of entanglement diverges



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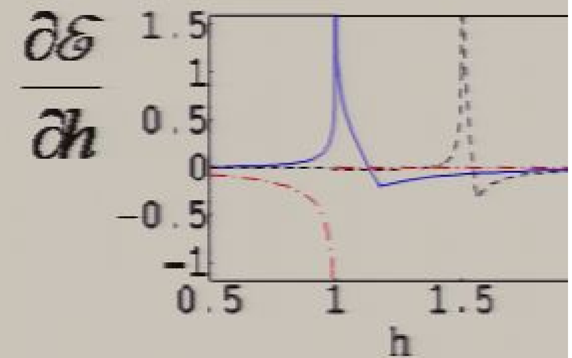

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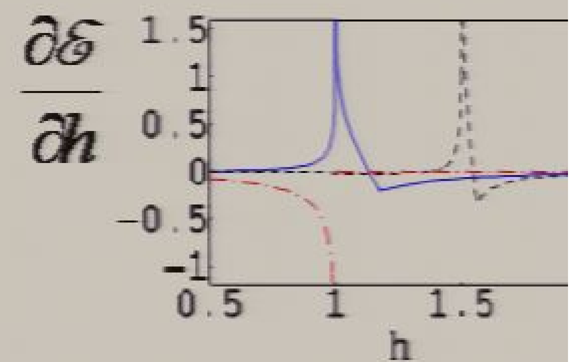

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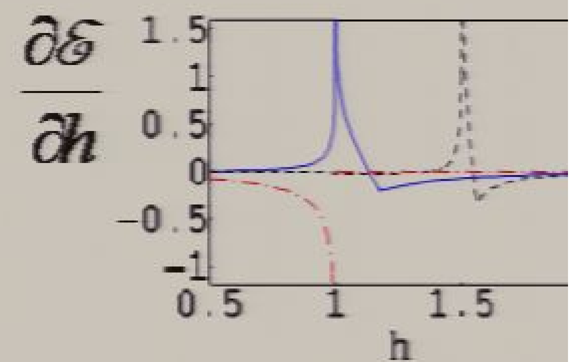
Singular behavior of entanglement: XX universality class



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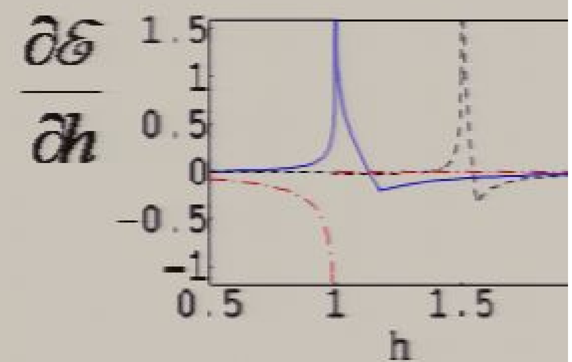

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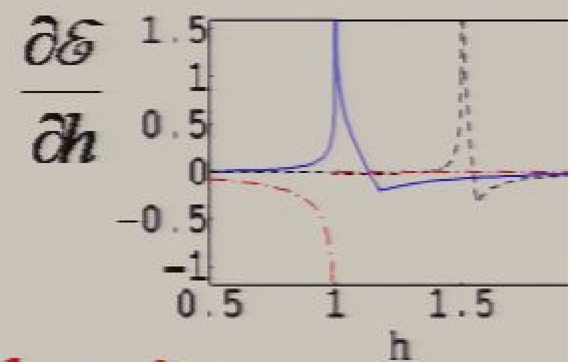

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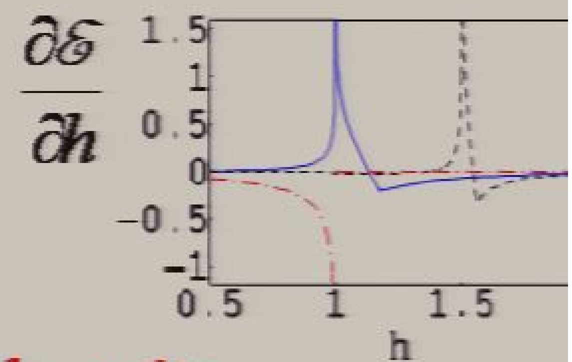

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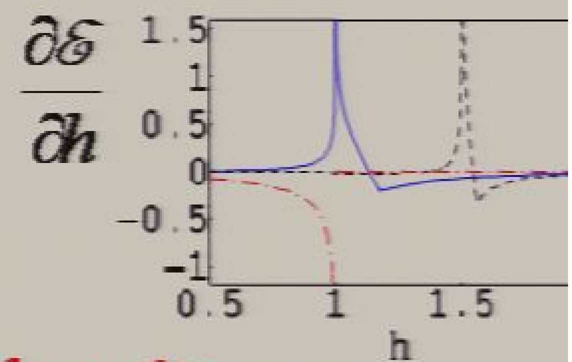

