Title: Aspects of Nonlinear Perturbations in Cosmological Models

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Abstract: I discuss two instances in which nonlinear perturbations in cosmological models are important. First, in de Sitter space-time, the bare necessity that the perturbations should be part of a consistent Taylor expansion of the field equations leads to the requirement, using the 'linearization stability' arguments of the '70's, that the quantum field theory of a scalar field on de Sitter space-time is manifestly de Sitter invariant (not covariant). Second, the concern that in slow-roll inflation the effect of second order perturbations on the long wavelength (super Hubble) perturbations could be much stronger than that of the first order perturbations, for a wide range of slow-roll conditions, is explored in the context of a linear inflation potential and chaotic inflation.

Aspects of nonlinear perturbations in cosmological models

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Outline of talk

- Are nonlinear perturbations/backreactions important during inflation or a de Sitter phase?
- Longwavelength backreactions during slow-roll inflation (gr-qc/0510078)
- Backreactions during no-roll inflation (de Sitter spacetime)(gr-qc/0604122 + J. Phys. A ?)
- Conclusions

2nd order theory? A heuristic look

For background scalar field a spacetime constant, the leading order contribution to T_{ab} occurs at *second* order in perturbation theory:

i.e., for $\phi(t,\vec{x}) = \bar{\phi}(t) + \epsilon\delta\phi(t,\vec{x}) \quad , \epsilon \ll 1$

$$T_{ab} = \nabla_a\phi\nabla_b\phi - g_{ab}\left(\tfrac{1}{2}\nabla^c\phi\nabla_c\phi + V(\phi)\right)$$

$$\bar{T}_{ab} \to 0$$

{Approaching de Sitter vacuum solution}

$$\delta T_{ab} \to 0$$

{No scalar or vector modes in vacuum background}

$$\delta^2 T_{ab} \neq 0$$

{Dominant term: Second order scalar, vector modes, mode mixing with TT gravity waves, etc.}

Is there a transition to nonlinear dominance in the gravitational field?

A deeper look

- Linear perturbations must consistently seed entire hierarchy of higher order perturbations
- In particular, initial value constraints

$$\mathcal{H}^a \approx 0$$

must be satisfied by all orders of perturbations

- At second order these constraints imply relations of the form

$$aL(\delta^2 g; \delta^2 \phi) + bF((\delta g)^2; (\delta\phi)^2) = 0$$

linear operator

linear operator on quadratic fluctuations

a, b $\in \Re$

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$$a\underset{\substack{\uparrow \\ \text{linear operator}}}{L}(\delta^2 g; \delta^2 \phi) + b\underset{\substack{\uparrow \\ \text{linear operator on} \\ \text{quadratic fluctuations}}}{F}((\delta g)^2; (\delta\phi)^2) = 0$$

a, b $\in \Re$

Large second order fluctuations near a de Sitter background?

- It turns out that |a| $\sim$ 0 and b $\neq$ 0 `very near' a de Sitter background

- Then the relation

$$aL(\delta^2 g; \delta^2 \phi) + bF((\delta g)^2; (\delta\phi)^2) = 0$$

implies some components of (δ^2 g, δ^2 ϕ) can get very large, maybe even dominate linear!

- Note that this line of argumentation is inherently gauge-invariant; minimal

The gory technical details

- Projecting the initial value constraints along X^a near a de Sitter background:

$$\int_{\Sigma_t} \delta^2 \hat{\mathcal{H}}^{\mathbf{a}}(\delta^2 \mathbf{h}; \delta^2 \pi) \mathbf{X_a} \mathbf{d^3x} = - \int_{\Sigma_t} \delta^2 \hat{\mathcal{H}}^{\mathbf{a}}((\delta \mathbf{h})^2; (\delta \pi)^2) \mathbf{X_a} \mathbf{d}$$

- Left hand side zero only when $\nabla_{(a} X_{b)} = 0$, i.e. exactly at a de Sitter background

- However, for $Y^a \ni \nabla_{(a} Y_{b)} \approx 0$ some components of $(\delta^2 h, \delta^2 \pi)$ must be huge

Some controversy about "backreactions" in the field

- Backreactions of inhomogeneous super-Hubble modes mimic Dark energy?

