Title: The Distinguishability of Random Quantum States

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Abstract: It is a fundamental property of quantum mechanics that non-orthogonal pure states cannot be distinguished with certainty, which leads to the following problem: Given a state picked at random from some ensemble, what is the maximum probability of success of determining which state we actually have? I will discuss two recently obtained analytic lower bounds on this optimal probability. An interesting case to which these bounds can be applied is that of ensembles consisting of states that are themselves picked at random. In this case, I will show that powerful results from random matrix theory may be used to give a strong lower bound on the probability of success, in the regime where the ratio of the number of states in the ensemble to the dimension of the states is constant. I will also briefly discuss applications to quantum computation (the oracle identification problem) and to the study of generic entanglement.

The distinguishability of random quantum states

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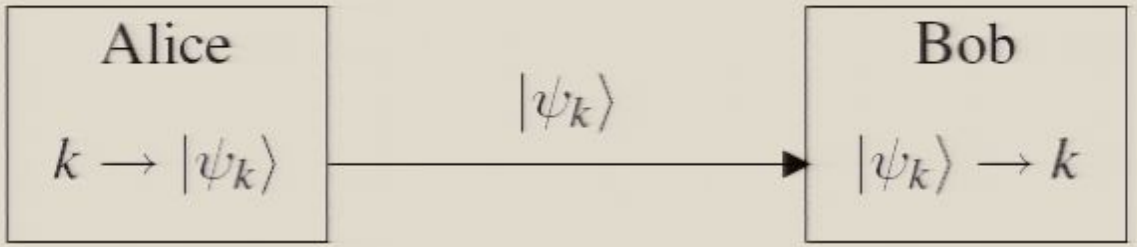
4th December 2006

University of BRISTOL

Distinguishing quantum states

We will consider a basic question in quantum measurement theory.



- Alice encodes a number k, $1 \leq k \leq n$, as a quantum state $|\psi_k\rangle$ picked from a known set of n states, and sends it to Bob.
- Bob measures $|\psi_k\rangle$ in the hope of determining k.
- What is Bob's optimal probability of success?

Distinguishing quantum states

More formally:

Question

Consider a known ensemble $\mathcal{E}$ of n quantum states $\{|\psi_i\rangle\}$ with known a priori probabilities p_i. Given an unknown state $|\psi_?\rangle$, picked at random from $\mathcal{E}$, what is the optimal probability $P^{opt}(\mathcal{E})$ of identifying $|\psi_?\rangle$? That is,

$$P^{opt}(\mathcal{E}) = \max_M \sum_{i=1}^{n} p_i \langle\psi_i|M_i|\psi_i\rangle$$

where we maximise over all POVMs $M = \{M_i\}$.

We think of $P^{opt}(\mathcal{E})$ as the *distinguishability* of the ensemble $\mathcal{E}$.

Previous work

- This problem has been considered by many authors since the 1970s, under titles like "quantum hypothesis testing", "quantum detection", "quantum state discrimination" etc.
- Many other optimality criteria have also been considered (e.g.: maximise information gain).

[1] C. Helstrom, *Quantum detection and estimation theory* (1978)
[2] Y. Eldar, A. Megretski, G. Verghese, quant-ph/0205178 (2002)

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- Helstrom derived an analytic expression for $P^{opt}(\mathcal{E})$ in the case where $\mathcal{E}$ contains 2 states [1].
- In general, producing an analytic expression for $P^{opt}(\mathcal{E})$ appears to be intractable (although good numerical solutions can be found[2])

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- In general, producing an analytic expression for $P^{opt}(\mathcal{E})$ appears to be intractable (although good numerical solutions can be found[2])
- We are therefore led to producing lower bounds on $P^{opt}(\mathcal{E})$.

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This talk

I will discuss:

1. Part I: the distinguishability of quantum states
   1. Using a specific measurement to lower bound $P^{opt}(\mathcal{E})$
   2. Two lower bounds on $P^{opt}(\mathcal{E})$: a "local" bound and a "global" bound
2. Part II: random quantum states
   1. Random quantum states and random matrix theory
   2. Lower bounds on the distinguishability of random quantum states
   3. Application: how mixed is my subsystem?
   4. Application: the "oracle identification problem" in quantum computation

Notation

I will use the following notation throughout the talk:

- $\mathcal{E} = \{|\psi_i\rangle\}$: the ensemble of states to distinguish
- p_i: the a priori probability of the i'th state
- $n = |\mathcal{E}|$: the number of states in $\mathcal{E}$
- d: the dimension of the states in $\mathcal{E}$

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- S: the $d \times n$ state matrix $S = (\sqrt{p_1}|\psi_1\rangle\ \sqrt{p_2}|\psi_2\rangle\ \cdots\ \sqrt{p_n}|\psi_n\rangle)$
- ρ: the density matrix $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$
- G: the Gram matrix $G_{ij} = \sqrt{p_i}\sqrt{p_j}\langle\psi_i|\psi_j\rangle$