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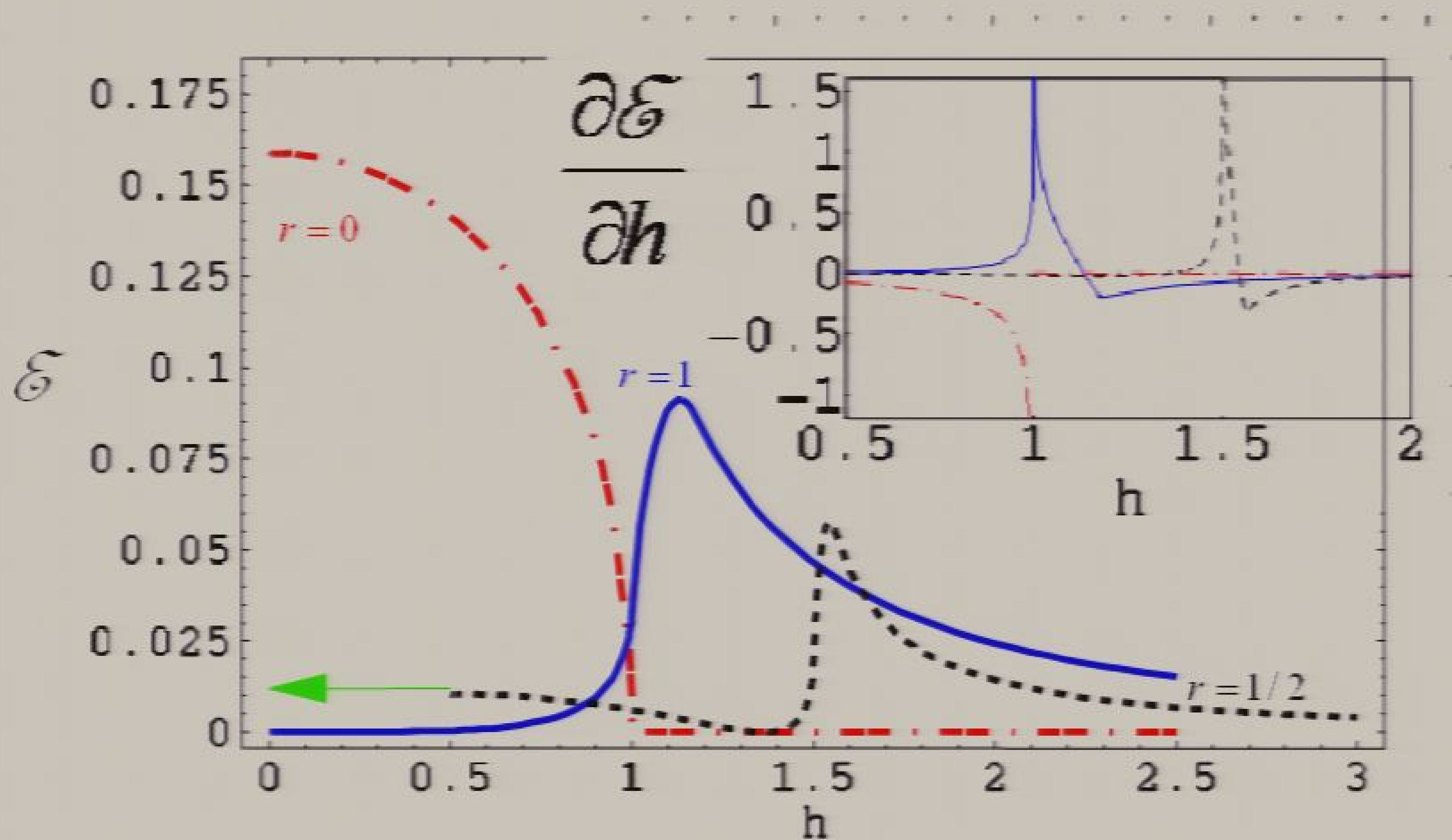