  (Barausse, Kolb et al, … vs. Geshnizjani, Flanagan, Wald, Hirata)

- Really hard calculations; poorly controlled approximations; gauge ambiguities

- No general consensus on importance of backreactions, especially during inflation

- I focus here only on whether or not linear perturbation theory runs into trouble during slow-roll inflation, for long wavelengths

Programme

- How and why can nonlinear perturbations/backreactions be important for cosmology?
- **Longwavelength backreactions during slow-roll inflation**
  - Slow-roll background; long wavelengths
  - Second order perturbations, gauge issues
  - Eigenvalues of total stress energy
  - Dispersion of eigenvalues at second order
- Backreactions during no-roll inflation (de Sitter spacetime)
- Conclusions

Basic setup

- Perturb slow-roll metric and inflaton to second order:

$$g_{ab} \equiv \bar{g}_{ab} + \epsilon\delta g_{ab} + \epsilon^2\delta^2 g_{ab}$$

$$\phi \equiv \bar{\phi} + \epsilon\delta\phi + \epsilon^2\delta^2\phi$$

- Einstein field equations:

$$\mathcal{L}(\delta g_{ab}, \delta\phi)_k = 0$$

$$\mathcal{L}(\delta^2 g_{ab}, \delta^2\phi)_k = \int {}^{(2)}S((\delta g_{ab})^2_k, (\delta\phi)^2_k)dk$$

Many fourier modes at linear order contribute to a given k at 2nd order…*cumulative effects are generically expected*

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Linear equations: Details well known

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$$\boxed{\mathcal{L}(\delta^2 g_{ab}, \delta^2\phi)_k = \int {}^{(2)}S((\delta g_{ab})^2_k, (\delta\phi)^2_k)dk}$$

How large, compared to the linear perturbations, do the second order perturbations get (during slow-roll)?

Many fourier modes at linear order contribute to a given k at 2nd order…*cumulative effects are generically expected*

perturbations in detail

- Scalar, Vector, Tensor perturbations at linear order and only scalar perturbations at second order:

$$\begin{aligned} ds^2 \;&=\; -(1+\epsilon A(t,\vec{x})+\epsilon^2\mathcal{A}(t,\vec{x}))dt^2 + 2(\epsilon B_i(t,\vec{x}) \overbrace{+\,\epsilon^2\mathcal{B}_i(t,\vec{x}))dt}^{\textbf{Scalar part called } \mathcal{B}} \\ &\quad + a^2(t)(\delta_{ij} + \epsilon h_{ij}(t,\vec{x}) + \underbrace{\epsilon^2 q_{ij}(t,\vec{x})}_{\textbf{Scalar parts called } {}^{(2)}E \textbf{ and } Q})dx^i dx^j \end{aligned}$$

Scalar parts called ψ **and** E

- Similarly perturb the scalar field

$$\phi(t,\vec{x}) = \bar{\phi}(t) + \epsilon\Phi(t,\vec{x}) + \epsilon^2\mathcal{F}(t,\vec{x})$$

Background scalar field

Slow-roll background spacetime

- Inflaton potential taken to be linear (for simplicity)

$$V(\bar{\phi}) = \Lambda + \beta\bar{\phi}\ , \beta \in \Re$$

- Associated (and only non-trivial) slow-roll condition is

$$\epsilon_{SR} \equiv \frac{1}{\kappa}\left(\frac{V_{,\bar{\phi}}}{V}\right)^2 = \frac{\kappa\beta^2}{H^4} \ll 1$$

