Title: Ancillae with Homogeneous Errors

Date: Dec 07, 2006 09:15 AM

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Abstract: Ancillary state construction is a necessary component of quantum computing.
Ancillae are required both for error correction and for performing universal computation in a fault-tolerant way. Computation to an arbitrary accuracy, however, is effectively achieved by increasing the number of qubits in order to suppress the variance in the expected number of errors. Thus, it is important to be able to construct very large ancillary states. Concatenated quantum coding provides a means of constructing ancillae of any size, but, this fact aside, concatenation is not a particularly efficient form of coding. More efficient codes exist, but these codes lack the substructure of concatenated codes that enables fault-tolerant preparation of large ancillae.
In this talk I will discuss the advantages of coding in large blocks, both from the perspective of efficiency and analysis, and I will describe my progress in developing construction procedures for moderately large ancillae.

Engineering a Quantum Computer

Today's themes

Threshold - The error rate below which an arbitrary quantum algorithm can be implemented efficiently.

Fault Tolerance - A design strategy that seeks to avoid the spread or compounding of errors.

Ancillae - Ancillary states that aid computation.

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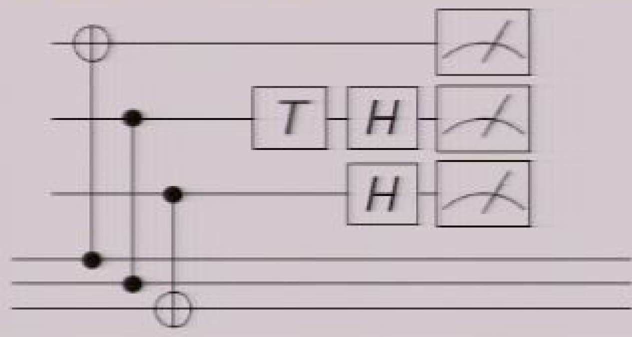
Ancillae - Ancillary states that aid computation.

Real themes

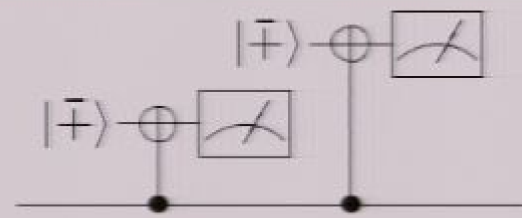
- Using large ancillae with homogeneous errors
- Constructing ancillae with homogeneous errors

Fault Tolerance

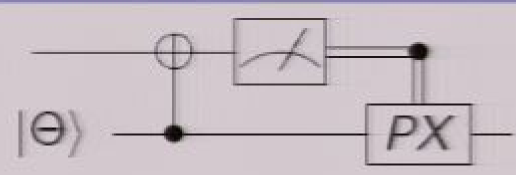
Transversal Gates



Repetition

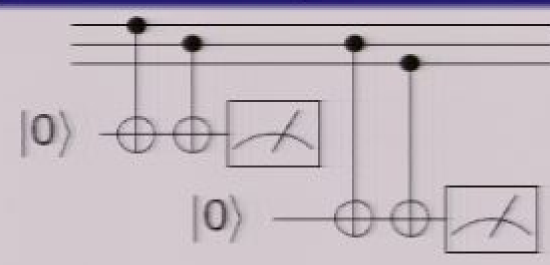


Teleportation



$$|\Theta\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \exp^{i\pi/4}|1\rangle)$$

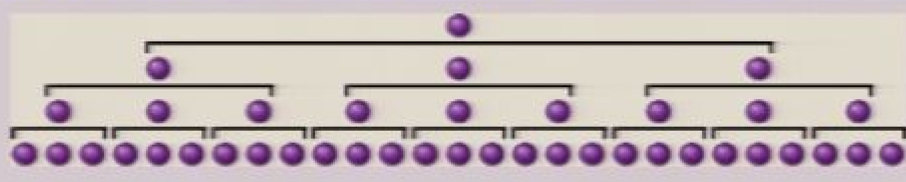
Expenditure of Qubits



Discard on 1

Concatenated vs. Block Coding

Concatenated coding encodes and corrects in layers.



Crash probability $\sim K_C^{-N^{\log_n(t+1)}}$ Preskill, quant-ph/9712048

Block coding utilizes a single layer of encoding.