- $P^M(\mathcal{E})$: the probability of success of measurement M applied to $\mathcal{E}$

Part I: the distinguishability of quantum states

1. Using a specific measurement to lower bound $P^{opt}(\mathcal{E})$
2. Two lower bounds on $P^{opt}(\mathcal{E})$: a "local" bound and a "global" bound

Methods

- The lower bounds are obtained by putting a lower bound on the probability of success of a specific measurement that can be defined for any ensemble of states, the *Pretty Good Measurement* (PGM)[3].
- For pure states, the PGM is defined by the set of measurement operators $\{|\mu_i\rangle\langle\mu_i|\}$, where $|\mu_i\rangle = \sqrt{p_i}\rho^{-1/2}|\psi_i\rangle$.
- It's easy to show that this always gives a valid measurement ($\sum_i |\mu_i\rangle\langle\mu_i| = I$)

[3] P. Hausladen, W. Wootters (1994)

The canonical nature of the PGM

The PGM has a number of desirable properties, including that:

- It can be defined analytically for any ensemble of states
- It's almost optimal for any ensemble $\mathcal{E}$ [4]:
  - $P^{pgm}(\mathcal{E}) \geq P^{opt}(\mathcal{E})^2$

For us, the important fact is that it's easy to analyse.

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Key fact

Let G be the rescaled Gram matrix of the ensemble $\mathcal{E}$, $G_{ij} = \sqrt{p_i p_j}\langle\psi_i|\psi_j\rangle$. Then the probability of success of the PGM is

$$P^{pgm}(\mathcal{E}) = \sum_{i=1}^{n} p_i |\langle\psi_i|\mu_i\rangle|^2 = \sum_{i=1}^{n} (\sqrt{G})_{ii}^2$$

Our two lower bounds will be based on lower bounding this sum.

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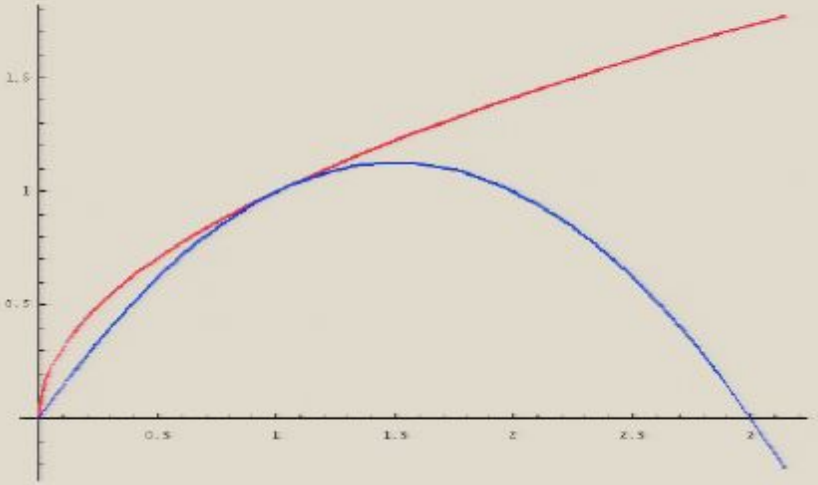
The pairwise inner product bound

- The first lower bound is based on a strategy used by Hausladen et al.[5] to get a bound in terms of the entries of the Gram matrix.
- A lower bound on the square root function by an "easier" function (a parabola) gives a lower bound on the $(\sqrt{G})_{ii}$.
- Works because $\sqrt{x} \geq ax + bx^2 \Rightarrow (\sqrt{G})_{ii} \geq aG_{ii} + b\sum_j |G_{ij}|^2$.



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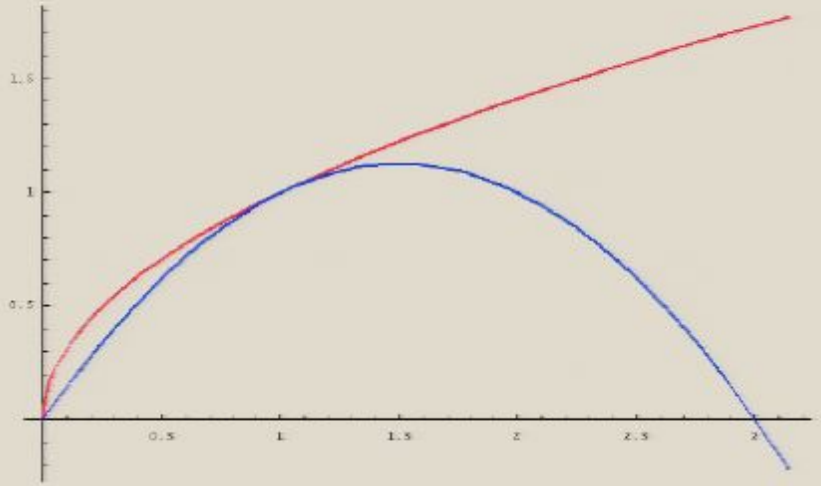

