Title: Kappa Deformed Field Theory

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Abstract: The description of noncommutative space will be given. I will show the relation between field theory on kappa-Minkowski space and the one in Minkowski. This construction leads to deformed energy momentum conservation law for energies close to the Planck scale.

Kappa-Deformed Field Theory

Plan of the talk

- Introduction
- Kappa-Minkowski quantum space-time and field theory
- Back to normal: Minkowski space, star product and all that.

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Collaborators

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- Laurent Freidel (PI)
- Jerzy Kowalski-Glikman (UWr)

Introduction

- In the recent years a class of theories dubbed DSR (Doubly Special Relativity or Deformed Special Relativity) has been investigated intensively…

What is DSR?

- DSR is an extension of Special Relativity, based on two principles*:

1. Relativity principle for inertial observers;
2. Existence of two observer-independent scales: one of velocity c (as in SR), and the second of mass κ (or, equivalently – length λ). This second scale is usually identified with Planck scale.

* Amelino-Camelia (2001)

But how to …

- Implement these postulates?
- One possibility is to make use of kappa-Poincare algebra* and its structures:

1. This algebra has two scales built in
2. It possesses Hopf algebra structure, which makes it possible to construct an associated (non-commutative) spacetime, and phase space.

Kappa-Poincare algebra

- Kappa-Poincare is a deformed algebra of symmetries with deformation parameter kappa, equipped with additional structures: co-product and antipode.
- With the help of some underlying group structure one can interpret

1. group elements composition as co-product
2. the inverse group element as antipode

Quantum (noncommuting) space-time

- With the help of co-algebra one can build the dual of algebra of translations which is called kappa-Minkowski space. Its coordinates are hermitian operators satisfying the following Lie algebra

$$[\hat{x}_0, \hat{x}_i] = -\frac{i}{\kappa}\hat{x}_i$$

in the limit $\kappa \longrightarrow \infty$ one gets Minkowski space.

Plane waves

- The plane wave on kappa-Minkowski space

$$\hat{e}_k \equiv e^{i\mathbf{k}\,\hat{\mathbf{x}}}\, e^{-ik_0\hat{x}_0}$$

Plane waves form Lie group with group composition law

$$\hat{e}_k\hat{e}_p = \hat{e}_{kp}, \quad kp \equiv (k_0 + p_0, \mathbf{k} + e^{-k_0/\kappa}\mathbf{p})$$

Hermitian conjugation

$$\hat{e}_k^\dagger = \hat{e}_k^{-1} = \hat{e}_{S(k)}$$

where S is the antipode

$$S(k_0) = -k_0, \quad S(k_i) = -k_i e^{k_0/\kappa}$$

Group theoretical interpretation

- The group formed by plane waves is a subgroup of SO(1,4) group and „momenta" k are coordinates on four-dimensional de Sitter space, being the quotient SO(1,4)/SO(1,3). The parameter of deformation κ is the radius of de Sitter (momentum) space.

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What does curved momenta mean?

- What is the interpretation of deformed adition of „momenta"?

$$kp \equiv (k_0 + p_0, \mathbf{k} + e^{-k_0/\kappa}\mathbf{p})$$

- Perhaps this should be interpreted as a quantity conserved during the interaction in particle interaction processes. But the answer requires constructing field theory.

Which „momenta" are the physical ones?

- We can introduce another „momenta" by intruducing different coordinate system on de Sitter space

$$\hat{e}_{k(P)}\hat{e}_{k(Q)} = \hat{e}_{k(PQ)},$$

which gives new composition law. To decide which momenta are physical we have to construct field theory.

Action of Poincare algebra

- One can introduce the action of Lorentz generators on kappa-Minkowski space

$$M_i \triangleright x_0 = 0, \;\; M_i \triangleright x_j = i\epsilon_{ijk} x_k$$

$$N_i \triangleright x_0 = i x_i, \;\; N_i \triangleright x_k = i\delta_{ik} x_0.$$

- Because of nontrivial co-product

$$\Delta h_i = \sum h_i^{(1)} \otimes h_i^{(2)}$$

the action on the product of coordinates reads

$$h_i \triangleright (xy) = (h_i^{(1)} \triangleright x)(h_i^{(2)} \triangleright y)$$