Crash probability $\sim K_B^{-N}$ Steane, quant-ph/9601029

$K_C, K_B > 1$ $N =$total # of qubits

Concatenation is performed using an $[[n, 1, 2t+1]]$ code.

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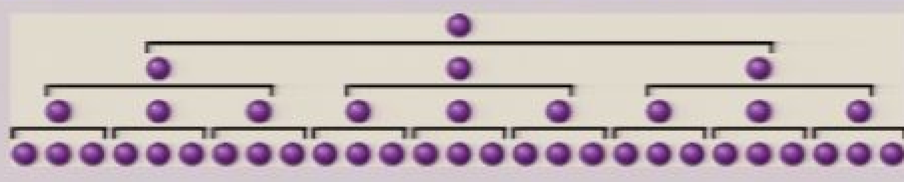

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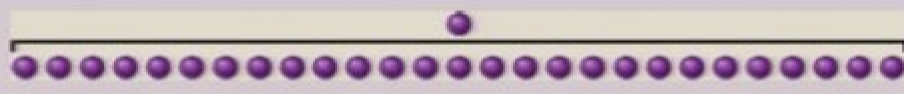

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Problem

No efficient way of directly making block code ancillae is known.

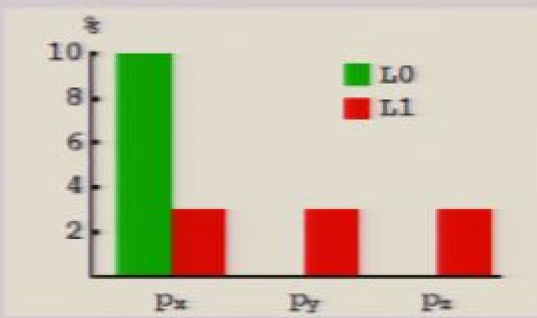
Concatenation Thresholds

Approximately, error rates are below threshold if

$$\left\{ \begin{array}{c} \text{Encoded} \\ \text{error rate} \end{array} \right\} < \left\{ \begin{array}{c} \text{Unencoded} \\ \text{error rate} \end{array} \right\}.$$

Complications, however, abound

Diverse species of error



Inexact mapping of good qubits

Rigorous thresholds bypass these problems by careful analysis of worst case scenarios.

Monte-Carlo routines, in conjunction with error propagation are used to produce threshold estimates.

Crashing the Infinite (and Uniform)

As the number of samples increases the frequency of errors approaches the expectation.

In the limit $n \to \infty$

$$\tau = \lim_{n\to\infty} t/n$$

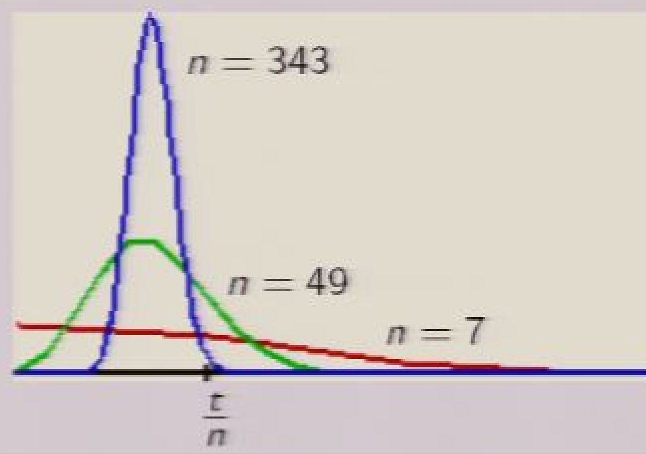
$$\left\{ \begin{array}{c} \text{Failure} \\ \text{probability} \end{array} \right\} \to \begin{cases} 1 & \text{if } p > \tau \\ 0 & \text{if } p < \tau \end{cases}$$

Failure becomes a deterministic function of the error probability.

