Title: A simplicial path to the quantum Hamiltonian of gravity

Date: Dec 06, 2006 01:00 PM

URL: http://pirsa.org/06120030

Abstract: TBA

A simplicial path to the quantum Hamiltonian of gravity

Dario Benedetti

Utrecht University

D. Benedetti, R. Loll, F. Zamponi: to appear soon

Plan of the talk

- Introduction
- Model description
- Model solution
- Continuum limit
- Analysis of the results
- Conclusions and outlook

Path integral from Causal Dynamical Triangulations

- In analogy with the piecewise linear paths of Feynman's path integral we have here piecewise flat manifolds, *i.e.* simplicial manifolds with a flat metric assignment inside each simplex.
  In Dynamical Triangulations such metric is chosen as to have equilateral simplices with edge length a.

$$\int [dg_{\mu\nu}] e^{iS_{EH}[g_{\mu\nu}]} \to \sum_T \frac{1}{C(T)} e^{iS_R[T]}$$

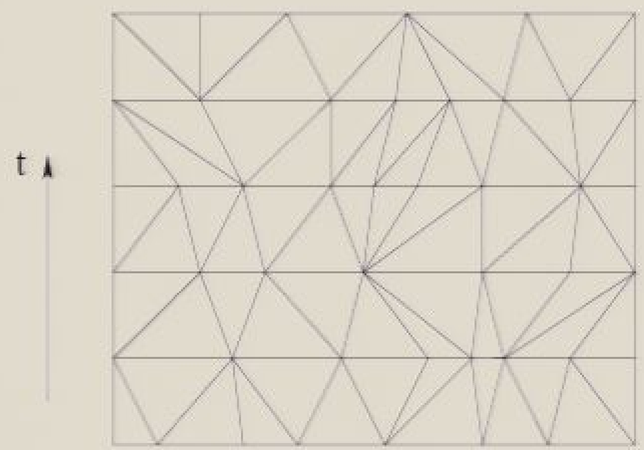
- The causal version (CDT) is obtained by restricting the class of triangulations

  (Ambjorn, Loll - 1998)

- Triangulations with product structure: discrete version of M$\simeq \mathrm{R}\times\Sigma$

- Global time and space slices

- It is possible to Wick rotate.
  The path integral becomes a partition function for random geometries (but still with product structure)



Transfer matrix and Hamiltonian

- The one-step (Euclidean) propagator from a geometry g_1 to a geometry g_2 defines an element of the transfer matrix:

$$\langle g_2|\hat{T}|g_1\rangle = \sum_{T:g_1\to g_2;\Delta t=a} \frac{1}{C(T)} e^{-S[T]}$$

- Analogue of the one-step evolution operator in quantum mechanics

- The transfer matrix can be shown to be symmetric, strictly positive and bounded.
  (Ambjorn, Jurkiewicz, Loll - 2001)

- This ensure the existence of a well defined Hamiltonian operator, which in principle can be extracted in the continuum limit:

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- This task has been accomplished in (1+1)-dimensions, in various versions of the model and with different techniques
  (Ambjorn, Loll – 1998; Di Francesco, Guitter – 2001; Loll, Westra, Zohren - 2005)

- Not many results in higher dimensions

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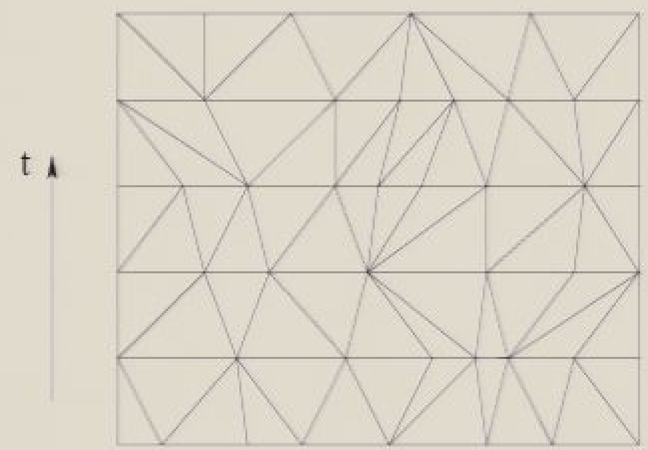

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Numerical results

- Monte Carlo simulations.

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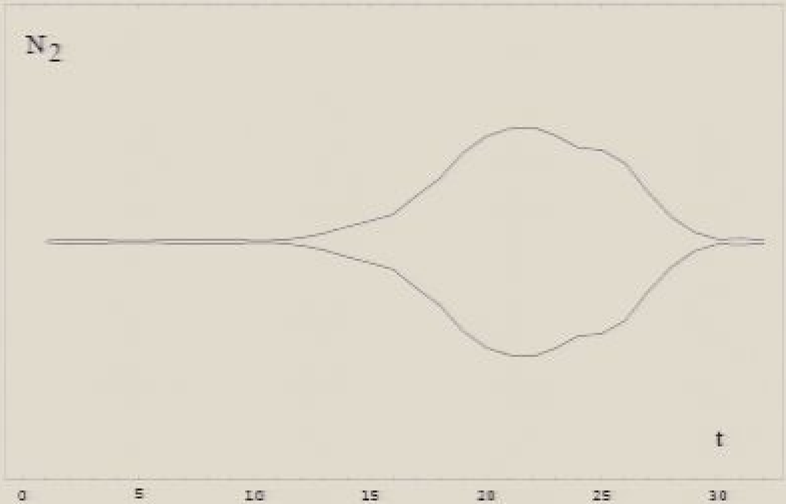
Fixed total volume N_3 and total time T (both large); $S^2\times[0,1]$ topology

It is observed only one phase.
The typical geometry shows a "semiclassical" (3-dimensional) lump of spacetime



- The fluctuations of successive spatial volumes have been studied and their distribution is very well described by

$$P(N_2(t), N_2(t+a)) \sim e^{-c(k_0)\frac{(N_2(t+a)-N_2(t))^2}{N_2(t+a)+N_2(t)}}$$

which points in the direction of an effective action for the spatial volumes of the form

$$S_{eff}(V_2) = \int dt \left(\frac{1}{G_N} \frac{\dot{V}_2^2(t)}{V_2(t)} + \Lambda V_2(t) \right)$$

and this is exactly the classical action for the spatial volume (for the S^2 case)

Gravity in (2+1)-dimensions – canonical quantization

- M$\simeq$ R$\times\Sigma$
- ADM decomposition: $g_{\mu\nu} \rightarrow \{N, N_i, h_{ij}\}$
- Any metric on a 2-dimensional manifold admits a decomposition like:

$$h_{ij}(x) = e^{\lambda(x)} f * \tilde{h}_{ij}(x)$$

conformal factor — diffeomorphism — constant curvature metric $\Rightarrow$ moduli space

- Choosing York slicing, N(x)=N(t) and imposing the momentum constraint one obtains:

(Hosoya, Nakao – 1989)

$$S = \int dt \left[P_{(\alpha)}\dot{\rho}^{(\alpha)} + \tau\dot{v} - N \underbrace{\left(G_N\left(\frac{g^{(\alpha)(\beta)}P_{(\alpha)}P_{(\beta)}}{2v} - \frac{1}{2}\tau^2 v\right) - \frac{1}{G_N}(4\pi\chi - 2\Lambda v)\right)}_{H(P_{(\alpha)},\rho^{(\alpha)},\tau,v)} \right]$$

- Canonical quantization in reduced phase space:

$$\tau \rightarrow -i\frac{\partial}{\partial v} \qquad P_{(\alpha)} \rightarrow -i\frac{\partial}{\partial \rho^{(\alpha)}}$$

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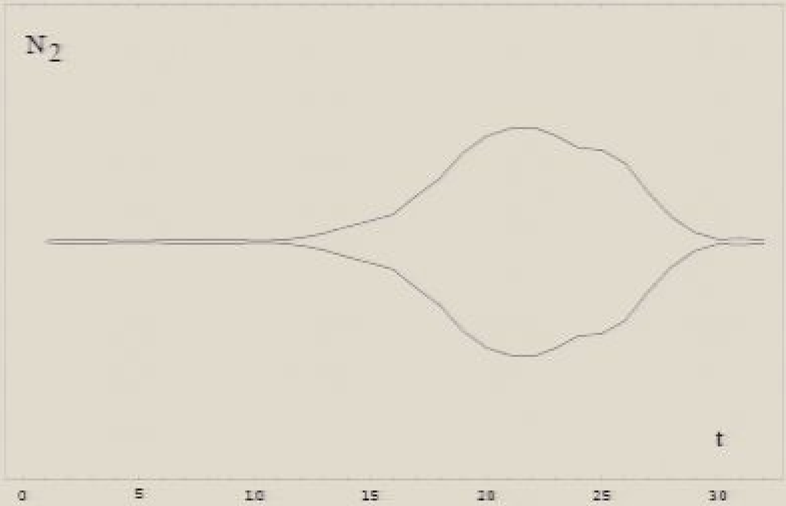

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Analytical results

- Mapping between transfer matrix and free energy of the ABAB matrix model

(Ambjorn, Jurkiewicz, Loll, Vernizzi - 2001)

$$Z = \lim_{N\to\infty} \frac{-1}{N^2} \log \mathcal{Z}$$

where:

$$Z = \sum_{N_1, N_2} e^{-z_1 N_1 - z_2 N_2} \sum_{g_1(N_1), g_2(N_2)} \langle g_2(N_2) | \hat{T} | g_1(N_1) \rangle$$

$$\mathcal{Z} = \int dA dB e^{-NTr[A^2 + B^2 - \alpha_1 A^4 - \alpha_2 B^4 - \beta ABAB]}$$

- The matrix model was solved (Kazakov, Zinn-Justin - 1998)
- But the intricate analytical structure of the model prevented from even finding the identity (0-th order) term
- Maybe this is because the model contains too many (and unwanted) configurations?

