Title: Non-Gaussianity in multi-field inflation

Date: Dec 12, 2006 02:00 PM

URL: http://pirsa.org/06120009

Abstract: During multi-field Inflation, the curvature perturbation can evovlve on superhorizon scales and will develop non-gaussianity due to non-linear interactions. In this talk I will discuss the calculation of this effect for models of inflation with two scalar fields.

Non-Gaussianity and its Evolution in Multi-field Inflation

Gerasimos Rigopoulos (Utrecht)

Work in collaboration with E.P.S. Shellard (Cambridge)
& B.J.W. van Tent (Orsay)

Perimeter 12 December 2006

Non-Gaussianity and its Evolution in Multi-field Inflation

Gerasimos Rigopoulos (Utrecht)

Work in collaboration with E.P.S. Shellard (Cambridge) & B.J.W. van Tent (Orsay)

Perimeter 12 December 2006

Recent Interest

Primordial non-Gaussianity has attracted a lot of interest over the past few years.

- Precision Cosmology: Non-Linearities may be observable
- Gravity is non-linear. Some non-Gaussianity will always be present
- Potentially useful for further testing inflation
  - Consistency check - Identification of new physics beyond inflation (which may produce stronger NG signals)
  - Discriminant among models

One more handle on the physics of Inflation. Primordial NG has been approached via various angles in an effort to compute it and relate it to observables.

Observations

- Various detections of non-Gaussian signals have been reported in the CMB. However, none has been linked to a primordial source.
- Observations focusing on the CMB measure $f_{NL} \sim \frac{\langle TTT \rangle}{\langle TT \rangle^2}$ or even $\tau_{NL} \sim \frac{\langle TTTT \rangle}{\langle TT \rangle^2}$. Current limits set $-54 < f_{NL} < 114$ (95%CL, WMAP). Planck is expected to reach $f_{NL} < 5$ at best, while an ideal experiment is limited to $f_{NL} < 3$.
- Recently $f_{NL} < 0.01$ has been claimed accessible via observations of the 21 cm radio background (astro-ph/0610257). If viable, it will set the whole discussion on inflationary NG under a totally different light.

Observations

- Various detections of non-Gaussian signals have been reported in the CMB. However, none has been linked to a primordial source.
- Observations focusing on the CMB measure $f_{NL} \sim \frac{\langle TTT \rangle}{\langle TT \rangle^2}$ or even $\tau_{NL} \sim \frac{\langle TTTT \rangle}{}$. Current limits set $-54 < f_{NL} < 114$ (95% … each $f_{NL} < 5$ at best, while … $_L < 3$.
- Rece… ccessible via observations of the 21 cm radio background (astro-ph/0610257). If viable, it will set the whole discussion on inflationary NG under a totally different light.

Automatic Updates

Updating your computer is almost complete. You must restart your computer for the updates to take effect.

Do you want to restart your computer now?

Restart Now | Restart Later

Observations

- Various detections of non-Gaussian signals have been reported in the CMB. However, none has been linked to a primordial source.
- Observations focusing on the CMB measure $f_{NL} \sim \frac{\langle TTT \rangle}{\langle TT \rangle^2}$ or even $\tau_{NL} \sim \frac{\langle TTTT \rangle}{\langle TT \rangle^2}$. Current limits set $-54 < f_{NL} < 114$ (95%CL, WMAP). Planck is expected to reach $f_{NL} < 5$ at best, while an ideal experiment is limited to $f_{NL} < 3$.
- Recently $f_{NL} < 0.01$ has been claimed accessible via observations of the 21 cm radio background (astro-ph/0610257). If viable, it will set the whole discussion on inflationary NG under a totally different light.

Calculating non-Gaussianity

Focusing on Inflation, there are four regimes relevant for NG generation and evolution

- Effects before or during horizon crossing.
  Calculating non-linear corrections from the inflationary mechanism for the generation of perturbations.
- Long wavelength evolution during inflation.
  The subject of this talk...
- Long wavelength evolution after inflation.
  E.g. Reheating, Preheating & The Curvaton scenario.
- The relation of this primordial NG to the observed CMB sky.
  Solving the full system of Boltzman equations at second order.

Calculational Approaches

- Straightforward second-order perturbation theory:
  Follow the route of linear theory by extending the perturbative analysis of the Einstein equations to second order. This seems essential for studying scales smaller or close to the horizon $\rightarrow$ proliferation of terms in the equations.
- Long wavelength approximations:
  First focus on long wavelengths, particularly relevant during inflation, where the dynamics simplifies. This is the approach of the δN formalism as well as the one taken in this talk.

Long wavelength approximation

$$ds^2 = -N^2(t,\mathbf{x})dt^2 + e^{2\alpha(t,\mathbf{x})}h_{ij}(\mathbf{x})dx^i dx^j$$

On long wavelengths ($\Delta\mathbf{x} > (aH)^{-1}$, ∇^2 dropped):

- $\frac{d}{N dt}H = -\frac{1}{2}\bar{K}^{ij}\bar{K}_{ij} - \frac{1}{2M_p^2}(\mathcal{E} + S/3)$
- $\frac{d}{N dt}\bar{K}^i_j = -3H\bar{K}^i_j \quad \Rightarrow \quad \bar{K}^i_j = C^i_j(\mathbf{x})e^{-3\alpha}$
- $\frac{\mathcal{D}}{N dt}\Pi^A + 3H\Pi^A + \partial^A V = 0$
- $H^2 = \frac{1}{3M_p^2}\mathcal{E} + \frac{1}{6}\bar{K}^{ij}\bar{K}_{ij}, \quad \partial_i H = -\frac{1}{2M_p^2}\Pi_A\partial_i\phi^A - \frac{1}{2}\nabla_j\bar{K}^j_i$

$$\{H = \frac{d}{N dt}\alpha, \quad \Pi^A = \frac{d}{N dt}\phi^A, \quad \bar{K}_{ij} = -e^{2\alpha}\frac{d}{2N dt}h_{ij}\}$$

'Separate universe' evolution $\quad \Rightarrow \quad$ ΔN formalism.

Long Wavelength Coordinate Transformations

Consider coordinate transformations which preserves $g_{0i} = 0$:

$$ds^2 = -\tilde{N}^2(T, \mathbf{X}) dT^2 + e^{2\tilde{\alpha}(T,\mathbf{X})} \tilde{h}_{ij}(T, \mathbf{X}) dX^i dX^j$$

$$T = T(t, x^l), \quad X^i = X^i(t, x^l)$$

Then, up to $\mathcal{O}(\nabla^2)$ the transformation matrix is:

- $\Lambda^\mu_T \equiv \frac{\partial x^\mu}{\partial T} = N \frac{\partial^\mu T}{(\partial_t T)^2}$
- $\Lambda^i{}_{\tilde{j}} \equiv \frac{\partial x^i}{\partial X^j} = \delta^i{}_{\tilde{j}} + \mathcal{O}\left[(\partial_i)^2\right]$
- $\Lambda^0{}_{\tilde{j}} \equiv \frac{\partial t}{\partial X^j} = -\delta^i{}_{\tilde{j}} \frac{\partial_i T}{\partial_t T} + \mathcal{O}\left[(\partial_i)^2\right]$

Using these we learn:

$$x^i = X^i + \int dT \frac{N \partial^i T}{(\partial_t T)^2}$$

- $ds^2 = -\tilde{N}^2(T, \mathbf{x}) dT^2 + e^{2\tilde{\alpha}(T,\mathbf{x})} h_{ij}(\mathbf{x}) dx^i dx^j$
- $\frac{1}{N}\frac{\partial}{\partial t} = \frac{1}{\tilde{N}}\frac{\partial}{\partial T} + \mathcal{O}(\nabla^2), \quad \frac{\partial}{\partial x^i} = \frac{\partial}{\partial X^i} + \partial_i T \frac{\partial}{\partial T}$

Long Wavelength Coordinate Transformations

Consider coordinate transformations which preserves $g_{0i} = 0$:

$$ds^2 = -\tilde{N}^2(T, \mathbf{X})dT^2 + e^{2\tilde{\alpha}(T,\mathbf{X})}\tilde{h}_{ij}(T, \mathbf{X})dX^i dX^j$$

$$T = T(t, x^l), \quad X^i = X^i(t, x^l)$$

Then, up to $\mathcal{O}(\nabla^2)$ the transformation matrix is:

- $\Lambda^{\mu}_{T} \equiv$

Automatic Updates

Updating your computer is almost complete. You must restart your computer for the updates to take effect.

Do you want to restart your computer now?

Restart Now | Restart Later

- $\Lambda^{i}_{\tilde{j}} \equiv$
- $\Lambda^{0}_{\tilde{j}} \equiv \frac{\partial t}{\partial X^j} = -\delta^{i}_{\tilde{j}} \frac{\partial_i T}{\partial_t T} + \mathcal{O}\left[(\partial_i)^2\right]$

Using these we learn:

$$x^i = X^i + \int dT \frac{N \partial^i T}{(\partial_t T)^2}$$

- $ds^2 = -\tilde{N}^2(T, \mathbf{x})dT^2 + e^{2\tilde{\alpha}(T,\mathbf{x})} h_{ij}(\mathbf{x}) dx^i dx^j$
- $\frac{1}{N}\frac{\partial}{\partial t} = \frac{1}{\tilde{N}}\frac{\partial}{\partial T} + \mathcal{O}(\nabla^2), \quad \frac{\partial}{\partial x^i} = \frac{\partial}{\partial X^i} + \partial_i T \frac{\partial}{\partial T}$

Long Wavelength Coordinate Transformations

Consider coordinate transformations which preserves $g_{0i} = 0$:

$$ds^2 = -\tilde{N}^2(T, \mathbf{X})dT^2 + e^{2\tilde{\alpha}(T,\mathbf{X})}\tilde{h}_{ij}(T, \mathbf{X})dX^i dX^j$$

$$T = T(t, x^l), \quad X^i = X^i(t, x^l)$$

Then, up to $\mathcal{O}(\nabla^2)$ the transformation matrix is:

- $\Lambda^\mu_T \equiv \frac{\partial x^\mu}{\partial T} = N \frac{\partial^\mu T}{(\partial_t T)^2}$
- $\Lambda^i{}_{\tilde{j}} \equiv \frac{\partial x^i}{\partial X^j} = \delta^i{}_{\tilde{j}} + \mathcal{O}\left[(\partial_i)^2\right]$
- $\Lambda^0{}_{\tilde{j}} \equiv \frac{\partial t}{\partial X^j} = -\delta^i{}_{\tilde{j}} \frac{\partial_i T}{\partial_t T} + \mathcal{O}\left[(\partial_i)^2\right]$

Using these we learn:

$$x^i = X^i + \int dT \frac{N \partial^i T}{(\partial_t T)^2}$$

- $ds^2 = -\tilde{N}^2(T, \mathbf{x})dT^2 + e^{2\tilde{\alpha}(T,\mathbf{x})}h_{ij}(\mathbf{x})dx^i dx^j$
- $\frac{1}{N}\frac{\partial}{\partial t} = \frac{1}{\tilde{N}}\frac{\partial}{\partial T} + \mathcal{O}(\nabla^2), \quad \frac{\partial}{\partial x^i} = \frac{\partial}{\partial X^i} + \partial_i T \frac{\partial}{\partial T}$

Inhomogeneity

Separate inhomogeneous evolution from the homogeneous background:

- Given a spacetime scalars $A(t, \mathbf{x}) = A(t) + \Delta A(t, \mathbf{x})$ one can always set $\Delta A = 0$ by a suitable choice of time slicing - no coordinate invariant meaning for ΔA
- However, given two scalars $A(t, \mathbf{x})$ and $B(t, \mathbf{x})$ one can construct a fully non-linear variable which encodes the inhomogeneity and is a scalar (invariant) under long wavelength transformations:

$$C_i(t, \mathbf{x}) \equiv \partial_i A - \frac{\partial_t A}{\partial_t B} \partial_i B = \tilde{C}_i(T, \mathbf{x})$$

For example:

$$\zeta_i = \partial_i \alpha - \frac{H}{\dot{\rho}} \partial_i \rho \;\Rightarrow\; \frac{d}{N dt} \zeta_i = -\frac{H}{\rho + P} \left(\partial_i P - \frac{\dot{P}}{\dot{\rho}} \partial_i \rho \right)$$

This is a fully non-linear statement formally similar to that of linear theory.

Long Wavelength Coordinate Transformations

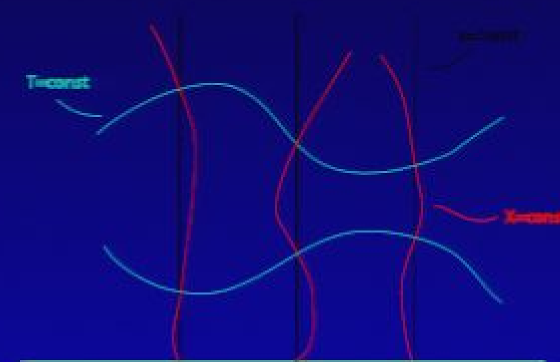
Consider coordinate transformations which preserves $g_{0i} = 0$:

$$ds^2 = -\tilde{N}^2(T,\mathbf{X})dT^2 + e^{2\tilde{\alpha}(T,\mathbf{X})}\tilde{h}_{ij}(T,\mathbf{X})dX^i dX^j$$

$$T = T(t,x^l), \quad X^i = X^i(t,x^l)$$

Then, up to $\mathcal{O}(\nabla^2)$ the transformation matrix is:

- $\Lambda^\mu_T \equiv \frac{\partial x^\mu}{\partial T} = N\frac{\partial^\mu T}{(\partial_t T)^2}$
- $\Lambda^i{}_{\tilde{j}} \equiv \frac{\partial x^i}{\partial X^j} = \delta^i{}_{\tilde{j}} + \mathcal{O}\left[(\partial_i)^2\right]$
- $\Lambda^0{}_{\tilde{j}} \equiv \frac{\partial t}{\partial X^j} = -\delta^i{}_{\tilde{j}}\frac{\partial_i T}{\partial_t T} + \mathcal{O}\left[(\partial_i)^2\right]$



Using these we learn:

$$x^i = X^i + \int dT\,\frac{N\partial^i T}{(\partial_t T)^2}$$

- $ds^2 = -\tilde{N}^2(T,\mathbf{x})dT^2 + e^{2\tilde{\alpha}(T,\mathbf{x})}h_{ij}(\mathbf{x})dx^i dx^j$
- $\frac{1}{N}\frac{\partial}{\partial t} = \frac{1}{\tilde{N}}\frac{\partial}{\partial T} + \mathcal{O}(\nabla^2)\,, \quad \frac{\partial}{\partial x^i} = \frac{\partial}{\partial X^i} + \partial_i T\frac{\partial}{\partial T}$

Inhomogeneity

Separate inhomogeneous evolution from the homogeneous background:

- Given a spacetime scalars $A(t, \mathbf{x}) = A(t) + \Delta A(t, \mathbf{x})$ one can always set $\Delta A = 0$ by a suitable choice of time slicing - no coordinate invariant meaning for ΔA
- However, given two scalars $A(t, \mathbf{x})$ and $B(t, \mathbf{x})$ one can construct a fully non-linear variable which encodes the inhomogeneity and is a scalar (invariant) under long wavelength transformations:

$$C_i(t, \mathbf{x}) \equiv \partial_i A - \frac{\partial_t A}{\partial_t B}\partial_i B = \tilde{C}_i(T, \mathbf{x})$$

For example:

$$\zeta_i = \partial_i \alpha - \frac{H}{\dot{\rho}}\partial_i \rho \;\Rightarrow\; \frac{d}{N dt}\zeta_i = -\frac{H}{\rho + P}\left(\partial_i P - \frac{\dot{P}}{\dot{\rho}}\partial_i \rho\right)$$

This is a fully non-linear statement formally similar to that of linear theory.

