Title: Einstein geometry and conformal field theory

Date: Nov 28, 2006 11:00 AM

URL: http://pirsa.org/06110036

Abstract: TBA

Einstein geometry and conformal field theory

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Based on work with J. Gauntlett, D. Martelli, S.-T. Yau

AdS/CFT correspondence:

Type IIB string theory on $AdS_5 \times L$ with N units of G_5 flux	$\Longleftrightarrow$	$d = 4$, $\mathcal{N} = 1$, superconformal field theory (SCFT)

Here (L, g_L) is a five-dimensional Sasaki-Einstein manifold.

Definition: A Riemannian manifold (L, g_L) is

- Sasakian iff its metric cone $(X_0 = \mathbb{R}_+ \times L, g = \mathrm{d}r^2 + r^2 g_L)$ is Kähler
- Sasaki-Einstein iff the cone is also Ricci-flat

Examples:

- $(X = \mathbb{C}^3, g = \text{flat metric}) \Longleftrightarrow \mathcal{N} = 4\ SU(N)\ SYM$
- $(X, g) = \text{conifold} \Longleftrightarrow SU(N) \times SU(N)$ Klebanov-Witten theory

It is remarkable that until 2004, these were essentially the **only** two examples where both sides of the correspondence were known explicitly.

Theorem (Gauntlett, Martelli, JFS, Waldram): $\exists$ infinitely many Sasaki-Einstein metrics $Y^{p,q}$ on $S^2 \times S^3$, labelled by $p, q \in \mathbb{N}$, hcf$(p,q) = 1$, $q < p$.

The metrics are completely explicit, cohomogeneity one under the isometric action of a Lie group with Lie algebra $\mathfrak{su}(2) \times \mathfrak{u}(1) \times \mathfrak{u}(1)$.

$$\frac{\text{vol}[Y^{p,q}]}{\pi^3} = \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}$$

Dual SCFTs: (Benvenuti, Franco, Hanany, Martelli, JFS): $SU(N)^{2p}$ quiver gauge theories (Moose theories), q determines the quiver and superpotential (interactions).

An important check on this duality is a-maximisation (Intriligator, Wecht).

The $\mathcal{N} = 1$ superconformal algebra contains $so(4,2) \times u(1)_R$.

The R-symmetry satisfies:

- conserved
- by definition, superpotential has R-charge 2

The exact R-symmetry may be computed by locally maximising

$$a(R) = \frac{3}{32}\left(3\mathrm{tr}R^3 - \mathrm{tr}R\right)$$

over all R satisfying the above constraints.

$a(R_*)$ at the critical point is the a central charge:

$$< T^\mu_\mu >= \frac{1}{120\ (4\pi)^2}\left(c(\text{Weyl})^2 - \frac{a}{4}(\text{Euler})\right)$$

Cardy: a believed to count massless degrees of freedom.

$a_{IR} < a_{UV}$ for any RG flow.

AdS/CFT (Henningson-Skenderis):

$$\frac{a}{a_{\mathcal{N}=4\ \text{SYM}}} = \frac{\text{vol}[S^5]}{\text{vol}[L, g_L]}$$

For $Y^{p,q}$ theories, this agrees with the earlier formula!

$$\frac{32a(R_1, R_2)}{9N^2} =$$
$$2p + (p-q)(R_1 - 1)^3 + (p+q)(R_2 - 1)^3 - \frac{p}{4}(R_1 + R_2)^3 + \frac{q}{4}(R_1 - R_2)^3$$

Questions:

- Geometrically, how do we determine a volume without solving the Einstein equations?
- a-maximisation implies that these volumes are always algebraic numbers. Why?

Rest of talk:

- The answers to these questions
- "Calabi-Yau's" X that do not admit Ricci-flat Kähler cone metrics $\Longleftrightarrow$ SQFTs that do not flow to dual IR fixed points

In particular, the second point disproves some claims made by (Cachazo, Fiol, Intriligator, Katz, Vafa) and (Gukov, Vafa, Witten).

Sasakian geometry

Definition: A Riemannian manifold (L, g_L) is Sasakian iff its metric cone $(X_0 = \mathbb{R}_+ \times L, g = \mathrm{d}r^2 + r^2 g_L)$ is Kähler

In particular X_0 is a complex manifold; metric

$$g = \frac{\partial^2 r^2}{\partial z_i \partial \bar{z}_j} \mathrm{d}z_i \mathrm{d}\bar{z}_j$$

$\exists$ complex structure tensor J with

$$\begin{aligned} J\left(\frac{\partial}{\partial z_i}\right) &= i\frac{\partial}{\partial z_i} \\ J\left(\frac{\partial}{\partial \bar{z}_i}\right) &= -i\frac{\partial}{\partial \bar{z}_i} \end{aligned}$$

Then a calculation shows that

$$\xi = J\left(r\frac{\partial}{\partial r}\right)$$

is a holomorphic Killing vector field (Reeb vector field).

This is dual to the R-symmetry in the SCFT.

In the SCFT, we had an optimisation problem for the R-symmetry, that determines the central charge at the critical point.

Idea: try to do the same in the geometry.

For simplicity, I'll focus on toric geometry here, since then I can draw 3d pictures.

We always have at least a holomorphic $U(1) = \mathbb{T}^1$ isometry for a Kähler cone (X_0, g). If ξ is to move, that means we have at least a $\mathbb{T}^2$.