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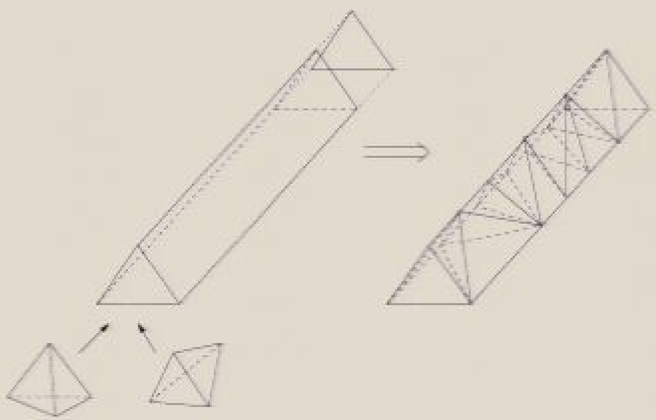
A different model

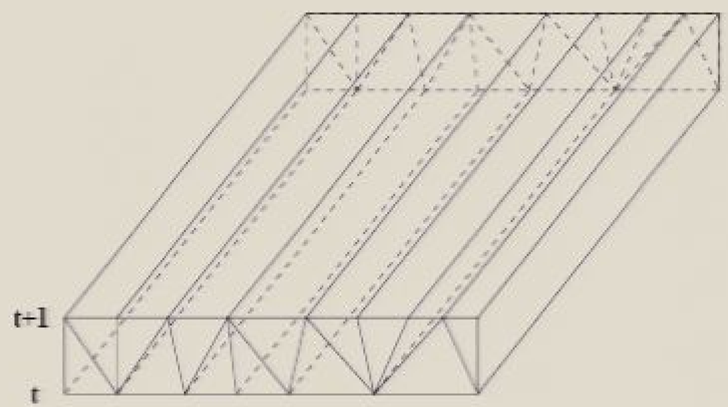
- Introduce a new class of simplicial manifolds generalizing the product structure of CDT, which we will then call "of product type". (Dittrich, Loll - 2005)

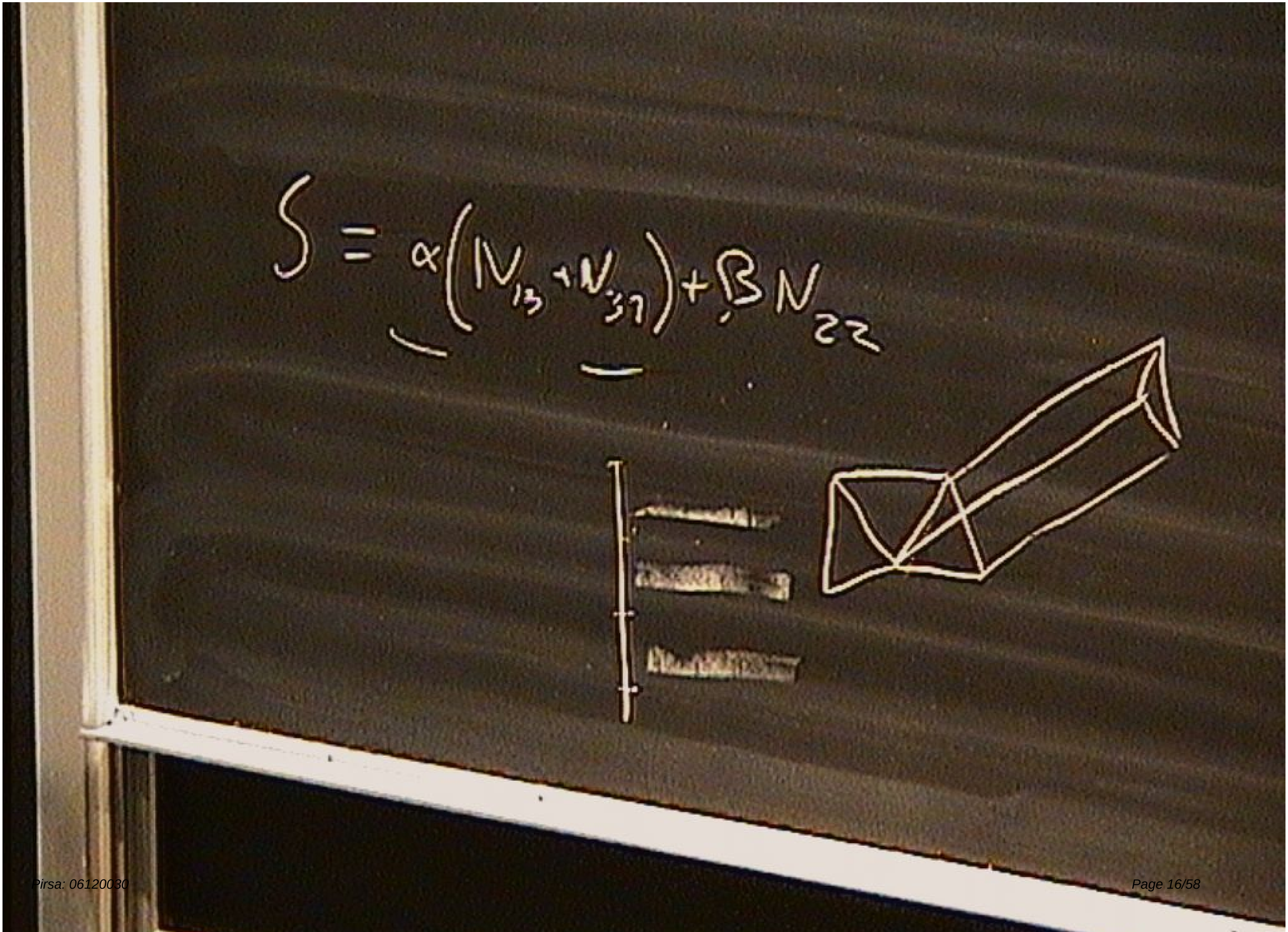
- "Base space" $\times$ "Fibers" (or "towers").

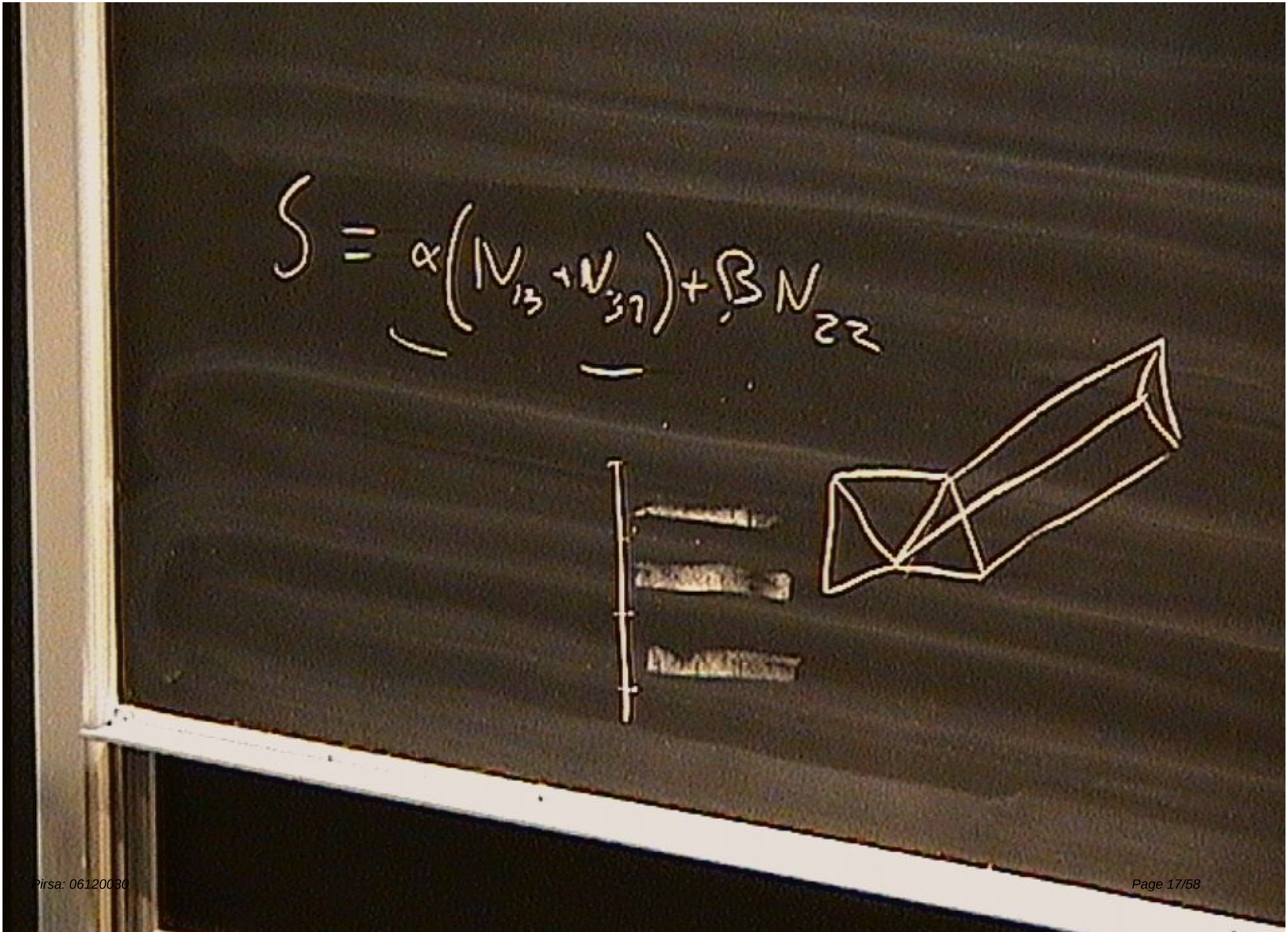
- It looks like if there is a second time, but the idea is actually to have a slicing also on the space slices, having in mind a "radial coordinate" $\times$ "spherical shells" decomposition of space in order to study black holes.