Inhomogeneity

Separate inhomogeneous evolution from the homogeneous background:

- Given a spacetime scalars $A(t,\mathbf{x}) = A(t) + \Delta A(t,\mathbf{x})$ one can always set $\Delta A = 0$ by a suitable choice of time slicing - no coordinate invariant meaning for ΔA
- However, given two scalars $A(t,\mathbf{x})$ and $B(t,\mathbf{x})$ one can construct a fully non-linear variable which encodes the inhomogeneity and is a scalar (invariant) under long wavelength transformations:

$$C_i(t,\mathbf{x}) \equiv \partial_i A - \frac{\partial_t A}{\partial_t B}\partial_i B = \tilde{C}_i(T,\mathbf{x})$$

For example:

$$\zeta_i = \partial_i \alpha - \frac{H}{\dot{\rho}}\partial_i \rho \;\Rightarrow\; \frac{d}{N dt}\zeta_i = -\frac{H}{\rho+P}\left(\partial_i P - \frac{\dot{P}}{\dot{\rho}}\partial_i \rho\right)$$

This is a fully non-linear statement formally similar to that of linear theory.

Long wavelength approximation

$$ds^2 = -N^2(t,\mathbf{x})dt^2 + e^{2\alpha(t,\mathbf{x})}h_{ij}(\mathbf{x})dx^i dx^j$$

On long wavelengths ($\Delta \mathbf{x} > (aH)^{-1}$, ∇^2 dropped):

- $\frac{d}{N dt} H = -\frac{1}{2}\bar{K}^{ij}\bar{K}_{ij} - \frac{1}{2M_p^2}(\mathcal{E} + S/3)$
- $\frac{d}{N dt}\bar{K}^i_j = -3H\bar{K}^i_j \quad \Rightarrow \quad \bar{K}^i_j = C^i_j(\mathbf{x})e^{-3\alpha}$
- $\frac{\mathcal{D}}{N dt}\Pi^A + 3H\Pi^A + \partial^A V = 0$
- $H^2 = \frac{1}{3M_p^2}\mathcal{E} + \frac{1}{6}\bar{K}^{ij}\bar{K}_{ij}, \quad \partial_i H = -\frac{1}{2M_p^2}\Pi_A \partial_i \phi^A - \frac{1}{2}\nabla_j \bar{K}^j_i$

$$\left\{H = \frac{d}{N dt}\alpha, \quad \Pi^A = \frac{d}{N dt}\phi^A, \quad \bar{K}_{ij} = -e^{2\alpha}\frac{d}{2N dt}h_{ij}\right\}$$

'Separate universe' evolution $\Rightarrow$ ΔN formalism.

Inhomogeneity

Separate inhomogeneous evolution from the homogeneous background:

- Given a spacetime scalars $A(t,\mathbf{x}) = A(t) + \Delta A(t,\mathbf{x})$ one can always set $\Delta A = 0$ by a suitable choice of time slicing - no coordinate invariant meaning for ΔA
- However, given two scalars $A(t,\mathbf{x})$ and $B(t,\mathbf{x})$ one can construct a fully non-linear variable which encodes the inhomogeneity and is a scalar (invariant) under long wavelength transformations:

$$C_i(t,\mathbf{x}) \equiv \partial_i A - \frac{\partial_t A}{\partial_t B}\partial_i B = \tilde{C}_i(T,\mathbf{x})$$

For example:

$$\zeta_i = \partial_i \alpha - \frac{H}{\dot{\rho}}\partial_i \rho \;\Rightarrow\; \frac{d}{N\,dt}\zeta_i = -\frac{H}{\rho+P}\left(\partial_i P - \frac{\dot{P}}{\dot{\rho}}\partial_i \rho\right)$$

This is a fully non-linear statement formally similar to that of linear theory.

For Inflation...

Consider a set of scalar fields during inflation

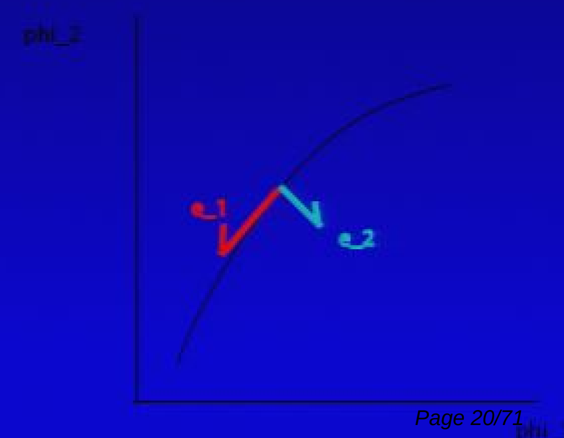
$$T_{\mu\nu} = G_{AB}\partial_\mu\phi^A\partial_\nu\phi^B - g_{\mu\nu}\left(\tfrac{1}{2}\partial_\lambda\phi_A\partial^\lambda\phi^A + V(\phi)\right)$$

The following spatial vectors are scalar invariants:

$$\mathcal{Q}_i^A = e^\alpha\left(\partial_i\phi^A - \frac{\partial_t\phi^A}{NH}\partial_i\alpha\right), \qquad \left(H \equiv \frac{\partial_t\alpha}{N\alpha}\right)$$

$$\zeta_i^A = -\frac{\kappa}{\sqrt{2\tilde\epsilon}}\partial_i\phi^A + \frac{\Pi^A}{\Pi}\partial_i\alpha, \quad \left(\Pi^A \equiv \frac{\partial_t\phi^A}{N},\ \tilde\epsilon \equiv \frac{\kappa^2}{2}\frac{\Pi^2}{H^2},\ \kappa^2 \equiv M_p^{-2}\right)$$

Note that: $\mathcal{Q}_i^A = \partial_i q^A + \dots$ and $\zeta_i^A = \partial_i\zeta^A + \dots$ where q^A and ζ^A well known linear gauge-invariant variables.



Define isocurvature and adiabatic directions $\hat e^1_A = \frac{\Pi_A}{\Pi}$, $\hat e^2_A$, …
with $\hat e^2_A \hat e^{1A} = 0, \dots$

... For more fields an iterative procedure will produce an orthonormal basis adapted to the trajectory with N-1 isocurvature directions.

$$\tilde{\eta}^A_{(n)} = \frac{\left(\frac{1}{N}\mathcal{D}_t\right)^{n-1}\Pi^A}{H^{n-1}\Pi} \qquad \hat{e}^A_n \equiv \frac{\tilde{\eta}^A_{(n)} - \sum_{B=1}^{n-1} \tilde{\eta}^B_{(n)} \hat{e}^A_B}{\tilde{\eta}^n_{(n)}}$$

where

$$\tilde{\eta}^n_{(n)} \equiv -\epsilon_{A_1 \cdots A_n} \hat{e}_1^{A_1} \ldots \hat{e}_{n-1}^{A_n - 1} \tilde{\eta}^{A_n}_{(n)}$$

which gives the basis a definite handedness.

Non-linear Isocurvature and Adiabatic Variables

- $\zeta_i \equiv \hat{e}^1_A \zeta^A_i = \partial_i \ln a - \frac{\kappa}{\sqrt{2\tilde{\epsilon}}}\hat{e}_{1A}\partial_i \phi^A = \partial_i \ln a - \frac{H}{\dot{\rho}}\partial_i \rho$
- $\sigma_i \equiv \hat{e}^2_A \zeta^A_i = -\frac{\kappa}{\sqrt{2\tilde{\epsilon}}}\hat{e}_{2A}\partial_i \phi^A$

Define Slow Roll parameters:

$$\tilde{\epsilon} \equiv \frac{\kappa^2}{2}\frac{\Pi^2}{H^2}, \quad \tilde{\eta}^A \equiv -\frac{3H\Pi^A + \partial^A V}{H\Pi}, \quad \tilde{\xi}^A \equiv 3\tilde{\epsilon}\,\hat{e}^{1A} - 3\tilde{\eta}^A - \frac{\hat{e}^{1B}V^A_B}{H^2}$$

Project isocurvature and adiabatic parts:

$$\tilde{\eta}^{\|} \equiv \hat{e}^1_A \tilde{\eta}^A, \quad \tilde{\eta}^{\perp} \equiv \hat{e}^2_A \tilde{\eta}^A, \quad \tilde{\xi}^{\|} \equiv \hat{e}^1_A \tilde{\xi}^A, \quad \tilde{\xi}^{\perp} \equiv \hat{e}^2_A \tilde{\xi}^A$$

The non-linear equations of motion are formally the same as those of linear perturbation theory with

$$\delta \to \partial_i$$
$$f(t) \to f(t, \mathbf{x})$$

Non-linear Isocurvature and Adiabatic Variables

- $\zeta_i \equiv \hat{e}^1_A \zeta^A_i = \partial_i \ln a - \frac{\kappa}{\sqrt{2\tilde{\epsilon}}}\hat{e}_{1A}\partial_i\phi^A = \partial_i \ln a - \frac{H}{\dot{\rho}}\partial_i\rho$
- $\sigma_i \equiv \hat{e}^2_A \zeta^A_i = -\frac{\kappa}{\sqrt{2\tilde{\epsilon}}}\hat{e}_{2A}\partial_i\phi^A$

Define Slow Roll parameters:

$\tilde{\epsilon} \equiv \frac{\kappa^2}{2}\frac{\Pi^2}{H^2},$ … $^A - 3\tilde{\eta}^A - \frac{\hat{e}^{1B}V^A_B}{H^2}$

Automatic Updates

Updating your computer is almost complete. You must restart your computer for the updates to take effect.

Do you want to restart your computer now?

Restart Now | Restart Later

Project isoc…

$\tilde{\eta}^{\parallel} \equiv \hat{e}^1_A\tilde{\eta}^A, \quad \tilde{\eta}^{\perp} \equiv \hat{e}^2_A\tilde{\eta}^A, \quad \xi^{\parallel} \equiv \hat{e}^1_A\xi^A, \quad \xi^{\perp} \equiv \hat{e}^2_A\tilde{\xi}^A$

The non-linear equations of motion are formally the same as those of linear perturbation theory with

$$\delta \rightarrow \partial_i$$
$$f(t) \rightarrow f(t, \mathbf{x})$$

Non-linear Isocurvature and Adiabatic Variables

- $\zeta_i \equiv \hat{e}^1_A \zeta^A_i = \partial_i \ln a - \frac{\kappa}{\sqrt{2\tilde{\epsilon}}} \hat{e}_{1A} \partial_i \phi^A = \partial_i \ln a - \frac{H}{\dot{\rho}} \partial_i \rho$
- $\sigma_i \equiv \hat{e}^2_A \zeta^A_i = -\frac{\kappa}{\sqrt{2\tilde{\epsilon}}} \hat{e}_{2A} \partial_i \phi^A$

Define Slow Roll parameters:

$$\tilde{\epsilon} \equiv \frac{\kappa^2}{2} \frac{\Pi^2}{H^2}, \quad \tilde{\eta}^A \equiv -\frac{3H\Pi^A + \partial^A V}{H\Pi}, \quad \tilde{\xi}^A \equiv 3\tilde{\epsilon}\, \hat{e}^{1A} - 3\tilde{\eta}^A - \frac{\hat{e}^{1B} V^A_B}{H^2}$$

Project isocurvature and adiabatic parts:

$$\tilde{\eta}^{\parallel} \equiv \hat{e}^1_A \tilde{\eta}^A, \quad \tilde{\eta}^{\perp} \equiv \hat{e}^2_A \tilde{\eta}^A, \quad \tilde{\xi}^{\parallel} \equiv \hat{e}^1_A \tilde{\xi}^A, \quad \tilde{\xi}^{\perp} \equiv \hat{e}^2_A \tilde{\xi}^A$$

The non-linear equations of motion are formally the same as those of linear perturbation theory with

$$\delta \to \partial_i$$
$$f(t) \to f(t, \mathbf{x})$$

Long wavelength equations of motion

Choose a gauge with $NH = 1$ ($\partial_i \alpha = 0$ - homogeneous expansion) to simplify expressions

$$\frac{d}{dt}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} + \begin{pmatrix}0 & -2\tilde\eta^{\perp} & 0\\ 0 & 0 & -1\\ 0 & \tilde\kappa & \tilde\lambda\end{pmatrix}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} = 0$$

where

$$\tilde\kappa(t,\mathbf{x}) = 3\left(\frac{V_{22}}{3H^2} + \tilde\epsilon + \tilde\eta^{\parallel}\right) + 2\tilde\epsilon^2 + 4\tilde\epsilon\tilde\eta^{\parallel} + 4(\tilde\eta^{\perp})^2 + \tilde\xi^{\parallel},$$

$$\tilde\lambda(t,\mathbf{x}) = 3 + \tilde\epsilon + 2\tilde\eta^{\parallel}$$

All local quantities are given by:

- $\partial_i \ln H = \tilde\epsilon\,\zeta_i, \quad e_{m\,A}\partial_i\phi^A = -\frac{\sqrt{2\tilde\epsilon}}{\kappa}\,\zeta_i^m$
- $e_1^A\partial_i\Pi_A = -\frac{H\sqrt{2\tilde\epsilon}}{\kappa}\left(\tilde\eta^{\parallel}\zeta_i + \tilde\eta^{\perp}\sigma_i\right)$
- $e_2^A\partial_i\Pi_A = -\frac{H\sqrt{2\tilde\epsilon}}{\kappa}\left(\dot\sigma_i + \tilde\eta^{\perp}\zeta_i + \left(\tilde\eta^{\parallel} + \tilde\epsilon\right)\sigma_i\right)$

"Initial Conditions"

Assuming that non-linearities are not important on short scales, one can include in a straightforward manner perturbations from shorter wavelengths. This amounts to adding "sources" on the rhs:

$$\frac{d}{dt}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix}+\begin{pmatrix}0 & -2\tilde\eta^{\perp} & 0\\ 0 & 0 & -1\\ 0 & \tilde\kappa & \tilde\lambda\end{pmatrix}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix}=\partial_i\int\frac{d^3k}{(2\pi)^{3/2}}\begin{pmatrix}\zeta_l(k)\,\hat\alpha_{\mathbf{k}}\\ \sigma_l(k)\,\hat\beta_{\mathbf{k}}\\ \dot\sigma_l(k)\,\hat\beta_{\mathbf{k}}\end{pmatrix}\dot{\mathcal{W}}(k)e^{i\mathbf{kx}}$$

where

$$\hat\alpha_{\mathbf{k}}=a^{\dagger}(\mathbf{k})+a(-\mathbf{k})\,,\quad \hat\beta_{\mathbf{k}}=b^{\dagger}(\mathbf{k})+b(-\mathbf{k})$$

with $\left[a(\mathbf{k}),a^{\dagger}(-\mathbf{k}')\right]=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$, e.t.c.

$$\zeta_l(k)=-\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k)=-\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k)=\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}\chi$$

$\mathcal{W}(k)$ cuts off short wavelength modes. Simplest choice: $\mathcal{W}(k)=\Theta(caH-k)$. Final results are *independent* of the form of $\mathcal{W}(k)$.

When linearized, these equations are exact and valid to all scales, simply being linear perturbation theory.

Perturbation Theory

We can now perturb the e.o.m. to directly obtain solutions at second order

- Perturbing $A(t,\mathbf{x}) = \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}$ represents non-linearities in the long wavelength evolution
- Perturbing $\frac{H}{\sqrt{\tilde{\epsilon}}}$ represents "initial" non-linearities of the modes at horizon crossing

Write $(\Delta A)_{ab} = \bar{A}_{abc}\, v_c$, with $\boxed{v_i = \begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix}}$.