Threshold criterion:

$$\lim_{n\to\infty} \left\{ \begin{array}{c} \text{Encoded} \\ \text{error rate} \end{array} \right\} \to 0.$$

Probability vs. error fraction

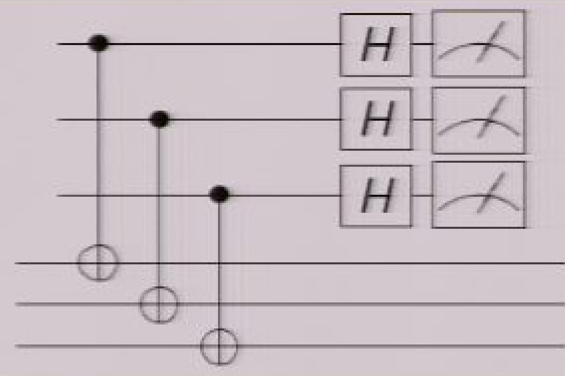


Ancillae with Homogeneous Errors

Homogeneous - Transversal and identical

Homogeneous errors are independent and identically distributed (i.i.d).

Homogeneous gates



I assume that ancillae errors are homogeneous and that they are trace preserving and unbiased, i.e. for an error operator $\mathcal{E}$ s.t.

$$\mathcal{E}(\rho) = \sum_j E_j \rho E_j^\dagger, \quad \sum_j E_j^\dagger E_j = I \quad \text{and} \quad \text{Conjugate}[\mathcal{E}] = \mathcal{E}$$

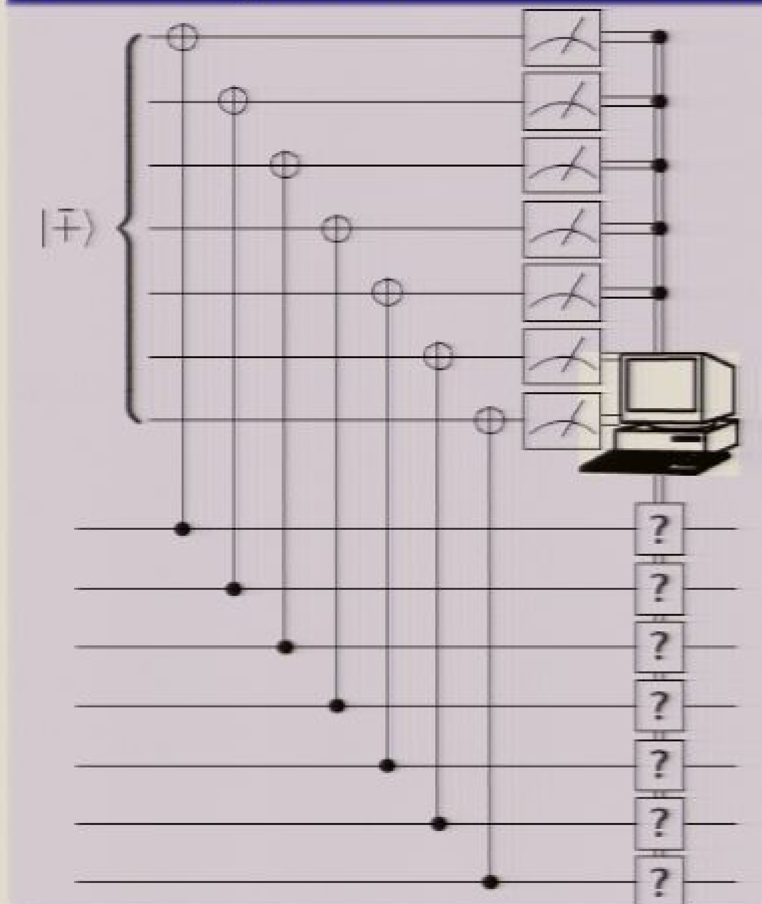
Error Finding in the Limit

Often, error diagnosis is homogeneous but for syndrome interpretation and recovery.

When syndrome extraction succeeds, the interpretation of the syndrome yields an error pattern equivalent to those that occurred.

Since success is deterministic, it is sufficient to reveal the error locations.