- A natural choice for the topology of the slices is that of a cylinder
  $\Rightarrow$ one Teichmuller parameter






$S = \alpha(N_{13} + N_{31}) + \beta N_{22}$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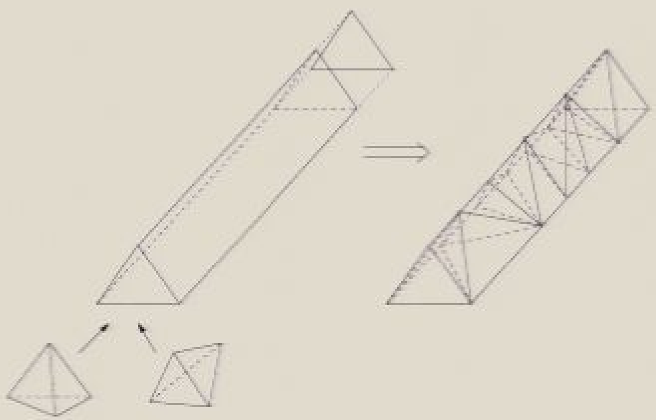

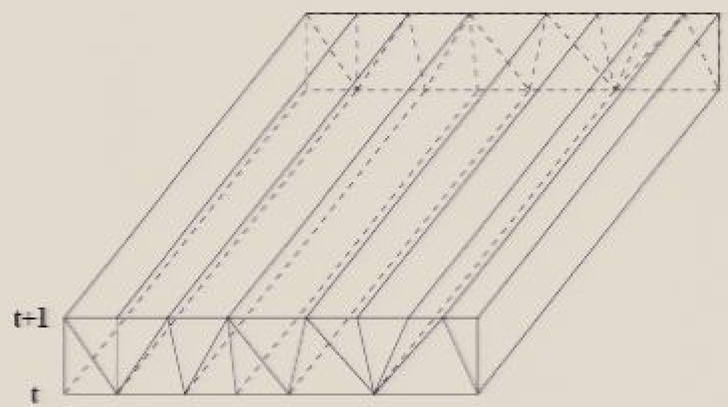

Maybe enough…

- …to capture the properties of full (2+1)D gravity?
- After all only area and Teichmuller parameters should be relevant
- But of course I don't want to start assuming this, I want to obtain it from a path integral over geometries
- The problem is then whether or not this class of geometries spans the configuration space densely enough to implement the reduction
- This we are not able to judge at the beginning
- What we can say is that for sure the model has enough entropy to suggest the existence of a continuum limit.
  (remember that the entropy of the triangulations has to compete with the cosmological weight which is an exponential in the volume)

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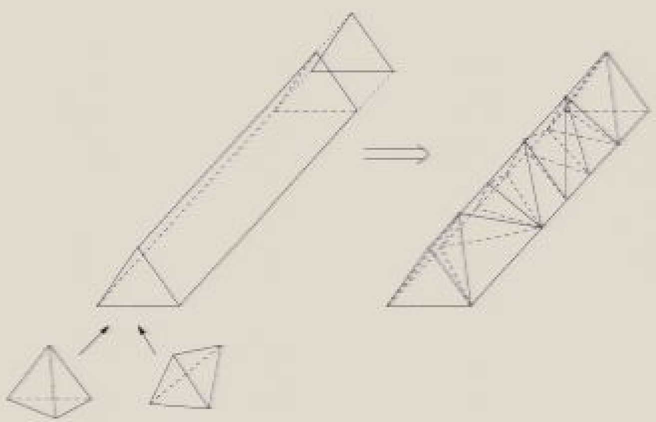
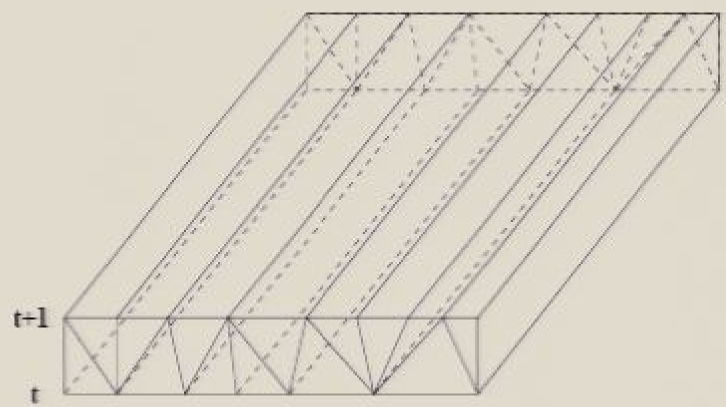
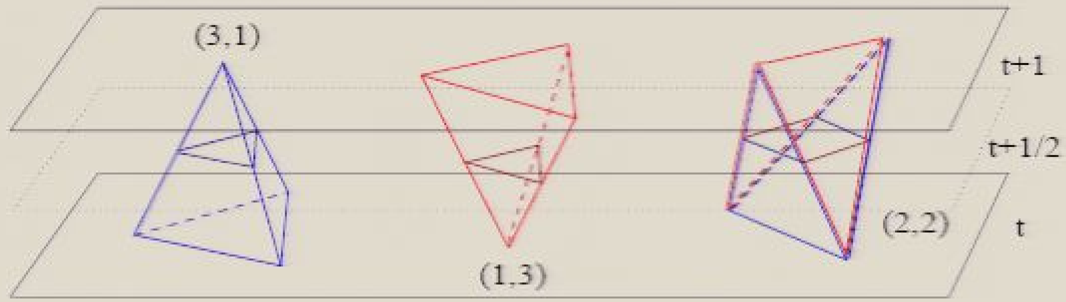
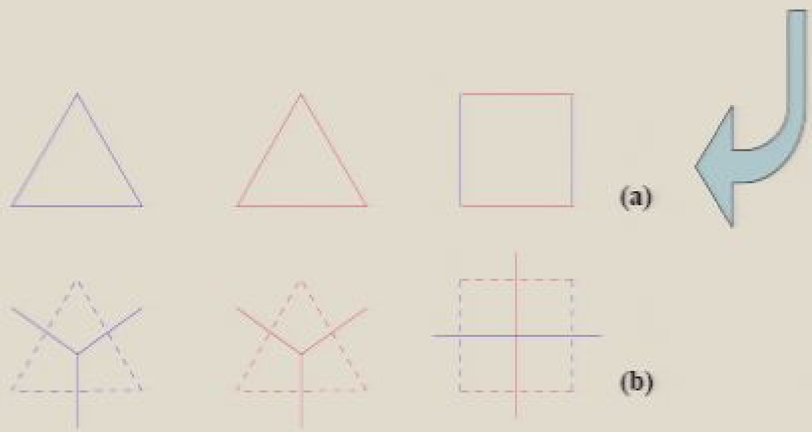




Dual mapping: in general

- Take a "sandwich" triangulation, *i.e.* the triangulation of the 3D space in between two adjacent slices with time separation a





- Colour code the triangular faces, and cut the sandwich at an intermediate time
- Draw the dual graph of the obtained tessellation

Action and partition function for the sandwich

- Starting from the Einstein-Hilbert action with Gibbons-Hawking boundary term

$$S_{EH}+S_{GH}=\int_{I\times\Sigma} d^3x\sqrt{g}\left(\frac{1}{2G_N}R-\Lambda\right)+\frac{1}{G_N}\int_{\Sigma} d^2x\sqrt{h}K$$

- The Regge prescription and topological relations give for our sandwich triangulation

$$S=\alpha(N_{13}+N_{31})+\beta N_{22}$$

where: $\alpha=(-\frac{5}{2}\pi+6\arccos\frac{1}{3})k+\frac{1}{6\sqrt{2}}\lambda$

$\beta=(-2\pi+6\arccos\frac{1}{3})k+\frac{1}{6\sqrt{2}}\lambda$

and k and λ are the bare inverse Newton and cosmological constants

- The grand canonical partition function (with sum over volume and boundary geometries) is

$$Z=\sum_{N_{13},N_{31}} x^{N_{13}}y^{N_{31}}\sum_{T_{|N_{13},N_{31}}} e^{-S}=$$

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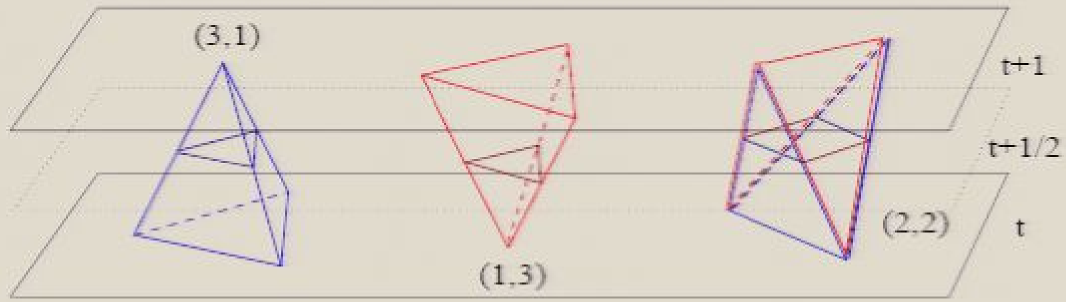

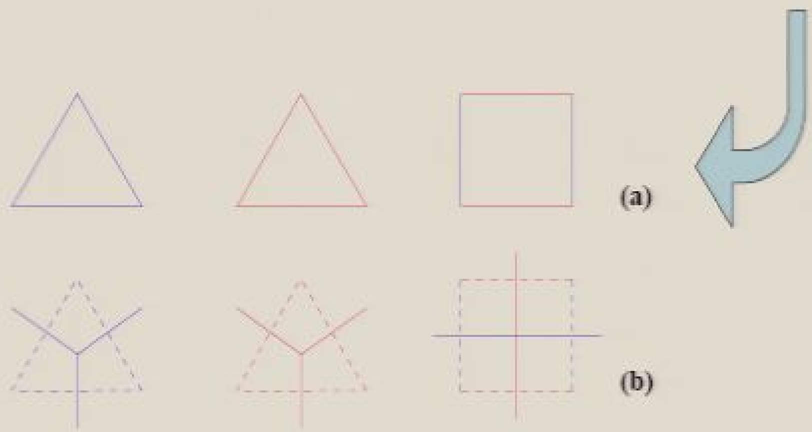

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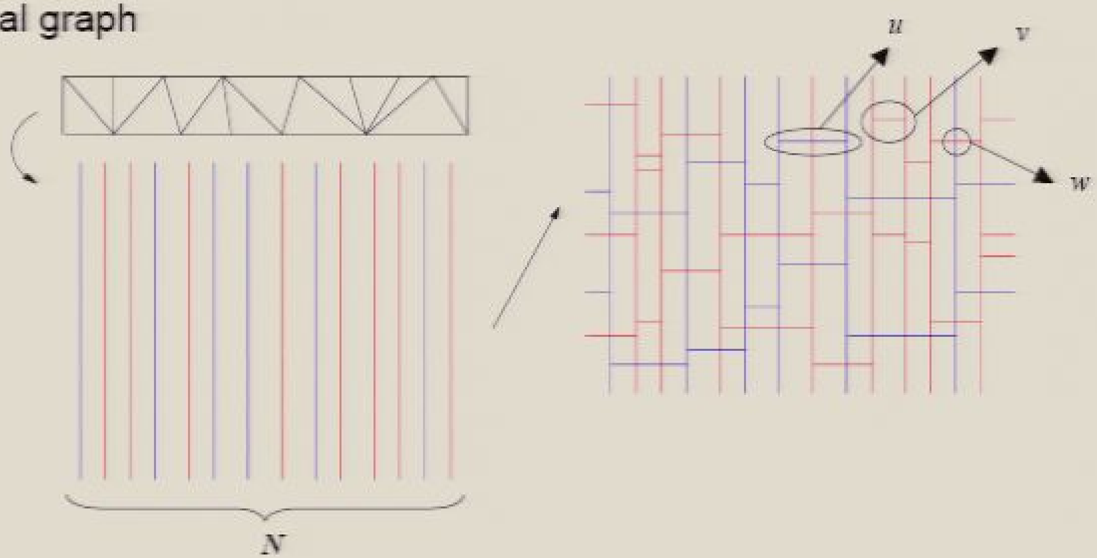
Dual mapping: in our specific case