- Evolution terms:

$$\bar{A}_{12c} = 2 \begin{pmatrix} (\tilde{\epsilon}-2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp}+\tilde{\xi}^{\perp} \\ -3\chi-(\tilde{\epsilon}+\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(\tilde{\eta}^{\perp})^2 \\ -3-\tilde{\eta}^{\parallel} \end{pmatrix}, \bar{A}_{33c} = \begin{pmatrix} -2\tilde{\epsilon}^2+(2\tilde{\epsilon}-\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}+(\tilde{\eta}^{\perp})^2+\tilde{\xi}^{\parallel} \\ 2\tilde{\epsilon}\tilde{\eta}^{\perp}+\tilde{\xi}^{\perp} \\ \tilde{\eta}^{\perp} \end{pmatrix}$$

"Initial Conditions"

Assuming that non-linearities are not important on short scales, one can include in a straightforward manner perturbations from shorter wavelengths. This amounts to adding "sources" on the rhs:

$$\frac{d}{dt}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} + \begin{pmatrix}0 & -2\tilde\eta^\perp & 0\\ 0 & 0 & -1\\ 0 & \tilde\kappa & \tilde\lambda\end{pmatrix}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} = \partial_i \int \frac{d^3k}{(2\pi)^{3/2}} \begin{pmatrix}\zeta_l(k)\,\hat\alpha_{\mathbf{k}}\\ \sigma_l(k)\,\hat\beta_{\mathbf{k}}\\ \dot\sigma_l(k)\,\hat\beta_{\mathbf{k}}\end{pmatrix} \dot{\mathcal{W}}(k)e^{i\mathbf{kx}}$$

where

$$\hat\alpha_{\mathbf{k}} = a^\dagger(\mathbf{k}) + a(-\mathbf{k})\,, \quad \hat\beta_{\mathbf{k}} = b^\dagger(\mathbf{k}) + b(-\mathbf{k})$$

with $\left[a(\mathbf{k}), a^\dagger(-\mathbf{k}')\right] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$, e.t.c.

$$\zeta_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k) = \frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}\chi$$

$\mathcal{W}(k)$ cuts off short wavelength modes. Simplest choice: $\mathcal{W}(k) = \Theta(caH - k)$. Final results are *independent* of the form of $\mathcal{W}(k)$.

When linearized, these equations are exact and valid to all scales, simply being linear perturbation theory.

Perturbation Theory

We can now perturb the e.o.m. to directly obtain solutions at second order

- Perturbing $A(t,\mathbf{x}) = \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}$ represents non-linearities in the long wavelength evolution
- Perturbing $\frac{H}{\sqrt{\tilde{\epsilon}}}$ represents "initial" non-linearities of the modes at horizon crossing

Write $(\Delta A)_{ab} = \bar{A}_{abc}\, v_c$, with $\boxed{v_i = \begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix}}$.

- Evolution terms:

$$\bar{A}_{12c} = 2 \begin{pmatrix} (\tilde{\epsilon}-2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp}+\tilde{\xi}^{\perp} \\ -3\chi-(\tilde{\epsilon}+\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(\tilde{\eta}^{\perp})^2 \\ -3-\tilde{\eta}^{\parallel} \end{pmatrix} , \bar{A}_{33c} = \begin{pmatrix} -2\tilde{\epsilon}^2+(2\tilde{\epsilon}-\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}+(\tilde{\eta}^{\perp})^2+\tilde{\xi}^{\parallel} \\ 2\tilde{\epsilon}\tilde{\eta}^{\perp}+\tilde{\xi}^{\perp} \\ \tilde{\eta}^{\perp} \end{pmatrix}$$

$\bar{A}_{32c} = 2\times$

$$\begin{pmatrix} (-6\tilde{\epsilon}-10\tilde{\epsilon}^2-2\tilde{\epsilon}\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(6+6\tilde{\epsilon}-8\tilde{\eta}^{\parallel})(\tilde{\eta}^{\perp})^2-\tilde{\epsilon}(3\chi+4\tilde{\epsilon}^2+3\xi^{\parallel})-6\tilde{\eta}^{\perp}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{111}-V_{22}}{3H^2} \\ (-12\tilde{\epsilon}-6\tilde{\eta}^{\parallel}+12\chi-6\tilde{\epsilon}^2)\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\parallel})^2\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\perp})^3-4\tilde{\epsilon}\xi^{\perp}-2\tilde{\eta}^{\parallel}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{211}-V_{222}}{3H^2} \\ 6\tilde{\eta}^{\perp}-2\tilde{\epsilon}\tilde{\eta}^{\perp}+4\tilde{\eta}^{\parallel}\tilde{\eta}^{\perp}-2\tilde{\xi}^{\perp} \end{pmatrix}$$

- "Initial Condition Terms": $\Delta(\frac{H}{\sqrt{\tilde{\epsilon}}}) = (2\tilde{\epsilon}+\tilde{\eta})\frac{H}{\sqrt{\tilde{\epsilon}}}\zeta$

We solve $\quad \frac{d}{dt}v_{ai}^{(2)} + A(t)v_{ai}^{(2)} = \bar{A}_{abc}(t')\, v_{bi}^{(1)}\, v_{c}^{(1)} + \Delta\, b_{ai} \quad \Rightarrow$

$$v_{ai}^{(2)} = \int_{-\infty}^{t} G_{ab}(t,t')\left(\bar{A}_{bcd}\, v_{ci}^{(1)}\, v_{d}^{(1)} + \Delta\, b_{bi}\right)dt'$$

$$\frac{d}{dt'}G_{ab}(t,t') - G_{ac}(t,t')A_{cb}(t') = 0,$$

$$\frac{d}{dt}G_{ab}(t,t') + A_{ac}(t)G_{cb}(t,t') = 0, \qquad G_{ab}(t_\star,t_\star) = \delta_{ab}$$

Perturbation Theory

We can now perturb the e.o.m. to directly obtain solutions at second order

- Perturbing $A(t,\mathbf{x}) = \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}$ represents non-linearities in the long wavelength evolution
- Perturbing $\frac{H}{\sqrt{\tilde{\epsilon}}}$ represents "initial" non-linearities of the modes at horizon crossing

Write $(\Delta A)_{ab} = \bar{A}_{abc} v_c$, with $\boxed{v_i = \begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix}}$.

- Evolution terms:

$$\bar{A}_{12c} = 2\begin{pmatrix} (\tilde{\epsilon}-2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp}+\tilde{\xi}^{\perp} \\ -3\chi-(\tilde{\epsilon}+\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(\tilde{\eta}^{\perp})^2 \\ -3-\tilde{\eta}^{\parallel} \end{pmatrix}, \bar{A}_{33c} = \begin{pmatrix} -2\tilde{\epsilon}^2+(2\tilde{\epsilon}-\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}+(\tilde{\eta}^{\perp})^2+\tilde{\xi}^{\parallel} \\ 2\tilde{\epsilon}\tilde{\eta}^{\perp}+\tilde{\xi}^{\perp} \\ \tilde{\eta}^{\perp} \end{pmatrix}$$

$$\bar{A}_{32c} = 2\times \begin{pmatrix} (-6\tilde\epsilon - 10\tilde\epsilon^2 - 2\tilde\epsilon\tilde\eta^\parallel)\tilde\eta^\parallel - (6 + 6\tilde\epsilon - 8\tilde\eta^\parallel)(\tilde\eta^\perp)^2 - \tilde\epsilon(3\chi + 4\tilde\epsilon^2 + 3\xi^\parallel) - 6\tilde\eta^\perp\xi^\perp + \frac{3\sqrt{\tilde\epsilon}}{\sqrt{2}\kappa}\frac{V_{111} - V_{22}}{3H^2} \\ (-12\tilde\epsilon - 6\tilde\eta^\parallel + 12\chi - 6\tilde\epsilon^2)\tilde\eta^\perp + 4(\tilde\eta^\parallel)^2\tilde\eta^\perp + 4(\tilde\eta^\perp)^3 - 4\tilde\epsilon\xi^\perp - 2\tilde\eta^\parallel\xi^\perp + \frac{3\sqrt{\tilde\epsilon}}{\sqrt{2}\kappa}\frac{V_{211} - V_{222}}{3H^2} \\ 6\tilde\eta^\perp - 2\tilde\epsilon\tilde\eta^\perp + 4\tilde\eta^\parallel\tilde\eta^\perp - 2\tilde\xi^\perp \end{pmatrix}$$

- "Initial Condition Terms": $\Delta(\frac{H}{\sqrt{\tilde\epsilon}}) = (2\tilde\epsilon + \tilde\eta)\frac{H}{\sqrt{\tilde\epsilon}}\zeta$

We solve $\quad \frac{d}{dt}v^{(2)}_{ai} + A(t)v^{(2)}_{ai} = \bar{A}_{abc}(t')\, v^{(1)}_{bi}\, v^{(1)}_{c} + \Delta\, b_{ai} \quad \Rightarrow$

$$v^{(2)}_{ai} = \int_{-\infty}^{t} G_{ab}(t,t')\left(\bar{A}_{bcd}\, v^{(1)}_{ci}\, v^{(1)}_{d} + \Delta\, b_{bi}\right) dt'$$

$$\frac{d}{dt'}G_{ab}(t,t') - G_{ac}(t,t')A_{cb}(t') = 0,$$

$$\frac{d}{dt}G_{ab}(t,t') + A_{ac}(t)G_{cb}(t,t') = 0, \qquad G_{ab}(t_\star,t_\star) = \delta_{ab}$$

Perturbation Theory

We can now perturb the e.o.m. to directly obtain solutions at second order

- Perturbing $A(t,\mathbf{x}) = \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}$ represents non-linearities in the long wavelength evolution
- Perturbing $\frac{H}{\sqrt{\tilde{\epsilon}}}$ represents "initial" non-linearities of the modes at horizon crossing

Write $(\Delta A)_{ab} = \bar{A}_{abc}\, v_c$, with $\boxed{v_i = \begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix}}$.

- Evolution terms:

$$\bar{A}_{12c} = 2\begin{pmatrix} (\tilde{\epsilon}-2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp}+\tilde{\xi}^{\perp} \\ -3\chi-(\tilde{\epsilon}+\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(\tilde{\eta}^{\perp})^2 \\ -3-\tilde{\eta}^{\parallel} \end{pmatrix}, \bar{A}_{33c} = \begin{pmatrix} -2\tilde{\epsilon}^2+(2\tilde{\epsilon}-\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}+(\tilde{\eta}^{\perp})^2+\tilde{\xi}^{\parallel} \\ 2\tilde{\epsilon}\tilde{\eta}^{\perp}+\tilde{\xi}^{\perp} \\ \tilde{\eta}^{\perp} \end{pmatrix}$$

"Initial Conditions"

Assuming that non-linearities are not important on short scales, one can include in a straightforward manner perturbations from shorter wavelengths. This amounts to adding "sources" on the rhs:

$$\frac{d}{dt}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} + \begin{pmatrix}0 & -2\tilde\eta^{\perp} & 0\\ 0 & 0 & -1\\ 0 & \tilde\kappa & \tilde\lambda\end{pmatrix}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} = \partial_i \int \frac{d^3k}{(2\pi)^{3/2}} \begin{pmatrix}\zeta_l(k)\,\hat\alpha_{\mathbf{k}}\\ \sigma_l(k)\,\hat\beta_{\mathbf{k}}\\ \dot\sigma_l(k)\,\hat\beta_{\mathbf{k}}\end{pmatrix}\dot{\mathcal{W}}(k)e^{i\mathbf{kx}}$$

where

$$\hat\alpha_{\mathbf{k}} = a^\dagger(\mathbf{k}) + a(-\mathbf{k})\,, \quad \hat\beta_{\mathbf{k}} = b^\dagger(\mathbf{k}) + b(-\mathbf{k})$$

with $\left[a(\mathbf{k}), a^\dagger(-\mathbf{k}')\right] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$, e.t.c.

$$\zeta_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k) = \frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}\chi$$

$\mathcal{W}(k)$ cuts off short wavelength modes. Simplest choice: $\mathcal{W}(k) = \Theta(caH - k)$. Final results are *independent* of the form of $\mathcal{W}(k)$.

When linearized, these equations are exact and valid to all scales, simply being linear perturbation theory.

$$\bar{A}_{32c} = 2\times$$

$$\left(\begin{array}{c} (-6\tilde{\epsilon}-10\tilde{\epsilon}^2-2\tilde{\epsilon}\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(6+6\tilde{\epsilon}-8\tilde{\eta}^{\parallel})(\tilde{\eta}^{\perp})^2-\tilde{\epsilon}(3\chi+4\tilde{\epsilon}^2+3\xi^{\parallel})-6\tilde{\eta}^{\perp}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{111}-V_{22}}{3H^2} \\ (-12\tilde{\epsilon}-6\tilde{\eta}^{\parallel}+12\chi-6\tilde{\epsilon}^2)\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\parallel})^2\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\perp})^3-4\tilde{\epsilon}\xi^{\perp}-2\tilde{\eta}^{\parallel}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{211}-V_{222}}{3H^2} \\ 6\tilde{\eta}^{\perp}-2\tilde{\epsilon}\tilde{\eta}^{\perp}+4\tilde{\eta}^{\parallel}\tilde{\eta}^{\perp}-2\tilde{\xi}^{\perp} \end{array} \right.$$

- "Initial Condition Terms": $\Delta(\frac{H}{\sqrt{\tilde{\epsilon}}}) = (2\tilde{\epsilon}+\tilde{\eta})\frac{H}{\sqrt{\tilde{\epsilon}}}\zeta$

We solve $\frac{d}{dt}v_{ai}^{(2)} + A(t)v_{ai}^{(2)} = \bar{A}_{abc}(t')\, v_{bi}^{(1)}\, v_c^{(1)} + \Delta\, b_{ai} \quad \Rightarrow$

$$v_{ai}^{(2)} = \int_{-\infty}^{t} G_{ab}(t,t') \left(\bar{A}_{bcd}\, v_{ci}^{(1)}\, v_d^{(1)} + \Delta\, b_{bi} \right) dt'$$

$$\frac{d}{dt'}G_{ab}(t,t') - G_{ac}(t,t')A_{cb}(t') = 0,$$

$$\frac{d}{dt}G_{ab}(t,t') + A_{ac}(t)G_{cb}(t,t') = 0, \qquad G_{ab}(t_\star,t_\star) = \delta_{ab}$$

“Initial Conditions”

Assuming that non-linearities are not important on short scales, one can include in a straightforward manner perturbations from shorter wavelengths. This amounts to adding “sources" on the rhs:

$$\frac{d}{dt}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} + \begin{pmatrix}0 & -2\tilde\eta^{\perp} & 0\\ 0 & 0 & -1\\ 0 & \tilde\kappa & \tilde\lambda\end{pmatrix}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} = \partial_i \int \frac{d^3k}{(2\pi)^{3/2}} \begin{pmatrix}\zeta_l(k)\,\hat\alpha_{\mathbf{k}}\\ \sigma_l(k)\,\hat\beta_{\mathbf{k}}\\ \dot\sigma_l(k)\,\hat\beta_{\mathbf{k}}\end{pmatrix} \dot{\mathcal{W}}(k)e^{i\mathbf{kx}}$$

where

$$\hat\alpha_{\mathbf{k}} = a^{\dagger}(\mathbf{k}) + a(-\mathbf{k}), \quad \hat\beta_{\mathbf{k}} = b^{\dagger}(\mathbf{k}) + b(-\mathbf{k})$$

with $\left[a(\mathbf{k}), a^{\dagger}(-\mathbf{k}')\right] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$, e.t.c.

$$\zeta_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k) = \frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}\chi$$

$\mathcal{W}(k)$ cuts off short wavelength modes. Simplest choice: $\mathcal{W}(k) = \Theta(caH - k)$. Final results are *independent* of the form of $\mathcal{W}(k)$.