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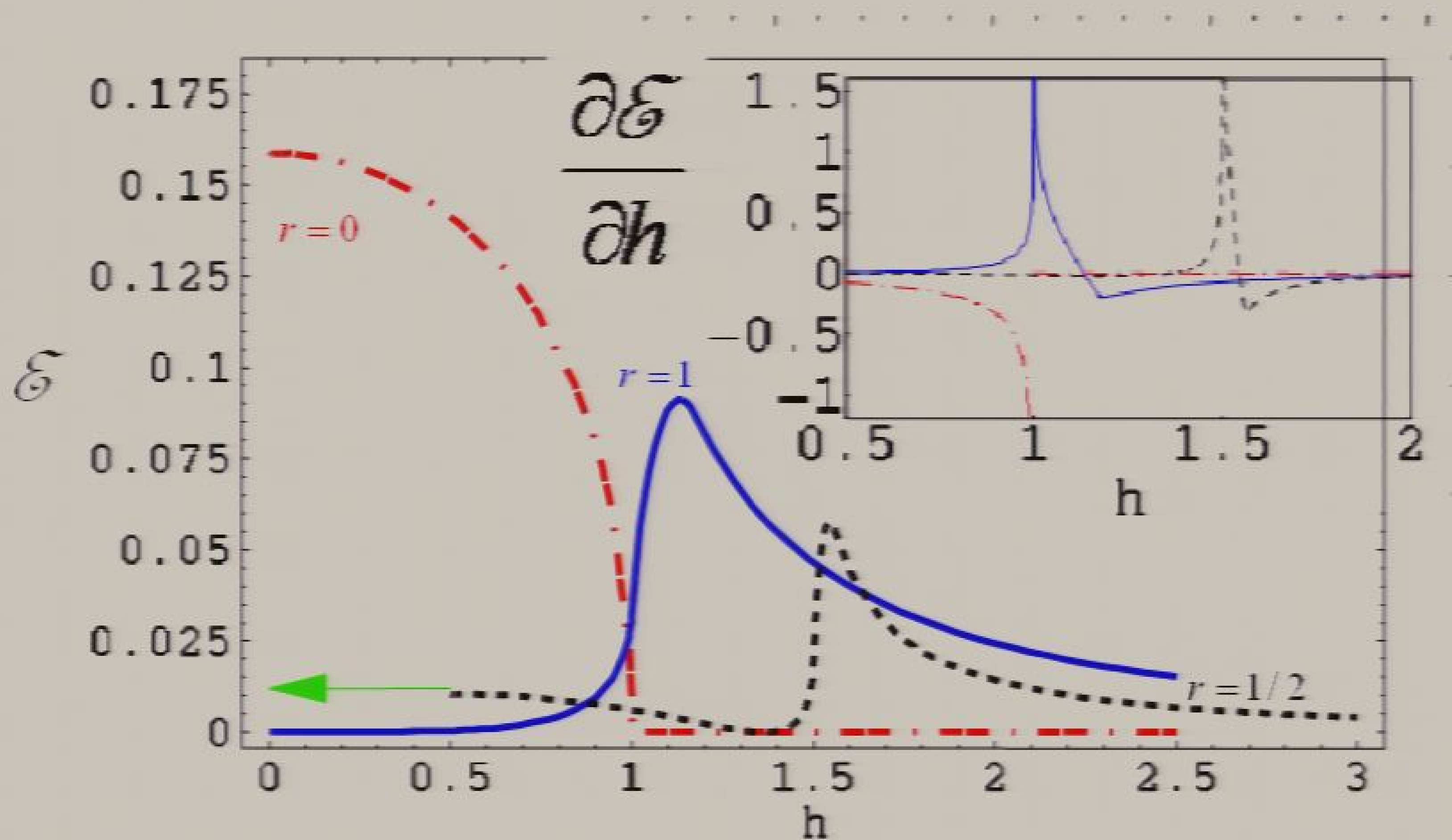