Let's assume we have $\mathbb{T}^n$, where $n = \dim_{\mathbb{C}} X_0$.

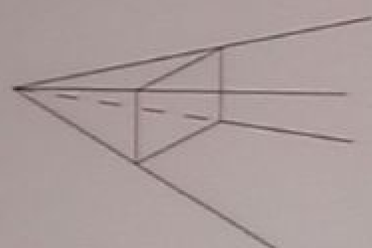
Let ϕ_i, $\phi_i \sim \phi_i + 2\pi$ $(i = 1, \ldots, n)$, be angular coordinates on $\mathbb{T}^n$.

Define

$$y^i = \frac{1}{2} g\left(\xi, \frac{\partial}{\partial \phi_i}\right)$$

We now have $2n$ real coordinates – enough to cover X_0.

In fact, X_0 is always a $\mathbb{T}^n$ fibration over a convex polyhedral cone $C^* \subset \mathbb{R}^n$:



$n = 3$, $d = 4$

Concretely, there are primitive vectors $\vec{v}_a \in \mathbb{Z}^n$, $a = 1, \ldots, d$, such that

$$C^* = \{\vec{y} \in \mathbb{R}^n \mid \vec{y} \cdot \vec{v}_a \geq 0, a = 1, \ldots, d\}$$

Above every point in the interior C^*_{int} of the cone, there is a copy of $\mathbb{T}^n$.

At each bounding face of the cone, one of the circles in $\mathbb{T}^n$ collapses, leaving $\mathbb{T}^{n-1} = \mathbb{T}^n / S^1$ fibred over the face.

Circle subgroup $S^1 \subset \mathbb{T}^n$ is specified by a charge vector $\vec{v} \in \mathbb{Z}^n$.

The normal vector $\vec{v}_a \in \mathbb{Z}^n$ to the ath bounding face specifies which S^1 collapses.

Examples:

- Think of $\mathbb{C} = \mathbb{R}^2$ in polar coordinates. This is S^1 fibred over $\mathbb{R}_+$, with S^1 collapsing at the origin
- Similarly, $\mathbb{C}^n$ is $\mathbb{T}^n$ fibred over $C^* = (\mathbb{R}_+)^n$ = positive orthant:

$$g_{\text{flat}} = \sum_{i=1}^{n} \mathrm{d}\rho_i^2 + \rho_i^2 \mathrm{d}\phi_i^2$$

where $y^i = \frac{1}{2}\rho_i^2 \geq 0$.

We may write

$$\xi = \sum_{i=1}^{n} b_i \frac{\partial}{\partial \phi_i}$$

where one can show that

$$\vec{b} \in \mathcal{C} = \{\vec{b} \in \mathbb{R}^n \mid \vec{b} \cdot \vec{y} \geq 0, \forall \vec{y} \in \mathcal{C}^*\}$$

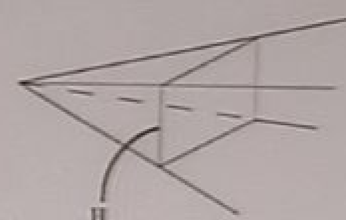
Dual cone to $\mathcal{C}^*$, a convex rational polyhedral cone by Farkas' Theorem.

Remember that $y^i = \frac{1}{2} g(\xi, \partial/\partial\phi_i)$. Contracting with b_i gives

$$\vec{b} \cdot \vec{y} = \frac{1}{2} g(\xi, \xi) = \frac{1}{2} r^2$$

so that the link $L = \{r = 1\}$ is $\mathbb{T}^n$ fibred over the intersection of $\mathcal{C}^*$ with the hyperplane

$\vec{b} \cdot \vec{y} = \frac{1}{2}$



We also must impose that X_0 is "Calabi-Yau": $c_1(X_0) = 0$.

It turns out this is equivalent to the existence of a basis for $\mathbb{T}^n$ in which

$$\vec{v}_a = (1, \vec{w}_a)$$

for some $\vec{w}_a \in \mathbb{Z}^{n-1}$.

This also means $\exists$ a nowhere zero holomorphic $(n, 0)$-form Ω.

Extremal problem: Einstein metrics g_L on L are critical points of

$$S[L, g_L] = \int_L [s(g_L) + 2(n-1)(3-2n)]\, \mathrm{d}\mu$$

$s(g_L) =$ Ricci scalar of g_L.

Amazing fact: for Sasakian metrics, the Einstein-Hilbert action depends only on the Reeb vector field $\xi = b_i \partial/\partial\phi_i$.

Reason: remember the metric is

$$g = \frac{\partial^2 r^2}{\partial z_i \partial \bar{z}_j} \mathrm{d}z_i \mathrm{d}\bar{z}_j$$

Changing $r^2 \to r^2 \exp(\varphi)$ changes the metric.

If $\mathcal{L}_{r\partial/\partial r}\varphi = 0 = \mathcal{L}_\xi \varphi$, then $r\partial/\partial r$ and ξ invariant.

$S[L, g_L]$ is invariant under the above change of metric, by explicit calculation.