\`slow-roll\` parameter

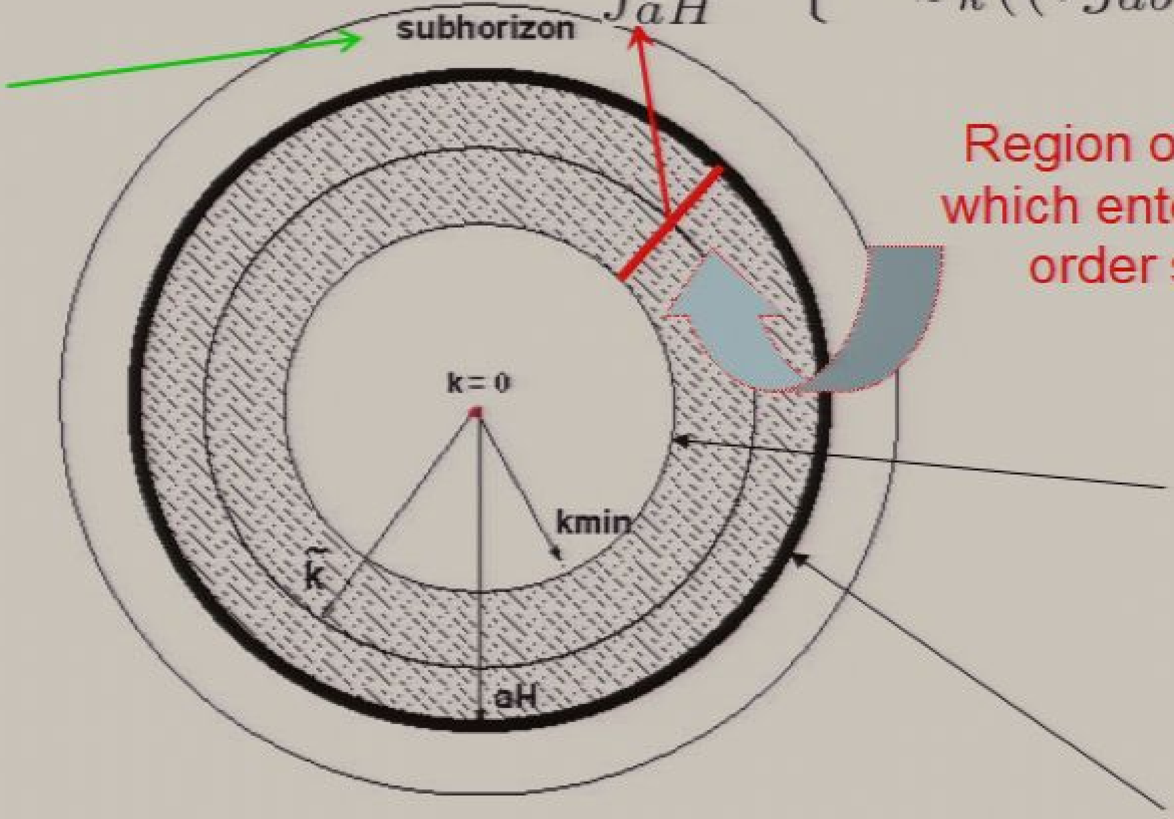
$8\pi G$

- Simple background scalar field: $\bar{\phi} = \bar{\phi}_0 - \frac{\beta t}{3H}$

Scales of (long wavelength) k

$$\int_{aH}^{k_{min}} \left\{ {}^{(2)}S_k((\delta g_{ab})^2_k, (\delta\phi)^2_k) \right\}$$

We make no comment on subhorizon backreactions



Region of k-space which enters the 2nd order source

IR Cutoff scale

Hubble radiu

$$\left(\frac{k}{aH}\right)^2 \ll 1$$

`Long-wavelength condition`

2nd order perturbations: a whole host of new problems

- Solving the constraint equations can, e.g. , introduce terms that go as

$$\left(\frac{k}{aH}\right)^2 \frac{1}{\epsilon_{SR}}$$

- The second order gauge choice will depend on the linear gauge choice as well:

$$\delta^2 \hat{g}'_{ab} = \delta^2 g_{ab} + \mathcal{L}_\chi \bar{g}_{ab} + \left(\mathcal{L}^2_\zeta \bar{g}_{ab} + 2\mathcal{L}_\zeta \delta g_{ab}\right)$$

- Regularization of UV fluctuations in CST

Effective equation of state @ second order

- We want to extract some gauge invariant information from the second order equations of motion
- Can we express this information in terms of an effective equation of state? Then we may construct

IR Cutoff

$$\begin{aligned}\rho &= \bar{\rho} + \epsilon\delta\rho + \epsilon^2 \int_{aH}^{k_{min}} \delta^2 \rho_k d^3 k \\ p &= \bar{p} + \epsilon \frac{1}{3} \sum_i \delta p_i + \epsilon^2 \int_{aH}^{k_{min}} \frac{1}{3} \sum_i \delta^2 p_i d^3 k\end{aligned}$$