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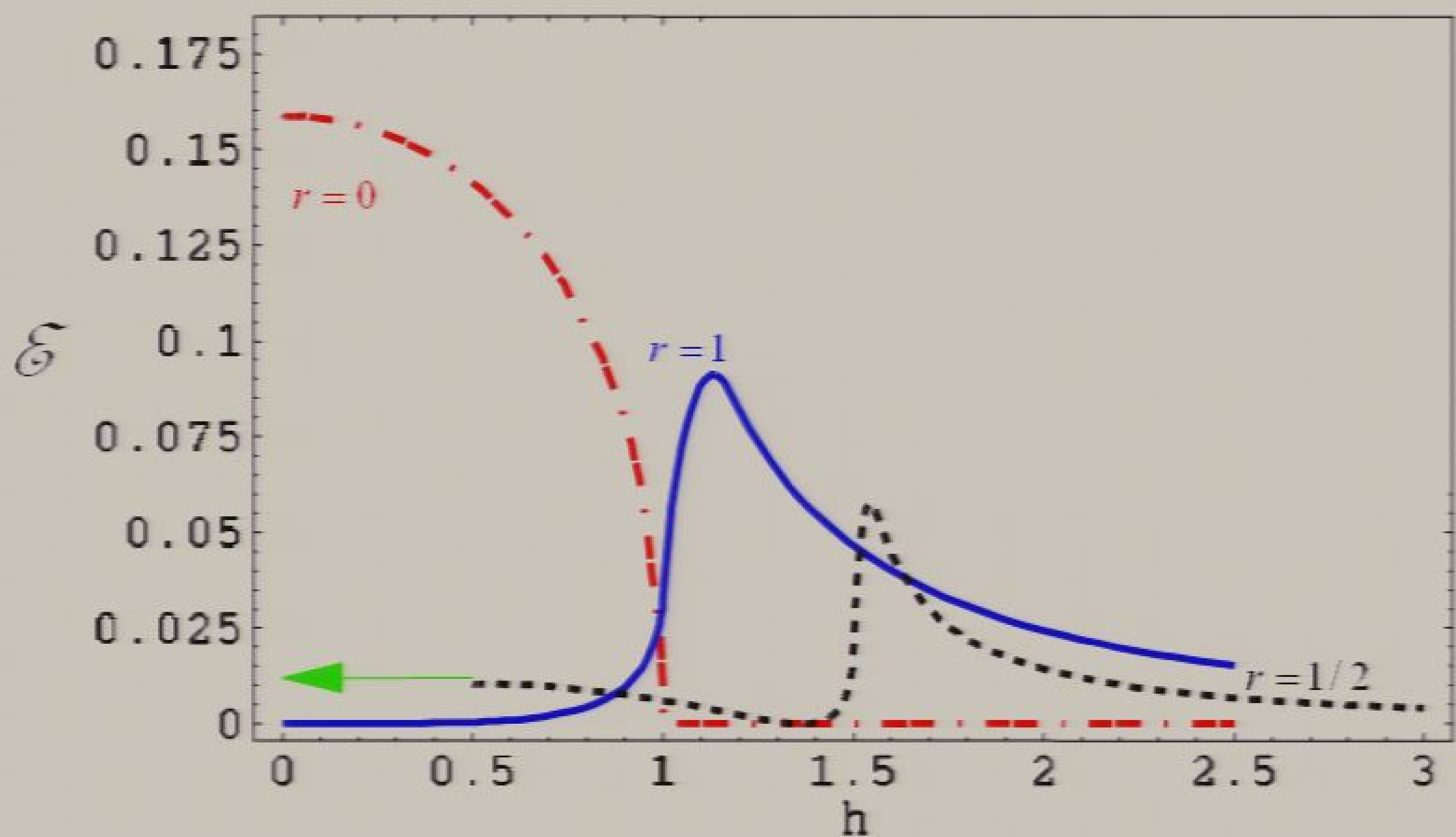

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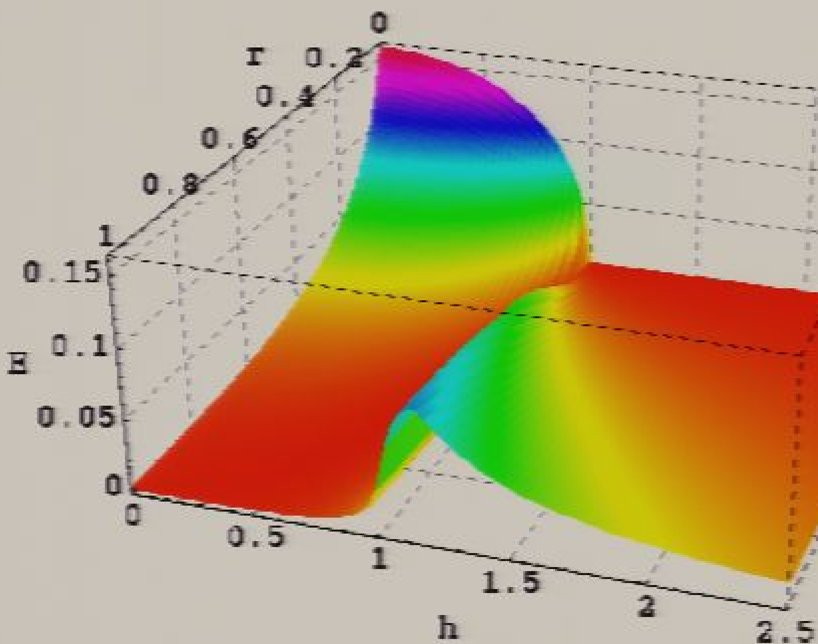