When linearized, these equations are exact and valid to all scales, simply being linear perturbation theory.

$$\bar{A}_{32c} = 2\times$$

$$\left(\begin{array}{c} (-6\tilde{\epsilon}-10\tilde{\epsilon}^2-2\tilde{\epsilon}\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(6+6\tilde{\epsilon}-8\tilde{\eta}^{\parallel})(\tilde{\eta}^{\perp})^2-\tilde{\epsilon}(3\chi+4\tilde{\epsilon}^2+3\xi^{\parallel})-6\tilde{\eta}^{\perp}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{111}-V_{22}}{3H^2} \\ (-12\tilde{\epsilon}-6\tilde{\eta}^{\parallel}+12\chi-6\tilde{\epsilon}^2)\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\parallel})^2\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\perp})^3-4\tilde{\epsilon}\xi^{\perp}-2\tilde{\eta}^{\parallel}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{211}-V_{222}}{3H^2} \\ 6\tilde{\eta}^{\perp}-2\tilde{\epsilon}\tilde{\eta}^{\perp}+4\tilde{\eta}^{\parallel}\tilde{\eta}^{\perp}-2\tilde{\xi}^{\perp} \end{array} \right.$$

- "Initial Condition Terms": $\Delta(\frac{H}{\sqrt{\tilde{\epsilon}}}) = (2\tilde{\epsilon}+\tilde{\eta})\frac{H}{\sqrt{\tilde{\epsilon}}}\zeta$

We solve $\quad \frac{d}{dt}v^{(2)}_{ai} + A(t)v^{(2)}_{ai} = \bar{A}_{abc}(t')\, v^{(1)}_{bi}\, v^{(1)}_{c} + \Delta\, b_{ai} \quad \Rightarrow$

$$v^{(2)}_{ai} = \int_{-\infty}^{t} G_{ab}(t,t') \left(\bar{A}_{bcd}\, v^{(1)}_{ci}\, v^{(1)}_{d} + \Delta\, b_{bi} \right) dt'$$

$$\frac{d}{dt'}G_{ab}(t,t') - G_{ac}(t,t')A_{cb}(t') = 0,$$

$$\frac{d}{dt}G_{ab}(t,t') + A_{ac}(t)G_{cb}(t,t') = 0, \qquad G_{ab}(t_\star,t_\star) = \delta_{ab}$$

$$\bar{A}_{32c} = 2\times$$

$$\left(\begin{array}{c} (-6\tilde{\epsilon}-10\tilde{\epsilon}^2-2\tilde{\epsilon}\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(6+6\tilde{\epsilon}-8\tilde{\eta}^{\parallel})(\tilde{\eta}^{\perp})^2-\tilde{\epsilon}(3\chi+4\tilde{\epsilon}^2+3\xi^{\parallel})-6\tilde{\eta}^{\perp}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2\kappa}}\frac{V_{111}-V_{22}}{3H^2} \\ (-12\tilde{\epsilon}-6\tilde{\eta}^{\parallel}+12\chi-6\tilde{\epsilon}^2)\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\parallel})^2\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\perp})^3-4\tilde{\epsilon}\xi^{\perp}-2\tilde{\eta}^{\parallel}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2\kappa}}\frac{V_{211}-V_{222}}{3H^2} \\ 6\tilde{\eta}^{\perp}-2\tilde{\epsilon}\tilde{\eta}^{\perp}+4\tilde{\eta}^{\parallel}\tilde{\eta}^{\perp}-2\tilde{\xi}^{\perp} \end{array} \right.$$

- "Initial Condition Terms": $\Delta(\frac{H}{\sqrt{\tilde{\epsilon}}}) = (2\tilde{\epsilon}+\tilde{\eta})\frac{H}{\sqrt{\tilde{\epsilon}}}\zeta$

We solve $\frac{d}{dt}v_{ai}^{(2)} + A(t)v_{ai}^{(2)} = \bar{A}_{abc}(t')\, v_{bi}^{(1)}\, v_c^{(1)} + \Delta\, b_{ai} \quad \Rightarrow$

$$v_{ai}^{(2)} = \int_{-\infty}^{t} G_{ab}(t,t')\left(\bar{A}_{bcd}\, v_{ci}^{(1)}\, v_d^{(1)} + \Delta\, b_{bi}\right) dt'$$

$$\frac{d}{dt'}G_{ab}(t,t') - G_{ac}(t,t')A_{cb}(t') = 0,$$

$$\frac{d}{dt}G_{ab}(t,t') + A_{ac}(t)G_{cb}(t,t') = 0, \qquad G_{ab}(t_\star,t_\star) = \delta_{ab}$$

The Bispectrum

Focus on the adiabatic perturbation $v_1 = \zeta$ and define

$$v_{1m}(k,t) = \frac{\kappa}{2}\frac{H(t_k)}{k^{3/2}}\int_{-\infty}^{t} dt' G_{1b}(t,t') X_{bm}(t')\dot{\mathcal{W}}(k,t')\,,$$

$$X_{bm} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \chi(t') \end{pmatrix}.$$

We have been working in a uniform expansion slicing:

$$\zeta_i^1 = -\frac{\kappa}{\sqrt{2\epsilon}}\left(e_{1A}\,\partial_i\phi^A\right) = \frac{1}{3\Pi^2}\,\partial_i\rho.$$

To get the curvature perturbation we transform to a uniform energy density slicing: $\tilde{\zeta}_i = \partial_i \ln \tilde{a} \equiv \partial_i\tilde{\alpha}$. Using $\tilde{\zeta}_i(T,\mathbf{x}) = \zeta_i(t,\mathbf{x})$ we find:

$$\partial_i\tilde{\alpha}^{(1)} = \tilde{\zeta}_i^{(1)} = \zeta_i^{(1)}$$

$$\partial_i\tilde{\alpha}^{(2)} = \tilde{\zeta}_i^{(2)} = \zeta_i^{(2)} + 2\eta^{\perp}\zeta^{(1)}\sigma_i^{(1)}$$

(Note: No non-local terms)

$$\bar{A}_{32c} = 2\times$$

$$\left(\begin{array}{c} (-6\tilde{\epsilon}-10\tilde{\epsilon}^2-2\tilde{\epsilon}\tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel}-(6+6\tilde{\epsilon}-8\tilde{\eta}^{\parallel})(\tilde{\eta}^{\perp})^2-\tilde{\epsilon}(3\chi+4\tilde{\epsilon}^2+3\xi^{\parallel})-6\tilde{\eta}^{\perp}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{111}-V_{22}}{3H^2} \\ (-12\tilde{\epsilon}-6\tilde{\eta}^{\parallel}+12\chi-6\tilde{\epsilon}^2)\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\parallel})^2\tilde{\eta}^{\perp}+4(\tilde{\eta}^{\perp})^3-4\tilde{\epsilon}\xi^{\perp}-2\tilde{\eta}^{\parallel}\xi^{\perp}+\frac{3\sqrt{\tilde{\epsilon}}}{\sqrt{2}\kappa}\frac{V_{211}-V_{222}}{3H^2} \\ 6\tilde{\eta}^{\perp}-2\tilde{\epsilon}\tilde{\eta}^{\perp}+4\tilde{\eta}^{\parallel}\tilde{\eta}^{\perp}-2\tilde{\xi}^{\perp} \end{array} \right.$$

- "Initial Condition Terms": $\Delta(\frac{H}{\sqrt{\tilde{\epsilon}}}) = (2\tilde{\epsilon}+\tilde{\eta})\frac{H}{\sqrt{\tilde{\epsilon}}}\zeta$

We solve $\frac{d}{dt}v_{ai}^{(2)} + A(t)v_{ai}^{(2)} = \bar{A}_{abc}(t')\, v_{bi}^{(1)}\, v_c^{(1)} + \Delta\, b_{ai} \quad \Rightarrow$

$$v_{ai}^{(2)} = \int_{-\infty}^{t} G_{ab}(t,t')\left(\bar{A}_{bcd}\, v_{ci}^{(1)}\, v_d^{(1)} + \Delta\, b_{bi}\right) dt'$$

$$\frac{d}{dt'}G_{ab}(t,t') - G_{ac}(t,t')A_{cb}(t') = 0,$$

$$\frac{d}{dt}G_{ab}(t,t') + A_{ac}(t)G_{cb}(t,t') = 0, \qquad G_{ab}(t_\star,t_\star) = \delta_{ab}$$

The Bispectrum

Focus on the adiabatic perturbation $v_1 = \zeta$ and define

$$v_{1m}(k,t) = \frac{\kappa}{2}\frac{H(t_k)}{k^{3/2}}\int_{-\infty}^{t} dt' G_{1b}(t,t')X_{bm}(t')\dot{\mathcal{W}}(k,t')\,,$$

$$X_{bm} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \chi(t') \end{pmatrix}.$$

We have been working in a uniform expansion slicing:

$$\zeta_i^1 = -\frac{\kappa}{\sqrt{2\epsilon}}\left(e_{1A}\,\partial_i\phi^A\right) = \frac{1}{3\Pi^2}\,\partial_i\rho.$$

To get the curvature perturbation we transform to a uniform energy density slicing: $\tilde{\zeta}_i = \partial_i \ln \tilde{a} \equiv \partial_i\tilde{\alpha}$. Using $\tilde{\zeta}_i(T,\mathbf{x}) = \zeta_i(t,\mathbf{x})$ we find:

$$\partial_i\tilde{\alpha}^{(1)} = \tilde{\zeta}_i^{(1)} = \zeta_i^{(1)}$$

$$\partial_i\tilde{\alpha}^{(2)} = \tilde{\zeta}_i^{(2)} = \zeta_i^{(2)} + 2\eta^{\perp}\zeta^{(1)}\sigma_i^{(1)}$$

(Note: No non-local terms)

The Bispectrum

Focus on the adiabatic perturbation $v_1 = \zeta$ and define

$$v_{1m}(k,t) = \frac{\kappa}{2}\frac{H(t_k)}{k^{3/2}}\int_{-\infty}^{t} dt' G_{1b}(t,t')X_{bm}(t')\dot{W}(k,t'),$$

$$X_{bm} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \chi(t') \end{pmatrix}.$$

We have be licing:

Automatic Updates

Updating your computer is almost complete. You must restart your computer for the updates to take effect.

Do you want to restart your computer now?

Restart Now Restart Later

$$\zeta_i^1 = -\frac{\kappa}{\sqrt{2\epsilon}}$$

To get the curvature perturbation we transform to a uniform energy density slicing: $\tilde{\zeta}_i = \partial_i \ln \tilde{a} \equiv \partial_i \tilde{\alpha}$. Using $\tilde{\zeta}_i(T,\mathbf{x}) = \zeta_i(t,\mathbf{x})$ we find:

$$\partial_i \tilde{\alpha}^{(1)} = \tilde{\zeta}_i^{(1)} = \zeta_i^{(1)}$$

$$\partial_i \tilde{\alpha}^{(2)} = \tilde{\zeta}_i^{(2)} = \zeta_i^{(2)} + 2\eta^{\perp}\zeta^{(1)}\sigma_i^{(1)}$$

(Note: No non-local terms)

The Bispectrum

Focus on the adiabatic perturbation $v_1 = \zeta$ and define

$$v_{1m}(k,t) = \frac{\kappa}{2}\frac{H(t_k)}{k^{3/2}}\int_{-\infty}^{t} dt' G_{1b}(t,t')X_{bm}(t')\dot{W}(k,t'),$$

$$X_{bm} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \chi(t') \end{pmatrix}.$$

We have been working in a uniform expansion slicing:

$$\zeta_i^1 = -\frac{\kappa}{\sqrt{2\epsilon}}(e_{1A}\,\partial_i\phi^A) = \frac{1}{3\Pi^2}\,\partial_i\rho.$$

To get the curvature perturbation we transform to a uniform energy density slicing: $\tilde{\zeta}_i = \partial_i \ln \tilde{a} \equiv \partial_i\tilde{\alpha}$. Using $\tilde{\zeta}_i(T,\mathbf{x}) = \zeta_i(t,\mathbf{x})$ we find:

$$\partial_i\tilde{\alpha}^{(1)} = \tilde{\zeta}_i^{(1)} = \zeta_i^{(1)}$$

$$\partial_i\tilde{\alpha}^{(2)} = \tilde{\zeta}_i^{(2)} = \zeta_i^{(2)} + 2\eta^{\perp}\zeta^{(1)}\sigma_i^{(1)}$$

(Note: No non-local terms)

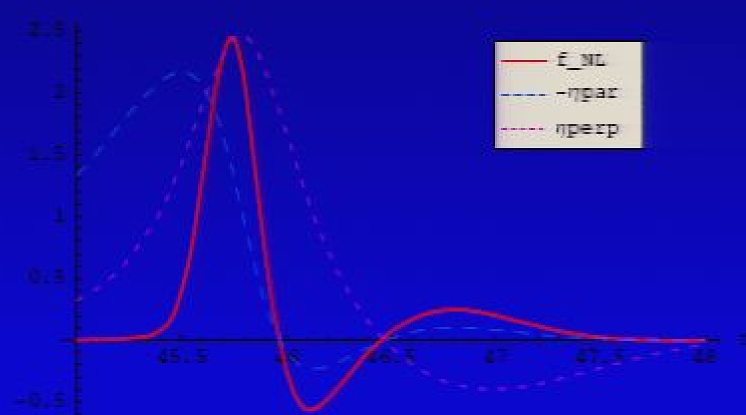
2-pt fn: $\langle\tilde{\alpha}\tilde{\alpha}\rangle^{(1)}(k,t) = v^{(1)}_{1m}(k,t)v^{(1)}_{1m}(k,t)$