- Applying these formulas one computes the action of Poincare generators on plane waves

$$N_i \triangleright \hat{e}_k = (\hat{x}_i \hat{\partial}_0 - \hat{x}_0 \hat{\partial}_i) e^{-\frac{1}{i}\partial_0}\, \hat{e}_k$$

$$M_i \triangleright \hat{e}_k = \epsilon_{ijk} \hat{x}_j \frac{1}{i} \partial_k\, \hat{e}_k$$

$$\hat{P}_\mu \triangleright \hat{e}_k = \hat{\partial}_\mu\, \hat{e}_k$$

Where $\hat{\partial}$ satisfies

$$\hat{\partial}_\mu \hat{e}_k(\hat{x}) = P_\mu \hat{e}_k(\hat{x})$$

$$\hat{\partial}_4 \hat{e}_k(\hat{x}) = (P_4 + \kappa) \hat{e}_k(\hat{x})$$

and

$$
\begin{aligned}
P_0(k_0, \mathbf{k}) &= \kappa \sinh \frac{k_0}{\kappa} + \frac{\mathbf{k}^2}{2\kappa} e^{\frac{k_0}{\kappa}} \\
P_i(k_0, \mathbf{k}) &= k_i \, e^{\frac{k_0}{\kappa}} \\
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\end{aligned}
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are coordinates of de Sitter space embedded in Minkowski space P_A

Bicovariant differential calculus

- Total differential of a function of quantum coordinates

$$d:f(\hat{x}) := i\, d\hat{x}_A\, \hat{\partial}^A : f(\hat{x}) :$$

where the basis of one forms satisfy the folowing consistency relations

$$[\hat{x}_\mu, d\hat{x}_4] = \frac{i}{\kappa}\, d\hat{x}_\mu, \quad [\hat{x}_0, d\hat{x}_0] = \frac{i}{\kappa}\, d\hat{x}_4, \quad [\hat{x}_0, d\hat{x}_i] = 0,$$

$$[\hat{x}_i, d\hat{x}_0] = \frac{i}{\kappa}\, d\hat{x}_i, \quad [\hat{x}_i, d\hat{x}_j] = \frac{i}{\kappa}\, \delta_{ij}(d\hat{x}_0 - d\hat{x}_4).$$

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Transformations

- Following the commutative case one can introduce transformations of functions on kappa-Minkowski

$$\delta f(\hat{x}) \equiv \delta_0 f(\hat{x}) + i\varepsilon T \triangleright f(\hat{x})$$

where δ_0 is a functional change and T is a generator of transformation

- For scalar field

$$\delta \phi(\hat{x}) = 0 \quad \delta_0 \phi(\hat{x}) = -i\varepsilon T \triangleright \phi(\hat{x})$$

Fields

- Field on quantum space can be defined as a Fourier transform

$$\phi(\hat{x}) = \frac{1}{(2\pi)^4} \int d\mu \phi(k)\, e^{ik_i \hat{x}_i}\, e^{-ik_0 \hat{x}_0}$$

- Hermitian conjugated field

$$\phi^\dagger(\hat{x}) = \frac{1}{(2\pi)^4} \int d\mu \phi^*(k)\, e^{iS(k_i)\hat{x}_i}\, e^{-iS(k_0)\hat{x}_0}$$

Invariant field action

- Having defined transformations together with action of Poincare generators we can introduce the invariant field action

$$S = \int d^4\hat{x}\, (\hat{\partial}_\mu \phi(\hat{x}))^\dagger \hat{\partial}^\mu \phi(\hat{x}) + m^2 \int d^4\hat{x}\, \phi^\dagger(\hat{x})\phi(\hat{x})$$

where the integration over noncommutative space-time is done by delta function

$$\frac{1}{2\pi^4} \int d^4\hat{x} e^{i\mathbf{k}\hat{\mathbf{x}}} e^{-ik_0\hat{x}_0} \equiv \delta^4(k)$$

…Summarizing

- We have nice mathematical structure:

1. Plane waves and fields
2. Action of Poincare generators on fields
3. Differential calculus on quantum space-time
4. Invariant field action

But how to introduce physical quantities?

- The field theory on noncommutative space-time can be associated with an appropriate one in Minkowski space.
- Then conserved charges can be found.
- But this unavoidably leads to the field theory with Lagrangian containing higher order derivatives.