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- We can improve their bound by producing (for a given set of states) a set of optimal parabolae.
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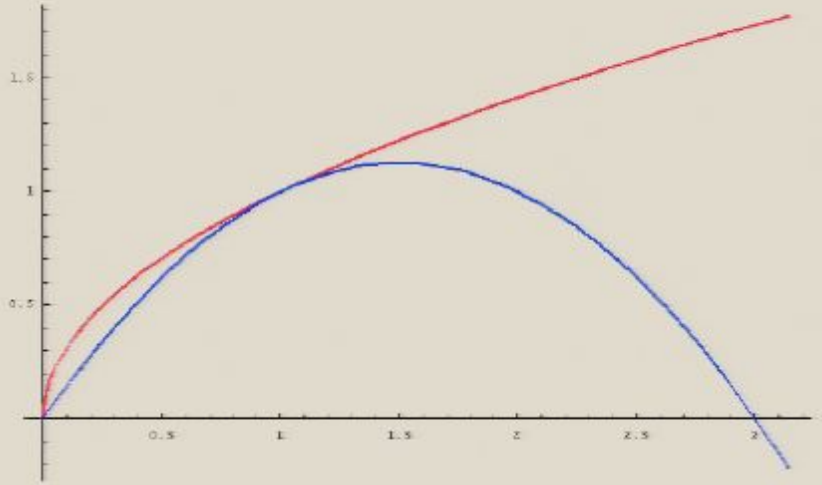

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- For each i, we look for a and b such that $\sqrt{x} \geq ax + bx^2$ for $x \geq 0$, and $aG_{ii} + b\sum_j |G_{ij}|^2$ is maximised.
- Only basic calculus is required to find these values of a and b, and substituting in gives the result:

Pairwise inner product bound

Let $\mathcal{E}$ be an ensemble of n states $\{|\psi_i\rangle\}$ with a priori probabilities p_i.

$$\text{Then} \quad P^{pgm}(\mathcal{E}) \geq \sum_{i=1}^{n} \frac{p_i^2}{\sum_{j=1}^{n} p_j |\langle\psi_i|\psi_j\rangle|^2}$$

The eigenvalue bound

The second lower bound is based on a global measure of distinguishability of the states in $\mathcal{E}$: the eigenvalues $\{\lambda_i\}$ of the Gram matrix G. The proof is simple:

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In terms of the trace norm, $P^{pgm}(\mathcal{E}) \geq \frac{1}{n}\|S\|_1^2 = \frac{1}{n}\left(\sum_i \sigma_i(S)\right)^2$.

Comparison with previous bounds

- Previous authors (e.g. Burnashev and Holevo [6]) have used bounds based on similar principles.
- But the bounds here are stronger, especially for low values of $P^{pgm}(\mathcal{E})$, and always give a non-trivial value.

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- Assuming the states in $\mathcal{E}$ have equal probabilities:

Comparison of bounds

Previously known lower bound	**New lower bound**
$P^{pgm}(\mathcal{E}) \geq 1 - \frac{1}{n}\sum_{i \neq j} \lvert\langle\psi_i\rvert\psi_j\rangle\rvert^2$	$P^{pgm}(\mathcal{E}) \geq \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\sum_{j=1}^{n} \lvert\langle\psi_i\rvert\psi_j\rangle\rvert^2}$
$P^{pgm}(\mathcal{E}) \geq \frac{2}{\sqrt{n}}\mathrm{tr}(\sqrt{G}) - 1$	$P^{pgm}(\mathcal{E}) \geq \frac{1}{n}\mathrm{tr}(\sqrt{G})^2$

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A local bound and a global bound

- It is interesting to note that the inner product bound only considers the pairwise distinguishability between states, while the eigenvalue bound is based on global features of the ensemble.
- We might therefore expect the latter to be stronger...

$$P^{pgm}(\mathcal{E}) \geq 1 - \frac{1}{n}\sum_{i\neq j}|\langle\psi_i|\psi_j\rangle|^2 \qquad P^{pgm}(\mathcal{E}) \geq \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sum_{j=1}^{n}|\langle\psi_i|\psi_j\rangle|^2}$$

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- Consider an ensemble of n states, each pair of which have the same inner product, $k \in \mathbb{R}^+$. Then it is possible to show that:
  - The inner product bound gives an almost trivial bound: $P^{pgm}(\mathcal{E}) \geq O(1/n)$
  - The eigenvalue bound gives a strong bound: $P^{pgm}(\mathcal{E}) \geq (1-k) - o(1)$

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  - The eigenvalue bound gives a strong bound: $P^{pgm}(\mathcal{E}) \geq (1-k) - o(1)$
- (NB: in this trivial case we can actually diagonalise the Gram matrix and calculate the probability of success of the PGM exactly)

Part II: random quantum states

1. Random quantum states and random matrix theory
2. Lower bounds on the distinguishability of random quantum states
3. Application: how mixed is my subsystem?
4. Application: the "oracle identification problem" in quantum computation

Random quantum states

- We will now apply the eigenvalue bound to the case where the states in $\mathcal{E}$ are random.
- To be precise, for all i:
  - $|\psi_i\rangle$ is distributed uniformly at random on the d-dimensional complex unit sphere (according to Haar measure)
  - $p_i = 1/n$ (the states are equiprobable)
- We will calculate the expected probability of success of identifying $|\psi_?\rangle$ in this case.