- Then we can succinctly demand *linearized consistency*:

$$\sqrt{\langle 0| \left(\frac{\delta p}{\bar{p}}\right)^2 \Big|_{k=aH} |0\rangle} > \sqrt{\langle 0| \left(\frac{\delta^2 p_{IR}}{\bar{p}}\right)^2 |0\rangle}\Bigg|_{k=\tilde{k}}$$

Standard
Hadamard
vacuum

Setting the 1st and 2nd order gauge exhaustively

- Overall strategy: Emulate harmonic gauge

$$\frac{1}{\sqrt{-|g|}}\partial_a\left(\sqrt{-|g|}g^{ab}\right) = 0$$

- Set the linear sector to obey

$$\begin{aligned} B^i_{,i} &= 0 \\ \partial^j\left(h_{ij} - \frac{\delta^{\ell m}h_{\ell m}}{3}\delta_{ij}\right) &= 0 \end{aligned}$$

Sometimes called the `Poisson gauge`

- Set the 2nd order *scalar* sector to obey

Polarizations of TT part of h

$$\mathcal{A} = -\mathcal{Q} + \underbrace{\frac{2}{3}\left[(h_+)^2 + h_+h_- + (h_-)^2\right] +}_{\text{TT-TT sector}} \underbrace{\psi^2 - 2\kappa\Phi^2}_{\text{Scalar-Scalar sector}}$$

$$\mathcal{B} =$$

$$^{(2)}E =$$

Eigenvalues of stress-energy

- Consider the eigenvalue problem

$$\det\left(\bar{T}^a_b + \delta T^a_b + \underbrace{\delta^2 T^a_b + \delta^2 \tau^a_b}_{\text{Matter + gravity parts}} - \lambda_i \delta^a_b\right) = 0$$

- Define the ‘energy density’ ρ as minus the timelike eigenvalue, similarl the averaged pressure in terms of the average of spacelike eigenvalues
- Obtain expressions in terms of scalars like Tr(T) and $S^b_a S^a_b$

where $S_{ab} \equiv T_{ab} - \frac{g_{ab}}{4} T^m_m$, e.g.

$$\delta^2(T^a_{\ a} + \tau^a_{\ a}) = -\delta^2\rho + \sum_1 \delta^2 p_i$$

$$\delta^2\left(\frac{4}{3} S^a_{\ b} S^b_{\ a}\right) = (\delta\rho)^2 + \left(\sum_i \delta p_i\right)^2 + \frac{2}{3}\delta\rho \sum_i \delta p_i - \frac{8}{3} \overbrace{\sum_{i\neq j} \delta p_i}^{\text{(Off diagonal) shears}} \delta p_j + 2(\bar{\rho} + \bar{p})\left(\delta^2\rho + \frac{1}{3}\sum_i \delta^2 p_i\right)$$

Reduction Strategy

- Eigenvalues $\delta^2\rho$ and $\frac{1}{3}\sum_i \delta^2 p_i$ are complicated functions of metric, matter variables.
- To reduce:

  - First order Poisson gauge fixing and second order scalar fixing
  - Solve the *constraint equations* for the matter (scalar field) fluctuations
  - Solve the *constrained evolution* equation for second order scalar fluctuations in terms of linear solutions

- Once $\delta^2\rho$ and $\frac{1}{3}\sum_i \delta^2 p_i$ are known in terms of linearized fluctuations

(e.g. ψ), we can calculate their dispersions using well-known expressions lik

$$k^3|\psi_k|^2 = \frac{1}{4}\frac{H^4}{(2\pi\dot{\phi})^2} = \frac{1}{4}\frac{9\kappa}{\epsilon_{SR}}\left(\frac{H}{2\pi}\right)^2$$

Dispersion of eigenvalues I

- Expanding out the modes ψ, we take $\psi_k = \omega_{\vec{k}} a_k + \omega^*_{\vec{k'}} a^\dagger_{k'}$
- One can show…

$$< \psi^2 > \equiv < \int_{\Omega_{k'}} \psi_{(k'-k)} \psi_{k'} d^3\vec{k'} > \; = \; \delta(-k) < \int_{\Omega_{k'}} \omega_{(\vec{k'}-\vec{k})} \omega^*_{\vec{k'}} d^3\vec{k'} >$$