Extremal problem: Einstein metrics g_L on L are critical points of

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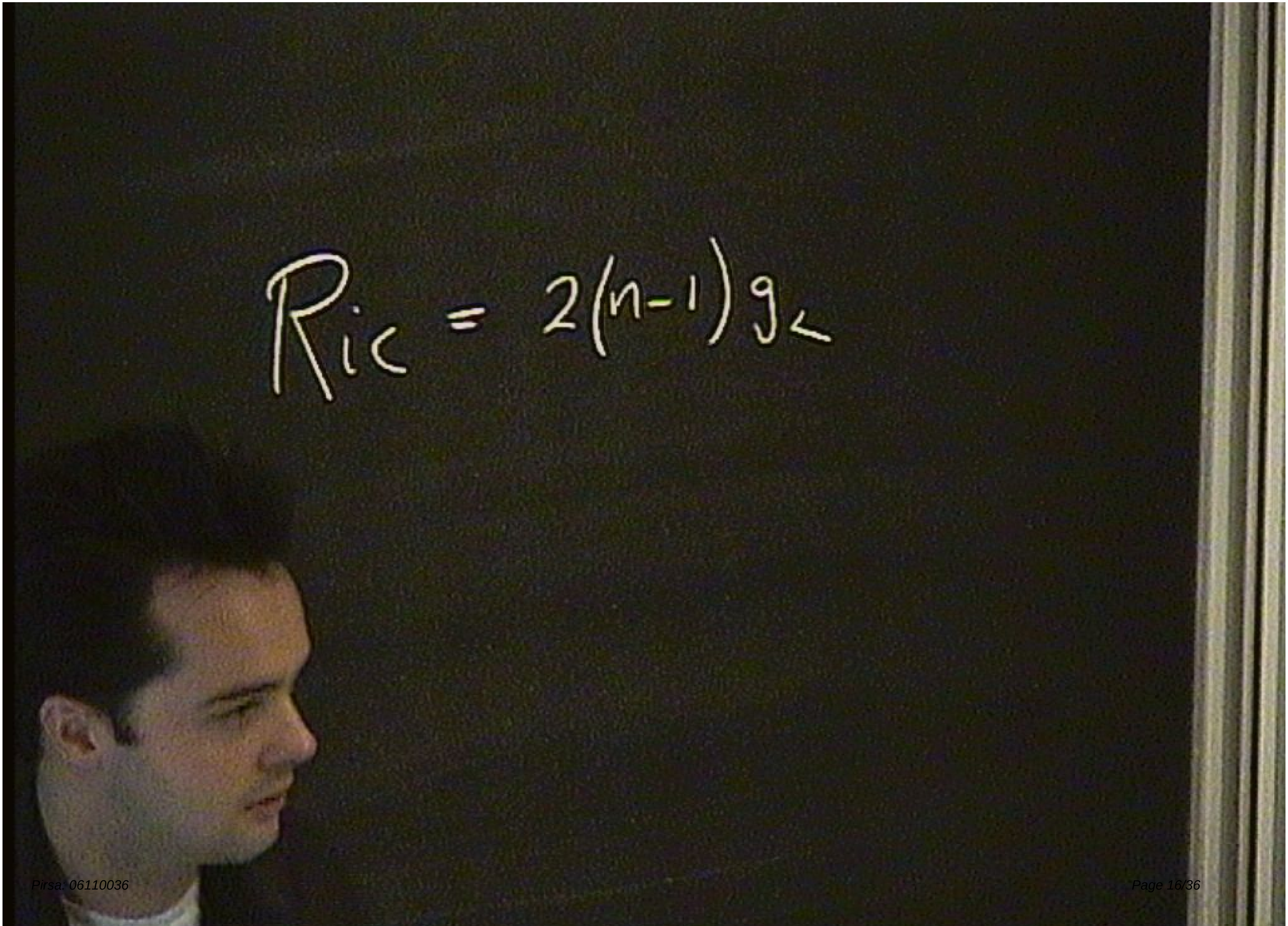
$S[L, g_L]$ is invariant under the above change of metric, by explicit calculation.


$Ric = 2(n-1)g$

Extremal problem: Einstein metrics g_L on L are critical points of

$$S[L, g_L] = \int_L [s(g_L) + 2(n-1)(3-2n)]\, \mathrm{d}\mu$$

$s(g_L) =$ Ricci scalar of g_L.

Amazing fact: for Sasakian metrics, the Einstein-Hilbert action depends only on the Reeb vector field $\xi = b_i \partial/\partial\phi_i$.

Reason: remember the metric is

$$g = \frac{\partial^2 r^2}{\partial z_i \partial \bar{z}_j} \mathrm{d}z_i \mathrm{d}\bar{z}_j$$

Changing $r^2 \to r^2 \exp(\varphi)$ changes the metric.

If $\mathcal{L}_{r\partial/\partial r}\varphi = 0 = \mathcal{L}_\xi \varphi$, then $r\partial/\partial r$ and ξ invariant.

$S[L, g_L]$ is invariant under the above change of metric, by explicit calculation.