Steane 7-qubit-code bit correction

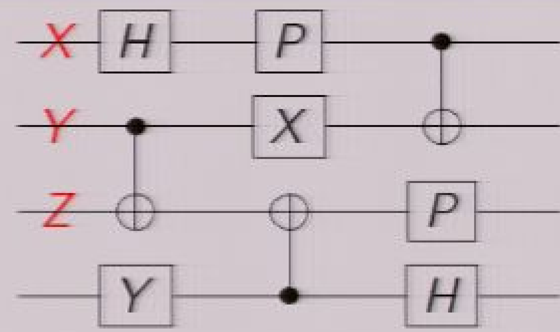


Error Recovery

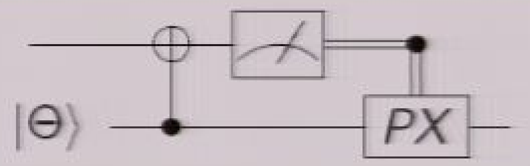
Error recovery can be dispensed with.

Once detected, the effect of errors is tracked by error propagation.

Error propagation



Teleporting on the $\pi/4$ Rotation



If non-Clifford gates are never directly applied, recovery operators are unnecessary.

Knill, quant-ph/0410199

Threshold Determination

Having eliminated the inhomogeneities, threshold analysis is particularly simple.

Method for Threshold Determination

1. By exhaustive counting and error propagation find the probability of error at every time in the circuit assuming that all syndrome decodings succeed.
2. Determine the maximum the probability of a bit-flip error for any measurement used in error diagnosis.
3. If this number is less than the correctable error fraction, then the error rate is below threshold.

What's it all for?

Review of assumptions:

- Logical ancillae of any size can be prepared.
- Ancillae can be prepared such that their error distribution is homogeneous.

Possible applications:

Obvious: Rigorous threshold surface given ancillae with homogeneous errors.

Error models:
#1 Full depolarizing gate errors
#2 Strong measurement errors
#3 Depolarizing 2-qubit gate errors
#4 Restricted 2-qubit gate errors

Thresholds for Homogeneous Ancillae in units of the correctable error fraction

Procedure	Error Model			
	#1	#2	#3	#4
Single Steane	0.15	0.06	0.24	0.29
Double Steane	0.16	0.10	0.18	0.29
Knill	0.35	0.15	0.50	0.67

What's it all for?

Less obvious: Simple, flexible method of generating threshold estimates.

Reichardt's Steane-procedure depolarizing threshold	$\sim .9\%$
My Steane-procedure depolarizing threshold	$\sim 1\%$
Knill's telecorrection depolarizing threshold	$\sim 3-5\%$
My telecorrection depolarizing threshold	$\sim 4\%$

Long shot: Recipe for achieving some threshold.

This last use would require a means of preparing ancillae in logical encoded states.

Reichardt, quant-ph/0406025

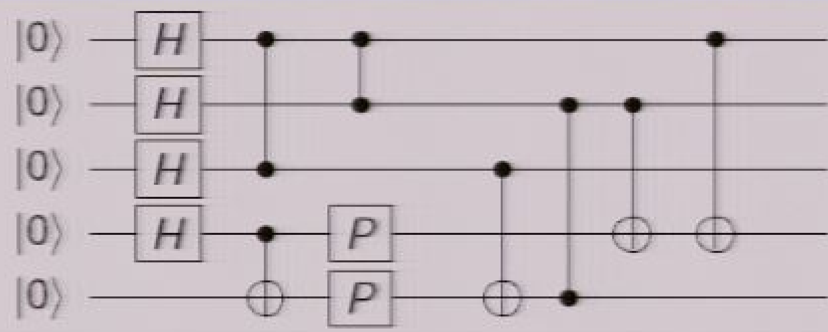
Knill, quant-ph/0410199

Making Large Ancillae

Direct preparation is problematic for two reasons.

First, the number of gates applied to each qubit typically scales with the size of the ancilla. Thus the final error probability on each qubit should likewise increase.

5-qubit $|\bar{0}\rangle$ preparation



Second, before the first problem can become an issue, the spread of errors becomes fatal.

Concatenated Coding for Ancilla Construction

Concatenated coding avoids the aforementioned difficulties by preparing ancillae in stages interspersed with error correction.

High level ancillae, that consist of many layers of encoding, are prepared starting from the lowest level of encoding and working up.

This method requires the code to have a concatenated substructure however.

How do we prepare an arbitrary ancilla?

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~~How do we prepare an arbitrary ancilla?~~

How do we prepare an arbitrary stabilizer state?