- The presence of the "triangle towers" is reflected in the presence of a sliced structure of the dual graph



- The partition function is the generating function for a combinatorial problem

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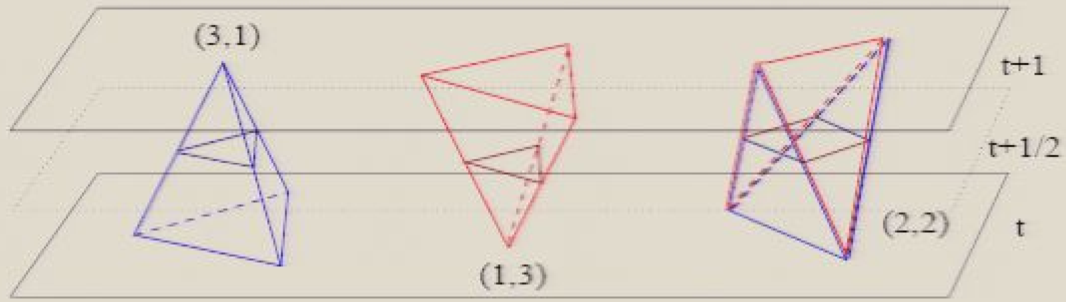

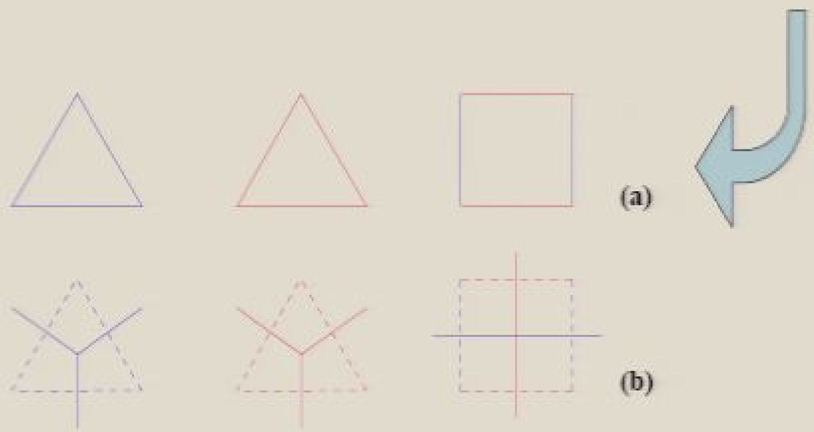

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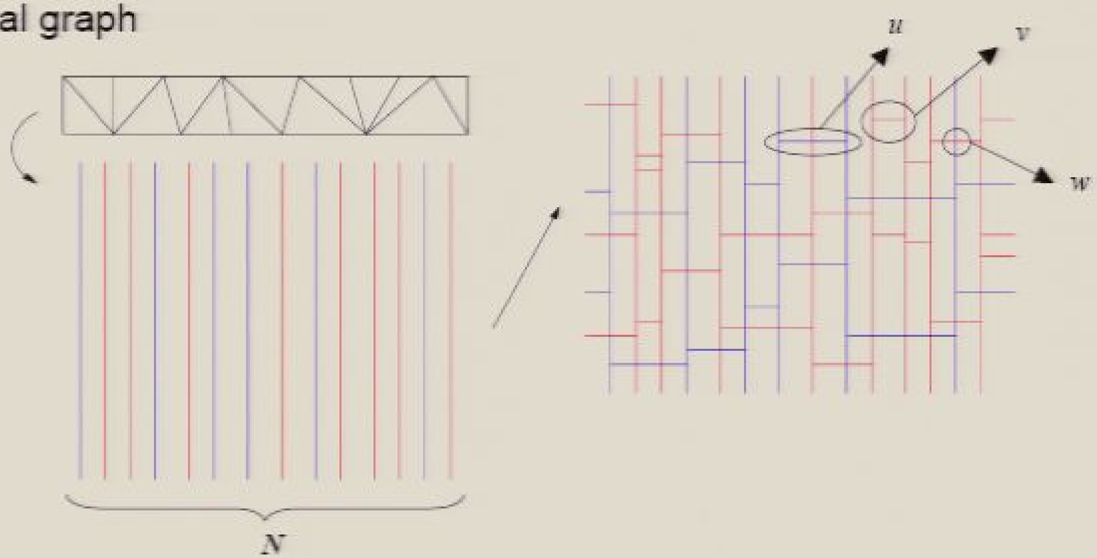

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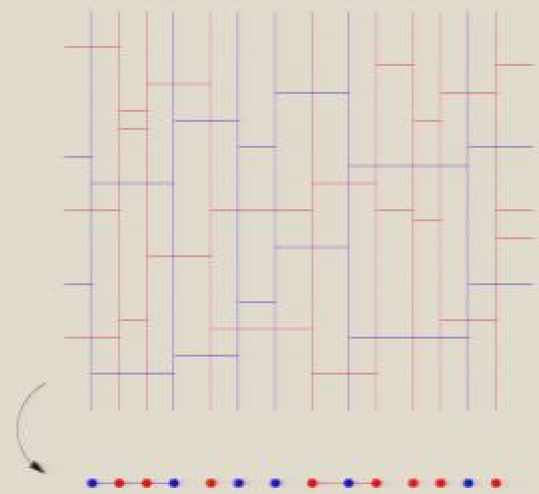
Heap of pieces and inversion formula

- The graph looks like a coloured version of what is called "heap of pieces" in combinatorics
- There exist a formula that relates heaps of pieces in *D* dimensions to hard objects in *D-1* dimensions (Viennot – 1986; Di Francesco, Guitter - 2001)
- It turns out that we can extend such a formula to the case with more colours, when we keep fixed the sequence of towers.
- The inversion formula looks like this

$$Z_{S_N}(u,v,w) = \frac{1}{Z^{h.d.}_{S_N}(-u,-v,w)}$$



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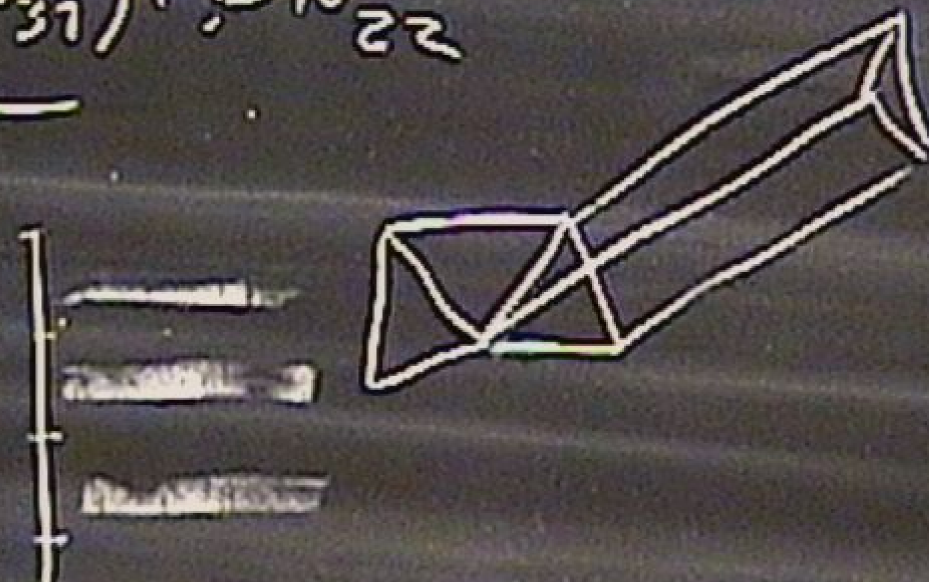

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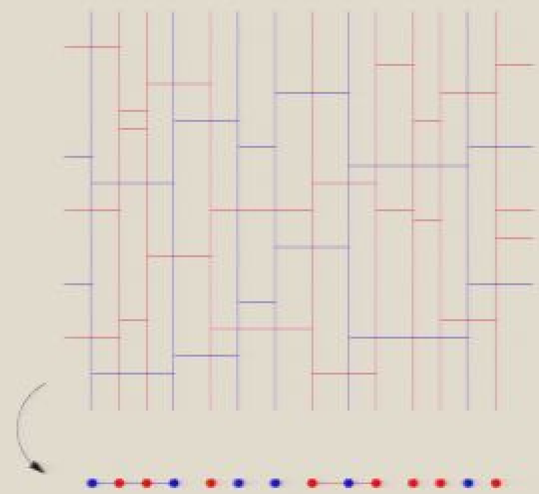

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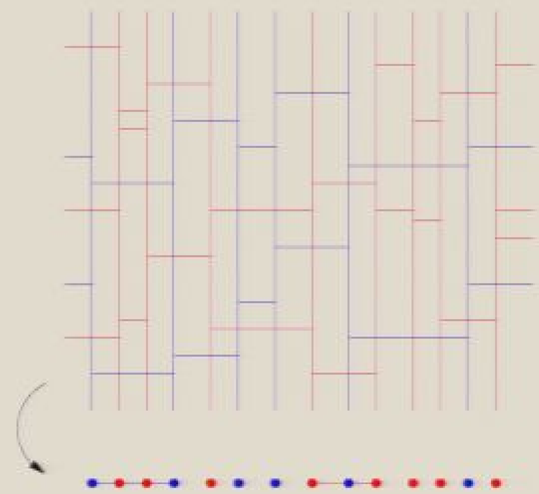