…Summarizing

- We have nice mathematical structure:

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Star product formalism

- The structure defined on kappa-Minkowski space can be equivalently expressed in Minkowski space by introducing the linear map $\mathcal{W}$:

$$\mathcal{W}\left(e^{i\mathbf{k}\,\hat{\mathbf{x}}}\,e^{-ik_0\hat{x}_0}\right) = e^{iP_\mu(k)\,x^\mu}$$

- This map is uniquely defined if we demand

$$\mathcal{W}\left(\hat{\partial}_\nu e^{i\mathbf{k}\,\hat{\mathbf{x}}}\,e^{-ik_0\hat{x}_0}\right) = \frac{1}{i}\,\partial_\nu e^{iP(k)_\mu\,x^\mu} = P_\nu\,e^{iP(k)_\mu\,x^\mu}$$

and it defines the following star product on Minkowski according to formula

$$\mathcal{W}\left(e^{i\mathbf{k}\,\hat{\mathbf{x}}}\,e^{-ik_0\hat{x}_0}\,e^{i\mathbf{l}\,\hat{\mathbf{x}}}\,e^{-il_0\hat{x}_0}\right) = e^{iP(k)_\mu\,x^\mu} \star e^{iP(l)_\mu\,x^\mu}$$

The left hand side is equal

$$e^{iP_\mu(kp)x^\mu}$$

and so the star product reflects the group structure of noncommutative plane waves

$$\hat{e}_k\hat{e}_p = \hat{e}_{kp}, \quad kp \equiv (k_0 + p_0, \mathbf{k} + e^{-k_0/\kappa}\mathbf{p})$$

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- Using the identity

$$\lim_{a\to 0} \exp\left\{ix\left[f\left(\frac{1}{i}\frac{\partial}{\partial a}\right) - \frac{1}{i}\frac{\partial}{\partial a}\right]\right\} e^{ip(x+a)} = e^{if(p)x}$$

we can generalize star to an operator acting on two arbitrary analytic functions. One checks

$$x_0 \star x_i - x_i \star x_0 = -\frac{1}{\kappa} x_i$$

- The explicit form of star operator is rather complicated and we limit our considerations to Fourier transforms what simplifies the problem a lot.

Transformed Poincare action

- The action on kappa-Minkowski can be translated to Minkowski using the map W

$$\mathcal{W}(N_i \triangleright \hat{e}_k) = \mathcal{W}((\hat{x}_i\hat{\partial}_0 - \hat{x}_0\hat{\partial}_i)e^{-\frac{1}{i}\partial_0}) \star \mathcal{W}(\hat{e}_k)$$

$$\mathcal{W}(M_i \triangleright \hat{e}_k) = \mathcal{W}(\epsilon_{ijk}\hat{x}_j\frac{1}{i}\partial_k) \star \mathcal{W}(\hat{e}_k)$$

which gives the usual differential representation of Poincare algebra

Field action

- The field action on kappa-Minkowski can be mapped to Minkowski space using the star product

$$S = \int \mathrm{d}^4x \, \frac{1}{2}(\partial_\mu \phi)^\dagger \star (\partial_\mu \phi)(x) + \frac{m^2}{2}\phi^\dagger \star \phi(x)$$

- Substituting Fourier transforms we can rewrite the action in the following simple form

$$S = \int \mathrm{d}^4x \, \frac{1}{2}(\partial_\mu \phi)^*(1 - \partial_4)(\partial_\mu \phi)(x) + \frac{m^2}{2}\phi^*(1 - \partial_4)\phi(x)$$

where

$$(1-\partial_4) = \sqrt{1-\frac{\Box}{\kappa^2}}$$

- This lagrangian is obviously invariant under Poincare transformations

$$\phi(x) \to \phi(x+a), \quad \phi(x) \to \phi(\Lambda x)$$

- Rescaling the field one finds the action is equivalent to the free scalar field action.

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And thus

- No deformation for free particle. This can be understood if we recall that P are variables for which algebraic sector of kappa-Poincare algebra is not deformed (this is coalgebra that is deformed). The momenta of free particle are the usual one ($P^2 = m^2$)

What can we say about interaction?

- Let us consider for simplicity the real field

$$\psi = \phi^\dagger + \phi$$

The interaction term imported from kappa-Minkowski space has the form

$$S_{int} \propto \int \mathrm{d}^4 x \, \psi \star \psi \star \psi \star \psi$$

- Using Fourier transforms we get the term of the form

$$S_{int} \propto \int \mathrm{d}^4x\,(\psi \star \psi)(1-\partial_4)(\psi \star \psi)$$

but

$$\psi \star \psi$$

produces exponent of the form

$$e^{iP_\mu(kp)x^\mu}$$

and in momentum space we get

$$S_{int} \propto \int \psi(P(k_1))\psi(P(k_2))\psi(P(k_3))\psi(P(k_4))$$

$$\delta^4(P(k_1k_2) + P(k_3k_4))$$

If we treat P as patricle momenta, this result can be interpreted as deformation of momentum conservation law.

conclusion

- The most important observation is that the noncommutativity (curved momentum space) has no impact on free particles. It is the interaction that might lead to deformed conservation law for energies close to the Planck scale. But without consistent quantum field theory one cannot conclude anything yet.

THE END