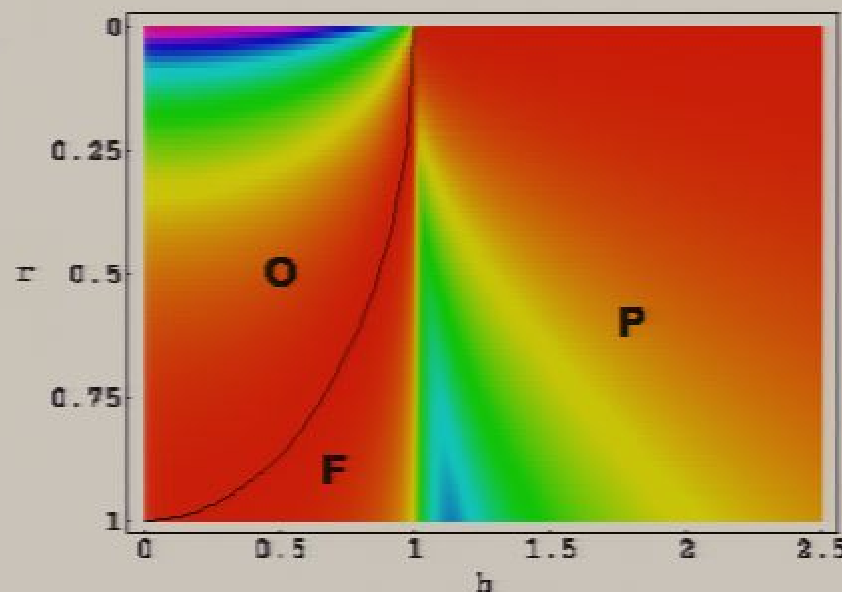

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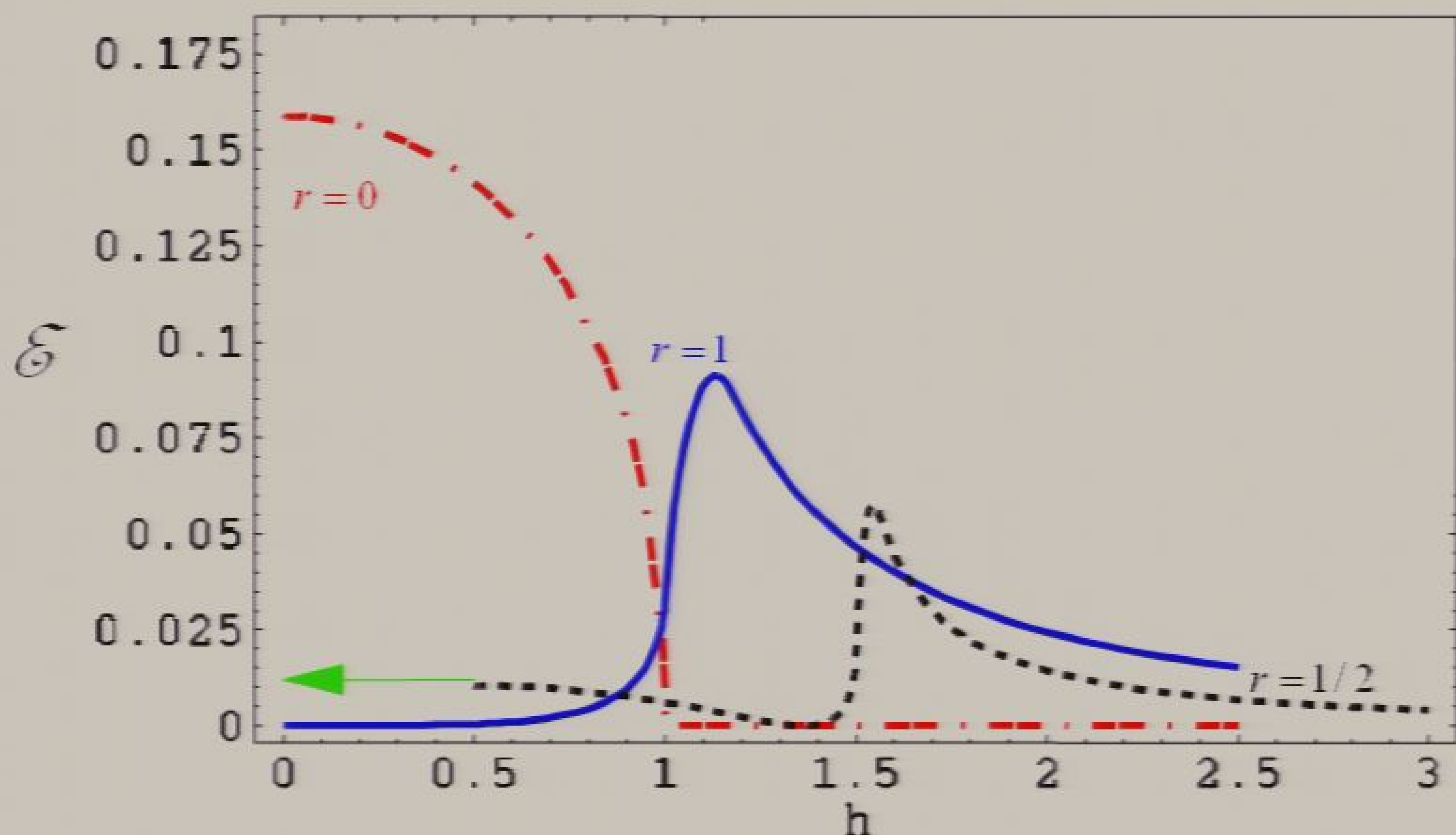