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- We will calculate the expected probability of success of identifying $|\psi_?\rangle$ in this case.

So we are given a state picked at random from a known set of states which are themselves randomly picked, and asked to determine which random state our randomly picked state actually is ☺

Random quantum states

How do we produce a state $|\psi\rangle$ distributed uniformly at random?

- Generate a vector v whose components v_i are complex Gaussians, then set $|\psi\rangle = v/\|v\|$.
  - i.e. v_i's real and complex parts are independently normally distributed with variance $1/2$; both parts have probability density function $\frac{1}{2\sqrt{\pi}} e^{-x^2/2}$ and $\mathbb{E}(|v_i|^2) = 1$.
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- This works because of the spherical symmetry of the multivariate normal distribution.

- It turns out that the normalisation step becomes "almost" unnecessary in high dimension (qv): rescaling v by $1/\sqrt{d}$ will give a complex vector whose norm is approximately 1.
- So the state matrix S is (almost!) a rescaled matrix of Gaussians: $S_{ij} \sim \widetilde{N}(0, 1/nd)$, and we need to calculate $\mathbb{E}(\frac{1}{n}\|S\|_1^2)$.

Random matrix theory

- Random matrix theory deals with the properties of matrices whose entries are random variables.

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- In particular, infinite-dimensional random matrix theory allows us to answer questions like "what is the limiting density of the eigenvalues of a family of $n \times n$ random matrices, as $n \to \infty$?".
  - By density, we mean the function $f(x)$ which integrates to $F(x) = \frac{1}{n}(\#eigenvalues < x)$
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- Statisticians have long studied the density of eigenvalues of the matrix $G = SS^\dagger$, where S is a random matrix: under certain conditions, it's given by the Marčenko-Pastur law [7].
  - This is the equivalent of the famous Wigner semicircle law for random Hermitian matrices...

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Marčenko-Pastur law

Let R_r be a family of $d \times n$ matrices with $n \geq d$ and $d/n \to r \in (0, 1]$ as $n, d \to \infty$, where the entries of R_r are i.i.d. complex random variables with mean 0 and variance 1. Then, as $n, d \to \infty$, the eigenvalues of the rescaled matrix $\frac{1}{n} R_r R_r^\dagger$ tend almost surely to a limiting distribution with density

$$p_r(x) = \frac{\sqrt{(x - A^2)(B^2 - x)}}{2\pi r x}$$

for $A^2 \leq x \leq B^2$ (where $A = 1 - \sqrt{r}$, $B = 1 + \sqrt{r}$), and density 0 elsewhere.