I.e., Poincare invariance of linearized fluctuations ↔ homogeneous support of 2 pt fcn

- We wish to measure the *fluctuations in* $< \psi^2 >$ *:*

$$\begin{aligned} < \psi^4 > \; = \; & \left(< \int_{\Omega_{k'}} \psi_{(k'-k)} \psi_{k'} d^3\vec{k'} > \right)^2 \\ & + < \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{(k'-k)} \psi_{(k''-k)} d^3\vec{k'} d^3\vec{k''} >< \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{k'} \psi_{k''} d^3\vec{k'} d^3\vec{k''} > \\ & + < \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{(k'-k)} \psi_{k''} d^3\vec{k'} d^3\vec{k''} >< \int_{\Omega_{k'}} \int_{\Omega_{k''}} \psi_{(k''-k)} \psi_{k'} d^3\vec{k'} d^3\vec{k''} > \end{aligned}$$

Dispersion of eigenvalues II

- One can show, e.g., that

$$< \delta^2 p_{IR}(k)\delta^2 p^\dagger_{IR}(k) > \;\approx\; \overbrace{< \int_{\Omega_{k'}}\int_{\Omega_{k''}} \left(\frac{54H^2}{\kappa\epsilon_{SR}}\right)^2 \psi_{(k'-k)}\psi_{k'}\psi_{(k''-k)}\psi_{k''}\, d^3\vec{k}'\, d^3\vec{k}'' >}^{\text{Contains linear fluctuations only}}$$

Use second order e.o.m. here

Pure second order term

$$+ < \int_{\Omega_{k'}} \left(\frac{3H}{\kappa}\right)^2 \mathcal{L}Q_{k'-k}\,\mathcal{L}Q_{k'}\, d^3\vec{k}' >$$

Second order and linear 2-pt term cross-term

$$+ < \int_{\Omega_{k'}}\int_{\Omega_{k''}} \left(\frac{6H}{\kappa}\frac{54H^2}{\kappa\epsilon_{SR}}\right) \mathcal{L}Q(t',k';k)\,\psi_{(k''-k)}\psi_{k''}\, d^3\vec{k}'\, d^3\vec{k}'' >$$

- It turns out that

Numbers

$$< \delta^2 p_{IR}(k)\delta^2 p^\dagger_{IR}(k) > \;\approx\; \left(\frac{H^2}{\kappa}\right)^2 \frac{\kappa^2 H^4}{\epsilon^4_{SR}\pi^2}\left(A_1\alpha^2 + \alpha\left(B_1\ln(\gamma) + C_1\ln(\sigma)\right)\right)$$

 # e-folds ↔ N/ϵ_{SR}

$\gamma \equiv 1 + \frac{2k}{k_{min}}$

$\sigma \equiv \frac{aH}{k_{mi}}$

`Linearized consistency`

- Finally we can use this expression to explore the consistency requirement

$$\sqrt{\langle 0| \left(\frac{\delta p}{\bar{p}}\right)^2 \Big|_{k=aH} |0\rangle} > \sqrt{\langle 0| \left(\frac{\delta^2 p_{IR}}{\bar{p}}\right)^2 |0\rangle}\,\Bigg|_{k=\tilde{k}}$$

- It turns out that this is equivalent to

$$\epsilon_{SR} > \frac{2}{3}(\kappa H^2)^{\frac{1}{4}}(\underset{\sim 200}{A_1}\overset{\text{\# e-folds \sim 70}}{N})^{\frac{1}{4}} \quad \underbrace{> \quad 1}_{\text{for } H^2 \sim {m_{planck}}^2}$$

- Somewhat worrisome; gauge effect, potential effect?

Gauge dependence of the second order terms

- Look at perturbations in θ

$$\theta \equiv \frac{4}{3} S^b_a S^a_b \overset{*}{=} (\rho + \frac{1}{3} \sum_i p_i)^2$$

- This scalar quantity is ‘almost’ gauge invariant:

$$\mathcal{L}_X \bar{\theta} = X^0 \partial_0 (\bar{\rho} + \bar{p})^2 \approx 0$$

- Dispersion of θ at second order still dominates linear dispersion.