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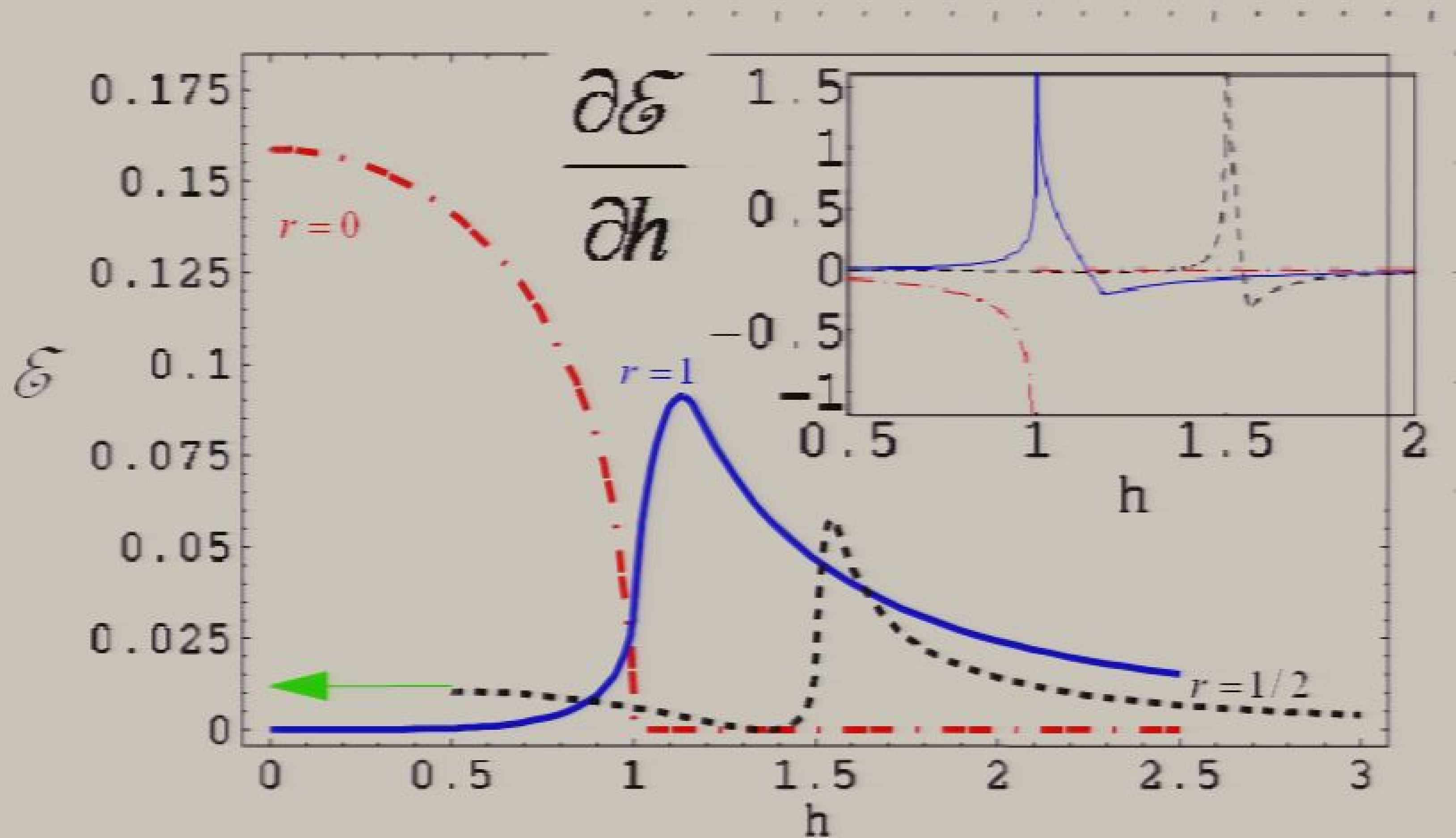

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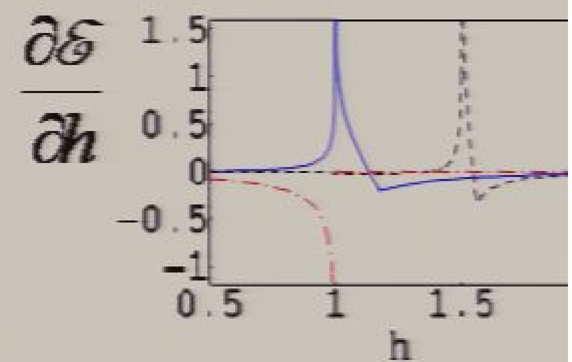

