Title: Additivity Conjectures in Quantum Information Theory

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Abstract: Quantum information theory has two equivalent mathematical conjectures concerning quantum channels, which are also equivalent to other important conjectures concerning the entanglement. In this talk I explain these conjectures and introduce recent results.

Additivity conjectures in quantum information theory

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Spectral decomposition of a state

Take a state ρ. Since ρ is Hermitian it has a spectral decomposition:

$$\rho = \sum_i \lambda_i |v_i\rangle\langle v_i|,$$

where λ_i are the eigenvalues and $|v_i\rangle$ are the normalized eigenvectors. The number of nonzero eigenvalues is the rank of ρ. When the rank is one, the state is called a pure state.

Hilbert spaces and quantum states

A quantum state is represented as a positive semi-definite Hermitian operator of trace one in a separable Hilbert space $\mathcal{H}$. The set of quantum states is written by $\mathcal{D}(\mathcal{H})$. (We write $\mathcal{B}(\mathcal{H})$ for the algebra of operators in $\mathcal{H}$.)

A separable Hilbert space $\mathcal{H}$ is a complex vector space equiped with an inner product $\langle , \rangle$ such that it is complete with this norm. When the dimension is finite it is same as $\mathbf{C}^d$. In the following we only consider finite dimensional Hilbert spaces.

An operator A is Hermitian if $A^* = A$. Here the adjoint A^* of A is defined as $\langle Av, v \rangle = \langle v, A^* v \rangle$ for any vecror v. A Hermitian operator is positive semi-definite if $\langle v, Av \rangle \geq 0$ for any vector $v \in \mathcal{H}$. The trace is defined as $\mathrm{tr} A = \sum_i \langle v_i, Av_i \rangle$, where $\{v_i\}_i$ is any orthonormal basis in $\mathcal{H}$.

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Quantum channels

Take a linear map

$$\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2).$$

Then, Φ is a (quantum) channel if it is completely positive (CP) trace-preserving (TP). A map Φ is CP if for any Hilbert space $\mathcal{K}$ the product $\Phi \otimes 1_{\mathcal{K}}$ is positive, where $1_{\mathcal{K}}$ is the identity map on $\mathcal{B}(\mathcal{K})$.

A CP map Φ can be written in the Kraus form:

$$\Phi(\rho) = \sum_{k=1}^{N} A_k \rho A_k^*$$

where $A_k : \mathcal{H}_1 \to \mathcal{H}_2$ are linear maps. Then, Φ is TP if

$$\sum_k A_k^* A_k = I_{\mathcal{H}_1}.$$

Stinespring dilation theorem

Take an N-dimensional space $\mathcal{A}$ to form the isometry:

$$\begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix} : \mathcal{H}_1 \to \mathcal{A} \otimes \mathcal{H}_2.$$

Here, $\dim \mathcal{A} = N$. Then the above channel Φ is

$$\Phi(\rho) = \mathrm{tr}_{\mathcal{A}} \begin{pmatrix} A_1 \\ \vdots \\ A_N \end{pmatrix} \rho \begin{pmatrix} A_1^* \dots A_N^* \end{pmatrix} = \mathrm{tr}_{\mathcal{A}} \begin{pmatrix} A_1 \rho A_1^* & \dots & A_1 \rho A_N^* \\ \vdots & \ddots & \vdots \\ A_N \rho A_1^* & \dots & A_N \rho A_N^* \end{pmatrix}.$$

Aside. Taking trace over $\mathcal{H}_2$, instead of $\mathcal{A}$, gives another channel:

$$(\tilde{\Phi}(\rho))_{ij} = \mathrm{tr} A_i \rho A_j^*.$$

This is called the complementary (conjugate) channel of Φ. (Holevo; King, Matsumoto, Nathanson, Ruskai)

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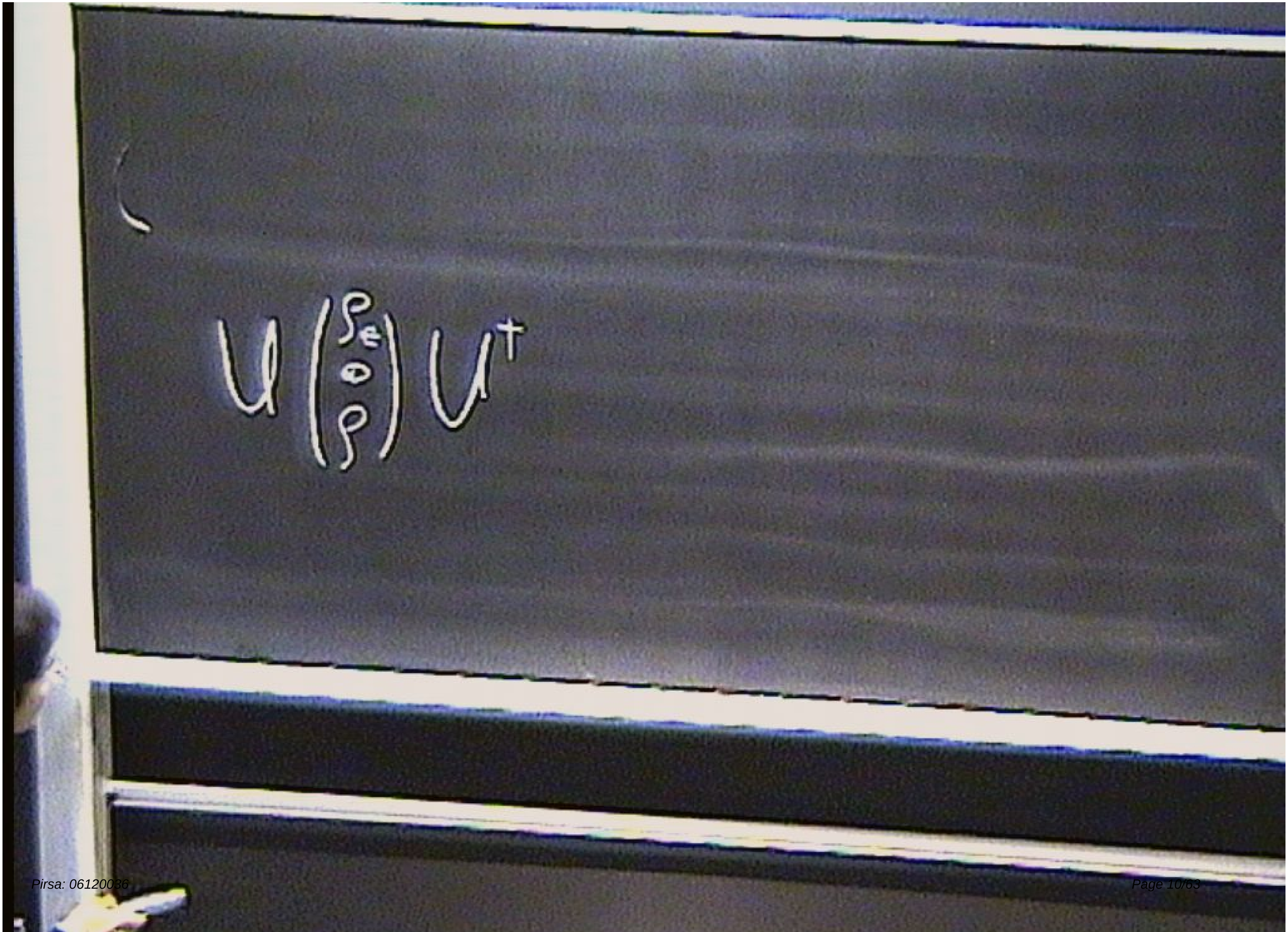
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Measure of the noisiness