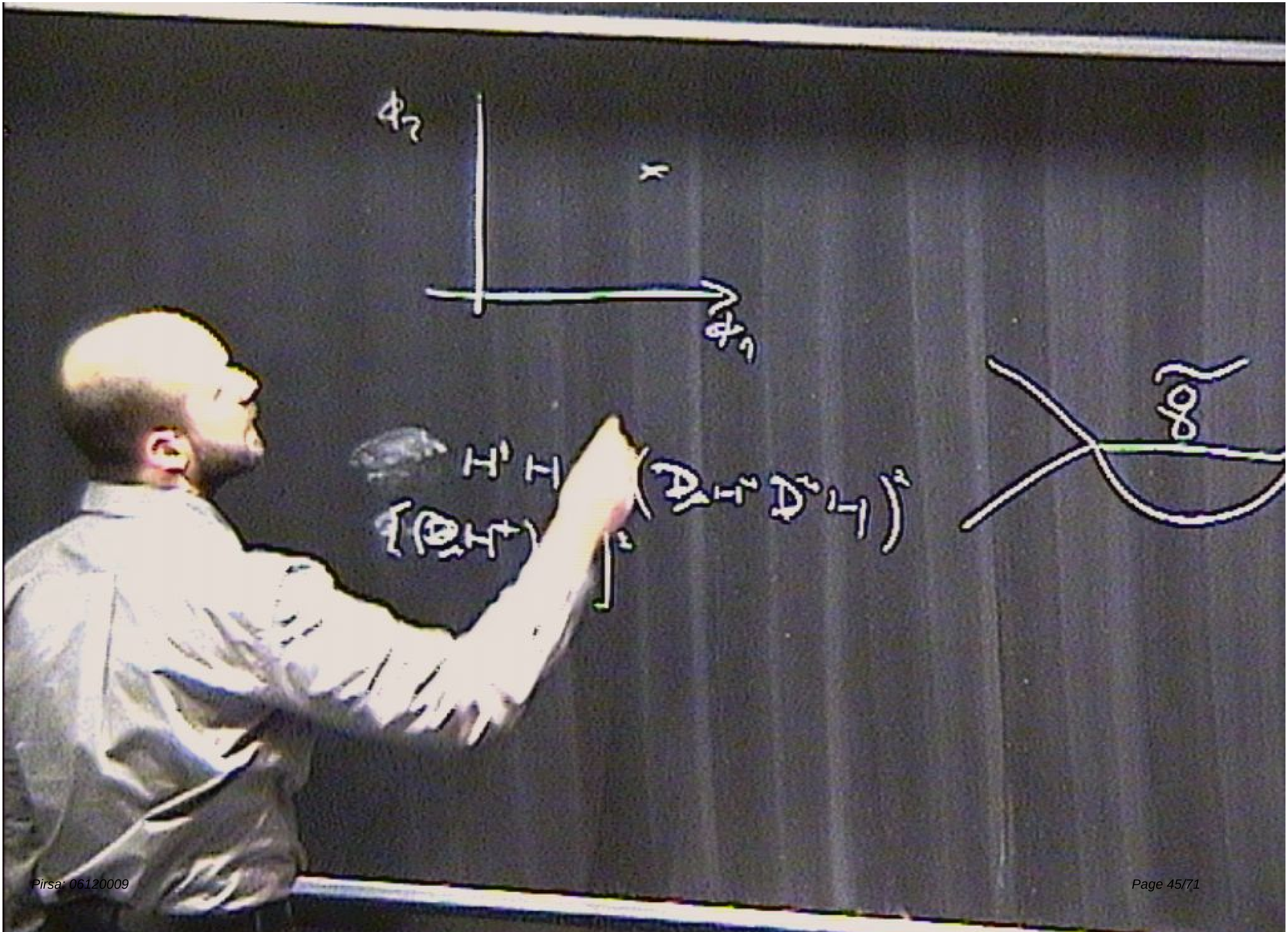
3-pt fn: $\langle\tilde{\alpha}\tilde{\alpha}\tilde{\alpha}\rangle^{(2)}(k_1,k_2,k_3,t) = (2\pi)^3\delta^3(\sum_s \mathbf{k}_s)\left[f(k_1,k_2)+\ldots\right]$

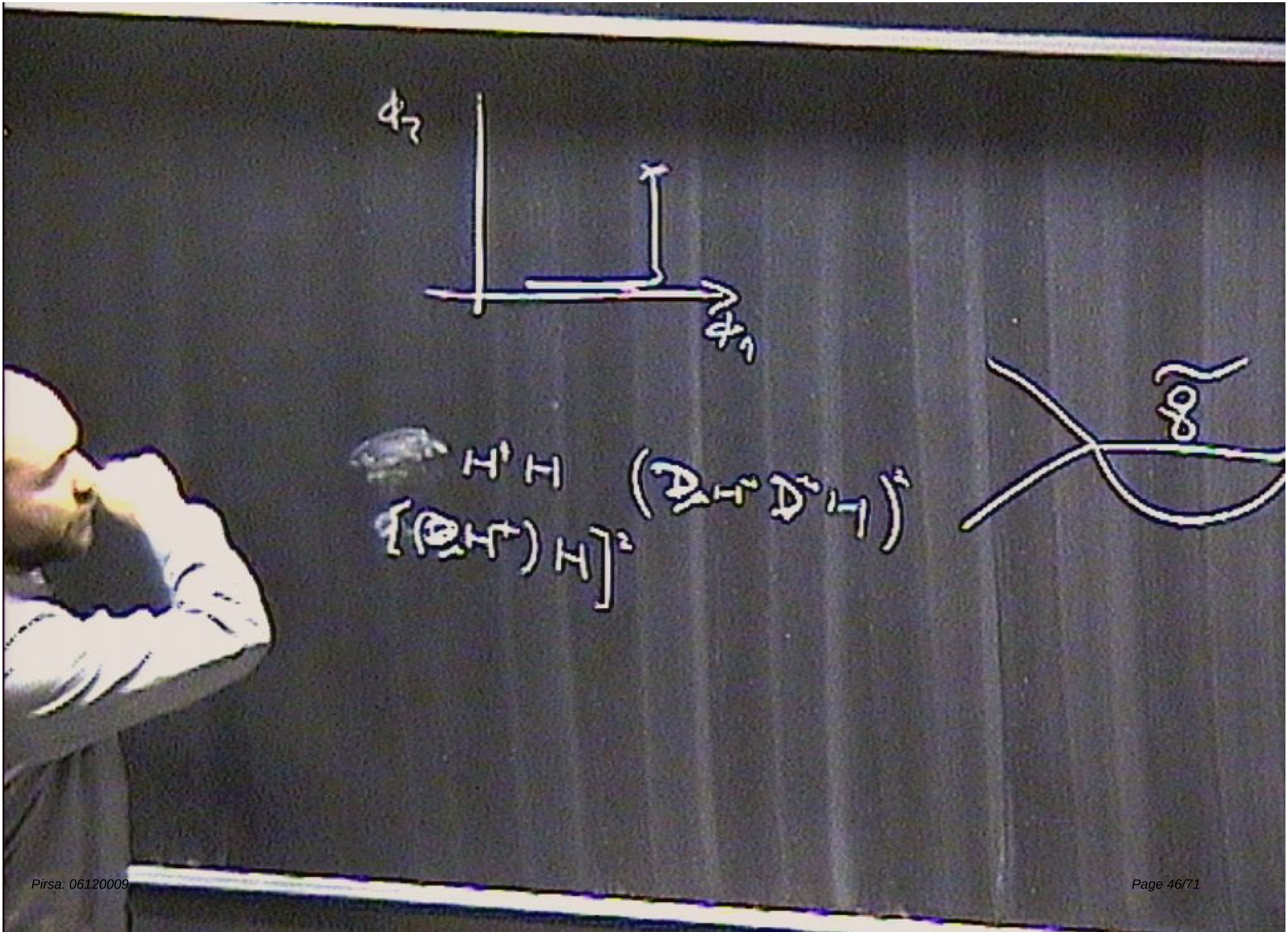
- $f(k,k') \equiv L_{mn}(k,k',t)v^{(1)}_{1m}(k,t)v^{(1)}_{1n}(k',t) + k \leftrightarrow k'$
- $L_{mn}(k,k',t) \equiv \eta^{\perp}v^{(1)}_{2m}(k,t)v^{(1)}_{1n}(k',t) -$
  $\frac{1}{2}\int_{-\infty}^{t}\mathrm{d}t'\, G_{1a}(t,t')\bar{A}^{(0)}_{abc}(t')v^{(1)}_{bm}(k,t')v^{(1)}_{cn}(k',t')$

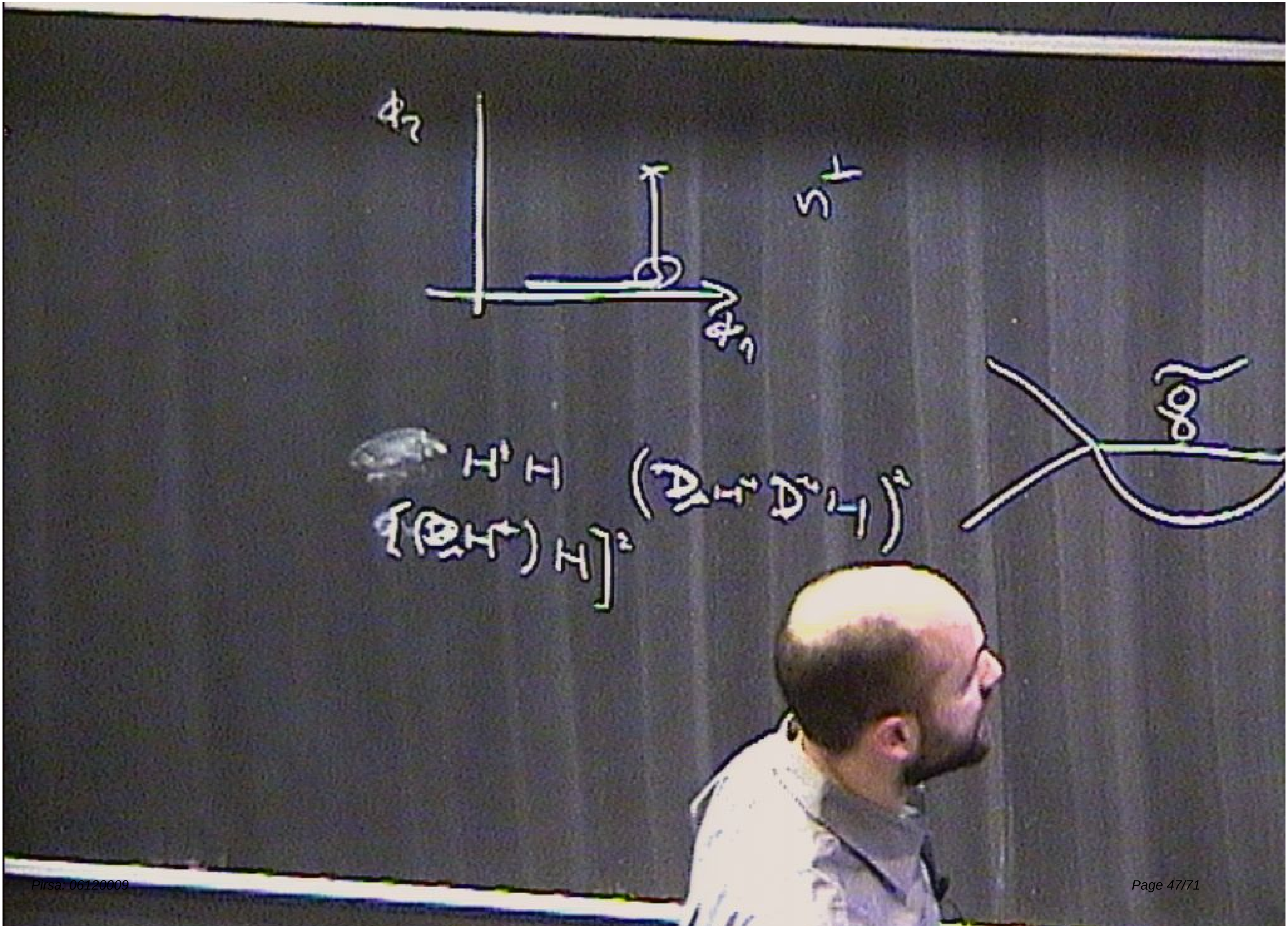
Parameterize non-Gaussianity by:

$$f_{NL} \equiv \frac{\langle\tilde{\alpha}\tilde{\alpha}\tilde{\alpha}\rangle(t)}{\langle\tilde{\alpha}\tilde{\alpha}\rangle(t)}$$









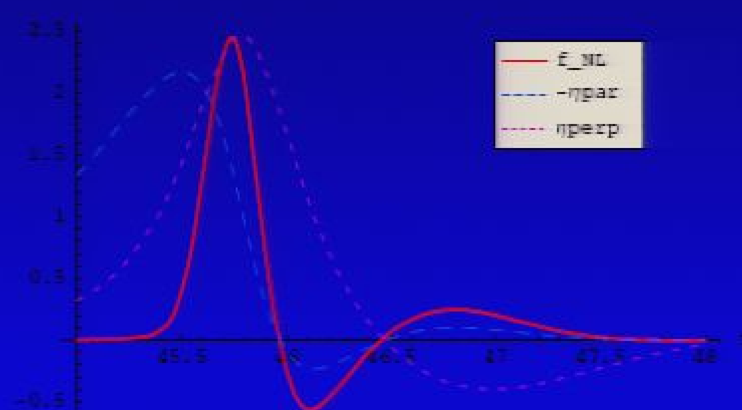
2-pt fn: $\langle\tilde{\alpha}\tilde{\alpha}\rangle^{(1)}(k,t) = v^{(1)}_{1m}(k,t)v^{(1)}_{1m}(k,t)$

3-pt fn: $\langle\tilde{\alpha}\tilde{\alpha}\tilde{\alpha}\rangle^{(2)}(k_1,k_2,k_3,t) = (2\pi)^3\delta^3(\sum_s \mathbf{k}_s)\Big[f(k_1,k_2)+\ldots\Big]$

- $f(k,k') \equiv L_{mn}(k,k',t)v^{(1)}_{1m}(k,t)v^{(1)}_{1n}(k',t) + k \leftrightarrow k'$
- $L_{mn}(k,k',t) \equiv \eta^{\perp}v^{(1)}_{2m}(k,t)v^{(1)}_{1n}(k',t) -$
  $\frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t'\, G_{1a}(t,t')\bar{A}^{(0)}_{abc}(t')v^{(1)}_{bm}(k,t')v^{(1)}_{cn}(k',t')$

Parameterize non-Gaussianity by:

$$f_{NL} \equiv \frac{\langle\tilde{\alpha}\tilde{\alpha}\tilde{\alpha}\rangle(t)}{\langle\tilde{\alpha}\tilde{\alpha}\rangle(t)}$$



Some analytic results

One can further process the second order solution analytically:

$$f_{\rm NL}(t) = \frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\left((\tilde{\epsilon}+\tilde{\eta}^{\parallel})(G_{22})^2 + G_{22}G_{32}\right)$$

$$\frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\int_{-\infty}^{t}{\rm d}t'\Bigg[-2(\tilde{\eta}^{\perp})^2(G_{22})^2 - (\tilde{\epsilon}+\tilde{\eta}^{\parallel})G_{22}G_{32} - (G_{32})^2 +$$

$$\bar{G}_{13}9\tilde{\eta}^{\perp}G_{22}G_{32} + \bar{G}_{13}\Big(12\tilde{\eta}^{\perp}\chi - 6\tilde{\eta}^{\parallel}\tilde{\eta}^{\perp} + 6(\tilde{\eta}^{\parallel})^2\tilde{\eta}^{\perp} + 6(\tilde{\eta}^{\perp})^3 -$$

$$2\tilde{\eta}^{\perp}\tilde{\xi}^{\parallel} - 2\tilde{\eta}^{\parallel}\tilde{\xi}^{\perp} - \frac{3}{2}\frac{\sqrt{2\tilde{\epsilon}}}{\kappa}\frac{V_{211}+V_{222}}{3H^2}\Big)(G_{22})^2\Bigg]$$

where $\bar{G} = G(t,t')$. For the models we have studied the first two terms are an excellent approximation to f_{NL}. Is the second term always small? (Work in progress)

(Note: The terms which would give rise to a curl have explicitly canceled.)