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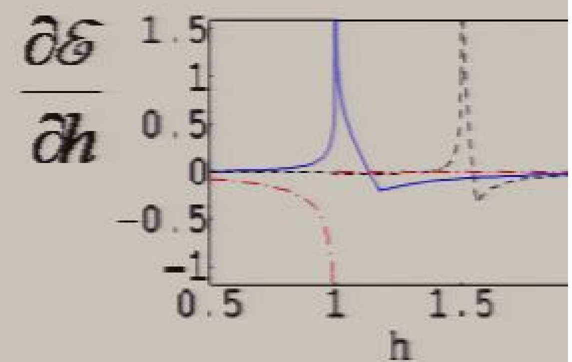

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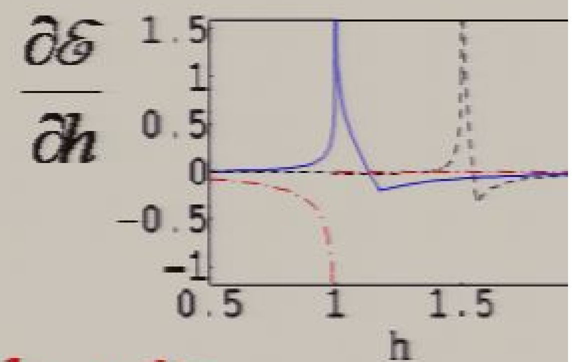

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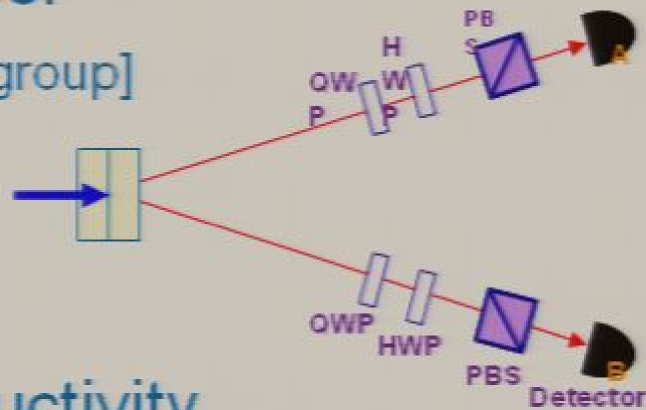
Other activities

Linear optics:

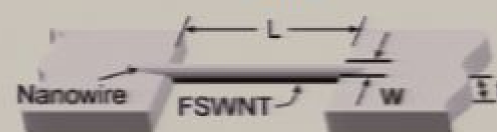
[with Kwiat's group]

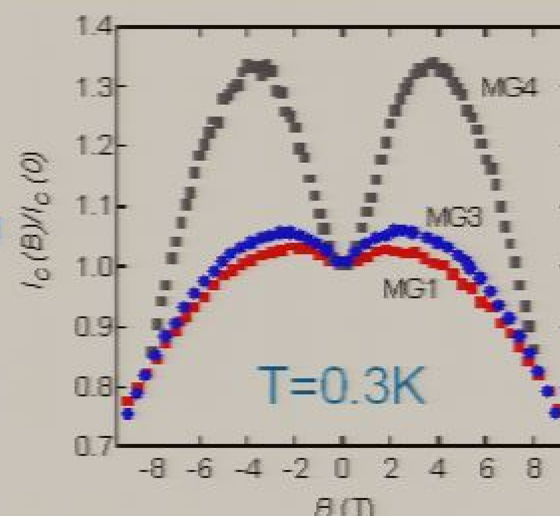


Superconductivity
& condensed matter:

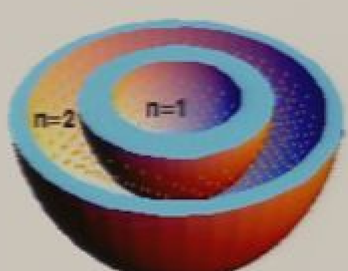
Carbon
Nanotube

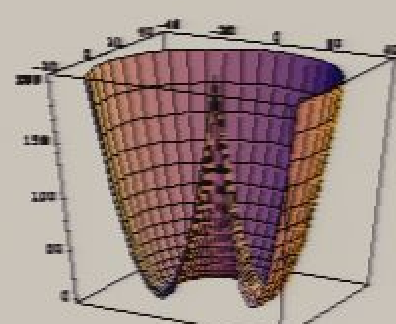
B Superconducting
wire

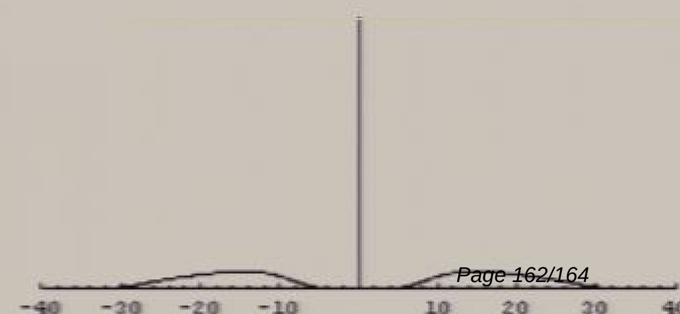




BEC and optical lattice:







Work in progress: entanglement in XXZ

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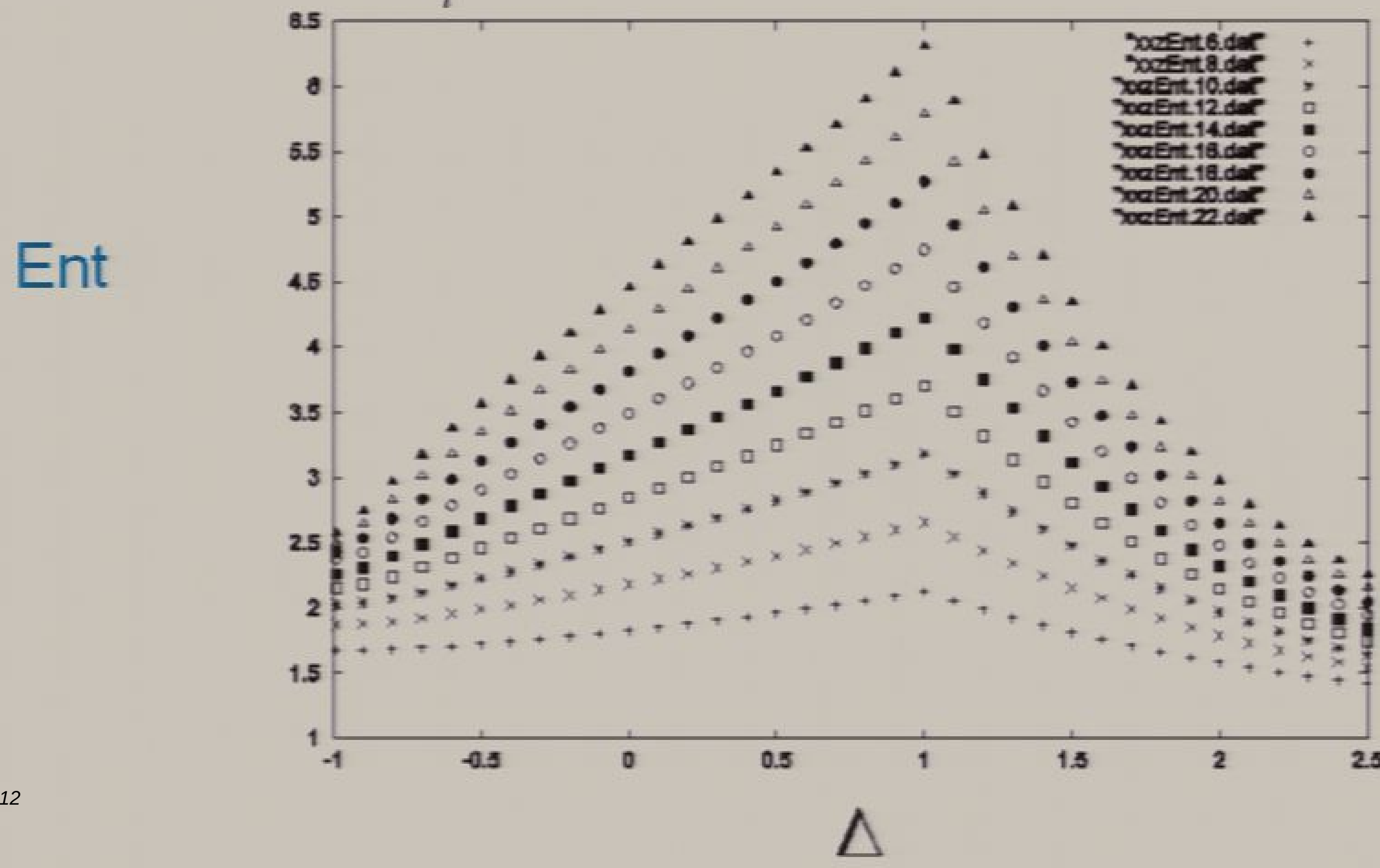

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