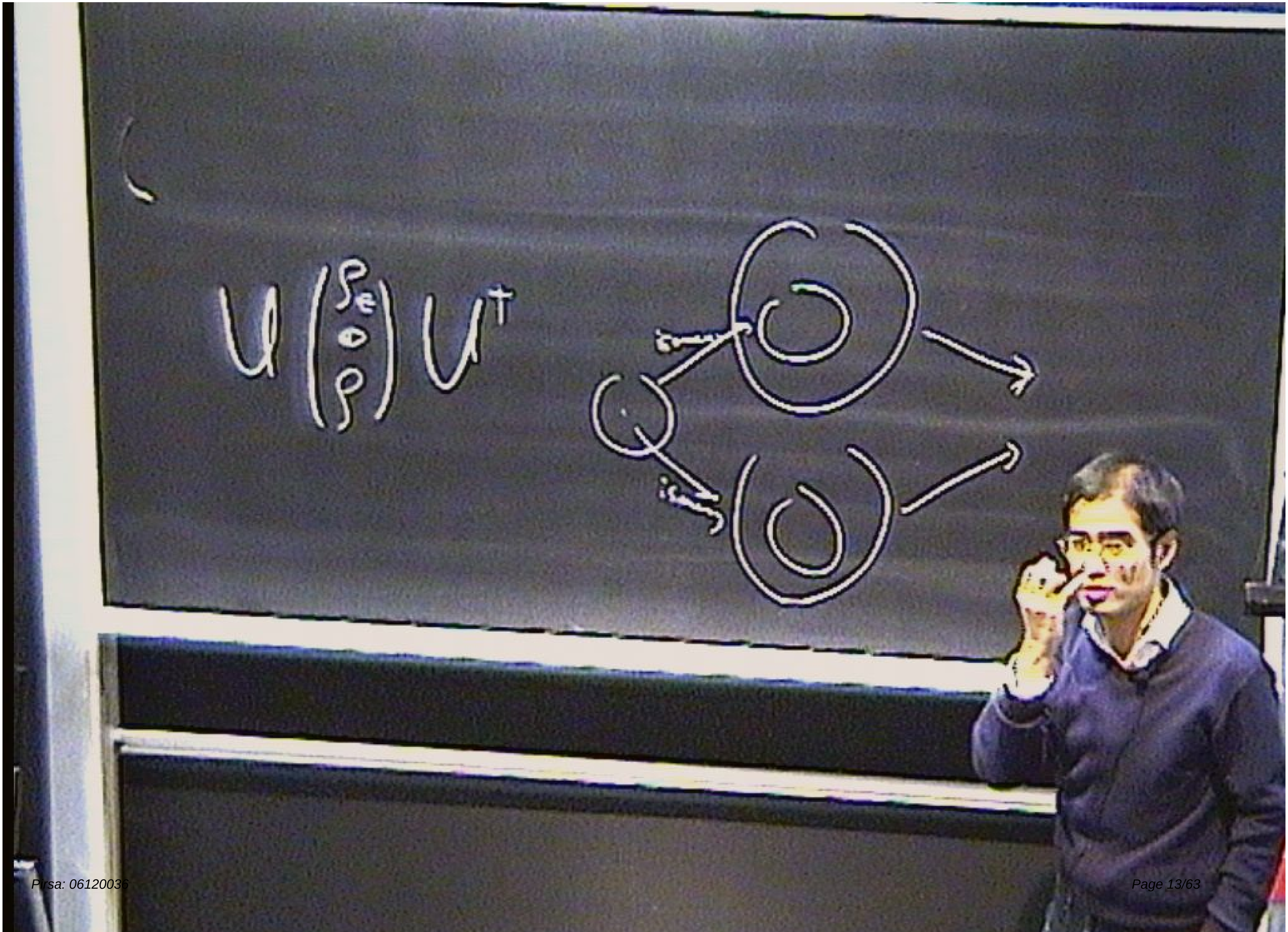
A natural class of measures of the noisiness of a quantum channel is how close can the output be to a pure state as measured by the minimal ouput entropy (MOE) or the maximal output p-norm (MOpN).

1) The MOE of Φ is defined as

$$S_{\min}(\Phi) = \inf_{\rho \in \mathcal{D}(\mathcal{H})} S(\Phi(\rho))$$

2) The MOpN of Φ is defined as

$$\nu_p(\Phi) = \sup_{\rho \in \mathcal{D}(\mathcal{H})} \|\Phi(\rho)\|_p.$$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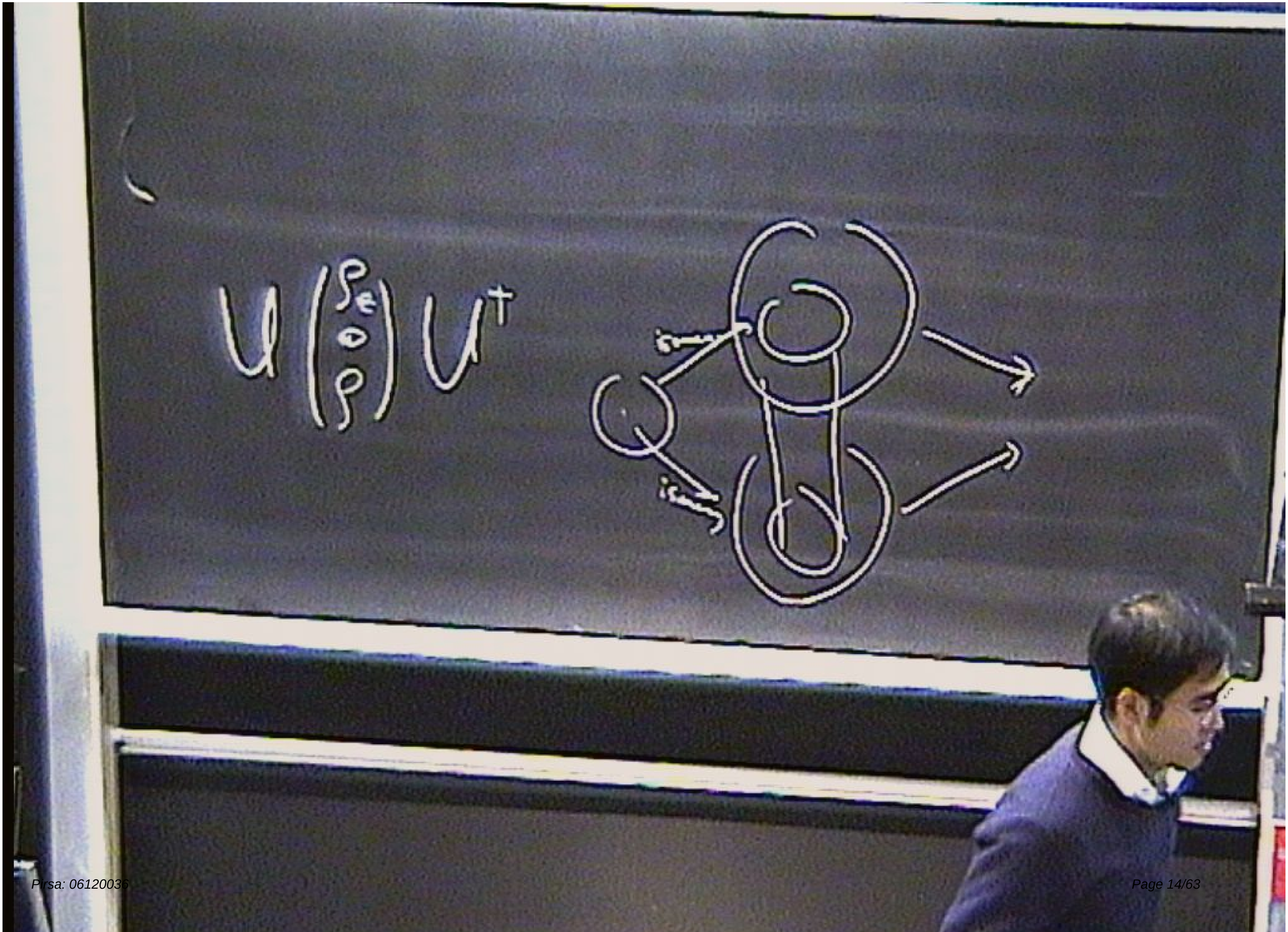
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Holevo capacity

The Holevo capacity of Φ is defined as

$$\chi(\Phi) = \sup_{\{p_i,\rho_i\}} \left\{ S\left(\sum_i p_i \Phi(\rho_i)\right) - \sum_i p_i S(\Phi(\rho_i)) \right\},$$

where $\{p_i\}_i$ is a probability distribution and $\{\rho_i\}_i$ are states.

The classical information capacity of a channel Φ is

$$C(\Phi) = \lim_{n\to\infty} \frac{1}{n}\chi(\Phi^{\otimes n}).$$

Conjectures

Suppose we have arbitrary two channels Φ and Ω.

1) The additivity conjecture of the MOE is that

$$S_{\text{min}}(\Phi \otimes \Omega) = S_{\text{min}}(\Phi) + S_{\text{min}}(\Omega).$$

2) The multiplicativity conjecture of the MOpN is that

$$\nu_p(\Phi \otimes \Omega) = \nu_p(\Phi)\nu_p(\Omega),$$

for any $p \in [1, 2]$. The multiplicativity was conjectured to be true for $p \in [1, \infty]$ before a counterexample was found by Holevo and Warner.

3) The additivity conjecture of the Holevo capacity is that

$$\chi(\Phi \otimes \Omega) = \chi(\Phi) + \chi(\Omega).$$

Entanglement

Think of the following two pure states:

$$\begin{pmatrix}1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \otimes \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$\begin{pmatrix}1/\sqrt{2} & 0 & 0 & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{pmatrix} = \begin{pmatrix}1/2 & 0 & 0 & 1/\sqrt{2}\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1/\sqrt{2} & 0 & 0 & 1/2\end{pmatrix}.$$

Definition. A state $\rho \in \mathcal{D}(\mathcal{H} \otimes \mathcal{K})$ is not entangled if

$$\rho = \sum_i p_i \left(\rho_{\mathcal{H}}^{(i)} \otimes \rho_{\mathcal{K}}^{(i)}\right)$$

for $\rho_{\mathcal{H}}^{(i)} \in \mathcal{D}(\mathcal{H}), \rho_{\mathcal{K}}^{(i)} \mathcal{D}(\mathcal{H})$ and a probability distribution $\{p_i\}$.