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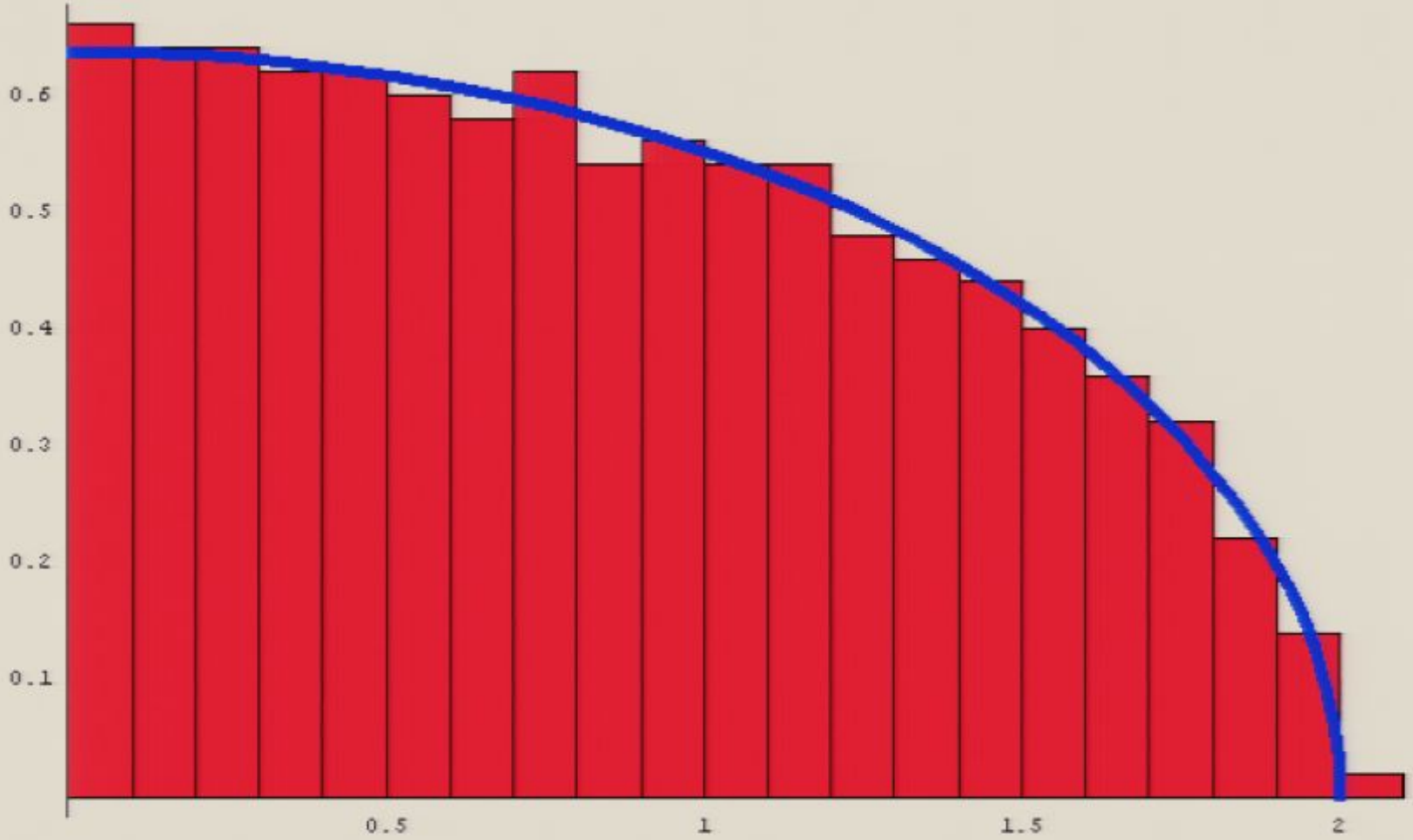
for $A^2 \leq x \leq B^2$ (where $A = 1 - \sqrt{r}$, $B = 1 + \sqrt{r}$), and density 0 elsewhere.

We can easily tweak this result to tell us the density of the singular values of R_r instead!

Experimental results



Blue: singular value density predicted by Marčenko-Pastur law

Red: empirical singular value distribution of a 500x500 matrix

Applying the Marčenko-Pastur law

We can use the M-P law to give us the expected trace norm of a random matrix, again under very weak conditions.

Expected trace norm

Let R_r be a family of $d \times n$ matrices with $k/m \to r \in (0, 1]$ as $n, d \to \infty$, where $k = \min(n, d)$ and $m = \max(n, d)$, and the entries of R_r are i.i.d. complex random variables with mean 0 and variance 1. Then, as $n, d \to \infty$, the expected trace norm of R_r tends almost surely to

$$\mathbb{E}(\|R_r\|_1) = \frac{m^{3/2}}{\pi} \int_A^B \sqrt{(y^2 - A^2)(B^2 - y^2)}\, dy$$

where $A = 1 - \sqrt{r}, B = 1 + \sqrt{r}$.

Applying the Marčenko-Pastur law (2)

We want to evaluate the following integral:

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Elliptic integral lower bound

Let $0 \leq r \leq 1$ and $A = 1 - \sqrt{r}$, $B = 1 + \sqrt{r}$. Then

$$\int_A^B \sqrt{(y^2 - A^2)(B^2 - y^2)}\, dy \geq r\pi\sqrt{1 - r\left(1 - \frac{64}{9\pi^2}\right)}$$

with equality at $r = 0$, $r = 1$.

(The proof is fairly long and involves representing the integral as the difference of two hypergeometric series and performing several transformations on these hypergeometric series...)

The asymptotic lower bound

Main theorem

Let $\mathcal{E}$ be an ensemble of n equiprobable d-dimensional quantum states $\{|\psi_i\rangle\}$ with $n/d \to r \in (0, \infty)$ as $n, d \to \infty$, and let the components of $|\psi_i\rangle$ in some basis be i.i.d. complex random variables with mean 0 and variance $1/d$. Then, as $n, d \to \infty$,

$$\mathbb{E}(P^{pgm}(\mathcal{E})) \geq \begin{cases} \frac{1}{r}\left(1 - \frac{1}{r}\left(1 - \frac{64}{9\pi^2}\right)\right) & \text{if } n \geq d \\ 1 - r\left(1 - \frac{64}{9\pi^2}\right) & \text{otherwise} \end{cases}$$

and in particular $\mathbb{E}(P^{pgm}(\mathcal{E})) > 0.720$ when $n \leq d$.