Aside: Graph States

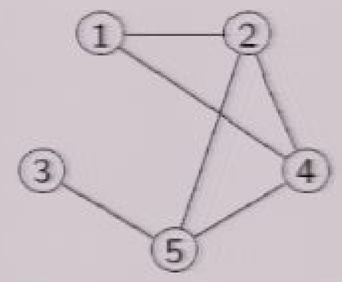
Stabilizer generator

$$\begin{bmatrix} Z & Z & I & X & Z \\ I & Y & I & Y & I \\ I & Z & Z & Z & X \\ Y & Y & X & I & I \\ Z & Z & X & X & I \end{bmatrix}$$

Graph-form generator

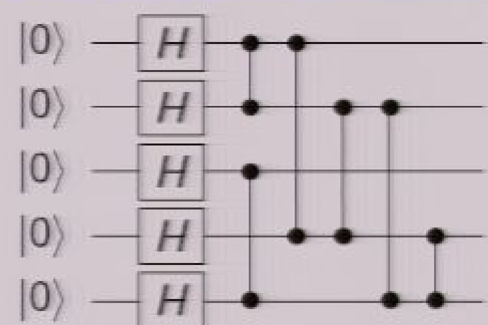
$$\begin{bmatrix} X & Z & I & Z & I \\ Z & X & I & Z & Z \\ I & I & X & I & Z \\ Z & Z & I & X & Z \\ I & Z & Z & Z & X \end{bmatrix}$$

Graph



Any stabilizer state is equivalent, up to local Clifford operations, to a graph state.

Preparation Circuit

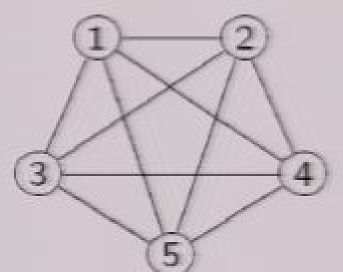


Compact Graph-state Construction

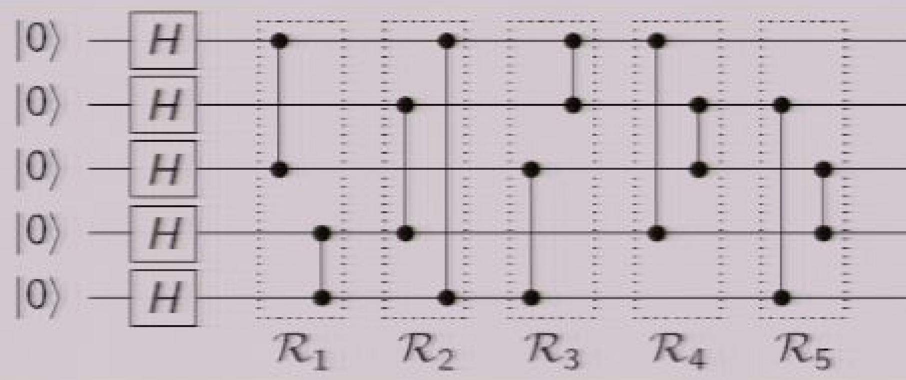
An n-qubit graph state can be constructed from the state $|0\rangle^{\otimes n}$ by applying H to every qubit and then applying ${}^{C}Z$s in $r = 2\lfloor \frac{n}{2} \rfloor + 1$ rounds.

This is proven by showing that the complete graph can be constructed in that way.

Complete graph



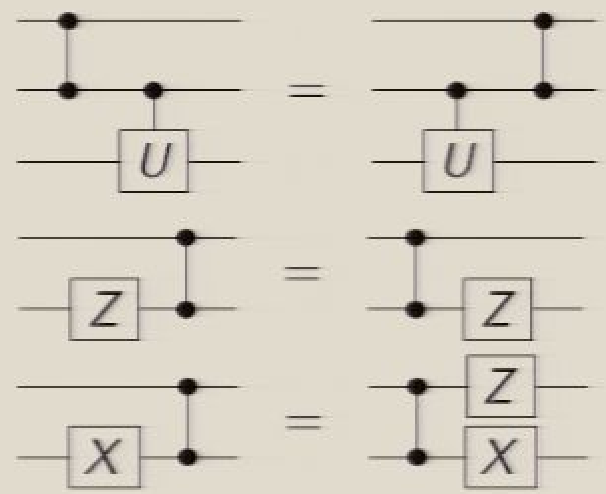
Compact preparation circuit



Aside: Controlled-Z Gates

Properties of the ${}^{C}Z$ gate:

- ${}^{C}Z_{12}\,{}^{C}U_{23} = {}^{C}U_{23}\,{}^{C}Z_{12}$
- $Z_2\,{}^{C}Z_{12} = {}^{C}Z_{12}Z_2$
- $X_2\,{}^{C}Z_{12} = {}^{C}Z_{12}X_2Z_1$



Z errors are not spread by ${}^{C}Z$ gates and X errors spread only Z errors.

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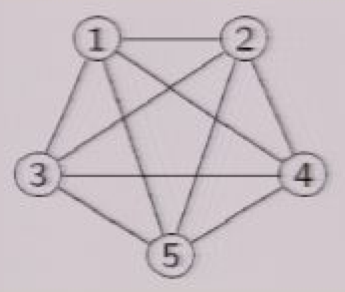

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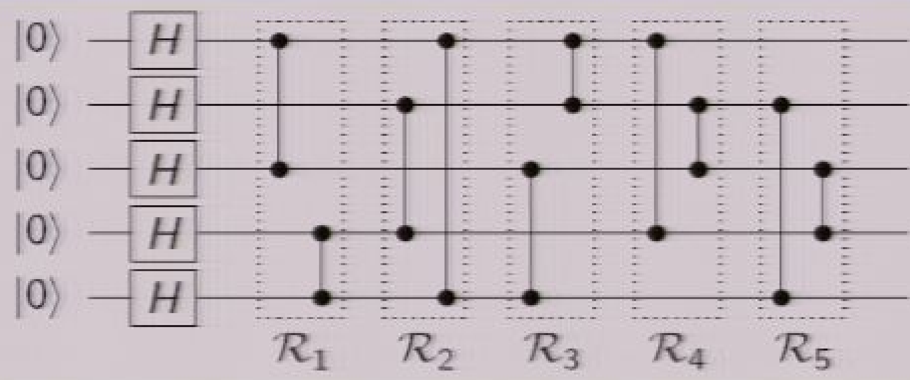

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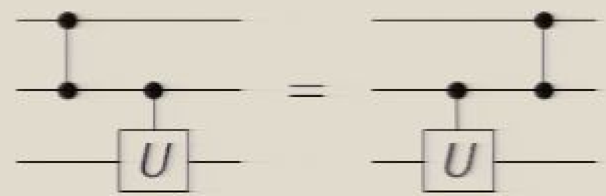
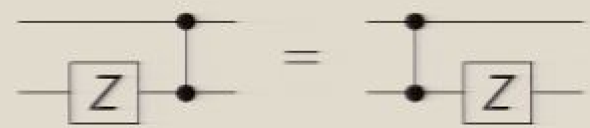


- $Z_2\,{}^{C}Z_{12} = {}^{C}Z_{12}\,Z_2$



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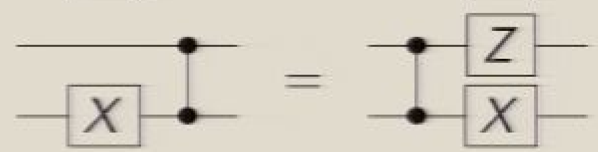