A computation gives

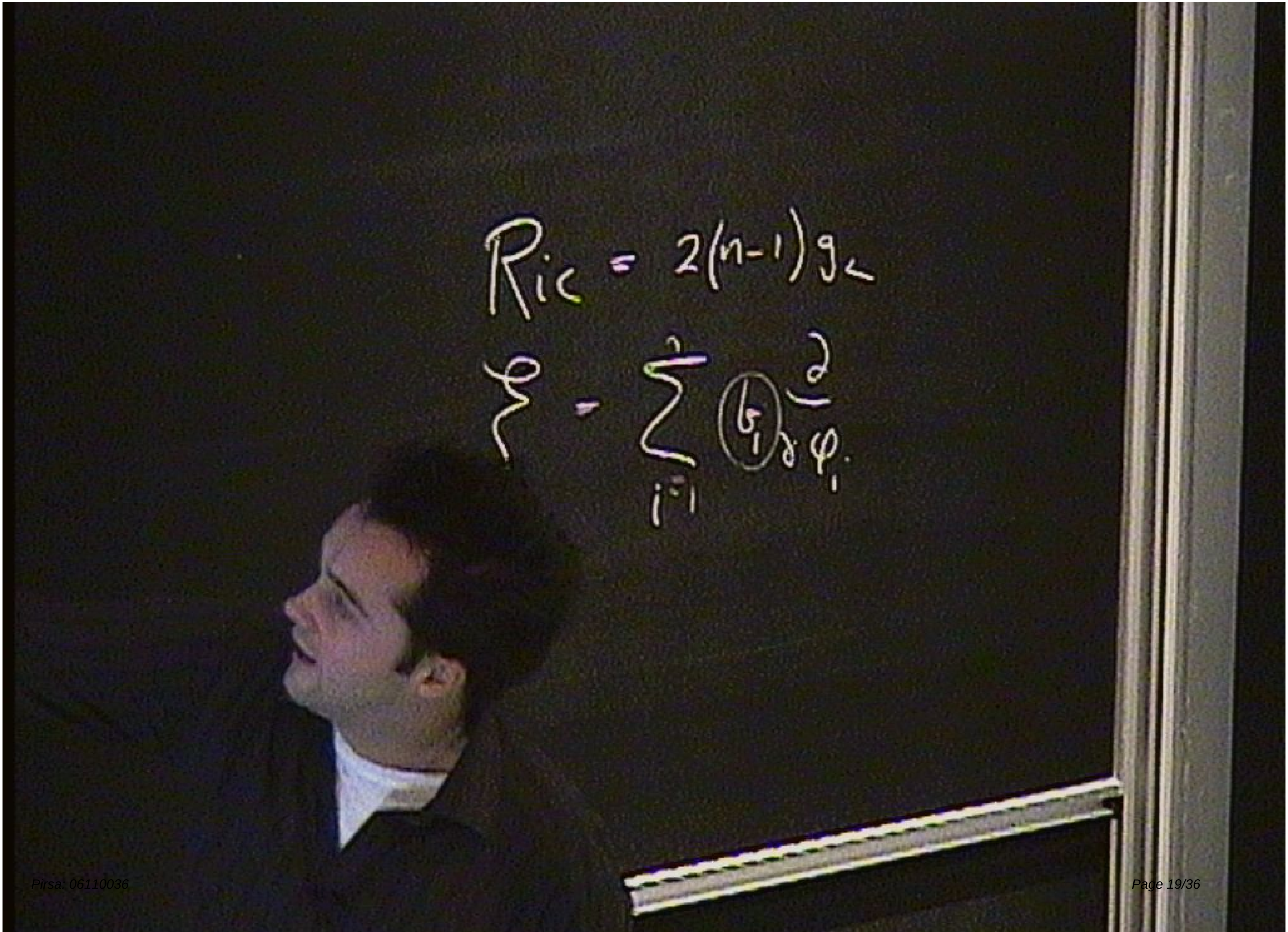
$$S[L, g_L] = 8n(n-1)(2\pi)^n[b_1 - (n-1)]\mathrm{vol}[\mathcal{P}(\vec{b})]$$