Extension to chaotic inflation

- Inflaton potential (inflation ends `properly` now)

$$V(\bar{\phi}) = \frac{m^2\bar{\phi}^2}{2}$$

- Compute $< F\left(\delta^2\theta, (\delta\theta)^2; \bar{\theta}\right)^2 >^{1/2}$ and compare to

$< \left(\frac{\delta\theta}{\bar{\theta}}\right)^2 >^{1/2}$

- Still obtain that $\varepsilon_{SR} > 1$ for linear dominance…(!)

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- How and why can nonlinear perturbations/backreactions be important for cosmology?
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  - Quantum anomalies
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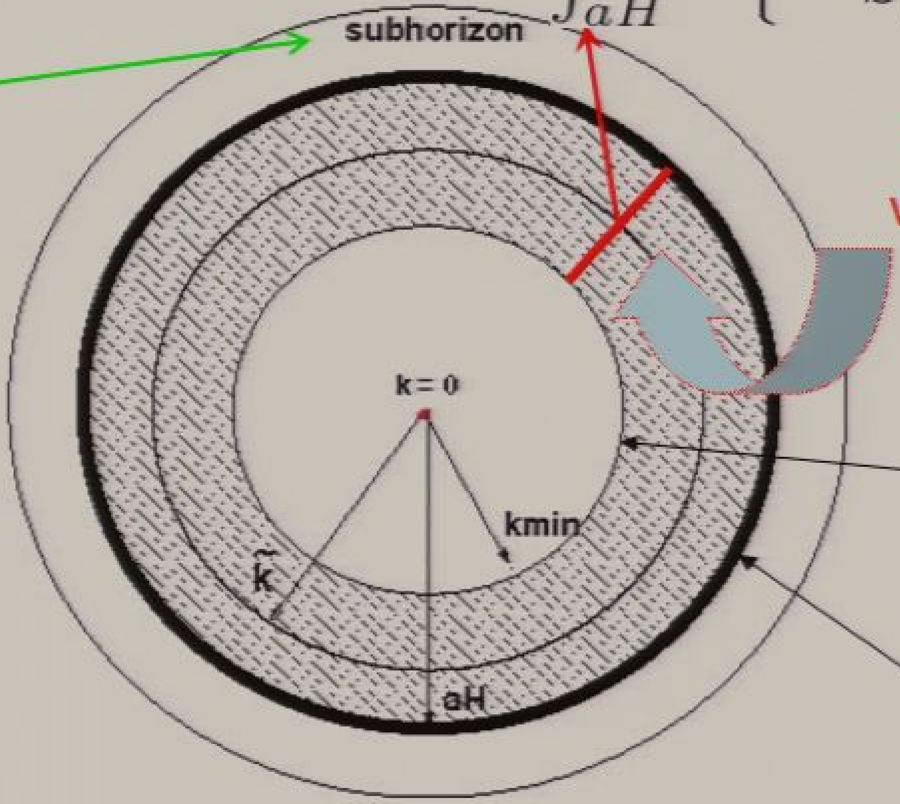
- Regularization of UV fluctuations in CST

Scales of (long wavelength) k

$\int_{aH}^{k_{min}} \{^{(2)}S_k((\delta g_{ab})_k^2, (\delta\phi)_k^2)$

We make no comment on subhorizon backreactions



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de Sitter spacetime perturbations

- Constant scalar field with constant potential ↔ de Sitter Spacetime
- Perturbation ansatz:

Leading order is second order

Overbar denotes `background`

$$g_{ab} = \bar{g}_{ab}(t,\chi,\theta,\phi) + \epsilon^2\delta^2 g_{ab}(t,\chi,\theta,\eta)$$

Background metric

$$ds^2 = \bar{g}_{ab}dx^a dx^b = \underbrace{-dt^2 + \cosh(t)^2(d\chi^2 + \sin(\chi)^2 d\Omega(\theta,\eta)^2)}_{\Re \times S^3 \text{ (closed) slicing}}$$

- Similarly perturb the `scalar field` $\phi = \bar{\phi} + \underbrace{\epsilon\delta\phi(t,\chi,\theta,\eta)}_{\text{Quantum perturbation}}$