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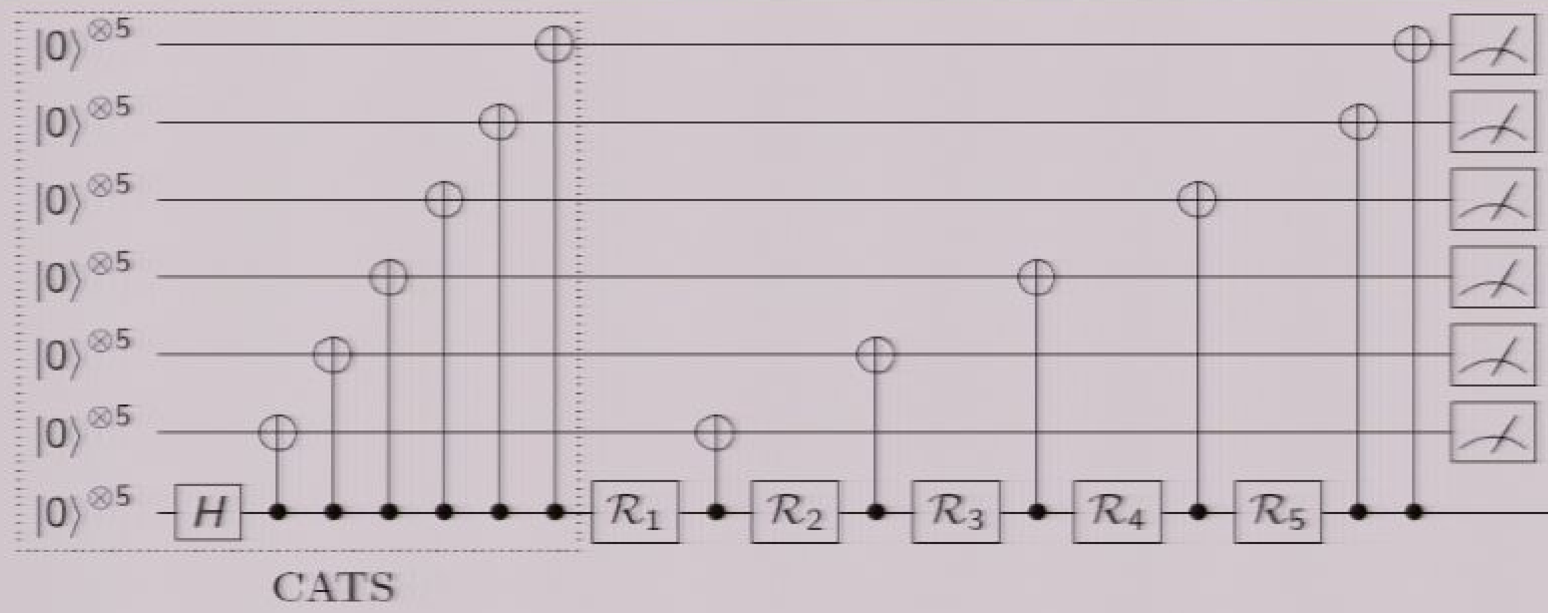
Fault-tolerant Graph-state Construction

Logical circuit for fault-tolerant 5-qubit-graph-state construction



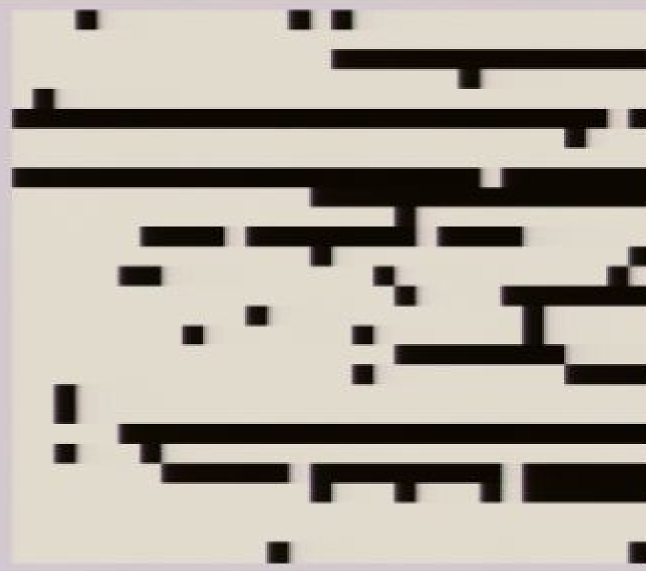
All qubits are initially in the $|0\rangle$ state, and $\mathcal{R}_i$ is the ith round of phase gates in graph state construction.

Error tracking fails if the CATS portion contains correlated errors, so the five 7-qubit cat states must be prepared separately.

Error Tracking

Examining the X-error tracks reveals noisy streaks.

Error tracks (30 qubits, Recorded)



Applying a filter based on the relative frequencies of error events yields a guess about the actual X-error track.

The location of X errors provides a clue to the location of Z errors.