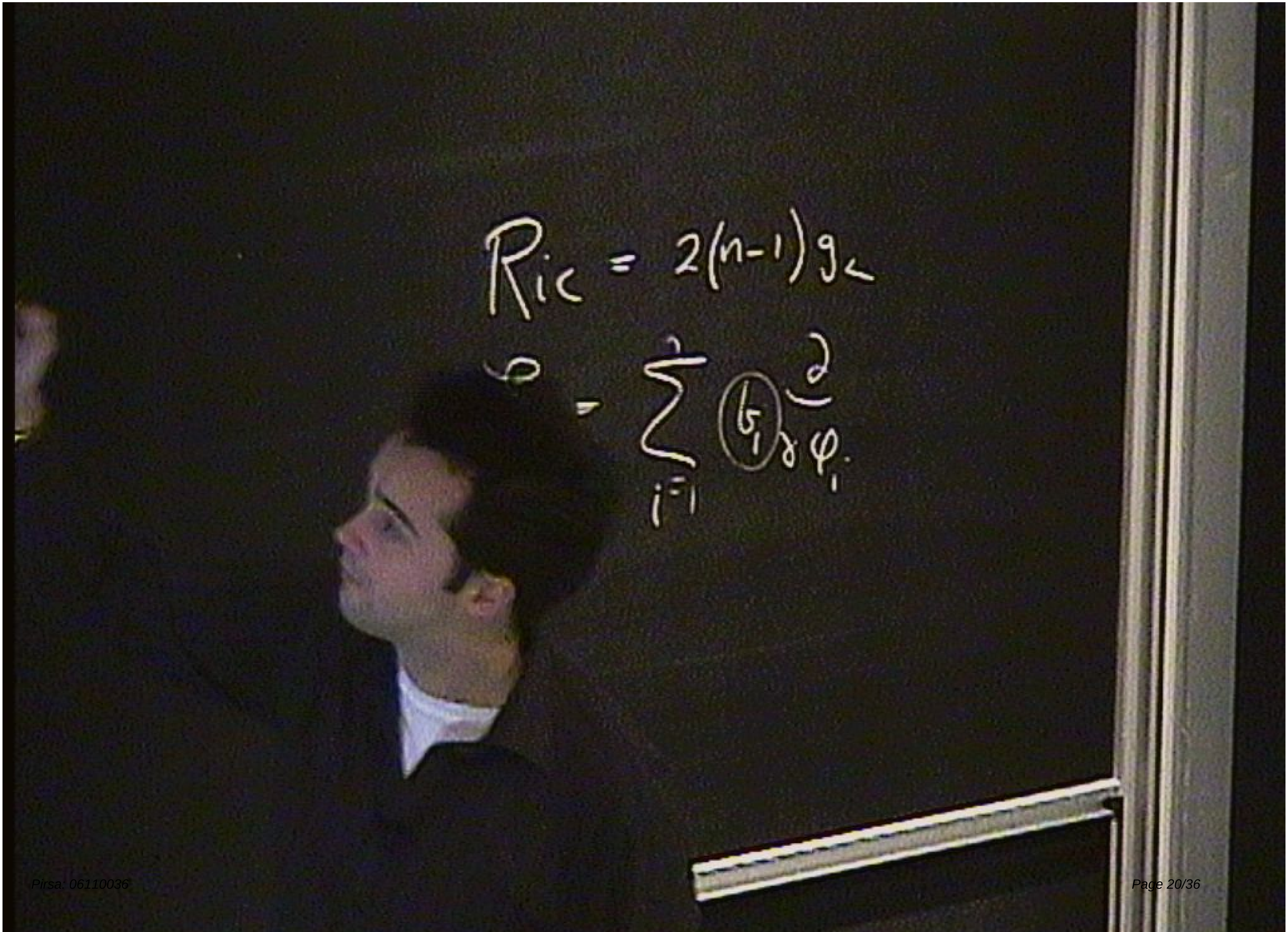
where $\mathrm{vol}[\mathcal{P}(\vec{b})]$ is the Euclidean volume of the finite polytope formed by $\mathcal{C}^*$ and $H_{\vec{b}}$.

The first component b_1 is singled out by the Calabi-Yau condition $\vec{v}_a = (1, \vec{w_a})$.

$$b_i \frac{\partial}{\partial b_i} S = 0 \rightarrow b_1 = n$$

Same as saying $\mathcal{L}_\xi \Omega = in\Omega$, or $\Omega \wedge \bar{\Omega} \sim r^{2n}$.