Some analytic results

One can further process the second order solution analytically:

$$f_{\rm NL}(t) = \frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\left((\tilde\epsilon + \tilde\eta^{\|})(G_{22})^2 + G_{22}G_{32}\right)$$

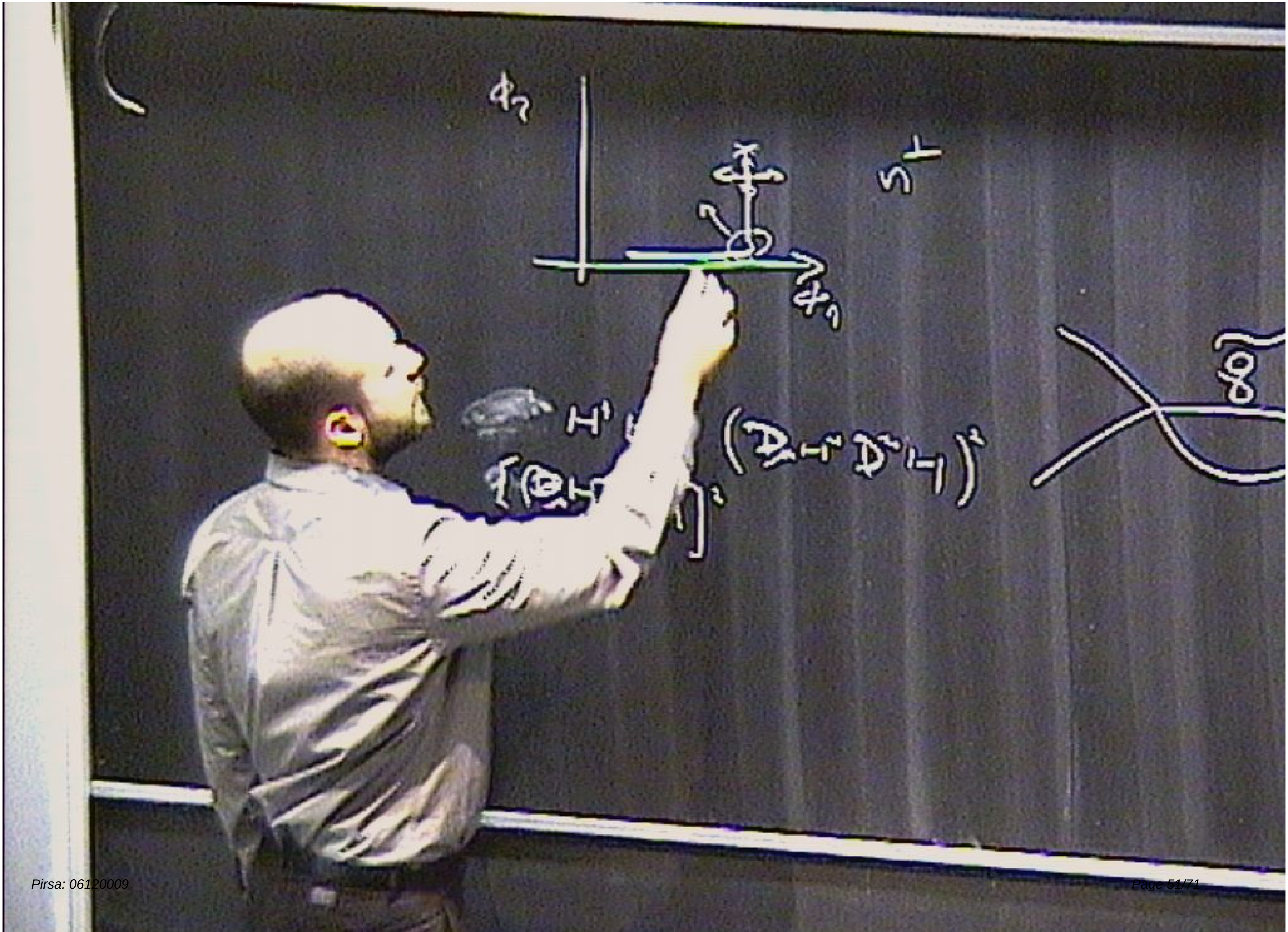
$$\frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\int_{-\infty}^{t} {\rm d}t' \Bigg[-2(\tilde\eta^{\perp})^2(G_{22})^2 - (\tilde\epsilon + \tilde\eta^{\|})G_{22}G_{32} - (G_{32})^2 +$$

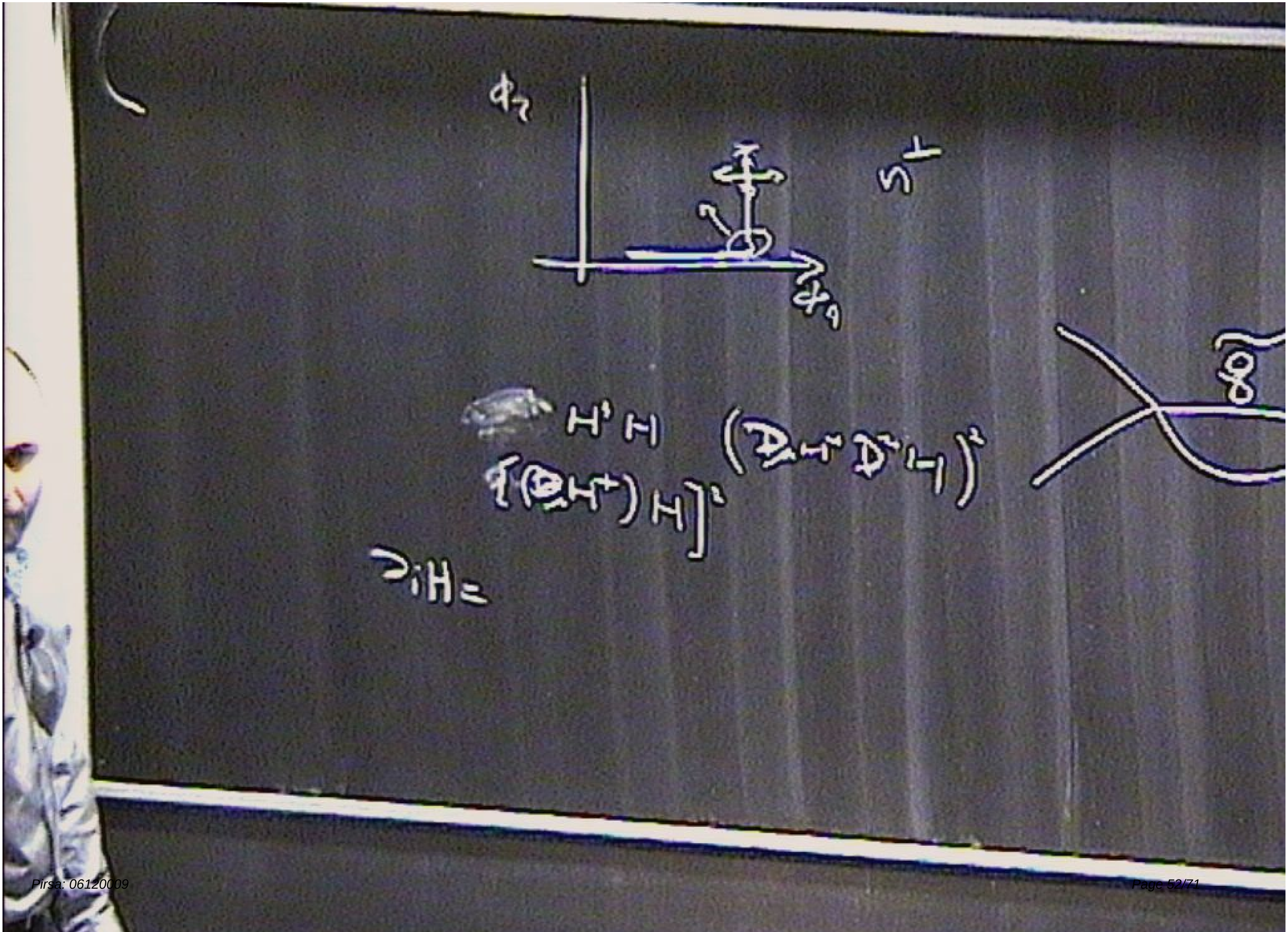
$$\bar G_{13} 9\tilde\eta^{\perp} G_{22}G_{32} + \bar G_{13}\Big(12\tilde\eta^{\perp}\chi - 6\tilde\eta^{\|}\tilde\eta^{\perp} + 6(\tilde\eta^{\|})^2\tilde\eta^{\perp} + 6(\tilde\eta^{\perp})^3 -$$

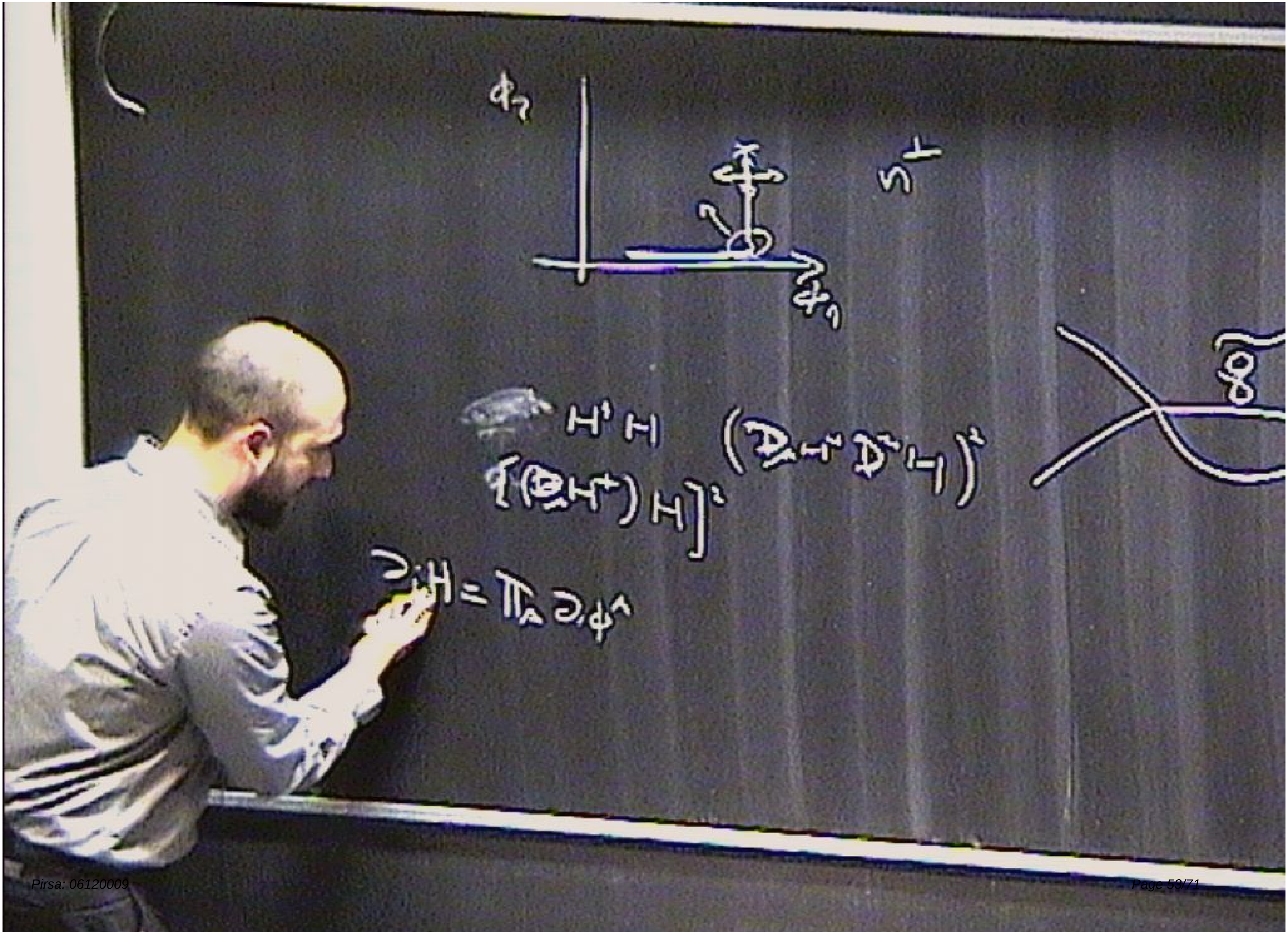
$$2\tilde\eta^{\perp}\tilde\xi^{\|} - 2\tilde\eta^{\|}\tilde\xi^{\perp} - \frac{3}{2}\frac{\sqrt{2\tilde\epsilon}}{\kappa}\frac{V_{211}+V_{222}}{3H^2}\Big)(G_{22})^2\Bigg]$$

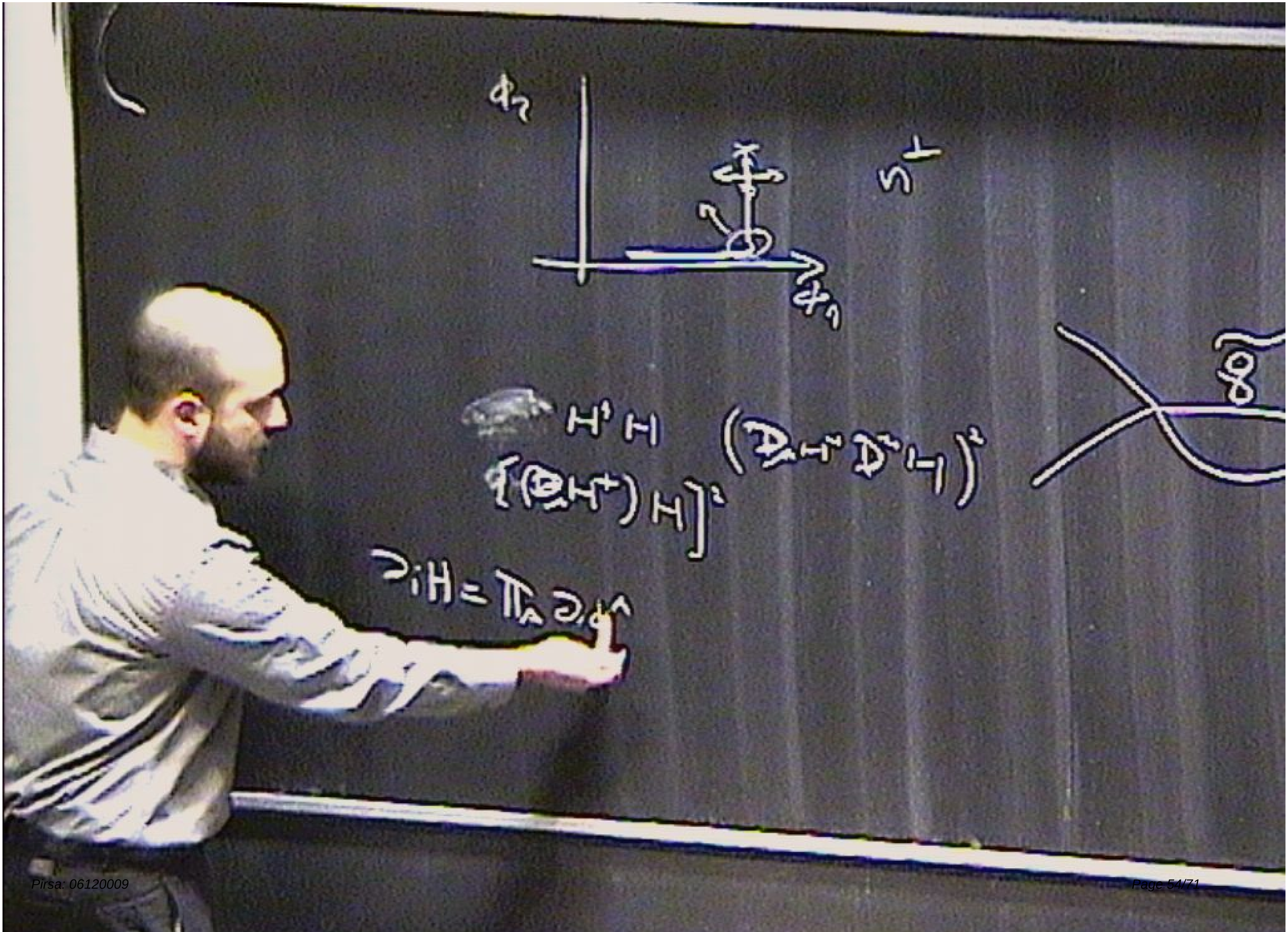
where $\bar G = G(t,t')$. For the models we have studied the first two terms are an excellent approximation to f_{NL}. Is the second term always small? (Work in progress)