Concentration of measure results can be used to show that for almost all ensembles $\mathcal{E}$, $P^{pgm}(\mathcal{E}) \approx \mathbb{E}(P^{pgm}(\mathcal{E}))$.

Comparison with numerical results (1)
$(0 \leq n \leq 2d)$

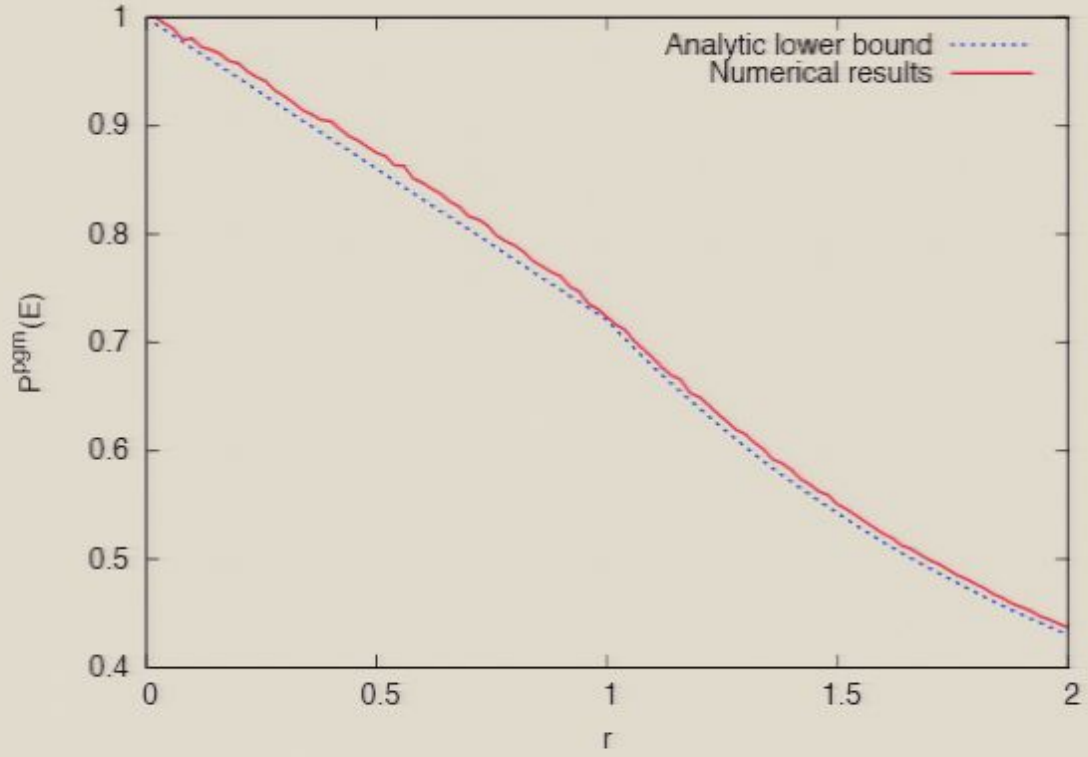
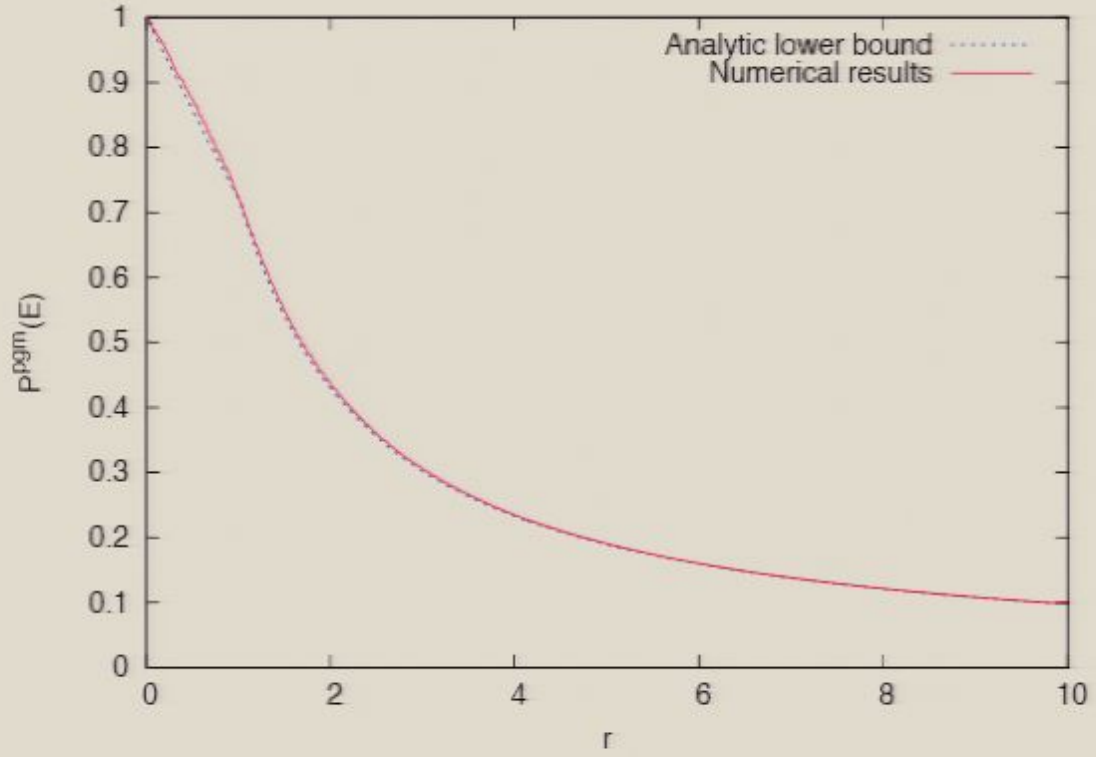