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Obvious inequalities

The following inequality is obvious:

$$S_{\min}(\Phi \otimes \Omega) \le S_{\min}(\Phi) + S_{\min}(\Omega).$$

To see this

$$\begin{aligned} S_{\min}(\Phi \otimes \Omega) &= \inf_{\hat{\rho} \in \mathcal{D}(\mathcal{H} \otimes \mathcal{K})} S((\Phi \otimes \Omega)(\hat{\rho})) \\ &\le \inf_{\hat{\rho} \in \mathcal{D}(\mathcal{H} \otimes \mathcal{K})} \left\{ S((\Phi \otimes \Omega)(\hat{\rho})) : \hat{\rho} = \sum_i p_i \left(\rho_{\mathcal{H}}^{(i)} \otimes \rho_{\mathcal{K}}^{(i)} \right) \right\} \\ &= \inf_{\rho_{\mathcal{H}} \otimes \rho_{\mathcal{K}}} S((\Phi \otimes \Omega)(\rho_{\mathcal{H}} \otimes \rho_{\mathcal{K}})) \\ &= \inf_{\rho_{\mathcal{H}} \otimes \rho_{\mathcal{K}}} S(\Phi(\rho_{\mathcal{H}}) \otimes \Omega(\rho_{\mathcal{K}})) = S_{\min}(\Phi) + S_{\min}(\Omega). \end{aligned}$$

Similarly we have

$$\nu_p(\Phi \otimes \Omega) \ge \nu_p(\Phi)\nu_p(\Omega); \qquad \chi(\Phi \otimes \Omega) \ge \chi(\Phi) + \chi(\Omega).$$

S. A. of convex closure of output entropy		**-Global view-**
$\Downarrow$		
S. A. of entanglement of formation	$\Leftarrow$	A. of entanglement of formation
$\Downarrow$ $\star$		$\Uparrow$
A. of Holevo capacity	$\Rightarrow$	A. of minimal output entropy
		$\Uparrow$ $*$
		M. of maximal output p-norm

$\star$: Shor; Pomeransky; Audenaert, Braunstein. $*$: Holevo.

Definition. A channel Φ is called bistochastic if

$$\Phi(\bar{I}_{\mathcal{H}_1}) = \bar{I}_{\mathcal{H}_2}.$$

Here $\bar{I}_{\mathcal{H}_1} = I_{\mathcal{H}_1}/\dim\mathcal{H}_1$, called the normalised identity, where $I_{\mathcal{H}_1}$ is the identity operator in $\mathcal{H}_1$ ($\bar{I}_{\mathcal{H}_2}$ is similarly defined). When $\mathcal{H}_1 = \mathcal{H}_2$ a bistochastic channel is called a unital channel.

Theorem 1. *Take a channel Ω. The additivity of MOE of $\Phi_b \otimes \Omega$ for any bistochastic channel Φ_b would imply that of $\Phi \otimes \Omega$ for any channel Φ.*

Outline of proof. Suppose we have a channel

$$\Phi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2).$$

Let $d = \dim \mathcal{H}_2$. Then we construct a new channel Φ':

$$\begin{aligned} \Phi' : \mathcal{B}(\mathbf{C}^{d^2} \otimes \mathcal{H}_1) &\longrightarrow \mathcal{B}(\mathcal{H}_2) \\ \tilde{\rho} &\longmapsto \sum_z W_z \Phi(E_z \tilde{\rho} E_z^*) W_z^*. \end{aligned}$$

Here W_z are the Weyl operators in $\mathcal{H}_2$ and $E_z = (\langle z| \otimes I_{\mathcal{H}_1})$, where $\{|z\rangle\}$ forms the standard basis for $\mathbf{C}^{d^2}$. Note that this channel is bistochastic.

To prove this theorem we show that

$$S_{\min}(\Phi' \otimes \Psi) = S_{\min}(\Phi \otimes \Psi)$$

for any channel Ψ. Indeed, it would show that

$$\begin{aligned} S_{\min}(\Phi \otimes \Omega) = S_{\min}(\Phi' \otimes \Omega) &= S_{\min}(\Phi') + S_{\min}(\Omega) \\ &= S_{\min}(\Phi) + S_{\min}(\Omega). \qquad \text{QED} \end{aligned}$$

Corollary 2. *The assumption of theorem 1 would be implied by proving the conjecture on $\Phi_u \otimes \Omega$ for all unital channels Φ_u.*

Proof. Consider a unital channel:

$$\Phi'' : \mathcal{B}(\mathbf{C}^{d^2} \otimes \mathcal{H}_1) \longrightarrow \mathcal{B}(\mathbf{C}^{cd} \otimes \mathcal{H}_2)$$
$$\tilde{\rho} \longmapsto \bar{I}_{\mathbf{C}^{cd}} \otimes \Phi'(\tilde{\rho}).$$

Here c is the dimension of $\mathcal{H}_1$. Then

$$S((\Phi'' \otimes \Omega)(\hat{\rho})) = \log cd + S((\Phi' \otimes \Omega)(\hat{\rho}))$$

for any channel Ω and any state $\hat{\rho} \in \mathcal{D}(\mathbf{C}^{d^2} \otimes \mathcal{H}_1 \otimes \mathcal{K}_1)$. The results follow obviously. QED

Corollary 3. *The additivity of MOE of $\Phi_\mathrm{u} \otimes \Omega_\mathrm{u}$ for any unital channels Φ_u and Ω_u would imply that of $\Phi \otimes \Omega$ for any channels Φ and Ω.*

Remark.

1) We get similar results for the multiplicativity of maximal output of p-norm.

2) The channel Φ' is a bistochastic extension of Φ and the channelΦ'' is a unital extension of Φ.

3) In order to prove the theorem we generalized the channel extension which Shor used to prove that the additivity of Holevo capacity implies the additivity of MOE. Reading his proof in view of our proof one can notice that proving the additivity of Holevo capacity for all unital channels would imply the additivity for all channels.

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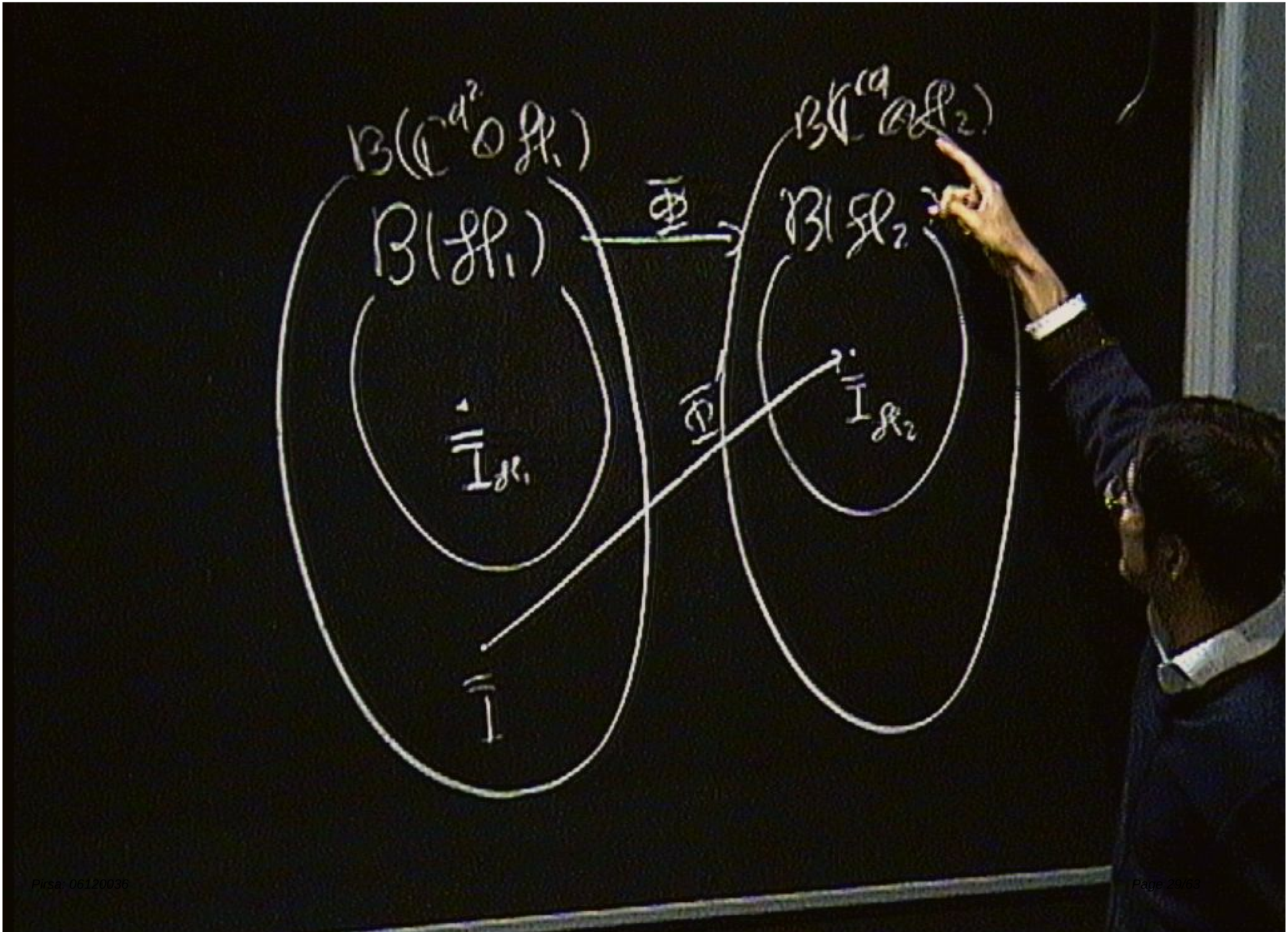
B(ℂ^{d²} ⊗ H₁)
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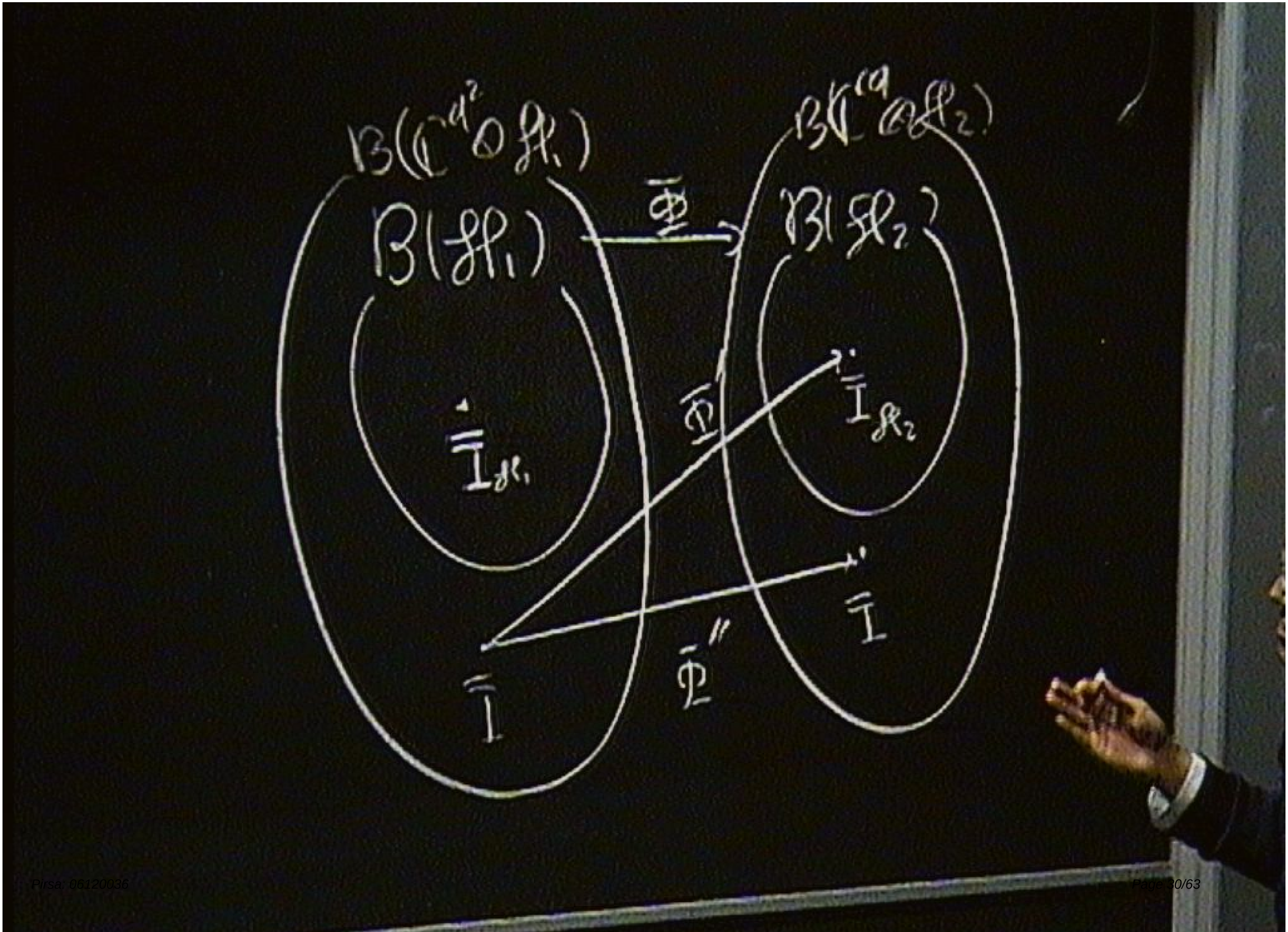
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Holevo capacity and minimal output entropy