Constant

Higher order equations

- Stress energy is quadratic in field → leading contribution in de Sitter spacetime at second order

- Defining the monomials (assuming Leibniz rule)

$$\Psi \equiv (\delta\phi)^2 \qquad \Psi_{ab} \equiv \delta\phi\bar{\nabla}_a\bar{\nabla}_b\delta\phi$$

we may write the leading order stress-energy as

$$T_{ab} = \frac{1}{2}\bar{\nabla}_a\bar{\nabla}_b\Psi - \Psi_{ab} - \frac{\bar{g}_{ab}}{4}\bar{\nabla}^c\bar{\nabla}_c\Psi$$

- Leading order Einstein equations are of the form

$$\underbrace{\mathcal{L}[\delta^2 g_{ab}]}_{\text{Linearized gravity}} \quad = \quad \underbrace{\kappa\mathcal{Q}_{ab}[(\delta\phi)(\delta\phi)]}_{\text{Nonlinear source}}$$

Interacting quantum fields in curved spacetime

- Regularization of nonlinear field products involving δ ϕ always has a renormalization ambiguity

- Hollands and Wald showed this ambiguity can be reduced to a finite # of parameters in CST using 'locality and covariance' requirement

- For the first time there is a rigorous manner in which to treat interacting fields in CST

Linearization instability I

- Vary the Bianchi identity around the de Sitter background

$$(G_a^b + \Lambda\delta_a^b)_{;b} = 0$$

Lambda constant, so drops out of variation

to obtain

$$\left(\overrightarrow{\frac{\delta G_a^b}{\delta^2 g_{\ell m}}}(\delta^2 g_{\ell m})\right)_{;b} = 0$$

- Now vary Bianchi identity times a Killing vector of the de Sitter background:

$$\left(X^a \overrightarrow{\frac{\delta G_a^b}{\delta^2 g_{\ell m}}}(\delta^2 g_{\ell m}))\right)_{;b} + (X^a G_a^b + \Lambda X^a \delta_a^b)_{\delta^2;b} = X^a_{;b}\overrightarrow{\frac{\delta G_a^b}{\delta^2 g_{\ell m}}}(\delta^2 g_{\ell m})$$

De Sitter Killing vector

Zero if Killing eqn. holds

Variation of
Christoffel symbols

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De Sitter Killing vector

Zero if Killing eqn. holds

Variation of Christoffel symbols

Integrate both sides and use Gauss' theorem

Linearization stability II

- The integral is independent of hypersurface and variation of metric. Thus ge

$$\int X^a n_b \frac{\overrightarrow{\delta G^b_a}}{\delta^2 g_{\ell m}}(\delta^2 g_{\ell m})\sqrt{|\bar{h}|}d^3x = 0$$

- However we want the fluctuations to obey the Einstein equations

$$\frac{\overrightarrow{\delta G^a_b}}{\delta^2 g_{cd}}(\delta^2 g_{cd}) \;=\; \kappa T^a_b(\delta\phi, \delta\phi)$$

- Thus we get an *integral constraint on the scalar field fluctuations:*

$$\int n_a X^b T^a_b(\delta\phi, \delta\phi)\sqrt{|\bar{h}|}d^3x \;=\; 0.$$

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$$\boxed{\int n_a X^b T^a_b(\delta\phi, \delta\phi)\sqrt{|\bar{h}|}d^3x \;=\; 0.}$$

Linearization
stability (LS)
condition

Anomalies in the LS conditions

- One cannot insist that the classical equations of motion and other conditions hold for nonlinear quantum fluctuations, in general there is a renormalization ambiguity

- One can redefine products of fields consistent with locality and covariance in Hollands' and Wald's sense:

$$\Psi \rightarrow \Psi + C$$

Curvature scalar, [length]$^{-2}$

Recall

$$\Psi \equiv (\delta\phi)^2$$

$$\Psi_{ab} \equiv \delta\phi \bar{\nabla}_a \bar{\nabla}_b \delta\phi$$

$$\Psi_{ab} \rightarrow \Psi_{ab} + C_{ab}$$

Curvature scalar, [length]$^{-4}$

- We show that the anomalies present in the LS conditions for de Sitter are of the form

$$-5\kappa Q \int_{\Sigma_t} X_\perp \sqrt{|\bar{h}|} d^3x$$

A number

Normal component of Killing vector

Volume measure of hypersurface

Anomalies in the LS conditions

- One cannot insist that the classical equations of motion and other conditions hold for nonlinear quantum fluctuations, in general there is a renormalization ambiguity