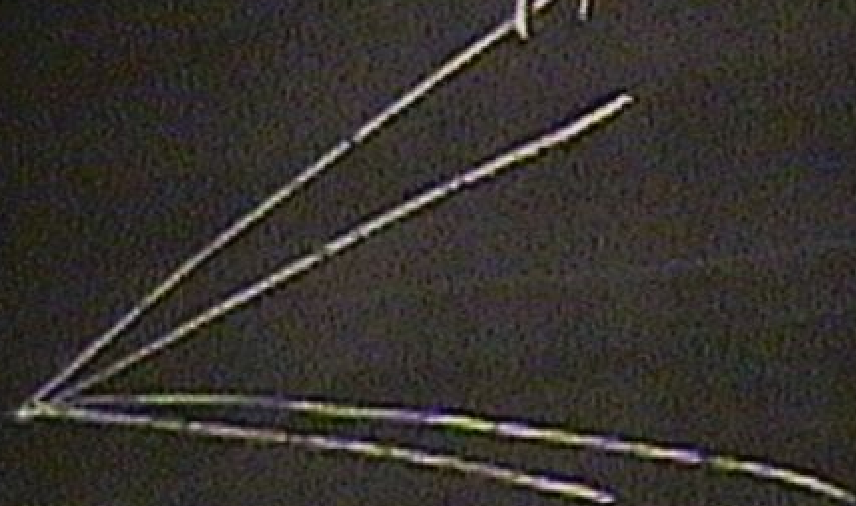

Ric = 2(n-1) g
ξ = Σ_{i=1}^{2} b_i ∂/∂φ_i

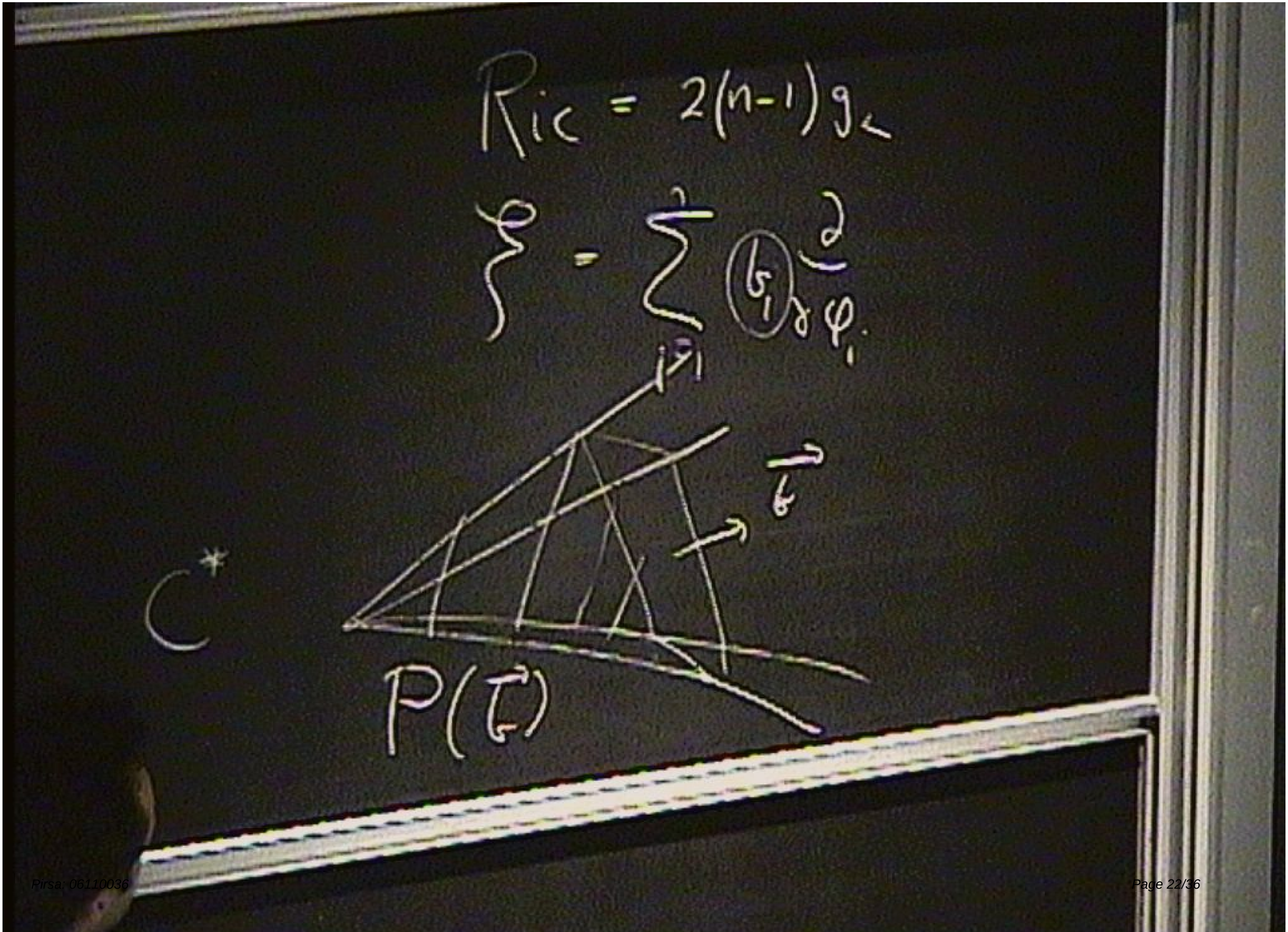

Ric = 2(n-1)g

$$\mathrm{Ric} = 2(n-1)\, g_c$$

$$\xi = \sum_{i=1}^{n} b_i \frac{\partial}{\partial \varphi_i}$$

C^*




Ric = 2(n-1) g
ξ = Σ b_i ∂/∂φ_i
C*
P(b)

A computation gives

$$S[L, g_L] = 8n(n-1)(2\pi)^n[b_1 - (n-1)]\text{vol}[\mathcal{P}(\vec{b})]$$

where vol$[\mathcal{P}(\vec{b})]$ is the Euclidean volume of the finite polytope formed by C^* and $H_{\vec{b}}$.

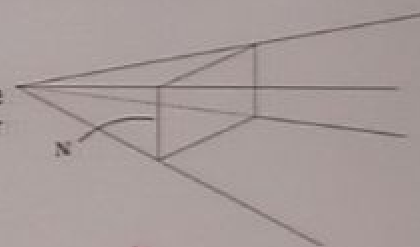
The first component b_1 is singled out by the Calabi-Yau condition $\vec{v}_a = (1, \vec{w_a})$.

$$b_i \frac{\partial}{\partial b_i} S = 0 \rightarrow b_1 = n$$

Same as saying $\mathcal{L}_\xi \Omega = in\Omega$, or $\Omega \wedge \bar{\Omega} \sim r^{2n}$.

Existence and uniqueness of an extremum:

$b_1 = n$ defines a polytope N in $\mathcal{C}$ (space of $\vec{b}$), rather than $\mathcal{C}^*$.



Set $V(\vec{b}) = \mathrm{vol}[\mathcal{P}(\vec{b})]$. Then

$$\frac{\partial V}{\partial b_i} = \frac{1}{2|\vec{b}|}\int_{H_\xi} y^i \mathrm{d}\sigma$$
$$\frac{\partial^2 V}{\partial b_i \partial b_j} = \frac{2(n+1)}{|\vec{b}|}\int_{H_\xi} y^i y^j \mathrm{d}\sigma$$

This shows that $V(\vec{b})$ is strictly convex on $\mathcal{C}$.