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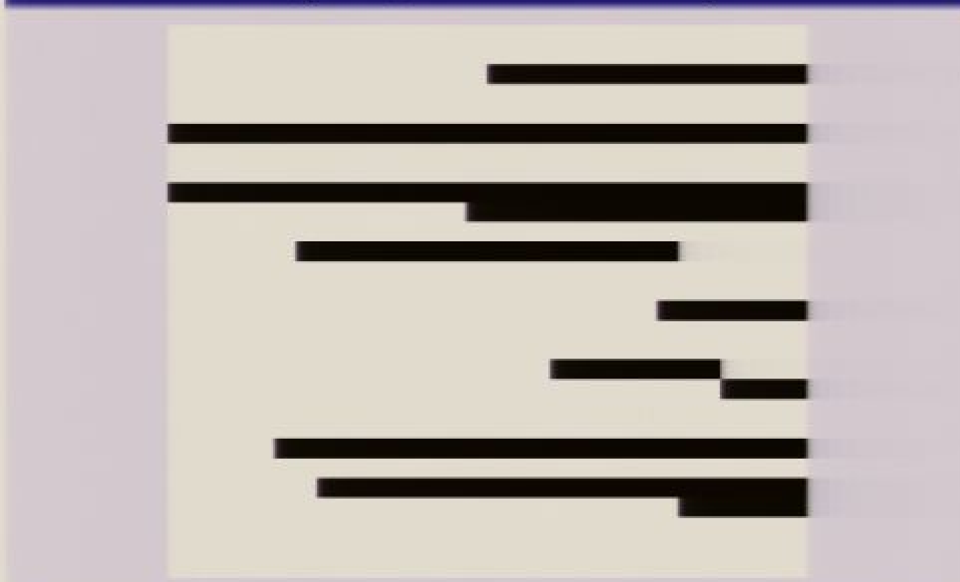
Error tracks (30 qubits, Filtered)



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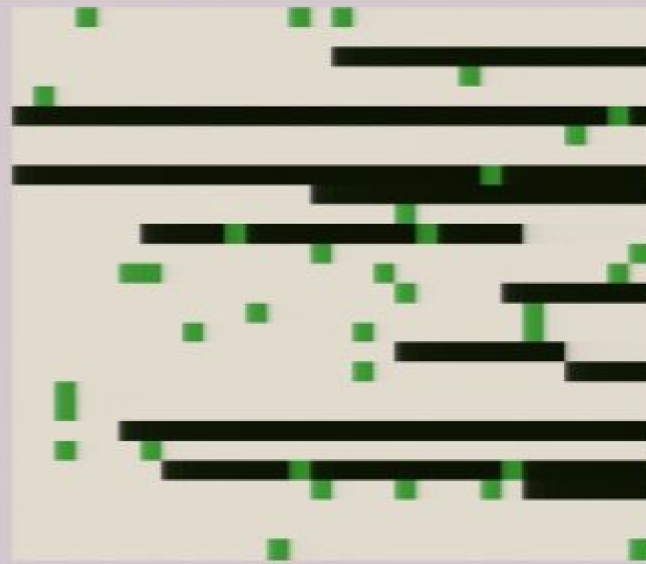
Error tracks (30 qubits, Change)



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Construction Summary

Graph-state construction procedure

1. Prepare n $(r+2)$-qubit cat states.
2. While $i \leq r+1$
    1. Apply the ith round of phase gates to the n qubit block consisting of the first qubit in every cat state.
    2. Apply ${}^{C}X$ from the first to the $(i+1)$st n qubit block of the cat states.
    3. Measure the target qubits.
3. Infer the error locations from the filtered error track.

Properties (preliminary)

- The total number of residual X and Z errors is no more than twice the number of errors that occurred.
- The residual error probability on each qubit scales like n.
- For base error probability p, construction fails when $n > \frac{1}{p}$.

Roadmap

- Thresholds
  - ✓ Thresholds for ancillae with homogeneous errors
  - ✓ Threshold estimates for single-level encoding with liberal qubit discard
  - ↗ Constructive method of block coding for moderately sized ancillae
- Graph-state preparation
  - ↗ Graph-state construction s.t. minimal correlations arise using i.i.d. cat states
  - ? Graph-state preparation for non-depolarizing Pauli channels
- Cat-state preparation
  - ↗ Improved cat-state verification

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