Theorem 4. *Take any channels* Φ *and* Ω*. Then*

$$\chi(\Phi' \otimes \Omega') = \log d_h d_k - S_{\text{min}}(\Phi' \otimes \Omega').$$

Remark. Note that

$$\chi(\Phi'' \otimes \Omega) = \chi(\Phi' \otimes \Omega), \qquad \forall \Omega.$$

Then we have the following local equivalence relations.

$$\begin{array}{ccc} S_{\text{min}}(\Phi'' \otimes \Omega'') & \Leftrightarrow & \chi(\Phi'' \otimes \Omega'') \\ \Updownarrow & & \Updownarrow \\ S_{\text{min}}(\Phi' \otimes \Omega') & \Leftrightarrow & \chi(\Phi' \otimes \Omega') \\ \Updownarrow & & \\ S_{\text{min}}(\Phi \otimes \Omega) & & \chi(\Phi \otimes \Omega) \end{array}$$

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Results for unital channels

1. Unital qubit channels.
   King (2002)

2. Depolarizing channel:

$$\Phi(\rho) = \lambda\rho + (1-\lambda)\mathrm{tr}[\rho]\bar{I}.$$

   King (2002); Fujiwara, Hashizumé (2002); Amosov (2004)

3. Transpose depolarizing channel:

$$\Phi(\rho) = \lambda\rho^{\mathsf{T}} + (1-\lambda)\mathrm{tr}[\rho]\bar{I}.$$

   Datta, Holevo, Suhov (2004)

Results for non-unital channels

1. Entanglement-breaking channels.
   Shor (2002); King (2002)

2. A class:

   $$\Phi(\rho) = \lambda M(\rho) + (1-\lambda)\mathrm{tr}[\rho]\bar{I}.$$

   Here, M is a channel which has a rank one output.
   Fukuda (2005)

3. Another class:

   $$\Phi(\rho) = -\frac{1}{d-1}M(\rho) + \frac{\mathrm{tr}\rho}{d-1}I.$$

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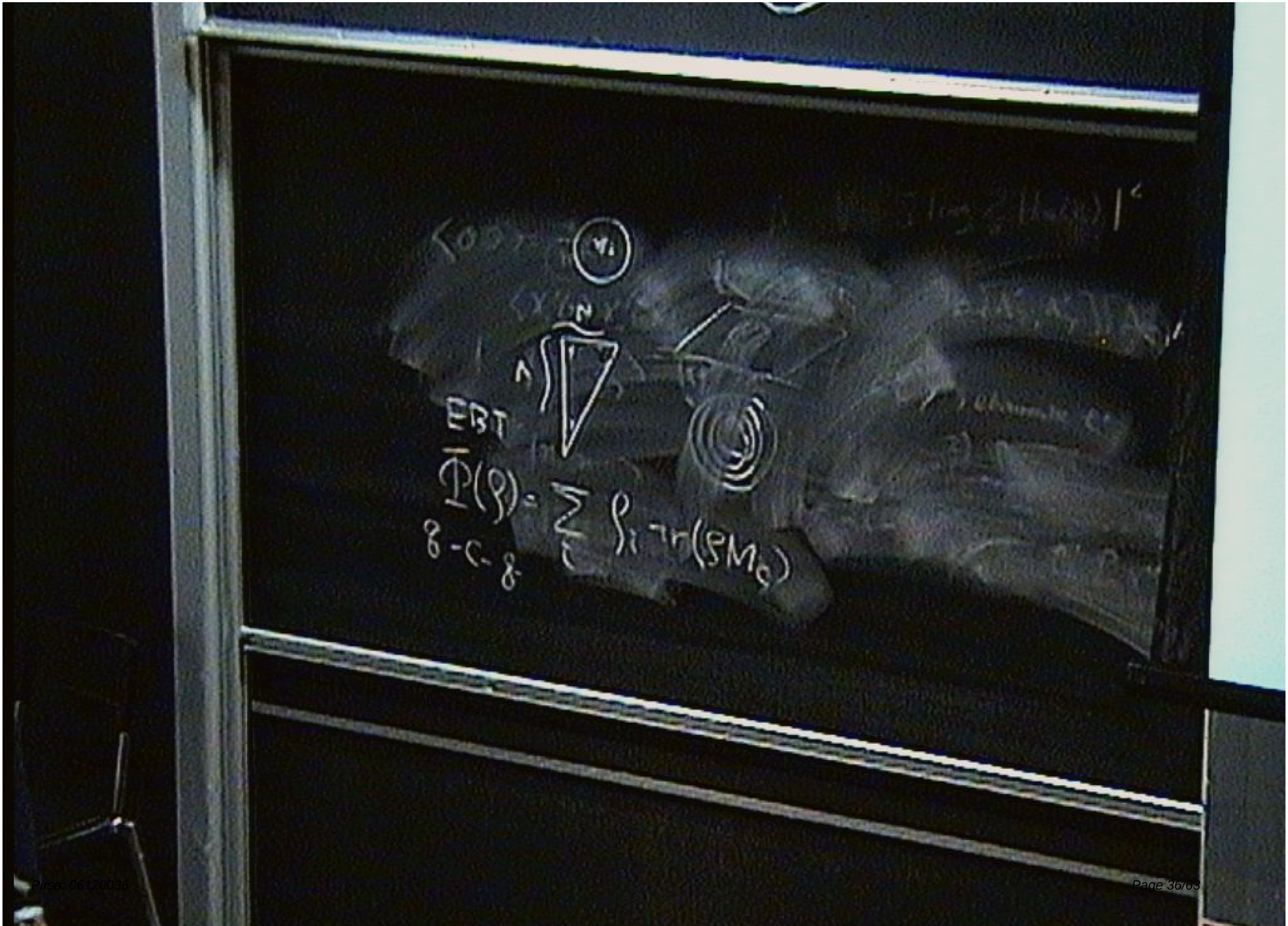
EBT
Φ(ρ) = Σ ρi tr(ρMi)
g-c-g

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Conclusion

To prove the additivity conjectures concerning quantum channels; the additivity of MOE, the additivity of Holevo capacity, the multiplicativity of MOpN we can focus on the special class of unital cases. The next step is to find examples of unital channels for which the conjecture is true so that we can find useful techniques to be generalized later. A candidate is the Weyl-covariant channel.