It is bounded below, and diverges to $+\infty$ on $\partial\mathcal{C}$ [Why?: Because $\xi \to 0$ somewhere on X_0 as ξ approaches the boundary of $\mathcal{C}$]

So there exists a unique minimum on N.

One can write a real Monge-Ampère equation on $\mathcal{C}^*$, equivalent to the Ricci-flat Kähler condition (Martelli, JFS, Yau). Recently solved, by Futaki, Ono, Wang.

Example: complex dimension $n = 3$:

Order the normals $v_1, v_2, \ldots, v_d, v_{d+1} \equiv v_1$ around the polyhedral cone.

Using GCSE maths:

$$V(\vec{b}) = \frac{1}{48 b_1} \sum_{a=1}^{d} \frac{(\vec{v}_{a-1}, \vec{v}_a, \vec{v}_{a+1})}{(\vec{b}, \vec{v}_{a-1}, \vec{v}_a)(\vec{b}, \vec{v}_a, \vec{v}_{a+1})}$$

volume of a 3d polytope, where $(\cdot,\cdot,\cdot)$ denotes a 3×3 determinant.

The toric data for the $Y^{p,q}$ singularities is $\vec{v}_1 = [1,0,0]$, $\vec{v}_2 = [1,1,0]$, $\vec{v}_3 = [1,p,p]$, $\vec{v}_4 = [1, p-q-1, p-q]$ (Martelli, JFS).

One finds the Einstein-Hilbert action

$$\frac{2S(\vec{b})}{3(2\pi)^3} = \frac{(b_1-2)p[p(p-q)b_1 + q(p-q)b_2 + q(2-p+q)b_3]}{b_3[pb_1 - pb_2 + (p-1)b_3)((p-q)b_2 + (1-p+q)b_3)(pb_1 + qb_2 - (q+1)b_3]}$$

Extremising gives $\bar{b}_*$ with volume

$$\mathrm{vol}[Y^{p,q}] = 6(2\pi)^3 V(\bar{b}_*) = \frac{q^2[2p + (4p^2-3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2-3q^2)^{1/2}]}\pi^3$$

Much of what I described generalises (Martelli, JFS, Yau). However, one needs to take a different approach to calculate the volume.

Localisation

Write

$$\mathrm{vol}[L, g_L] = \frac{1}{2^{n-1}(n-1)!}\int_X e^{-r^2/2}\frac{\omega^n}{n!}$$

where

$$\omega = \frac{i}{2}\frac{\partial^2 r^2}{\partial z_i \partial \bar{z}_j}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$$

is the Kähler form.

Then $H = r^2/2$ is the Hamiltonian function for ξ: $\mathrm{d}H = -\xi \lrcorner \omega$.

This looks like a classical partition function, with phase space (X, ω).

It is, for a BPS D3-brane wrapping the $S^3 \subset \mathrm{AdS}_5$ (Martelli, JFS).

Duistermaat-Heckman formula says this localises where $\xi = 0$.

But $\|\xi\|^2 = r^2$, so this is the singular point of the Calabi-Yau cone X

$\rightarrow$ must (partially) resolve the singularity.

upshot: rational function of ξ, with rational coefficients.

Unique critical point $\rightarrow \xi = \sum_{i=1}^{n} b_i \partial / \partial \phi_i$ with $\vec{b}$ an algebraic vector.

Technical slide:

Let $\pi : W \to X$ be a $\mathbb{T}^*$-equivariant partial resolution of X, exceptional set E.

$W \setminus E \cong X_0$ equivariant biholomorphism.

Note fixed point set is entirely in E. Then

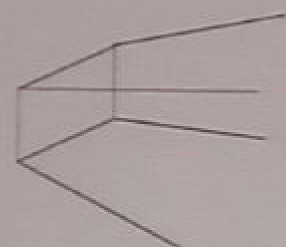
$$\frac{\mathrm{vol}[L, g_L]}{\mathrm{vol}[S^{2n-1}]} = \sum_{\{F\}} \frac{1}{d_F} \int_F \prod_{m=1}^{R} \frac{1}{(\xi, u_m)^{n_m}} \left[\sum_{a \geq 0} \frac{c_a(\mathcal{E}_m)}{(\xi, u_m)^a} \right]^{-1}$$

- $E \supset \{F\} =$ set of connected components of the fixed point set of generic $\xi \in \mathfrak{t}$.
- For fixed F, normal bundle $\mathcal{E}$ in W splits $\mathcal{E} = \bigoplus_{m=1}^{R} \mathcal{E}_m$ where rank $\mathcal{E}_m = n_m$ and $\sum_{m=1}^{R} n_m = \mathrm{rank}(\mathcal{E})$
- Splitting determined by linearised $\mathbb{T}^*$ action on $\mathcal{E}$: weights, $u_1, \ldots, u_R \in \mathbb{Q}^s \subset \mathfrak{t}_s^*$.
- $c_a(\mathcal{E}_m)$ are the Chern classes of $\mathcal{E}_m$.
- When W has orbifold singularities, normal fibre to a generic point on F is not a complex vector space, but rather an orbifold $\mathbb{C}^k/\Gamma$. Then $\mathcal{E}$ is more generally an orbibundle. $d_F = |\Gamma|$ is the order of Γ.

For our toric pictures, this different formula works as follows.