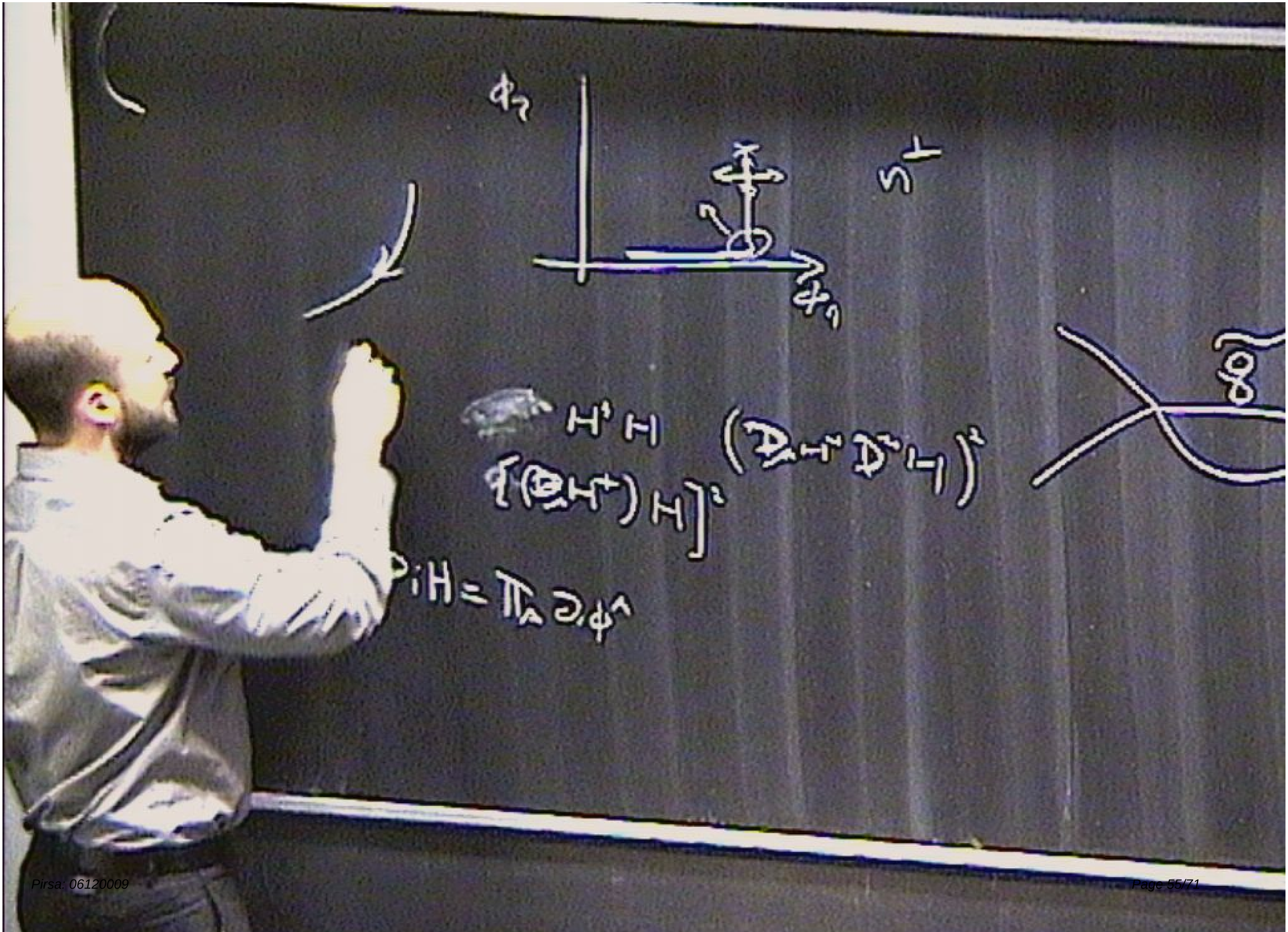
(Note: The terms which would give rise to a curl have explicitly canceled.)

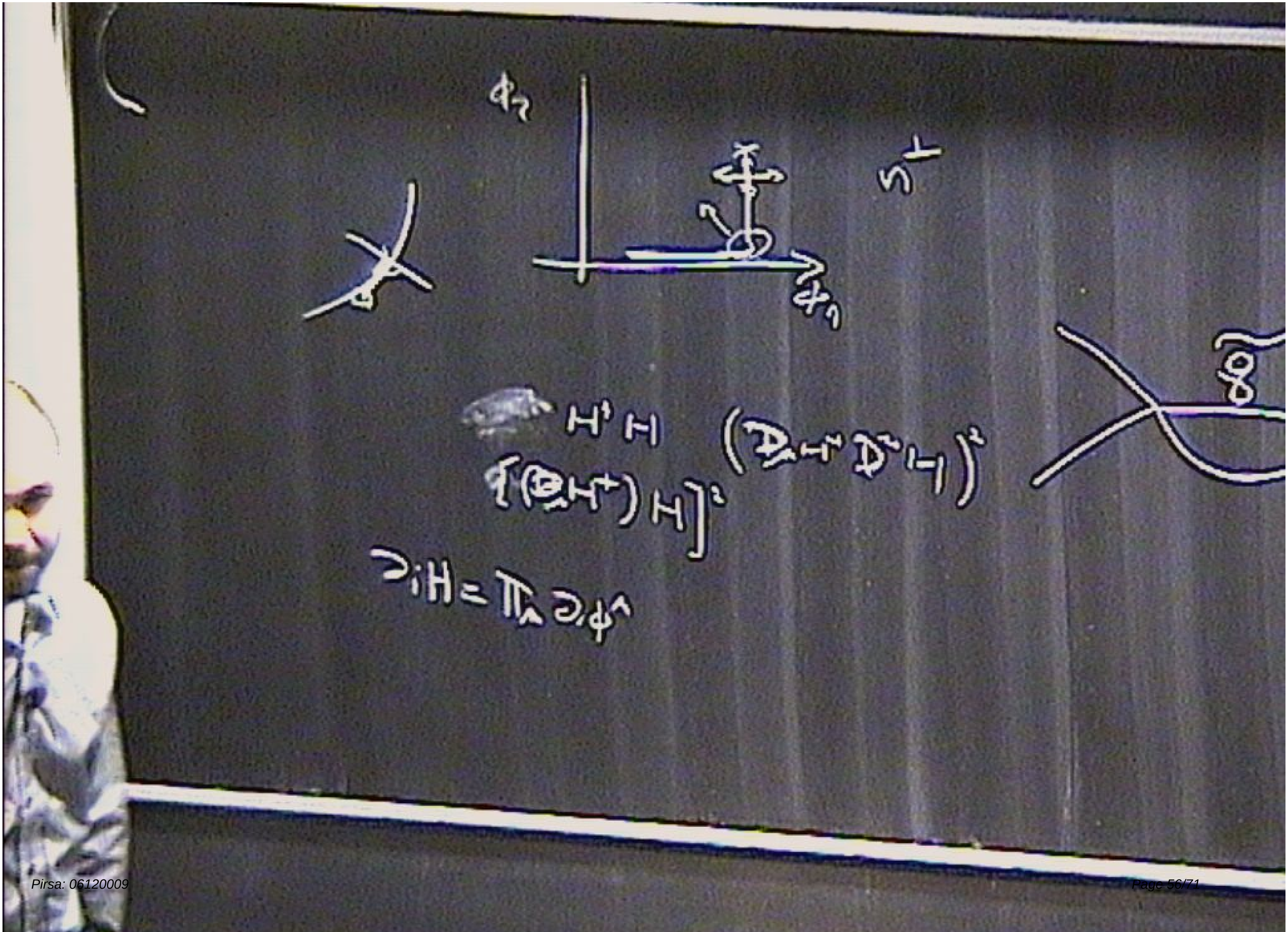



H†H
∂iH =









Some analytic results

One can further process the second order solution analytically:

$$f_{\rm NL}(t) = \frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\left((\tilde{\epsilon}+\tilde{\eta}^{\|})(G_{22})^2 + G_{22}G_{32}\right)$$

$$\frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\int_{-\infty}^{t} \mathrm{d}t' \Big[-2(\tilde{\eta}^{\perp})^2(G_{22})^2 - (\tilde{\epsilon}+\tilde{\eta}^{\|})G_{22}G_{32} - (G_{32})^2 +$$

$$\bar{G}_{13} 9\tilde{\eta}^{\perp} G_{22}G_{32} + \bar{G}_{13}\Big(12\tilde{\eta}^{\perp}\chi - 6\tilde{\eta}^{\|}\tilde{\eta}^{\perp} + 6(\tilde{\eta}^{\|})^2\tilde{\eta}^{\perp} + 6(\tilde{\eta}^{\perp})^3 -$$

$$2\tilde{\eta}^{\perp}\tilde{\xi}^{\|} - 2\tilde{\eta}^{\|}\tilde{\xi}^{\perp} - \frac{3}{2}\frac{\sqrt{2\tilde{\epsilon}}}{\kappa}\frac{V_{211}+V_{222}}{3H^2}\Big)(G_{22})^2\Big]$$

where $\bar{G} = G(t,t')$. For the models we have studied the first two terms are an excellent approximation to f_{NL}. Is the second term always small? (Work in progress)

(Note: The terms which would give rise to a curl have explicitly canceled.)

Large non-Gaussianity?

Simple inflationary models give $f_{NL} \sim \mathcal{O}(0.01) = \mathcal{O}(\tilde{\epsilon}, \tilde{\eta})$ for both single and multi field inflation.

- Less symmetric potentials?
- Non-separable potentials?
- Hybrid inflation
- Reheating, preheating
- Later dominance of another field (Curvaton)

Large NG: both $G_{22} \neq 0$ and $G_{12} \neq 0$ at the end of inflation.

Of course this primordial input must be connected to the observable CMB sky and this connection adds one more NG component which may be dominant.

Conclusions

- Non-Gaussianity has been the focus of many studies over the past few years.
- It provides another observable that links the present to the early universe.
- Simple inflationary models, both single and multi field, predict it to be very small, $f_{NL} \sim \mathcal{O}(\tilde{\epsilon}, \tilde{\eta})$.
- However, later processes and/or more complicated models may yield larger primordial non-Gaussianity.
- An interesting observable for the future, especially if $f_{NL} \sim 0.01$ is accessible to future observations.

Some analytic results

One can further process the second order solution analytically:

$$f_{\rm NL}(t) = \frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\left((\tilde\epsilon + \tilde\eta^{\|})(G_{22})^2 + G_{22}G_{32}\right)$$
$$\frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\int_{-\infty}^{t} {\rm d}t' \Big[-2(\tilde\eta^{\perp})^2(G_{22})^2 - (\tilde\epsilon+\tilde\eta^{\|})G_{22}G_{32} - (G_{32})^2 +$$
$$\bar G_{13} 9\tilde\eta^{\perp} G_{22}G_{32} + \bar G_{13}\Big(12\tilde\eta^{\perp}\chi - 6\tilde\eta^{\|}\tilde\eta^{\perp} + 6(\tilde\eta^{\|})^2\tilde\eta^{\perp} + 6(\tilde\eta^{\perp})^3 -$$
$$2\tilde\eta^{\perp}\tilde\xi^{\|} - 2\tilde\eta^{\|}\tilde\xi^{\perp} - \frac{3}{2}\frac{\sqrt{2\tilde\epsilon}}{\kappa}\frac{V_{211}+V_{222}}{3H^2}\Big)(G_{22})^2\Big]$$

where $\bar G = G(t,t')$. For the models we have studied the first two terms are an excellent approximation to f_{NL}. Is the second term always small? (Work in progress)

(Note: The terms which would give rise to a curl have explicitly canceled.)

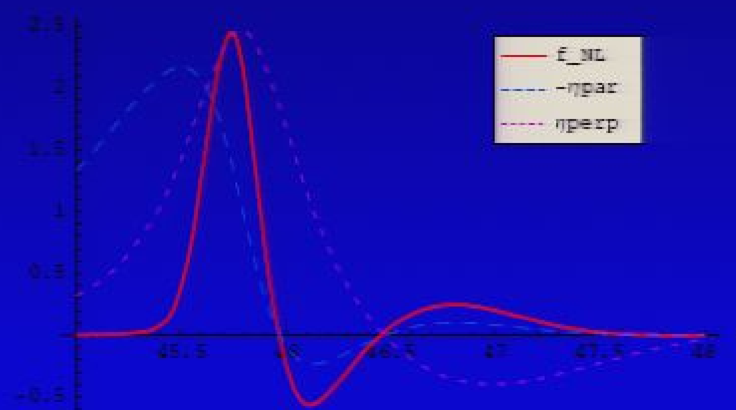
2-pt fn: $\langle\tilde{\alpha}\tilde{\alpha}\rangle^{(1)}(k,t) = v^{(1)}_{1m}(k,t)v^{(1)}_{1m}(k,t)$

3-pt fn: $\langle\tilde{\alpha}\tilde{\alpha}\tilde{\alpha}\rangle^{(2)}(k_1,k_2,k_3,t) = (2\pi)^3\delta^3(\sum_s \mathbf{k}_s)\Big[f(k_1,k_2)+\ldots\Big]$

- $f(k,k') \equiv L_{mn}(k,k',t)v^{(1)}_{1m}(k,t)v^{(1)}_{1n}(k',t) + k \leftrightarrow k'$
- $L_{mn}(k,k',t) \equiv \eta^{\perp}v^{(1)}_{2m}(k,t)v^{(1)}_{1n}(k',t) -$
  $\frac{1}{2}\int_{-\infty}^{t} \mathrm{d}t'\, G_{1a}(t,t')\bar{A}^{(0)}_{abc}(t')v^{(1)}_{bm}(k,t')v^{(1)}_{cn}(k',t')$

Parameterize non-Gaussianity by:

$$f_{NL} \equiv \frac{\langle\tilde{\alpha}\tilde{\alpha}\tilde{\alpha}\rangle(t)}{\langle\tilde{\alpha}\tilde{\alpha}\rangle(t)}$$



Some analytic results

One can further process the second order solution analytically:

$$f_{\rm NL}(t) = \frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\left((\tilde\epsilon+\tilde\eta^{\parallel})(G_{22})^2+G_{22}G_{32}\right)$$

$$\frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\int_{-\infty}^{t}{\rm d}t'\Bigg[-2(\tilde\eta^{\perp})^2(G_{22})^2-(\tilde\epsilon+\tilde\eta^{\parallel})G_{22}G_{32}-(G_{32})^2+$$

$$\bar G_{13}9\tilde\eta^{\perp}G_{22}G_{32}+\bar G_{13}\Big(12\tilde\eta^{\perp}\chi-6\tilde\eta^{\parallel}\tilde\eta^{\perp}+6(\tilde\eta^{\parallel})^2\tilde\eta^{\perp}+6(\tilde\eta^{\perp})^3-$$

$$2\tilde\eta^{\perp}\tilde\xi^{\parallel}-2\tilde\eta^{\parallel}\tilde\xi^{\perp}-\frac{3}{2}\frac{\sqrt{2\tilde\epsilon}}{\kappa}\frac{V_{211}+V_{222}}{3H^2}\Big)(G_{22})^2\Bigg]$$

where $\bar G = G(t,t')$. For the models we have studied the first two terms are an excellent approximation to f_{NL}. Is the second term always small? (Work in progress)

(Note: The terms which would give rise to a curl have explicitly canceled.)

Some analytic results

One can further process the second order solution analytically:

$$f_{\rm NL}(t) = \frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\left((\tilde\epsilon + \tilde\eta^{\|})(G_{22})^2 + G_{22}G_{32}\right)$$

$$\frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\int_{-\infty}^{t} dt' \Big[-2(\tilde\eta^{\perp})^2(G_{22})^2 - (\tilde\epsilon + \tilde\eta^{\|})G_{22}G_{32} - (G_{32})^2 +$$

$$\bar G_{13} 9\tilde\eta^{\perp} G_{2} \ldots \|)^2\tilde\eta^{\perp} + 6(\tilde\eta^{\perp})^3 -$$

$$2\tilde\eta^{\perp}\tilde\xi^{\|} - 2\tilde\eta \ldots \Big]$$

Automatic Updates

Updating your computer is almost complete. You must restart your computer for the updates to take effect.