Figure: Asymptotic bound on $P^{pgm}(\mathcal{E})$ vs. numerical results (averaged over 10 runs) for ensembles of $n = 50r$ 50-dimensional uniformly random states.

Comparison with numerical results (2)
$(0 \leq n \leq 10d)$



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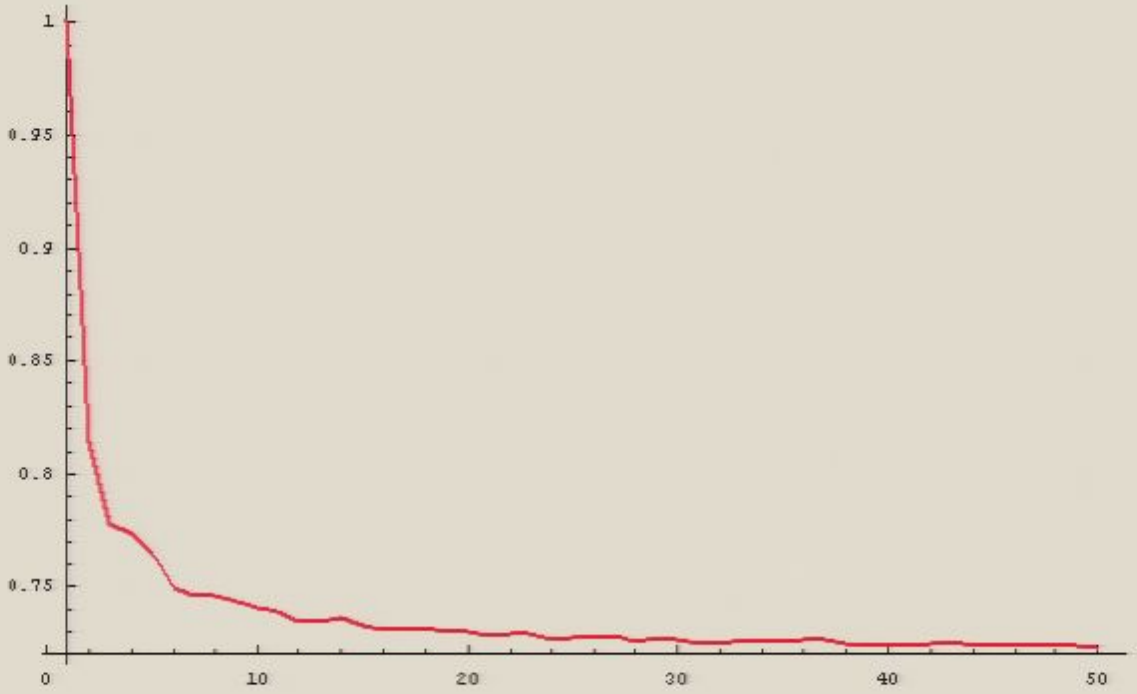
Convergence in low dimension



Empirical probability of success of the PGM applied to n states in n dimensions (averaged over 100 runs).

Why study random states anyway?

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Almost all states are random!

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But what if we don't care about quantum measurement theory?

Application: how mixed is my subsystem?

- There is another interpretation of these results which doesn't come from quantum measurement.
- It turns out that $\frac{1}{n}\|S\|_1^2$ gives the fidelity of the Gram matrix G with the n-dimensional maximally mixed state I/n.
  - where the fidelity $F(\rho,\sigma) = (\mathrm{tr}\sqrt{\rho^{1/2}\,\sigma\,\rho^{1/2}})^2$
- We may thus interpret the lower bound on the distinguishability of a set of states as how close its Gram matrix is to the maximally mixed state.

Application: how mixed is my subsystem?

- Let $\rho_{n,d}$ be the density matrix obtained by picking a pure state uniformly at random from a $n \times d$-dimensional Hilbert space, and tracing out the n-dimensional portion of it.
  - It's easy to show that $\rho_{n,d} \approx \frac{1}{n} \sum_{i=1}^{n} |\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle$ is picked uniformly at random in the d-dimensional space

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- Using this, one can show that $\frac{1}{d}\|S\|_1^2$ gives the approximate fidelity of $\rho_{n,d}$ with I/d!