Acknowledgement

I would like to thank my supervisor Yuri Suhov for constant encouragement and numerous discussions. I also would like to thank Alexander Holevo and Mary Beth Ruskai for giving useful comments. Nilanjana Datta is also thanked for useful discussions.

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$$\begin{array}{ccc} S_{\min}(\Phi'' \otimes \Omega'') & \Leftrightarrow & \chi(\Phi'' \otimes \Omega'') \\ \Updownarrow & & \Updownarrow \\ S_{\min}(\Phi' \otimes \Omega') & \Leftrightarrow & \chi(\Phi' \otimes \Omega') \\ \Updownarrow & & \\ S_{\min}(\Phi \otimes \Omega) & & \chi(\Phi \otimes \Omega) \end{array}$$

Outline of proof. Suppose we have a channel

$$\Phi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_2).$$

Let $d = \dim \mathcal{H}_2$. Then we construct a new channel Φ':

$$\begin{aligned} \Phi' : \mathcal{B}(\mathbf{C}^{d^2} \otimes \mathcal{H}_1) &\longrightarrow \mathcal{B}(\mathcal{H}_2) \\ \tilde{\rho} &\longmapsto \sum_z W_z \Phi(E_z \tilde{\rho} E_z^*) W_z^*. \end{aligned}$$

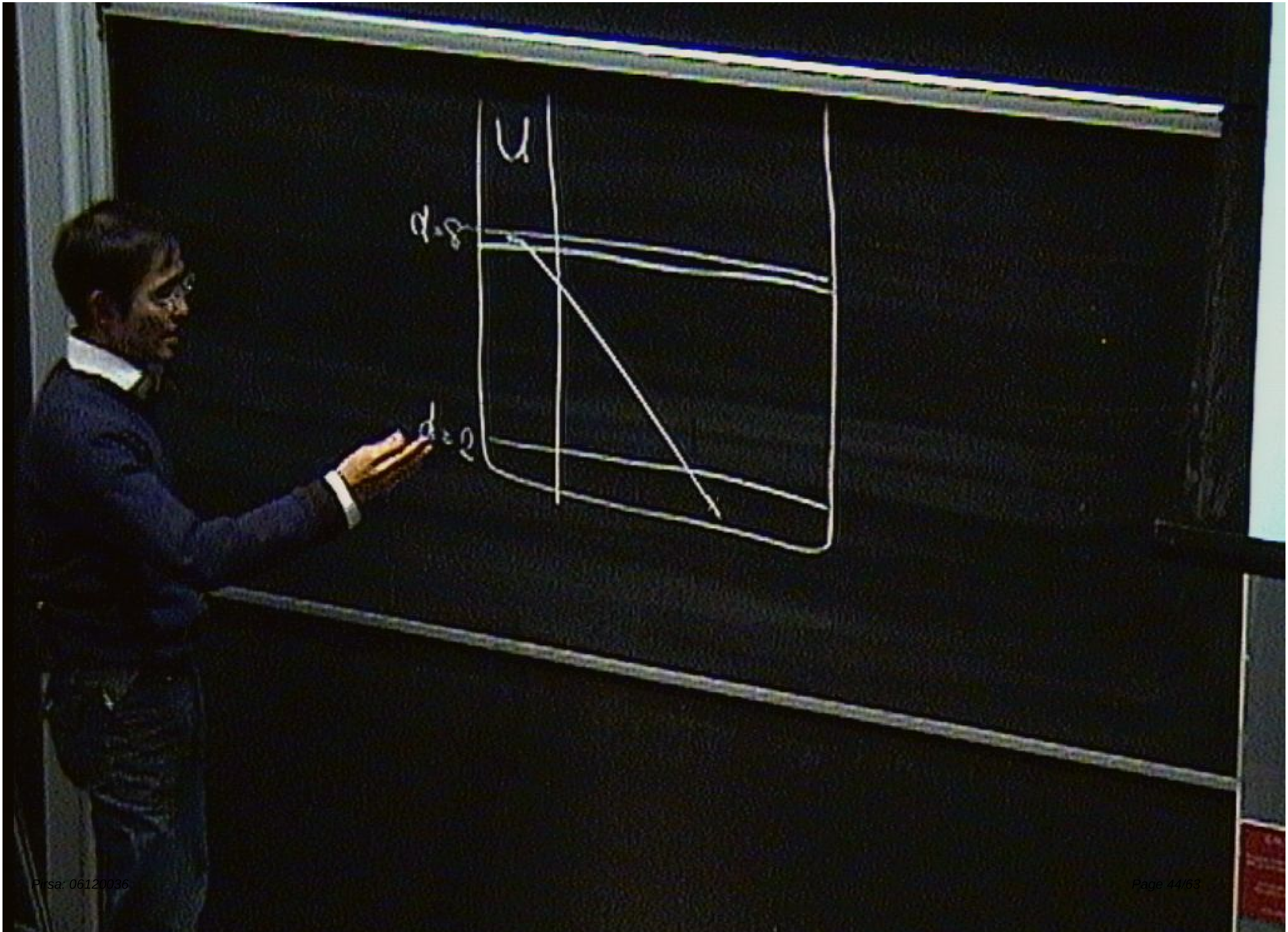
Here W_z are the Weyl operators in $\mathcal{H}_2$ and $E_z = (\langle z| \otimes I_{\mathcal{H}_1})$, where $\{|z\rangle\}$ forms the standard basis for $\mathbf{C}^{d^2}$. Note that this channel is bistochastic.

To prove this theorem we show that

$$S_{\min}(\Phi' \otimes \Psi) = S_{\min}(\Phi \otimes \Psi)$$

for any channel Ψ. Indeed, it would show that

$$\begin{aligned} S_{\min}(\Phi \otimes \Omega) &= S_{\min}(\Phi' \otimes \Omega) = S_{\min}(\Phi') + S_{\min}(\Omega) \\ &= S_{\min}(\Phi) + S_{\min}(\Omega). \qquad \text{QED} \end{aligned}$$



U
d=8
d=2

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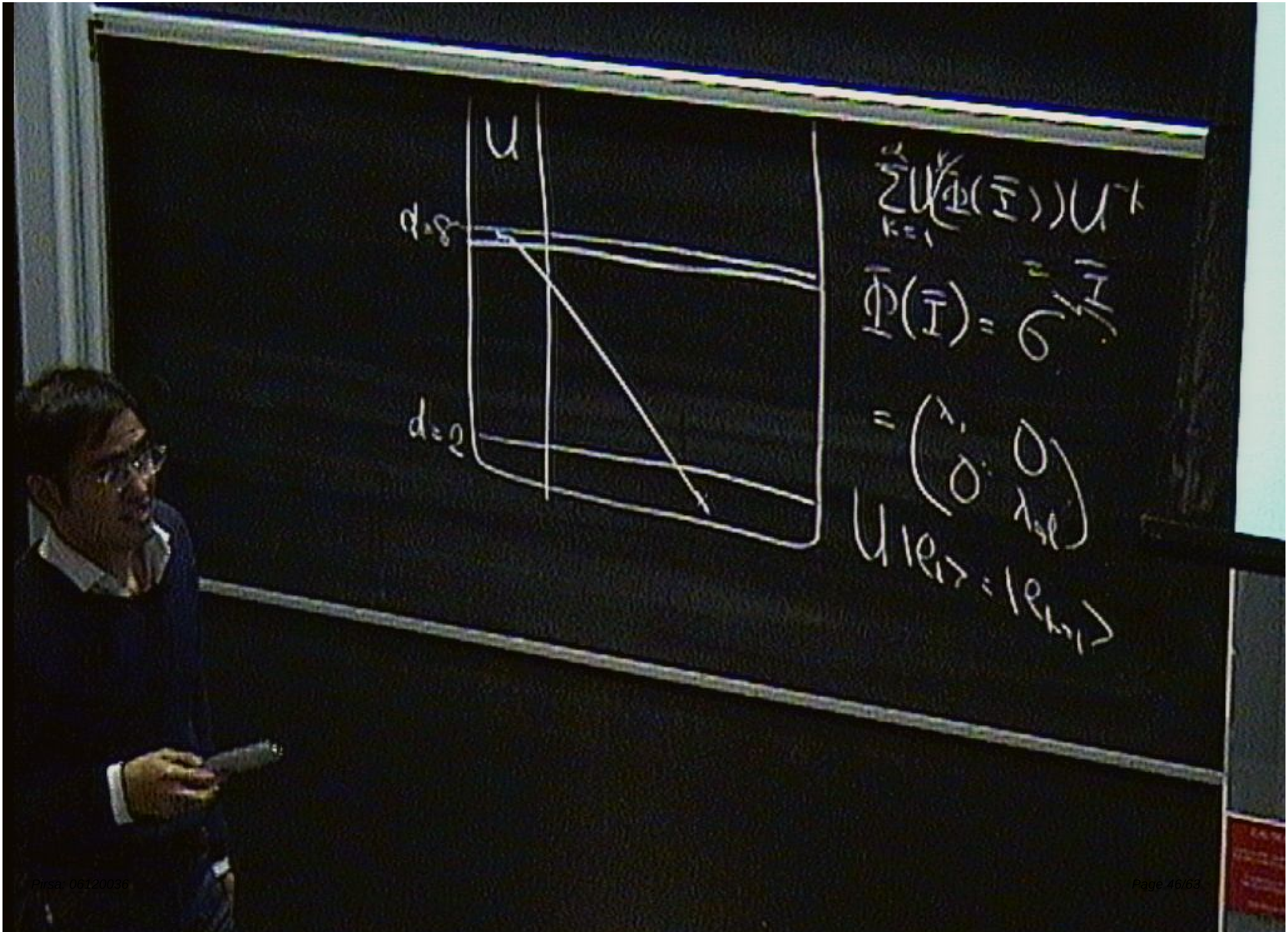
U
d=8
d=2

Definition. A channel Φ is called bistochastic if

$$\Phi(\bar{I}_{\mathcal{H}_1}) = \bar{I}_{\mathcal{H}_2}.$$

Here $\bar{I}_{\mathcal{H}_1} = I_{\mathcal{H}_1}/\dim\mathcal{H}_1$, called the normalised identity, where $I_{\mathcal{H}_1}$ is the identity operator in $\mathcal{H}_1$ ($\bar{I}_{\mathcal{H}_2}$ is similarly defined). When $\mathcal{H}_1 = \mathcal{H}_2$ a bistochastic channel is called a unital channel.