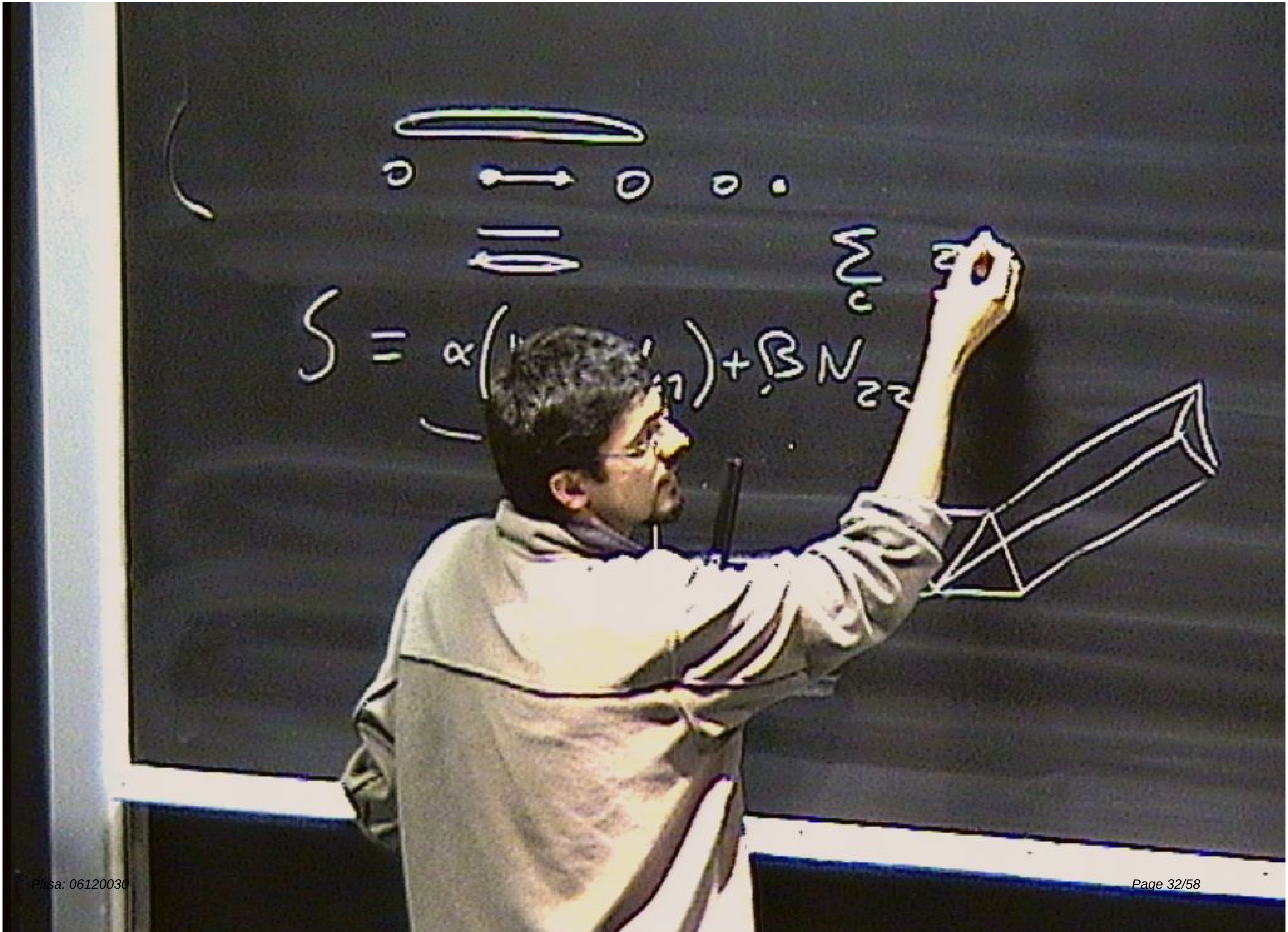

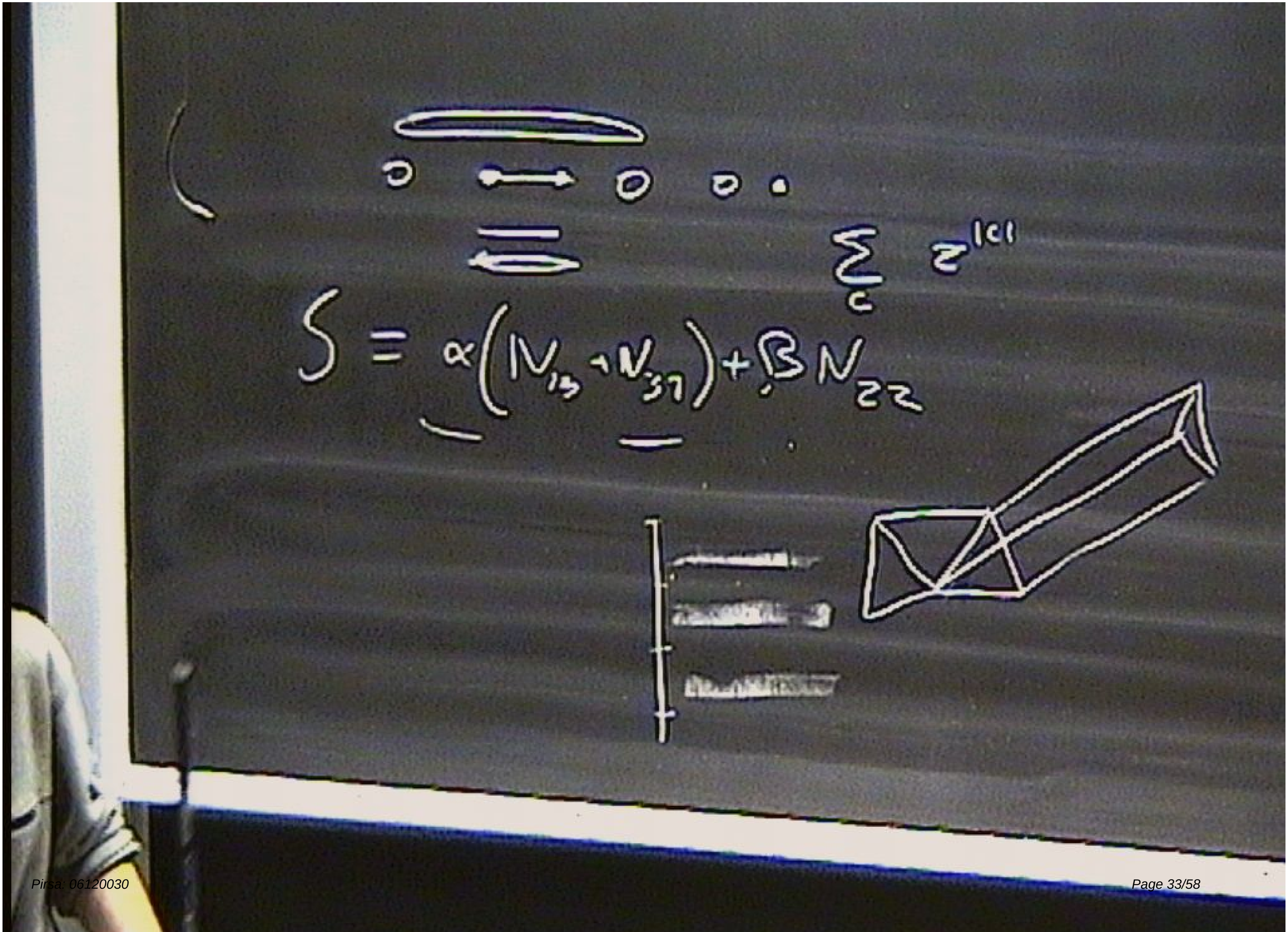
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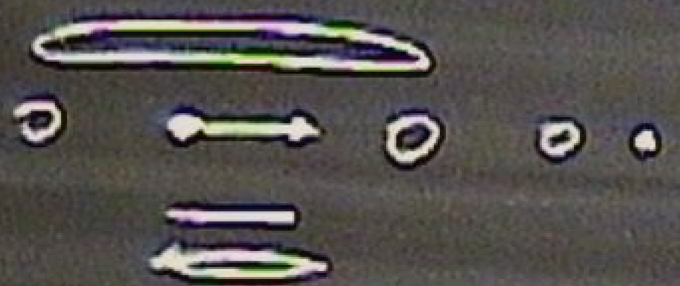

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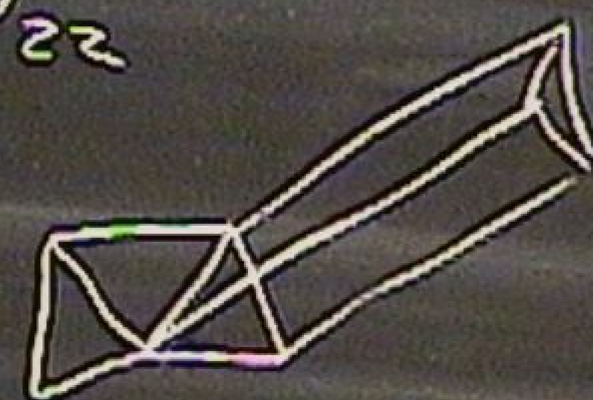