- One can redefine products of fields consistent with locality and covariance in Hollands' and Wald's sense:

Recall

$$\Psi \equiv (\delta\phi)^2$$

$$\Psi_{ab} \equiv \delta\phi\bar{\nabla}_a\bar{\nabla}_b\delta\phi$$

$$\Psi \to \Psi + C$$

Curvature scalar, [length]$^{-2}$

$$\Psi_{ab} \to \Psi_{ab} + C_{ab}$$

Curvature scalar, [length]$^{-4}$

- We show that the anomalies present in the LS conditions for de Sitter are of the form

$$-5\kappa Q\int_{\Sigma_t} X_\perp\sqrt{|\bar{h}|}d^3x$$

A number

Normal component of Killing vector

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LS conditions and SO(4,1) symmetry

- It turns out that the LS conditions form a Lie algebra

LS condition

$$\underbrace{[\delta^2 P(X_a), \delta^2 P(X_a)]}_{\text{No quantum anomalies in commutator}} = i A^c_{ab} \delta^2 P(X_c) \quad \text{holds}$$

Structure constants

- But it also turns out that the Killing vectors form the same algebra

$$[X_a, X_b] = A^c_{ab} X_c$$

The *same* structure constants

- The LS conditions demand that *all* physical states are SO(4,1) invariant

$$\delta^2 P(X)|\Psi\rangle = 0$$

A mini ‘problem of time’?

- Allen showed no SO(4,1) invariant states for massless scalar field:

$$^{(2)}S_M = -\frac{1}{2}\int \sqrt{-|\bar{g}|}\,\left[\bar{g}^{ab}\delta\phi_{,a}\delta\phi_{,b}\right] d^4x$$

Massless scalar field action with zero mode

- How are dynamics possible with such symmetric states?

- How do we understand the flat (Minkowski) limit?

Extension to include subleading backreactions

- If we include the gravity waves at subleading order, is the SO(4,1) invariance requirement lost?

- It turns out that the LS conditions are still the SO(4,1) generators in this case

- Reason? The gravity waves \`almost\` act like polarization scalar fields $h_\times$, h_+, which allows the results to go through with

$$\Psi \longrightarrow \Psi + \Psi_{h_\times} + \Psi_{h_+}$$

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Linearization stability II

- The integral is independent of hypersurface and variation of metric. Thus ge

$$\int X^a n_b \frac{\overrightarrow{\delta G^b_a}}{\delta^2 g_{\ell m}}(\delta^2 g_{\ell m})\sqrt{|\bar{h}|}\,d^3x = 0$$

- However we want the fluctuations to obey the Einstein equations

$$\frac{\overrightarrow{\delta G^a_b}}{\delta^2 g_{cd}}(\delta^2 g_{cd}) \;=\; \kappa T^a_b(\delta\phi, \delta\phi)$$

- Thus we get an *integral constraint on the scalar field fluctuations:*

$$\boxed{\int n_a X^b T^a_b(\delta\phi, \delta\phi)\sqrt{|\bar{h}|}\,d^3x \;=\; 0.}$$

Linearization stability (LS) condition

What are the consequences of this constraint?

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Conclusions/Future directions

- Second order IR fluctuations during slow-roll inflation may become large, especially as one tends towards de Sitter spacetime
- The QFT of a scalar field in de Sitter coupled to leading (and subleading) order to gravity has a mutilated space of states.
- UV (subhorizon) effects?
- Construction of nontrivial SO(4,1) invariant states?

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Nonuniform limit as G $\rightarrow$ 0, $\Lambda \rightarrow$ 0 ?

- For G $\rightarrow$ 0, $\Lambda \rightarrow$ 0, there are no SO(4,1) constraints
- In fact the constraints are severe enough to suggest that the G $\rightarrow$ 0 limit is not uniform
- Perhaps keep \frac{G}{\Lambda} constant in the limit?

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