Theorem 1. *Take a channel Ω. The additivity of MOE of $\Phi_b \otimes \Omega$ for any bistochastic channel Φ_b would imply that of $\Phi \otimes \Omega$ for any channel Φ.*

Conclusion

To prove the additivity conjectures concerning quantum channels; the additivity of MOE, the additivity of Holevo capacity, the multiplicativity of MOpN we can focus on the special class of unital cases. The next step is to find examples of unital channels for which the conjecture is true so that we can find useful techniques to be generalized later. A candidate is the Weyl-covariant channel.

Acknowledgement

I would like to thank my supervisor Yuri Suhov for constant encouragement and numerous discussions. I also would like to thank Alexander Holevo and Mary Beth Ruskai for giving useful comments. Nilanjana Datta is also thanked for useful discussions.

Results for unital channels

1. Unital qubit channels.
   King (2002)

2. Depolarizing channel:

$$\Phi(\rho) = \lambda\rho + (1-\lambda)\mathrm{tr}[\rho]\bar{I}.$$

   King (2002); Fujiwara, Hashizumé (2002); Amosov (2004)

3. Transpose depolarizing channel:

$$\Phi(\rho) = \lambda\rho^{\top} + (1-\lambda)\mathrm{tr}[\rho]\bar{I}.$$

   Datta, Holevo, Suhov (2004)

$|e_k\rangle$

$U|e_k\rangle \mapsto |e_{k+1}\rangle$

$V|e_k\rangle \mapsto \left(\exp \frac{2\pi i}{d} k\right)|e_k\rangle$

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$$W_{(x,y)} = U^x V^y$$

$$\frac{1}{d^2} \sum_{x,y} W_{(x,y)} \, \rho \, W^{\dagger}_{(x,y)} = \bar{I}$$

$|e_k\rangle$ $\quad \mathrm{tr}[W_{xy}^{*} W_{x'y'}]$

$U|e_k\rangle \mapsto |e_{k+1}\rangle \quad = d\,\delta_{xx'}\delta_{yy'}$

$x, y \in \{1, \ldots, d-1\}$ $\quad V|e_k\rangle \mapsto (\exp \frac{2\pi i}{d} k)|e_k\rangle$

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$U|e_k\rangle \mapsto |e_{k+1}\rangle$

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$W_{(x,y)} = U^x V^y \qquad = c_{\dots} W_{(\dots)}$

$\frac{1}{d^2}\sum_{x,y} W_{(x,y)}\, \rho\, W^*_{(x,y)} = \bar{I}$

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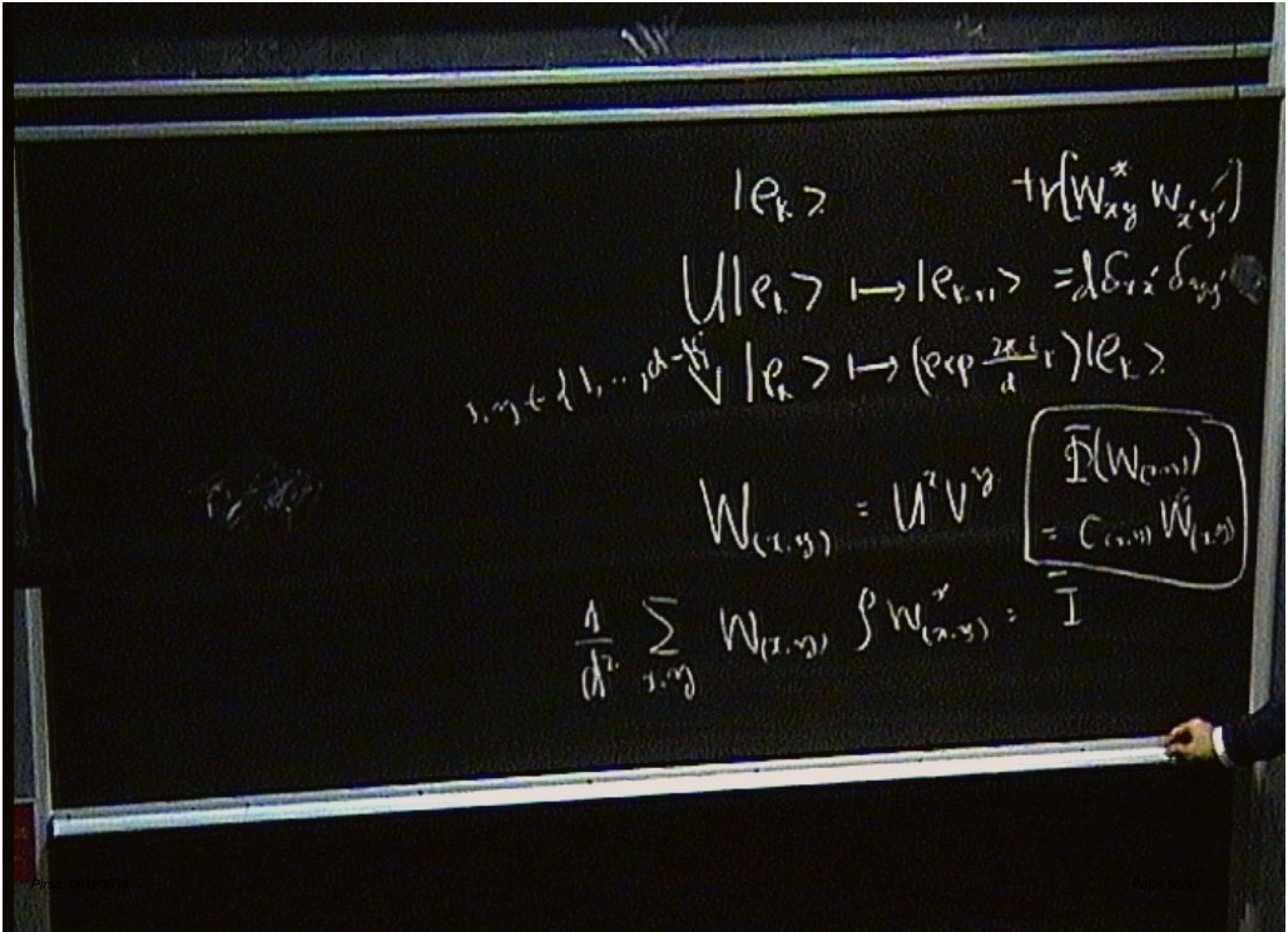
$W_{(x,y)} = U^x V^y$

$\Phi(W_{(x,y)}) = c_{(x,y)} W_{(x,y)}$

$\frac{1}{d^2}\sum_{x,y} W_{(x,y)}\, \rho\, W_{(x,y)}^* = \bar{I}$


|e_k>
tr(W*_{xy} W_{x'y'})
U|e_k> ↦ |e_{k+1}> = d δ_{xx'} δ_{yy'}
V|e_k> ↦ (exp 2πi/d k)|e_k>
x,y ∈ {1,…,d-1}
W_(x,y) = U^x V^y
Φ(W_(x,y)) = c_(x,y) W_(x,y)
1/d² Σ_{x,y} W_(x,y) ρ W*_(x,y) = I