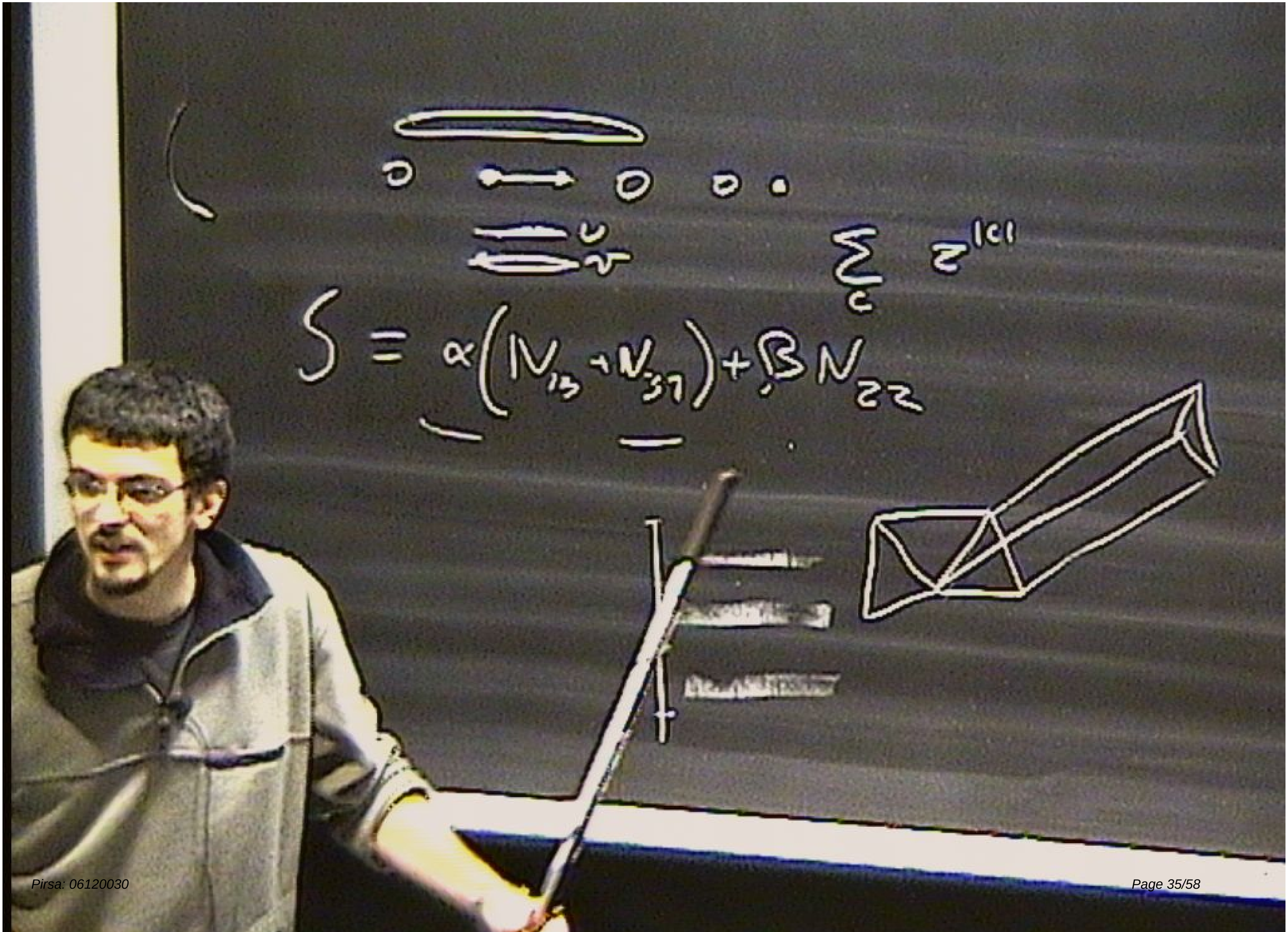
$\sum_{c} z^{|c|}$
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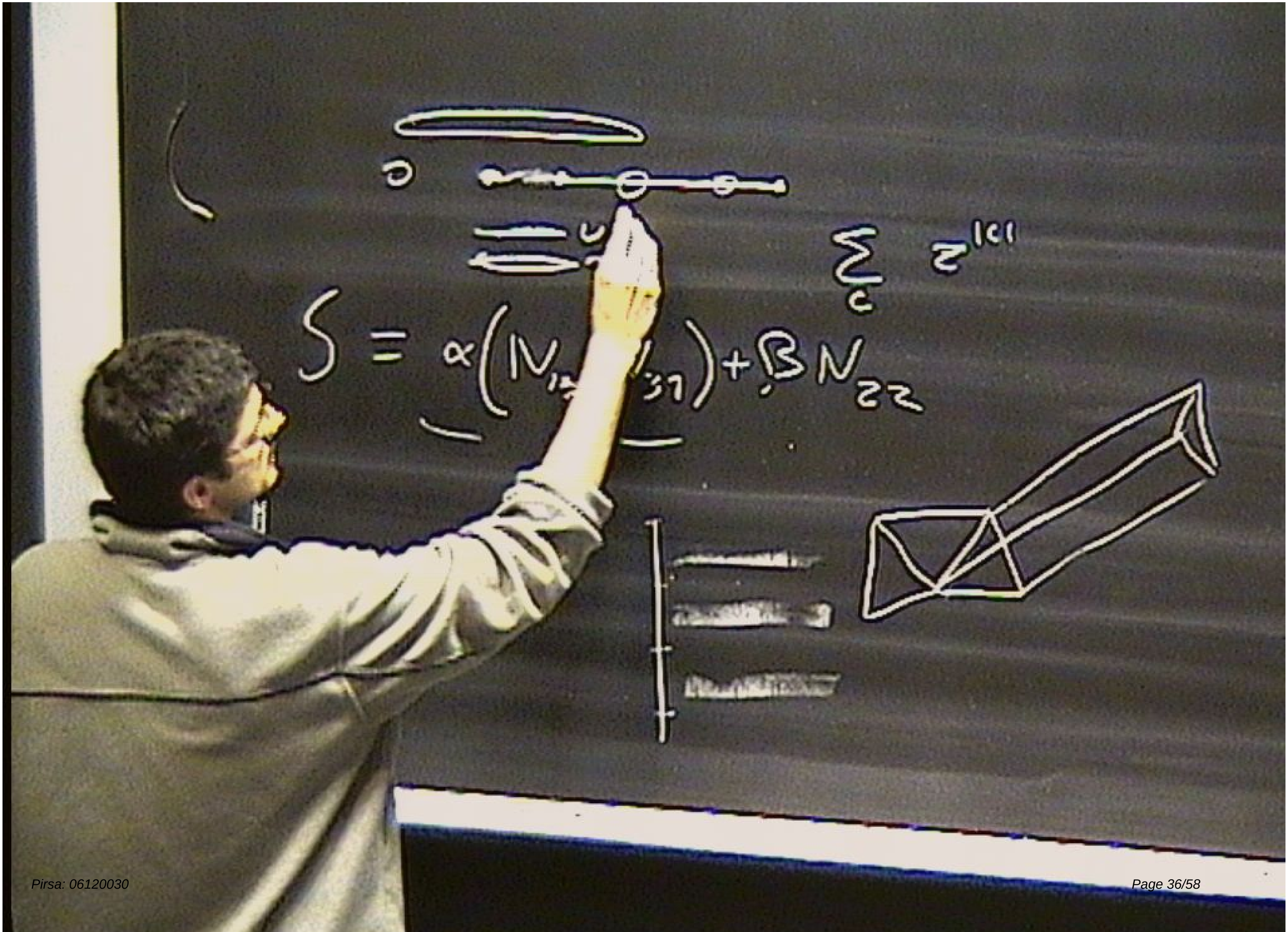
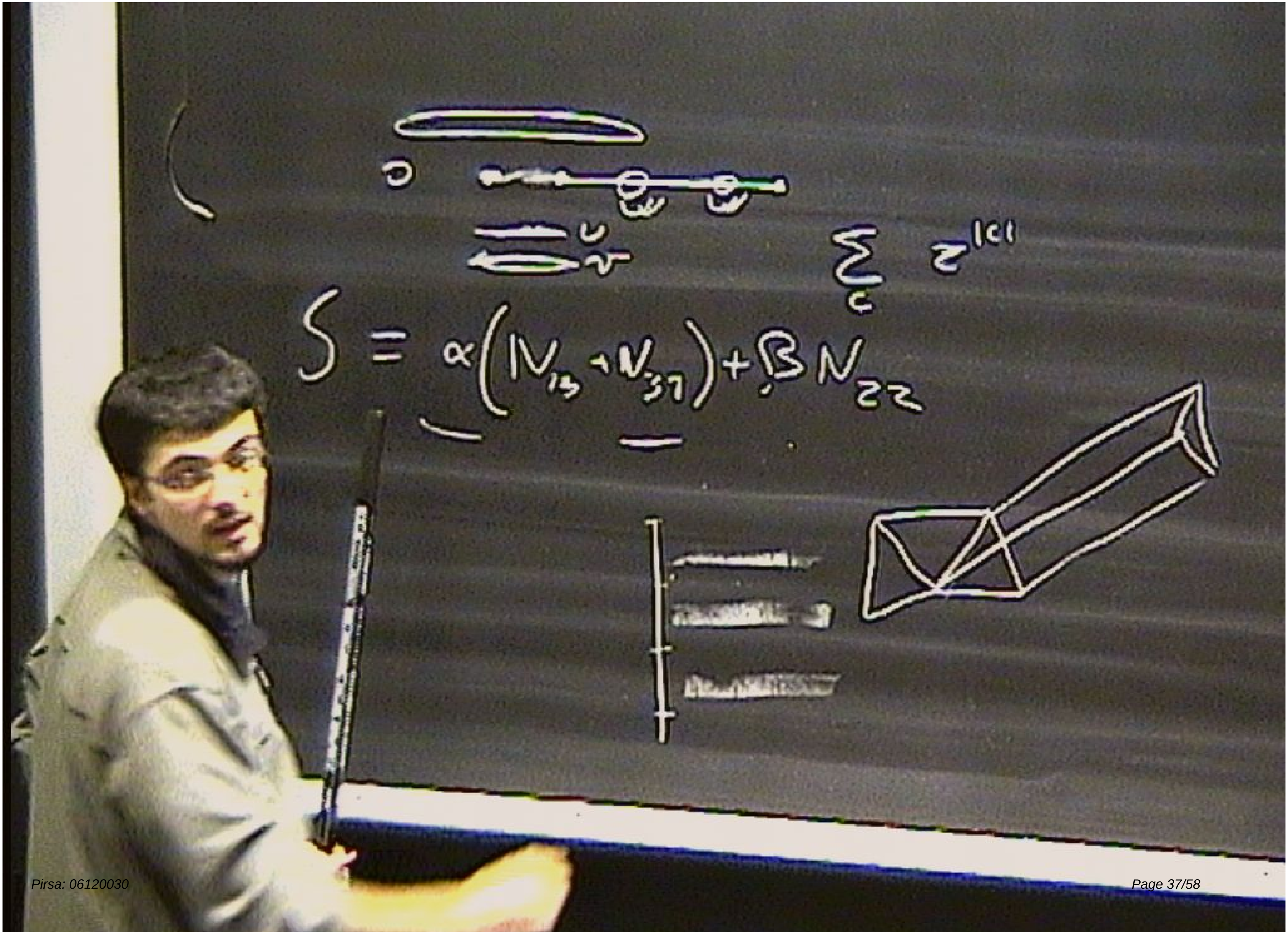