Chop the polyhedral cone C^* with enough rational hyperplanes so that every vertex of the resulting non-compact polytope P satisfies:

- precisely n edges meet at the vertex
- if $\vec{u}_i^A \in \mathbb{Z}^n$ denotes the n outward-pointing primitive edges at vertex A, then these span $\mathbb{Z}^n$ over $\mathbb{Z}$



4-sided cone C^* in dimension $n = 3$ cut with a single hyperplane. $A = 1, 2, 3, 4$.

This can always be done.

Then (cf. the topological string)

$$\frac{\text{vol}[L, g_L]}{\text{vol}[S^{2n-1}]} = \sum_{A \in P} \prod_{i=1}^{n} \frac{1}{\vec{b} \cdot \vec{u}_i^A}$$

Obstructions: (Gauntlett, Martelli, JFS, Yau)

Let (X, Ω) be a compact Calabi-Yau manifold, $\Omega =$ nowhere zero holomorphic $(n, 0)$-form.

Remember, this means that X is complex, admits a Kähler metric, and has $c_1(X) = 0$.

Yau's theorem: such an X always admits a unique Ricci-flat Kähler metric in a given Kähler class $[\omega] \in H^{1,1}(X, \mathbb{R})$.

For non-compact manifolds, this theorem can fail.

For cones, this is related to the IR behaviour of geometrically engineered $\mathcal{N} = 1$ QFTs at the singularity.

Let (L, g_L) be an Einstein manifold with

$$\text{Ric}(g_L) = (2n-2)g_L$$

Then

Bishop's Theorem: $\text{vol}[L, g_L] \leq \text{vol}[S^{2n-1}]$

Lichnerowicz's Theorem: The smallest positive eigenvalue E_1 of $\Delta_L =$ scalar Laplacian is bounded from below by $E_1 \geq 2n-1$, with equality iff (L, g_L) is the round sphere.

Recall $\Delta_L = -\nabla^\mu \nabla_\mu$.

Lichnerowicz: Let f be a holomorphic function on $X_0 = \mathbb{R}_+ \times L$, and an eigenfunction of $\mathcal{L}_\xi$:

- $\partial f / \partial \bar{z}_i = 0$
- $\mathcal{L}_\xi f = i\lambda f$, with $\lambda > 0$

Then

$$f = r^\lambda \tilde{f}$$

with $\tilde{f}$ a function on L and

$$\Delta_L \tilde{f} = E \tilde{f}$$

with $E = \lambda(\lambda + (2n-2))$.

Thus Lichnerowicz requires $\lambda \geq 1$.

Idea: both vol$[L, g_L]$ and holomorphic spectrum $\{\lambda\}$ are holomorphic invariants of X_0, for fixed ξ.

If vol$[L, g_L] >$ vol$[S^{2n-1}]$, or $\exists \lambda < 1$, then contradiction.

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Physics: very simple

Lichnerowicz

Suppose f is an eigenfunction of Δ_L with eigenvalue $E = \lambda(\lambda + 4)$.

There is an associated massive Kaluza-Klein state in AdS_5.

By AdS/CFT, this is dual to a scalar chiral primary operator $\mathcal{O}$ in the dual SCFT.

It has conformal dimension $\Delta(\mathcal{O}) = \lambda$.

Unitarity bound: $\Delta(\mathcal{O}) \geq 1$.

So Lichnerowicz bound = unitarity bound.

Bishop

By giving vevs and integrating out massive fields $\longrightarrow \mathcal{N} =$ 4 SYM.

Moves N D-branes to a smooth point of X.

By earlier remarks, a should decrease under this process.

So

$$a_{\mathcal{N}=4\ \mathrm{SYM}} \leq a_{\mathrm{Sasaki-Einstein}}$$

which is Bishop.

So Bishop $\Leftarrow$ a-theorem and intuitions about D-branes

Nice set of examples: ADE singularities

Define polynomials

$$
\begin{aligned}
H &= z_1^k + z_2^2 + z_3^2 && A_{k-1} \\
H &= z_1^k + z_1 z_2^2 + z_3^2 && D_{k+1} \\
H &= z_1^3 + z_2^4 + z_3^2 && E_6 \\
H &= z_1^3 + z_1 z_2^3 + z_3^2 && E_7 \\
H &= z_1^3 + z_2^5 + z_3^2 && E_8
\end{aligned}
$$

and

$$F = H + \sum_{i=4}^{n+1} z_i^2$$

Then set

$$X = \{F = 0\} \subset \mathbb{C}^{n+1}$$

Claim: for $n \geq 2$ these are Calabi-Yau singularities with isolated singularity at $z_1 = \ldots = z_{n+1} = 0$.

A_k 3-folds: For $k = 2p$ even, (Cachazo, Fiol, Intriligator, Katz, Vafa) constructed a family of $\mathcal{N} = 1$ SQFTs on D3-branes at the A_{2p} 3-fold singularities.

Their vacuum moduli spaces are precisely the A_{2p} 3-fold singularities.

a-maximisation gives a central charge that satisfies

$$\frac{a}{a_{\mathcal{N}=4 \text{ SYM}}} = \frac{\text{vol}[S^5]}{\text{vol}[L^k]}$$

assuming that the Sasaki-Einstein metric exists.

But it doesn't exist: all $k > 3$ violate Lichnerowicz's theorem. $k = 3$ recently ruled out by a different argument (Conti).

Moral: metrics don't always exist, and this reflects physics.