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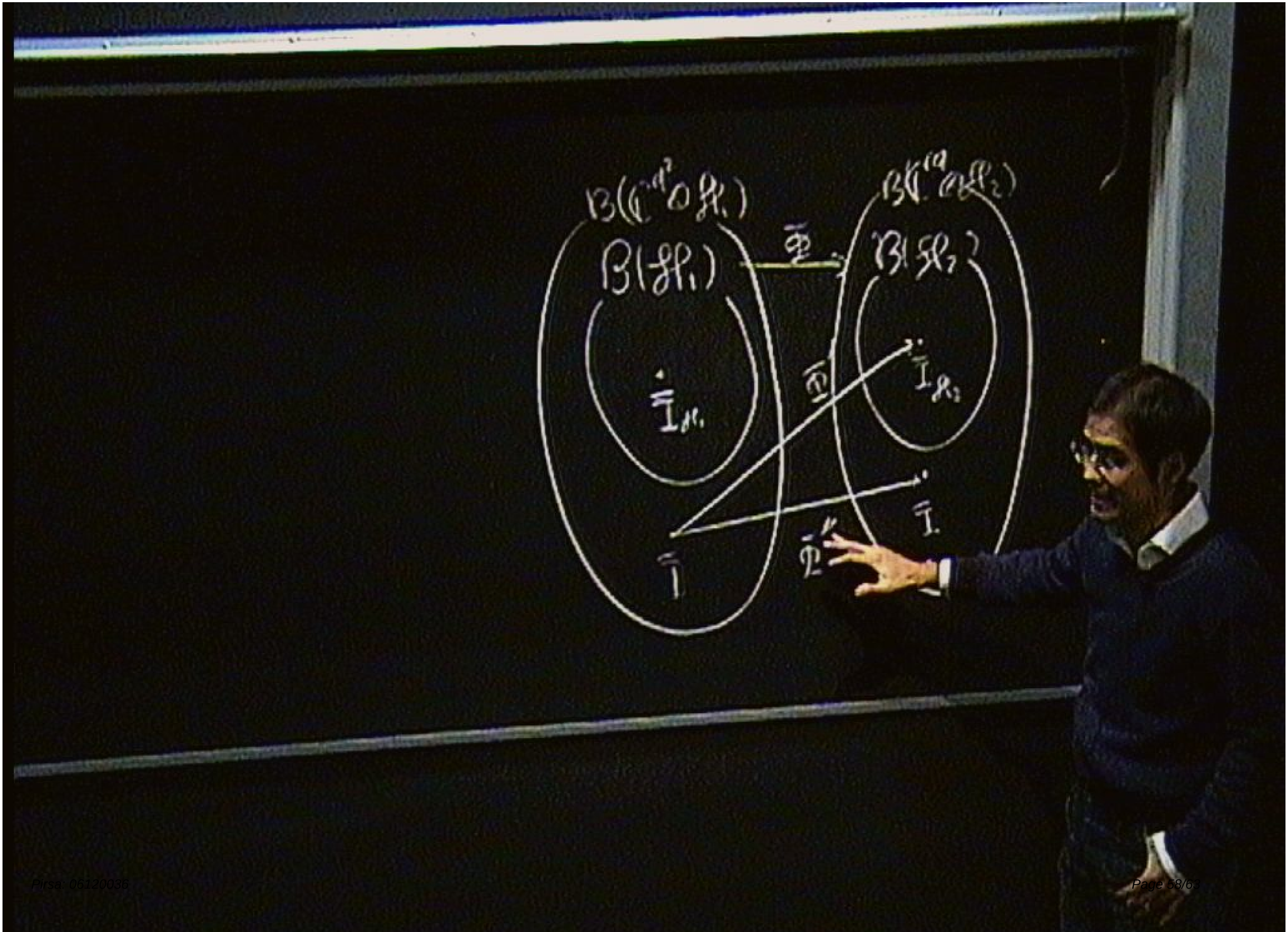