S = α(N
)+βN22
Σ z|c|
c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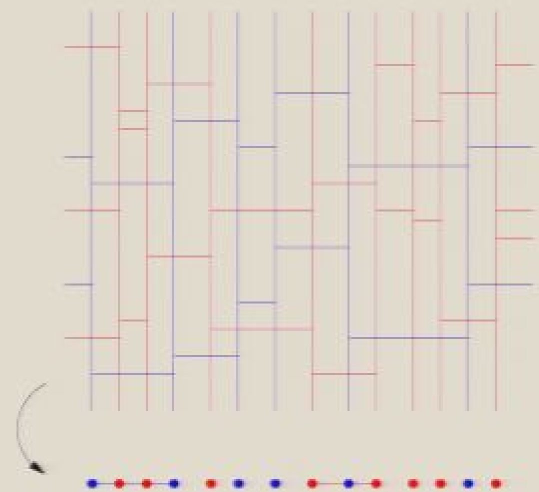
Heap of pieces and inversion formula

- The graph looks like a coloured version of what is called "heap of pieces" in combinatorics
- There exist a formula that relates heaps of pieces in *D* dimensions to hard objects in *D-1* dimensions
  (Viennot – 1986; Di Francesco, Guitter - 2001)
- It turns out that we can extend such a formula to the case with more colours, when we keep fixed the sequence of towers.
- The inversion formula looks like this

$$Z_{S_N}(u,v,w) = \frac{1}{Z^{h.d.}_{S_N}(-u,-v,w)}$$



Product of random matrices

- We can write the partition function for the hard dimers model in terms of transfer matrices

$$Z^{h.d.}_{S_N}(-u,-v,w) = Tr(\underbrace{ABBAB...}_{S_N})$$

$$A = \begin{pmatrix} 1 & i\sqrt{u} & 0 \\ i\sqrt{u} & 0 & 0 \\ 0 & 0 & w \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & i\sqrt{v} \\ 0 & w & 0 \\ i\sqrt{v} & 0 & 0 \end{pmatrix}$$

- We then have to do a sum over words of length N

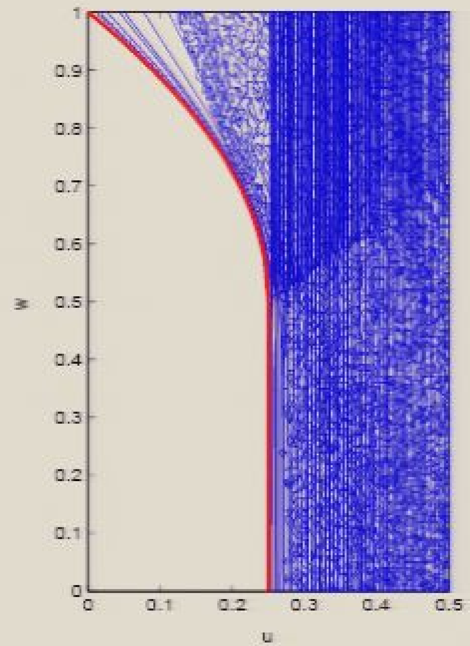
- In order to do this it is convenient to see the problem as a product of random matrices, *i.e.* a product N matrices picked up at random in a two-matrices ensembleeach with probability 1/2

$$Z = \sum_N e^{-\gamma N} \langle \frac{1}{Tr \prod_{j=1}^N M_j(u,v,w)} \rangle$$

- But before…

Avoiding the poles

- Because of the negative weights the partition function in the denominator can have zeros
- Different sequences have different zeros
- Such zeros accumulate for large N on a line which is then the critical line of our model
- We then have to find out what is the critical behaviour along that line
- It turns out that there is only one point along the critical line where our partition function has a singular behaviour and where then it is possible to define a non trivial continuum limit
(only at that point the average volume diverges)
- How do we see that? Next slide…



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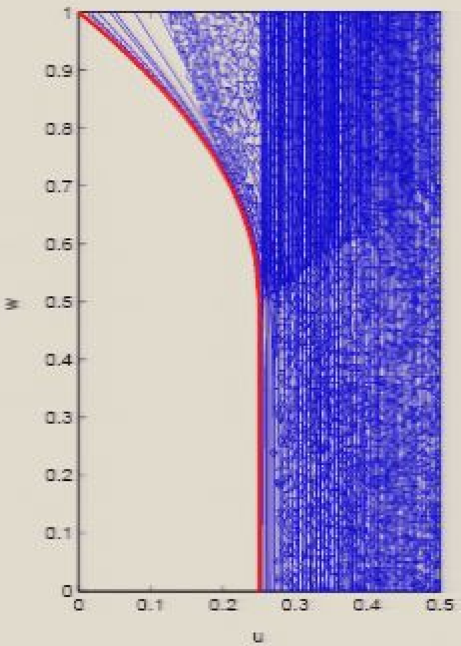

Replica trick

- In performing the sum comes in our help the so called "replica trick" widely used in spin glass theory
- For positive integer n the following formula holds:

$$\left\langle \left(Tr \prod_{j=1}^{N} M_j \right)^n \right\rangle = Tr \left\langle \prod_{j=1}^{N} M_j^{\otimes n} \right\rangle = Tr \prod_{j=1}^{N} \langle M_j^{\otimes n} \rangle = Tr \langle M^{\otimes n} \rangle^N \sim \nu_n^N$$

- The trick consists in using the above formula to compute

$$L_n = \lim_{N \to \infty} \frac{1}{N} \ln \left\langle \left(Tr \prod_{j=1}^{N} M_j \right)^n \right\rangle$$

and then in analytically continuing L_n to non integer values of n

- We are interested in $n = -1$
- The only problem with a negative value could be the appearance of poles, but we have already taken care of that.

Continuum limit

- We want to take the continuum limit tuning the coupling constants to the critical point
- Canonical scaling:

$$u = \frac{2}{9} e^{-2Xa^2 - 2b_1 \Lambda a^3 + 2c_1 ka}$$
$$v = \frac{2}{9} e^{-2Ya^2 - 2b_1 \Lambda a^3 + 2c_1 ka}$$
$$w = \frac{2}{3} e^{-b_2 \Lambda a^3 - c_2 ka}$$

- We need to insert these scalings in L_{-1}, computed with the replica trick and sum over N
- There's a problem with the scaling of k $\Longrightarrow$ take $c_1 = c_2 = 0$ for the time being
- We obtain:

$$Z = \frac{a}{\sqrt{X+Y}} + a^2 \left(\frac{5}{6} - \frac{XY}{(X+Y)^2} - \frac{\Lambda}{(X+Y)^{3/2}} \right) + O(a^3)$$

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Not yet there…fractional derivative

- We are computing

$$Z = \sum_{g_1, g_2} x^{A_1} y^{A_2} \langle g_2 | \widehat{T} | g_1 \rangle = \sum_{A_1, A_2} x^{A_1} y^{A_2} \langle A_2 | \widehat{T} | A_1 \rangle \sqrt{\mathcal{N}(A_1)\mathcal{N}(A_2)}$$

$$|A\rangle = \frac{1}{\sqrt{\mathcal{N}(A)}} \sum_{g|A} |g\rangle$$

- In order to extract the transfer matrix by Inverse Laplace transform we need to get rid of the entropy factor

- From (1+1)-dimensional CDT calculations we find: $\mathcal{N}(A) \sim A^{-\frac{1}{2}}$

- We can get rid of it by use of a fractional derivative of order ½ acting on the logarithm of x and y

- To cut it short, a fractional derivative is an operator which acts on exponentials like

$$\frac{\partial^\alpha}{\partial x^\alpha} e^{\gamma x} = \gamma^\alpha e^{\gamma x}$$

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The Hamiltonian

- Applying the fractional derivative we finally get

$$Z = \frac{1}{X-Y} + a\underbrace{\left(\frac{Y(2X+Y)}{(X-Y)^{5/2}} - \frac{\Lambda}{(X-Y)^2}\right)} + O(a^3)$$

$$\delta(A_1 - A_2)$$

$$\boxed{\hat{H} = -A^{3/2}\frac{\partial^2}{\partial A^2} - A^{1/2}\frac{\partial}{\partial A} + \Lambda A}$$

compare with: $\hat{H} = -G_N\left(A\frac{\partial^2}{\partial A^2} + \frac{\partial}{\partial A}\right) + \Lambda A$

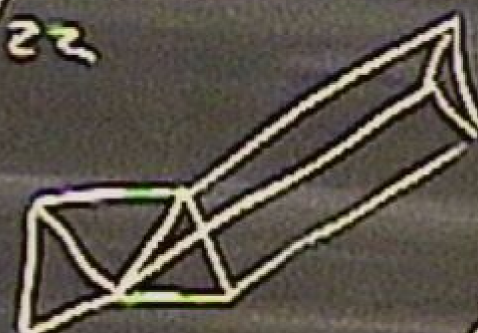
- Reintroduce Newton's constant? $\longrightarrow$ problems

$$\sum_C (-z)^{|C|}$$

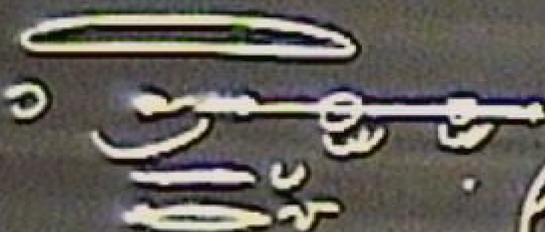
$$\langle A_1 | e^{-aH} | A_1 \rangle \sim \delta(A_2 - A_1) - a\,\hat{H}$$

$$S \equiv \alpha(N_{13} + N_{31}) + \beta N_{22}$$

$$\frac{1}{\sqrt{\ldots}}$$



$$Z_N \sim e^{N\Lambda_1}$$



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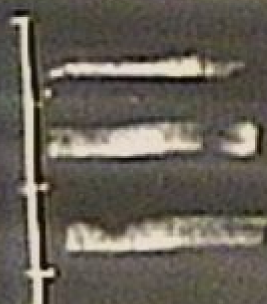