Do you want to restart your computer now?

Restart Now Restart Later

where $\bar G = G(t,t')$. For the models we have studied the first two terms are an excellent approximation to f_{NL}. Is the second term always small? (Work in progress)

(Note: The terms which would give rise to a curl have explicitly canceled.)

Some analytic results

One can further process the second order solution analytically:

$$f_{\rm NL}(t) = \frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\left((\tilde\epsilon + \tilde\eta^{\parallel})(G_{22})^2 + G_{22}G_{32}\right)$$

$$\frac{-6(G_{12})^2}{[1+(G_{12})^2]^2}\int_{-\infty}^{t} dt' \Big[-2(\tilde\eta^{\perp})^2(G_{22})^2 - (\tilde\epsilon + \tilde\eta^{\parallel})G_{22}G_{32} - (G_{32})^2 +$$

$$\bar G_{13} 9\tilde\eta^{\perp} G_{22}G_{32} + \bar G_{13}\Big(12\tilde\eta^{\perp}\chi - 6\tilde\eta^{\parallel}\tilde\eta^{\perp} + 6(\tilde\eta^{\parallel})^2\tilde\eta^{\perp} + 6(\tilde\eta^{\perp})^3 -$$

$$2\tilde\eta^{\perp}\tilde\xi^{\parallel} - 2\tilde\eta^{\parallel}\tilde\xi^{\perp} - \frac{3}{2}\frac{\sqrt{2\tilde\epsilon}}{\kappa}\frac{V_{211}+V_{222}}{3H^2}\Big)(G_{22})^2\Big]$$

where $\bar G = G(t,t')$. For the models we have studied the first two terms are an excellent approximation to f_{NL}. Is the second term always small? (Work in progress)

(Note: The terms which would give rise to a curl have explicitly canceled.)

The Bispectrum

Focus on the adiabatic perturbation $v_1 = \zeta$ and define

$$v_{1m}(k,t) = \frac{\kappa}{2}\frac{H(t_k)}{k^{3/2}}\int_{-\infty}^{t} dt' G_{1b}(t,t') X_{bm}(t') \dot{W}(k,t'),$$

$$X_{bm} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \chi(t') \end{pmatrix}.$$

We have been working in a uniform expansion slicing:

$$\zeta_i^1 = -\frac{\kappa}{\sqrt{2\epsilon}}\left(e_{1A}\,\partial_i\phi^A\right) = \frac{1}{3\Pi^2}\,\partial_i\rho.$$

To get the curvature perturbation we transform to a uniform energy density slicing: $\tilde{\zeta}_i = \partial_i \ln\tilde{a} \equiv \partial_i\tilde{\alpha}$. Using $\tilde{\zeta}_i(T,\mathbf{x}) = \zeta_i(t,\mathbf{x})$ we find:

$$\partial_i\tilde{\alpha}^{(1)} = \tilde{\zeta}_i^{(1)} = \zeta_i^{(1)}$$

$$\partial_i\tilde{\alpha}^{(2)} = \tilde{\zeta}_i^{(2)} = \zeta_i^{(2)} + 2\eta^{\perp}\zeta^{(1)}\sigma_i^{(1)}$$

(Note: No non-local terms)

“Initial Conditions”

Assuming that non-linearities are not important on short scales, one can include in a straightforward manner perturbations from shorter wavelengths. This amounts to adding “sources" on the rhs:

$$\frac{d}{dt}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} + \begin{pmatrix}0 & -2\tilde\eta^\perp & 0\\ 0 & 0 & -1\\ 0 & \tilde\kappa & \tilde\lambda\end{pmatrix}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} = \partial_i \int \frac{d^3k}{(2\pi)^{3/2}} \begin{pmatrix}\zeta_l(k)\,\hat\alpha_{\mathbf{k}}\\ \sigma_l(k)\,\hat\beta_{\mathbf{k}}\\ \dot\sigma_l(k)\,\hat\beta_{\mathbf{k}}\end{pmatrix} \dot{\mathcal{W}}(k) e^{i\mathbf{kx}}$$

where

$$\hat\alpha_{\mathbf{k}} = a^\dagger(\mathbf{k}) + a(-\mathbf{k})\,, \quad \hat\beta_{\mathbf{k}} = b^\dagger(\mathbf{k}) + b(-\mathbf{k})$$

with $\left[a(\mathbf{k}), a^\dagger(-\mathbf{k}')\right] = (2\pi)^3\delta(\mathbf{k}-\mathbf{k}')$, e.t.c.

$$\zeta_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k) = -\frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}$$
$$\sigma_l(k) = \frac{\kappa}{\sqrt{2k^3}}\frac{H}{\sqrt{2\tilde\epsilon}}\chi$$

$\mathcal{W}(k)$ cuts off short wavelength modes. Simplest choice: $\mathcal{W}(k) = \Theta(caH - k)$. Final results are *independent* of the form of $\mathcal{W}(k)$.

When linearized, these equations are exact and valid to all scales, simply being linear perturbation theory.

Perturbation Theory

We can now perturb the e.o.m. to directly obtain solutions at second order

- Perturbing $A(t,\mathbf{x}) = \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}$ represents non-linearities in the long wavelength evolution
- Perturbing $\frac{H}{\sqrt{\tilde{\epsilon}}}$ represents "initial" non-linearities of the modes at horizon crossing

Write $(\Delta A)_{ab} = \bar{A}_{abc}\, v_c$, with $\boxed{v_i = \begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix}}$.

- Evolution terms:

$$\bar{A}_{12c} = 2 \begin{pmatrix} (\tilde{\epsilon} - 2\tilde{\eta}^{\parallel})\tilde{\eta}^{\perp} + \tilde{\xi}^{\perp} \\ -3\chi - (\tilde{\epsilon} + \tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} - (\tilde{\eta}^{\perp})^2 \\ -3 - \tilde{\eta}^{\parallel} \end{pmatrix} , \bar{A}_{33c} = \begin{pmatrix} -2\tilde{\epsilon}^2 + (2\tilde{\epsilon} - \tilde{\eta}^{\parallel})\tilde{\eta}^{\parallel} + (\tilde{\eta}^{\perp})^2 + \tilde{\xi}^{\parallel} \\ 2\tilde{\epsilon}\tilde{\eta}^{\perp} + \tilde{\xi}^{\perp} \\ \tilde{\eta}^{\perp} \end{pmatrix}$$

Long wavelength equations of motion

Choose a gauge with $NH = 1$ ($\partial_i \alpha = 0$ - homogeneous expansion) to simplify expressions

$$\frac{d}{dt}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} + \begin{pmatrix}0 & -2\tilde\eta^\perp & 0\\ 0 & 0 & -1\\ 0 & \tilde\kappa & \tilde\lambda\end{pmatrix}\begin{pmatrix}\zeta_i\\ \sigma_i\\ \dot\sigma_i\end{pmatrix} = 0$$

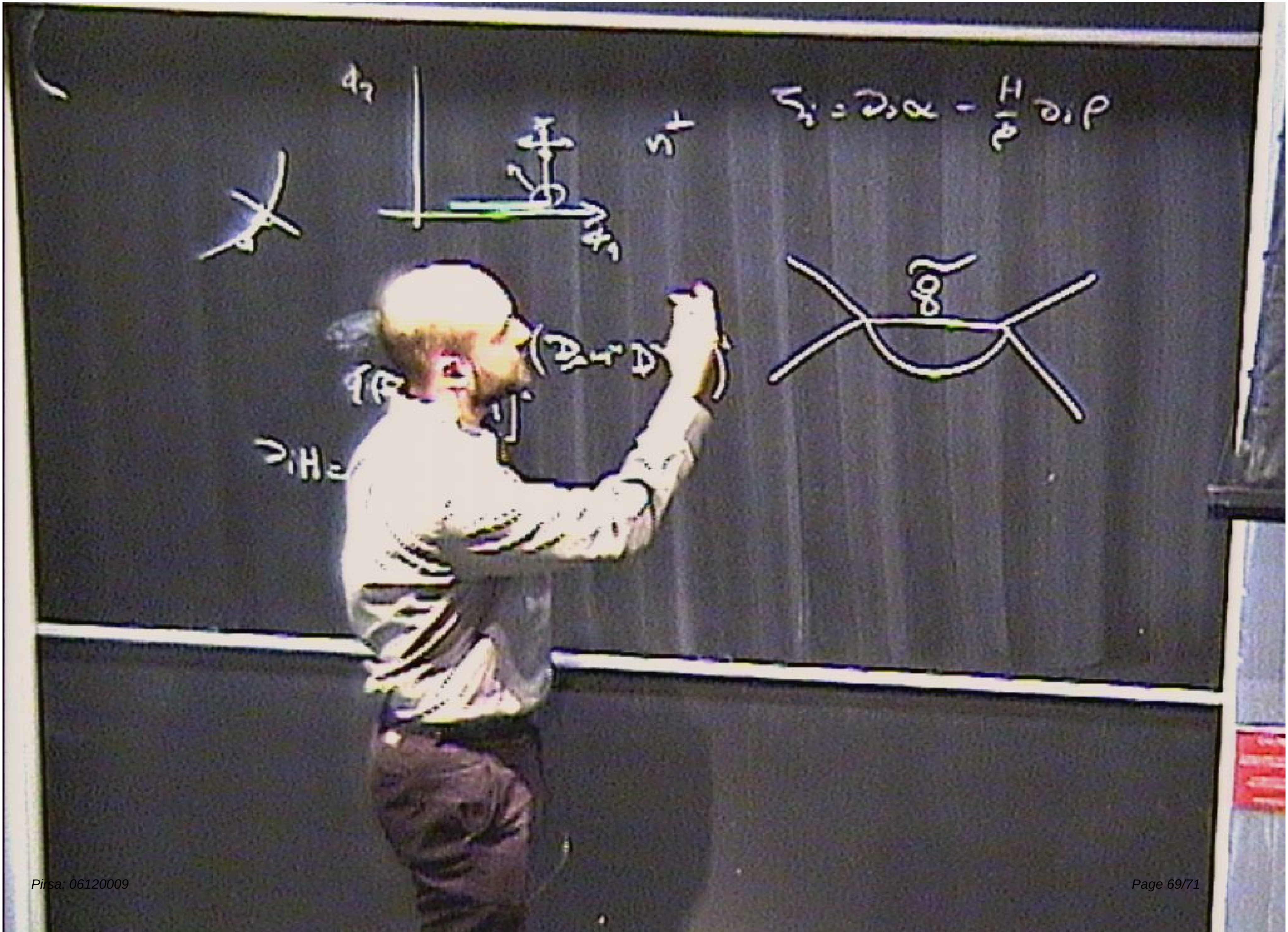
where

$$\tilde\kappa(t,\mathbf{x}) = 3\left(\frac{V_{22}}{3H^2} + \tilde\epsilon + \tilde\eta^\parallel\right) + 2\tilde\epsilon^2 + 4\tilde\epsilon\tilde\eta^\parallel + 4(\tilde\eta^\perp)^2 + \tilde\xi^\parallel,$$

$$\tilde\lambda(t,\mathbf{x}) = 3 + \tilde\epsilon + 2\tilde\eta^\parallel$$

All local quantities are given by:

- $\partial_i \ln H = \tilde\epsilon\,\zeta_i, \quad e_{m\,A}\partial_i\phi^A = -\frac{\sqrt{2\tilde\epsilon}}{\kappa}\,\zeta_i^m$
- $e_1^A\partial_i\Pi_A = -\frac{H\sqrt{2\tilde\epsilon}}{\kappa}\left(\tilde\eta^\parallel\zeta_i + \tilde\eta^\perp\sigma_i\right)$
- $e_2^A\partial_i\Pi_A = -\frac{H\sqrt{2\tilde\epsilon}}{\kappa}\left(\dot\sigma_i + \tilde\eta^\perp\zeta_i + \left(\tilde\eta^\parallel + \tilde\epsilon\right)\sigma_i\right)$



Long wavelength equations of motion

Choose a gauge with $NH = 1$ ($\partial_i \alpha = 0$ - homogeneous expansion) to simplify expressions

$$\frac{d}{dt}\begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix} + \begin{pmatrix} 0 & -2\tilde{\eta}^{\perp} & 0 \\ 0 & 0 & -1 \\ 0 & \tilde{\kappa} & \tilde{\lambda} \end{pmatrix}\begin{pmatrix} \zeta_i \\ \sigma_i \\ \dot{\sigma}_i \end{pmatrix} = 0$$

where

$$\tilde{\kappa}(t, \mathbf{x}) = 3\left(\frac{V_{22}}{3H^2} + \tilde{\epsilon} + \tilde{\eta}^{\parallel}\right) + 2\tilde{\epsilon}^2 + 4\tilde{\epsilon}\tilde{\eta}^{\parallel} + 4(\tilde{\eta}^{\perp})^2 + \tilde{\xi}^{\parallel},$$

$$\tilde{\lambda}(t, \mathbf{x}) = 3 + \tilde{\epsilon} + 2\tilde{\eta}^{\parallel}$$

All local quantities are given by:

- $\partial_i \ln H = \tilde{\epsilon}\,\zeta_i, \quad e_{m\,A}\partial_i \phi^A = -\frac{\sqrt{2\tilde{\epsilon}}}{\kappa}\,\zeta_i^m$
- $e_1^A \partial_i \Pi_A = -\frac{H\sqrt{2\tilde{\epsilon}}}{\kappa}\left(\tilde{\eta}^{\parallel}\zeta_i + \tilde{\eta}^{\perp}\sigma_i\right)$
- $e_2^A \partial_i \Pi_A = -\frac{H\sqrt{2\tilde{\epsilon}}}{\kappa}\left(\dot{\sigma}_i + \tilde{\eta}^{\perp}\zeta_i + \left(\tilde{\eta}^{\parallel} + \tilde{\epsilon}\right)\sigma_i\right)$

Non-linear Isocurvature and Adiabatic Variables

- $\zeta_i \equiv \hat{e}^1_A \zeta^A_i = \partial_i \ln a - \frac{\kappa}{\sqrt{2\tilde{\epsilon}}} \hat{e}_{1A} \partial_i \phi^A = \partial_i \ln a - \frac{H}{\dot{\rho}} \partial_i \rho$
- $\sigma_i \equiv \hat{e}^2_A \zeta^A_i = -\frac{\kappa}{\sqrt{2\tilde{\epsilon}}} \hat{e}_{2A} \partial_i \phi^A$

Define Slow Roll parameters:

$$\tilde{\epsilon} \equiv \frac{\kappa^2}{2} \frac{\Pi^2}{H^2}, \quad \tilde{\eta}^A \equiv -\frac{3H\Pi^A + \partial^A V}{H\Pi}, \quad \tilde{\xi}^A \equiv 3\tilde{\epsilon}\, \hat{e}^{1A} - 3\tilde{\eta}^A - \frac{\hat{e}^{1B} V^A_B}{H^2}$$

Project isocurvature and adiabatic parts:

$$\tilde{\eta}^{\parallel} \equiv \hat{e}^1_A \tilde{\eta}^A, \quad \tilde{\eta}^{\perp} \equiv \hat{e}^2_A \tilde{\eta}^A, \quad \tilde{\xi}^{\parallel} \equiv \hat{e}^1_A \tilde{\xi}^A, \quad \tilde{\xi}^{\perp} \equiv \hat{e}^2_A \tilde{\xi}^A$$

The non-linear equations of motion are formally the same as those of linear perturbation theory with

$$\delta \rightarrow \partial_i$$
$$f(t) \rightarrow f(t, \mathbf{x})$$