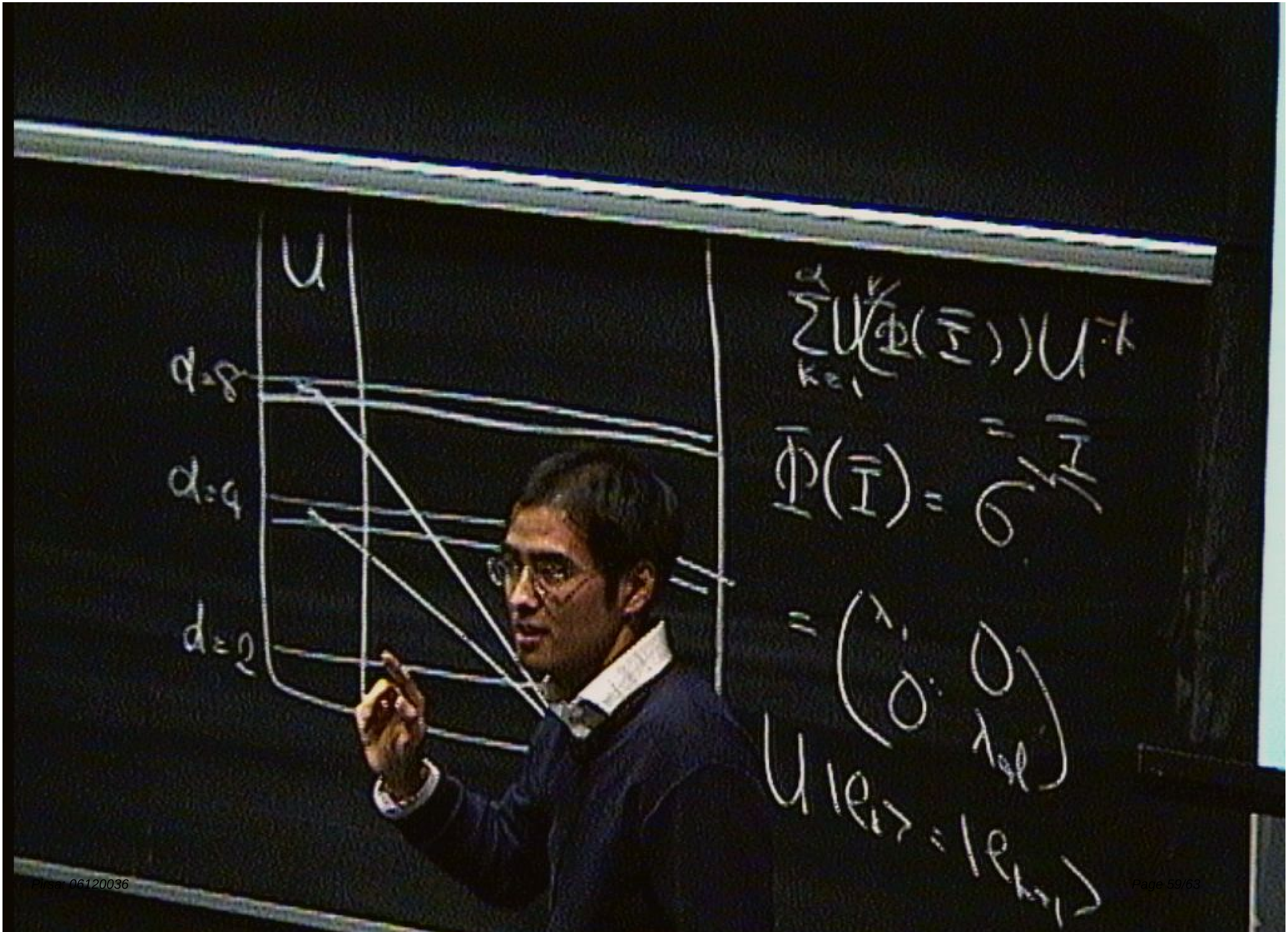
U
d=8
d=4
d=2

Results for unital channels

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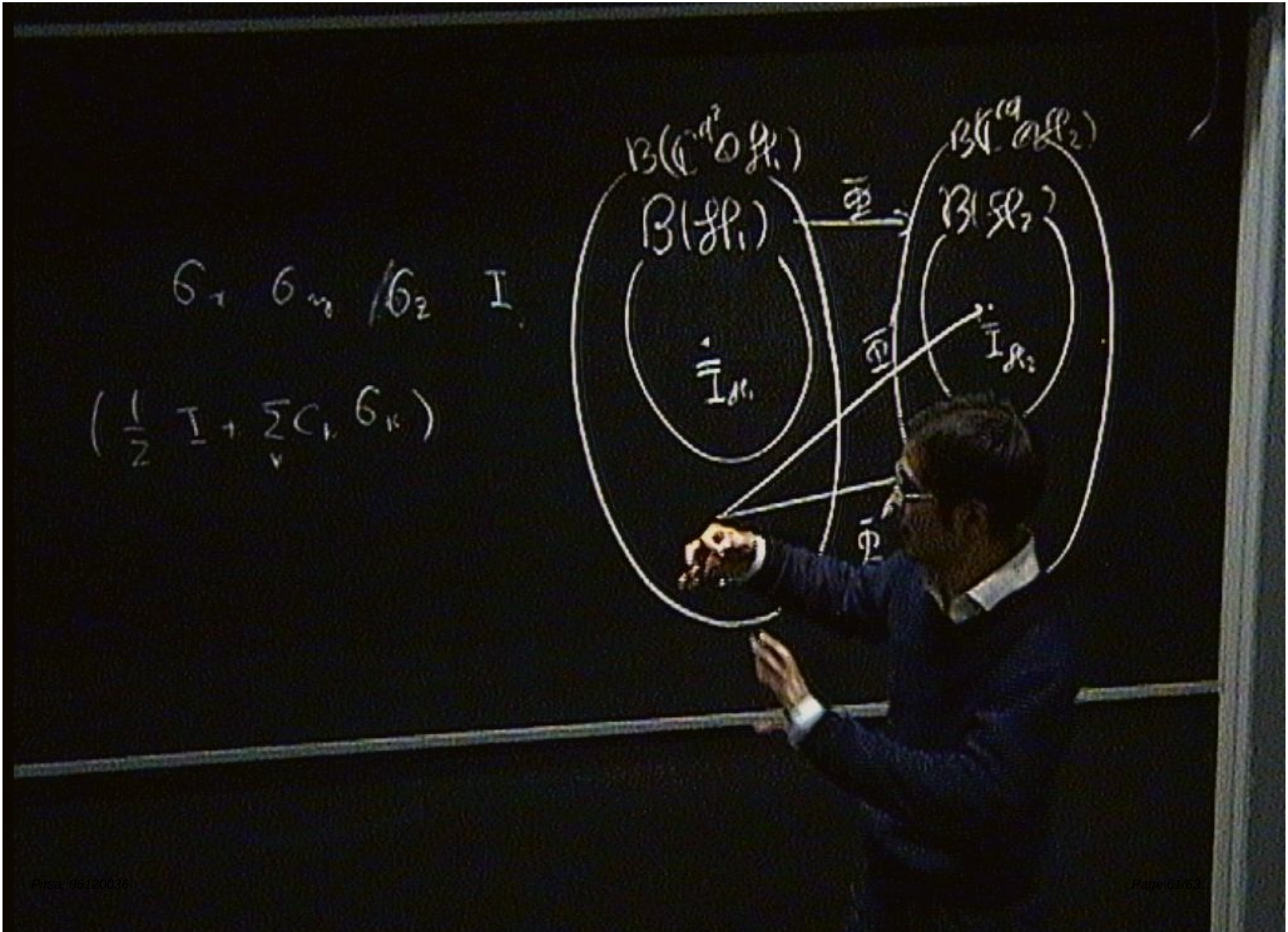
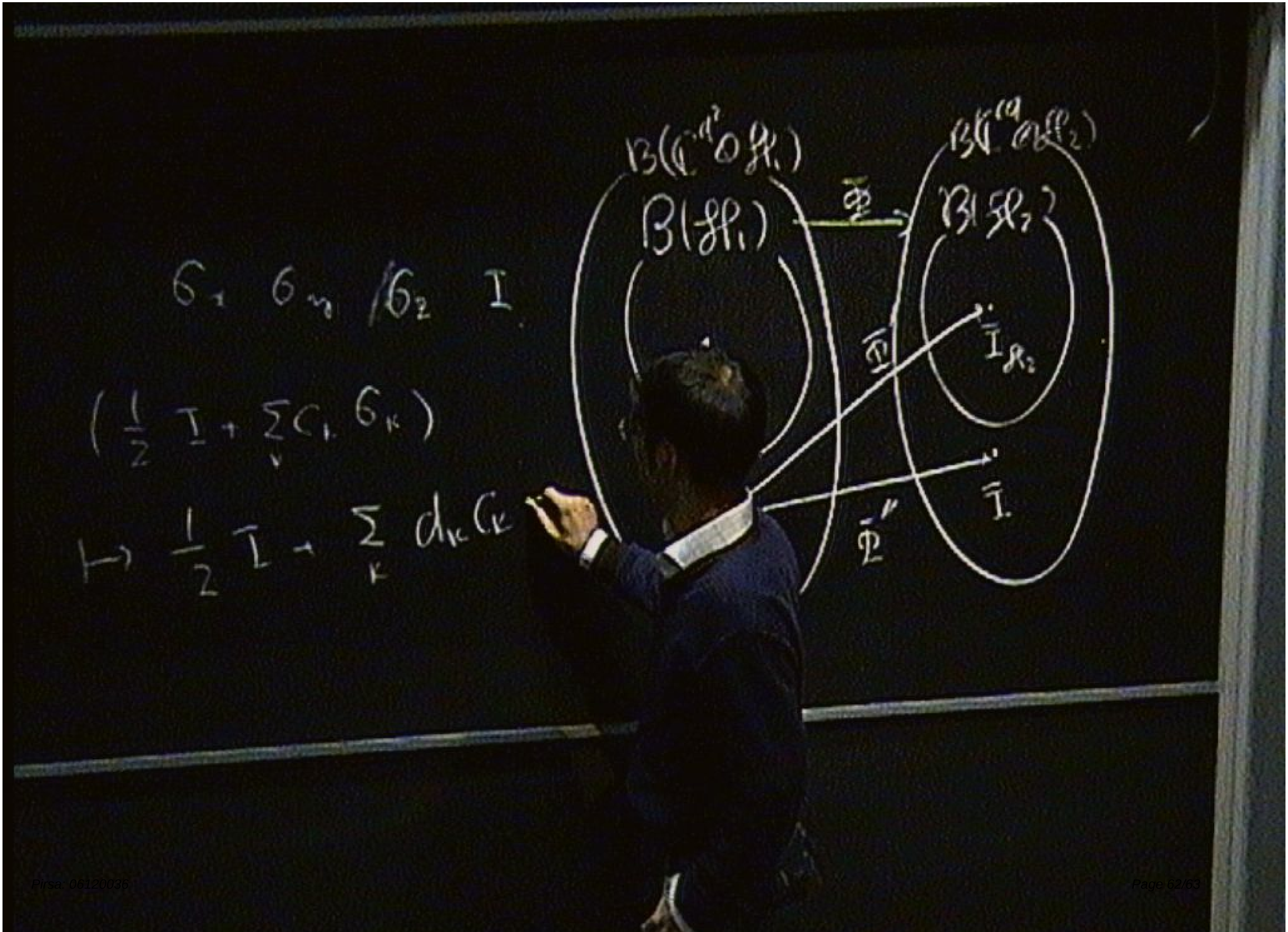
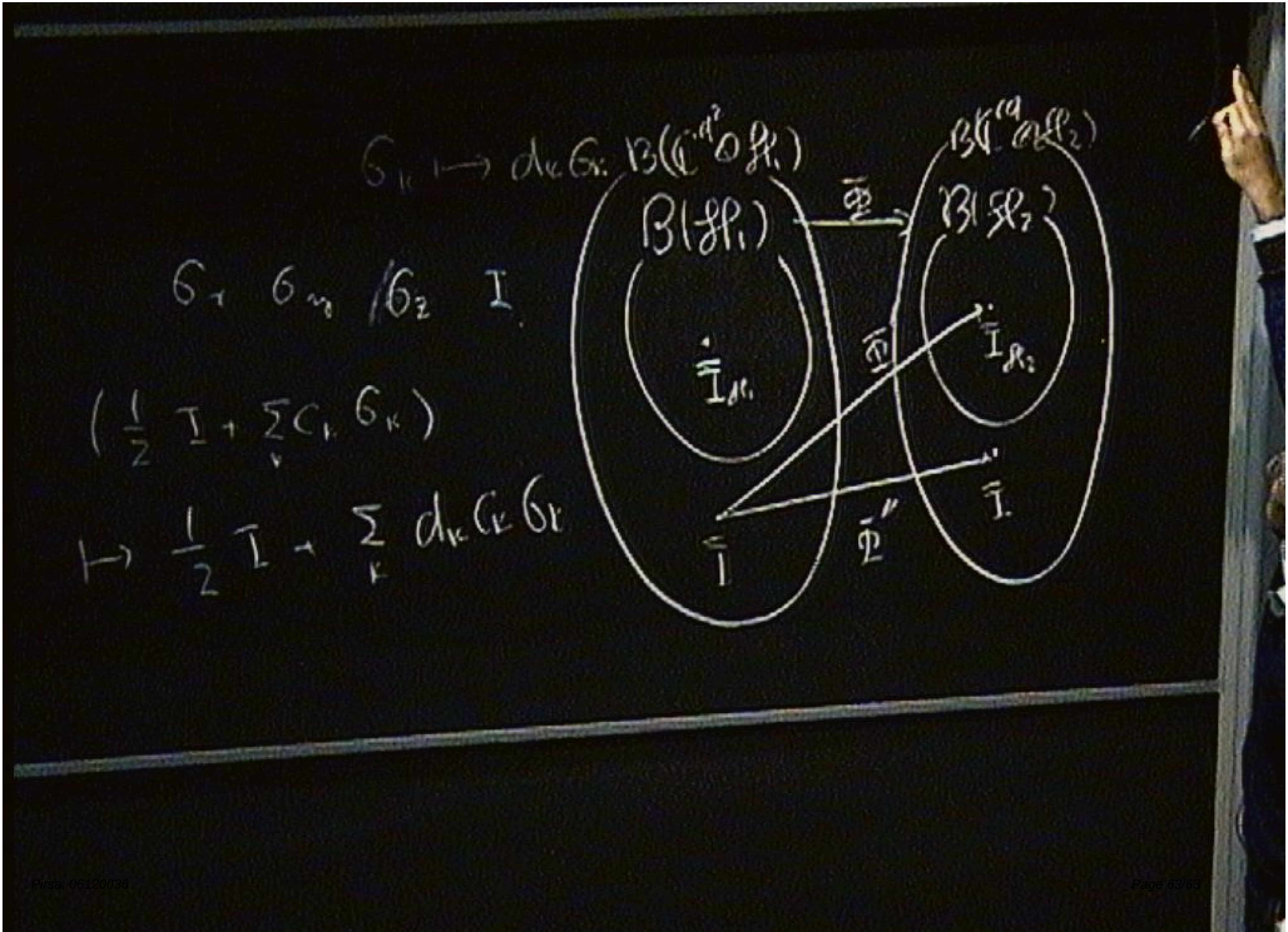

σ1 σ3 σ2 I
(1/2 I + Σ c_k σ_k)
B(ℂ^d ⊗ H1)
B(H1)
I_H1
B(ℂ^d ⊗ H2)
B(H2)
I_H2
Φ


σ_x σ_y σ_z I
(1/2 I + Σ_k c_k σ_k)
↦ 1/2 I + Σ_k d_k c_k
B(C^d ⊗ ℋ_1)
B(ℋ_1)
Φ
B(C^d ⊗ ℋ_2)
B(ℋ_2)
I_{ℋ_2}
I
Φ''


σ_k ↦ d_k σ_k
B(ℂ^d ⊗ H_1)
B(ℂ^d ⊗ H_2)
B(H_1)
B(H_2)
Φ
Φ'
Φ''
I_{H_1}
I_{H_2}
I
σ_x σ_y σ_z I
(1/2 I + Σ_k c_k σ_k)
↦ 1/2 I + Σ_k d_k c_k σ_k