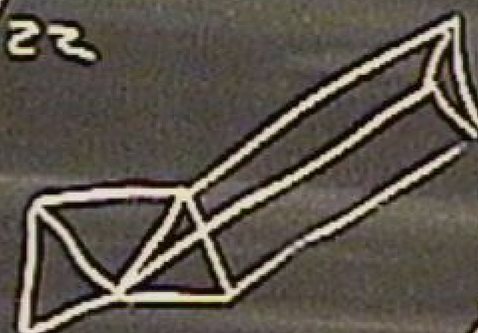

$$Z_N \sim e^{N L_1}$$

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$$\langle A_1 | e^{-\epsilon H} | A_1 \rangle$$

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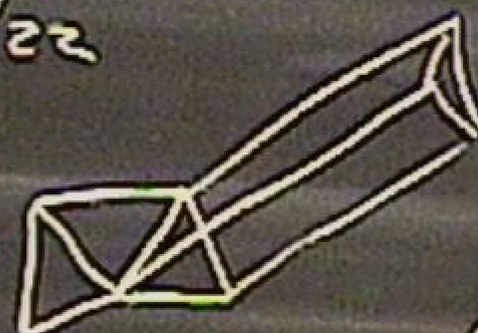
$$\sim \delta(A_2 - A_1) - a \dot{H}$$

$$\frac{1}{\sqrt{u}}$$



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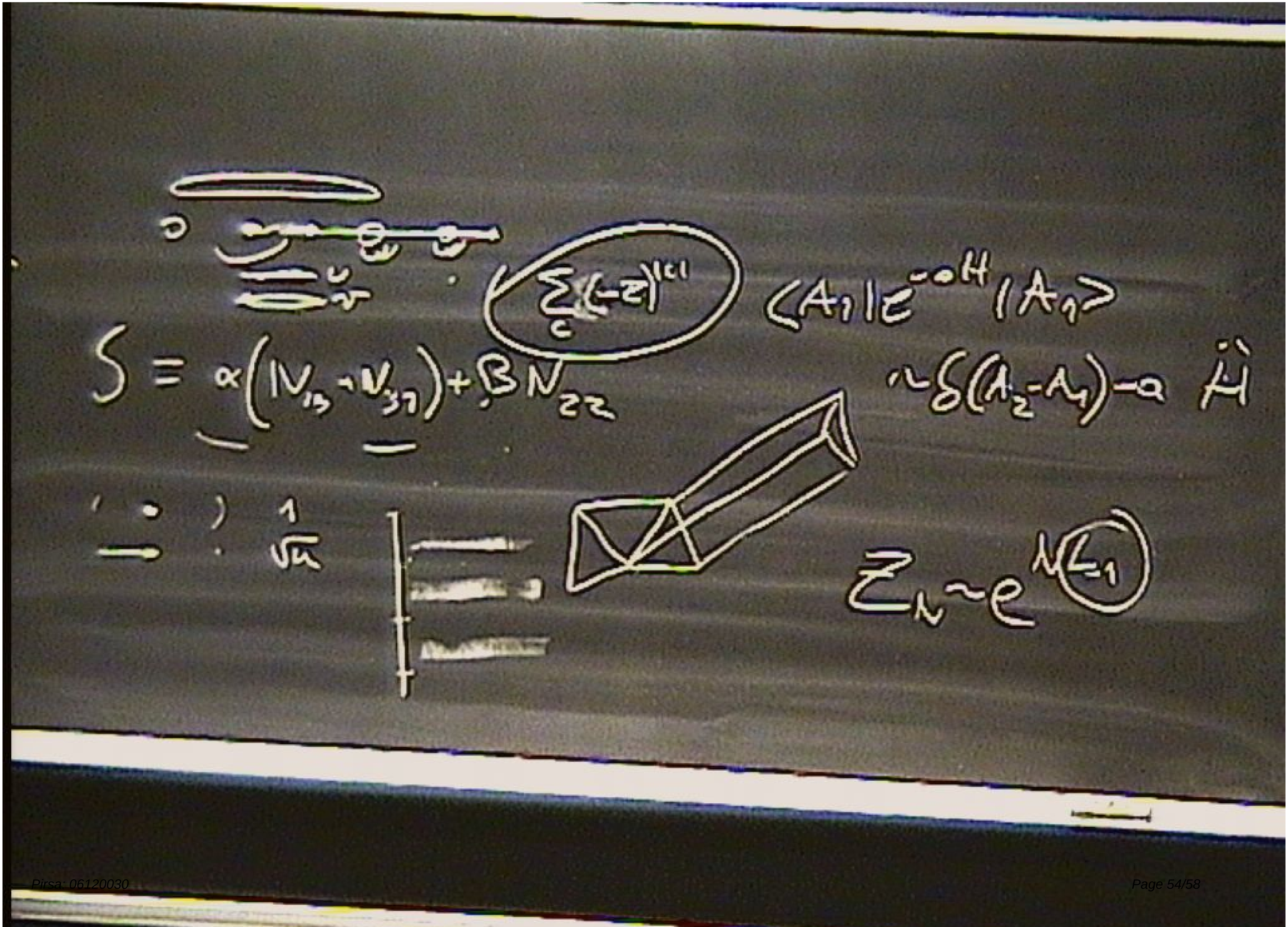
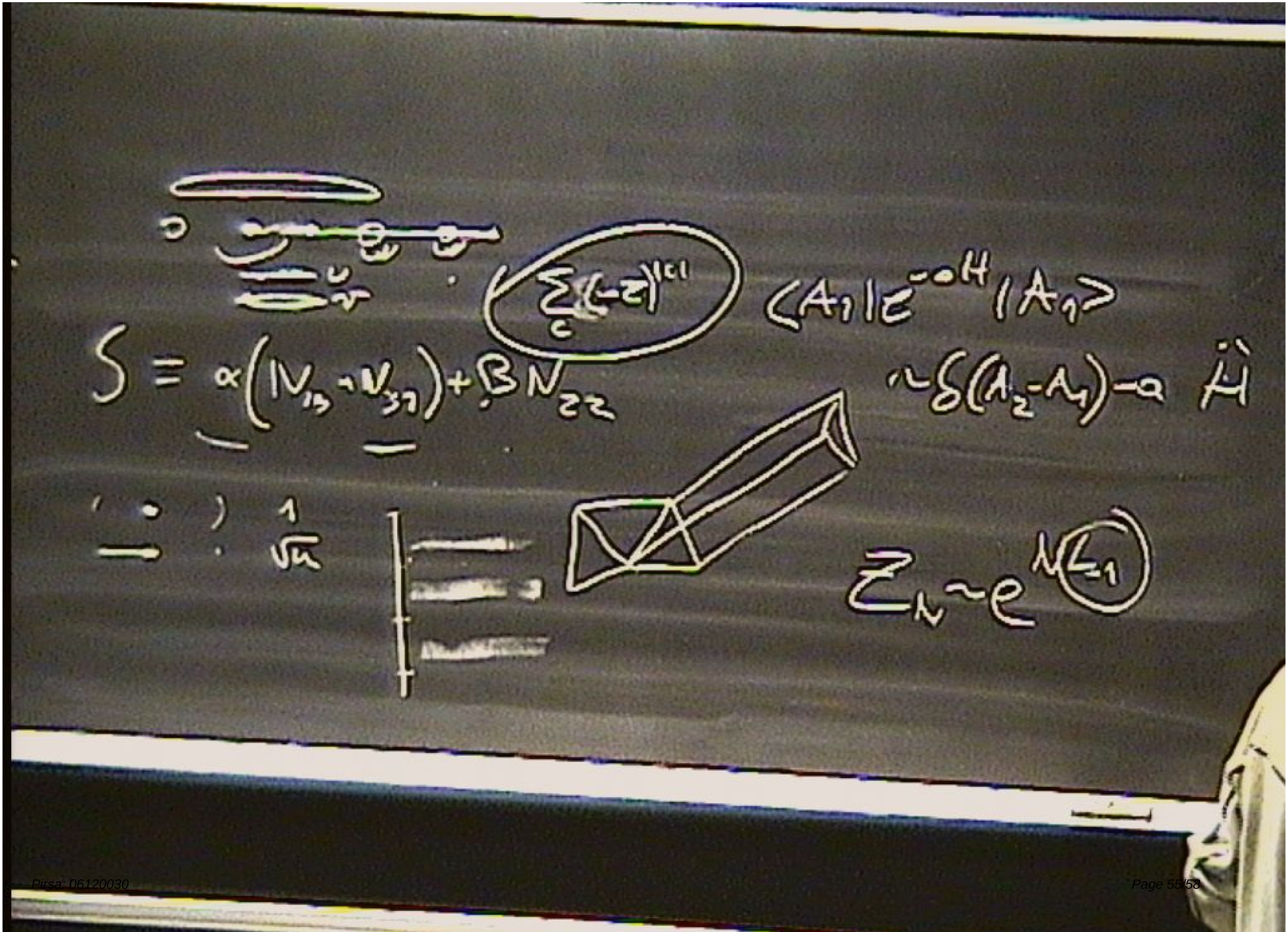

S ≡ α(N13 + N31) + β N22
⟨A1| e^-aH |A1⟩
~ δ(A2 - A1) - a H
Z_N ~ e^N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Who's to blame for this result?

- CDT and the idea of getting a Hamiltonian?
- This specific model?
- Some non-trivial step in the solution?
- Need to keep track of the Teichmuller parameter?
- Is it the non-renormalizability problem in disguise?
- Canonical scaling?

Under investigation…

Conclusions and outlook

- We have shown a full calculation of a CDT partition function in dimensions greater than 2.
- We have shown for the first time that a continuum limit with a well defined Hamiltonian exists in a (2+1)-dimensional CDT model
- Open issue about G_N must be understood
- Teichmuller part from "microcanonical" method
- Go back to the ABAB model with some of the lessons learned in this model