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- The previous results thus predict the distance of $\rho_{n,d}$ from the maximally mixed state very closely.
  - (Popescu, Short, and Winter previously obtained a similar result by different methods)

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Application: oracle identification

Problem

Given an unknown Boolean function f as a black box, picked uniformly at random from a set S of N Boolean functions on n bits, identify f with the minimum number of uses of f.

[9] A. Ambainis et al, Quantum identification of Boolean oracles, quant-ph/0403056

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- Many important problems fit into this framework (e.g. unstructured search as in Grover's algorithm).

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- A classical algorithm must make at least $\log N$ queries
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- When the functions are random, the state matrix $S = (\{|\psi_f\rangle / \sqrt{N}\})$ is random, in the sense that the M-P law can be applied to it.
- Why? Because each entry of $\sqrt{N\,2^n} S$ is i.i.d. with expectation 0 and variance 1.

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- So the results here can be used to put the same lower bound on the probability of success of distinguishing these states.
- And in particular, the input size and the number of functions determine this probability (unlike the classical case where we can't use all the input)...

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  - In fact, the proof is easier as there is no difficulty with normalisation.
- When the probability of success is a constant $> 1/2$, we can repeat the algorithm a constant number of times for an arbitrarily good probability of success.

Summary and further work

- Good lower bounds have been obtained on the probability of distinguishing pure quantum states.
- These bounds can be applied to distinguishing random quantum states. For example:
  - For large n, n random states in n dimensions can be distinguished with probability > 0.72.
  - Almost all sets of 2^n Boolean functions on n bits can be distinguished with a constant number of quantum queries.

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Possible future directions:

- Upper bounds on $P^{pgm}(\mathcal{E})$?
- Multiple copies?
- Further applications to quantum computation?

The End

- Further reading:
  "On the distinguishability of random quantum states"
  Communications in Mathematical Physics, to appear
  quant-ph/0607011

- Thanks for your time!

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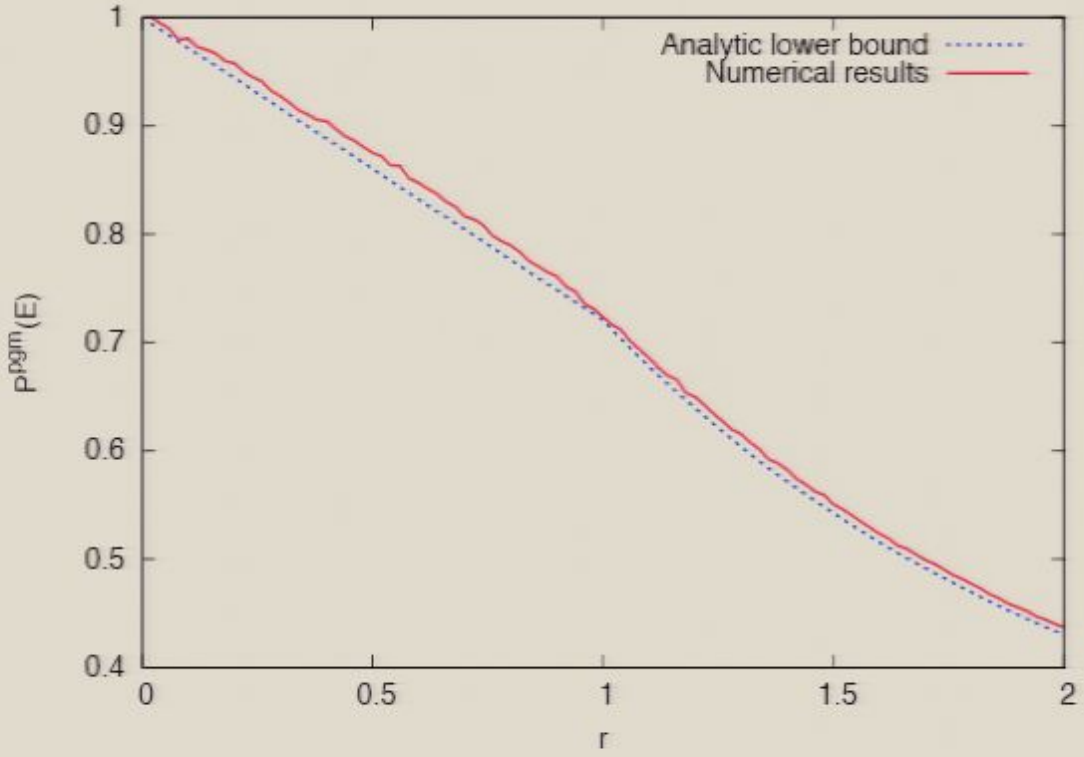

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